ABANT İZZET BAYSAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MASSIVE AND MASSLESS GAUGE THEORIES WITH FLAT CONNECTION

DOCTORAL THESIS

İBRAHİM ŞENER

BOLU, JANUARY 2016

ABANT İZZET BAYSAL UNIVERSITY
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APPROVAL OF THE THESIS.

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DECLARATION

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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İbrahim ŞENER

ABSTRACT

MASSIVE AND MASSLESS GAUGE THEORIES
WITH FLAT CONNECTION
PHD THESIS

İBRAHİM ŞENER
ABANT IZZET BAYSAL UNIVERSITY GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
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Yang - Mills type massive and massless gauge theories are interpreted in the geometrical frame of holomorphic principal bundles over a complex 2 - manifold. It is seen in this formalism that, the component $(1, 1)$ of curvature of this connection appears because of flat connections generated by holomorphic structure although connection is flat. Thus it is possible to write a Lagrangian for a Yang - Mills theory including massive and massless gauge fields. However, the mass matrix of a massive gauge field on such a bundle isn't nilpotent and this field is generated by a noncommutative flat connection on the same bundle, then the structure group of this bundle is non - Abelian complex Lie group. However, if gauge field is massless, then this is generated by commutative flat connection, and so the structure group of bundle is Abelian complex Lie group. Also one sees that second Chern number or topological charge is proportional to the total volume of base manifold for each massless and massive gauge theories and Abelian (massless) gauge theories which is indeed the theories of Kähler potential on complex projective space $\mathbb{C}P^2$.

KEYWORDS: Massive and Massless Gauge Fields, Holomorphic Principal Bundle, Complex Projective Space, Kähler Potential, Flat Connection, Mass Matrix, Abelian and Nonabelian Gauge Group, Topological Charge, Chern Classes.

ÖZET

DÜZ BAGLANTILI KÜTLELİ VE KÜTLESİZ AYAR TEORILERI
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Bu tezde, Yang - Mills tipi kütleli ve kütlesiz ayar teorileri kompleks 2 - manifold üzerindeki holomorfik asli demetlerin geometrik çerçevesinde değerlendirilmektedir. Bu geometrik yapıda görülmektedir ki, holomorfik yapi tarafindan indüklenen düz bağlantinin eğriliğinin $(1,1)$ - bileşeni, bağlanti düz olmasina ragmen yok olmaz. Böylece kütleli ve kütlesiz ayar alanlarını içeren bir Yang - Mills Lagrangian formu yazılabilir. Fakat kütleli ayar alanın kütle matrisi nilpotent olamaz ve bu alan demet üzerindeki komutatif olmayan bir düz baglanti tarafindan üretilir ve demetin yapı grubu Abelian olmayan Lie grubu olur. Ancak, eğer ayar alan kütlesiz ise, o zaman bu alanı komutatif bir düz bağlantı üretir ve yapı grubu Abelian Lie grubu olur. Ayrica görülmektedir ki, hem kütleli hem de kütlesiz ayar alanlar için topolojik yük ya da ikinci Chern sayısı taban manifoldun hacmi ile orantılıdır ve kütlesiz ayar alanlar esasinda $\mathbb{C}P^2$ kompleks projektif uzayının Kähler potansiyelinin bir teorisidir.

ANAHTAR KELiMELER: Kütleli ve Kütlesiz Ayar Alanlar, Holomorfik Asli Demet, Kompleks Projektif Uzay, Kähler Potansiyeli, Düz Bağlantı, Kütle Matrisi, Abelian and Abelian Olmayan Ayar Groupları, Topolojik Yük, Chern Sınıfları.

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1. INTRODUCTION

According to Georgi and Glashow, all fundamental forces are severally representations of a unique group of the unified fields (Georgi & Glashow, 1974). The fundamental interactions except for the gravity have internal symmetry group structures via the Hermitian structure in the quantum regime. As known that these groups correspond to the $U(1)$ for the electromagnetic force, the $SU(2)$ for the weak force and the $SU(3)$ for the strong force. In this context, the field theories which are invariant under the local transformations of these group are called Weyl (for the electromagnetic force) (Weyl, 1918) and Yang - Mills type (for the weak and strong forces) theories (Yang & Mills, 1954), (Ne'eman, 1961), (Gell-Mann, 1961). However, there doesn't exist a gauge group ansatz similar to that of other forces. On the other hand, a more general theory which is invariant under the general linear Lie groups was given by Utiyama (Utiyama, 1956) to be included the gravity. In the Utiyama' s formalism the gravitational interaction isn't a gauge theory like the Yang - Mills type theories. If we use a geometrical language, each Maxwell and Yang - Mills theory are the Lagrangian theories of the connections on some principal bundles whose the structure groups are the $U(1)$ and $SU(N)$. Therefore there is a near analogy between these theories and the differential geometry of the fiber bundles. Hence, if we show by A^a_μ the gauge potential (that is source potential) of an force, by $F^a_{\mu\nu}$ its gauge field stress and by G the gauge group, where a labels the generator indices of the gauge group and μ, ν the local coordinates, they are associated to following geometrical notions:

$$
\begin{aligned}
&Gauge\ potential\ A^a_\mu && \longleftrightarrow\ Lie\ algebra\ valued\ connection\ 1\ -\ form\ A\\
&Gauge\ field\ stress\ F^a_{\mu\nu} && \longleftrightarrow\ Lie\ algebra\ valued\ curvature\ 2\ -\ form\ F\\
&Gauge\ group\ G && \longleftrightarrow\ Structure\ group\ G\\
&Gauge\ transformation && \longleftrightarrow\ Automorphism\ of\ fibers
\end{aligned}
$$

1.1 Fundamental Geometric Structures

Definition 1.1. *(Manifold) The manifolds are topological spaces with Hausdorff structure and are the second countable, so that these spaces are locally homeomorphic to the Euclidean space. Therefore, let M be a topological space with n - dimension corresponding to $\mathbb{R}^n$ and U_i and U_j two opens neighborhood of a point p, where $U_i \cap U_j \neq \emptyset$. Then, $M = \bigcup_i U_i$ and a coordinate set is a map $x_i : U_i \to \mathbb{R}^n$, where $(i = 1, \cdots, n)$. Thus a manifold is called a collection of charts (U_i, x_i). If mapping $x_i \circ x_j^{-1} : x_i(U_i \cap U_j) \to x_j(U_i \cap U_j)$ is differentiable, called it transition function, then manifold M is called a differentiable or smooth manifold (we will use smooth manifold term for differentiable manifolds).*

