POISSON BI-VECTORS IN FOUR DIMENSIONS

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Mehmet Dinç

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APPROVED BY:

Prof. Dr. Ender Abadoğlu . . . .
(Thesis Supervisor)
(Yeditepe University)

Assoc. Prof. Dr. Nurettin Cenk Turgay . . . .
(Istanbul Technical Univesity)

Assist. Prof. Dr. Barış Efe . . . .
(Yeditepe University)

Assist. Prof. Dr. Cihangir Özemir . . . .
(Istanbul Technical Univesity)

Assist. Prof. Dr. İlker Savaş Yüce . . . .
(Yeditepe University)

DATE OF APPROVAL: . . . ./. . . ./2021

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ABSTRACT

POISSON BI-VECTORS IN FOUR DIMENSIONS

In this work, it is shown that a non-vanishing vector field on a four dimensional manifold is locally bi-Hamiltonian. We construct a method for computing the symplectic structures which make the given non-vanishing vector field Hamiltonian. The method uses solely the knowledge of the vector field but not its integral curves. Therefore the method proposes a way of constructing a bi-Hamiltonian structure for the dynamical system defined by the given vector field without requiring the solution of the dynamical system.

ÖZET

DÖRT BOYUTTAKİ POİSSON Bİ-VEKTÖRLER

Bu çalışmada dört boyutlu bir manifold üzerinde sıfır değeri almayan her vektör alanının yerel olarak bi-Hamiltonyen olduğu gösterilmiştir. Sözkonusu vektör alanını Hamiltonyen vektör alanları haline getiren iki simplektik yapının nasıl hesaplanacağına dair bir yöntem geliştirilimiştir. Bu yöntem sadece vektör alanının bilgisini gerektirmekte ancak integral eğrilerinin bilgisine ihtiyaç duymamaktadır. Dolayısıyla yöntem verili vektör alanının tanımladığı dinamik sistem için bir bi-Hamiltonyen yapının inşasına dair bir yöntem önermekte ancak dinamik sistemin çözümünün bilgisine ihtiyaç duymamaktadır.

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LIST OF SYMBOLS/ABBREVIATIONS

df	:	Directional derivative of f
$\Rightarrow$	:	To imply
$\Longleftrightarrow$	:	If and only if
$i_X f$	:	Interior derivative of f in the direction of X vector field
δ^i_j	:	Kronecker delta, $\delta^i_j = 1$ if $i = j$ and $\delta^i_j = 0$ if $i \neq j$
$\mathcal{L}_X f$	:	Lie derivative of f in the direction of X vector field
$\wedge$	:	Wedge product
ϵ_{ijk} or ϵ^{ijk}	:	Permutation $\epsilon_{ijk} = \begin{pmatrix} 1 & 2 & 3 \\ \mathrm{i} & \mathrm{j} & \mathrm{k} \end{pmatrix}$ and $\epsilon^{ijk} = \begin{pmatrix} 1 & 2 & 3 \\ \mathrm{i} & \mathrm{j} & \mathrm{k} \end{pmatrix}$
$\sharp$	:	Sharp operator
$\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$	:	Tensor product of $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$

1. INTRODUCTION

Poisson structures are special types of geometric structures that appears in various parts of mathematical physics, e.g. plasma physics [11], fluid dynamics [10], field theory [12], population dynamics [13], optics [14], electromagnetism [15] and mechanics [1]. Aside from their importance in a purely geometric perspective Poisson structures were investigated systematically also for investigating the dynamics produced by Hamiltonian vector fields defined with respect to a Poisson structure. In other words, Poisson structures are quite important in Hamiltonianization of a vector field and investigating the invariants of the flow generated by the vector field in question.

The usual approach is to start with a given Poisson manifold and then investigate Hamiltonian vector fields that can be defined by the Poisson structure defined on the manifold. The perspective of this thesis is quite a bit different from the usual approach. Instead of starting with a Poisson manifold, namely with a specific Poisson structure, we rather start with a manifold and a non-zero vector field on it. Or equivalently we may consider a vector field on an arbitrary four manifold whose support is also a four dimensional manifold. Then our aim is to construct a Poisson structure on the manifold by using the vector field alone. This is called the Hamiltonianization of a vector field in four dimensions.

In [5] it is shown that in three dimensions it is always possible to construct two Poisson structures which Hamiltonianize a non-vanishing vector field locally, and the topological obstructions for the global existence is presented. In this work we extended the local existence theorem to four dimensions and proved that for a given nonvanishing vector field e_0 on a four dimensional manifold, it is possible find two independent Poisson structures J_1 and J_2 which make this vector field bi-Hamiltonian. Our strategy is as follows: First we are going to write down equations determining a Poisson structure in four dimensions. We explicitly determine these equations in vector form as well as in terms of differential forms as in [6]. Then assuming nondegeneracy, it is shown that these equations amount to

$$i_{e_0}J \wedge di_{e_0}J = 0 \tag{1.1}$$

which gives rise to a Ricatti equation in a certain local frame $\{e_0, e_1, e_2, e_3\}$ constructed by the information provided by the vector field. To be able handle this equation we transform it into,

$$\alpha \wedge d\alpha = 0 \tag{1.2}$$

which is equivalent to

$$\mathcal{L}_{e_i}\alpha \wedge \alpha = 0 \tag{1.3}$$

for i=0,1. Then we are going to demonstrate that these equations have two independent solutions in α's and also it will be shown that there are two different J's satisfying

$$i_{e_0}J = \alpha \tag{1.4}$$

and there are two functions f such that

$$d\left(\frac{J}{f}\right) = 0 \tag{1.5}$$

In this way, we construct a method to find how to make a nonvanishing vector field Hamiltonian in four dimensions, using merely existence of solutions without explicitly finding them.

2. PRELIMINARIES

For the following definitions, let M be a smooth manifold.

Definition 2.0.1. ***Symplectic form*** *is a nondegenerate two form J on M such that J is closed, that is,* $dJ = 0$*. By nondegeneracy, we mean that the matrix representing J is nonsingular. A smooth manifold endowed with symplectic form is called a* ***symplectic manifold*** *.*

Definition 2.0.2. *[8] Let* $C^{\infty}(M)$ *denote smooth real-valued functions on* M *where the multiplication is defined pointwise.*
A Poisson bracket (or Poisson structure) *on* M *is a bilinear map*

$$\{.,.\} : C^{\infty}(M) \times C^{\infty}(M) \to C^{\infty}(M)$$

satisfying the following conditions:
i. ***Skew symmetry***

$$\{f, g\} = -\{g, f\}$$

ii. ***Jacobi identity***

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

iii. ***Leibniz's Rule***

$$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

Definition 2.0.3. *[8] Schouten- Nijenhuis bracket is a type of graded Lie bracket defined on multivector fields on a smooth manifold extending the Lie bracket of vector fields.*

$$[.,.] : \mathfrak{X}^{p}(M) \times \mathfrak{X}^{q}(M) \to \mathfrak{X}^{p+q-1}(M)$$

$$(v, \xi) \to [v, \xi]$$

which is defined as

$$[v, \xi] = v \circ \xi - (-1)^{(k-1)(l-1)} \xi \circ v \tag{2.1}$$

where

$$\xi \circ v\left(df_1, df_2, ..., df_{k+l-1}\right) := \sum_{\sigma} sgn\sigma \bar{\xi}\left(\bar{v}\left(f_{\sigma(1)}, ..., f_{\sigma(k)}\right), f_{\sigma(k+1)}, ..., f_{\sigma(k+l-1)}\right)$$

such that σ *runs over* $(k, l-1)$ *−shuffles and*

$$\bar{v}\left(f_1, ..., f_k\right) = v\left(df_1, ..., df_k\right)$$

and $\bar{\xi}$ *is also similar.*

Definition 2.0.4. *[8] Let* $(M, \{.,.\})$ *be a Poisson manifold. The Hamiltonian vector field of* $f \in C^\infty(M)$ *is the vector field* $X_f \in \mathfrak{X}(M)$ *defined by*

$$X_f(g) = \{g, f\} \qquad \text{for all } g \in C^\infty(M)$$

The function f *is called the Hamiltonian function.*

Definition 2.0.5. *Let* (M, θ) *be a symplectic manifold. For any smooth function* $f \in C^\infty(M)$*, the Hamiltonian vector field of f,* X_f *is defined by*

$$i_{X_f}\theta = df \quad \text{or equivalently} \quad \theta\left(X_f, X\right) = df\left(X\right) = Xf \tag{2.2}$$

for any smooth vector field X.

Last two definitions are equivalent if M is symplectic manifold. The following remark 3.1.5 reveals the equivalance.

Definition 2.0.6. *[9] A system of differential equations is called bi-Hamiltonian , if it can be written in Hamiltonian form in two distinct ways:*

$$v = J_1 \nabla H_1 = J_2 \nabla H_0$$

H_0*,* H_1 *are the two Hamiltonian functions, and* J_1*,* J_2 *are skew symmetric* nxn *Hamiltonian matrices, not constant multiples of each other, determining Poisson brackets. Entries of* J_i*'s are given* $J_i^{jk} = \left\{x^j, x^k\right\}_i$ *where* x^i *'s are coordinate functions and* $\{\}_i$ *is the poisson bracket.*

3. POISSON STRUCTURE ON A SMOOTH MANIFOLD

The first two conditions of the Poisson bracket, namely skew symmetry and Jacobi identity, ensure that $\{.,.\}$ defines a Lie algebra structure on $C^{\infty}(M)$. The third condition guarantees that for each $f \in C^{\infty}(M)$, the map

$$\{.\,,f\} : C^{\infty}(M) \to C^{\infty}(M)$$

is a derivation of the commutative product on $C^{\infty}(M)$, i.e, is a vector field X_f. Using properties above, we may think of the bracket as defining a contravariant tensor π as

$$\{f, g\} = \pi(df, dg)$$

where $\pi \in \Gamma(\Lambda^2 TM)$ is a smooth bi-vector field. Conversely, given any smooth bi-vector field π on M the formula

$$\{f, g\} = \pi(df, dg)$$

defines a bilinear skew-symmetric bracket $\{.,.\}$ that obeys the Leibniz's rule. To ensure that $\{.,.\}$ is a Poisson bracket, Jacobi identity must hold. Jacobi identity can be characterized by the partial differential equation

$$[\pi, \pi] = 0$$

where $[.,.]$ is Shouten- Nijenhuis bracket. Poisson structure on a smooth manifold M is defined in some local coordinates $(x^a) \in M$ by a skew symmetric contravariant tensor π^{ab} which forms the components of a bi-vector

$$\pi = \frac{1}{2}\pi^{ab}\frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b}; a, b = 0, 1, ..., n-1$$

satisfying

$$[\pi, \pi] = \pi^{da}\frac{\partial \pi^{bc}}{\partial x^d}\frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b} \wedge \frac{\partial}{\partial x^c} = 0$$

where the bracket is the Schouten- Nijenhuis bracket of algebra of multi-vectors.

Lemma 3.0.1. *If*

$$\pi = \frac{1}{2}\pi^{ab}\frac{\partial}{\partial x^a}\wedge\frac{\partial}{\partial x^b};\, a,b = 0,1,......,n-1$$

then Jacobi identity corresponds to

$$[\pi,\pi] = \pi^{da}\frac{\partial \pi^{bc}}{\partial x^d}\frac{\partial}{\partial x^a}\wedge\frac{\partial}{\partial x^b}\wedge\frac{\partial}{\partial x^c} = 0$$

Proof. Replace ξ, v in (2.1) by π then

$$[\pi,\pi] = \pi\circ\pi - (-1)^{1.1}\pi\circ\pi = 2(\pi\circ\pi) \tag{3.1}$$

$$\{\{f_1,f_2\},f_3\} + \{\{f_2,f_3\},f_1\} + \{\{f_3,f_1\},f_2\} = \frac{1}{2}\epsilon^{i_1i_2i_3}\bar{\pi}\left(\bar{\pi}\left(f_{i_1},f_{i_2}\right),f_{i_3}\right) \tag{3.2}$$

$$[\pi,\pi] = 2(\pi\circ\pi) = 0 \iff 2(\pi\circ\pi)(df_1,df_2,df_3) = 0 \text{ for all } f_1,f_2,f_3 \tag{3.3}$$

$$2(\pi\circ\pi)(df_1,df_2,df_3) = 2\left(\frac{1}{2}\epsilon^{i_1i_2i_3}\bar{\pi}\left(\bar{\pi}\left(f_{i_1},f_{i_2}\right),f_{i_3}\right)\right) \tag{3.4}$$

$$\text{Since } \bar{\pi}(f_i,f_j) = \pi(df_i,df_j) = \pi^{bc}\frac{\partial f_i}{\partial x^b}\frac{\partial f_j}{\partial x^c} \text{ then} \tag{3.5}$$

$$\epsilon^{i_1i_2i_3}\bar{\pi}\left(\bar{\pi}\left(f_{i_1},f_{i_2}\right),f_{i_3}\right) = \epsilon^{i_1i_2i_3}\bar{\pi}\left(\pi^{bc}\frac{\partial f_{i_1}}{\partial x^b}\frac{\partial f_{i_2}}{\partial x^c},f_{i_3}\right) \tag{3.6}$$

$$= \epsilon^{i_1i_2i_3}\pi^{da}\frac{\partial\left(\pi^{bc}\frac{\partial f_{i_1}}{\partial x^b}\frac{\partial f_{i_2}}{\partial x^c}\right)}{\partial x^d}\frac{\partial f_{i_3}}{\partial x^a} \tag{3.7}$$

$$= \epsilon^{i_1i_2i_3}\pi^{da}\frac{\partial \pi^{bc}}{\partial x^d}\frac{\partial f_{i_1}}{\partial x^b}\frac{\partial f_{i_2}}{\partial x^c}\frac{\partial f_{i_3}}{\partial x^a} + \epsilon^{i_1i_2i_3}\pi^{da}\pi^{bc}\frac{\partial\left(\frac{\partial f_{i_1}}{\partial x^b}\frac{\partial f_{i_2}}{\partial x^c}\right)}{\partial x^d}\frac{\partial f_{i_3}}{\partial x^a} \tag{3.8}$$

