CLASSICAL ZARISKI PAIRS WITH NODES

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By
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this thesis we study complex projective sextic curves with simple singularities. All curves constituting classical Zariski pairs, especially those with nodes, are enumerated and classified up to equisingular deformation. Every set of singularities constituting a classical Zariski pair gives rise to at most two families, called abundant and non-abundant except for one which gives rise to three families, one abundant and two conjugate non-abundant. This classification is done arithmetically with the aid of integral lattices and quadratic forms.

Keywords: Sextic curves, simple singularity, classical Zariski pairs, integral lattice, quadratic forms.

ÖZET

DÜĞÜMLÜ KLASİK ZARİSKİ ÇİFTLERİ

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Bu tezde basit singülerliği olan kompleks izdüşel 6.dereceden eğriler çalışıldı. Klasik Zariski çiftlerini oluşturan eğrilerin, özellikle de düğüm içerenlerin, singülerliği koruyan deformasyona göre sınıflandırması yapıldı. Klasik Zariski çiftlerini oluşturan singülerlik kümelerinin, bir tanesinin dışında, her biri en fazla iki aileye karşılık gelir ki bunlar, abundant ve abundant olmayandır. Diğer bir tanesi üç aileye karşılık gelir ki bunlar da abundant ve iki eşlenik abundant olmayandır. Bu sınıflandırma latisler ve ikinci dereceden formlar kullanılarak aritmetik bir yolla yapılmaktadır.

Anahtar sözcükler: 6.dereceden eğriler, basit singülerlikler, klasik Zariski çiftleri, integral latisler, 2.dereceden formlar.

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Contents

Chapter 1

Introduction

Rigid isotopy is an equisingular deformation of complex plane projective algebraic curves. If two curves C_1, $C_2 \subset \mathbb{P}^2$ are rigidly isotopic, then the pairs $(\mathbb{P}^2, C_i)$, $(i = 1, 2)$ are homeomorphic. The name Zariski pair originates from O.Zariski, who constructed a pair of irreducible curves C_1, C_2 of degree six having the same set of singularities, but not rigidly isotopic [16].

A.Degtyarev ([5] and [7]) generalized Zariski's example and found all pairs of irreducible sextics $C \subset \mathbb{P}^2$ which have the same singularities but differ by their Alexander polynomial and he called such curves classical Zariski pairs. He also enumerated all curves constituting classical Zariski pairs without nodes and proved [10] that any set of singularities Σ of the form

$$\Sigma = e\mathbf{E}_6 \oplus \bigoplus_{i=1}^{6} a_i \mathbf{A}_{3i-1} \ , \qquad 2e + \sum i a_i = 6$$

is realized by exactly two rigid isotopy classes of irreducible plane sextics, one abundant and one not.

The main result of this thesis is the classification of the sets of singularities in (5.1) with nodes (see theorem(5.0.5)). Enumerating and classifying classical Zariski pairs with nodes is based on some descriptions of the moduli space of sextics which are proved by Degtyarev [10]. By theorem 3.2.1, rigid isotopy classification of plane sextics is reduced to an arithmetic matter. By the way outlined in [10] and theorems 2.1.3, 2.1.4, 2.1.5 of Nikulin [12], we enumerated abstract homological types. In some set of singularities theorem 2.1.5 was failed to apply, which was about the the surjective canonical homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$. Then we considered these cases one-by-one investigating this canonical homomorphism.

With these calculations we found that the set of singularities in (5.1) are not always realized by exactly two rigid isotopic classes of irreducible plane sextics. As is conjectured in [7], some singularities having the virtual genus as zero are realized by only one rigid isotopy classes of irreducible sextics which is abundant. The others except for the set $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ are realized by exactly two rigid isotopy classes. This remaining set of singularity is so different than all other classical Zariski pairs. The set of singularity $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ is realized by three different rigid isotopy classes of sextics, one abundant and two conjugate non-abundant.

In a K3-surface there exist two opposite orientations of periods. In general these two orientations are interchanged by sending a K3-surface to its conjugate. In the case of asymmetric homological type we consider K3 surfaces with fixed polarization.

Examples of asymmetric homological types are given in Artal [1] and then in Degtyarev [10]. However, in all these examples a set of singularities has the maximal value of the total Milnor number $\mu = \mathrm{rk}\Sigma$ is 19, *i.e.*, the moduli spaces of these set of singularities are discrete. Thus, the set of singularity $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ with the total Milnor number $\mu = 18$ seems to be the first known example of a moduli space of sextic curves (having positive dimension) consisting of two connected components interchanged by the complex conjugation.

The contents of this thesis can be summarized as follows: Chapter 2 is based on the definitions and statements which are necessary for the construction of moduli space. Chapter 3 is devoted to the relation between the topology of a curve and moduli space constructed by the double covering of the curve. Chapter 4 defines classical Zariski pairs and states some results about them. The second chapter is largely based on Nikulin's paper [12] and generally up to last chapter statements and constructions depend on Degtyarev's paper [10]. Finally, in the last chapter we prove the main theorem by enumerating all curves one-by-one and obtain the classification of all of them.

Chapter 2

Notations and Basic Facts

2.1 Integral lattices and Discriminant forms

We shall use the standard notations and some facts specialized in [10] and [12].

By a lattice (L, φ) we shall mean a finitely generated free Abelian group, equipped with a symmetric integral bilinear form $\varphi : L \otimes L \rightarrow \mathbb{Z}$. The signature of a lattice is denoted by $(\sigma_+ L, \sigma_- L)$ where $\sigma_+ L$ is the dimension of any maximal positive definite subspace of the vector space $L \otimes \mathbb{R}$ and $\sigma_- L$ is the dimension of any maximal negative definite subspace of $L \otimes \mathbb{R}$. Hence one can define the rank of lattice as $\mathrm{rk} L = \sigma_+ L + \sigma_- L$. A lattice L is called even if $u^2 = 0 \bmod 2$ for all $u \in L$; otherwise odd.

Let (L, φ) be a lattice with $\det L \neq 0$ and $L^* = \mathrm{Hom}(L, \mathbb{Z})$ the dual group of L identified with the subgroup $\{x \in L \otimes \mathbb{Q} | x \cdot y \in \mathbb{Z}$ for all $y \in L\}$. The discriminant group of L is the quotient of L^*/L and it is denoted by $\mathcal{L}$ or $\mathrm{disc} L$. If L is even, the discriminant form of it is finite quadratic form. A finite quadratic form is a finite abelian group $\mathcal{L}$ equipped with a map $q : \mathcal{L} \rightarrow \mathbb{Q}/2\mathbb{Z}$ satisfying $q(x+y) = q(x) + q(y) + 2b(x, y)$ for all $x, y \in \mathcal{L}$ and some nonsingular symmetric bilinear form $b : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathbb{Q}/\mathbb{Z}$ where x in mod L has the congruence $q(x) = x^2$ mod $2\mathbb{Z}$.

Given a lattice L, we denote by $O(L)$ the group of isometries of L, and by $O^+(L) \subset O(L)$ its subgroup consisting of the isometries preserving the orientation of maximal positive definite subspaces. Clearly, either $O^+(L) = O(L)$ or $O^+(L) \subset O(L)$ is a subgroup of index 2. In the second case, each element of $O(L)/O^+(L)$ is called a +disorienting isometry.

Since the construction of the discriminant form $\mathcal{L}$ is natural, there is a canonical homomorphism $O(L) \to \mathrm{Aut}\mathcal{L}$. Its image is denoted by $\mathrm{Aut}_L\mathcal{L}$. Reflections of L have great importance. For any vector $a \in L$, the reflection against the hyperplane orthogonal to a is the automorphism

$$t_a : L \to L, \qquad x \mapsto x - 2\frac{a \cdot x}{a^2}a \quad .$$

It is easy to see that t_a is an involution, *i.e.*, t_a^2 =id. The reflection t_a is well defined whenever $a \in (a^2/2)L^*$. In particular, t_a is well defined if $a^2 = \pm 1$ or ± 2; in this case the induced automorphism of the discriminant group $\mathcal{L}$ is the identity and t_a extends to any lattice containing L.

Discriminant form $\mathcal{L}$ splits into orthogonal sum of its primary components as $\mathcal{L} = \bigoplus \mathcal{L}_p = \bigoplus \mathcal{L} \otimes \mathbb{Z}_p$. For an integral lattice L one has $\mathrm{disc}(L \otimes \mathbb{Z}_p) = (\mathrm{disc}L) \otimes \mathbb{Z}_p = \mathcal{L}_p$. Minimal number of generators of $\mathcal{L}$ is denoted by $\ell(\mathcal{L})$, so that $\ell(\mathcal{L}_p) = \ell_p(\mathcal{L}_p)$.

For a fraction $\frac{m}{n} \in \mathbb{Q}/2\mathbb{Z}$ with $(m,n) = 1$ and $m \cdot n = 0 \bmod 2$, let $\langle \frac{m}{n} \rangle$ be the nondegenerate quadratic form on $\mathbb{Z}/n\mathbb{Z}$ sending the generator to $\frac{m}{n}$. For an integer $k \geq 1$, let $\mathcal{U}_{2^k}$ and $\mathcal{V}_{2^k}$ be the quadratic forms on the group $(\mathbb{Z}/2^k\mathbb{Z})^2$ defined by the matrices

$$\mathcal{U}_{2^k} = \begin{pmatrix} 0 & \alpha_k \\ \alpha_k & 0 \end{pmatrix}, \qquad \mathcal{V}_{2^k} = \begin{pmatrix} \alpha_{k-1} & \alpha_k \\ \alpha_k & \alpha_{k-1} \end{pmatrix} \qquad \text{where} \quad \alpha_k = \tfrac{1}{2^k}$$

According to Nikulin [12], each finite quadratic form is an orthogonal sum of cyclic summands $\langle \frac{m}{n} \rangle$ and summands of the form $\mathcal{U}_{2^k}$, $\mathcal{V}_{2^k}$.

Another important invariant of discriminant group is Brown invariant which is the residue Br$\mathcal{L} \in \mathbb{Z}/8\mathbb{Z}$ defined via the Gauss sum

$$exp\ (\frac{1}{4} i\pi Br\mathcal{L}) = |\mathcal{L}|^{\frac{-1}{2}} \sum_{i \in \mathcal{L}} exp\ (i\pi x^2)$$

Brown invariant is additive: Br$(\mathcal{L}_1 \oplus \mathcal{L}_2)$ =Br$\mathcal{L}_1$+Br$\mathcal{L}_2$.

Brown invariant of a discriminant form $\mathcal{L}$ having the lattice L can be found by the following theorem.

Theorem 2.1.1 *(van der Blij formula, see [3]) For any nondegenerate even integral lattice L one has $Br\mathcal{L} = \sigma_+ L - \sigma_- L$ mod 8.*

This invariant also can be calculated as listed below.

Br$\langle \frac{2a}{p^{2s-1}} \rangle = 2(\frac{a}{p}) - (\frac{-1}{p}) - 1$, Br$\langle \frac{2a}{p^{2s}} \rangle = 0$ (for $s \geq 1$ and $(a, p) = 1$)

Br$\langle \frac{a}{2^k} \rangle = a + \frac{1}{2}k(a^2 - 1)$ mod 8 (for $k \geq 1$, $p = 2$ and odd $a \in \mathbb{Z}$)

Br$\mathcal{U}_{2^k} = 0$ Br$\mathcal{V}_{2^k} = 4k$ mod 8 (for all $k \geq 1$)

After stating the facts about Brown invariant and other invariants, we shall focus on how they determine discriminant form of a lattice.

Proposition 2.1.1 *Let $p \neq 2$ be an odd prime. Then a quadratic form on a group $\mathcal{L}$ of exponent p is determined by its rank $\ell(\mathcal{L}) = \ell_p(\mathcal{L})$ and Brown invariant $Br\mathcal{L}$.*

A finite quadratic form is called even if x^2 is an integer for each element $x \in \mathcal{L}$ of order 2; otherwise, it is called odd.

Proposition 2.1.2 *A quadratic form on a group $\mathcal{L}$ of exponent 2 is determined by its rank $\ell(\mathcal{L}) = \ell_2(\mathcal{L})$, parity (even or odd), and Brown invariant $Br\mathcal{L}$.*

Two integral lattice L_1, L_2 are said to have the same genus if all their localizations $L_i \otimes \mathbb{R}$ and $L_i \otimes \mathbb{Q}_p$ are isomorphic (over $\mathbb{R}$ and $\mathbb{Q}_p$, respectively). Each genus contains finitely many isomorphism classes.

Theorem 2.1.2 *(see [12]) The genus of an even integral lattice L is determined by its signature (σ_+L, σ_-L) and discriminant form discL.*

Nikulin had some results about existence of a lattice for a known discriminant form and signature, and about having this lattice unique in its genus.

Theorem 2.1.3 *(see Theorem 1.10.1 in [12]) Let $\mathcal{L}$ be a finite quadratic form and let $\sigma_\pm$ be a pair of integers. Then, the following four conditions are necessary and sufficient for the existence of an even integral lattice L whose signature is (σ_+, σ_-) and whose discriminant form is $\mathcal{L}$:*

(1) $\sigma_\pm \geq 0$ and $\sigma_+ + \sigma_- \geq \ell(\mathcal{L})$;

(2) $\sigma_+ - \sigma_- = Br\mathcal{L}$ mod 8;

(3) for each $p \neq 2$, either $\sigma_+ + \sigma_- > \ell_p(\mathcal{L})$ or

$$det_p\mathcal{L}_p \cdot |\mathcal{L}_{\neq p}| = (-1)^{\sigma_-} \ mod \ (\mathbb{Z}_p^*)^2;$$

(4) either $\sigma_+ + \sigma_- > \ell_2(\mathcal{L})$, or $\mathcal{L}_2$ is odd, or $det_2\mathcal{L}_2 = \pm 1$ mod $(\mathbb{Z}_2^*)^2$.

Theorem 2.1.4 *(see Theorem 1.13.2 in [12]) Let L be an indefinite even integral lattice, rk≥ 3. The following two conditions are sufficient for L to be unique in its genus:*

(1) for each $p \neq 2$, either rk$L \geq \ell_p(\mathcal{L}) + 2$ or $\mathcal{L}_p$ contains a subform isomorphic to $\langle \frac{a}{p^k} \rangle \oplus \langle \frac{b}{p^k} \rangle$, $k \geq 1$, as an orthogonal summand;

(2) either rk$L \geq \ell_2(\mathcal{L}) + 2$ or $\mathcal{L}_2$ contains a subform isomorphic to $\mathcal{U}_{2^k}$, $\mathcal{V}_{2^k}$, or $\langle \frac{a}{2^k} \rangle \oplus \langle \frac{b}{2^k} \rangle$, $k \geq 1$, as an orthogonal summand.

Theorem 2.1.5 *(see Theorem 1.14.2 in [12]). Let L be an indefinite even integral lattice, rk$L \geq 3$. The following two conditions are sufficient for L to be unique in its genus and for the canonical homomorphism $Aut(L) \to Aut(\mathcal{L})$ to be onto:*

(1) for each $p \neq 2$, either $rkL \geq \ell_p(\mathcal{L}) + 2$

(2) either $rkL \geq \ell_2(\mathcal{L}) + 2$ or $\mathcal{L}_2$ contains a subform isomorphic to $\mathcal{U}_{2^k}$, $\mathcal{V}_{2^k}$ as an orthogonal summand.

Definite lattices of rank 2 doesn't satisfy these theorems. Each positive definite even lattice N of rank 2 has a unique representation by a matrix of the form

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \qquad 0 \leq a \leq c, \qquad 0 \leq b \leq a \tag{2.1}$$

Denote the lattice represented by (2.1) by $M(a, b, c)$. Any lattice $N = M(a, b, c)$ with a given discriminant form $\mathcal{N}$ must satisfy $|\mathcal{N}|/4 \leq ac \leq |\mathcal{N}|/3$.

2.2 Extensions

An extension of even lattice S is an even lattice L containing S. Two extensions $L_1 \supset S$ and $L_2 \supset S$ are called isomorphic if there is an isomorphism $L_1 \to L_2$ preserving S. For any subgroup $A \subset O(S)$, A-isomorphism or A-automorphisms of extension is isometries whose restriction to S belongs to A.

Any extension $L \supset S$ of finite index admits a unique embedding $L \subset S \otimes \mathbb{Q}$. If $\det S \neq 0$, then L belongs to S^* and thus defines a subgroup $\mathcal{K} = L/S \subset \mathcal{S}$, called the kernel of the extension. Since L is integral lattice, the kernel $\mathcal{K}$ is isotropic, *i.e.*, the restriction to $\mathcal{K}$ of the discriminant quadratic form is identically zero.

Proposition 2.2.1 *(see [12]). Let S be a nondegenerate even lattice, and fix a subgroup $A \subset O(S)$. The map $L \to \mathcal{K} = L/S \subset \mathcal{S}$establishes a one-to-one correspondence between the set of A-isomorphism classes of finite index extensions $L \supset S$ and the set of A-orbits of isotropic subgroups $\mathcal{K} \subset \mathcal{S}$. Under this correspondence, one has disc$L = \mathcal{K}^\perp/\mathcal{K}$.*

An isometry $f : S \to S$ extends to a finite index extension $L \supset S$ defined by an isotropic subgroup $\mathcal{K} \subset \mathcal{S}$ if and only if the automorphism $\mathcal{S} \to \mathcal{S}$ induced by f preserves $\mathcal{K}$ (as a set).

Corollary 2.2.1 *(see[10]) Any impirimitive extension of a root system $S = 3\mathbf{A}_2$, $\mathbf{A}_5 \oplus \mathbf{A}_2 \oplus$, $\mathbf{A}_8$, $\mathbf{E}_6 \oplus \mathbf{A}_2$, $2\mathbf{A}_4$, $\mathbf{A}_5 \oplus \mathbf{A}_1$, $\mathbf{A}_7$, $\mathbf{D}_8$, $\mathbf{E}_7 \oplus \mathbf{A}_1$, $4\mathbf{A}_1$, $\mathbf{A}_3 \oplus 2\mathbf{A}_1$, or $\mathbf{D}_q \oplus 2\mathbf{A}_1$ with $q < 12$ or $q \neq 0$ mod 4 contains a finite index extension $R \supset S$, where R is a root system strictly larger than S.*

Another extreme case is when $S \subset L$ is a primitive sublattice with $\det S \neq 0$ (S is nondegenerate) and $\det L = \pm 1$ (L is unimodular). Then the following proposition is obtained in [12].

Proposition 2.2.2 *Let S be a nondegenerate even lattice, and let s_+, s_- be non-negative integers. Fix a subgroup $A \subset O(S)$. Then the A-isomorphism class of a primitive extension $L \supset S$ of S to a unimodular lattice L of signature (s_+, s_-) is determined by*

(1) a choice of a lattice N in the genus $(s_+ - \sigma_+ S,\ s_- - \sigma_- S; -\mathcal{S})$, and

(2) a choice of a bi-coset of the canonical left-right action of $A \times$ $Aut_N \mathcal{N}$ on the set of anti-isometries $\mathcal{S} \to \mathcal{N}$.

If a lattice N and an isometry $\kappa : \mathcal{S} \to \mathcal{N}$ as above are chosen (and thus an extension L is fixed), an isometry $t : N \to N$ extends to an A-automorphism of L if and only if the composition $\kappa^{-1} \circ t \circ \kappa \in Aut\mathcal{S}$ is in the image of A.

2.3 Root Systems

A root in a lattice L is an element $v \in L$ of square -2. We denote the sublattice generated by all roots of L by $rL \subset L$. A root system is a negative definite lattice generated by its roots. Every root system admits a unique decomposition into an orthogonal sum of irreducible root systems. These irreducible systems are $\mathbf{A}_p$, $p \geq 1$; $\mathbf{D}_q$, $q \geq 4$; $\mathbf{E}_6$; $\mathbf{E}_7$; $\mathbf{E}_8$. However in this thesis we will deal only with $\mathbf{A}_p$, $p \geq 1$; and $\mathbf{E}_6$. Their discriminant forms are as follows:

$$\mathrm{disc}\mathbf{A}_p = \langle -\tfrac{p}{p+1} \rangle, \quad \mathrm{disc}\mathbf{E}_6 = \langle \tfrac{2}{3} \rangle.$$

Chapter 3

The Moduli Space

3.1 Basic Concept

A rigid isotopy of plane projective algebraic curve is an equisingular deformation in the class of algebraic curves. Curves with simple singularities are our interest in this thesis, especially the curves with singularities reduced to root systems $\mathbf{A}_p$, $p \geq 1$ and $\mathbf{E}_6$. Simple singularities, (see[11]) are said to be 0-modal, *i.e.*, their differential type is determined by their topological type.