Definition 1.2. *(Tangent and Cotangent Bundles) Let $\{x^\mu\} \in \mathbb{R}^n$ be the local coordinates neighborhood of a point $p \in M$, where ($\mu = 1, \cdots, n$). Then we can show by T_pM a tangent space at $p \in M$ and its cotangent by T_p^*M. The tangent and cotangent bundles of these space together with basis are locally written, respectively, as*

$$\{\partial_\mu\} \in TM \cong \bigcup_{p\in M} \{p\} \times T_pM, \tag{1.1}$$

$$\{dx^\mu\} \in T^*M \cong \bigcup_{p\in M} \{p\} \times T_p^*M, \tag{1.2}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

Definition 1.3. *(Vector Field and Its Dual (Nakahara, 2003)) A vector over a manifold is defined as a tangent vector to a curve over M and if a vector can be defined smoothly to each point of manifold M, it is called a vector field over M. The dual of a vector over a manifold is called 1 - form and if a 1 - form can be defined smoothly to each point of manifold M, it called differentiable 1 - form.*

Definition 1.4. *(Tensor Product (Nakahara, 2003)) Let $\mathbb{V}^k$ and $\mathbb{V}^l$ be two vector spaces on the same field, i.e. $\mathbb{V} = \mathbb{R}$ or $\mathbb{V} = \mathbb{C}$. Define a $\mathbb{V}$ - linear map as follow*

$$\otimes : \mathbb{V}^k \times \mathbb{V}^l \to \mathbb{V}^{kl}, \tag{1.3}$$

so that this map is multi-linear and distributive, respectively,

$$
\begin{aligned}
a(\mathrm{v}) \otimes \mathrm{u} &= a(\mathrm{v} \otimes \mathrm{u}), &(1.4)\\
a\mathrm{u} \otimes (b_1\mathrm{v}_1 + s + b_m\mathrm{v}_m) &= ab_1\mathrm{u} \otimes \mathrm{v}_1 + s + ab_m\mathrm{u} \otimes \mathrm{v}_m, &(1.5)\\
(b_1\mathrm{v}_1 + s + b_m\mathrm{v}_m) \otimes (a\mathrm{u}) &= ab_1\mathrm{v}_1 \otimes \mathrm{u} + s + ab_m\mathrm{v}_m \otimes \mathrm{u}, &(1.6)
\end{aligned}
$$

where $a, b_1, \cdots, b_m \in \mathbb{V}$, $\mathrm{u} \in \mathbb{V}^k$, $\mathrm{v}, \mathrm{v}_1, \cdots, \mathrm{v}_m \in \mathbb{V}^l$. *Then map* $\otimes$ *is called tensor product.*

Definition 1.5. *(Tensor Field (Nakahara, 2003)) A tensor of type* (q, r) *is a multilinear object which maps the* q *elements of* T_p^*M *and the* r *elements of* T_pM *to a real number. An element of a tensor is written in the terms of local basis as*

$$T = T^{\mu_1\cdots\mu_q}_{\nu_1\cdots\nu_r}\frac{\partial}{\partial x^{\mu_1}}\cdots\frac{\partial}{\partial x^{\mu_q}}dx^{\nu_1}\cdots dx^{\nu_r} \in \otimes^q_r|_p(M), \tag{1.7}$$

where we show the space of tensors by

$$\otimes^q_r|_p(M) = \underbrace{T_pM \otimes \cdots \otimes T_pM}_{q\ times} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{r\ times}. \tag{1.8}$$

If a tensor can be defined smoothly to each point over manifold M*, then it is called tensor field.*

Definition 1.6. *(Metric (Nakahara, 2003)) Let* M *be a differentiable manifold and* TM *a tangent bundle with local coordinates basis* $\{\partial_\mu\} \in TM$ *and* T^*M *a cotangent bundle with local dual basis* $dx^\mu \in T^*M$ *over this manifold. A Riemannian metric* g *over* M *is a tensor field of type* $(0, 2)$ *which satisfies following axioms at each point* $p \in M$*:*

$$
\begin{aligned}
\mathrm{g}_p(X, Y) &= \mathrm{g}_p(Y, X), &(1.9)\\
\mathrm{g}_p(X, X) &\geq 0, \quad if\ only\ if\ X = 0, &(1.10)
\end{aligned}
$$

where $X, Y \in T_pM$. *If* $\{x^\mu\}$ *is local coordinates at a point* $p \in M$*, metric is expanded in the terms of* $dx^\mu \otimes dx^\nu$ *as*

$$\mathrm{g}_p = g_{\mu\nu}(p)dx^\mu \otimes dx^\nu, \tag{1.11}$$

where

$$g_{\mu\nu}(p) = \mathrm{g}_p(\partial_\mu, \partial_\nu) = g_{\nu\mu}(p). \tag{1.12}$$

Since $(g_{\mu\nu})$ is a symmetric matrix, its eigenvalues are real. Therefore, If there exist i positive and j negative eigenvalues, pair (i, j) is called the index of metric. If $j = 1$, metric is called a Lorentz metric. If a smooth manifold M admits a Riemannian metric g, pair (M, g) is called a Riemannian manifold. If g is Lorentzian, (M, g) is called a Lorentz manifold.

Definition 1.7. *(Wedge Product (Şuhubi, 2013)) Let M be a manifold of real n - dimension. Wedge product $\wedge$ is totally antisymmetric tensor product for local basis $\{e^\mu\} \in T^*M$ given as*

$$e^\mu \wedge e^\nu = e^\mu \otimes e^\nu - e^\nu \otimes e^\mu \tag{1.13}$$

and more generally

$$e^{\mu_1} \wedge \cdots \wedge e^{\mu_r} = \delta^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_r} e^{\nu_1} \otimes \cdots \otimes e^{\nu_r}, \tag{1.14}$$

where $\delta^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_r}$ is generalised Kronecker delta.

Definition 1.8. *(Differential Forms (Şuhubi, 2013), (Nakahara, 2003)) A differential r - form is a totally antisymmetric tensor of type $(0, r)$ and is written such that*

$$\omega = \frac{1}{r!}\omega_{\mu_1 \cdots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}. \tag{1.15}$$

Thus the bundle of exterior r - forms over manifold M is shown by

$$\Lambda^r(M) = \underbrace{T^*M \wedge \cdots \wedge T^*M}_{r\ times}, \quad (0 \leq r \leq n) \tag{1.16}$$

and the dimension of $\Lambda^r(M)$ is

$$\dim \Lambda^r(M) = \binom{n}{r} = \frac{n!}{(n-r)!r!} \tag{1.17}$$

Equality $\binom{n}{r} = \binom{n}{n-r}$ implies that $\dim \Lambda^r(M) = \dim \Lambda^{n-r}(M)$.