Observe that second part of the sum in (3.8) is zero. That is,

$$\epsilon^{i_1i_2i_3}\pi^{da}\pi^{bc}\frac{\partial\left(\frac{\partial f_{i_1}}{\partial x^b}\frac{\partial f_{i_2}}{\partial x^c}\right)}{\partial x^d}\frac{\partial f_{i_3}}{\partial x^a} = 0. \tag{3.9}$$

To be able to see it, (3.9) can be written explicitly then indices are rearranged. That is, the following sum is zero.

$$\epsilon^{i_1i_2i_3}\pi^{da}\pi^{bc}\frac{\partial^2 f_{i_1}}{\partial x^d\partial x^b}\frac{\partial f_{i_3}}{\partial x^a}\frac{\partial f_{i_2}}{\partial x^c} + \epsilon^{i_1i_2i_3}\pi^{da}\pi^{bc}\frac{\partial^2 f_{i_2}}{\partial x^d\partial x^c}\frac{\partial f_{i_3}}{\partial x^a}\frac{\partial f_{i_1}}{\partial x^b} \tag{3.10}$$

One can swap i_1 and i_3 and interchange b and c then

$$\epsilon^{i_1 i_2 i_3}\pi^{da}\pi^{bc}\frac{\partial^2 f_{i_1}}{\partial x^d \partial x^b}\frac{\partial f_{i_3}}{\partial x^a}\frac{\partial f_{i_2}}{\partial x^c} = -\epsilon^{i_1 i_2 i_3}\pi^{da}\pi^{bc}\frac{\partial^2 f_{i_3}}{\partial x^d \partial x^b}\frac{\partial f_{i_1}}{\partial x^a}\frac{\partial f_{i_2}}{\partial x^c} \tag{3.11}$$

$$= \epsilon^{i_1 i_2 i_3}\pi^{da}\pi^{bc}\frac{\partial^2 f_{i_3}}{\partial x^d \partial x^c}\frac{\partial f_{i_1}}{\partial x^a}\frac{\partial f_{i_2}}{\partial x^b} \quad \text{then interchange a and b} \tag{3.12}$$

$$= \epsilon^{i_1 i_2 i_3}\pi^{da}\pi^{bc}\frac{\partial^2 f_{i_3}}{\partial x^d \partial x^c}\frac{\partial f_{i_1}}{\partial x^b}\frac{\partial f_{i_2}}{\partial x^a} \tag{3.13}$$

For the second part of the sum in (3.10), swap i_2 and i_3 , that is,

$$\epsilon^{i_1 i_2 i_3}\pi^{da}\pi^{bc}\frac{\partial^2 f_{i_2}}{\partial x^d \partial x^c}\frac{\partial f_{i_3}}{\partial x^a}\frac{\partial f_{i_1}}{\partial x^b} = -\epsilon^{i_1 i_2 i_3}\pi^{da}\pi^{bc}\frac{\partial^2 f_{i_3}}{\partial x^d \partial x^c}\frac{\partial f_{i_2}}{\partial x^a}\frac{\partial f_{i_1}}{\partial x^b} \tag{3.14}$$

Adding the findings (3.13) and (3.14) present the sum (3.10) as 0. That is, (3.8) becomes the following :

$$[\pi,\pi](df_1, df_2, df_3) = \epsilon^{i_1 i_2 i_3}\pi^{da}\tfrac{\partial \pi^{bc}}{\partial x^d}\tfrac{\partial f_{i_1}}{\partial x^b}\tfrac{\partial f_{i_2}}{\partial x^c}\tfrac{\partial f_{i_3}}{\partial x^a} \tag{3.15}$$

$$= \epsilon^{i_1 i_2 i_3}\pi^{da}\tfrac{\partial \pi^{bc}}{\partial x^d}\frac{\partial}{\partial x^a}\otimes\frac{\partial}{\partial x^b}\otimes\frac{\partial}{\partial x^c}(df_{i_3}, df_{i_1}, df_{i_2}) \tag{3.16}$$

$$= \epsilon^{i_3 i_1 i_2}\pi^{da}\tfrac{\partial \pi^{bc}}{\partial x^d}\frac{\partial}{\partial x^a}\otimes\frac{\partial}{\partial x^b}\otimes\frac{\partial}{\partial x^c}(df_{i_3}, df_{i_1}, df_{i_2}) \tag{3.17}$$

$$= \epsilon^{i_1 i_2 i_3}\pi^{da}\tfrac{\partial \pi^{bc}}{\partial x^d}\frac{\partial}{\partial x^a}\otimes\frac{\partial}{\partial x^b}\otimes\frac{\partial}{\partial x^c}(df_{i_1}, df_{i_2}, df_{i_3}) \tag{3.18}$$

$$= \pi^{da}\tfrac{\partial \pi^{bc}}{\partial x^d}\frac{\partial}{\partial x^a}\wedge\frac{\partial}{\partial x^b}\wedge\frac{\partial}{\partial x^c}(df_{i_1}, df_{i_2}, df_{i_3}) \text{ then} \tag{3.19}$$

$$[\pi,\pi] = \pi^{da}\tfrac{\partial \pi^{bc}}{\partial x^d}\frac{\partial}{\partial x^a}\wedge\frac{\partial}{\partial x^b}\wedge\frac{\partial}{\partial x^c} = 0. \tag{3.20}$$

□

The following partial differential equations have dimensional dependence:

$$[\pi,\pi] = \pi^{da}\frac{\partial \pi^{bc}}{\partial x^d}\frac{\partial}{\partial x^a}\wedge\frac{\partial}{\partial x^b}\wedge\frac{\partial}{\partial x^c} = 0 \tag{3.21}$$

holds identically in two dimensions, meaning that any bi-vector is Poisson.

3.1. POISSON STRUCTURE IN FOUR DIMENSIONS IN VECTOR FORM

In this section, we closely follow [6]. For $n = 3$, it gives one scalar equation for three functions of three variables. As for $n = 4$, there are four equations for six unknowns of four variables.

(3.21) can be identified with the following : One can use coordinate functions x^i instead of f_i in (3.3) then relations regarding π^{ij} are obtained. That is,

$$\pi^{dj}\frac{\partial \pi^{ki}}{\partial x^d}+\pi^{dk}\frac{\partial \pi^{ij}}{\partial x^d}+\pi^{di}\frac{\partial \pi^{jk}}{\partial x^d}=0 \tag{3.22}$$

where d,i,j,k=0,1,2,3. Indeed, (3.22) and skew symmetries of π lead to be equivalent to (3.21). Since (3.22) is symmetric, there are four outcomes for $\{i,j,k\}$ as follows :

$$\begin{Bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 2 & 3 \end{Bmatrix} \tag{3.23}$$

Let $\pi^{0i}=-u^i, \pi^{ij}=-\epsilon^{ijk}v_k$ where i,j,k=1,2,3. We are dealing with case $n=4$. If $(i,j,k)=(1,2,3)$, that is,

$$\begin{Bmatrix} i & j & k \\ 1 & 2 & 3 \end{Bmatrix} \tag{3.24}$$

then (3.22) reveals that

$$u\cdot\frac{\partial v}{\partial t}+v\cdot\nabla\times v=0 \iff \tag{3.25}$$

$$\frac{\partial\left(u\cdot v\right)}{\partial t}=v\cdot\left(\frac{\partial u}{\partial t}-\nabla\times v\right). \tag{3.26}$$

For the the following cases :

$$\begin{Bmatrix} i & j & k \\ 0 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 2 & 3 \end{Bmatrix} \tag{3.27}$$

Equation (3.22) suggests that

$$-u^2\frac{\partial u^3}{\partial t}+u^3\frac{\partial u^2}{\partial t}+\pi^{d2}\frac{\partial u^3}{\partial x^d}-\pi^{d3}\frac{\partial u^2}{\partial x^d}-u^d\frac{\partial v_1}{\partial x^d}=0 \tag{3.28}$$

$$-u^3\frac{\partial u^1}{\partial t}+u^1\frac{\partial u^3}{\partial t}+\pi^{d3}\frac{\partial u^1}{\partial x^d}-\pi^{d1}\frac{\partial u^3}{\partial x^d}-u^d\frac{\partial v_2}{\partial x^d}=0 \tag{3.29}$$

$$-u^1\frac{\partial u^2}{\partial t}+u^2\frac{\partial u^1}{\partial t}+\pi^{d1}\frac{\partial u^2}{\partial x^d}-\pi^{d2}\frac{\partial u^1}{\partial x^d}-u^d\frac{\partial v_3}{\partial x^d}=0 \tag{3.30}$$

Using the sum on d $(d = 1, 2, 3)$, these equations suggest that

$$v_1\nabla\cdot u-v_1\frac{\partial u^1}{\partial x^1}-v_2\frac{\partial u^2}{\partial x^1}-v_3\frac{\partial u^3}{\partial x^1}-((u\cdot\nabla)\,v)_i=\left(u\times\frac{\partial u}{\partial t}\right)_i \tag{3.31}$$

$$v_2\nabla\cdot u-v_1\frac{\partial u^1}{\partial x^2}-v_2\frac{\partial u^2}{\partial x^2}-v_3\frac{\partial u^3}{\partial x^2}-((u\cdot\nabla)\,v)_j=\left(u\times\frac{\partial u}{\partial t}\right)_j \tag{3.32}$$

$$v_3\nabla\cdot u-v_1\frac{\partial u^1}{\partial x^3}-v_2\frac{\partial u^2}{\partial x^3}-v_3\frac{\partial u^3}{\partial x^3}-((u\cdot\nabla)\,v)_k=\left(u\times\frac{\partial u}{\partial t}\right)_k \tag{3.33}$$

Paying attention to the following,

$$[(v\cdot\nabla)\,u+v\times(\nabla\times u)]=\begin{bmatrix} v_1\frac{\partial u^1}{\partial x^1}+v_2\frac{\partial u^2}{\partial x^1}+v_3\frac{\partial u^3}{\partial x^1} \\ v_1\frac{\partial u^1}{\partial x^2}+v_2\frac{\partial u^2}{\partial x^2}+v_3\frac{\partial u^3}{\partial x^2} \\ v_1\frac{\partial u^1}{\partial x^3}+v_2\frac{\partial u^2}{\partial x^3}+v_3\frac{\partial u^3}{\partial x^3}\end{bmatrix}$$

The equation above indicate that

$$v\nabla\cdot u-(v\cdot\nabla)\,u-v\times(\nabla\times u)-(u\cdot\nabla)\,v=u\times\frac{\partial u}{\partial t} \tag{3.34}$$

Using the following equality,

$$\nabla\,(u\cdot v)=u\times(\nabla\times v)+v\times(\nabla\times u)+(u\cdot\nabla)\,v+(v\cdot\nabla)\,u$$

One can reach that

$$\nabla\,(u\cdot v)=v\,(\nabla\cdot u)-u\times\left(\frac{\partial u}{\partial t}-\nabla\times v\right)=0. \tag{3.35}$$

The matrix representation of π is denoted by $[\pi]$.

$$[\pi] = \begin{bmatrix} 0 & -u^1 & -u^2 & -u^3 \\ u^1 & 0 & -v_3 & v_2 \\ u^2 & v_3 & 0 & -v_1 \\ u^3 & -v_2 & v_1 & 0 \end{bmatrix}$$

Put $u = (u^1, u^2, u^3)$, $v = (v_1, v_2, v_3)$ and **x**=(t, x, y, z)

$$\det[\pi] = (u.v)^2 = (\rho(\pi))^2$$

where

$$\rho(\pi) = u.v$$

In four dimensions, the Jacobi identity becomes

$$\frac{\partial(\rho(\pi))}{\partial t} = v.\left(\frac{\partial u}{\partial t} - \nabla \times v\right) \tag{3.36}$$

$$\nabla(\rho(\pi)) = v(\nabla.u) - u \times \left(\frac{\partial u}{\partial t} - \nabla \times v\right) \tag{3.37}$$

In case $\rho(\pi) = 0$, if $u = B$ and $v = E$ then the Jacobi identity can also be written as follows:

$$E \cdot (\frac{\partial B}{\partial t} - \nabla \times E) = 0 \tag{3.38}$$

$$E(\nabla \cdot B) - B \times (\frac{\partial B}{\partial t} - \nabla \times E) = 0 \tag{3.39}$$

(3.38), (3.39) correspond to Jacobi identity for a degenerate Poisson bi-vector. Nondegenerate bi-vector means that matrix representation of the tensor has full rank, otherwise it is called degenerate.

Lemma 3.1.1. *Jacobi identity con be identified as the solutions of* (3.38), (3.39) *for a degenerate and nondegenerate Poisson bi-vector.*

Proof. Since we have already observed that (3.38), (3.39) correspond to Jacobi identity for a degenerate Poisson bi-vector. Consider the case $\rho(\pi) \neq 0$. Put $B = \dfrac{u}{\rho(\pi)}$, $E = \dfrac{v}{\rho(\pi)}$.