Let $C \subset \mathbb{P}^2$ be a reduced sextic with simple singularities. Consider the following diagram:

$$\begin{array}{ccc} X & \longleftarrow & \overline{X} \\ p \downarrow & & \bar{p} \downarrow \\ \mathbb{P}^2 & \longleftarrow & \overline{Y} \\ & \pi & \end{array}$$

where X is the double covering of $\mathbb{P}^2$ branched at C, $\overline{X}$ is the minimal resolution of singularities of X, and $\overline{Y}$ is the minimal embedded resolution of singularities of C such that all odd order components of the divisoral pull-back π^*C of

C are nonsingular and disjoint. Here, X is K3-surface.

Let $L_X = \mathrm{H}_2(\overline{X})$ which is a lattice isomorphic to $2\mathbf{E}_8 \oplus 3\mathbf{U}$. Then introduce the following vectors and sublattices:

- $\sigma_X \subset L_X$, the set of the classes of the exceptional divisors appearing in the blow-up $\overline{X} \longrightarrow X$;
- $\Sigma_X \subset L_X$, the sublattice generated by σ_X;
- $h_X \in L_X$, the pull-back of the hyperplane section class $[\mathbb{P}^1] \in \mathrm{H}_2(\mathbb{P}^2)$;
- $S_X = \Sigma_X \oplus \langle h_X \rangle \subset L_X$;
- $\widetilde{\Sigma}_X \subset \widetilde{S}_X \subset L_X$, the primitive hulls of Σ_X and S_X, respectively;
- $w_X \subset L_X \otimes \mathbb{R}$, the oriented 2-subspace spanned by the real and imaginary parts of the class of a holomorphic 2-form on $\overline{X}$.

The isomorphism class of the collection (L_X, h_X, σ_X) is both a deformation invariant of a curve C and a topological invariant of pair $(Y, \pi^* C)$ which is called homological type of C. An isomorphism between two collections (L', h', σ') and (L'', h'', σ'') means that an isometry $L' \longrightarrow L''$ takes h' and σ' to h'' and σ'', respectively.

The following definitions are essential for classifying isotopic algebraic curves with simple singularity.

Definition 3.1.1 *Let Σ and h be as above. A configuration is a finite index $\widetilde{S} \supset S = \Sigma \oplus \langle h \rangle$ satisfying the following conditions:*

(1) $\mathrm{r}\widetilde{\Sigma} = \Sigma$ where $\widetilde{\Sigma} = h^{\perp}_{\widetilde{S}}$ is the primitive hull of Σ in $\widetilde{S}$ and $\mathrm{r}\widetilde{\Sigma} \subset \widetilde{\Sigma}$ is the sublattice generated by the roots of $\widetilde{\Sigma}$;

(2) there is no root $r \in \Sigma$ such that $\frac{1}{2}(r + h) \in \widetilde{S}$.

Definition 3.1.2 *An abstract homological type (extending a fixed set of simple* *singularities* (Σ, σ)*) is an extension of the orthogonal sum* $S = \Sigma \oplus \langle h \rangle$, $h^2 = 2$, *to a lattice* L *isomorphic to* $2\mathbf{E}_8 \oplus 3\mathbf{U}$ *so that the primitive hull* $\widetilde{S}$ *of* S *in* L *is a configuration.*

Definition 3.1.3 *An orientation of an abstract homological type* $\mathcal{H} = (L, h, \sigma)$ *is a choice of one of the two orientations of positive definite 2-subspaces of* $S^\perp \otimes \mathbb{R}$. *(Recall that* $\sigma_+ \widetilde{S}^\perp = 2$ *and, hence , all positive definite 2-subspaces of* $S^\perp \otimes \mathbb{R}$ *can be oriented in a coherent way.)*

3.2 Rigid Isotopy of Curves

Consider the lattice $S = \Sigma \oplus \langle h \rangle$, $h^2 = 2$. One has $\mathcal{S} =$disc$\Sigma \oplus \langle \frac{1}{2} \rangle$. An isometry of S is called admissible if it preserves both h and σ (as a set) and an isometry of a configuration $\widetilde{S}$ is called admissible if it preserves S and induces an admissible isometry of S.

The group of admissible isometries of $\widetilde{S}$ and its image in Aut$\widetilde{\mathcal{S}}$ are denoted by $O_h(\widetilde{S})$ and Aut$_h\widetilde{\mathcal{S}}$, respectively. Any isometry of $\widetilde{S}$ preserving h preserves Σ. Hence one has Aut$_h\widetilde{\mathcal{S}} = \{s \in$Aut$_h\mathcal{S} \mid\ s(\mathcal{K}) \subset \mathcal{K}\}$ where $\mathcal{K}$ is the kernel of the extension $\widetilde{S} \supset S$.

Recall that an abstract homological type is uniquely determined by the collection (L, h, σ); then Σ is the sublattice spanned by σ, and $S = \Sigma \oplus \langle h \rangle$. An abstract homological type $\mathcal{H} = (L, h, \sigma)$ is called symmetric if $(\mathcal{H}, \theta)$ is isomorphic to $(\mathcal{H}, -\theta)$ (for some orientation θ of $\mathcal{H}$). In other words, $\mathcal{H}$ is symmetric if it has an automorphism whose restriction to $\widetilde{S}^\perp$ is +disorienting.

The following statement, based on the isotopy classes of sextics, is essentially contained in Degtyarev[10].

Theorem 3.2.1 *The map sending a plane sextic* $C \subset \mathbb{P}^2$ *to the pair consisting of its homological type* (L_X, h_X, σ_X) *and the orientation of the space* w_X *establishes a one-to-one correspondence between the set of rigid isotopy classes of sextics with* *a given set of singularities (Σ, σ) and the set of isomorphism classes of oriented abstract homological type extending (Σ, σ).*

To enumerate an abstract homological type of a fixed set of singularities Σ, the following statements are listed in Degtyarev[10]. The classification of oriented abstract homological types extending Σ is done in four steps:

(1) Enumerating the configurations $\widetilde{S}$ extending Σ;

(2) Enumerating the isomorphism classes of $\widetilde{S}^{\perp}$;

(3) Enumerating the bi-cosets of $\mathrm{Aut}_h\widetilde{\mathcal{S}}\times\ \mathrm{Aut}_N\mathcal{N}$;

(4) Detecting whether the abstract homological types are symmetric.

Below is a sufficient condition for an abstract homological type to be symmetric.

Proposition 3.2.1 *Let $\mathcal{H} = (L, h, \sigma)$ be an abstract homological type. If the lattice $\widetilde{S}^{\perp}$ contains a vector* v *of square 2, then $\mathcal{H}$ is symmetric.*

Furthermore, the following theorem confirm the uniqueness of rigid isotopy classes of plane sextics which satisfy the conditions of the theorem, consequently it is not needed to be checked the fourth step for these sextics.

Theorem 3.2.2 *Each configuration extending a set of singularities Σ satisfying the inequality $\ell(disc\Sigma)+rk\ \Sigma \leq 19$ is realized by a unique rigid isotopy class of plane sextics.*

Chapter 4

Classical Zariski Pairs

At this point we know how to construct lattices according to the singularities of curves. After all, we will discuss the relation between the geometry of a plane sextic and its homological type, especially in the case of classical Zariski pairs.

In general, Zariski pairs are defined as follows:

Definition 4.0.1 *Two reduced curves C_1, $C_2 \subset \mathbb{P}^2$ are said to form Zariski pair if*

> *(1) C_1 and C_2 have the same combinatorial data, and*
>
> *(2) the pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic.*

By a classical Zariski pair we mean a pair of curves that have the same combinatorial data and differ by their Alexander polynomial.

For an irreducible curve C, its combinatorial data are determined by the degree $\deg C$ and the set of topological types of the singularities of C.

The Alexander polynomials $\Delta_c(t)$ of all irreducible sextics C are found in [5], where it is shown that $\Delta_c(t) = (t^2 - t + 1)^d$ and the exponent d is determined by the set of singularities of C unless the latter has the form

$$\Sigma = e\mathbf{E}_6 \oplus \bigoplus_{i=1}^{6} a_i \mathbf{A}_{3i-1} \oplus n\mathbf{A}_1 \ , \qquad 2e + \sum ia_i = 6. \tag{4.1}$$

If the set of singularities is as in (4.1), then d take values 0 or 1; in the case of $d = 1$ the curve is called abundant. The following statement is proved in [7].

Theorem 4.0.3 *For an irreducible plane sextic C with a set of singularities Σ as in (4.1), the following three conditions are equivalent:*

1. *C is abundant;*
2. *C is tame,* i.e., *it is given by an equation of the form $f_2^3 + f_3^2 = 0$, where f_2 and f_3 are some polynomials of degree 2 and 3, respectively;*
3. *there is a conic Q whose local intersection index with C at each singular point of C of type $\mathbf{A}_{3i-1}$ (respectively, $\mathbf{E}_6$) is $2i$ (respectively, 4).*

Observe that the discriminant group of each lattice $\mathbf{A}_{3i-1}$ or $\mathbf{E}_6$ has a unique subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$ where its nontrivial elements are the residues of $\bar{\beta}^{(1)}$, $\bar{\beta}^{(2)}$ such that

$\bar{\beta}^{(1)} = \frac{1}{3}(2e_1 + 4e_2 + ... + 2ie_i + (2i-1)e_{i+1} + ... + e_{2i-1}) \in (\mathbf{A}_{3i-1})^*$,

$\bar{\beta}^{(1)} = \frac{1}{3}(4e_1 + 5e_2 + 6e_3 + 4e_4 + 2e_5 + 3e_6) \in (\mathbf{E}_6)^*$

and $\bar{\beta}^{(2)}$ is obtained from $\bar{\beta}^{(1)}$ by the nontrivial symmetry of the Dynkin graph. Using these elements the following theorem is proven in [10] for the characterization of abundant curves.

Theorem 4.0.4 *Let C be a plane sextic with a set of singularities Σ as in (4.1). Then a reduced conic Q as in (4.0.3)(3) exists if and only if the kernel $\mathcal{K}$ of the extension $\widetilde{S}_X \supset S_X$ has 3-torsion. If this is the case, the 3-primary part of $\mathcal{K}$ is a cyclic group of order 3 generated by a residue of the form $\sum \bar{\beta}_i^{(1,2)}$ mod S_X, where*

$\bar{\beta}_i^{(1,2)}$ *are the elements defined above and the sum contains exactly one element for each singular point of* C *other than* $\mathbf{A}_1$.

Hence the following corollary is resulted in [10].

Corollary 4.0.1 *Each set of singularities* Σ *as in (4.1) extends to two isomorphism classes of configurations* $\widetilde{S} \supset S = \Sigma \oplus \langle h \rangle$ *that may correspond to irreducible sextics, one abundant* ($\mathcal{K} = \mathbb{Z}/3\mathbb{Z}$) *and one not* ($\mathcal{K} = 0$).

Chapter 5

Main Theorem

Theorem 5.0.5 *Except for the set $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$, any set of singularities Σ of the form*

$$\Sigma = e\mathbf{E}_6 \oplus \bigoplus_{i=1}^{6} a_i \mathbf{A}_{3i-1} \oplus n\mathbf{A}_1 \ , \qquad 2e + \sum i a_i = 6 \tag{5.1}$$

is realized at most by two rigid isotopy classes of irreducible plane sextics, one abundant and one not.

Remark: The set of singularity $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ admits two different configurations $\widetilde{S}$ as abundant and non-abundant type. Abundant homological type is symmetric, the other is not, so that there are three rigid isotopy classes of sextics with this set of singularity.

Remark: Each set of singularities $\Sigma = 6\mathbf{A}_2 \oplus 4\mathbf{A}_1$, $2\mathbf{A}_2 \oplus 2\mathbf{E}_6 \oplus 2\mathbf{A}_1$, $4\mathbf{A}_2 \oplus \mathbf{E}_6 \oplus 3\mathbf{A}_1$, $3\mathbf{E}_6 \oplus \mathbf{A}_1$ is realized by only abundant type of irreducible plane sextics. Any other set of singularities from the below list is realized by exactly two rigid isotopy classes of irreducible sextics, one abundant and one not.

The case of assuming that the number of nodes $n = 0$ was proved in [10]. A general proof can not be given for the remaining part of theorem. Each curve having at least one node should be investigated one by one. Here is the list of singularities attainable from (5.1) having at least one node.

(1) $\Sigma = \mathbf{A}_{17} \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 18)$

(2) $\Sigma = \mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(3) $\Sigma = \mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 18)$

(4) $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 18)$

(5) $\Sigma = \mathbf{A}_{11} \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 16)$

(6) $\Sigma = \mathbf{A}_{11} \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(7) $\Sigma = \mathbf{A}_{11} \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(8) $\Sigma = \mathbf{A}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(9) $\Sigma = \mathbf{A}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 18)$

(10) $\Sigma = \mathbf{A}_8 \oplus 3\mathbf{A}_2 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 15)$

(11) $\Sigma = \mathbf{A}_8 \oplus 3\mathbf{A}_2 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 16)$

(12) $\Sigma = \mathbf{A}_8 \oplus 3\mathbf{A}_2 \oplus 3\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(13) $\Sigma = \mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 16)$

(14) $\Sigma = \mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(15) $\Sigma = 2\mathbf{A}_8 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(16) $\Sigma = 2\mathbf{A}_8 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 18)$

(17) $\Sigma = 3\mathbf{A}_5 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 16)$

(18) $\Sigma = 2\mathbf{A}_5 \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(19) $\Sigma = 2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 15)$

(20) $\Sigma = 2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 16)$

(21) $\Sigma = \mathbf{A}_5 \oplus 2\mathbf{E}_6 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 18)$

(22) $\Sigma = \mathbf{A}_5 \oplus 4\mathbf{A}_2 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 14)$

(23) $\Sigma = \mathbf{A}_5 \oplus 4\mathbf{A}_2 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 15)$

(24) $\Sigma = \mathbf{A}_5 \oplus 4\mathbf{A}_2 \oplus 3\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 16)$

(25) $\Sigma = \mathbf{A}_5 \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 16)$

(26) $\Sigma = \mathbf{A}_5 \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(27) $\Sigma = 6\mathbf{A}_2 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 13)$

(28) $\Sigma = 6\mathbf{A}_2 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 14)$

(29) $\Sigma = 6\mathbf{A}_2 \oplus 3\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 15)$

(30) $\Sigma = 6\mathbf{A}_2 \oplus 4\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 16)$

(31) $\Sigma = 2\mathbf{A}_2 \oplus 2\mathbf{E}_6 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(32) $\Sigma = 2\mathbf{A}_2 \oplus 2\mathbf{E}_6 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 18)$

(33) $\Sigma = 4\mathbf{A}_2 \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 15)$

(34) $\Sigma = 4\mathbf{A}_2 \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 16)$

(35) $\Sigma = 4\mathbf{A}_2 \oplus \mathbf{E}_6 \oplus 3\mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 17)$

(36) $\Sigma = 3\mathbf{E}_6 \oplus \mathbf{A}_1$, $(\sigma_+, \sigma_-) = (0, 19)$

To realize these set of singularities to rigid isotopy classes of irreducible plane sextics we would consider configurations of these Σs' to have combinatorial data and furthermore we would extend these configurations to abstract homological types in order to get rigid isotopy classes. Steps of classifying oriented abstract homological types extending any Σ are declared in [10].

5.1 Enumerating $\widetilde{S}$ by $\mathcal{K}$

As a first step of classification of oriented abstract homological types, isomorphism class of $\widetilde{S}$ should be defined. Recall that $S = \Sigma \oplus \langle h \rangle$ where $\langle h \rangle$ is the hyperplane class with $h^2 = 2$ and one has $\mathcal{S} = \mathrm{disc}\Sigma \oplus \langle \frac{1}{2} \rangle$. Hence by confirming configuration by $\mathcal{K}$, existence and uniqueness of $\widetilde{S}$ can be investigated.

By the corollary (4.0.1) each set of singularities Σ as in (5.1) extends to two isomorphism classes of configurations $\widetilde{S}$ that may correspond to irreducible sextics, one abundant and one non-abundant type.

A configuration $\widetilde{S}$ is determined by a choice of isotropic subgroup $\mathcal{K} \subset \mathcal{S}$. Depending on the Corollary (4.0.1) for a non-abundant ($\mathcal{K} = 0$) irreducible sextic, the configuration $\widetilde{S}$ is the same with $S = \Sigma \oplus < h >$. Hence the following list of discriminants and minimal number of generators ℓ of non-abundant configurations extending set of singularities Σ are attained.

(1) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle \frac{2}{9} \rangle$, $(\ell = 3)$

(2) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{2}{5} \rangle$, $(\ell = 2)$

(3) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{2}{5} \rangle$, $(\ell = 3)$

(4) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{1}{4} \rangle$, $(\ell = 3)$

(5) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 3\langle -\frac{2}{3} \rangle \oplus \langle -\frac{1}{4} \rangle$, $(\ell = 3)$

(6) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus 3\langle -\frac{2}{3} \rangle \oplus \langle -\frac{1}{4} \rangle$, $(\ell = 4)$

(7) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{1}{4} \rangle$, $(\ell = 4)$

(8) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle$, $(\ell = 3)$

(9) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle$, $(\ell = 3)$

(10) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 3\langle -\frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle$, $(\ell = 4)$

(11) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus 3\langle -\frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle \ , \qquad (\ell = 4)$

(12) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 3\langle -\frac{1}{2} \rangle \oplus 3\langle -\frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle \ , \qquad (\ell = 4)$

(13) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle \ , \qquad (\ell = 3)$

(14) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle \ , \qquad (\ell = 4)$

(15) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 2\langle -\frac{8}{9} \rangle \ , \qquad (\ell = 2)$

(16) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus 2\langle -\frac{8}{9} \rangle \ , \qquad (\ell = 3)$

(17) $\widetilde{\mathcal{S}} = 4\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 3\langle \frac{2}{3} \rangle \ , \qquad (\ell = 5)$

(18) $\widetilde{\mathcal{S}} = 3\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 3\langle \frac{2}{3} \rangle \ , \qquad (\ell = 4)$

(19) $\widetilde{\mathcal{S}} = 3\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 2\langle -\frac{2}{3} \rangle \oplus 2\langle \frac{2}{3} \rangle \ , \qquad (\ell = 4)$

(20) $\widetilde{\mathcal{S}} = 3\langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus 2\langle -\frac{2}{3} \rangle \oplus 2\langle \frac{2}{3} \rangle \ , \qquad (\ell = 5)$

(21) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 3\langle \frac{2}{3} \rangle \ , \qquad (\ell = 3)$

(22) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 4\langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \ , \qquad (\ell = 5)$

(23) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus 4\langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \ , \qquad (\ell = 5)$

(24) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus 3\langle -\frac{1}{2} \rangle \oplus 4\langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \ , \qquad (\ell = 5)$

(25) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 2\langle -\frac{2}{3} \rangle \oplus 2\langle \frac{2}{3} \rangle \ , \qquad (\ell = 4)$

(26) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus 2\langle -\frac{2}{3} \rangle \oplus 2\langle \frac{2}{3} \rangle \ , \qquad (\ell = 4)$

(27) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 6\langle -\frac{2}{3} \rangle \ , \qquad (\ell = 6)$

(28) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus 6\langle -\frac{2}{3} \rangle \ , \qquad (\ell = 6)$

(29) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 3\langle -\frac{1}{2} \rangle \oplus 6\langle -\frac{2}{3} \rangle \ , \qquad (\ell = 6)$

(30) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 4\langle -\frac{1}{2} \rangle \oplus 6\langle -\frac{2}{3} \rangle \ , \qquad (\ell = 6)$

(31) $\widetilde{\mathcal{S}} = \langle\frac{1}{2}\rangle \oplus \langle-\frac{1}{2}\rangle \oplus 2\langle-\frac{2}{3}\rangle \oplus 2\langle\frac{2}{3}\rangle$, $(\ell = 4)$

(32) $\widetilde{\mathcal{S}} = \langle\frac{1}{2}\rangle \oplus 2\langle-\frac{1}{2}\rangle \oplus 2\langle-\frac{2}{3}\rangle \oplus 2\langle\frac{2}{3}\rangle$, $(\ell = 4)$

(33) $\widetilde{\mathcal{S}} = \langle\frac{1}{2}\rangle \oplus \langle-\frac{1}{2}\rangle \oplus 4\langle-\frac{2}{3}\rangle \oplus \langle\frac{2}{3}\rangle$, $(\ell = 5)$

(34) $\widetilde{\mathcal{S}} = \langle\frac{1}{2}\rangle \oplus 2\langle-\frac{1}{2}\rangle \oplus 4\langle-\frac{2}{3}\rangle \oplus \langle\frac{2}{3}\rangle$, $(\ell = 5)$

(35) $\widetilde{\mathcal{S}} = \langle\frac{1}{2}\rangle \oplus 3\langle-\frac{1}{2}\rangle \oplus 4\langle-\frac{2}{3}\rangle \oplus \langle\frac{2}{3}\rangle$, $(\ell = 5)$

(36) $\widetilde{\mathcal{S}} = \langle\frac{1}{2}\rangle \oplus \langle-\frac{1}{2}\rangle \oplus 3\langle\frac{2}{3}\rangle$, $(\ell = 3)$

If the singularities in (5.1) are in abundant type ($\mathcal{K} = \mathbb{Z}_3$) then $\widetilde{\mathcal{S}}$ would be different than $\mathcal{S}$. Observe that disc$\widetilde{S} = \mathcal{K}^{\perp}/\mathcal{K}$ with $\mathcal{K} = \mathbb{Z}_3$ has 9 less order than $\mathcal{S}$ has and discriminant of configurations of abundant types can be obtained by removing some parts with order 9 from 3-primary part of $\mathcal{S}$.