Definition 1.9. *(Exterior or Wedge Product of Two Forms (Şuhubi, 2013), (Nakahara, 2003)) Let $\omega \in \Lambda^r(M)$ and $\sigma \in \Lambda^s(M)$ be two differential forms. The wedge product of them is*

$$\omega \wedge \sigma = \frac{1}{(r+s)!}\gamma_{\mu_1\cdots\mu_r\nu_1\cdots\nu_s} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_s}, \tag{1.18}$$

where

$$\gamma_{\mu_1\cdots\mu_r\nu_1\cdots\nu_s} = \omega_{[\nu_1\cdots\nu_s}\sigma_{\nu_1\cdots\nu_s]} \tag{1.19}$$

Let $\xi \in \Lambda^q(M)$, $\eta \in \Lambda^r(M)$ and $\omega \in \Lambda^s(M)$, then

$$\xi \wedge \xi = 0, \ if\ q\ is\ odd \tag{1.20}$$

$$\xi \wedge \eta = (-1)^{qr}\eta \wedge \xi, \tag{1.21}$$

$$(\xi \wedge \eta) \wedge \omega = \xi \wedge (\eta \wedge \omega). \tag{1.22}$$

Definition 1.10. *(Exterior Derivative (Şuhubi, 2013), (Nakahara, 2003)) The exterior differentiation d over a manifold M is defined as a map*

$$\mathrm{d} : \Lambda^r(M) \to \Lambda^{r+1}(M), \quad 0 \leq r < n. \tag{1.23}$$

Leibnitz rules for $\alpha \in \Lambda^r(M)$, $\beta \in \Lambda^q(M)$ and $f \in \mathcal{C}^\infty(M)$ are

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \mathrm{d}\beta, \tag{1.24}$$

$$\mathrm{d}(f\beta) = \mathrm{d}f \wedge \beta + f\mathrm{d}\beta. \tag{1.25}$$

Definition 1.11. *(Volume Element Form) The volume element form over a smooth manifold M of real n dimension is defined for local orthogonal coordinates basis $\{dx^\mu\} \in T^*M$ as follow*

$$\mathrm{dvol} = dx^1 \wedge \cdots \wedge dx^n \in \Lambda^n(M). \tag{1.26}$$

Definition 1.12. *(Hodge Duality (De Azcárrage & Izquirdo, 1995)) The Hodge duality operator over a manifold M of real n - dimension is defines as*

$$* : \Lambda^r(M) \to \Lambda^{n-r}(M) \tag{1.27}$$

and it acts on a r - form as follow

$$*\omega = (*\omega)_{\mu_{r+1}} \cdots_{\mu_n} dx^{\mu_{r+1}} \wedge \cdots \wedge dx^{\mu_n} \in \Lambda^r\,(M), \tag{1.28}$$

where

$$(*\omega)_{\mu_{r+1}} \cdots_{\mu_n} = \frac{\sqrt{|\mathrm{detg}^{\mu\nu}|}}{(n-r)!r!} \mathrm{g}^{\mu_1\nu_1} \cdots \mathrm{g}^{\mu_r\nu_r} \varepsilon_{\nu_1\cdots\nu_r\nu_{r+1}\cdots\nu_n} \omega_{\mu_1} \cdots_{\mu_r}\,, \tag{1.29}$$

so one gets

$$\omega \wedge *\omega = \omega_{\mu_1\cdots\mu_r} (*\omega)_{\mu_{r+1}\cdots\mu_n} \mathrm{dvol}, \tag{1.30}$$

where g *is metric over manifold* M *and* $\varepsilon_{\nu_1\cdots\nu_n}$ *is Levi - Civita exact antisymmetric tensor .*

Definition 1.13. *(Inner Product of Exterior Forms) Let* $\alpha, \beta \in \Lambda^K(M)$. *Then inner product of them is defined as*

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta. \tag{1.31}$$

Definition 1.14. *(Lie Groups) A Lie group is a set* G *with two structures:* G *is a group and a (smooth) manifold. These structures agree in following sense: multiplication and inversion are smooth. A Lie group is also defined as a differentiable manifold having group structure. Some examples of (matrix) Lie groups using mostly in physics are generally are summarized as follow.*

Groups of General Linear Lie Group: $GL(n, \mathbb{F})$
Special Linear Group: $SL(n, \mathbb{F}) = \{A \in GL(n, \mathbb{F}) \,|\, det(A) = 1\}$
Orthogonal Grup: $O(n; \mathbb{F}) = \{A \in GL(n, \mathbb{F}) \,|\, A^\top A = 1\}$
Special Orthogonal Grup: $SO(n, \mathbb{F}) = \{A \in O(n, \mathbb{F}) \,|\, det(A) = 1\}$
Unitary Group: $U(n) = \{A \in GL(n, \mathbb{C}) \,|\, A^\dagger A = 1\}$
Special Unitary Group: $SU(n) = \{A \in U(n) \,|\, A^\dagger A = 1 \,|\, det(A) = 1\}$

Here $\mathbb{F}$ *denotes* $\mathbb{R}$ *or* $\mathbb{C}$ *and* $\top, \dagger$ *the transpose and the hermitian conjugate of a square matrix, respectively.*

Definition 1.15. *(Connection on A Tangent Bundle) Let $\Gamma(TM)$ be the set of the local sections on the tangent bundle TM over a Riemannian manifold. A $\mathfrak{g}$ valued connection D is a linear map*

$$D : \Gamma(TM) \to \Gamma(TM \otimes \mathfrak{g} \otimes T^*M) \tag{1.32}$$

which satisfies following conditions:

$$\begin{aligned} D(X+Y) &= DX + DY, & (1.33) \\ D(fX) &= (\mathrm{d}f)Y + fDX & (1.34) \end{aligned}$$

where $\mathfrak{g}$ is the Lie algebra of group $G \in \mathrm{End}(TM)$, $f \in \mathcal{C}^\infty(M)$, $X, Y \in \Gamma(TM)$. Thus the covariant derivative of connection D is