Then $u=\dfrac{B}{E\cdot B}, v=\dfrac{E}{E\cdot B}$ and $u\cdot v=\dfrac{1}{E\cdot B}$. (3.36) suggests that

$$\frac{\partial\left(\frac{1}{E\cdot B}\right)}{\partial t}=\frac{E}{E\cdot B}\cdot\left(\frac{\partial\left(\frac{1}{E\cdot B}\right)}{\partial t}-\nabla\times\left(\frac{E}{E\cdot B}\right)\right) \tag{3.40}$$

$$\text{Using } \nabla\times\left(\frac{E}{E\cdot B}\right)=\frac{E}{E\cdot B}\nabla\times E+\frac{E\times\nabla\left(E\cdot B\right)}{(E\cdot B)^2} \tag{3.41}$$

$$\frac{-1}{(E\cdot B)^2}\frac{\partial(E\cdot B)}{\partial t}=\frac{E}{(E\cdot B)^3}\cdot\left[(E\cdot B)\frac{\partial B}{\partial t}-\frac{\partial\left(E\cdot B\right)}{\partial t}B\right]-\frac{E\cdot\nabla\times E}{(E\cdot B)^2} \tag{3.42}$$

(3.42) is multiplied by $(E\cdot B)^2$, (3.38) is obtained. As for (3.39), firstly observe the followings :

$$\nabla\left(\frac{1}{E\cdot B}\right) = -\frac{\nabla\left(E\cdot B\right)}{(E\cdot B)^2} \tag{3.43}$$

$$\nabla\left(\frac{B}{E\cdot B}\right) = \frac{\nabla\cdot B}{E\cdot B}-\frac{B\cdot\nabla\left(E\cdot B\right)}{(E\cdot B)^2} \tag{3.44}$$

Using (3.41),(3.43) and (3.44), (3.37) suggests that

$$\begin{aligned}-\frac{\nabla\cdot B}{(E\cdot B)^2}=\frac{E}{E\cdot B}\left(\frac{\nabla\cdot B}{E\cdot B}-\frac{B\cdot\nabla\left(E\cdot B\right)}{(E\cdot B)^2}\right)-\frac{B\times\frac{\partial B}{\partial t}}{(E\cdot B)^2}\\+\frac{B}{(E\cdot B)^2}\times\left(\nabla\times E+\frac{E\times\nabla(E\cdot B)}{E\cdot B}\right)\end{aligned} \tag{3.45}$$

$$\text{Using } B\times\left(E\times\nabla(E\cdot B)\right)=\left(B\cdot\nabla(E\cdot B)\right)E-(E\cdot B)\nabla(E\cdot B) \tag{3.46}$$

and (3.45) multiplied by $(E\cdot B)^2$, (3.39) is obtained. (3.47)

□

Proposition 3.1.2. *If* **x**$=(t,x,y,z)\in\mathbb{R}^4$*, the rank of Poisson tensor* Λ *in* **x**$=(t,x,y,z)$ *takes*

$$rank\Lambda=\begin{cases}0 & \textit{iff} \quad E=B=0\\ 2 & \textit{iff} \quad \|E\|^2+\|B\|^2\neq 0 \textit{ and } \quad E\cdot B=0\\ 4 & \textit{iff} \quad E\cdot B\neq 0\end{cases}$$

Proof. Since $rank\Lambda+dimker\Lambda=4$, dimker$\Lambda$ is determined to find rankΛ. Let a $\in\mathbb{R}$ and

$\alpha \in \mathbb{R}^3$. To find kernel :

$$\begin{bmatrix} 0 & -B^t \\ B & -\epsilon^{ijk}E_k \end{bmatrix} \begin{bmatrix} a \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.48}$$

Then we have

$$\begin{bmatrix} -B \cdot \alpha \\ aB - \alpha \times E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \alpha \cdot B = 0 \\ aB = \alpha \times E \end{cases} \tag{3.49}$$

There are two cases: $E \cdot B \neq 0$ and $E \cdot B = 0$.

- $E \cdot B \neq 0$ implies that

$$aB \cdot E = (\alpha \times E) \cdot E = 0 \Rightarrow a = 0 \Rightarrow \alpha \times E = 0.$$

$\Rightarrow \alpha = \lambda E$ for some function λ.

$$\alpha \cdot B = \lambda(B \cdot E) = 0 \Rightarrow \lambda = 0 \Rightarrow \alpha = 0.$$

Thus

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} = 0 \; i.e, \ker\Lambda = 0 \Rightarrow \operatorname{rank}\Lambda = 4. \tag{3.50}$$

- It is clear that rank$\Lambda = 0$ iff $E = B = 0$.

- For $E \cdot B = 0$:

Case 1: Assume that E$\neq 0$, B$\neq 0$. Let α=$d_1E + d_2B + d_3B$×E.

$$\left.\begin{array}{r} \alpha \cdot B = 0 \\ aB = \alpha \times E \end{array}\right\} \Rightarrow \begin{array}{c} d_2\|B\|^2 = 0 \Rightarrow d_2 = 0. \\ (d_1E + d_3B \times E) \times E = -d_3E \times (B \times E) = -d_3\|E\|^2B \\ \Rightarrow a = -d_3\|E\|^2 \Rightarrow d_3 = -\dfrac{a}{\|E\|^2} \end{array} \tag{3.51}$$

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} = \begin{bmatrix} a \\ d_1 \\ 0 \\ -\dfrac{a}{\|E\|^2} \end{bmatrix} = d_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\dfrac{1}{\|E\|^2} \end{bmatrix}. \tag{3.52}$$

That is, kerΛ=2 $\Rightarrow$ rankΛ=2.

Case 2: Assume that E=0 and B$\neq$ 0 and E·B=0. We have

$$\left.\begin{aligned} \alpha \cdot B = 0 \\ aB = \alpha \times E \end{aligned}\right\} \Rightarrow a = 0 \; and \; \alpha \cdot B = 0. \tag{3.53}$$

Since B$\neq$0, at least one component of B must be nonzero so without lost of generality,say B_1 $\neq$0. In this case ,

$$\alpha \cdot B = 0 \Rightarrow \alpha_1 = -\frac{B_2}{B_1}\alpha_2 - \frac{B_3}{B_1}\alpha_3$$

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} = \alpha_2 \begin{bmatrix} 0 \\ -\frac{B_2}{B_1} \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ -\frac{B_3}{B_1} \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{rank}\Lambda = 2 \text{ (since dimker}\Lambda = 2). \tag{3.54}$$

Case 3: Assume that E$\neq$0, B=0 and E·B=0 :

In this case, $\alpha \times E = aB = 0 \Rightarrow \alpha$=$\lambda E$ for some function λ.

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix} \Rightarrow \text{dimker}\Lambda = 2 \; i.e, \text{rank}\Lambda = 2. \tag{3.55}$$

We have covered all cases regarding E·B=0 and $\|E\|^2 + \|B\|^2$ $\neq$0.

If rankΛ $= 4$, suppose that E·B=0 but E·B=0 implies that rankΛ=2 or 0, contradiction. If rankΛ $= 2$, suppose that E·B$\neq$0 but E·B$\neq$0 implies that rank$\Lambda = 4$, contradiction. □

The nondegeneracy and degeneracy of π occurs whenever $E \cdot B \neq 0$ and $E \cdot B = 0$, respectively.

Considering (3.38) and (3.39) (We convert u and v into E and B thus $[\pi]$ changes), then Poisson tensor can be written as

$$\Lambda = \begin{bmatrix} 0 & -B^t \\ B & -\epsilon^{ijk} E_k \end{bmatrix} \quad for \quad rank\Lambda = 2 \tag{3.56}$$

$$\Lambda = \frac{1}{E \cdot B} \begin{bmatrix} 0 & -B^t \\ B & -\epsilon^{ijk} E_k \end{bmatrix} \quad for \quad rank\Lambda = 4 \tag{3.57}$$

If the rank of Λ is 4 then $B \cdot E \neq 0$ (Symplectic case) then (3.39) is multiplied by B then $(E \cdot B)\,(\nabla \cdot B) = 0$ and since $E \cdot B \neq 0$, one reaches that $\nabla \cdot B = 0$. Since $\nabla \cdot B = 0$, (3.39) implies that $\frac{\partial B}{\partial t} - \nabla \times E = \lambda B$ then (3.38) indicates $E \cdot \left(\frac{\partial B}{\partial t} - \nabla \times E\right) = 0 \Rightarrow \lambda E \cdot B = 0 \Rightarrow \lambda = 0$.

Hence equations can be reduced into

$$\frac{\partial B}{\partial t} - \nabla \times E = 0 \tag{3.58}$$

$$\nabla \cdot B = 0 \tag{3.59}$$

Remark 3.1.3. *Observe that*

$$\{f, g\} = \pi\,(df, dg) = -X_f(g) = -X_f(dg) \tag{3.60}$$

$$\pi^\sharp(df)(dg) = \pi\,(df, dg) \tag{3.61}$$

Therefore, $\pi^\sharp(df) = -X_f$ *and*

$$i_{X_f} df = X_f f = \pi\,(df, df) = 0. \tag{3.62}$$

Remark 3.1.4. *If* $X_f = a^i \frac{\partial}{\partial x^i}$ *and* $dg = \dfrac{\partial g}{\partial x^i} dx^i$ *, the followings are also hold.*

$$[X_f(dg)] = \begin{bmatrix} a^1 & a^2 & \dots & a^n \end{bmatrix} \begin{bmatrix} \dfrac{\partial g}{\partial x^1} \\ \\ \dfrac{\partial g}{\partial x^2} \\ \dots \\ \dfrac{\partial g}{\partial x^n} \end{bmatrix}, \text{ that is }, [X_f(dg)] = [X_f]^T [dg] \quad (3.63)$$

Since π *is skew-symmetric,*

$$[\pi(df, dg)] = [df]^T [\pi][dg] = -([\pi][df])^T [dg] = -[X_f]^T [dg]. \quad (3.64)$$

Therefore,

$$[\pi][df] = [X_f]. \quad (3.65)$$

Remark 3.1.5. *If* π *is* ***full rank***, *let* $[\theta]_{ij} = -[\pi]^{-1}_{ij}$ *then* $[\theta]_{ij} = -[\theta]_{ji}$. *Put* $\theta = \dfrac{1}{2}\theta_{ij} dx^i \wedge dx^j$ *whenever* $\pi = \frac{1}{2}\pi^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$.
Observe that

$$[\theta(X_f, X_g)] = [X_f]^T [\theta][X_g] = [df]^T [\pi]^T [\theta][\pi][dg] = -[df]^T [\pi]^T [dg] \quad (3.66)$$

$$\text{Since} \quad \pi \text{ is skew symmetric,} \quad (3.67)$$

$$[\pi(df, dg)] = [df]^T [\pi][dg] = -[df]^T [\pi]^T [dg] \quad (3.68)$$

$$\{f, g\} = \pi(df, dg) = \theta(X_f, X_g) \quad (3.69)$$

$$[X_f]^T [\theta][X_g] = [\theta(X_f, X_g)] = \left[i_{X_f}\theta\right]^T [X_g] \quad (3.70)$$

$$\left[i_{X_f}\theta\right]^T = [X_f]^T [\theta] = -[df]^T [\pi][\theta] = [df]^T \quad (3.71)$$

$$i_{X_f}\theta = df \quad (3.72)$$

When manifold is ***symplectic****, we see that* (3.72) *defines* ***Hamitonian*** *vector field of f. Here, as usual [.] is used to denote matrix representation and T represents transposition of the matrix.*

Theorem 3.1.6. *If θ is defined as in remark 3.1.5 then*

$$\frac{\partial\theta_{ij}}{\partial x^k}+\frac{\partial\theta_{ki}}{\partial x^j}+\frac{\partial\theta_{jk}}{\partial x^i}=0 \iff \pi^{dj}\frac{\partial\pi^{ki}}{\partial x^d}+\pi^{dk}\frac{\partial\pi^{ij}}{\partial x^d}+\pi^{di}\frac{\partial\pi^{jk}}{\partial x^d}=0 \tag{3.73}$$

$$d\theta=0 \iff \frac{\partial\theta_{ij}}{\partial x^k}+\frac{\partial\theta_{ki}}{\partial x^j}+\frac{\partial\theta_{jk}}{\partial x^i}=0 \tag{3.74}$$

Since (3.22) *is equivalent to Jacobi identity, Jacobi identity is equivalent to* $d\theta=0$. *More explicitly, if there is a* ***Symplectic manifold*** *M then there is a* ***Poisson structure*** *on M. In this case,* $\mathcal{L}_{X_f}\theta=i_{X_f}d\theta+di_{X_f}\theta=0$.

Proof. Observe that

$$0=-\frac{\partial\,[I]}{\partial x^d}=\frac{\partial\,([\theta]\cdot[\pi])}{\partial x^d}=\frac{\partial\,[\theta]}{\partial x^d}\cdot[\pi]+[\theta]\cdot\frac{\partial\,[\pi]}{\partial x^d} \tag{3.75}$$

(3.75) implies that

$$\frac{\partial\,[\theta]}{\partial x^d}=[\theta]\cdot\frac{\partial\,[\pi]}{\partial x^d}\cdot[\theta] \tag{3.76}$$

$$\frac{\partial\,[\pi]}{\partial x^d}=[\pi]\cdot\frac{\partial\,[\pi]}{\partial x^d}\cdot[\pi] \tag{3.77}$$

(3.76) explicitly implies that

$$\frac{\partial\theta_{jk}}{\partial x^i}=\theta_{jl}\frac{\partial\pi^{lm}}{\partial x^i}\theta_{mk} \qquad \frac{\partial\theta_{ki}}{\partial x^j}=\theta_{kl}\frac{\partial\pi^{lm}}{\partial x^j}\theta_{mi} \qquad \frac{\partial\theta_{ij}}{\partial x^k}=\theta_{il}\frac{\partial\pi^{lm}}{\partial x^k}\theta_{mj} \tag{3.78}$$

(3.77) explicitly implies that

$$\frac{\partial\pi^{ki}}{\partial x^d}=\pi^{kl}\frac{\partial\theta_{lm}}{\partial x^d}\pi^{mi} \qquad \frac{\partial\pi^{ij}}{\partial x^d}=\pi^{il}\frac{\partial\theta_{lm}}{\partial x^d}\pi^{mj} \qquad \frac{\partial\pi^{jk}}{\partial x^d}=\pi^{jl}\frac{\partial\theta_{lm}}{\partial x^d}\pi^{mk} \tag{3.79}$$