Looking at the list of $\mathcal{S}$, one can see the fact that if $\mathcal{S}$ has only 9 order in 3-primary part (only $\langle\frac{2}{3}\rangle \oplus \langle-\frac{2}{3}\rangle$ or $\langle\frac{a}{9}\rangle$) then abundant $\widetilde{\mathcal{S}}$ is the discriminant form of S without 3-primary part. If $\mathcal{S}$ has 3-primary part without $\mathbb{Z}_9$, to preserve the Brown invariant, $\widetilde{\mathcal{S}}$ is the discriminant form obtained by removing $\langle\frac{2}{3}\rangle \oplus \langle-\frac{2}{3}\rangle$ part of $\mathcal{S}$.

In the remaining case, if $\mathcal{S}$ has both $\langle\frac{2}{3}\rangle \oplus \langle-\frac{2}{3}\rangle$ and $\langle-\frac{8}{9}\rangle$, then $\langle\frac{2}{3}\rangle \oplus \langle-\frac{2}{3}\rangle$ part should be removed. Indeed let α,β and γ be generators of $\langle\frac{2}{3}\rangle$, $\langle-\frac{2}{3}\rangle$ and $\langle-\frac{8}{9}\rangle$, respectively. One can take $a = \alpha+\beta+3\gamma$ as an element of $\mathcal{K}$ and $b = \beta+\gamma$ as an element of $\mathcal{K}^{\perp}$ since $a.b = 0$. Since $b \in \mathbb{Z}_9$ and $b \notin \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\langle-\frac{8}{9}\rangle$ part should stay in abundant $\widetilde{\mathcal{S}}$.

Finally, the following list of discriminants and minimal number of generators ℓ' of abundant configurations extending set of singularities Σ are obtained.

(1) $\widetilde{\mathcal{S}} = \langle\frac{1}{2}\rangle \oplus 2\langle-\frac{1}{2}\rangle$, $(\ell' = 3)$

(2) $\widetilde{\mathcal{S}} = \langle\frac{1}{2}\rangle \oplus \langle-\frac{1}{2}\rangle \oplus \langle-\frac{2}{5}\rangle$, $(\ell' = 2)$

(3) $\widetilde{\mathcal{S}} = \langle\frac{1}{2}\rangle \oplus 2\langle-\frac{1}{2}\rangle \oplus \langle-\frac{2}{5}\rangle$, $(\ell' = 3)$

(4) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{1}{4} \rangle$, $\quad (\ell' = 3)$

(5) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{1}{4} \rangle$, $\quad (\ell' = 3)$

(6) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{1}{4} \rangle$, $\quad (\ell' = 4)$

(7) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{1}{4} \rangle$, $\quad (\ell' = 4)$

(8) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{8}{9} \rangle$, $\quad (\ell' = 2)$

(9) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle -\frac{8}{9} \rangle$, $\quad (\ell' = 3)$

(10) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle$, $\quad (\ell' = 2)$

(11) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle$, $\quad (\ell' = 3)$

(12) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 3\langle -\frac{1}{2} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle -\frac{8}{9} \rangle$, $\quad (\ell' = 4)$

(13) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{8}{9} \rangle$, $\quad (\ell' = 3)$

(14) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle -\frac{8}{9} \rangle$, $\quad (\ell' = 4)$

(15) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{8}{9} \rangle$, $\quad (\ell' = 2)$

(16) $\widetilde{\mathcal{S}} = \langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle -\frac{8}{9} \rangle$, $\quad (\ell' = 3)$

(17) $\widetilde{\mathcal{S}} = 4\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle$, $\quad (\ell' = 5)$

(18) $\widetilde{\mathcal{S}} = 3\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle$, $\quad (\ell' = 4)$

(19) $\widetilde{\mathcal{S}} = 3\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle$, $\quad (\ell' = 4)$

(20) $\widetilde{\mathcal{S}} = 3\langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle \oplus \langle \frac{2}{3} \rangle$, $\quad (\ell' = 5)$

(21) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus \langle -\frac{2}{3} \rangle$, $\quad (\ell' = 3)$

(22) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle \oplus 3\langle -\frac{2}{3} \rangle$, $\quad (\ell' = 3)$

(23) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2} \rangle \oplus 2\langle -\frac{1}{2} \rangle \oplus 3\langle -\frac{2}{3} \rangle$, $\quad (\ell' = 4)$

(24) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2}\rangle \oplus 3\langle -\frac{1}{2}\rangle \oplus 3\langle -\frac{2}{3}\rangle$, $(\ell' = 5)$

(25) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2}\rangle \oplus \langle -\frac{1}{2}\rangle \oplus \langle -\frac{2}{3}\rangle \oplus \langle \frac{2}{3}\rangle$, $(\ell' = 3)$

(26) $\widetilde{\mathcal{S}} = 2\langle \frac{1}{2}\rangle \oplus 2\langle -\frac{1}{2}\rangle \oplus \langle -\frac{2}{3}\rangle \oplus \langle \frac{2}{3}\rangle$, $(\ell' = 4)$

(27) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus \langle -\frac{1}{2}\rangle \oplus 3\langle -\frac{2}{3}\rangle \oplus \langle \frac{2}{3}\rangle$, $(\ell' = 4)$

(28) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus 2\langle -\frac{1}{2}\rangle \oplus 3\langle -\frac{2}{3}\rangle \oplus \langle \frac{2}{3}\rangle$, $(\ell' = 4)$

(29) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus 3\langle -\frac{1}{2}\rangle \oplus 3\langle -\frac{2}{3}\rangle \oplus \langle \frac{2}{3}\rangle$, $(\ell' = 4)$

(30) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus 4\langle -\frac{1}{2}\rangle \oplus 3\langle -\frac{2}{3}\rangle \oplus \langle \frac{2}{3}\rangle$, $(\ell' = 5)$

(31) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus \langle -\frac{1}{2}\rangle \oplus \langle -\frac{2}{3}\rangle \oplus \langle \frac{2}{3}\rangle$, $(\ell' = 2)$

(32) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus 2\langle -\frac{1}{2}\rangle \oplus \langle -\frac{2}{3}\rangle \oplus \langle \frac{2}{3}\rangle$, $(\ell' = 3)$

(33) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus \langle -\frac{1}{2}\rangle \oplus 3\langle -\frac{2}{3}\rangle$, $(\ell' = 3)$

(34) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus 2\langle -\frac{1}{2}\rangle \oplus 3\langle -\frac{2}{3}\rangle$, $(\ell' = 3)$

(35) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus 3\langle -\frac{1}{2}\rangle \oplus 3\langle -\frac{2}{3}\rangle$, $(\ell' = 4)$

(36) $\widetilde{\mathcal{S}} = \langle \frac{1}{2}\rangle \oplus \langle -\frac{1}{2}\rangle \oplus \langle -\frac{2}{3}\rangle$, $(\ell' = 2)$

Abundant and non-abundant forms of a curve are not rigidly isotopic. They have different homological types since they have different configurations. But the problem is whether they exist and whether they are rigidly isotopic in themselves as a class.

Existence and uniqueness of $\widetilde{S}^{\perp}$ and uniqueness of $\mathrm{Aut}_h\widetilde{\mathcal{S}}\times\mathrm{Aut}_N\mathcal{N}$ should be investigated to conclude that this is the classification of abstract homological types. To be sure that the curves in this class are rigidly isotopic, also we would check that these abstract homological types are symmetric.

5.2 Enumerating the isomorphism classes of $\widetilde{S}^{\perp}$

5.2.1 Existence of $\widetilde{S}^{\perp}$ according to Nikulin's theorem in [12]

Starting with the first condition of the theorem (2.1.3), signature (σ_+N,σ_-N) of $\widetilde{S}^{\perp}$ would be considered. Recall that, since the signature of whole space is (3,19), one has (σ_+N,σ_-N)=($3-\sigma_+$,$19-\sigma_-$) where (σ_+,σ_-) is signature of $\widetilde{S}^{\perp}$. On the other hand, disc$\widetilde{S}^{\perp}$=$-\widetilde{\mathcal{S}}$, then disc$\widetilde{S}^{\perp}$ has the same minimal number of generators with $\widetilde{\mathcal{S}}$. Looking at the lists of data, one can see that non-abundant curves numbered (30),(32),(35) and (36) doesn't satisfy the condition $\sigma_+N+\sigma_-N \geq \ell(\text{disc}\widetilde{S}^{\perp})$. It is easy to see that $\text{Br}(\text{disc}\widetilde{S}^{\perp})=\sigma_+(\widetilde{S}^{\perp})-\sigma_-(\widetilde{S}^{\perp})$ satisfying for all curves by the additivity property of Brown invariant and existence of $\widetilde{S}$.

The other condition has two subconditions where satisfying one of them is sufficient. All abundant $\widetilde{S}^{\perp}$'s and most of non-abundant ones satisfy $\sigma_+N+\sigma_-N \geq \ell_p(\text{disc}\widetilde{S}^{\perp})$, $[p \neq 2]$. Curves numerated (9), (12), (21), (24), (26), (29), (31), (34) doesn't satisfy $\sigma_+N, \sigma_-N > \ell_3(\text{disc}\widetilde{S}^{\perp})$. However these curves satisfy the second subcondition $\det_3(\text{disc}\widetilde{S}^{\perp})_3.\ |\text{disc}\widetilde{S}^{\perp}_{\neq 3}|=(-1)^{\sigma_-N} \bmod (\mathbb{Z}_3^*)^2$.

In checking last condition a consideration similar to the previous case shows that the abundant curves numbered (2), (4), (5), (8), (10), (11), (13), (15), (19), (22), (23), (25), (27), (28), (29), (31), (33), (34) satisfy the subcondition $\sigma_+N, \sigma_-N > \ell_2(\text{disc}\widetilde{S}^{\perp})$. Notice that 2-primary part in a configuration's disc$\widetilde{S}^{\perp}$ doesn't change with abundant and non-abundant forms. It means that same numbered non-abundant curves have the desired results for this subcondition,too. Likewise it was stated as second subcondition that being $(\text{disc}\widetilde{S}^{\perp})_2$ is odd. All $\widetilde{\mathcal{S}}^{\perp}$s contain $\langle\frac{1}{2}\rangle$ in 2-primary part and one can take the multiple of $\langle\frac{1}{2}\rangle$ as 1 and multiple of others as 0. Then an element a of 2-primary part is obtained where $a^2 \neq 0 \bmod 2$. So, being odd of $(\text{disc}\widetilde{S}^{\perp})_2$ confirms the existence of the remaining abundant and non-abundant $\widetilde{S}^{\perp}$s.

As a consequence, all curves reach the standards needed for the last three conditions, but the first condition forms an obstacle for the existence of some $\widetilde{S}^{\perp}$s. That is to say, non-abundant curves with singularities $6\mathbf{A}_2 \oplus 4\mathbf{A}_1$, $2\mathbf{A}_2 \oplus 2\mathbf{E}_6 \oplus 2\mathbf{A}_1$, $4\mathbf{A}_2 \oplus \mathbf{E}_6 \oplus 3\mathbf{A}_1$, $3\mathbf{E}_6 \oplus \mathbf{A}_1$ doesn't exist and the others with abundant and non-abundant types exists certainly.

5.2.2 Uniqueness of $\widetilde{S}^{\perp}$ according to Nikulin's theorem in [12]

From now on, we look for the uniqueness of existing $\widetilde{S}^{\perp}$ obtained from the previous subsection. There are two theorems for checking uniqueness where one of them is stronger. The stronger one includes confirming surjectivity of the canonical homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{\mathcal{S}}^{\perp})$. Firstly it is better to check the strong theorem and then to check the weak theorem together with surjectivity for the remaining ones.

One can not check the uniqueness of abundant curve numbered (36) by the theorems (2.1.4) and (2.1.5), since it has $\mathrm{rk}\widetilde{S}^{\perp} = 2$. For others, theorem (2.1.5) includes two conditions, one is for p-primary part [p$\neq$ 2] and the other is for 2-primary part. Abundant and non-abundant curves numbered (4) doesn't satisfy both subconditions for 2-primary part. Notice that the condition for 2-primary part is the same for abundant and non-abundant type of a curve. All other curves satisfy the condition for 2-primary part. Indeed, the curves other than abundant, non-abundant curves numbered (2), (5), (8), (10), (15), (27), (31), (33) have $\langle \frac{1}{2} \rangle \oplus \langle \frac{-1}{2} \rangle \oplus \langle \frac{\pm 1}{2} \rangle \cong \mathcal{U}_2 \oplus \langle \frac{\pm 1}{2} \rangle$ as an orthogonal summand in $\mathrm{disc}\widetilde{S}^{\perp}$ which satisfies the condition for 2-primary part and the curves numbered above satisfy the first subcondition $\mathrm{rk}(\widetilde{S}^{\perp}) \geq \ell_2(\mathrm{disc}\widetilde{S}^{\perp}) + 2$.

On the other hand, non-abundant $\widetilde{S}^{\perp}$s numbered (3), (4), (6), (8), (9), (11), (12), (14), (16), (18), (20), (21), (23), (24), (25), (26), (28), (29), (31), (33), (34) and abundant $\widetilde{S}^{\perp}$s numbered (30), (32), (35), (36) don't satisfy the condition $\mathrm{rk}\widetilde{S}^{\perp} \geq \ell_3(\mathrm{disc}\widetilde{S}^{\perp}) + 2$. All curves satisfy the inequality $\mathrm{rk}\widetilde{S}^{\perp} \geq \ell_p(\mathrm{disc}\widetilde{S}^{\perp}) + 2$ for all $p \neq 2, 3$. As a conclusion, curves other than the last listed ones and abundant curve numbered (4) satisfy both conditions of theorem (2.1.5) and then their uniqueness and surjectivity of their canonical maps are non-debateable.

At this point one has to check the weak theorem (2.1.4) for $\widetilde{S}^{\perp}$s numbered above. Abundant type of disc$\widetilde{S}^{\perp}$ numbered (4) contains $\langle\frac{1}{2}\rangle \oplus \langle\frac{1}{4}\rangle$ which makes the condition for 2-primary part satisfied.

It is known that rk$\widetilde{S}^{\perp} \geq 3$ except for the last $\Sigma = 3\mathbf{E}_6 \oplus \mathbf{A}_1$. These numbered $\widetilde{\mathcal{S}}^{\perp}$s not satisfying rk$\widetilde{S}^{\perp} \geq \ell_3(\text{disc}\widetilde{S}^{\perp}) + 2$ should have $\ell_3(\text{disc}\widetilde{S}^{\perp})$ at least 2. By reason that there is no disc$\widetilde{S}^{\perp}$ having only $\langle\frac{a}{3}\rangle \oplus \langle\frac{b}{9}\rangle$ as 3-primary part, each disc$\widetilde{S}^{\perp}$ has either $\langle\frac{a}{3}\rangle \oplus \langle\frac{b}{3}\rangle$ or $\langle\frac{a}{9}\rangle \oplus \langle\frac{b}{9}\rangle$.

After all, thinking about the remaining one, one can observe that $\widetilde{S}^{\perp}$ of $\Sigma = 3\mathbf{E}_6 \oplus \mathbf{A}_1$ is the definite lattice of rank 2. Discriminant of $\widetilde{S}^{\perp}$ must satisfy $|\text{disc}\widetilde{S}^{\perp}|/4 \leq ac \leq |\text{disc}\widetilde{S}^{\perp}|/3$ and then one has the lattices $M(1,0,3)$, $M(2,0,2)$, $M(2,1,2)$, $M(2,2,2)$, $M(1,0,4)$, $M(1,1,4)$, $M(1,1,3)$ as $\widetilde{S}^{\perp}$. However only $M(1,0,3)$, $M(2,2,2)$ is appropriate to have det$M = |\text{disc}\widetilde{S}^{\perp}|$. Additionally, disc$M$ should have the same group with $\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle$. This is corrected only by the lattice $M(1,0,3)$ which shows that abundant $\widetilde{S}^{\perp}$ of $\Sigma = 3\mathbf{E}_6 \oplus \mathbf{A}_1$ is unique in its genus up to isomorphism.

In conclusion, by uniqueness theorems every existing $\widetilde{S}^{\perp}$ is unique and the canonical homomorphism $\text{Aut}(\widetilde{S}^{\perp}) \to \text{Aut}(\text{disc}\widetilde{S}^{\perp})$ is onto for non-abundant $\widetilde{S}^{\perp}$s numbered (1), (2), (5), (7), (10), (13), (15), (17), (19), (22), (27) and for all abundant $\widetilde{S}^{\perp}$s except for the curves numbered (4), (30), (32), (35), (36). Surjectiveness of $\text{Aut}(\widetilde{S}) \to \text{Aut}(\widetilde{\mathcal{S}})$ was obtained by uniqueness theorems. This means that $\text{Aut}_h\widetilde{\mathcal{S}} = \text{Aut}\widetilde{\mathcal{S}}$. When we will show that the homomorphism $\text{Aut}(\widetilde{S}^{\perp}) \to \text{Aut}(\text{disc}\widetilde{S}^{\perp})$ is onto for all remaining curves, $\text{Aut}_{\widetilde{S}^{\perp}}(\text{disc}\widetilde{S}^{\perp}) = \text{Aut}(\text{disc}\widetilde{S}^{\perp})$ will be attained. Hence the quotient $\text{Aut}_h\widetilde{\mathcal{S}}\backslash\text{Aut}\widetilde{\mathcal{S}}/\text{Aut}_{\widetilde{S}^{\perp}}\text{disc}\widetilde{S}^{\perp}$ will consist only one coset, giving rise to one abstract homological type if it is symmetric and two conjugate abstract homological types if it is asymmetric. To check the symmetry, it is enough to show that $\widetilde{S}^{\perp}$s have a $+$disorienting isometry t whose image in $\text{Aut}(\text{disc}\widetilde{S}^{\perp})$ belongs to the product of $\text{Aut}_h\widetilde{S}$ and the image of $\text{Aut}^{+}\widetilde{S}^{\perp}$.

Firstly, a lemma will be stated that will be useful in most of the proofs of these maps' surjectiveness.

Lemma 5.2.1 *Let L be an indefinite integral lattice with $rkL \geq 4$ and let the discriminant group $\mathcal{L}$ of L be a direct sum of copies of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ and assume further that $\mathcal{L}$ is odd. Then the conditions for $a, b \in L$*

(1) $a^2 = b^2 = +6$ with $\sigma_+ L \geq 2, \quad \sigma_- L \geq 1$, or

$a^2 = b^2 = -6$ with $\sigma_+ L \geq 1, \quad \sigma_- L \geq 2$,

(2) $(x, a) = (x, b) = 0$ mod 6 for any $x \in L$,

(3) $[\frac{a}{2}], [\frac{b}{2}]$ are not characteristic elements in $\mathcal{L} \otimes \mathbb{Z}_2$

are sufficient to have an isomorphism sending $\langle a \rangle$ to $\langle b \rangle$.

Proof:Assume that for such an L conditions are satisfied for two elements $a, b \in L$. Since one has the projection $\varphi : L \otimes \mathbb{Q} \to \langle a \rangle \otimes \mathbb{Q}$ such that every $x \in L$ has $\varphi : x \mapsto \frac{(x,a)}{(a,a)} \cdot a$. By the second condition $(x, a) = 0$ mod 6, x is taken to $n \cdot a$ for some $n \in \mathbb{Z}$. Then one obtains the projection on a which means that $\langle a \rangle$ is direct summand of N. In the same way $\langle b \rangle$ is a direct summand of N. Hence one has the isomorphism $\langle a \rangle \oplus \langle a \rangle^\perp \cong \langle b \rangle \oplus \langle b \rangle^\perp$.