$$DX = \mathrm{d}X + \Gamma X, \tag{1.35}$$

where $\Gamma \in \Lambda^1(M, \mathfrak{g})$ is called $\mathfrak{g}$ - valued connection 1 - form. Given a smooth manifold M of real n - dimension with a tangent bundle TM and cotangent bundle T^*M. Let $\{e_a\} \in TM$ be local basis and $e^a \in T^*M$ dual base 1 - forms. Then a $\mathfrak{g}$ - valued connection 1 - form is defined as follow

$$De_a = \Gamma^b_{\ a} e_b, \tag{1.36}$$

where $\theta_i \in \mathfrak{g}$ is the generators of Lie algebra $\mathfrak{g}$ and $i = 1 \cdots \dim[G]$ and

$$\Gamma^b_{\ a} = (\Gamma^b_{\ a})^i_\mu \theta_i dx^\mu. \tag{1.37}$$

Definition 1.16. *(Curvature and Torsion of A Connection on A Tangent Bundle) The $\mathfrak{g}$ valued curvature 2 - form of a connection on tangent bundle TM is given by*

$$\begin{aligned} R^a_b &= \mathrm{d}\Gamma^a_{\ b} + \Gamma^a_{\ c} \wedge \Gamma^c_{\ b} \\ &= \sum_{\mu<\nu} (R^a_b)^i_{\mu\nu} \theta_i dx^\mu \wedge dx^\nu. & (1.38) \end{aligned}$$

and its torsion 2 - form is also

$$\begin{aligned} T^a &= \mathrm{d}e^a + \Gamma^a_{\ b} \wedge e^b \\ &= \sum_{\mu<\nu} (T^a)^i_{\mu\nu} \theta_i dx^\mu \wedge dx^\nu. & (1.39) \end{aligned}$$

Definition 1.17. *(Levi - Civita Connection and Riemann Curvature) If connection matrix Γ^a_b is symmetric, that is $\Gamma_{ab} = \Gamma_{ba}$, where $\Gamma_{ab} = g_{ac}\Gamma^c_b$, then torsion vanishes and this connection is called Levi - Civita connection. The curvature of Levi - Civita connection is called Riemann Curvature.*

1.2 Complex Manifolds

More detailed knowledge about complex manifolds may be founded in those references (Kobayashi & Nomizu, 1969), (Griffiths & Harris, 1978), (Kodaira, 1986), (Huybrechts, 2005). Here we give fundamental notations which will request in thesis.

Definition 1.18. *(Holomorphic (Complex) Function) Any complex number consists of two real numbers. If $x, y \in \mathbb{R}$, then a complex number is $z = x + iy \in \mathbb{C}$ and its complex conjugate $\bar{z} = x - iy$. Consider a function $f(z) \in \mathcal{C}^\infty(\mathbb{C})$ depending only on $z \in \mathbb{C}$, that is not $\bar{z}$. Decompose function $f(z)$ such that $f(z) = g(x, y) + ih(x, y)$, where $g(x, y), h(x, y) \in \mathcal{C}^\infty(\mathbb{R}^2)$. If*

$$\frac{df(z)}{d\bar{z}} = 0, \tag{1.40}$$

function $f(z)$ is holomorphic. This is also called Cauchy - Riemann condition

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}, \tag{1.41}$$

$$\frac{\partial g}{\partial y} = \frac{\partial h}{\partial x} \tag{1.42}$$

Then, a multi - components complex number over a complex space $\mathbb{C}^{n=2m}$ occur from two multi - components real numbers and its complex conjugate: $z^\mu = x^\mu + iy^\mu$ $\bar{z}^\mu = x^\mu - iy^\mu$, $(\mu = 1, \cdots, n)$, respectively. Therefore, let z^μ and z'^μ be two local complex coordinates. A coordinate transformation $z'^\mu = z'^\nu(z^\nu)$ between them is holomorphic if

$$\frac{\partial z'^\mu}{\partial \bar{z}^\nu} = 0. \tag{1.43}$$

Thus we can show complex differentiations shortly by

$$\begin{aligned} dz^{\mu} &= dx^{\mu} + idy^{\mu}, && (1.44) \\ d\bar{z}^{\mu} &= dx^{\mu} - idy^{\mu} && (1.45) \end{aligned}$$

and

$$\begin{aligned} \partial_{\mu} &= \frac{\partial}{\partial z^{\mu}} = \frac{1}{2}\left(\frac{\partial}{\partial x^{\mu}} - i\frac{\partial}{\partial y^{\mu}}\right), && (1.46) \\ \partial_{\overline{\mu}} &= \frac{\partial}{\partial \bar{z}^{\mu}} = \frac{1}{2}\left(\frac{\partial}{\partial x^{\mu}} + i\frac{\partial}{\partial x^{\mu}}\right). && (1.47) \end{aligned}$$

Also, there exist following relations

$$\begin{aligned} dz^{\mu}(\partial_{\nu}) &= \delta^{\mu}_{\nu}, && (1.48) \\ dz^{\mu}(\partial_{\overline{\nu}}) &= 0. && (1.49) \end{aligned}$$

Then, the exterior differentiation of a smooth function $f(z, \bar{z})$ is written as

$$\begin{aligned} \mathrm{d}f &= \frac{\partial f}{\partial z^{\mu}}dz^{\mu} + \frac{\partial f}{\partial \bar{z}^{\mu}}d\bar{z}^{\mu} \\ &= \partial f + \overline{\partial} f, && (1.50) \end{aligned}$$

where ∂ and $\overline{\partial}$ are called holomorphic and antiholomorphic derivatives, respectively, so we have two exterior derivative operators

$$\begin{aligned} \partial f &= \frac{\partial f}{\partial z^{\mu}}dz^{\mu}, && (1.51) \\ \overline{\partial} f &= \frac{\partial f}{\partial \bar{z}^{\mu}}d\bar{z}^{\mu}. && (1.52) \end{aligned}$$

Thus, if a smooth function $f(z^{\mu})$ is complex or holomorphic, then

$$\overline{\partial} f = \frac{\partial f}{\partial \bar{z}^{\mu}}d\bar{z}^{\mu} = 0. \tag{1.53}$$

Definition 1.19. *(Complex Manifolds) Let M be a smooth manifold of real $2n$ dimension. If each tangent space over this manifold is locally homeomorphic* *to $\mathbb{C}^n$ and if the transition functions of all coordinate charts are holomorphic, then manifold M is called complex manifold. Therefore, let $U_\alpha, U_\beta \subset M$ be two opens neighborhood of a point $p \in M$, $U_\alpha \cap U_\beta \neq \emptyset$ and given a local coordinate chart as pair (U_α, ϕ_α) over M by a homeomorphism $\phi_\alpha : U_\alpha \to \phi(U_\alpha) \subset \mathbb{R}^{2n}$. Two coordinate charts (U_α, ϕ_α) and (U_β, ϕ_β) give a transition function such that $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$. If this transition function is holomorphic, then manifold M is a complex manifold.*

Definition 1.20. *(Decomposition of Tangent and Cotangent Bundles) Let M be a complex manifold and TM and T^*M complexified tangent and cotangent bundles over this manifold. Then the decomposition of them are*

$$\begin{aligned} TM &= T^{(1,0)}M \oplus T^{(0,1)}M, & (1.54)\\ T^*M &= T^{*(1,0)}M \oplus T^{*(0,1)}M, & (1.55)\end{aligned}$$

where $(1,0)$ and $(0,1)$ denote holomorphic and antiholomorphic subbundles, respectively.