Using (3.22) and (3.79) suggest that

$$\pi^{dj}\pi^{kl}\frac{\partial\theta_{lm}}{\partial x^d}\pi^{mi}+\pi^{dk}\pi^{il}\frac{\partial\theta_{lm}}{\partial x^d}\pi^{mj}+\pi^{di}\pi^{jl}\frac{\partial\theta_{lm}}{\partial x^d}\pi^{mk}=0 \tag{3.80}$$

if (3.80) is multiplied by $\theta_{i\tilde{i}}\theta_{j\tilde{j}}\theta_{k\tilde{k}}$ and summed over i,j,k=0,1,2,3 then

$$\delta^{\tilde{j}d}\delta^{\tilde{k}l}\delta^{\tilde{i}m}\frac{\partial\theta_{lm}}{\partial x^d}+\delta^{\tilde{k}d}\delta^{\tilde{i}l}\delta^{\tilde{j}m}\frac{\partial\theta_{lm}}{\partial x^d}+\delta^{\tilde{i}d}\delta^{\tilde{j}l}\delta^{\tilde{k}m}\frac{\partial\theta_{lm}}{\partial x^d}=0 \tag{3.81}$$

$$\left(\delta^{\tilde{i}m}\delta^{\tilde{j}d}\delta^{\tilde{k}l}+\delta^{\tilde{i}l}\delta^{\tilde{j}m}\delta^{\tilde{k}d}+\delta^{\tilde{i}d}\delta^{\tilde{j}l}\delta^{\tilde{k}m}\right)\frac{\partial\theta_{lm}}{\partial x^d}=0 \tag{3.82}$$

$$\frac{\partial\theta_{\tilde{j}\tilde{k}}}{\partial x^{\tilde{i}}}+\frac{\partial\theta_{\tilde{k}\tilde{i}}}{\partial x^{\tilde{j}}}+\frac{\partial\theta_{\tilde{i}\tilde{j}}}{\partial x^{\tilde{k}}}=0 \tag{3.83}$$

Reindexing,

$$\frac{\partial\theta_{jk}}{\partial x^i}+\frac{\partial\theta_{ki}}{\partial x^j}+\frac{\partial\theta_{ij}}{\partial x^k}=0 \tag{3.84}$$

Using (3.78) and (3.84), we are going to show (3.22). More explicitly, If (3.84) is multiplied by $\pi^{i\tilde{i}}\pi^{j\tilde{j}}\pi^{k\tilde{k}}$ and summed over i,j,k then the followings appear :

$$\left(\frac{\partial\theta_{jk}}{\partial x^i}+\frac{\partial\theta_{ki}}{\partial x^j}+\frac{\partial\theta_{ij}}{\partial x^k}\right)\pi^{i\tilde{i}}\pi^{j\tilde{j}}\pi^{k\tilde{k}}=0 \tag{3.85}$$

$$\left(\theta_{jl}\frac{\partial\pi^{lm}}{\partial x^i}\theta_{mk}+\theta_{kl}\frac{\partial\pi^{lm}}{\partial x^j}\theta_{mi}+\theta_{il}\frac{\partial\pi^{lm}}{\partial x^k}\theta_{mj}\right)\pi^{i\tilde{i}}\pi^{j\tilde{j}}\pi^{k\tilde{k}}=0 \tag{3.86}$$

$$-\delta^{l\tilde{j}}\delta^{m\tilde{k}}\frac{\partial\theta_{lm}}{\partial x^i}\pi^{i\tilde{i}}-\delta^{l\tilde{k}}\delta^{m\tilde{i}}\frac{\partial\theta_{lm}}{\partial x^j}\pi^{j\tilde{j}}-\delta^{l\tilde{i}}\delta^{m\tilde{j}}\frac{\partial\theta_{lm}}{\partial x^k}\pi^{k\tilde{k}}=0 \tag{3.87}$$

$$-\pi^{i\tilde{i}}\frac{\partial\pi^{\tilde{j}\tilde{k}}}{\partial x^i}-\pi^{j\tilde{j}}\frac{\partial\pi^{\tilde{k}\tilde{i}}}{\partial x^j}-\pi^{k\tilde{k}}\frac{\partial\pi^{\tilde{i}\tilde{j}}}{\partial x^k}=0 \tag{3.88}$$

Reindexing,

$$\pi^{di}\frac{\partial\pi^{jk}}{\partial x^d}+\pi^{dj}\frac{\partial\pi^{ki}}{\partial x^d}+\pi^{dk}\frac{\partial\pi^{ij}}{\partial x^d}=0 \tag{3.89}$$

The first line of the theorem is proved, now consider $\theta=\frac{1}{2}\theta_{ij}dx^i\wedge dx^j$ then

$$2d\theta = \frac{\partial\theta_{ij}}{\partial x^k}dx^i\wedge dx^j\wedge dx^k \tag{3.90}$$

$$6d\theta = \frac{\partial\theta_{ij}}{\partial x^k}dx^i\wedge dx^j\wedge dx^k+\frac{\partial\theta_{ki}}{\partial x^j}dx^i\wedge dx^j\wedge dx^k+\frac{\partial\theta_{jk}}{\partial x^i}dx^i\wedge dx^j\wedge dx^k \tag{3.91}$$

$$6d\theta = \left[\frac{\partial\theta_{jk}}{\partial x^i}+\frac{\partial\theta_{ki}}{\partial x^j}+\frac{\partial\theta_{ij}}{\partial x^k}\right]dx^i\wedge dx^j\wedge dx^k \tag{3.92}$$

Hence $d\theta=0$ because of (3.84). As last comment, $d\theta=0$ and $i_{X_f}\theta=df$ lead to

$$\mathcal{L}_{X_f}\theta=i_{X_f}d\theta+di_{X_f}\theta=0 \tag{3.93}$$

□

Main concern of this thesis is determining the conditions of being Hamiltonian (and bi-Hamiltonian) for a given ω=(V_0,V) $\in TM$, we try to find conditions on ω to be Hamiltonian (and bi-Hamiltonian)

$$\Lambda : T^{*}M \rightarrow TM$$

$$\Lambda(dH) = \begin{bmatrix} 0 & -B^t \\ B & -\epsilon^{ijk}E_k \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial t} \\ \nabla H \end{bmatrix} = \begin{bmatrix} V_0 \\ V \end{bmatrix} \tag{3.94}$$

$$\Lambda(dH) = \frac{1}{E \cdot B} \begin{bmatrix} 0 & -B^t \\ B & -\epsilon^{ijk}E_k \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial t} \\ \nabla H \end{bmatrix} = \begin{bmatrix} V_0 \\ V \end{bmatrix} \tag{3.95}$$

These equations are valid for rank 2 and rank 4 of Λ, respectively. That is,

$$\begin{bmatrix} -B \cdot \nabla H \\ \frac{\partial H}{\partial t} B + E \times \nabla H \end{bmatrix} = \begin{bmatrix} V_0 \\ V \end{bmatrix} for\, rank \Lambda = 2 \tag{3.96}$$

$$\frac{1}{E \cdot B} \begin{bmatrix} -B \cdot \nabla H \\ \frac{\partial H}{\partial t} B + E \times \nabla H \end{bmatrix} = \begin{bmatrix} V_0 \\ V \end{bmatrix} for\, rank \Lambda = 4 \tag{3.97}$$

Our aim is to find Λ and H using the data provided by ω. Therefore, these equations below must be hold.

For $rank\Lambda = 2$,

System 1	**System 2**
$E \cdot (\dfrac{\partial B}{\partial t} - \nabla \times E) = 0$	$-B \cdot \nabla H = V_0$
$E(\nabla \cdot B) - B \times (\dfrac{\partial B}{\partial t} - \nabla \times E) = 0$	$\frac{\partial H}{\partial t} B + E \times \nabla H = V$

For rank $\Lambda = 4$,

We can take inverse of Λ then we can find that equations that should be hold as follows :

System 1	**System 2**
$\dfrac{\partial B}{\partial t} - \nabla \times E = 0$	$E \cdot V = \partial_t H$
$\nabla \cdot B = 0$	$-V_0 E + V \times B = \nabla H$

3.2. POISSON STRUCTURE IN FOUR DIMENSIONS IN DIFFERENTIAL FORM

Let

$$\begin{aligned} u = E_1 dx^1 \wedge dt + E_2 dx^2 \wedge dt + E_3 dx^3 \wedge dt \\ -B_1 dx^2 \wedge dx^3 - B_2 dx^3 \wedge dx^1 - B_3 dx^1 \wedge dx^2 \end{aligned} \tag{3.98}$$

and $(dt, dt) = 1, (dx^i, dx^j) = \delta^{ij}, (dx^i, dt) = 0, \Omega = dt \wedge dx^1 \wedge dx^2 \wedge dx^3$.

We can see

$$\begin{aligned} du = -\nabla \cdot B dx^1 \wedge dx^2 \wedge dx^3 - (\partial_t B_1 - (\nabla \times E)^1) dt \wedge dx^2 \wedge dx^3 \\ -(\partial_t B_2 - (\nabla \times E)^2) dt \wedge dx^3 \wedge dx^1 - (\partial_t B_3 - (\nabla \times E)^3) dt \wedge dx^1 \wedge dx^2 \end{aligned} \tag{3.99}$$

$du = 0$ means the following equations :

$$\begin{aligned} \nabla \cdot B &= 0 \\ \partial_t B - \nabla \times E &= 0. \end{aligned}$$

For rank $\Lambda = 2$,

Consider $i_w u$,

$$\begin{aligned} i_w u &= -E_i dt \wedge dx^i (V_0 \frac{\partial}{\partial t} + V_j \frac{\partial}{\partial x^j}) - \frac{1}{2} \epsilon_{ijk} B_i dx^j \wedge dx^k (V_0 \frac{\partial}{\partial t} + V_l \frac{\partial}{\partial x^l}) \\ &= -V_0 E_i dx^i + E \cdot V dt + \frac{1}{2} \epsilon_{ijk} B_i V_j dx^k - \frac{1}{2} \epsilon_{ijk} B_i V_k dx^j \\ &= -V_0 E_i dx^i + E \cdot V dt - (B \times V)_i dx^i \end{aligned}$$

For rank $\Lambda = 4$,

$$i_w u = dH \text{ and } du = 0$$

if and only if the systems 1,2 are hold.

Considering $dH \wedge u$,

$$\begin{aligned}
dH \wedge u &= -\left(\partial_t H dt + \partial_i H dx^i\right) \wedge E_j dt \wedge dx^j - \left(\partial_t H dt + \partial_l H dx^l\right) \wedge \frac{1}{2}\epsilon_{ijk} B_i dx^j \wedge dx^k \\
&= (\partial_j H) E_k dt \wedge dx^j \wedge dx^k - \frac{1}{2}\epsilon_{ijk}(\partial_t H) B_i dt \wedge dx^j \wedge dx^k \\
&- \frac{1}{2}\epsilon_{ijk}\epsilon^{ljk} B_i (\partial_l H) dx^1 \wedge dx^2 \wedge dx^3 \\
&= -\left(E \times \nabla H + (\partial_t H) B\right)_1 dt \wedge dx^2 \wedge dx^3 \\
&- \left(E \times \nabla H + (\partial_t H) B\right)_2 dt \wedge dx^3 \wedge dx^1 \\
&- \left(E \times \nabla H + (\partial_t H) B\right)_3 dt \wedge dx^1 \wedge dx^2 - B \cdot \nabla H dx^1 \wedge dx^2 \wedge dx^3.
\end{aligned}$$

$$i_w \Omega = V_0 dx^1 \wedge dx^2 \wedge dx^3 - V_1 dt \wedge dx^2 \wedge dx^3 - V_2 dt \wedge dx^3 \wedge dx^1 - V_3 dt \wedge dx^1 \wedge dx^2 \quad (3.100)$$

Thus,

$$i_w \Omega = u \wedge dH$$

corresponds to the system 2 equations for rank $\Lambda = 2$.

$$\frac{1}{E \cdot B} i_w \Omega = u \wedge dH$$

corresponds to the system 2 equations for rank $\Lambda = 4$. The results above give us a motivation to suggest that $i_w \Omega = J \wedge dH$ may hold for some two form J .

4. THE LINE OF REASONING

We show that the followings are equivalent ($A \iff B \iff C \iff D \iff X$):

$X : i_w\Omega = J \wedge dH$; $\quad A : i_w J = f dH$; $\quad B : i_{\frac{w}{f}} J = dH$; $\quad C : i_{gw} J \wedge d i_{gw} J = 0$;

$D : i_w J \wedge d i_w J = 0$; $\qquad$ E: Poisson Structure.

$$\begin{tikzcd}
X \arrow[r, leftrightarrow] & A \arrow[r, leftrightarrow] & B \arrow[r, "\text{dJ=0}"'] \arrow[d, leftrightarrow, "\text{Frobenius}"] & E \\
& D \arrow[r, leftrightarrow] & C &
\end{tikzcd}$$

Here, suppose that J is a nondegenerate closed two form and and w nonvanishing vector field and Ω is volume form.

Proposition 4.0.1. $(X \Rightarrow A)$
If $i_w\Omega = J \wedge dH$ then $i_w dH = 0$ and $i_w J = f dH$ for some smooth function f.

Proof. Since $i_w\Omega \wedge dH = 0$ and $w.(\partial_t H, \nabla H) = i_w dH$,

$$\begin{aligned}
i_w\Omega \wedge dH &= -\left(\partial_t H V_0 + \partial_i H V_i\right) dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&= -\left(w.(\partial_t H, \nabla H)\right)\Omega = -\left(i_w dH\right)\Omega. \\
&= 0 \\
\Rightarrow i_w dH &= 0
\end{aligned}$$

$i_w i_w \Omega = 0 \; and \; i_w dH = 0,$

$$\begin{aligned}
i_w i_w \Omega &= i_w\left(J \wedge dH\right) = i_w J \wedge dH + J \wedge i_w dH = i_w J \wedge dH \\
&= 0, \text{ then} \\
i_w J &= f dH
\end{aligned}$$

for some smooth function f.

$\square$

Proposition 4.0.2. $(X \text{ and } dJ = 0 \Rightarrow E)$

If $i_w\Omega = J \wedge dH$ and J is nondegenerate two form and $dJ = 0$ then there is a poisson structure.

Proof. Using previous lemma we have $i_wJ = fdH$.