To determine the genus of $\langle a \rangle^\perp$ suppose that $a^2 = b^2 = +6$, then the new lattice $\langle a \rangle^\perp$ has the same summands with L other than $\langle 6 \rangle$. Assume that L has the signature (σ_+, σ_-), then one has $(\sigma_+ \langle a \rangle^\perp, \sigma_- \langle a \rangle^\perp) = (\sigma_+ - 1, \sigma_-)$. Since disc$\langle a \rangle$ is $\langle \frac{-1}{2} \rangle \oplus \langle \frac{2}{3} \rangle$, by the proposition(2.1.1), one can determine 3-primary part of disc$\langle a \rangle^\perp$ by $\ell_3(\text{disc}\langle a \rangle^\perp) = \ell_3(\mathcal{L}) - 1$ and $\text{Br}((\text{disc}\langle a \rangle^\perp)_3) = \text{Br}\mathcal{L} - 2$. By the proposition(2.1.2), $(\text{disc}\langle a \rangle^\perp)_2$ is determined by $\ell_2(\text{disc}\langle a \rangle^\perp) = \ell_2(\mathcal{L}) - 1$, $\text{Br}((\text{disc}\langle a \rangle^\perp)_2) = \text{Br}\mathcal{L} + 1$ and parity. Parity of $(\text{disc}\langle a \rangle^\perp)_2$ is evenness or oddness of $(\text{disc}\langle a \rangle^\perp)_2$. Indeed, 2-primary part of $\mathcal{L}$ is odd and if $[\frac{a}{2}]$ is not characteristic element then $(\text{disc}\langle a \rangle^\perp)_2$ is odd, if $[\frac{a}{2}]$ is characteristic element then $(\text{disc}\langle a \rangle^\perp)_2$ is even. Hence $[\frac{a}{2}]$ and shouldn't be characteristic element in $\mathcal{L}_2$ to have the same parity with $(\text{disc}\langle b \rangle^\perp)_2$. Thus for $a^2 = b^2 = +6$ the genus of the integral lattice $\langle a \rangle^\perp$ is $(\sigma_+ - 1, \sigma_-;\text{disc}\langle a \rangle^\perp)$ with Brown invariant $\text{Br}(\mathcal{L}) - 1$. If $a^2 = -6$, then the genus of the integral lattice $\langle a\rangle^{\perp}$ is $(\sigma_{+},\sigma_{-}-1;\mathrm{disc}\langle a\rangle^{\perp})$ with Brown invariant $\mathrm{Br}(\mathcal{L})+1$.

By showing that $\langle a\rangle^{\perp}$ is unique up to isomorphism, one can say that there is an automorphism sending $\langle a\rangle^{\perp}$ to $\langle b\rangle^{\perp}$ and $\langle a\rangle$ to $\langle b\rangle$.

For the uniqueness of the lattice $\langle a\rangle^{\perp}$ with $a^2=+6$, by the theorem(2.1.4), either $(\mathrm{disc}\langle a\rangle^{\perp})_3$ should contain a subform isomorphic to $\langle\frac{a'}{3}\rangle\oplus\langle\frac{b'}{3}\rangle$ as an orthogonal summand or $\mathrm{rk}\langle a\rangle^{\perp}\geq\ell_3(\mathrm{disc}\langle a\rangle^{\perp})+2$. If $\mathrm{rk}L\geq\ell_3(\mathcal{L})+2$ then $\mathrm{rk}\langle a\rangle^{\perp}=\mathrm{rk}L-1\geq\ell_3(\mathcal{L})-1+2=\ell_3(\mathrm{disc}\langle a\rangle^{\perp})+2$ holds, too. Otherwise, since rank of L is at least 4, to disturb this inequality $\ell_3(\mathcal{L})$ is at least 3. Removing one of them by taking out $\mathrm{disc}\langle a\rangle$, one has $\langle\frac{a'}{3}\rangle\oplus\langle\frac{b'}{3}\rangle$ as an orthogonal summand in $\mathrm{disc}\langle a\rangle^{\perp}$ which satisfies the first condition of uniqueness theorem. In the same way, if $\ell_2(\mathcal{L})$ is at most 2, then one has $\mathrm{rk}\langle a\rangle^{\perp}\geq\ell_2(\mathrm{disc}\langle a\rangle^{\perp})+2$. If one has $\ell_2(\mathcal{L})=3$, then $\ell_2(\mathrm{disc}\langle a\rangle^{\perp})$ is 2. At this point there is a congruence $\ell_2(\mathrm{disc}\langle a\rangle^{\perp})=\mathrm{rk}(\langle a\rangle^{\perp})=\mathrm{Br}(\mathrm{disc}\langle a\rangle^{\perp}) \bmod 2$. Also having $\mathrm{rk}(\langle a\rangle^{\perp})\geq 3$ and $\ell_2(\mathrm{disc}\langle a\rangle^{\perp})=2$ cause that the rank of $\langle a\rangle^{\perp}$ is at least 4 which corrects the inequality $\mathrm{rk}(\langle a\rangle^{\perp})\geq\ell_2(\mathrm{disc}\langle a\rangle^{\perp})+2$. If $\ell_2(\mathcal{L})$ is strictly greater than 3, then $\ell_2(\mathrm{disc}\langle a\rangle^{\perp})$ is at least 3. Hence one has $\mathcal{U}_2$ or $\mathcal{V}_2$ as an orthogonal summand in $\mathrm{disc}\langle a\rangle^{\perp}$. Thus $\langle a\rangle^{\perp}$ is unique in its genus. For the lattice $\langle a\rangle^{\perp}$ with $a^2=-6$, the same procedure occurs and $\langle a\rangle^{\perp}$ is obtained unique in its genus.

$\square$

It is better to start with abundant ones numbered (4), (30), (32), (35), (36) to show the surjective homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp})\to\mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ and symmetry.

- *The case abundant $\Sigma=\mathbf{A}_{11}\oplus\mathbf{E}_6\oplus\mathbf{A}_1$ numbered (4).* One has $\mathrm{disc}\widetilde{S}^{\perp}=\langle\frac{1}{2}\rangle\oplus\langle\frac{-1}{2}\rangle\oplus\langle\frac{1}{4}\rangle\cong 2\langle\frac{1}{2}\rangle\oplus\langle\frac{-1}{4}\rangle$ and signature (2,1), so that one can take $\widetilde{S}^{\perp}=\langle 2\rangle\oplus\langle -2\rangle\oplus\langle 4\rangle\cong 2\langle 2\rangle\oplus\langle -4\rangle$. Let a_1,a_2,b be generators of the $2\langle 2\rangle$ and $\langle -4\rangle$-summands, respectively. Consider $\alpha_1=[\frac{a_1}{2}]$, $\alpha_2=[\frac{a_2}{2}]$, $\beta=[\frac{b}{2}]$, as generators of $2\langle\frac{1}{2}\rangle$, $\langle\frac{1}{4}\rangle$-summands.

Automorphisms on $\mathrm{disc}\widetilde{S}^{\perp}$ are the reflections $t_{a_2-a_1}$ transposing α_1, α_2 and the reflection t_b multiplying β by (-1).

Since $\widetilde{S}^{\perp}$ contains a vector of square 2, abstract homological type extending this Σ is symmetric by the proposition(3.2.1).

- *The case abundant* $\Sigma = 6\mathbf{A}_2 \oplus 4\mathbf{A}_1$ *numbered (30).* One has disc$\widetilde{S}^{\perp} = 4\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle \oplus 3\langle\frac{-2}{3}\rangle$ and signature (2,3), so that one can take $\widetilde{S}^{\perp} = 3\langle -6\rangle \oplus \langle 6\rangle \oplus \langle 2\rangle$. Let a_1, a_2, a_3 be generators of the $3\langle -6\rangle$-summands and b, c be generators of the $\langle 6\rangle$ and $\langle 2\rangle$-summands, respectively. Consider $\delta_i = [\frac{a_i}{2}]$ $(i = 1, 2, 3)$, $\xi = [\frac{c}{2}]$, $\theta = [\frac{b}{2}]$ as generators of $4\langle\frac{1}{2}\rangle$, $\langle\frac{-1}{2}\rangle$-summands and $\beta = [\frac{b}{3}]$, $\alpha_i = [\frac{a_i}{3}]$ $(i = 1, 2, 3)$ as generators of $\langle\frac{2}{3}\rangle$, $3\langle\frac{-2}{3}\rangle$-summands, respectively.

The reflections t_{a_i}, t_b are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part and act identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_{a_i}, t_b, this automorphism h takes α_1 to either α_i or $(\alpha_i + \alpha_j + \beta)$, $(i, j = 1, 2, 3$ and $i \neq j)$, where according to lemma (5.2.1) they can be realized by the automorphism h' in $\widetilde{S}^{\perp}$, taking a_1 to a_i or $(a_i + a_j + b)$, $(i, j = 1, 2, 3$ and $i \neq j)$, respectively. Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes α_1.

Similarly, the automorphism h on the 2-primary part sends δ_1 to either δ_i, or ξ, or $(\delta_i + \delta_j + \theta)$, or $(\delta_i + \xi + \theta)$,$(i, j = 1, 2, 3,\ i \neq j)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending a_1 to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(a_1) \in \widetilde{S}^{\perp}$
δ_1	a_1
δ_2	$(-2a_1 + 3a_2 + 6c)$
δ_3	$(-2a_1 + 3a_3 + 6c)$
ξ	$(-2a_1 + 3c)$
$(\delta_1 + \delta_i + \theta)$, $(i \neq 1)$	$(a_1 + 3a_i + 3b)$, $(i \neq 1)$
$(\delta_1 + \xi + \theta)$	$(a_1 + 6a_2 + 3b + 9c)$
$(\delta_2 + \delta_3 + \theta)$	$(-8a_1 + 3a_2 + 3a_3 + 9b)$
$(\delta_j + \xi + \theta)$, $(j \neq 1)$	$(-2a_1 + 3a_j + 3b + 3c)$, $(j \neq 1)$

Notice that the elements $h'(a_1)$ and a_1 are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of Aut(disc$\widetilde{S}^{\perp}$) leaves α_1 fixed. Thus, one can assume that h fixes both α_1 and δ_1 and consider the smaller form $\mathcal{N}_1 = 3\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle \oplus 2\langle\frac{-2}{3}\rangle =$disc$N_1$, where $N_1 = 2\langle-6\rangle \oplus \langle 6\rangle \oplus \langle 2\rangle$.

The automorphism h on $\mathcal{N}_1$, modulo t_{a_2}, t_{a_3}, t_b sends α_2 to either α_2, or α_3, or $(\alpha_2 + \alpha_3 + \beta)$. For these automorphisms, by lemma(5.2.1), one can find automorphism h' on $\widetilde{S}^{\perp}$ sending a_2 to a_2 or a_3 or $(a_2 + a_3 + b)$. Then any automorphism fixing α_2 can be realized by some automorphisms on $\widetilde{S}^{\perp}$. Then, modulo $O(N_1)$, one can assume that h fixes α_2.

Besides, the automorphism h sends δ_2 to either δ_i with $(i = 2, 3)$ or ξ or $(\delta_2 + \delta_3 + \theta)$ or $(\delta_2 + \xi + \theta)$ or $(\delta_3 + \xi + \theta)$. By lemma (5.2.1) one can find automorphisms on $\widetilde{S}^{\perp}$ sending a_2 to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_2) \in$disc$\widetilde{S}^{\perp}$	$h'(a_2) \in \widetilde{S}^{\perp}$
δ_2	a_2
δ_3	$(-2a_2 + 3a_3 + 6c)$
ξ	$(-2a_2 + 3c)$
$(\delta_2 + \delta_3 + \theta)$	$(a_2 + 3a_3 + 3b)$
$(\delta_2 + \xi + \theta)$	$(a_2 + 6a_3 + 3b + 9c)$
$(\delta_3 + \xi + \theta)$	$(2a_2 + 3a_3 + 3b + 3c)$

Similar to the case of investigating the act of automorphism h on δ_1 and α_1, the elements $h'(a_2)$ and a_2 are congruent in mod $3N_1$ which means that the induced automorphism elements of Aut($\mathcal{N}_1$) leaves α_2 fixed. Thus, one can assume that h fixes both α_2 and δ_2 and consider the smaller form $\mathcal{N}_2 = 2\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle \oplus \langle\frac{-2}{3}\rangle =$disc$N_2$, where $N_2 = \langle-6\rangle \oplus \langle 6\rangle \oplus \langle 2\rangle$ with generators a_3, b, c, respectively.

On the 3-primary part of $\mathcal{N}_2$ the automorphism h is trivial, modulo the reflections t_{a_3}, t_b. On the 2-primary part of $\mathcal{N}_2$ if h is nontrivial, then it is realized by the reflection t_{c-a_3} transposing δ_3 and ξ.

Additionally, notice that $\widetilde{S}^{\perp}$ contains a vector of square 2 which makes the abstract homological type extended by this Σ symmetric by the proposition(3.2.1)

- *The case abundant $\Sigma = 2\mathbf{A}_2 \oplus 2\mathbf{E}_6 \oplus 2\mathbf{A}_1$ numbered (32).* One has $\mathrm{disc}\widetilde{S}^{\perp} = \langle\frac{-1}{2}\rangle \oplus 2\langle\frac{1}{2}\rangle \oplus \langle\frac{2}{3}\rangle \oplus \langle\frac{-2}{3}\rangle$ and signature (2,1), so that one can take $\widetilde{S}^{\perp} = \langle 6\rangle \oplus \langle -6\rangle \oplus \langle 2\rangle$.

This is the same case with the discriminant $\mathcal{N}_2$ and lattice N_2 in the case of abundant $\Sigma = 6\mathbf{A}_2 \oplus 4\mathbf{A}_1$ numbered (30). It is shown that all automorphisms on $\mathrm{disc}\widetilde{S}^{\perp}$ can be realized by automorphisms on $\widetilde{S}^{\perp}$. Then, it is known that $O(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto.

Since $\widetilde{S}^{\perp}$ contains a vector of square 2, abstract homological type extended by this Σ is symmetric by the proposition(3.2.1)

- *The case abundant $\Sigma = 4\mathbf{A}_2 \oplus \mathbf{E}_6 \oplus 3\mathbf{A}_1$ numbered (35).* One has $\mathrm{disc}\widetilde{S}^{\perp} = 3\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle \oplus 2\langle\frac{-2}{3}\rangle$ and signature $(2,2)$, so that one can take $\widetilde{S}^{\perp} = 2\langle -6\rangle \oplus \langle 6\rangle \oplus \langle 2\rangle$.

This is the same case with the discriminant $\mathcal{N}_1$ and lattice N_1 in the case of abundant $\Sigma = 6\mathbf{A}_2 \oplus 4\mathbf{A}_1$. It is shown that all automorphisms on $\mathrm{disc}\widetilde{S}^{\perp}$ can be realized by automorphisms on $\widetilde{S}^{\perp}$. Then, it is known that $O(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto.

Since $\widetilde{S}^{\perp}$ contains a vector of square 2, abstract homological type extended by this Σ is symmetric by the proposition 5.1.3.

- *The case abundant $\Sigma = 3\mathbf{E}_6 \oplus \mathbf{A}_1$ numbered (36).* One has $\mathrm{disc}\widetilde{S}^{\perp} = \langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle$ with signature $(2,0)$ and one can take $\widetilde{S}^{\perp} = \langle 6\rangle \oplus \langle 2\rangle$. Let a, b be the generators of the $\langle 6\rangle$, $\langle 2\rangle$-summands. Consider $\delta = [\frac{b}{2}]$, $\theta = [\frac{a}{2}]$, $\alpha = [\frac{a}{3}]$ as generators of $\langle\frac{1}{2}\rangle$, $\langle\frac{-1}{2}\rangle$, $\langle\frac{2}{3}\rangle$-summands, respectively. The only nontrivial automorphism can be generated by the reflection t_a which multiplies α by (-1).

Since $\widetilde{S}^{\perp}$ contains a vector of square 2, abstract homological type extended by this Σ is symmetric by the proposition 5.1.3.

We have shown that each singularity Σ from the list with abundant type numbered (4), (30), (32), (35), (36) has one abstract homological type which is symmetric. Remember that abundant ones other than these five numbered satisfy the theorem (2.1.5) and each of them has one abstract homological type. Furthermore, homological type of each remaining curve is symmetric by theorem (3.2.1) or theorem (3.2.2). Indeed, abundant configurations extending the set of singularities Σ numbered (2), (5), (8), (10), (11), (13), (15), (19), (22), (23), (25), (27), (28), (29), (31), (33), (34) satisfy theorem (3.2.2) and abstract homological types extending the set of singularities Σ numbered (1), (3), (6), (7), (9), (12), (14), (16), (17), (18), (20), (21), (24), (26) satisfy theorem (3.2.1). Configurations satisfying theorem (3.2.2) are realized by a unique rigid isotopy class of plane sextics and each of the other numbered abstract homological types has $\widetilde{S}^{\perp}$ containing a vector of square 2 as listed below.

(1) $\widetilde{S}^{\perp} = 2\langle 2\rangle \oplus \langle -2\rangle$

(3) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle 10\rangle$

(6) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle 4\rangle \oplus \langle -6\rangle$

(7) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus 2\langle -2\rangle \oplus \langle 4\rangle$

(9) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle 18\rangle$

(12) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle -6\rangle \oplus \langle 18\rangle$

(14) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus 2\langle -2\rangle \oplus \langle 18\rangle$

(16) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle 18\rangle$

(17) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus 3\langle -2\rangle \oplus \langle 6\rangle$

(18) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus 2\langle -2\rangle \oplus \langle 6\rangle$

(20) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus 2\langle -2\rangle \oplus \langle 6\rangle \oplus \langle -6\rangle$

(21) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle 6\rangle$

(24) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle 6\rangle \oplus 2\langle -6\rangle$

(26) $\widetilde{S}^{\perp} = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle 6\rangle \oplus \langle -6\rangle$

As a conclusion each abundant sextic extending Σ in the form 5.1 is realized by a rigid isotopy class.

For the same classification of non-abundant sextics extending Σ in the form 5.1, it is needed to prove the surjectiveness of $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ for the remaining non-abundant curves, numbered (3), (4), (6), (8), (9), (11), (12), (14), (16), (18), (20), (21), (23), (24), (25), (26), (28), (29), (31), (33), (34), in the same way done for abundant ones.

- *The case non-abundant* $\Sigma = \mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$ *numbered (3).* One has $\mathrm{disc}\widetilde{S}^{\perp} = 2\langle \frac{1}{2}\rangle \oplus \langle \frac{-1}{2}\rangle \oplus \langle \frac{2}{3}\rangle \oplus \langle \frac{-2}{3}\rangle \oplus \langle \frac{2}{5}\rangle$ with signature $(2,1)$ and one can take $\widetilde{S}^{\perp} = \langle 6\rangle \oplus \langle -6\rangle \oplus \langle 10\rangle$. Let a, b, c be the generators of the $\langle 6\rangle$, $\langle -6\rangle$, $\langle 10\rangle$-summands, respectively. Consider $\delta_1 = [\frac{b}{2}]$, $\delta_2 = [\frac{c}{2}]$, $\theta = [\frac{a}{2}]$ as generators of the 2-primary part and $\alpha = [\frac{a}{3}]$, $\beta = [\frac{b}{3}]$, $\gamma = [\frac{c}{5}]$ as generators of $\langle \frac{2}{3}\rangle$, $\langle \frac{-2}{3}\rangle$, $\langle \frac{2}{5}\rangle$-summands, respectively.

The reflections t_a, t_b are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part, they act identically on the 2 and 5-primary part. Similarly, t_c is the automorphism multiplying γ by (-1) where it acts identically on the 2 and 3-primary part. On the 2-primary part the reflection t_{b-c} transposes δ_1, δ_2 and acts identically on 3 and 5-primary parts.