Definition 1.21. *(Exterior Forms on Complex Manifolds) The bundle of exterior (r,q) - forms over a complex manifold is*

$$\begin{gathered} \Lambda_M^{(r,q)} = \Lambda^r T^{*(1,0)}M \otimes \Lambda^q T^{*(0,1)}M, \\ r, q \leq \dim_{\mathbb{C}}(M), \quad r+q \leq \dim_{\mathbb{R}}(M), \end{gathered} \tag{1.56}$$

where $\Lambda_M^{(r,q)} = \overline{\Lambda_M^{(q,r)}}$. Then the exterior $K = r+q$ - form over complex manifold M is written as

$$\Lambda_M^{(K)} = \oplus_{(K=r+q)} \Lambda^{*(r,q)} M. \tag{1.57}$$

Any exterior form $\omega \in \Lambda^{(r,q)}(M)$ is written locally as follow

$$\omega = \omega_{r\bar{q}} dz^r \wedge dz^{\bar{q}}, \tag{1.58}$$

where $\omega_{r\bar{q}} \in \mathcal{C}^\infty(M)$.

Hereafter we will show by $\Lambda^{(r,q)}$ and $\Lambda^{(K)}$ the bundles of the exterior forms over complex manifold M.

Definition 1.22. *(Wedge Product on Complex Manifolds) For some exterior forms $\omega \in \Lambda^{(r,q)}$, $\mu \in \Lambda^{(t,s)}$ and $\eta \in \Lambda^{(u,v)}$ over a complex manifold M the commutation relations are*

$$\omega \wedge \mu = (-1)^{(a)} \mu \wedge \omega \tag{1.59}$$

$$[\omega, \mu]_{\wedge} = \omega \wedge \mu - (-1)^{(a)} \mu \wedge \omega \tag{1.60}$$

and the Jacobi identity is

$$[\omega, [\mu, \eta]_{\wedge}]_{\wedge} + (-1)^{(a+b)} [\mu, [\eta, \omega]_{\wedge}]_{\wedge} + (-1)^{(b+c)} [\eta, [\omega, \mu]_{\wedge}]_{\wedge} = 0, \tag{1.61}$$

where

$$a = deg(\omega) = r + q, \tag{1.62}$$

$$b = deg(\mu) = t + s, \tag{1.63}$$

$$c = deg(\eta) = u + v. \tag{1.64}$$

Definition 1.23. *(Hodge Duality on Complex Manifolds) The Hodge duality operators over a complex n - manifold M acting forms on $\Lambda^{(K)}_M = \oplus_{(K=r+q)} \Lambda^{*(r,q)} M$ are defined as*

$$* \; : \; \Lambda^{(r,q)} \to \Lambda^{(n-r,n-q)}, \tag{1.65}$$

$$* \; : \; \Lambda^{(K)} \to \Lambda^{(2n-K)}. \tag{1.66}$$

Definition 1.24. *(Exterior Derivatives on Complex Manifolds) The Exterior derivative over a complex manifold is a linear map $\mathrm{d} : \Lambda^{(r,q)} \to \Lambda^{(r+1,q+1)}$ and it decomposes into holomorphic and antiholomorphic parts $\partial : \Lambda^{(r,q)} \to \Lambda^{(r+1,q)}$ and $\bar{\partial} : \Lambda^{(r,q)} \to \Lambda^{(r,q+1)}$, respectively, such that*

$$\mathrm{d} = \partial \oplus \bar{\partial}. \tag{1.67}$$

Poincarè lemma, $\mathrm{d}^2 = 0$*, is written as*

$$\partial^2 = 0, \tag{1.68}$$

$$\bar{\partial}^2 = 0, \tag{1.69}$$

$$\partial\bar{\partial} + \partial\bar{\partial} = 0. \tag{1.70}$$

Definition 1.25. *(Leibniz Rule) Let* $\alpha \in \Lambda^{(r,q)}$ *and* $\beta \in \Lambda^{(t,s)}$ *be two differential forms. Then Leibniz rules for the wedge product of them are*

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{deg\alpha}\alpha \wedge \mathrm{d}\beta, \tag{1.71}$$

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{deg\alpha}\alpha \wedge \partial\beta, \tag{1.72}$$

$$\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^{deg\alpha}\alpha \wedge \overline{\partial}\beta. \tag{1.73}$$

Definition 1.26. *(Hermitian Structure) Let* V *be a complex vector space. For* $u \neq 0, v \neq 0, w \neq 0 \in V$ *and* $a, b \in \mathbb{R}$*, a Hermitian structure, under a linear mapping* $H : V \times V \to \mathbb{C}$*, is defined as*

$$H(au + bv, \overline{w}) = aH(u, \overline{w}) + bH(v, \overline{w}) \tag{1.74}$$

$$H(u, \overline{w}) = \overline{H(w, \overline{u})} \tag{1.75}$$

$$H(Iu, \overline{w}) = iH(w, \overline{u}) \tag{1.76}$$

$$H(u, \overline{u}) > 0, \tag{1.77}$$

where I *is the complex structure on* V*,* $I : V \to V,\ I \circ I = -id$*.*

Definition 1.27. *(Hermitian Metric) Let* M *be a complex* n *- manifold with local holomorphic coordinates* $\{z^\mu\} \in \mathbb{C}^n$*. If* $\{e_\mu\}$ *are any local basis over this manifold and*

$$h_{\mu\overline{\nu}}(z) = \langle e_\mu | e_\nu \rangle \tag{1.78}$$

is a hermitian inner product, the hermitian metric over this manifold is defined by

$$h = h_{\mu\overline{\nu}}(dz^\mu \otimes d\bar{z}^\nu) > 0, \tag{1.79}$$

$$h_{\mu\overline{\nu}} = \overline{h_{\overline{\mu}\nu}}. \tag{1.80}$$

Definition 1.28. *(Kähler Form) For a hermitian metric over a complex manifold the Kähler of* $(1,1)$ *- form is given by*

$$K = \frac{i}{2} h_{\mu\overline{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu} \tag{1.81}$$

and this form satisfies

$$\begin{aligned} \overline{K} &= -\frac{i}{2}\overline{h_{\mu\overline{\nu}}} dz^{\mu} \wedge d\bar{z}^{\nu} \\ &= \frac{i}{2} h_{\mu\overline{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu} = K \end{aligned} \tag{1.82}$$