$fi_{\frac{1}{f}w}J = fdH \Rightarrow i_{\frac{1}{f}w}J = dH$ and $dJ = 0$ give symplectic structure $\Rightarrow$ Poisson structure.
If $i_wJ = fdH$ then,

$$i_wJ \wedge di_wJ = fdH \wedge (df \wedge dH) = 0. \tag{4.1}$$

□

Proposition 4.0.3. $(C \iff D)$

If g is a nonvanishing function then

$$i_wJ \wedge di_wJ = 0 \iff i_{gw}J \wedge di_{gw}J = 0. \tag{4.2}$$

Proof.

$$i_{gw}J \wedge di_{gw}J = gi_wJ \wedge d\,(gi_wJ) \tag{4.3}$$

$$= gi_wJ \wedge (dg \wedge i_wJ + gdi_wJ) = g^2 i_wJ \wedge di_wJ \tag{4.4}$$

□

Proposition 4.0.4. $(B \iff C)$

$i_{gw}J \wedge di_{gw}J = 0$ holds if and only if $i_{\frac{g}{f}w}J = dh$ for some smooth function f.

Proof. For any one form λ,

$\lambda \wedge d\lambda = 0 \iff \lambda = fdh$ for some functions f,h by Frobenius integration theorem, (See [16], p:318). Particularly, if $\lambda = i_{gw}J$ then

$$i_{gw}J \wedge di_{gw}J = 0 \iff i_{gw}J = fdh \iff i_{\frac{g}{f}w}J = dh.$$

where f can not be zero because J is nondegenerate, $i_wJ \neq 0 \iff w \neq 0$. □

Proposition 4.0.5. $(A \Rightarrow X)$

If $i_w J = f dH$ *then* $i_w(h\Omega) = f J \wedge dH$ *for some smooth function h.*

Proof.

$$i_{\frac{w}{f}} dH = i_{\frac{w}{f}} i_{\frac{w}{f}} J = 0 \tag{4.5}$$

$$i_{\frac{w}{f}} J \wedge dH = 0 \tag{4.6}$$

$$i_{\frac{w}{f}} (J \wedge dH) = i_{\frac{w}{f}} J \wedge dH + J \wedge i_{\frac{w}{f}} dH = 0 \tag{4.7}$$

$$i_{\frac{w}{f}} (J \wedge dH) = 0 \Rightarrow J \wedge dH = i_{\frac{w}{f}} (h\Omega) = \frac{1}{f} i_w (h\Omega) \tag{4.8}$$

is found. □

Comment: Instead of starting with $i_w\Omega = J \wedge dH$ if we started with $i_w(h\Omega) = J \wedge dH$ the same line of reasoning would not change. There is a relation $i_{gw}J = f dh$ provided that $i_w\Omega = J \wedge dH$. If it can be shown that $dJ = 0$ also holds, this means that there is a symplectic structure which is Poisson.

4.1. RICCATI EQUATIONS RELATED TO A POISSON STRUCTURE

We consider that

$$i_w J \wedge d i_w J = 0.$$

We are going to investigate under what conditions that equation holds. If we find nondegenerate J so that $i_w J \wedge d i_w J = 0$ holds, then we can find a function h such that $i_{\tilde{g}w} J = dh$ for some integrating factor function $\tilde{g}$ by using lemma above and Frobenius Integration theorem. For next step, if it can be shown $dJ = 0$, then Poisson structure is constructed. This is the strategy we are going to follow. Let us choose $e_0 = \frac{w}{|w|}$ extend it to a basis such that $i_w J = a_i e^i$ holds, say $\{e_0, e_1, e_2, e_3\}$. Let $de^i = -\frac{1}{2} C^i_{jk} e^j \wedge e^k$ where $C^i_{jk} = -C^i_{kj}$ for $j, k = 0, 1, 2, 3$ and $da_i = a_{i0}e^0 + a_{ij}e^j$ for $i = 1, 2, 3$.

($i_w J = a_0 e^0 + a_i e^i$ so $i_w J(e_0) = J(w, \frac{w}{|w|}) = 0 \Rightarrow a_0 = 0$ and $i_w J = a_i e^i$)

$$\begin{aligned} d i_w J &= d(a_i de^i) = da_i \wedge e^i + a_i de^i &(4.9)\\ &= \left(a_{i0}e^0 + a_{ij}e^j\right) \wedge e^i + a_i \left(-C^i_{0k} e^0 \wedge e^k - \frac{1}{2} C^i_{jk} e^j \wedge e^k\right) &(4.10) \end{aligned}$$

where $i, j, k = 1, 2, 3$. Now, we consider the following equation

$$i_w J \wedge d i_w J = 0.$$

$$\begin{aligned} i_w J \wedge d i_w J &= a_l e^l \wedge \left[\left(a_{i0} e^0 + a_{ij} e^j\right) \wedge e^i - a_i \left(C^i_{0k} e^0 \wedge e^k + \frac{1}{2} C^i_{jk} e^j \wedge e^k \right) \right] \\ &= a_l a_{k0} e^0 \wedge e^k \wedge e^l + a_l a_{kj} e^j \wedge e^k \wedge e^l - a_l a_i C^i_{0k} e^0 \wedge e^k \wedge e^l - \frac{1}{2} a_i a_l C^i_{jk} e^j \wedge e^k \wedge e^l \\ &= a_l \left(-a_{k0} + a_i C^i_{0k}\right) e^0 \wedge e^l \wedge e^k + a_l \left(a_{kj} - \frac{1}{2} a_i C^i_{jk} \right) e^j \wedge e^k \wedge e^l \\ &= a_l \epsilon^{jkl} \left(a_{k0} + a_i C^i_{0k}\right) * e^j - a_l \epsilon^{jkl} \left(a_{kj} - \frac{1}{2} a_i C^i_{jk} \right) * e^0. \end{aligned}$$

Then, we obtain that

$$a_l \epsilon^{jkl} \left(a_{k0} + a_i C^i_{0k}\right) = 0 \tag{4.11}$$

$$a_l \epsilon^{jkl} \left(a_{jk} + \frac{1}{2} a_i C^i_{jk} \right) = 0. \tag{4.12}$$

Since $i_w J = a_1 e^1 + a_2 e^2 + a_3 e^3$ is not zero then at least one of coefficients is nonzero, say $a_2 \neq 0$. Without lost of generality, we can choose e_i's such that $a_1 = 0, a_2 = 1$ and demonstrate $a_3 = \mu$. (If necessary, replace e^3 with $a_1 e^1 + a_3 e^3$.) We have a new basis. Sticking to the old notation, we use same letters. In matrix notation,

$$[a_i] = \begin{bmatrix} 0 \\ 1 \\ \mu \end{bmatrix} \tag{4.13}$$

$$d a_i = a_{i0} e^0 + a_{ij} e^j$$

$$[d a_i] = \begin{bmatrix} 0 \\ 0 \\ \mu_0 \end{bmatrix} e_0 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu_1 & \mu_2 & \mu_3 \end{bmatrix} \begin{bmatrix} e^1 \\ e^2 \\ e^3 \end{bmatrix} \tag{4.14}$$

where $\mu_0 = a_{30}$ and $\mu_i = a_{3i}$ or $d\mu = \mu_0 e^0 + \mu_i e^i$.

Let's go back to equations (4.11) and (4.12) to see them in terms of new coefficients.

These two equations generate the following equations:

$$-\mu_0 = -C^2_{03} + \mu\left(C^2_{02} - C^3_{03}\right) + \mu^2 C^3_{02} \tag{4.15}$$

$$\mu_1 = -C^2_{31} + \mu\left(C^2_{21} - C^3_{31}\right) + \mu^2 C^3_{21} \tag{4.16}$$

$$C^2_{01} + \mu C^3_{01} = 0 \tag{4.17}$$

Since $d\mu = \mu_0 e^0 + \mu_i e^i$ and $\mathcal{L}_{e_0}\mu = i_{e_0} d\mu$, $\mathcal{L}_{e_0}\mu = \mu_0$ is obtained and similarly, $\mathcal{L}_{e_i}\mu = \mu_i$ for $i = 1, 2, 3$. Therefore, Riccati equations are obtained, in addition to one linear equation.

$$-\mathcal{L}_{e_0}\mu = -C^2_{03} + \mu\left(C^2_{02} - C^3_{03}\right) + \mu^2 C^3_{02} \tag{4.18}$$

$$\mathcal{L}_{e_1}\mu = -C^2_{31} + \mu\left(C^2_{21} - C^3_{31}\right) + \mu^2 C^3_{21} \tag{4.19}$$

$$C^2_{01} + \mu C^3_{01} = 0 \tag{4.20}$$

Here, equations (4.18) and (4.19) lead one more equation to consider

$$\mathcal{L}_{[e_0,e_1]}\mu = \mathcal{L}_{e_0}\mathcal{L}_{e_1}\mu - \mathcal{L}_{e_1}\mathcal{L}_{e_0}\mu. \tag{4.21}$$

$de^i = -\dfrac{1}{2}C^i_{jk} e^j \wedge e^k$ is indicated above hence, $de^i(e_j, e_k) = -C^i_{jk}$. Using the following formula,

$$d\omega(X,Y) = [X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])]$$

In particular,

$$\begin{aligned} de^i(e_j, e_k) &= \left[e_j(e^i(e_k)) - e_k(e^i(e_j)) - e^i([e_j, e_k])\right] \\ &= -e^i([e_j, e_k]) \end{aligned}$$

holds for i=1,2,3. Therefore, $[e_j, e_k] = C^l_{jk} e_l$, then

$$[e_0, e_1] = C^0_{01} e_0 + C^1_{01} e_1 + C^2_{01} e_2 + C^3_{01} e_3 \tag{4.22}$$

We have forced equations to be $C^2_{01} = C^3_{01} = 0$ then (4.20) holds.

5. CONSTRUCTION OF BI-HAMILTONIAN STRUCTURE IN FOUR DIMENSIONS

We claim that given nonvanishing vector field X can make Hamiltonian by scaling some smooth function f. Explicitly, there corresponds to a symplectic manifold so that fX is Hamiltonian vector field.

Now we are going to elaborate the details about what are trying to do. Let $e_0 = \frac{w}{\|w\|}$. Choose e_1 and e_3 so that $[e_0, e_1] = C_{01}^0 e_0 + C_{01}^1 e_1$ and $e^3 \wedge de^3 \neq 0$. Let $\alpha = \frac{1}{g}\left(e^2 + \mu e^3\right)$. Observe that $g\alpha \wedge d(g\alpha) = 0$ has same solution with $\alpha \wedge d\alpha = 0$ (See proposition 4.0.3). That is, $\alpha \wedge d\alpha = 0$ corresponds to equations (4.18), (4.19), (4.20). Firstly observe that $e^3 \wedge de^3 = e^3 \wedge \left(\frac{-1}{2} C_{ij}^3 e^i \wedge e^j\right) = -C_{01}^3 e^0 \wedge e^1 \wedge e^3 - C_{02}^3 e^0 \wedge e^2 \wedge e^3 - C_{12}^3 e^1 \wedge e^2 \wedge e^3$. Since $[e_0, e_1] = C_{01}^0 e_0 + C_{01}^1 e_1$, we infer that either C_{02}^3 or C_{12}^3 should be nonzero.

Lemma 5.0.1. *$\mathcal{L}_{e_0}\alpha \wedge \alpha = 0$ and $\mathcal{L}_{e_1}\alpha \wedge \alpha = 0$ are equivalent to $\alpha \wedge d\alpha = 0$.*

Proof. Observe that if g is a function which never attends zero, then

$$\mathcal{L}_{e_i}(g\alpha) \wedge g\alpha = 0 \Rightarrow \left(\left(\mathcal{L}_{e_i} g\right)\alpha + g\mathcal{L}_{e_i}\alpha\right) \wedge g\alpha = 0 \Rightarrow g^2 \mathcal{L}_{e_i}\alpha \wedge \alpha = 0 \tag{5.1}$$

$$\text{That is, } \mathcal{L}_{e_i}\alpha \wedge \alpha = 0 \text{ is equivalent to} \tag{5.2}$$

$$\mathcal{L}_{e_i}\left(e^2 + \mu e^3\right) \wedge \left(e^2 + \mu e^3\right) = 0 \text{ for } i = 1, 2. \tag{5.3}$$

$\mathcal{L}_{e_0} e^2 = di_{e_0} e^2 + i_{e_0} de^2 = i_{e_0} de^2 = i_{e_0}\left(-\frac{1}{2} C_{ij}^2 e^i \wedge e^j\right) = -C_{0k}^2 e^k$ for $k = 1, 2, 3$.
$\mathcal{L}_{e_0}\left(\mu e^3\right) = di_{e_0}\left(\mu e^3\right) + i_{e_0} d\left(\mu e^3\right) = i_{e_0} d\left(\mu e^3\right) = i_{e_0}\left(d\mu \wedge e^3 + \mu de^3\right) = \left(i_{e_0} d\mu\right) e^3 + \mu i_{e_0}\left(-\frac{1}{2} C_{ij}^3 e^i \wedge e^j\right) = \left(\mathcal{L}_{e_0}\mu\right) e^3 - \mu C_{0k}^3 e^k$. Then,
$\mathcal{L}_{e_0}\left(e^2 + \mu e^3\right) = -C_{0k}^2 e^k + \left(\mathcal{L}_{e_0}\mu e^3 - \mu C_{0k}^3 e^k\right)$. Explicitly,

$$\mathcal{L}_{e_0}\left(e^2 + \mu e^3\right) \wedge \left(e^2 + \mu e^3\right) = 0 \tag{5.4}$$

$$\iff -\left(C_{01}^2 + \mu C_{01}^3\right) e^1 \wedge e^2 - \mu\left(C_{01}^2 + \mu C_{01}^3\right) e^1 \wedge e^3 \tag{5.5}$$

$$-\left(\mu C_{02}^2 - C_{03}^2 + \mathcal{L}_{e_0}\mu + \mu^2 C_{02}^3 - \mu C_{03}^3\right) e^2 \wedge e^3 = 0 \tag{5.6}$$

$$\iff C_{01}^2 + \mu C_{01}^3 = 0 \text{ and } -\mathcal{L}_{e_0}\mu = -C_{03}^2 + \mu\left(C_{02}^2 - C_{03}^3\right) + \mu^2 C_{02}^3. \tag{5.7}$$

If we can do same calculations for e_1, that is,

$\mathcal{L}_{e_1} e^2 = di_{e_1} e^2 + i_{e_1} de^2 = i_{e_1} de^2 = i_{e_1} \left(-\frac{1}{2} C^2_{ij} e^i \wedge e^j\right) = -C^2_{1k} e^k.$

$\mathcal{L}_{e_1} (\mu e^3) = di_{e_1} e^3 + i_{e_1} d(\mu e^3) = i_{e_1} d(\mu e^3) = i_{e_1} (d\mu \wedge e^3 + \mu de^3)$

$= \mathcal{L}_{e_1} \mu e^3 + \mu i_{e_1} \left(-\frac{1}{2} C^3_{ij} e^i \wedge e^j\right) = \mathcal{L}_{e_1} \mu e^3 - \mu C^3_{1k} e^k$ for $k = 0, 2, 3$. Thus,

$\mathcal{L}_{e_1} (e^2 + \mu e^3) = -C^2_{1k} e^k + \mathcal{L}_{e_1}(\mu) e^3 - \mu C^3_{1k} e^k$ for $k = 0, 2, 3$.