On the 3-primary part any automorphism is trivial, modulo the reflections t_a, t_b and similarly on the 5-primary part any automorphism is trivial, modulo the reflections t_c. On the 2-primary part any nontrivial automorphism can be realized by the reflection t_{b-c}.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and $\mathrm{disc}\widetilde{S}^{\perp}$ can be identified such that δ_1 and δ_2 correspond to the generators of $\mathrm{disc}(2\mathbf{A}_1)$. In $\widetilde{S}$, δ_1 and δ_2 can be transposed by an admissible isometry where transposition of them is realized by a $+$disorienting automorphism on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ *numbered (4).* One has

$\mathrm{disc}\widetilde{S}^{\perp} = \langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle \oplus \langle\frac{-2}{3}\rangle \oplus \langle\frac{1}{4}\rangle$ with signature $(2,1)$ and one can take $\widetilde{S}^{\perp} = \langle 6\rangle \oplus \langle -6\rangle \oplus \langle 4\rangle$. Let a, b, c be the generators of the $\langle 6\rangle$, $\langle -6\rangle$, $\langle 4\rangle$-summands, respectively. Consider $\delta = [\frac{b}{2}]$, $\theta = [\frac{a}{2}]$, $\alpha = [\frac{a}{3}]$, $\beta = [\frac{b}{3}]$, $\gamma = [\frac{c}{4}]$ as generators of the $\langle\frac{1}{2}\rangle$, $\langle\frac{-1}{2}\rangle$, $\langle\frac{2}{3}\rangle$, $\langle\frac{-2}{3}\rangle$, $\langle\frac{1}{4}\rangle$-summands, respectively.

The reflections t_a, t_b are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part, they act identically on the 2-primary part. Similarly, on the 2-primary part t_c is the automorphism multiplying γ by (-1) and acts identically on the 3-primary parts.

On the 3-primary part any automorphism is trivial, modulo the reflections t_a, t_b. Modulo the reflection t_c, any automorphism h acting on the 2-primary part of $\mathrm{disc}\widetilde{S}^{\perp}$ is either trivial or takes γ to $(\delta + \theta + \gamma)$. By the same calculations done in the proof of the lemma (5.2.1), automorphism sending γ to $\delta + \theta + \gamma$ can be realized by an automorphism on $\widetilde{S}^{\perp}$ sending c to $(a + b + c)$. Then, it is clear that $O(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto for this case.

The principle significance of this case is that the singularity $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ admits a non-abundant type which is asymmetric.

We claim that there does not exist any +disorienting automorphism in $\widetilde{S}^{\perp}$ such that image of it in $\mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp}) = \mathrm{Aut}\widetilde{\mathcal{S}}$ corresponds to an admissible isometry of $\widetilde{S}$. The proof consists in the constructions of automorphism groups of $\mathrm{disc}\widetilde{S}^{\perp}$, $\widetilde{S}^{\perp}$ and $\widetilde{S}$.

One has $\mathrm{Aut}\ \mathrm{disc}\widetilde{S}^{\perp} \cong \mathbb{Z}^4$ with generators g_1, g_2, g_3, g_4 such that g_1 corresponds to the automorphism multiplying α by (-1), g_2 multiplying β by (-1), g_3 multiplying γ by (-1) and g_4 corresponds to the automorphism taking $\delta + \theta + \gamma$ to γ. All admissible isometries of $\widetilde{S}$ is either induced by the nontrivial symmetry of the Dynkin graph of $\mathbf{A}_{11}$ or by the nontrivial symmetry of the Dynkin graph of $\mathbf{E}_6$ or by the nontrivial symmetry of the Dynkin graph of both $\mathbf{A}_{11}$ and $\mathbf{E}_6$. Hence, $\mathrm{Im}\ O(\widetilde{S})$ in $\mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is a combination of an automorphism multiplying both α, γ by (-1) and an automorphism multiplying β by (-1).

It remains to describe the images of $O(\widetilde{S}^{\perp})$ and $O_+(\widetilde{S}^{\perp})$. To find all possible isometries in $\widetilde{S}^{\perp}$, one can make use of a larger lattice $N_2 = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle 4\rangle$. Notice that discriminant of N_2, $\mathcal{N}_2 = \langle \frac{1}{2}\rangle \oplus \langle \frac{-1}{2}\rangle \oplus \langle \frac{1}{4}\rangle$ is the same of disc$\widetilde{S}^{\perp}$ without 3-primary part. Let us call this part $\langle \frac{2}{3}\rangle \oplus \langle \frac{-2}{3}\rangle = \mathcal{K}$. Then automorphisms of $\mathcal{N}_2$ are among the automorphisms of disc$\widetilde{S}^{\perp}$ which stabilize 3-primary part and $\mathrm{Aut}_{\mathcal{K}}\mathrm{disc}\widetilde{S}^{\perp} \subset \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$. On the lattice form, $O(\widetilde{S}^{\perp}) \supset O_{\mathcal{K}}\widetilde{S}^{\perp} \subset O(N_2)$. Let y_1, y_2, y_3 be the generators of $\langle 2\rangle$, $\langle -2\rangle$, $\langle 4\rangle$ summands in $N_2 = \langle 2\rangle \oplus \langle -2\rangle \oplus \langle 4\rangle$. Then definition of $\widetilde{S}^{\perp}$ can be determined as $\widetilde{S}^{\perp} = \{v | v\cdot(y_1 - y_2) = 0 \ \ mod\, 3,\ v \in N_2\}$. Thus, automorphisms of $\widetilde{S}^{\perp}$ preserving $\mathcal{K}$, $O_{\mathcal{K}}\widetilde{S}^{\perp}$ is generated by automorphisms of N_2 preserving $(y_1 - y_2)$ mod $3N_2$ and $O(\widetilde{S}^{\perp})$ is generated by the elements of $O_{\mathcal{K}}(\widetilde{S}^{\perp})$ and t_b.

Obviously $O(N_2) = O(N_1)$, where $N_1 = N_2(\frac{1}{2}) = \langle 1\rangle \oplus \langle -1\rangle \oplus \langle 2\rangle \cong \langle 1\rangle \oplus \langle 1\rangle \oplus \langle -2\rangle$. Let $x_1 = y_2 + y_3$, $x_2 = y_1$, $x_3 = 2y_2 + y_3$ be the generators of $N_1 = \langle 1\rangle \oplus \langle 1\rangle \oplus \langle -2\rangle$. According to Vinberg [14], any automorphism on $N_1 = \langle 1\rangle \oplus \langle 1\rangle \oplus \langle -2\rangle$ is generated by the reflections $\langle t_{x_2},\ t_{x_1-x_2},\ t_{x_3-2x_1}\rangle$ which can be converted to $\langle t_{y_1},\ t_{-y_1+y_2+y_3},\ t_{-y_3}\rangle$ as the generators of automorphisms on $N_1 = \langle 1\rangle \oplus \langle -1\rangle \oplus \langle 2\rangle$ in terms of y_1, y_2, y_3.

Since $(y_1 - y_2)^2 = 0$ mod 3, the image of $(y_1 - y_2)$ mod 3 is a (mod 3)-isotopic element in $N_2/3N_2$, all such elements are $\langle y_1 \pm y_2\rangle$, $\langle y_1 \pm y_3\rangle$ in mod $3N_1$. Action of $O(N_2)$ over these elements is given by the following table:

	$y_1 - y_2$	$y_1 + y_2$	$y_1 + y_3$	$y_1 - y_3$
t_{y_1}	$y_1 + y_2$	$y_1 - y_2$	$y_1 - y_3$	$y_1 + y_3$
$t_{-y_1+y_2+y_3}$	$y_1 - y_2$	$y_1 + y_3$	$y_1 + y_2$	$y_1 - y_3$
t_{-y_3}	$y_1 - y_2$	$y_1 + y_2$	$y_1 - y_3$	$y_1 + y_3$

Say $G = O(N_2)$ and $H = \mathrm{stab}_G(y_1 - y_2) = \langle t_{-y_1+y_2+y_3},\ t_{-y_3}\rangle$, and hence $O_{\mathcal{K}}\widetilde{S}^{\perp} = G/H = \mathrm{orbit}\{y_1 - y_2\}$. The reflections $t_{y_1}, t_{-y_1+y_2+y_3}, t_{-y_3}$ were denoted as the generators of $G = \mathrm{Aut}N_2$. Fix $\{1,\ t_{y_1},\ t_{-y_1+y_2+y_3} \circ t_{y_1}, t_{y_1} \circ t_{-y_1+y_2+y_3} \circ t_{y_1}\}$ as the representatives of the cosets mod H. Then $H = \langle t_{-y_1+y_2+y_3},\ t_{-y_3},\ t_{y_1} \circ t_{-y_1+y_2+y_3} \circ t_{y_1} \circ t_{-y_1+y_2+y_3} \circ t_{y_1} \circ t_{-y_1+y_2+y_3} \circ t_{y_1},\ t_{y_1} \circ t_{-y_1+y_2+y_3} \circ t_{-y_3} \circ t_{y_1} \circ t_{-y_1+y_2+y_3} \circ t_{y_1}\rangle$. If the generators of N_2 are chosen as $x = \frac{b-2a}{3}$, $y = \frac{a-2b}{3}$, $z = c$, then $O_{\mathcal{K}}(\widetilde{S}^{\perp}) = H = \langle t_c,\ t_{a-b+c},\ X = t_{b-2a} \circ t_{a-b+c} \circ t_{b-2a} \circ t_{a-b+c} \circ t_{b-2a} \circ t_{a-b+c} \circ t_{b-2a},\ Y = t_{b-2a} \circ t_{a-b+c} \circ t_c \circ t_{b-2a} \circ t_{a-b+c} \circ t_{b-2a}\rangle$. In the matrix form one can set these reflections as; $t_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $t_{a-b+c} = \begin{pmatrix} -2 & -3 & -2 \\ 3 & 4 & 2 \\ -3 & -3 & -1 \end{pmatrix}$, $X = \begin{pmatrix} -2 & -3 & -2 \\ 3 & 4 & 2 \\ -3 & -3 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & -6 & -5 \end{pmatrix}$. The equality $t_{a-b+c} = X$ implies that $O_{\mathcal{K}}(\widetilde{S}^{\perp}) = \langle t_c,\ t_{a-b+c},\ Y\rangle$ and $O(\widetilde{S}^{\perp}) = \langle t_b, t_c,\ t_{a-b+c},\ Y\rangle$. Among these generators t_c and t_{a-b+c} are +disorienting and $O_+(\widetilde{S}^{\perp}) = \langle t_b,\ Y\rangle$. It remains to investigate the images of +disorienting automorphisms in $\widetilde{S}^{\perp}$ and Im$O(\widetilde{S})$.

Choose $s_1 : \alpha \mapsto -\alpha$, $s_2 : \beta \mapsto -\beta$, $s_3 : \gamma \mapsto -\gamma$, $s_4 : \gamma \mapsto \delta+\theta+\gamma$ as elements of Aut(disc$\widetilde{S}^{\perp}$).Then Im$\{t_b\}$=$\{s_2\}$, Im$\{t_c\}$=$\{s_3\}$, Im$\{t_{a-b+c}\}$=$\{s_3+s_4\}$ and Im$\{Y\}$=$\{s_1+s_2+s_3\}$.Therefore Im$O(\widetilde{S}^{\perp}) = \langle s_1, s_2, s_3, s_4\rangle$ and Im$O_+(\widetilde{S}^{\perp}) = \langle s_2, s_1+s_3\rangle$. On the other hand, Im$O(\widetilde{S}) = \langle s_2, s_1+s_3\rangle$. It is easily seen that (Im$O(\widetilde{S}^{\perp})\setminus$ Im$O_+(\widetilde{S}^{\perp})$)$\cap$ Im$O(\widetilde{S}) = \emptyset$ and this makes impossible to find any +disorienting automorphism in Aut(disc$\widetilde{S}^{\perp}$)=Aut($\widetilde{\mathcal{S}}$) corresponding to an admissible isometry in $O(\widetilde{S})$. Thus the proof of asymmetry of this case is completed.

- *The case non-abundant* $\Sigma = \mathbf{A}_{11} \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ *numbered (6).* One has disc$\widetilde{S}^{\perp} = 2\langle \frac{1}{2}\rangle \oplus \langle \frac{-1}{2}\rangle \oplus 2\langle \frac{-2}{3}\rangle \oplus \langle \frac{2}{3}\rangle \oplus \langle \frac{1}{4}\rangle$ with signature $(2,2)$ and one can take $\widetilde{S}^{\perp} = 2\langle -6\rangle \oplus \langle 6\rangle \oplus \langle 4\rangle$. Let a_1, a_2 be generators of $2\langle -6\rangle$-summands and b, c be generators of $\langle 6\rangle$ and $\langle 4\rangle$-summands, respectively. Consider $\delta_1 = [\frac{a_1}{2}]$, $\delta_2 = [\frac{a_2}{2}]$, $\theta = [\frac{b}{2}]$, $\alpha_1 = [\frac{a_1}{3}]$, $\alpha_2 = [\frac{a_2}{3}]$, $\beta = [\frac{b}{3}]$, $\gamma = [\frac{c}{4}]$ as generators of $2\langle \frac{1}{2}\rangle$, $\langle \frac{-1}{2}\rangle$, $2\langle \frac{-2}{3}\rangle$, $\langle \frac{2}{3}\rangle$, $\langle \frac{1}{4}\rangle$-summands, respectively.

The reflections t_{a_1}, t_{a_2}, t_b are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part and act identically on the 2-primary part. Similarly, on the 2-primary part t_c is the automorphism multiplying γ by (-1) and acts identically on 3-primary parts.

Pick an automorphism h acting on disc$\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_{a_1}, t_{a_2}, t_b, this automorphism h takes α_1 to either α_2 or $(\alpha_1+\alpha_2+\beta)$, where by the same calculations done in the proof of lemma (5.2.1) they can be realized by the automorphism h' in $\widetilde{S}^{\perp}$, taking a_1 to a_2 or (a_1+a_2+b), respectively.

Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes α_1.

Similarly, the automorphism h on the 2-primary part sends δ_1 to either δ_2, or $\delta_1 + \delta_2 + \theta$. By the same calculations done in the proof of lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending a_1 to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(a_1) \in \widetilde{S}^{\perp}$
δ_1	a_1
δ_2	$(4a_1 + 3a_2 + 6c)$
$(\delta_1 + \delta_2 + \theta)$	$(a_1 + 3a_2 + 3b)$

Notice that the elements $h'(a_1)$ and a_1 are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of Aut(disc$\widetilde{S}^{\perp}$) leaves α_1 fixed. Thus, one can assume that h fixes both α_1 and δ_1 and consider the smaller form $\mathcal{N}_1 = \langle \frac{1}{2} \rangle \oplus \langle \frac{-1}{2} \rangle \oplus \langle \frac{-2}{3} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle \frac{1}{4} \rangle =$disc$N_1$, where $N_1 = \langle -6 \rangle \oplus \langle 6 \rangle \oplus \langle 4 \rangle$.

This is the same with the disc$\widetilde{S}^{\perp}$ and the lattice $\widetilde{S}^{\perp}$ in the case of non-abundant $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ numbered (4). It is shown that all automorphisms on non-abundant disc$\widetilde{S}^{\perp}$ in numbered (4) can be realized by automorphisms of non-abundant $\widetilde{S}^{\perp}$ in numbered (4). Then, it is clear that $O(\widetilde{S}^{\perp}) \to$ Aut(disc$\widetilde{S}^{\perp}$) is onto for this case, too.

Since the homomorphism Aut$(\widetilde{S}^{\perp}) \to$ Aut(disc$\widetilde{S}^{\perp}$) is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^{\perp}$ can be identified such that β corresponds to the generator of disc$\mathbf{A}_2$. Sending the generator of disc$\mathbf{A}_2$ to the nontrivial symmetry of its Dynkin graph is an admissible isometry of $\widetilde{S}$ where transposition of them is realized by a +disorienting automorphism t_b on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

Now, we need one more statement which will be useful in cases with singularities containing one copy of the lattice $\mathbf{A}_8$.

Lemma 5.2.2 *Let L be an indefinite integral lattice with $rkL \geq 4$ and let the* *discriminant group $\mathcal{L}$ of L be a direct sum of copies of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and one copy of $\mathbb{Z}_9$ and assume further that $\mathcal{L}$ is odd. Then the conditions for $a, b \in L$*

(1) $a^2 = b^2 = 18$ with $\sigma_+ L \geq 2, \quad \sigma_- L \geq 1$, or

(2) $(x, a) = (x, b) = 0$ mod 18 for any $x \in L$,

(3) $[\frac{a}{2}], [\frac{b}{2}]$ are not characteristic elements in $\mathcal{L} \otimes \mathbb{Z}_2$

are sufficient to have an isomorphism sending $\langle a \rangle$ to $\langle b \rangle$.

Proof:

Similarly to the proof of lemma(5.2.1), if $a^2 = b^2 = 18$, then one has the isomorphism $\langle a \rangle \oplus \langle a \rangle^\perp \cong \langle b \rangle \oplus \langle b \rangle^\perp$ and the genus of the integral lattice $\langle a \rangle^\perp$ is $(\sigma_+ - 1, \sigma_-;\text{disc}\langle a \rangle^\perp)$ with Brown invariant $\text{Br}(\mathcal{L}) + 1$ where (σ_+, σ_-) is the signature of $\mathcal{L}$. The uniqueness of $\langle a \rangle^\perp$ in its genus concludes that there exists an isomorphism sending $\langle a \rangle^\perp$ to $\langle b \rangle^\perp$ and $\langle a \rangle$ to $\langle b \rangle$.

- *The case non-abundant $\Sigma = \mathbf{A}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ numbered (8).* One has $\text{disc}\widetilde{S}^\perp = \langle \frac{1}{2} \rangle \oplus \langle \frac{-1}{2} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle \frac{-2}{3} \rangle \oplus \langle \frac{8}{9} \rangle$ with signature $(2, 2)$ and one can take $\widetilde{S}^\perp = \langle 18 \rangle \oplus \langle -2 \rangle \oplus \mathbf{U}_3$. Let a, b be generators of $\langle 18 \rangle$ and $\langle -2 \rangle$-summands and u, v be standard basis for the $\mathbf{U}_3$-summand. Consider $\delta = [\frac{a}{2}]$, $\theta = [\frac{b}{2}]$, $\alpha = [\frac{u+v}{3}]$, $\beta = [\frac{u-v}{3}]$ and $\gamma = [\frac{2a}{9}]$ as generators of $\langle \frac{1}{2} \rangle$, $\langle \frac{-1}{2} \rangle$, $\langle \frac{2}{3} \rangle$, $\langle \frac{-2}{3} \rangle$ and $\langle \frac{8}{9} \rangle$-summands, respectively.

Every automorphism acting on 2-primary part is trivial.

The reflections t_{u+v}, t_{u-v}, t_a are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part and act identically on the 2-primary part.

Modulo the reflections t_{u+v}, t_{u-v}, t_a, any automorphism h acting on $\text{disc}\widetilde{S}^\perp$ is either trivial or takes γ to $\alpha + \beta + \gamma$ on the 3-primary part. According to lemma (5.2.2) they can be realized by the automorphism h' in $\widetilde{S}^\perp$, taking a to a or $(2u + a = u + v + u - v + a)$, respectively. Thus, modulo $O(\widetilde{S}^\perp)$, one can assume that h fixes γ.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, one can assume that γ in $\mathrm{disc}\widetilde{S}^{\perp}$ corresponds to the generator of $\mathbf{A}_8$ in $\widetilde{\mathcal{S}}$. Sending the generator of $\mathrm{disc}\mathbf{A}_8$ to the nontrivial symmetry of its Dynkin graph is an admissible isometry of $\widetilde{S}$. On the other hand multiplying the corresponding generator γ by (-1) is +disorienting automorphism in $\widetilde{S}^{\perp}$. So the abstract homological type is symmetric.

- *The case non-abundant* $\Sigma = \mathbf{A}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$ *numbered (9).* One has $\mathrm{disc}\widetilde{S}^{\perp} = 2\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle \oplus \langle\frac{-2}{3}\rangle \oplus \langle\frac{8}{9}\rangle$ with signature $(2,1)$ and one can take $\widetilde{S}^{\perp} = \langle 18\rangle \oplus \langle -6\rangle \oplus \langle 6\rangle$. Let a, b, c be the generators of $\langle 18\rangle$, $\langle -6\rangle$, $\langle 6\rangle$-summands, respectively. Consider $\delta_1 = [\frac{b}{2}]$, $\delta_2 = [\frac{a}{2}]$, $\theta = [\frac{c}{2}]$, $\alpha = [\frac{c}{3}]$, $\beta = [\frac{b}{3}]$ and $\gamma = [\frac{2a}{9}]$ as generators of $2\langle\frac{1}{2}\rangle$, $\langle\frac{-1}{2}\rangle$, $\langle\frac{2}{3}\rangle$, $\langle\frac{-2}{3}\rangle$ and $\langle\frac{8}{9}\rangle$-summands, respectively.