Definition 1.29. *(Kähler Manifold) If one can define a Kähler form over a complex manifold, then this manifold is called Kähler manifold.*

Definition 1.30. *(Kähler Potential And Kähler Metric) Let* M *be a Kähler manifold and* $\phi \in \mathcal{C}^{\infty}(M)$ *be a smooth scalar over this manifold. If this scalar satisfies*

$$\begin{aligned} \partial\overline{\partial}\phi &= \frac{i}{2}\frac{\partial^2 \phi}{\partial z^{\mu}\partial \bar{z}^{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu} \\ &= \frac{i}{2} h_{\mu\overline{\nu}} (dz^{\mu} \wedge d\bar{z}^{\nu}) = K, \end{aligned} \tag{1.83}$$

then it is called Kähler potential, and so term $\partial\overline{\partial}\phi$ *becomes a Kähler form. Therefore, since it becomes* $h_{\mu\overline{\nu}} = \frac{\partial^2 K}{\partial z^{\mu}\partial \bar{z}^{\nu}}$, *Kähler metric is given by*

$$h_K = \frac{\partial^2 K}{\partial z^{\mu}\partial \bar{z}^{\nu}} (dz^{\mu} \otimes d\bar{z}^{\nu}). \tag{1.84}$$

1.3 Fiber Bundles

Fiber bundles are quite useful geometrical objects for the formulation of classical fields, i.e gravitation, electromagnetism and Yang - Mills forces, and classical mechanics, i.e Hamiltonian dynamics, continuum mechanics. After the Yang - Mills theory was introduced, the components of principal bundles, i.e structure group, connection and curvature, was identified to non - Abelian gauge field, local gauge transformation, gauge potential and gauge stress (Wu & Yang, 1975), (Honda, 1979), (Yang, 1995).

Definition 1.31. *(Fiber Bundles) Given a topological space $\mathfrak{F}$ called typical fiber at a point p of a smooth manifold M called base manifold. Act a group G on $\mathfrak{F}$ called the structure group of bundle, so that $G \times \mathfrak{F} \to \mathfrak{F}$. Define a (differentiable) projection map by homeomorphism $\pi : E \to M$ called fiber projection, where E is called fiber bundle. Given an open U neighborhood of a point $p \in M$. Then, inverse bundle projection is defined by $\pi^{-1}(U) : U \to U \times \mathfrak{F}$, so any fiber bundle must be trivial locally and it is represented by a diffeomorphism $\pi \circ \phi(p, f) = p$, where $\phi : U \times \mathfrak{F} \to \pi^{-1}(U)$ is a diffeomorphism and $f \in \mathfrak{F}$ is an element of typical fiber space. Here, because of ϕ, a structure group can be defined. Let U and V be two opens neighborhood of a point $p \in M$, so that they obey $U \cap V \neq \emptyset$. Then, a transition function induced by a diffeomorphism $\phi_U : U \times \mathfrak{F} \to \pi^{-1}(U)$ is defined as follow*

$$\Psi_{UV} = \phi_U \circ \phi_V^{-1} : (U \cap V) \times \mathfrak{F} \to (U \cap V) \times \mathfrak{F}. \tag{1.85}$$

If $U, V, W, \cdots, Y, Z$ are some opens neighborhood of a point $p \in M$ obeying $U \cap V \cap W \cap \cdots \cap Y \cap Z \neq \emptyset$, then we have as follow relations known as reflection, symmetry and transition, respectively,

$$\Psi_{UU} = \mathbb{I}_M \tag{1.86}$$

$$\Psi_{UV} \circ \Psi_{VU} = \mathbb{I}_M \tag{1.87}$$

$$\Psi_{UV} \circ \Psi_{VW} \cdots \circ \Psi_{YZ} \circ \Psi_{ZU} = \mathbb{I}_M, \tag{1.88}$$

where $\mathbb{I}_M$ is unit transformation function. Since transition function is an automorphism $Aut(\mathfrak{F}) : \mathfrak{F} \to \mathfrak{F}$, as well, this function is called the structure group of fiber bundle E.

Therefore, we can summarize some particular bundle definitions as follow.

Summary 1.32. *(Summarization of Fiber Bundles)*

Component of Bundle	***Vector Bundle***	***Principal Bundle***	***Holomorphic Bundle***
Base Manifold	$\mathbb{R}^n/\mathbb{C}^m$	$\mathbb{R}^n/\mathbb{C}^m$	$\mathbb{C}^m$ *(complex manifold)*
Typical Fiber Space	$\mathbb{R}^k/\mathbb{C}^l$	*Lie Group*	$\mathbb{C}^k$
Bundle Projection	$E \to M$	$P \to M$	$E \to M$
Transition Function	$\psi_U \circ \psi_V^{-1}$	$\psi_U \circ \psi_V^{-1}$	$\psi_U \circ \psi_V^{-1}$ *(holomorphic)*
Local Section	$U \to E$	$U \to P$	$U \to E$
Structure Group	*Lie Group*	*Lie Group*	*Complex Lie Group*

Definition 1.33. *(G - Structure (Joyce, 2000)) Let M be a n - manifold and F a frame bundle over M, as a principal bundle. Then F is a principal bundle over M with fibre $GL(n, \mathbb{R})$. Let G be a Lie subgroup of $GL(n, \mathbb{R})$. Then a G - structure on M is a principal subbundle P of F, with fibre G. Thus, some geometrical objects such as Riemannian metrics and complex structures can also be interpreted as G - structures.*