Then, we get

$$\mathcal{L}_{e_1} \left(e^2 + \mu e^3\right) \wedge \left(e^2 + \mu e^3\right) = 0 \tag{5.8}$$

$$\iff -\left(C^2_{10} + \mu C^3_{10}\right) e^0 \wedge e^2 - \mu \left(C^2_{10} + \mu C^3_{10}\right) e^0 \wedge e^3 \tag{5.9}$$

$$-\left(\mu C^2_{12} - C^2_{13} + \mathcal{L}_{e_1} \mu + \mu^2 C^3_{12} - \mu C^3_{13}\right) e^2 \wedge e^3 = 0 \tag{5.10}$$

$$\iff C^2_{10} + \mu C^3_{10} = 0 \text{ and } \mathcal{L}_{e_1} \mu = -C^2_{31} + \mu \left(C^2_{21} - C^3_{31}\right) + \mu^2 C^3_{21}. \tag{5.11}$$

Since solution of $\alpha \wedge d\alpha = 0$ corresponds to equations (4.18),(4.19),(4.20). We conclude that they are equivalent.

□

Lemma 5.0.2. *Let* $\alpha = \frac{1}{g}(e^2 + \mu e^3)$. *Then*

$$\mathcal{L}_{e_0} \alpha = f_0 \alpha \iff \begin{cases} \mathcal{L}_{e_0} \log g + C^2_{02} + \mu C^3_{02} + f_0 = 0 \\ C^2_{03} + \mu C^3_{03} - \mathcal{L}_{e_0} \mu + \mu \mathcal{L}_{e_0} \log g + f_0 \mu = 0 \end{cases} \tag{5.12}$$

$$\mathcal{L}_{e_1} \alpha = f_1 \alpha \iff \begin{cases} \mathcal{L}_{e_1} \log g + C^2_{12} + \mu C^3_{12} + f_1 = 0 \\ C^2_{13} + \mu C^3_{13} - \mathcal{L}_{e_1} \mu + \mu \mathcal{L}_{e_1} \log g + f_1 \mu = 0 \end{cases} \tag{5.13}$$

Proof.

$$\mathcal{L}_{e_0} \alpha = f_0 \alpha \iff \mathcal{L}_{e_0} \left(\frac{1}{g}\left(e^2 + \mu e^3\right)\right) = \frac{f_0}{g}\left(e^2 + \mu e^3\right) \tag{5.14}$$

$$\mathcal{L}_{e_1} \alpha = f_1 \alpha \iff \mathcal{L}_{e_1} \left(\frac{1}{g}\left(e^2 + \mu e^3\right)\right) = \frac{f_1}{g}\left(e^2 + \mu e^3\right) \tag{5.15}$$

Equation (5.14) can be unwound as follows:

$$\mathcal{L}_{e_0}\left(\frac{1}{g}\right)e^2+\frac{1}{g}\mathcal{L}_{e_0}e^2+\mathcal{L}_{e_0}\left(\frac{\mu}{g}\right)e^3+\frac{\mu}{g}\mathcal{L}_{e_0}e^3=\frac{f_0}{g}\left(e^2+\mu e^3\right) \tag{5.16}$$

$$\frac{-\mathcal{L}_{e_0}\log g}{g}e^2-\frac{1}{g}C^2_{0k}e^k+\frac{g\mathcal{L}_{e_0}\mu-\left(\mathcal{L}_{e_0}g\right)\mu}{g^2}e^3-\frac{\mu}{g}C^3_{0k}e^k=\frac{f_0}{g}\left(e^2+\mu e^3\right) \tag{5.17}$$

If we multiply last equation by g and consider the sum on $k=1,2,3$ then

$$\begin{aligned}\left(C^2_{01}+\mu C^3_{01}\right)e^1+\left(\mathcal{L}_{e_0}\log g+C^2_{02}+\mu C^3_{02}+f_0\right)e^2+\\ \left(C^2_{03}+\mu C^3_{03}-\mathcal{L}_{e_0}\mu+\mu\mathcal{L}_{e_0}\log g+f_0\mu\right)e^3=0\end{aligned} \tag{5.18}$$

That is,

$$\mathcal{L}_{e_0}\log g+C^2_{02}+\mu C^3_{02}+f_0=0 \tag{5.19}$$

$$C^2_{03}+\mu C^3_{03}-\mathcal{L}_{e_0}\mu+\mu\mathcal{L}_{e_0}\log g+f_0\mu=0 \tag{5.20}$$

Coefficient of e^1 are already zero by construction.

Equation (5.15) can be unwound as follows:

$$\mathcal{L}_{e_1}\left(\frac{1}{g}\right)e^2+\frac{1}{g}\mathcal{L}_{e_1}e^2+\mathcal{L}_{e_1}\left(\frac{\mu}{g}\right)e^3+\frac{\mu}{g}\mathcal{L}_{e_1}e^3=\frac{f_1}{g}\left(e^2+\mu e^3\right) \tag{5.21}$$

$$\frac{-\mathcal{L}_{e_1}\log g}{g}e^2-\frac{1}{g}C^2_{1k}e^k+\frac{g\mathcal{L}_{e_1}\mu-\left(\mathcal{L}_{e_1}g\right)\mu}{g^2}e^3-\frac{\mu}{g}C^3_{1k}e^k=\frac{f_1}{g}\left(e^2+\mu e^3\right) \tag{5.22}$$

If last equation is multiplied by g and consider the sum on $k=0,2,3$, we obtain the following:

$$\begin{aligned}\left(C^2_{10}+\mu C^3_{10}\right)e^1+\left(\mathcal{L}_{e_1}\log g+C^2_{12}+\mu C^3_{12}+f_1\right)e^2+\\ \left(C^2_{13}+\mu C^3_{13}-\mathcal{L}_{e_1}\mu+\mu\mathcal{L}_{e_1}\log g+f_1\mu\right)e^3=0\end{aligned} \tag{5.23}$$

That is,

$$\mathcal{L}_{e_1}\log g+C^2_{12}+\mu C^3_{12}+f_1=0 \tag{5.24}$$

$$C^2_{13}+\mu C^3_{13}-\mathcal{L}_{e_1}\mu+\mu\mathcal{L}_{e_1}\log g+f_1\mu=0 \tag{5.25}$$

□

Lemma 5.0.3. *There are two α satifying $\alpha \wedge d\alpha = 0$.*

Proof. Firstly, observe that

$$\mathcal{L}_{e_0}\alpha \wedge \alpha = 0 \tag{5.26}$$

$$\mathcal{L}_{e_1}\alpha \wedge \alpha = 0 \tag{5.27}$$

equation (5.26) implies that equations (4.18), (4.20). Equation (5.27) implies that equations (4.19), (4.20). On the other hand,

$$\mathcal{L}_{e_0}\alpha \wedge \alpha = 0 \iff \mathcal{L}_{e_0}\alpha = f_0\alpha \tag{5.28}$$

$$\mathcal{L}_{e_1}\alpha \wedge \alpha = 0 \iff \mathcal{L}_{e_1}\alpha = f_1\alpha \tag{5.29}$$

$$\mathcal{L}_{e_1}\left(\mathcal{L}_{e_0}\alpha\right) = \mathcal{L}_{e_1}\left(f_0\alpha\right) = \left(\mathcal{L}_{e_1}f_0\right)\alpha + f_0\mathcal{L}_{e_1}\alpha = \left(\mathcal{L}_{e_1}f_0\right)\alpha + f_0f_1\alpha \tag{5.30}$$

$$\mathcal{L}_{e_0}\left(\mathcal{L}_{e_1}\alpha\right) = \mathcal{L}_{e_0}\left(f_1\alpha\right) = \left(\mathcal{L}_{e_0}f_1\right)\alpha + f_1\mathcal{L}_{e_0}\alpha = \left(\mathcal{L}_{e_0}f_1\right)\alpha + f_1f_0\alpha \tag{5.31}$$

If previous two equations are subtracted from each other, four equations above generate the following (5.32), which is integrability condition. More explicitly,

$\mathcal{L}_{[e_0,e_1]}\alpha = \mathcal{L}_{e_0}\mathcal{L}_{e_1}\alpha - \mathcal{L}_{e_1}\mathcal{L}_{e_0}\alpha = \mathcal{L}_{e_0}\left(f_1\alpha\right) - \mathcal{L}_{e_1}\left(f_0\alpha\right) = \left(\mathcal{L}_{e_0}f_1\right)\alpha - \left(\mathcal{L}_{e_1}f_0\right)\alpha$

Since $\mathcal{L}_{fX}\alpha = f\mathcal{L}_X\alpha + df \wedge i_X\alpha$ and $i_{e_0}\alpha = i_{e_1}\alpha = 0$,

$\mathcal{L}_{[e_0,e_1]}\alpha = C^0_{01}\mathcal{L}_{e_0}\alpha + C^1_{01}\mathcal{L}_{e_1}\alpha = C^0_{01}f_0\alpha + C^1_{01}f_1\alpha$

$C^0_{01}f_0\alpha + C^1_{01}f_1\alpha = \left(\mathcal{L}_{e_0}f_1\right)\alpha - \left(\mathcal{L}_{e_1}f_0\right)\alpha$ where $\alpha \neq 0$ by construction, thus

$$C^0_{01}f_0 + C^1_{01}f_1 + \mathcal{L}_{e_1}f_0 - \mathcal{L}_{e_0}f_1 = 0 \tag{5.32}$$

If we plug $\alpha = \frac{1}{g}\left(e^2 + \mu e^3\right)$ into equations (5.28) and (5.29), they suggest that

$$\mathcal{L}_{e_0}\log g + C^2_{02} + \mu C^3_{02} = -f_0 \tag{5.33}$$

$$C^2_{03} + \mu C^3_{03} + \mu\mathcal{L}_{e_0}\log g - \mathcal{L}_{e_0}\mu = -f_0\mu \tag{5.34}$$

$$\mathcal{L}_{e_1}\log g + C^2_{12} + \mu C^3_{12} = -f_1 \tag{5.35}$$

$$C^2_{13} + \mu C^3_{13} + \mu\mathcal{L}_{e_1}\log g - \mathcal{L}_{e_1}\mu = -f_1\mu \tag{5.36}$$

However $e^3 \wedge de^3 \neq 0 \Rightarrow C^3_{12} \neq 0$ or $C^3_{02} \neq 0$. Unless $C^3_{12} = 0$, arrange g, set $f_0 = 0$.

Then, integrability condition (See equation (5.32)) indicates that $\mathcal{L}_{e_0} f_1 = C^1_{01} f_1$. Actually this equation suggests two solutions trivial and nontrivial solution for f_1. Going back to the equation (5.35) we can determine μ . If $C^3_{02} \neq 0$ then arrange g to make $f_1 = 0$ the integrability condition (see equation (5.32)) indicates that $\mathcal{L}_{e_1} f_0 = -C^0_{01} f_0$ which suggests trivial and nontrivial solution for f_0. Going back to the equation (5.33), we obtain two solutions for μ. Since both cases reveal two μ's, we get two different α's satisfying $\alpha \wedge d\alpha = 0$. □

Lemma 5.0.4. *There exist nondegenerate* J_1 *and* J_2 *satisfying* $i_{e_0} J_i = \alpha_i$ *for* $i = 1, 2$.