The reflections t_a, t_b, t_c are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part and act identically on the 2-primary part. On the 2-primary part the reflection t_{a-b} transposes δ_1, δ_2 and acts identically on the 3-primary part.

Similarly to the previous case, modulo the reflections t_a, t_b, t_c and t_{a-b}, any automorphism h acting on $\mathrm{disc}\widetilde{S}^{\perp}$ is either trivial or takes γ to $\alpha + \beta + \gamma$ on the 3-primary part. According to lemma (5.2.2) they can be realized by the automorphism h' in $\widetilde{S}^{\perp}$, taking a to a or $(a+b+c)$, respectively. Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes γ.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and $\mathrm{disc}\widetilde{S}^{\perp}$ can be identified such that δ_1 and δ_2 correspond to the generators of $\mathrm{disc}(2\mathbf{A}_1)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_1)$ can be transposed by an admissible isometry where transposition of the corresponding generators δ_1 and δ_2 is realized by a +disorienting automorphism t_{a-b} on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = \mathbf{A}_8 \oplus 3\mathbf{A}_2 \oplus 2\mathbf{A}_1$ *numbered (11).* One has $\mathrm{disc}\widetilde{S}^{\perp} = 2\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle \oplus 2\langle\frac{-2}{3}\rangle \oplus \langle\frac{8}{9}\rangle$ with signature $(2,3)$ and one can take

$\widetilde{S}^{\perp} = \langle 18\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle \oplus \mathbf{U}_3$. Consider the non-abundant case numbered (8) which has the same $\widetilde{S}^{\perp}$ without $\langle -6\rangle$-summand. Then take the same generators in the non-abundant case numbered (8) and let c be the generator of $\langle -6\rangle$-summand and take $\delta' = [\frac{c}{2}]$, $\beta' = [\frac{c}{3}]$ as generators of $\langle \frac{1}{2}\rangle$, $\langle \frac{-2}{3}\rangle$-summands, respectively.

The same reflections and automorphisms in the non-abundant case numbered (8) acts here. Additionally, the reflection t_{a-c} transposes δ, δ' and acts identically on the 3-primary part and the reflection t_{u-v-c} transposes β, β' and acts identically on the 2-primary part.

On the 3-primary part, modulo the reflections t_{u+v}, t_{u-v}, t_a, t_{a-c}, t_{u-v-c}, any automorphism h acting on disc$\widetilde{S}^{\perp}$ is either trivial or takes γ to $\alpha+\beta+\gamma$ which was realized by h' as mentioned in the case numbered (8). Additionally automorphism h can take β' to $(\alpha+\beta+\beta')$ on the 3-primary part. By the same calculations done in the proof of the lemma (5.2.1), automorphism sending β' to $\alpha+\beta+\beta'$ can be realized by an automorphism on $\widetilde{S}^{\perp}$ sending c to $((u+v)+c+(u-v))$.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^{\perp}$ can be identified such that δ and δ' correspond to the generators of disc$(2\mathbf{A}_1)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_1)$ can be transposed by an admissible isometry where transposition of the corresponding generators δ and δ' is realized by a +disorienting automorphism t_{a-c} on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = \mathbf{A}_8 \oplus 3\mathbf{A}_2 \oplus 3\mathbf{A}_1$ *numbered (12).* One has disc$\widetilde{S}^{\perp} = 3\langle \frac{1}{2}\rangle \oplus \langle \frac{-1}{2}\rangle \oplus \langle \frac{2}{3}\rangle \oplus 2\langle \frac{-2}{3}\rangle \oplus \langle \frac{8}{9}\rangle$ with signature $(2,2)$ and one can take $\widetilde{S}^{\perp} = \langle 18\rangle \oplus 2\langle -6\rangle \oplus \langle 6\rangle$. Let a, b_1, b_2, c be the generators of $\langle 18\rangle$, $2\langle -6\rangle$ and $\langle 6\rangle$-summands, respectively. Consider $\delta_1 = [\frac{b_1}{2}]$, $\delta_2 = [\frac{b_2}{2}]$, $\delta_3 = [\frac{-a}{2}]$, $\theta = [\frac{c}{2}]$, $\alpha = [\frac{c}{3}]$, $\beta_1 = [\frac{b_1}{3}]$, $\beta_2 = [\frac{b_2}{3}]$ and $\gamma = [\frac{2a}{9}]$ as generators of $3\langle \frac{1}{2}\rangle$, $\langle \frac{-1}{2}\rangle$, $\langle \frac{2}{3}\rangle$, $2\langle \frac{-2}{3}\rangle$ and $\langle \frac{8}{9}\rangle$-summands, respectively.

The reflections t_a, t_{b_1}, t_{b_2}, t_c are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part and act identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^\perp$. On the 3-primary part, modulo the reflections t_a, t_{b_1}, t_{b_2}, t_c, this automorphism h takes β_1 to either β_2 or $(\alpha+\beta_1+\beta_2)$, where by the same calculations done in the proof of lemma (5.2.1) they can be realized by the automorphism h' in $\widetilde{S}^\perp$, taking b_1 to b_2 or $(a+b_1+b_2)$, respectively. Thus, modulo $O(\widetilde{S}^\perp)$, one can assume that h fixes β_1.

Similarly, the automorphism h on the 2-primary part sends δ_1 to either δ_i, or $(\delta_i + \delta_j + \theta)$. By the same calculations done in the proof of lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^\perp$ sending b_1 to the destinations listed in the table below which are realized on disc$\widetilde{S}^\perp$ as shown on the table.

$h(\delta_1) \in$disc$\widetilde{S}^\perp$	$h'(b_1) \in \widetilde{S}^\perp$
δ_1	b_1
δ_2	$(-8b_1 + 9b_2 + 12c)$
δ_3	$(3a - 8b_1 + 6c)$
$(\delta_1 + \delta_2 + \theta)$	$(b_1 + 3b_2 + 3c)$
$(\delta_1 + \delta_3 + \theta)$	$(3a + b_1 + 6b_2 + 3c)$
$(\delta_2 + \delta_3 + \theta)$	$(3a + 10b_1 + 3b_2 + 9c)$

Notice that the elements $h'(b_1)$ and b_1 are congruent in mod $3\widetilde{S}^\perp$ which means that the induced automorphism elements of Aut(disc$\widetilde{S}^\perp$) leaves β_1 fixed. Thus, one can assume that h fixes both β_1 and δ_1 and consider the smaller form $\mathcal{N}_1 = 2\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus \langle\frac{2}{3}\rangle \oplus \langle\frac{-2}{3}\rangle \oplus \langle\frac{8}{9}\rangle$, where $N_1 = \langle 18\rangle \oplus \langle -6\rangle \oplus \langle 6\rangle$.

This is the same with the disc$\widetilde{S}^\perp$ and the lattice $\widetilde{S}^\perp$ in the case of non-abundant $\Sigma = \mathbf{A}_{11} \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ numbered (9). It is shown that all automorphisms on non-abundant disc$\widetilde{S}^\perp$ in numbered (9) can be realized by automorphisms of non-abundant $\widetilde{S}^\perp$ in numbered (9). Then, it is clear that $O(\widetilde{S}^\perp) \to$ Aut(disc$\widetilde{S}^\perp$) is onto for this case, too.

Since the homomorphism Aut$(\widetilde{S}^\perp) \to$ Aut(disc$\widetilde{S}^\perp$) is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^\perp$ can be identified such that δ_1 and δ_3 correspond to the generators of disc$(2\mathbf{A}_1)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_1)$ can be transposed by an admissible isometry where transposition of the corresponding generators δ_1 and δ_3 is realized by a +disorienting automorphism t_{b_1-a} on $\widetilde{S}^\perp$. So the abstract homological type extended by this Σ

is symmetric.

- *The case non-abundant* $\Sigma = \mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$ *numbered (14).* One has $\mathrm{disc}\widetilde{S}^{\perp} = 2\langle \frac{1}{2} \rangle \oplus 2\langle \frac{-1}{2} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle \frac{-2}{3} \rangle \oplus \langle \frac{8}{9} \rangle$ with signature $(2,2)$ and one can take $\widetilde{S}^{\perp} = \langle 18 \rangle \oplus \langle -6 \rangle \oplus \langle 6 \rangle \oplus \langle -2 \rangle$. Consider the non-abundant case numbered (9) which has the same $\widetilde{S}^{\perp}$ without $\langle -2 \rangle$-summand. Then take the same generators in the non-abundant case numbered (9) and let d be the generator of $\langle -2 \rangle$-summand and take $\xi = [\frac{d}{2}]$ as generator of $\langle \frac{-1}{2} \rangle$-summand.

The same reflections and automorphisms in the non-abundant case numbered (9) acts here. Additionally, the reflection t_{d-c} transposes θ, ξ and acts identically on the 3-primary part.

On the 3-primary part, modulo the reflections t_a, t_b, t_c, any automorphism h acting on $\mathrm{disc}\widetilde{S}^{\perp}$ is either trivial or takes γ to $\alpha + \beta + \gamma$ which was realized by h' as mentioned in the case numbered (9). Additionally, modulo the reflection t_{d-c} automorphism h can take ξ to either ξ or to $(\delta_1 + \theta + \xi)$ on the 2-primary part. By the same calculations done in the proof of the lemma (5.2.1), automorphism sending ξ to $(\delta_1 + \theta + \xi)$ can be realized by an automorphism on $\widetilde{S}^{\perp}$ sending c to $(b + c + d)$.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and $\mathrm{disc}\widetilde{S}^{\perp}$ can be identified such that δ_1 and δ_2 correspond to the generators of $\mathrm{disc}(2\mathbf{A}_1)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_1)$ can be transposed by an admissible isometry where transposition of the corresponding generators δ_1 and δ_2 is realized by a +disorienting automorphism t_{b-a} on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = 2\mathbf{A}_8 \oplus 2\mathbf{A}_1$ *numbered (16).* One has $\mathrm{disc}\widetilde{S}^{\perp} = 2\langle \frac{1}{2} \rangle \oplus \langle \frac{-1}{2} \rangle \oplus 2\langle \frac{8}{9} \rangle$ with signature $(2,1)$ and one can take $\widetilde{S}^{\perp} = 2\langle 18 \rangle \oplus \langle -2 \rangle$. Let a_1, a_2, b be the generators of $2\langle 18 \rangle$ and $\langle -2 \rangle$-summands, respectively. Consider $\delta_i = [\frac{a_i}{2}]$ with $(i = 1,2)$, $\theta = [\frac{b}{2}]$ and $\gamma_i = [\frac{2a_i}{9}]$ with $(i = 1,2)$ as generators of $2\langle \frac{1}{2} \rangle$, $\langle \frac{-1}{2} \rangle$ and $2\langle \frac{8}{9} \rangle$-summands, respectively.

The reflections t_{a_1}, t_{a_2} are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part and act identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_{a_1}, t_{a_2}, this automorphism h takes γ_1 to either γ_1 or γ_2, where by the same calculations done in the proof of lemma (5.2.1) they can be realized by the automorphism h' in $\widetilde{S}^{\perp}$, taking a_1 to a_1 or a_2, respectively. Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes γ_1.

Similarly, the automorphism h on the 2-primary part sends δ_1 to either δ_1, or δ_2. By the same calculations done in the proof of lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending a_1 to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(a_1) \in \widetilde{S}^{\perp}$
δ_1	a_1
δ_2	$(-8a_1 + 9a_2 + 36b)$

Notice that the elements $h'(a_1)$ and a_1 are congruent in mod $9\widetilde{S}^{\perp}$ which means that the induced automorphism elements of Aut(disc$\widetilde{S}^{\perp}$) leaves γ_1 fixed. Thus, one can assume that h fixes both γ_1 and δ_1 and consider the smaller form $\mathcal{N}_1 = \langle \frac{1}{2} \rangle \oplus \langle \frac{-1}{2} \rangle \oplus \langle \frac{8}{9} \rangle =$disc$N_1$, where $N_1 = \langle 18 \rangle \oplus \langle -2 \rangle$.

On the 3-primary part of $\mathcal{N}_1$ the automorphism h is trivial, modulo the reflection t_{a_2}. On the 2-primary part of $\mathcal{N}_1$ every automorphism is trivial.

Since the homomorphism $\text{Aut}(\widetilde{S}^{\perp}) \to \text{Aut}(\text{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^{\perp}$ can be identified such that δ_1 and δ_2 correspond to the generators of disc$(2\mathbf{A}_1)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_1)$ can be transposed by an admissible isometry where transposition of the corresponding generators δ_1 and δ_2 is realized by a +disorienting automorphism $t_{a_1-a_2}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = 2\mathbf{A}_5 \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ *numbered (18).* One has disc$\widetilde{S}^{\perp} = \langle \frac{1}{2} \rangle \oplus 3\langle \frac{-1}{2} \rangle \oplus 2\langle \frac{2}{3} \rangle \oplus \langle \frac{-2}{3} \rangle$ with signature $(2,2)$ and one can take $\widetilde{S}^{\perp} =$

$2\langle 6\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle$. Let a_1, a_2, b, c be the generators of $2\langle 6\rangle$, $\langle -6\rangle$ and $\langle -2\rangle$-summands, respectively. Consider $\delta = [\frac{b}{2}]$, $\theta_1 = [\frac{a_1}{2}]$, $\theta_2 = [\frac{a_2}{2}]$, $\theta_3 = [\frac{c}{2}]$, $\alpha_1 = [\frac{a_1}{3}]$, $\alpha_2 = [\frac{a_2}{3}]$ and $\beta = [\frac{b}{3}]$ as generators of $\langle \frac{1}{2}\rangle$, $3\langle \frac{-1}{2}\rangle$, $2\langle \frac{2}{3}\rangle$ and $\langle \frac{-2}{3}\rangle$-summands, respectively.

The reflections t_{a_1}, t_{a_2}, t_b are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part and act identically on the 2-primary part.

Pick an automorphism h acting on $\mathrm{disc}\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_{a_1}, t_{a_2}, t_b, this automorphism h takes α_1 to either α_1 or α_2 or $\alpha_1 + \alpha_2 + \beta$, where by lemma (5.2.1) they can be realized by the automorphism h' in $\widetilde{S}^{\perp}$, taking a_1 to a_1 or a_2 or $a_1 + a_2 + b$, respectively. Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes α_1.

Similarly, the automorphism h on the 2-primary part sends θ_1 to either θ_i, or $(\theta_i + \theta_j + \delta)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending a_1 to the destinations listed in the table below which are realized on $\mathrm{disc}\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in \mathrm{disc}\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
θ_1	a_1
θ_2	$(2a_1 + 3a_2 + 6c)$
θ_3	$(2a_1 + 3c)$
$(\theta_1 + \theta_2 + \delta)$	$(a_1 + 3a_2 + 3b)$
$(\theta_1 + \theta_3 + \delta)$	$(a_1 + 6a_2 + 3b + 9c)$
$(\theta_2 + \theta_3 + \delta)$	$(2a_1 + 3a_2 + 3b + 3c)$

Notice that the elements $h'(a_1)$ and a_1 are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of $\mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ leaves θ_1 fixed. Thus, one can assume that h fixes both α_1 and θ_1 and consider the smaller form $\mathcal{N}_1 = \langle \frac{1}{2}\rangle \oplus 2\langle \frac{-1}{2}\rangle \oplus \langle \frac{2}{3}\rangle \oplus \langle \frac{-2}{3}\rangle$, where $N_1 = \langle 6\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle$ still having the same generators except for a_1.

On the 3-primary part of $\mathcal{N}_1$ the automorphism h is trivial, modulo the reflection t_{a_2}, t_b. On the 2-primary part of $\mathcal{N}_1$ if h is nontrivial, then it is realized by the reflection t_{c-a_2} transposing δ_2 and δ_3 which acts identically on 3-primary part.

One can assume that $\alpha_1 \oplus \alpha_2$ and $\theta_1 \oplus \theta_2$ in disc$\widetilde{S}^{\perp}$ corresponds to the generator of two copies of disc$\mathbf{A}_5$ in $\widetilde{\mathcal{S}}$. Transposing $2\mathbf{A}_5$ in $\widetilde{S}$ corresponds to the +disorienting reflection $t_{a_1-a_2}$ which transposes α_1, α_2 and δ_1, δ_2 together. So the abstract homological type is symmetric.

- *The case non-abundant* $\Sigma = 2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ *numbered (20).* One has disc$\widetilde{S}^{\perp} = 2\langle\frac{1}{2}\rangle \oplus 3\langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 2\langle\frac{-2}{3}\rangle$ with signature $(2,3)$. Then one can take $\widetilde{S}^{\perp} = 2\langle -6\rangle \oplus 2\langle 6\rangle \oplus \langle -2\rangle$. Consider the non-abundant case numbered (18) which has the same $\widetilde{S}^{\perp}$ without $\langle -6\rangle$-summand. Then take the same generators in the non-abundant case numbered (18) and let a_2 be the generator of $\langle -6\rangle$-summand and take $\delta_2 = [\frac{a_2}{2}]$, $\alpha_2 = [\frac{a_2}{3}]$ as generators of $\langle\frac{1}{2}\rangle$, $\langle\frac{-2}{3}\rangle$-summands, respectively.

The same reflections in the non-abundant case numbered (18) acts here. Additionally, the reflection t_{a_2} multiplies α_2 by (-1) and acts identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_{a_1}, t_{a_2}, t_{b_1}, t_{b_2}, this automorphism h takes α_2 to either α_2 or α_1 or $\alpha_1 + \alpha_2 + \beta_i$ or $\beta_1 + \beta_2$, where by lemma (5.2.1) they can be realized by the automorphism h' in $\widetilde{S}^{\perp}$, taking a_2 to a_2 or a_1 or $a_1 + a_2 + b_i$ or $2b_1 + 2b_2 + 3a_2$, respectively. Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes α_2.

Similarly, the automorphism h on the 2-primary part sends δ_2 to either δ_i or $(\delta_1 + \delta_2 + \theta_i)$ or $(\theta_1 + \theta_2 + \theta_3)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending a_2 to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in \mathrm{disc}\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
δ_2	a_2
δ_1	$(9a_1 + 8a_2 + 12b_1)$
$(\delta_1 + \delta_2 + \theta_1)$	$(3a_1 + a_2 + 3b_1)$
$(\delta_1 + \delta_2 + \theta_2)$	$(3a_1 + a_2 + 3b_2)$
$(\delta_1 + \delta_2 + \theta_3)$	$(3a_1 + a_2 + 6b_1 + 9c)$
$(\theta_1 + \theta_3 + \theta_3)$	$(8a_2 + 3b_1 + 9b_2 + 9c)$

Notice that the elements $h'(a_2)$ and a_2 are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of $\mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ leaves δ_2 fixed. Thus, one can assume that h fixes both α_2 and δ_2 and consider the smaller form $\mathcal{N}_1 = \langle \frac{1}{2} \rangle \oplus 3\langle \frac{-1}{2} \rangle \oplus 2\langle \frac{2}{3} \rangle \oplus \langle \frac{-2}{3} \rangle$, where $N_1 = \langle -6 \rangle \oplus 2\langle 6 \rangle \oplus \langle -2 \rangle$ still having the same generators except for a_2.

This is the same with the $\mathrm{disc}\widetilde{S}^{\perp}$ and the lattice $\widetilde{S}^{\perp}$ in the case of non-abundant $\Sigma = 2\mathbf{A}_5 \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ numbered (18). It is shown that all automorphisms on non-abundant $\mathrm{disc}\widetilde{S}^{\perp}$ in numbered (18) can be realized by automorphisms of non-abundant $\widetilde{S}^{\perp}$ in numbered (18). Then, it is clear that $O(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto for this case, too.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and $\mathrm{disc}\widetilde{S}^{\perp}$ can be identified such that β_1 and β_2 correspond to the generators of $\mathrm{disc}(2\mathbf{A}_2)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_2)$ can be transposed by an admissible isometry where transposition of the corresponding generators β_1 and β_2 is realized by a +disorienting automorphism $t_{b_1-b_2}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = \mathbf{A}_5 \oplus 2\mathbf{E}_6 \oplus \mathbf{A}_1$ *numbered (21).* One has $\mathrm{disc}\widetilde{S}^{\perp} = \langle \frac{1}{2} \rangle \oplus 2\langle \frac{-1}{2} \rangle \oplus 2\langle \frac{2}{3} \rangle \oplus \langle \frac{-2}{3} \rangle$ with signature $(2, 1)$ and one can take $\widetilde{S}^{\perp} = 2\langle 6 \rangle \oplus \langle 6 \rangle$. Let a_1, a_2, b be the generators of $2\langle 6 \rangle$ and $\langle 6 \rangle$-summands, respectively. Consider $\delta = [\frac{b}{2}]$, $\theta_1 = [\frac{a_1}{2}]$, $\theta_2 = [\frac{a_2}{2}]$, $\alpha_1 = [\frac{a_1}{3}]$, $\alpha_2 = [\frac{a_2}{3}]$, $\beta = [\frac{b}{3}]$ as generators of $\langle \frac{1}{2} \rangle$, $2\langle \frac{-1}{2} \rangle$, $2\langle \frac{2}{3} \rangle$ and $\langle \frac{-2}{3} \rangle$-summands, respectively.