Definition 1.34. *(Connection on A Fiber Bundle) Let E be a vector bundle with typical fiber $\mathbb{R}^k/\mathbb{C}^k$ and with structure group G over a manifold M of real n - dimension. If we show the local basis by $\{e_a\} \in E$ on vector bundle E, where $a = 1, \cdots, k$, a Lie algebra $\mathfrak{g}$ - valued connection on this bundle is defined as follow*

$$\nabla : \Gamma(E) \to \Gamma(E \otimes \mathfrak{g} \otimes T^*M), \tag{1.89}$$

which satisfies following conditions:

$$\begin{aligned} \nabla(X + Y) &= \nabla X + \nabla Y, & (1.90) \\ \nabla(fX) &= (\mathrm{d}f)Y + f\nabla X & (1.91) \end{aligned}$$

where $\mathfrak{g}$ is the Lie algebra of structure group G, $f \in \mathcal{C}^\infty(M)$ and $X, Y \in \Gamma(E)$. This connection acts on local basis e_a such that

$$\nabla e_a = A_a^b e_b. \tag{1.92}$$

The covariant derivative of connection ∇, by the same symbol, is as follow

$$\nabla X = \mathrm{d}X + AX, \tag{1.93}$$

where A_a^b is $\mathfrak{g}$ - valued connection 1 - form:

$$A = (A_a^b)_\mu^i \theta_i dx^\mu \in \Lambda^1(M, \mathfrak{g}). \tag{1.94}$$

The exterior covariant derivative of a differential form $\xi \in \Lambda^r(M)$ or $\xi \in \Lambda^r(\mathfrak{g})$ or $\xi \in \Lambda^r(E)$ is such that

$$\mathrm{d}^\nabla \xi = \mathrm{d}\xi + \frac{1}{2}[A, \xi]. \tag{1.95}$$

Definition 1.35. *(Curvature of A Connection on A Fiber Bundle) The* $\mathfrak{g}$ *valued curvature* 2 *- form of a connection on a fiber bundle* E *is given by*

$$\begin{aligned} F = F^a_b &= \mathrm{d}A^a_b + A^a_c \wedge A^c_b \\ &= \sum_{\mu<\nu} (F^a_b)^i_{\mu\nu} \theta_i dx^\mu \wedge dx^\nu \in \Lambda^2(\mathfrak{g}). \end{aligned} \tag{1.96}$$

Shortly it is shown by

$$F = \mathrm{d}^\nabla A = \mathrm{d}A + A \wedge A. \tag{1.97}$$

The Bianchi identity of the curvature is also

$$\mathrm{d}^\nabla F = 0 \tag{1.98}$$

Remark 1.36. *Since the dual basis of a dual vector bundle isn't generally a* 1 *- form, torsion cannot be defined on the vector bundles, except for cotangent bundle.*

1.4 Characteristic Classes

The curvature of a connection on a fiber bundle gives information whether this bundle is twisted or not. If this bundle is trivial, then it has the constant transition functions between all fibers, and so its connection is flat, that is curvature vanishes. Thus it is called that the curvature of a trivial bundle is globally defined. However, if a bundle is non trivial, its curvature can only be defined locally. Thus, the compare of the curvatures of two different connections on the same bundle doesn't be possible in global sense. However, since curvatures are matrix valued 2-forms, the invariant polynomials of the curvature matrices are globally good defined and they have physical meaning, i.e. topological charge.

Definition 1.37. *(Characteristic Class) Let* $\mathfrak{g}$ *be the Lie algebra of a matrix Lie group* $G \in GL(N, \mathbb{R}/\mathbb{C})$*. If* θ_a *are the generators of Lie algebra* $\mathfrak{g}$*, where* $a = 1, \cdots, \dim(G)$*, then the curvature* 2*-form* F *of a connection on a fiber bundle with structure group* G

can be expressed with respect to the generators θ_a as follow

$$F = (F^a)\theta_a, \tag{1.99}$$

where

$$F^a = F^a_{\mu\nu} dx^\mu \wedge dx^\nu \in \Lambda^1(M), \tag{1.100}$$

so we show curvature matrix as

$$F = (F^a)\theta_a = \begin{pmatrix} F^1_1 & F^1_2 & \cdots & F^1_N \\ F^2_1 & F^2_2 & \cdots & F^2_N \\ \vdots & \vdots & \ddots & \vdots \\ F^N_1 & F^N_2 & \cdots & F^N_N \end{pmatrix}, \quad F^i_j \in \Lambda^2(M), \quad i, j = 1, \cdots, N. \tag{1.101}$$

Then, the $k-$th. exterior power of curvature 2-form defines the invariant polynomial of curvature 2-form with $N \times N$ matrix valued such that

$$P(F) = \det[F + \lambda \mathbb{I}] = \sum_{k=0}^{N} \lambda^{N-k} P_k(F), \tag{1.102}$$

where $\mathbb{I}$ is the identity matrix of $N \times N$, λ is a parameter and

$$P_k(F) = \mathrm{Tr}[\underbrace{F \wedge \cdots \wedge F}_{k\ times}]. \tag{1.103}$$

Some values of $P_k(F)$ are as follow

$$\begin{aligned}
P_0(F) &= \mathbb{I} \quad \in \Lambda^0(M), & (1.104)\\
P_1(F) &= \mathrm{Tr}[F] \quad \in \Lambda^2(M), & (1.105)\\
2!P_2(F) &= (\mathrm{Tr}[F])^2 - \mathrm{Tr}[F^2] \quad \in \Lambda^4(M), & (1.106)\\
3!P_3(F) &= (\mathrm{Tr}[F])^3 - 3\mathrm{Tr}[F^2] \wedge \mathrm{Tr}[F] + 2\mathrm{Tr}[F^3] \quad \in \Lambda^6(M), & (1.107)\\
4!P_4(F) &= (\mathrm{Tr}[F])^4 - 6\mathrm{Tr}[F^2] \wedge (\mathrm{Tr}[F])^2 + 3(\mathrm{Tr}[F]^2)^2 \\
&\quad +8\mathrm{Tr}[F^3] \wedge \mathrm{Tr}[F] - 6\mathrm{Tr}[F^4] \quad \in \Lambda^8(M), & (1.108)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Tr}[F^2] &= F^i_j \wedge F^j_i, & (1.109)\\
(\mathrm{Tr}[F]^2)^2 &= (F^i_j \wedge F^j_i) \wedge (F^p_r \wedge F^r_p), & (1.110)\\
\mathrm{Tr}[F^4] &= F^i_j \wedge F^j_p \wedge F^p_r \wedge F^r_i. & (1.111)
\end{aligned}$$