Proof. For solely existence, put $J_i = e^0 \wedge \alpha_i + i_{e_0} \Gamma_i$ where Γ is completely arbitrary three form. As for the nondegeneracy, let $\phi(v) = i_v J$. It is clear that ϕ is a linear map. Observe that $[i_v J(w)] = [J(v,w)] = [v]^T [J] [w]$ and $[i_v J(w)] = [i_v J]^T [w]$. Using skew symmetry of J, $[\phi(v)] = [i_v J] = -[J][v]$, that is, $[\phi] = -[J]$. Here, $[.,.]$ and T represent matrix and transpose of the matrix respectively. Since rank of J is even, rank of ϕ can be 0,2,4. Suppose that dim(Ker ϕ) $= 2$ say basis of Ker $\phi = \{v_1, v_2\}$ and $v_1 \perp v_2$. Using existence $i_{e_0} J = \alpha \Rightarrow i_{v_i} \alpha = i_{v_i} i_{e_0} J = -i_{e_0} i_{v_i} J = 0$ because $v_i \epsilon$Ker ϕ that is $i_{v_i} J = 0$. Put $\alpha^* = g(e_2 + \mu e_3)$. On the other hand, e_0 can not be linear combination of v_1 and v_2 assume otherwise, say $e_0 = a_1 v_1 + a_2 v_2$, then $\alpha = i_{e_0} J = a_1 i_{v_1} J + a_2 i_{v_2} J = 0$ which contradicts with being $\alpha = \frac{1}{g}(e^2 + \mu e^3)$ which is nonzero. Assume that α^* is a linear combination of e_0, v_1, v_2 ,i.e, $\alpha^* = a e_0 + a_1 v_1 + a_2 v_2$. Then $1 + \mu^2 = \alpha(\alpha^*) = a i_{e_0}\alpha + a_1 i_{v_1}\alpha + a_2 i_{v_2}\alpha$ but last term should be zero because $i_{e_0}\alpha = i_{v_1}\alpha = i_{v_2}\alpha = 0$ however $1 + \mu^2 \neq 0$. Therefore, $\{\alpha^*, v_1, v_2, e_0\}$ is a basis. Consider the dual basis $\left\{\tilde{e}^0, v^1, v^2, \frac{\alpha}{1+\mu^2}\right\}$ for $\{\alpha^*, v_1, v_2, e_0\}$. One can take $\{\tilde{e}^0, v^1, v^2, \alpha\}$as a basis. Let $e^0 = a\tilde{e}^0 + b_1 v^1 + b_2 v^2 + c\alpha$. Since $i_{e_0}\tilde{e}^0 = 1$, $e^0 = \tilde{e}^0 + b_1 v^1 + b_2 v^2 + c\alpha$. Put $dV = v^1 \wedge v^2 \wedge \alpha \wedge \tilde{e}^0 = v^1 \wedge v^2 \wedge \alpha \wedge e^0$ and $\Gamma = i_u dV$.

$$i_{v_1} J = i_{v_1}\left(e^0 \wedge \alpha\right) + i_{v_1} i_{e_0} i_u \left(v^1 \wedge v^2 \wedge \alpha \wedge e^0\right) = 0$$

$$\left(i_{v_1} e^0\right)\alpha - i_u\left(v^2 \wedge \alpha\right) = b_1\alpha - \left(i_u v^2\right)\alpha + \left(i_u\alpha\right) v^2 = \left(b_1 - i_u v^2\right)\alpha + \left(i_u\alpha\right) v^2 = 0$$

$$\Rightarrow i_u\alpha = 0 \text{ and } b_1 - i_u v^2 = 0$$

Similarly,

$$i_{v_2}J = i_{v_2}\left(e^0 \wedge \alpha\right) + i_{v_2}i_{e_0}i_u\left(v^1 \wedge v^2 \wedge \alpha \wedge e^0\right) = 0$$
$$\left(i_{v_2}e^0\right)\alpha + i_u\left(v^1 \wedge \alpha\right) = b_2\alpha + \left(i_u v^1\right)\alpha - \left(i_u\alpha\right)v^1 = \left(b_2 + i_u v^1\right)\alpha - \left(i_u\alpha\right)v^1 = 0$$
$$\Rightarrow i_u\alpha = 0 \text{ and } b_2 + i_u v^1 = 0$$

$i_u\alpha = 0 \Rightarrow u = c_1v_1 + c_2v_2 + c_3e_0.$

$$c_2 + c_3 i_{e_0}v^2 = \quad i_u v^2 \quad = b_1 \tag{5.37}$$
$$c_1 + c_3 i_{e_0}v^1 = \quad i_u v^1 \quad = -b_2 \tag{5.38}$$

Since $i_{e_0}v^1 = i_{e_0}v^2 = 0$, equations above suggest $c_1 = -b_2$ and $c_2 = b_1$. Hence, $u = -b_2v_1 + b_1v_2 + c_3e_0$. Since $\Gamma = i_u dV$ is completely arbitrary and if c_i's are changed, for example $c_1 \neq -b_2$, then J may have full rank. If $i_{v_1}e^0 = i_{v_2}e^0 = 0$, simply put $u = v_1$. More accurately, if rank(J) is two, then $c_1 = -b_2$, $c_2 = b_1$ hold otherwise nonzero J has full rank. □

Lemma 5.0.5. *There exists β satisfying $i_{e_0}d\beta = \frac{\alpha}{f}$.*

Proof. Let $\beta_i = b_{i1}dh_1 + b_{i2}dh_2$ where $\alpha_i = i_{e_0}J_i = f_i dh_i$, $i = 1,2$. We are going to determine b_{i1} and b_{i2}. Consider $d\beta_i = db_{i1} \wedge dh_1 + db_{i2} \wedge dh_2$, $i = 1,2$. Applying i_{e_0},

$$\frac{\alpha_1}{f_1} = \quad i_{e_0}d\beta_1 = \left(i_{e_0}db_{11}\right)dh_1 + \left(i_{e_0}db_{12}\right)dh_2 \quad = dh_1 \tag{5.39}$$
$$\frac{\alpha_2}{f_2} = \quad i_{e_0}d\beta_2 = \left(i_{e_0}db_{21}\right)dh_1 + \left(i_{e_0}db_{22}\right)dh_2 \quad = dh_2. \tag{5.40}$$

Since dh_1 and dh_2 are linearly independent, they suggest $i_{e_0}db_{11} = i_{e_0}db_{22} = 1$ and $i_{e_0}db_{12} = i_{e_0}db_{21} = 0$. Choose coordinates x^1 and x^2 so that span$\{dx^1, dx^2\}$=span$\{e^0, e^1\}$. Then

$$dx^1 = a_0^1e^0 + a_1^1e^1 \tag{5.41}$$
$$dx^2 = a_0^2e^0 + a_1^2e^1 \tag{5.42}$$

Since both a_0^1 and a_1^1 can not be zero, say $a_0^1 \neq 0$ then $i_{e_0}\left(\frac{dx^1}{a_0^1}\right) = 1$.
Let $db_{11} = \frac{dx^1}{a_0^1} \Rightarrow b_{11} = \int \frac{1}{a_0^1}dx^1$. If $a_0^1 = 0$, replace x^1 by x^2 then new a_0^1 should be different

than 0, otherwise dx^1 and dx^2 are independent of e^0, which is false. Thus either case suggest that there is b_{11} so that $i_{e_1} db_{11} = 1$. Put $b_{22} = b_{11}$ then $i_{e_0} db_{11} = i_{e_0} db_{22} = 1$. Choose b_{12} such that $\mathcal{L}_{e_0} b_{12} = 0$ and put $b_{21} = -b_{12}$,so $i_{e_0} db_{12} = i_{e_0} db_{21} = 0$ is found. That is,

$$\beta_1 = b_{11} dh_1 + b_{12} dh_2 \tag{5.43}$$

$$\beta_2 = -b_{12} dh_1 + b_{11} dh_2 \tag{5.44}$$

Therefore, proof is completed. However, equations above reveal that

$$d\beta_1 \wedge dh_1 = d\beta_2 \wedge dh_2 \tag{5.45}$$

Since $i_{e_0} d\beta_i = dh_i$, $i_{e_0} (d\beta_1 \wedge dh_1) = i_{e_0} (d\beta_2 \wedge dh_2) = 0$, there is some volume form Ω such that $i_{e_0} \Omega = d\beta_1 \wedge dh_1 = d\beta_2 \wedge dh_2$. □

Lemma 5.0.6. *If $J = e^0 \wedge \alpha + i_{e_0} \Gamma$ where $\alpha \wedge d\alpha = 0$ and α is defined as above then there exist a suitable three form Γ and a nonzero function f such that $d\left(\frac{J}{f}\right) = 0$.*

Proof. $\alpha \wedge d\alpha = 0$ implies that $\frac{\alpha}{f} = dh$ by Frobenius theorem. That is, $d\left(\frac{\alpha}{f}\right) = 0$.
$\frac{J}{f} = e^0 \wedge \frac{\alpha}{f} + \frac{i_{e_0}\Gamma}{f} \Rightarrow d\left(\frac{J}{f}\right) = de^0 \wedge \frac{\alpha}{f} + d\left(\frac{i_{e_0}\Gamma}{f}\right) = 0$
$\Rightarrow d\left(\frac{i_{e_0}\Gamma}{f}\right) = -d\left(e^0 \wedge \frac{\alpha}{f}\right) \Rightarrow i_{e_0}\Gamma = -e^0 \wedge \alpha + f d\beta$ where β is arbitrary 1-form.
$0 = i_{e_0} i_{e_0} \Gamma = -\alpha + f i_{e_0} d\beta$. Then $i_{e_0} d\beta = \frac{\alpha}{f}$ determines β.
Put $\Gamma = f e^0 \wedge d\beta + i_{e_0}\gamma$ where γ is arbitrary 4-form, in particular put $\gamma = 0$. Then $\frac{J}{f} = d\beta$, that is, $d\left(\frac{J}{f}\right) = 0$. □

Up to this point, we have shown that there are two α's satisfying $\alpha \wedge d\alpha = 0$. Then we have shown that nondegenerate J exists as a solution of $i_{e_0} J = \alpha$ where α satisfies $\alpha \wedge d\alpha = 0$. Note that $i_{fe_0} \frac{J}{f} = i_{e_0} J = \alpha$. Then, we have shown that it is possible to choose J satifying $i_{e_0} J = \alpha$ (i.e, $J = e^0 \wedge \alpha + i_{e_0}\Gamma$) and $d\left(\frac{J}{f}\right) = 0$. More explicitly, we have shown that nondegeneracy and closedness of J are revealed seperately. Now, we are going to suggest that there exist J so that $d\left(\frac{J}{f}\right) = 0$ and rank$\left(\frac{J}{f}\right) = 4$. We need the following propositions:

Proposition 5.0.7. *Let $J = e^0 \wedge \alpha + i_{e_0}\Gamma$ where $\alpha = f dh$ as above. If rank$(J) = 2$ then $J \wedge dh = 0$.*

Proof. Without lost of generality, we may take J, Γ as $\frac{J}{f}$ and $\frac{\Gamma}{f}$, respectively. That is, $J = e^0 \wedge dh + i_{e_0}\Gamma$. Let $\phi(v) = i_v J$ and $\mathrm{Ker}(\phi) = \{v_1, v_2\}$ where v_1 and v_2 are orthogonal unit vectors. We have already observed that $\{\alpha^*, v_1, v_2, e_0\}$ is a basis. $\{\tilde{e}^0, v^1, v^2, \alpha\}$ is considered as a dual basis for $\{\alpha^*, v_1, v_2, e_0\}$. We keep the same notation as in the lemma 5.0.4, that is, $\tilde{e}^0 = e^0 + b_1 v^1 + b_2 v^2 + c\alpha$ and $dV = v^1 \wedge v^2 \wedge \alpha \wedge e^0$ and $\Gamma = i_u dV$.
Observe that

$$i_{v_1} J = \left(i_{v_1} e^0\right) dh - \left(i_{v_1} dh\right) e^0 + i_{v_1} i_{e_0} \Gamma = 0$$

$$i_{v_2} J = \left(i_{v_2} e^0\right) dh - \left(i_{v_2} dh\right) e^0 + i_{v_2} i_{e_0} \Gamma = 0$$

$$i_{e_0} J = dh \Rightarrow i_{v_1} dh = i_{v_1} i_{e_0} J = -i_{e_0} i_{v_1} J = 0$$

$$i_{e_0} J = dh \Rightarrow i_{v_2} dh = i_{v_2} i_{e_0} J = -i_{e_0} i_{v_2} J = 0$$

Since $i_{v_1} dh = i_{v_2} dh = 0,$ the first two equations becomes,

$$i_{v_1} J = \left(i_{v_1} e^0\right) dh + i_{v_1} i_{e_0} \Gamma = 0$$

$$i_{v_2} J = \left(i_{v_2} e^0\right) dh + i_{v_2} i_{e_0} \Gamma = 0$$

We infer that $\left(i_{v_2} e^0\right) i_{v_1} i_{e_0} \Gamma - \left(i_{v_1} e^0\right) i_{v_2} i_{e_0} \Gamma = 0$

$$i_{\left(i_{v_2} e^0\right) v_1 - \left(i_{v_1} e^0\right) v_2} i_{e_0} \Gamma = 0$$

Put $v_3 = \left(i_{v_2} e^0\right) v_1 - \left(i_{v_1} e^0\right) v_2 \Rightarrow i_{v_3} i_{e_0} \Gamma = 0$

Put $v^3 = \left(i_{v_2} e^0\right) v^1 - \left(i_{v_1} e^0\right) v^2$ then, $i_{e_0} v^3 = \left(i_{v_2} e^0\right) i_{e_0} v^1 - \left(i_{v_1} e^0\right) i_{e_0} v^2 = 0$

Since $i_{v_1} dh = i_{v_2} dh = 0,\ i_{v_3} dh = 0$

$$i_{e_0} dh = i_{e_0} i_{e_0} J = 0.$$

Note that

$$i_{v_3} v^3 = \left(i_{v_1} e^0\right)^2 + \left(i_{v_2} e^0\right)^2 = 0 \iff i_{v_1} e^0 = i_{v_2} e^0 \ = \ 0 \tag{5.46}$$

$$\iff b_1 = b_2 = 0 \iff v^3 \ = \ 0. \tag{5.47}$$

Assume that v^3 is nonvanishing. Observe that $\{dh, v^3\}$ are linearly independent because $i_{\alpha^*} v^3 = 0$ but $i_{\alpha^*} dh \neq 0$. We can not write e^0 in terms of dh and v^3, if we could write that $1 = i_{e_0} e^0 = k_1 i_{e_0} dh + k_2 i_{e_0} v^3$ but $i_{e_0} dh = i_{e_0} v^3 = 0$ thus $1 = 0$,i.e, $\{e^0, dh, v^3\}$ is linearly independent. Using Hodge star operator, define $v^4 = *\left(e^0 \wedge dh \wedge v^3\right)$ where * represents Hodge star operator. By definition of Hodge star operator reveals that v^4 is a linear combination of v^1 and v^2 because of the fact that

$$e^0 \wedge dh \wedge v^3 \wedge \lambda = \langle v^4, \lambda \rangle dV \tag{5.48}$$

holds for all one form λ's. If v^4 is the linear combination of e^0, dh, v^3 then this means that $\langle v^4, v^4 \rangle = 0$. Then one understands that $\{e^0, dh, v^3, v^4\}$ is a basis. Since $v^3 = -b_2 v^1 + b_1 v^2$, the equation (5.48) suggests the $v^4 = \frac{b_1}{f} v^1 + \frac{b_2}{f} v^2$ where $\alpha = f dh$. Observe that $i_{v_3} v^4 = 0$ and there is a function γ satisfying $dV = \gamma e^0 \wedge dh \wedge v^3 \wedge v^4$. Since we have found $i_{v_3} i_{e_0} \Gamma = 0$ above,

$$\begin{aligned} i_{v_3} i_{e_0} \Gamma = i_{v_3} i_{e_0} i_u \left(\gamma e^0 \wedge dh \wedge v^3 \wedge v^4\right) &= 0 && (5.49)\\ -i_{v_3} i_u \left(dh \wedge v^3 \wedge v^4\right) &= 0 && (5.50)\\ i_u \Big(\left(i_{v_3} v^3\right) dh \wedge v^4 - i_{v_3} \left(dh \wedge v^4\right) \wedge v^3 \Big) &= 0. && (5.51) \end{aligned}$$