The reflections t_{a_1}, t_{a_2}, t_b are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part and act identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_{a_1}, t_{a_2}, t_b, this automorphism h takes α_1 to either α_1 or α_2 or $\alpha_1 + \alpha_2 + \beta$, where by lemma (5.2.1) they can be realized by the automorphism h' in $\widetilde{S}^{\perp}$, taking a_1 to a_1 or a_2 or $a_1 + a_2 + b$, respectively. Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes α_1.

Similarly, the automorphism h on the 2-primary part is either trivial or sends θ_1 to θ_2. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending θ_1 to θ_2 which is realized on disc$\widetilde{S}^{\perp}$ as taking a_1 to $(8a_1 + 9a_2 + 12b)$.

Notice that the elements $h'(a_1)$ and a_1 are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of Aut(disc$\widetilde{S}^{\perp}$) leaves θ_1 fixed. Thus, one can assume that h fixes both α_1 and θ_1 and consider the smaller form $\mathcal{N}_1 = \langle \frac{1}{2} \rangle \oplus \langle \frac{-1}{2} \rangle \oplus \langle \frac{2}{3} \rangle \oplus \langle \frac{-2}{3} \rangle$, where $N_1 = \langle 6 \rangle \oplus \langle 6 \rangle$ still having the same generators except for a_1.

On the 3-primary part of $\mathcal{N}_1$ the automorphism h is trivial, modulo the reflection t_{a_2}, t_b. On the 2-primary part of $\mathcal{N}_1$ every automorphism is trivial.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^{\perp}$ can be identified such that α_1 and α_2 correspond to the generators of disc$(2\mathbf{E}_6)$. In $\widetilde{S}$, generators of $(2\mathbf{E}_6)$ can be transposed by an admissible isometry where transposition of the corresponding generators α_1 and α_2 is realized by a +disorienting automorphism $t_{a_1 - a_2}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

Now, some little changes in the order of cases can be done to make calculations easier. Indeed, non-abundant case numbered (24) follows directly from the non-abundant case numbered (26) where it follows directly from the non-abundant case numbered (21). In the same way, non-abundant case numbered (23) follows from the non-abundant case numbered (25) and it follows from the non-abundant case numbered (31). Similarly, the non-abundant case numbered (29) follows from the non-abundant case numbered (34) and it follows from the non-abundant case numbered (31). Lastly, the non-abundant case numbered (28) follows from the non-abundant case numbered (33).

- *The case non-abundant* $\Sigma = \mathbf{A}_5 \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ *numbered (26).* One has disc$\widetilde{S}^{\perp} = 2\langle\frac{1}{2}\rangle \oplus 2\langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 2\langle\frac{-2}{3}\rangle$ with signature $(2,2)$. Then one can take $\widetilde{S}^{\perp} = 2\langle 6\rangle \oplus 2\langle -6\rangle$. Consider the non-abundant case numbered (21) which has the same $\widetilde{S}^{\perp}$ without one more $\langle -6\rangle$-summand. Then take the same generators in the non-abundant case numbered (21) and let b_2 be the generator of $\langle -6\rangle$-summand and take $\delta_2 = [\frac{b_2}{2}]$, $\beta_2 = [\frac{b_2}{3}]$ as generators of $\langle\frac{1}{2}\rangle$, $\langle\frac{-2}{3}\rangle$-summands, respectively.

The same reflections in the non-abundant case numbered (21) acts here. Additionally, the reflection t_{b_2} multiplies β_2 by (-1) and acts identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_{a_1}, t_{a_2}, t_{b_1}, t_{b_2}, this automorphism h takes β_2 to either β_2 or β_1 or $\beta_1 + \beta_2 + \alpha_i$ or $\alpha_1 + \alpha_2$, where by lemma (5.2.1) they can be realized by the automorphism h' in $\widetilde{S}^{\perp}$, taking b_2 to b_2 or b_1 or $b_1 + b_2 + a_i$ or $2a_1 + 2a_2 + 3b_1$, respectively. Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes β_2.

Similarly, the automorphism h on the 2-primary part sends δ_2 to either δ_i or $(\delta_1 + \delta_2 + \theta_i)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending b_2 to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
δ_2	b_2
δ_1	$(9b_1 + 8b_2 + 12a_1)$
$(\delta_1 + \delta_2 + \theta_1)$	$(3b_1 + b_2 + 3a_1)$
$(\delta_1 + \delta_2 + \theta_2)$	$(3b_1 + b_2 + 3a_2)$

Notice that the elements $h'(b_2)$ and b_2 are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of Aut(disc$\widetilde{S}^{\perp}$) leaves δ_2 fixed. Thus, one can assume that h fixes both β_2 and δ_2 and consider the smaller form $\mathcal{N}_1 =$

$\langle\frac{1}{2}\rangle\oplus 2\langle\frac{-1}{2}\rangle\oplus 2\langle\frac{2}{3}\rangle\oplus\langle\frac{-2}{3}\rangle$, where $N_1 = 2\langle 6\rangle\oplus\langle -6\rangle$ still having the same generators except for b_2.

This is the same with the disc$\widetilde{S}^\perp$ and the lattice $\widetilde{S}^\perp$ in the case of non-abundant $\Sigma = \mathbf{A}_5\oplus 2\mathbf{E}_6\oplus\mathbf{A}_1$ numbered (21). It is shown that all automorphisms on non-abundant disc$\widetilde{S}^\perp$ in numbered (21) can be realized by automorphisms of non-abundant $\widetilde{S}^\perp$ in numbered (21). Then, it is clear that $O(\widetilde{S}^\perp)\to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^\perp)$ is onto for this case, too.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^\perp)\to\mathrm{Aut}(\mathrm{disc}\widetilde{S}^\perp)$ is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^\perp$ can be identified such that α_1 and α_2 correspond to the generators of disc$(2\mathbf{A}_2)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_2)$ can be transposed by an admissible isometry where transposition of the corresponding generators α_1 and α_2 is realized by a +disorienting automorphism $t_{a_1-a_2}$ on $\widetilde{S}^\perp$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = \mathbf{A}_5\oplus 4\mathbf{A}_2\oplus 3\mathbf{A}_1$ *numbered (24).* One has disc$\widetilde{S}^\perp = 3\langle\frac{1}{2}\rangle\oplus 2\langle\frac{-1}{2}\rangle\oplus 2\langle\frac{2}{3}\rangle\oplus 3\langle\frac{-2}{3}\rangle$ with signature $(2,3)$. Then one can take $\widetilde{S}^\perp = 2\langle 6\rangle\oplus 3\langle -6\rangle$. Consider the non-abundant case numbered (26) which has the same $\widetilde{S}^\perp$ without one more $\langle -6\rangle$-summand. Then take the same generators in the non-abundant case numbered (26) and let b_3 be the generator of $\langle -6\rangle$-summand and take $\delta_3 = [\frac{b_3}{2}]$, $\beta_3 = [\frac{b_3}{3}]$ as generators of $\langle\frac{1}{2}\rangle$, $\langle\frac{-2}{3}\rangle$-summands, respectively.

The same reflections in the non-abundant case numbered (26) acts here. Additionally, the reflection t_{b_3} multiplies β_3 by (-1) and acts identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^\perp$. On the 3-primary part, modulo the reflections t_{a_1}, t_{a_2}, t_{b_1}, t_{b_2}, t_{b_3}, this automorphism h takes β_3 to either β_i or $\beta_i+\beta_j+\alpha_k$, $(i\neq j)$ or $\alpha_1+\alpha_2$, or $\alpha_1+\alpha_2+\beta_1+\beta_1+\beta_3$ where by lemma (5.2.1) they can be realized by the automorphism h' in $\widetilde{S}^\perp$, taking b_3 to b_i or $b_i+b_j+a_k$, $(i\neq j)$ or $2a_1+2a_2+3b_1$, $a_1+a_2+b_1+b_2+b_3$, respectively. Thus, modulo $O(\widetilde{S}^\perp)$, one can assume that h fixes β_3.

Similarly, the automorphism h on the 2-primary part sends δ_3 to either δ_i

or $(\delta_i + \delta_j + \theta_k)$, $(i \neq j)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending b_3 to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
δ_3	b_3
δ_1	$(9b_1 + 8b_3 + 12a_1)$
δ_2	$(9b_2 + 8b_3 + 12a_1)$
$(\delta_1 + \delta_2 + \theta_1)$	$(3b_1 + 3b_2 + 8b_3 + 9a_1)$
$(\delta_1 + \delta_3 + \theta_1)$	$(3b_1 + b_3 + 3a_1)$
$(\delta_2 + \delta_3 + \theta_1)$	$(3b_2 + b_3 + 3a_1)$
$(\delta_1 + \delta_2 + \theta_2)$	$(3b_1 + 3b_2 + 8b_3 + 9a_2)$
$(\delta_1 + \delta_3 + \theta_2)$	$(3b_1 + b_3 + 3a_2)$
$(\delta_2 + \delta_3 + \theta_2)$	$(3b_2 + b_3 + 3a_2)$

Notice that the elements $h'(b_3)$ and b_3 are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of Aut(disc$\widetilde{S}^{\perp}$) leaves δ_3 fixed. Thus, one can assume that h fixes both β_3 and δ_3 and consider the smaller form $\mathcal{N}_1 = 2\langle\frac{1}{2}\rangle \oplus 2\langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 2\langle\frac{-2}{3}\rangle$, where $N_1 = 2\langle 6\rangle \oplus 2\langle -6\rangle$ still having the same generators except for b_3.

This is the same with the disc$\widetilde{S}^{\perp}$ and the lattice $\widetilde{S}^{\perp}$ in the case of non-abundant $\Sigma = \mathbf{A}_5 \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ numbered (26). It is shown that all automorphisms on non-abundant disc$\widetilde{S}^{\perp}$ in numbered (26) can be realized by automorphisms of non-abundant $\widetilde{S}^{\perp}$ in numbered (26). Then, it is clear that $O(\widetilde{S}^{\perp}) \rightarrow$ Aut(disc$\widetilde{S}^{\perp}$) is onto for this case, too.

Since the homomorphism Aut$(\widetilde{S}^{\perp}) \rightarrow$ Aut(disc$\widetilde{S}^{\perp}$) is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^{\perp}$ can be identified such that α_1 and α_2 correspond to the generators of disc$(2\mathbf{A}_2)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_2)$ can be transposed by an admissible isometry where transposition of the corresponding generators α_1 and α_2 is realized by a +disorienting automorphism $t_{a_1-a_2}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = 2\mathbf{A}_2 \oplus 2\mathbf{E}_6 \oplus \mathbf{A}_1$ *numbered (31).* One has disc$\widetilde{S}^{\perp} = \langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 2\langle\frac{-2}{3}\rangle$ with signature $(2,2)$ and one can take $\widetilde{S}^{\perp} = \mathbf{U}_3 \oplus \langle 6\rangle \oplus \langle -6\rangle$. Let a, b be generators of $\langle 6\rangle$ and $\langle -6\rangle$-summands and u, v be standard basis for the $\mathbf{U}_3$-summand. Consider $\delta = [\frac{b}{2}]$, $\theta = [\frac{a}{2}]$, $\alpha_1 = [\frac{a}{3}]$, $\alpha_2 = [\frac{u+v}{3}]$, $\beta_1 = [\frac{b}{2}]$ and $\beta_2 = [\frac{u-v}{3}]$ as generators of $\langle\frac{1}{2}\rangle$, $\langle\frac{-1}{2}\rangle$, $2\langle\frac{2}{3}\rangle$ and $2\langle\frac{-2}{3}\rangle$-summands, respectively.

Every automorphism acting on 2-primary part is trivial.

On the 3-primary-part, the reflections t_a, t_b, t_{u+v}, t_{u-v} are the automorphisms multiplying the corresponding generator by (-1) and the reflections $t_{a-(u+v)}$, $t_{b-(u-v)}$ transposes the generators α_1, α_2 and β_1, β_2. These automorphisms act identically on the 2-primary part.

Modulo the reflections t_a, t_b, t_{u+v}, t_{u-v} and $t_{a-(u+v)}$, $t_{b-(u-v)}$, any automorphism h acting on disc$\widetilde{S}^{\perp}$ is either trivial or takes α_1 to $\alpha_1 + \alpha_2 + \beta_i$ or $\beta_1 + \beta_2$ or takes β_1 to $\alpha_1 + \alpha_2$ on the 3-primary part. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending a_1 to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table. Similarly, by lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending b to $(2a + 2(u+v) + 3b)$.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
α_1	a
$(\beta_1 + \beta_2)$	$(2b + 2(u-v) + 3a)$
$(\alpha_1 + \alpha_2 + \beta_1)$	$(a + (u+v) + b)$
$(\alpha_1 + \alpha_2 + \beta_2)$	$(a + (u+v) + (u-v))$

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\text{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^{\perp}$ can be identified such that α_1 and α_2 correspond to the generators of disc$(2\mathbf{A}_2)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_2)$ can be transposed by an admissible isometry where transposition of the corresponding generators α_1 and α_2 is realized by a +disorienting automorphism $t_{a-(u+v)}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = \mathbf{A}_5 \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ *numbered (25).* One has disc$\widetilde{S}^{\perp} = \langle\frac{1}{2}\rangle \oplus 2\langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 2\langle\frac{-2}{3}\rangle$ with signature $(2,3)$. Then one can take

$\widetilde{S}^{\perp} = \mathbf{U}_3 \oplus \langle 6\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle$. Consider the non-abundant case numbered (31) which has the same $\widetilde{S}^{\perp}$ without $\langle -2\rangle$-summand. Then take the same generators in the non-abundant case numbered (31) and let c be the generator of $\langle -2\rangle$-summand and take $\theta_2 = [\frac{c}{2}]$ as generator of corresponding $\langle \frac{-1}{2}\rangle$-summand.

The same reflections in the non-abundant case numbered (31) acts here. Additionally, the reflection t_{b-c} transposes θ_1 and θ_2 and acts identically on the 3-primary part. On the 3-primary part, any automorphism can be realized by automorphisms of $\widetilde{S}^{\perp}$ as listed in the non-abundant case numbered (31).

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and $\mathrm{disc}\widetilde{S}^{\perp}$ can be identified such that α_1 and α_2 correspond to the generators of $\mathrm{disc}(2\mathbf{A}_2)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_2)$ can be transposed by an admissible isometry where transposition of the corresponding generators α_1 and α_2 is realized by a +disorienting automorphism $t_{(b-(u+v))}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = \mathbf{A}_5 \oplus 4\mathbf{A}_2 \oplus 2\mathbf{A}_1$ *numbered (23).* One has $\mathrm{disc}\widetilde{S}^{\perp} = 2\langle \frac{1}{2}\rangle \oplus 2\langle \frac{-1}{2}\rangle \oplus 2\langle \frac{2}{3}\rangle \oplus 3\langle \frac{-2}{3}\rangle$ with signature $(2,4)$. Then one can take $\widetilde{S}^{\perp} = \mathbf{U}_3 \oplus \langle 6\rangle \oplus 2\langle -6\rangle \oplus \langle -2\rangle$. Consider the non-abundant case numbered (25) which has the same $\widetilde{S}^{\perp}$ without one more $\langle -6\rangle$-summand. Then take the same generators in the non-abundant case numbered (25) and let $b^{'}$ be the generator of $\langle -6\rangle$-summand and take $\delta^{'} = [\frac{b^{'}}{2}]$, $\beta_3 = [\frac{b^{'}}{3}]$ as generators of $\langle \frac{1}{2}\rangle$, $\langle \frac{-2}{3}\rangle$-summands, respectively.

The same reflections in the non-abundant case numbered (25) acts here. Additionally, the reflection $t_{b'}$ multiplies β_3 by (-1) and acts identically on the 2-primary part.

Pick an automorphism h acting on $\mathrm{disc}\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_a, t_b, t_{u+v}, t_{u-v}, $t_{b'}$, this automorphism h takes β_3 to either β_i or $\beta_i + \beta_j + \alpha_k$, $(i \neq j)$ or $\alpha_1 + \alpha_2$, or $\alpha_1 + \alpha_2 + \beta_1 + \beta_1 + \beta_3$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending $b^{'}$ to the destinations listed in the table below which are realized on $\mathrm{disc}\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in \mathrm{disc}\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
β_3	$b^{'}$
β_1	b
β_2	$(u-v)$
$(\beta_1+\beta_2+\alpha_1)$	$(b+(u-v)+a)$
$(\beta_1+\beta_3+\alpha_1)$	$(b+b^{'}+a)$
$(\beta_2+\beta_3+\alpha_1)$	$((u-v)+b^{'}+a)$
$(\beta_1+\beta_2+\alpha_2)$	$(b+(u-v)+(u+v))$
$(\beta_1+\beta_3+\alpha_2)$	$(b+b^{'}+(u+v))$
$(\beta_2+\beta_3+\alpha_2)$	$((u-v)+b^{'}+(u+v))$
$(\alpha_1+\alpha_2)$	$(2a+2(u-v)+3b^{'})$
$(\alpha_1+\alpha_2+\beta_1+\beta_1+\beta_3)$	$(a+(u+v)+b+(u-v)+b^{'})$

Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes β_3.

Similarly, the automorphism h on the 2-primary part sends $\delta^{'}$ to either $\delta^{'}$ or δ or $(\delta^{'}+\delta+\theta_i)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending $b^{'}$ to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in \mathrm{disc}\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
$\delta^{'}$	$b^{'}$
δ	$(12a+9b+8b^{'})$
$(\delta^{'}+\delta+\theta_1)$	$(3a+3b+b^{'})$
$(\delta^{'}+\delta+\theta_2)$	$(6a+3b+b^{'}+9c)$

Notice that the elements $h'(b^{'})$ and $b^{'}$ are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of Aut(disc$\widetilde{S}^{\perp}$) leaves $\delta^{'}$ fixed. Thus, one can assume that h fixes both β_3 and $\delta^{'}$ and consider the smaller form $\mathcal{N}_1 = \langle\frac{1}{2}\rangle \oplus 2\langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 2\langle\frac{-2}{3}\rangle$, where $N_1 = \mathbf{U}_3 \oplus \langle 6\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle$ still having the same generators except for $b^{'}$.

This is the same with the disc$\widetilde{S}^{\perp}$ and the lattice $\widetilde{S}^{\perp}$ in the case of non-abundant $\Sigma = \mathbf{A}_5 \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ numbered (25). It is shown that all automorphisms on non-abundant disc$\widetilde{S}^{\perp}$ in numbered (25) can be realized by automorphisms of non-abundant $\widetilde{S}^{\perp}$ in numbered (25). Then, it is clear that $O(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto for this case, too.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^{\perp}$ can be identified such that α_1 and α_2 correspond to the generators of disc$(2\mathbf{A}_2)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_2)$ can be transposed by an admissible isometry where transposition of the corresponding generators α_1 and α_2 is realized by a +disorienting automorphism $t_{a-(u+v)}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = 4\mathbf{A}_2 \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_1$ *numbered (34).* One has disc$\widetilde{S}^{\perp} = 2\langle \frac{1}{2} \rangle \oplus \langle \frac{-1}{2} \rangle \oplus 2\langle \frac{2}{3} \rangle \oplus 3\langle \frac{-2}{3} \rangle$ with signature $(2,3)$. Then one can take $\widetilde{S}^{\perp} = \mathbf{U}_3 \oplus \langle 6 \rangle \oplus 2\langle -6 \rangle$. Consider the non-abundant case numbered (31) which has the same $\widetilde{S}^{\perp}$ without one more $\langle -6 \rangle$-summand. Then take the same generators in the non-abundant case numbered (31) and let b' be the generator of $\langle -6 \rangle$-summand and take $\delta' = [\frac{b'}{2}]$, $\beta_3 = [\frac{b'}{3}]$ as generators of $\langle \frac{1}{2} \rangle$, $\langle \frac{-2}{3} \rangle$-summands, respectively.