Since $i_{v_3} dh = 0$ and $i_{v_3} v^4 = 0$ and $i_{v_3} v^3 = b_1^2 + b_2^2 \neq 0$, equation (5.51) can be reduced to

$$i_u \left(dh \wedge v^4\right) = 0. \tag{5.52}$$

Since $i_{e_0} v^1 = i_{e_0} v^2 = 0$, one can reach that $i_{e_0} v^3 = i_{e_0} v^4 = 0$. Therefore, the following equations reveal that

$$\begin{aligned} i_{e_0} \Gamma &= i_{e_0} i_u dV = \gamma i_{e_0} i_u \left(e^0 \wedge dh \wedge v^3 \wedge v^4\right) && (5.53)\\ -\gamma i_u \left(dh \wedge v^3 \wedge v^4\right) &= \gamma \left(i_u v^3\right) dh \wedge v^4 + \gamma i_u \left(dh \wedge v^4\right) \wedge v^3 && (5.54)\\ &= \gamma \left(i_u v^3\right) dh \wedge v^4. && (5.55) \end{aligned}$$

Last equality occurs by equations (5.52). Since $J = e^0 \wedge dh + i_{e_0} \Gamma$,

$$J \wedge dh = 0$$

is obtained. If $v^3 = 0$, we have observed that $i_{v_1} e^0 = i_{v_2} e^0 = 0$. In this case, $\{e^0, v^1, v^2, \alpha\}$ is dual to $\{e_0, v_1, v_2, \alpha^*\}$. In particular, $i_{v_1} dh = i_{v_2} dh = 0$. Then see the first two equation at the beginning of the lemma, that is ,

$$i_{v_1}J = \quad i_{v_1}i_{e_0}\Gamma \quad = 0 \tag{5.56}$$

$$i_{v_2}J = \quad i_{v_2}i_{e_0}\Gamma \quad = 0. \tag{5.57}$$

More explicitly,

$$i_{v_1}J = i_{v_1}i_{e_0}i_u dV = i_u\left(v^2 \wedge \alpha\right) = 0 \tag{5.58}$$

$$i_{v_2}J = i_{v_2}i_{e_0}i_u dV = i_u\left(v^1 \wedge \alpha\right) = 0 \tag{5.59}$$

If we take a closer look to the following equations, we see that $i_{e_0}\Gamma = 0$.

$$i_{e_0}\Gamma = i_{e_0}i_u dV \quad = \quad -f i_{e_0}i_u\left(e^0 \wedge v^1 \wedge v^2 \wedge dh\right) = f i_u\left(v^1 \wedge v^2 \wedge dh\right) \tag{5.60}$$

$$= \quad f\left(i_u v^1\right) v^2 \wedge dh + f i_u\left(v^2 \wedge dh\right) \wedge v^1 \tag{5.61}$$

$$= \quad f\left(i_u v^1\right) v^2 \wedge dh. \tag{5.62}$$

Last equality occurs by equation (5.58). Similarly,

$$i_{e_0}\Gamma \quad = \quad -f i_u\left(v^2 \wedge v^1 \wedge dh\right) = -f\left(i_u v^2\right) v^1 \wedge dh + f v^2 \wedge i_u\left(v^1 \wedge dh\right) \tag{5.63}$$

$$= \quad -f\left(i_u v^2\right) v^1 \wedge dh \tag{5.64}$$

Last equality occurs by equation (5.59). Since equations (5.62) and (5.64) represent same form but different base elements, which leads to conclude that $i_{e_0}\Gamma = 0$. That is, $J = e^0 \wedge dh$. Hence,

$$J \wedge dh = 0.$$

□

Proposition 5.0.8. *If J, dh $(J \neq 0)$ are defined as above then*

$$rank(J) = \begin{cases} 2 & \textit{iff} \quad J \wedge dh = 0 \\ 4 & \textit{iff} \quad J \wedge dh \neq 0 \end{cases}$$

Proof. Put $f^4 = dh$. Choose a basis $\{f^1, f^2, f^3, f^4\}$ and corresponding dual basis

$\{f_1, f_2, f_3, f_4\}$ so that $i_{f_k} f^j = \delta^j_k$. Suppose that J has full rank and $J \wedge dh = 0$ then $J = \theta \wedge dh$ for some one form θ. Explicitly, $\theta = a_1 f^1 + a_2 f^2 + a_3 f^3$. Since J has full rank , $i_v J = i_v (\theta \wedge dh) = i_v (a_1 f^1 \wedge dh + a_2 f^2 \wedge dh + a_3 f^3 \wedge dh) \neq 0$ for all non-vanishing v. However, put $v = a_2 f_1 - a_1 f_2$ then $i_v J = (a_1 a_2 - a_1 a_2)\, dh = 0$. If $a_1 = a_2 = 0$, simply put $v = f_1$ then $i_v J = 0$. We get a contradiction. Since rank of J may be 2 or 4, we infer that rank of J is two if $J \wedge dh = 0$. □

Proposition 5.0.9. *There are two nondegenerate closed form* $\dfrac{J_i}{f_i}$ *so that* $i_{e_0} \dfrac{J_i}{f_i} = dh_i$ *for* $i = 1, 2$.

Proof. Let β is defined as in (5.43) or (5.44), i.e, $i_{e_0} d\beta_i = \dfrac{\alpha_i}{f_i} = dh_i$, put $J_i = e^0 \wedge \alpha_i + i_{e_0} \Gamma_i$ where $\Gamma_i = f_i e^0 \wedge d\beta_i$ (Γ is defined as in lemma 5.0.6). Then $\dfrac{J_i}{f_i} = d\beta_i, i = 1, 2$. Instead of $\dfrac{J_i}{f_i}$, write J_i for simplicity. In equation (5.43), $\beta_1 = b_{11} dh_1 + b_{12} dh_2$ where $\mathcal{L}_{e_0} db_{12} = i_{e_0} db_{12} = 0$. Note that α_i's are expressed in terms of e^2 and e^3 and dh_i's are solely obtained those α_i's. Thus, span$\{e^2, e^3\}$=span$\{dh_1, dh_2\}$. Since $\{e^0, e^1, e^2, e^3\}$ is a basis, $\{e^0, e^1, dh_1, dh_2\}$ is also a basis. $i_{e_0} db_{12} = 0 \Rightarrow db_{12} = a_1 e^1 + a_2 dh_1 + a_3 dh_2$. Nondegeneracy of J_1 is guaranteed by $J_1 \wedge dh_1 \neq 0$. Since $J_1 = d\beta_1$ and $d\beta_1 = d_{11} \wedge dh_1 + db_{12} \wedge dh_2$, we infer that

$$J_1 \wedge dh_1 = db_{12} \wedge dh_2 \wedge dh_1 = a_1 e^1 \wedge dh_2 \wedge dh_1 \neq 0$$

if and only if $i_{e_1} db_{12} \neq 0$. We do not know whether $i_{e_1} db_{12} \neq 0$. Therefore, we have to choose b_{12} more carefully than one in lemma 5.0.5. Firstly since $[e_0, e_1] = C^0_{01} e_0 + C^1_{01} e_1$, we can find coordinates $\{x^1, x^2, x^3, x^4\}$ such that span$\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ =span$\{e_0, e_1\}$ by Frobenius theorem. Let S be the surface whose coordinates $\{x^1, x^2\}$. Let C be a curve in S such that e_0 is tangent to C. Let $F(x^1, x^2) = 0$ be the equation of C. Then, $\dfrac{dF}{dt} = 0 \Rightarrow \nabla F(x^1, x^2) \cdot e_0 = i_{e_0} dF = 0$. In general $dF = \frac{\partial F}{\partial x^1} dx^1 + \frac{\partial F}{\partial x^2} dx^2 = c_0 e^0 + c_1 e^1$. Since $i_{e_0} dF = 0$, $c_0 = 0$, that is, $dF = c_1 e^1$. In particular put $b_{12} = F$ then $db_{12} = c_1 e^1$. On the other hand $c_1 \neq 0$ because otherwise $dF = 0$, which means that F is independent of $\{x^1, x^2\}$, i.e, $F = 0$ can not represent surface. To sum up, $i_{e_0} dF = 0$ and $i_{e_1} dF = i_{e_1} db_{12} \neq 0$, thus we reach nondegenerate and closed form $\dfrac{J_1}{f_1}$. Since $\beta_2 = -b_{12} dh_1 + b_{11} dh_2$, same fashion is followed to exract the existence of $\dfrac{J_2}{f_2}$. □

6. CONCLUSION

In this work, it is shown that a non-vanishing vector field on a four dimensional manifold is locally bi-Hamiltonian. We construct a method for computing the symplectic structures which make the given non-vanishing vector field Hamiltonian. The method uses solely the knowledge of the vector field but not its integral curves. Therefore the method proposes a way of constructing a bi-Hamiltonian structure for the dynamical system defined by the given vector field without requiring the solution of the dynamical system.

For this purpose, firstly we worked on the local properties of Poisson and symplectic structures on a four dimensional manifold M, and the equations governing the components of a symplectic structure are obtained. Given an injective skew-symmetric map $\Lambda : T^*M \to TM$ whose components are given by

$$\Lambda = \begin{bmatrix} 0 & -B^t \\ B & \epsilon^{ijk}E_k \end{bmatrix}$$

with the nondegeneracy condition

$$E \cdot B \neq 0$$

the condition defining the Poisson structure

$$[\Lambda, \Lambda] = 0$$

amounts to

$$\begin{aligned} \tfrac{\partial B}{\partial t} - \nabla \times E &= 0 \\ \nabla \cdot B &= 0 \end{aligned}$$

Then, a four dimensional Hamiltonian vector field $\mathbf{w} = \begin{bmatrix} V_0 \\ V \end{bmatrix}$ satisfies

$$\mathbf{w} = \Lambda\,(dH)$$

is given explicitly by

$$\begin{aligned} E \cdot V &= \partial_t H \\ -V_0 E + V \times B &= \nabla H \end{aligned}$$

These equations can also be written in differential form notation as

$$i_u \Omega = J \wedge dH$$

where Ω is the volume form on M, and

$$J = E_i dx^i \wedge dt - \epsilon^{ijk} B_i dx^j \wedge dx^k$$

satisfying

$$dJ = 0$$

which implies

$$i_w J = dH$$

which is nothing but the definition of a Hamiltonian vector field.

Then we proved that

$$i_u \Omega = J \wedge dH \Longleftrightarrow i_w J = dH \Longleftrightarrow i_w J \wedge d i_w J = 0$$

Finally we obtained two solutions for J in

$$i_w J \wedge d i_w J = 0$$

satisfying

$$dJ = 0$$

and proved that these solutions are nondegenerate. Therefore they defined the symplectic structures we are looking for.

Since the topology of four dimensional manifolds are quite complicated in comparison with three dimensional manifolds, we could not obtain a necessary and sufficient condition for the global existence of bi-Hamiltonian structures on four dimensional manifolds. However, for our future work we are planning to work on some necessary conditions (without looking for sufficiency) of the global existence of bi-Hamiltonian structures on four dimensional manifolds. We believe our method will play a significant role in investigating global properties of such structures, as it is based solely on the non-vanishing vector field defined on a four manifold.

REFERENCES

1. R. Abraham, Foundations of Mechanics (Benjamin, New York, 1967).

2. J. Grabowski, G. Marmo and A. M. Peremelov. *Poisson Structures: Towards a Classification,* Modern Phys. Lett. A. 8:1719-1733, 1993.

3. R. Flores-Espinoza. *The Jacobi Equation and Poisson Geometry on* R^4 , arXiv:math.ph/1306.5254v1.

4. F. Hernandez-Bermejo. *New Four Dimensional Solution of the Jacobi equations for Poisson Structures*, J. Math. Phys. 47 (2006) 022901

5. M. Işim Efe, E. Abadoğlu. *Global Existence of Bi-Hamiltonian Structures on Orientable Three Dimensional Manifolds,* SIGMA 13:55-72, 2017.

6. Hasan Gümral. *Poisson Bi-Vectors in Four Dimensions,* http://www.researchgate.net/publication/263735268

7. Luis C. Garcia-Naranjo, Pablo Suarez-Serrato, Ramon Vera. *Poisson Structures on Smooth 4-Manifolds*, Lett Math Phys 105:1533-1550, 2015.

8. Rui Loja Fernandes, Ioan Marcut. *Lectures on Poisson Geometry,* https://faculty.math.illinois.edu/ ruiloja/Math595/Spring14/book.pdf, 2015.

9. Peter J.Olver. *Canonical forms and integrability of bi-Hamiltonian systems,* Physics Letters A. 148: 177-187, 1990.

10. P.J. Morrison. *Hamiltonian description of the ideal fluid* Rev. Mod. Phys. 70: 467, 1998.

11. R.D. Hazeltine, D. D. Holm and P. J. Morrison. *Electromagnetic solitary waves in magnetized plasmas,* J. Plasma Phys. 34: 103-111, 1985.

12. J. E. Marsden, R. Montgomery, P. J. Morrison and W. B. Thompson *Covariant poisson brackets for classical fields ,* Ann. Phys. 169: 29-47, 1986.

13. Y. Nutku.*Bi-Hamiltonian structure of the Kermack-McKendrick model for epidemics,* J.

Phys. A 23 L1145, 1990.

14. D. David, D.D Holm and Tratnik. *Hamiltonian chaos in nonlinear optical polarization dynamics,* Phys. Rep. 187: 281-367, 1990.

15. R. Littlejohn. *A guiding center Hamiltonian: A new approach ,* J. Math. Phys. 20: 2445, 1979.

16. Wendell Fleming. Functions of Several Variables, Springer, 1977.