The same reflections in the non-abundant case numbered (31) acts here. Additionally, the reflection $t_{b'}$ multiplies β_3 by (-1) and acts identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_a, t_b, t_{u+v}, t_{u-v}, $t_{b'}$, this automorphism h takes β_3 to either β_i or $\beta_i + \beta_j + \alpha_k$, $(i \neq j)$ or $\alpha_1 + \alpha_2$, or $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \beta_3$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending b' to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
β_3	$b^{'}$
β_1	b
β_2	$(u-v)$
$(\beta_1+\beta_2+\alpha_1)$	$(b+(u-v)+a)$
$(\beta_1+\beta_3+\alpha_1)$	$(b+b^{'}+a)$
$(\beta_2+\beta_3+\alpha_1)$	$((u-v)+b^{'}+a)$
$(\beta_1+\beta_2+\alpha_2)$	$(b+(u-v)+(u+v))$
$(\beta_1+\beta_3+\alpha_2)$	$(b+b^{'}+(u+v))$
$(\beta_2+\beta_3+\alpha_2)$	$((u-v)+b^{'}+(u+v))$
$(\alpha_1+\alpha_2)$	$(2a+2(u-v)+3b^{'})$
$(\alpha_1+\alpha_2+\beta_1+\beta_2+\beta_3)$	$(a+(u+v)+b+(u-v)+b^{'})$

Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes β_3.

Similarly, the automorphism h on the 2-primary part sends $\delta^{'}$ to either $\delta^{'}$ or δ or $(\delta^{'}+\delta+\theta)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending $b^{'}$ to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
$\delta^{'}$	$b^{'}$
δ	$(12a+9b+8b^{'})$
$(\delta^{'}+\delta+\theta)$	$(3a+3b+b^{'})$

Notice that the elements $h'(b^{'})$ and $b^{'}$ are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of Aut(disc$\widetilde{S}^{\perp}$) leaves $\delta^{'}$ fixed. Thus, one can assume that h fixes both β_3 and $\delta^{'}$ and consider the smaller form $\mathcal{N}_1 = \langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus \langle\frac{-2}{3}\rangle$, where $N_1 = \mathbf{U}_3 \oplus \langle 6\rangle \oplus \langle -6\rangle$ still having the same generators except for $b^{'}$.

This is the same with the disc$\widetilde{S}^{\perp}$ and the lattice $\widetilde{S}^{\perp}$ in the case of non-abundant $\Sigma = 2\mathbf{A}_2 \oplus 2\mathbf{E}_6 \oplus \mathbf{A}_1$ numbered (31). It is shown that all automorphisms on non-abundant disc$\widetilde{S}^{\perp}$ in numbered (31) can be realized by automorphisms of non-abundant $\widetilde{S}^{\perp}$ in numbered (31). Then, it is clear that $O(\widetilde{S}^{\perp}) \to$ Aut(disc$\widetilde{S}^{\perp}$)

is onto for this case, too.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and $\mathrm{disc}\widetilde{S}^{\perp}$ can be identified such that α_1 and α_2 correspond to the generators of $\mathrm{disc}(2\mathbf{A}_2)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_2)$ can be transposed by an admissible isometry where transposition of the corresponding generators α_1 and α_2 is realized by a +disorienting automorphism $t_{a-(u+v)}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = 6\mathbf{A}_2 \oplus 3\mathbf{A}_1$ *numbered (29).* One has $\mathrm{disc}\widetilde{S}^{\perp} = 3\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 4\langle\frac{-2}{3}\rangle$ with signature $(2,4)$. Then one can take $\widetilde{S}^{\perp} = \mathbf{U}_3 \oplus \langle 6\rangle \oplus 3\langle -6\rangle$. Consider the non-abundant case numbered (34) which has the same $\widetilde{S}^{\perp}$ without one more $\langle -6\rangle$-summand. Then take the same generators in the non-abundant case numbered (31) and let $b^{"}$ be the generator of $\langle -6\rangle$-summand and take $\delta^{"} = [\frac{b^{"}}{2}]$, $\beta_4 = [\frac{b^{"}}{3}]$ as generators of $\langle\frac{1}{2}\rangle$, $\langle\frac{-2}{3}\rangle$-summands, respectively.

The same reflections in the non-abundant case numbered (34) acts here. Additionally, the reflection $t_{b^{"}}$ multiplies β_4 by (-1) and acts identically on the 2-primary part.

Pick an automorphism h acting on $\mathrm{disc}\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_a, t_b, t_{u+v}, t_{u-v}, $t_{b'}$, $t_{b^{"}}$, this automorphism h takes β_4 to either β_i or $\beta_i + \beta_j + \alpha_k$, $(i \neq j)$ or $\alpha_1 + \alpha_2$, or $\alpha_1 + \alpha_2 + \beta_i + \beta_j + \beta_k$, $(i \neq j \neq k)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending b' to the destinations listed in the table below which are realized on $\mathrm{disc}\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in \mathrm{disc}\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
β_4	$b^{"}$
β_1	b
β_2	$(u-v)$
β_3	$b^{'}$
$(\beta_1+\beta_2+\alpha_1)$	$(b+(u-v)+a)$
$(\beta_1+\beta_3+\alpha_1)$	$(b+b^{'}+a)$
$(\beta_2+\beta_3+\alpha_1)$	$((u-v)+b^{'}+a)$
$(\beta_1+\beta_4+\alpha_1)$	$(b+b^{"}+a)$
$(\beta_2+\beta_4+\alpha_1)$	$((u-v)+b^{"}+a)$
$(\beta_3+\beta_4+\alpha_1)$	$(b^{'}+b^{"}+a)$
$(\beta_1+\beta_2+\alpha_2)$	$(b+(u-v)+(u+v))$
$(\beta_1+\beta_3+\alpha_2)$	$(b+b^{'}+(u+v))$
$(\beta_2+\beta_3+\alpha_2)$	$((u-v)+b^{'}+(u+v))$
$(\beta_1+\beta_4+\alpha_2)$	$(b+b^{"}+(u+v))$
$(\beta_2+\beta_4+\alpha_2)$	$((u-v)+b^{"}+(u+v))$
$(\beta_3+\beta_4+\alpha_2)$	$(b^{'}+b^{"}+(u+v))$
$(\alpha_1+\alpha_2)$	$(2a+2(u-v)+3b^{'})$
$(\alpha_1+\alpha_2+\beta_1+\beta_2+\beta_3)$	$(a+(u+v)+b+(u-v)+b^{'})$
$(\alpha_1+\alpha_2+\beta_1+\beta_2+\beta_4)$	$(a+(u+v)+b+(u-v)+b^{"})$
$(\alpha_1+\alpha_2+\beta_1+\beta_3+\beta_4)$	$(a+(u+v)+b+b^{'}+b^{"})$
$(\alpha_1+\alpha_2+\beta_2+\beta_3+\beta_4)$	$(a+(u+v)+(u-v)+b^{'})+b^{"}$

Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes β_4.

Similarly, the automorphism h on the 2-primary part sends $\delta^{"}$ to either $\delta^{"}$ or $\delta^{'}$ or δ or $(\delta^{'}+\delta+\theta)$ or $(\delta^{"}+\delta+\theta)$ or $(\delta^{'}+\delta^{"}+\theta)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending $b^{"}$ to the destinations listed in the table below which are realized on $\mathrm{disc}\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in \text{disc}\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
δ''	b''
δ'	$12a + 9b' + 8b''$
δ	$(12a + 9b + 8b'')$
$(\delta' + \delta + \theta)$	$(3a + 3b + b')$
$(\delta'' + \delta + \theta)$	$(3a + 3b + b'')$
$(\delta' + \delta'' + \theta)$	$(3a + 3b' + b'')$

Notice that the elements $h'(b'')$ and b'' are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of $\text{Aut}(\text{disc}\widetilde{S}^{\perp})$ leaves δ'' fixed. Thus, one can assume that h fixes both β_4 and δ'' and consider the smaller form $\mathcal{N}_1 = 2\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 3\langle\frac{-2}{3}\rangle$, where $N_1 = \mathbf{U}_3 \oplus \langle 6\rangle \oplus 2\langle -6\rangle$ still having the same generators except for b''.

This is the same with the $\text{disc}\widetilde{S}^{\perp}$ and the lattice $\widetilde{S}^{\perp}$ in the case of non-abundant $\Sigma = 4\mathbf{A}_2 \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_1$ numbered (34). It is shown that all automorphisms on non-abundant $\text{disc}\widetilde{S}^{\perp}$ in numbered (34) can be realized by automorphisms of non-abundant $\widetilde{S}^{\perp}$ in numbered (34). Then, it is clear that $O(\widetilde{S}^{\perp}) \to \text{Aut}(\text{disc}\widetilde{S}^{\perp})$ is onto for this case, too.

Since the homomorphism $\text{Aut}(\widetilde{S}^{\perp}) \to \text{Aut}(\text{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and $\text{disc}\widetilde{S}^{\perp}$ can be identified such that α_1 and α_2 correspond to the generators of $\text{disc}(2\mathbf{A}_2)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_2)$ can be transposed by an admissible isometry where transposition of the corresponding generators α_1 and α_2 is realized by a +disorienting automorphism $t_{a-(u+v)}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = 4\mathbf{A}_2 \oplus 2\mathbf{E}_6 \oplus \mathbf{A}_1$ *numbered (33).* One has $\text{disc}\widetilde{S}^{\perp} = \langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 3\langle\frac{-2}{3}\rangle$ with signature $(2, 3)$ and one can take $\widetilde{S}^{\perp} = 2\mathbf{U}_3 \oplus \langle -6\rangle \oplus \langle -2\rangle$. Let a, b be generators of $\langle -6\rangle$ and $\langle -2\rangle$-summands and, u_1, v_1 and u_2, v_2 be standard basis for the $2\mathbf{U}_3$-summands, respectively. Consider $\delta = [\frac{a}{2}]$, $\theta = [\frac{b}{2}]$, $\alpha_1 = [\frac{u_1+v_1}{3}]$, $\alpha_2 = [\frac{u_2+v_2}{3}]$, $\beta_1 = [\frac{a}{3}]$, $\beta_2 = [\frac{u_1-v_1}{3}]$ and $\beta_3 = [\frac{u_2-v_2}{3}]$ as generators of $\langle\frac{1}{2}\rangle$, $\langle\frac{-1}{2}\rangle$, $2\langle\frac{2}{3}\rangle$ and $3\langle\frac{-2}{3}\rangle$-summands, respectively.

Every automorphism acting on 2-primary part is trivial.

The reflections t_a, $t_{u_1+v_1}$, $t_{u_2+v_2}$, $t_{u_1-v_1}$ and $t_{u_2-v_2}$ are the automorphisms multiplying the corresponding generator by (-1) on the 3-primary-part, they act identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_a, $t_{u_1+v_1}$, $t_{u_2+v_2}$, $t_{u_1-v_1}$, $t_{u_2-v_2}$, this automorphism h takes β_1 to either β_i or $\beta_i + \beta_j + \alpha_k$, $(i \neq j)$ or $\alpha_1 + \alpha_2$, or $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \beta_3$ where by lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending a to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
β_1	a
β_2	$(u_1 - v_1)$
β_3	$(u_2 - v_2)$
$(\beta_1 + \beta_2 + \alpha_1)$	$(a + (u_1 - v_1) + (u_1 + v_1))$
$(\beta_1 + \beta_3 + \alpha_1)$	$(a + (u_2 - v_2) + (u_1 + v_1))$
$(\beta_2 + \beta_3 + \alpha_1)$	$((u_1 - v_1) + (u_2 - v_2) + (u_1 + v_1))$
$(\beta_1 + \beta_2 + \alpha_2)$	$(a + (u_1 - v_1) + (u_2 + v_2))$
$(\beta_1 + \beta_3 + \alpha_2)$	$(a + (u_2 - v_2) + (u_2 + v_2))$
$(\beta_2 + \beta_3 + \alpha_2)$	$((u_1 - v_1) + (u_2 - v_2) + (u_2 + v_2))$
$(\alpha_1 + \alpha_2)$	$(2(u_1 + v_1) + 2(u_2 + v_2) + 3a)$
$(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \beta_3)$	$((u_1 + v_1) + (u_2 + v_2) + a + (u_1 - v_1) + (u_2 - v_2))$

Modulo the reflections t_a, $t_{u_1+v_1}$, $t_{u_2+v_2}$, $t_{u_1-v_1}$, $t_{u_2-v_2}$ and the automorphisms listed above, any automorphism h acting on disc$\widetilde{S}^{\perp}$ is either trivial or takes β_2 to β_1 or $\beta_1 + \beta_2 + \alpha_1$ or $\beta_1 + \beta_2 + \alpha_2$ or $\alpha_1 + \alpha_2$ or takes β_3 to $\alpha_1 + \alpha_2$ on the 3-primary part or as the last possibility it transposes α_1 and α_2 which can be realized by the reflection $t_{(u_1+v_1)-(u_2+v_2)}$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending $(u_1 - v_1)$ to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table. Similarly, by lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending $(u_2 - v_2)$ to $(2(u_1 + v_1) + 2(u_2 + v_2) + 3(u_2 - v_2))$.

$h(\delta_1) \in$disc$\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
β_2	$(u_1 - v_1)$
$(\alpha_1 + \alpha_2)$	$(2(u_1 + v_1) + 2(u_2 + v_2) + 3(u_1 - v_1))$
$(\beta_2 + \beta_3 + \alpha_1)$	$((u_1 - v_1) + (u_2 - v_2) + (u_1 + v_1))$
$(\beta_2 + \beta_3 + \alpha_2)$	$((u_1 v_1) + (u_2 - v_2) + (u_2 + v_2))$

Since the homomorphism $\text{Aut}(\widetilde{S}^{\perp}) \to \text{Aut}(\text{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and disc$\widetilde{S}^{\perp}$ can be identified such that α_1 and α_2 correspond to the generators of disc$(2\mathbf{A}_2)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_2)$ can be transposed by an admissible isometry where transposition of the corresponding generators α_1 and α_2 is realized by a +disorienting automorphism $t_{(u_1+v_2)-(u_2+v_2)}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

- *The case non-abundant* $\Sigma = 6\mathbf{A}_2 \oplus 2\mathbf{A}_1$ *numbered (28).* One has disc$\widetilde{S}^{\perp} = 2\langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 4\langle\frac{-2}{3}\rangle$ with signature $(2,5)$. Then one can take $\widetilde{S}^{\perp} = 2\mathbf{U}_3 \oplus \langle -6\rangle \oplus \langle -2\rangle$. Consider the non-abundant case numbered (33) which has the same $\widetilde{S}^{\perp}$ without one more $\langle -6\rangle$-summand. Then take the same generators in the non-abundant case numbered (33) and let $a^{'}$ be the generator of $\langle -6\rangle$-summand and take $\delta^{'} = [\frac{a^{'}}{2}]$, $\beta_4 = [\frac{a^{'}}{3}]$ as generators of $\langle\frac{1}{2}\rangle$, $\langle\frac{-2}{3}\rangle$-summands, respectively.

The same reflections in the non-abundant case numbered (34) acts here. Additionally, the reflection $t_{a^{'}}$ multiplies β_4 by (-1) and acts identically on the 2-primary part.

Pick an automorphism h acting on disc$\widetilde{S}^{\perp}$. On the 3-primary part, modulo the reflections t_a, $t_{u_1+v_1}$, $t_{u_2+v_2}$, $t_{u_1-v_1}$, $t_{u_2-v_2}$, this automorphism h takes β_4 to either β_i or $\beta_i+\beta_j+\alpha_k$, $(i \neq j)$ or $\alpha_1+\alpha_2$, or $\alpha_1+\alpha_2+\beta_i+\beta_j+\beta_k$, $(i \neq j \neq k)$. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending $a^{'}$ to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in \text{disc}\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
β_4	$a^{'}$
β_1	a
β_2	$(u_1 - v_1)$
β_3	$(u_2 - v_2)$
$(\beta_1 + \beta_2 + \alpha_1)$	$(a + (u_1 - v_1) + (u_1 + v_1))$
$(\beta_1 + \beta_3 + \alpha_1)$	$(a + (u_2 - v_2) + (u_1 + v_1))$
$(\beta_2 + \beta_3 + \alpha_1)$	$((u_1 - v_1) + (u_2 - v_2) + (u_1 + v_1))$
$(\beta_1 + \beta_4 + \alpha_1)$	$(a + a^{'} + (u_1 + v_1))$
$(\beta_2 + \beta_4 + \alpha_1)$	$((u_1 - v_1) + a^{'} + (u_1 + v_1))$
$(\beta_3 + \beta_4 + \alpha_1)$	$((u_2 - v_2) + a^{'} + (u_1 + v_1))$
$(\beta_1 + \beta_2 + \alpha_2)$	$(a + (u_1 - v_1) + (u_2 + v_2))$
$(\beta_1 + \beta_3 + \alpha_2)$	$(a + (u_2 - v_2) + (u_2 + v_2))$
$(\beta_2 + \beta_3 + \alpha_2)$	$((u_1 - v_1) + (u_2 - v_2) + (u_2 + v_2))$
$(\beta_1 + \beta_4 + \alpha_2)$	$(a + a^{'} + (u_2 + v_2))$
$(\beta_2 + \beta_4 + \alpha_2)$	$((u_1 - v_1) + a^{'} + (u_2 + v_2))$
$(\beta_3 + \beta_4 + \alpha_2)$	$((u_2 - v_2) + a^{'} + (u_2 + v_2))$
$(\alpha_1 + \alpha_2)$	$(2(u_1 - v_1) + 2(u_2 - v_2) + 3a^{'})$
$(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \beta_3)$	$((u_1 + v_1) + (u_2 + v_2) + a + (u_1 - v_1) + (u_2 - v_2))$
$(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \beta_4)$	$((u_1 + v_1) + (u_2 + v_2) + a + (u_1 - v_1) + a^{'})$
$(\alpha_1 + \alpha_2 + \beta_1 + \beta_3 + \beta_4)$	$((u_1 + v_1) + (u_2 + v_2) + a + (u_2 - v_2) + a^{'})$
$(\alpha_1 + \alpha_2 + \beta_2 + \beta_3 + \beta_4)$	$((u_1 + v_1) + (u_2 + v_2) + (u_1 - v_1) + (u_2 - v_2) + a^{'})$

Thus, modulo $O(\widetilde{S}^{\perp})$, one can assume that h fixes β_4.

Similarly, the automorphism h on the 2-primary part sends $\delta^{'}$ to either $\delta^{'}$ or δ. By lemma (5.2.1) one can find automorphism h' of $\widetilde{S}^{\perp}$ sending $a^{'}$ to the destinations listed in the table below which are realized on disc$\widetilde{S}^{\perp}$ as shown on the table.

$h(\delta_1) \in \text{disc}\widetilde{S}^{\perp}$	$h'(b_1) \in \widetilde{S}^{\perp}$
$\delta^{'}$	$a^{'}$
δ	$(12a + 9b + 8a^{'})$

Notice that the elements $h'(a^{'})$ and $a^{'}$ are congruent in mod $3\widetilde{S}^{\perp}$ which means that the induced automorphism elements of $\mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ leaves δ' fixed. Thus, one can assume that h fixes both β_4 and δ' and consider the smaller form $\mathcal{N}_1 = \langle\frac{1}{2}\rangle \oplus \langle\frac{-1}{2}\rangle \oplus 2\langle\frac{2}{3}\rangle \oplus 3\langle\frac{-2}{3}\rangle$, where $N_1 = \mathbf{U}_3 \oplus \langle -6\rangle \oplus \langle -2\rangle$ still having the same generators except for a'.

This is the same with the $\mathrm{disc}\widetilde{S}^{\perp}$ and the lattice $\widetilde{S}^{\perp}$ in the case of non-abundant $\Sigma = 4\mathbf{A}_2 \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ numbered (33). It is shown that all automorphisms on non-abundant $\mathrm{disc}\widetilde{S}^{\perp}$ in numbered (33) can be realized by automorphisms of non-abundant $\widetilde{S}^{\perp}$ in numbered (33). Then, it is clear that $O(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto for this case, too.

Since the homomorphism $\mathrm{Aut}(\widetilde{S}^{\perp}) \to \mathrm{Aut}(\mathrm{disc}\widetilde{S}^{\perp})$ is onto, $\widetilde{\mathcal{S}}$ and $\mathrm{disc}\widetilde{S}^{\perp}$ can be identified such that δ and δ' correspond to the generators of $\mathrm{disc}(2\mathbf{A}_1)$. In $\widetilde{S}$, generators of $(2\mathbf{A}_1)$ can be transposed by an admissible isometry where transposition of the corresponding generators δ and δ' is realized by a +disorienting automorphism $t_{a-a'}$ on $\widetilde{S}^{\perp}$. So the abstract homological type extended by this Σ is symmetric.

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