Theorem 1.38. *(Chern - Weil)*

i) $P(F)$ is closed: $\mathrm{d}P(F) = 0$,

ii) If F and F' are the curvatures of two different connections on the same fiber bundle, then $P(F) - P(F')$ is exact:

$$P(F) - P(F') = \mathrm{d}\alpha, \qquad \alpha \in \Lambda^{2k-1}(M). \tag{1.112}$$

Chern Classes: These are defined on complex vector bundles. Let M be a smooth manifold of real n - dimension and E is a complex vector bundle with typical fiber $\mathbb{C}^N$ and structure group $G \in GL(N, \mathbb{C})$. Chern classes are written as follow

$$\begin{aligned} C(E) &= \det\left(\mathbb{I} + \frac{i}{2\pi}F\right) \\ &= \sum_{k=0}^{N} C_k(E) = C_0(E) + C_1(E) + C_2(E) + s, \end{aligned} \tag{1.113}$$

where $C_k(E)$ and $C(E)$ are called Chern classes and complete Chern class, respectively. $C_k(E)$ are generally defined as

$$C_0(E) = \mathbb{I}_{N\times N}, \tag{1.114}$$

$$C_k(E) = \left(\frac{i}{2\pi}\right)^k S_k(F), \tag{1.115}$$

where

$$\begin{aligned} S_k(F) &= \mathrm{Tr}[\underbrace{F \wedge \cdots \wedge F}_{k\ times}] \\ &= \frac{1}{k!} \sum_{1 \le i_k, j_k \le N} \delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} F^{j_1}_{i_1} \wedge F^{j_1}_{i_1} \wedge \cdots \wedge F^{j_k}_{i_k}. \end{aligned} \tag{1.116}$$

Also one defines Chern character

$$ch_k(E) = \mathrm{Tr}\left[\exp\left(\frac{i}{2\pi}\right)\right]^k S_k(F) \tag{1.117}$$

Hence, If S_{2k} is a closed surface of $2k$-dimension over a base manifold M, then the topological invariant of a fiber bundle (also called $k-$th. Chern number) is shown by integral

$$c_k = \int_{S_{2k}} ch_k(E) = Topological\ invariant \tag{1.118}$$

1.5 Geometrical Concepts In Gauge Theories

Let M be a smooth manifold and P a principal G - bundle over this manifold. The curvature 2 - form of a connection ∇ with connection 1 - form A on this bundle is defined as

$$F = \mathrm{d}A + A \wedge A. \tag{1.119}$$

This curvature 2 - form satisfied naturally Bianchi identity

$$\mathrm{d}^{\nabla} F = \mathrm{d}F + A \wedge F - F \wedge A = 0. \tag{1.120}$$

Then, the Yang - Mills action integral is given by

$$S_{YM}[A] = \int_M \mathrm{Tr}[F \wedge *F], \tag{1.121}$$

and Yang - Mills connection (or potential) is the solution of the field equation called Yang - Mills equation which makes extremum this action

$$\mathrm{d}^{\nabla}(*F) = \mathrm{d}(*F) + A \wedge *F - *F \wedge A = 0 \tag{1.122}$$

This equation proves also (anti-) self duality over real 4 - dimension,

$$*F = \pm F, \tag{1.123}$$

because the Bianchi identity of the curvature 2 - form is $\mathrm{d}^{\nabla} F = 0$, so

$$\mathrm{d}^{\nabla}(*F) = 0. \tag{1.124}$$

Particle (matter) fields are interpreted as the local section of a vector bundle and the interaction of matter field by the Yang - Mills field is given by a vector bundle associated to a principle bundle, so that, this principal bundle is naturally generated by the vector bundle. Let E be a vector bundle with typical fiber V and structure group G over a smooth manifold M. Then, this bundle generates a principal G - bundle with the same transition functions and the same structure group over the same manifold M (Kobayashi, 1987), (Steenrod, 1951). Thus, we write this associated bundle (showing by the same symbol that of the vector bundle E) as follow

$$E = P \times_{\Pi} V, \qquad \Pi : G \to GL(V). \tag{1.125}$$

Thus, a matter field interacting by a Yang - Mills field is the section $\phi \in \Gamma(E)$ and its field equations is obtained by an action, i.e

$$S[\phi, A] = \int_M \left(\mathrm{Tr}[F \wedge *F] + \mathcal{L}_{mat}(\phi, A)\right), \tag{1.126}$$

where $F = \mathrm{d}A + A \wedge A$ is the curvature of a connection ∇ on the associated vector bundle E, $*$ the Hodge duality operator and $\mathcal{L}_{mat}$ the matter Lagrangian interacting to Yang - Mills potential. The field equations with respect to the A and ϕ, respectively, are written as

$$\mathrm{d}^{\nabla}(*F) + \frac{\delta \mathcal{L}_{mat}}{\delta A} = 0, \tag{1.127}$$

$$\frac{\delta \mathcal{L}_{mat}}{\delta \phi} = 0. \tag{1.128}$$

1.5.1 Flat Connection Notion In Some Gauge Theories

The best known examples of the gauge theories with the flat connection are Chern - Simons and BF (Background Field) theories. These theories may be summarized from (Popov, 2000). Given a smooth manifold M and a principal G - bundle P over this manifold. Consider a complex vector space V and define a vector bundle associated to principle G - bundle P such that

$$Z = P \times_{\Pi} V \tag{1.129}$$

where $\Pi : \mathfrak{g} \to GL(V)$ and $\mathfrak{g}$ is the Lie algebra of the group G. Let ∇ be a connection on this bundle with connection 1 - form A and its curvature

$$F = \mathrm{d}^{\nabla} A. \tag{1.130}$$

In 3 - dimension the Chern - Simons action is

$$S_{CS}[A] = \int_M \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) \tag{1.131}$$

and the extremum of its variation with respect to the A serves a flat connection such that

$$F = \mathrm{d}A + A \wedge A = 0. \tag{1.132}$$

On the other hand, let M be a smooth manifold of real n - dimension, and given a principal G - bundle. Consider an exterior $n-2$ form, called background field, $B \in \Lambda^{n-2}(M, ad(P))$, where

$$ad(P) = P \times \mathfrak{g}/G \tag{1.133}$$

is the adjoint bundle of the principal G - bundle P. Then a BF action is written as

$$S_{BF}[A, B] = \int_M B \wedge F \tag{1.134}$$

and this action reads the field equations with respect to the B and A such that

$$F = \mathrm{d}^{\nabla} A = 0, \quad and \quad \mathrm{d}^{\nabla} B = 0. \tag{1.135}$$

2. AIM AND SCOPE OF THE STUDY

Physical theories are generally obtained by a variational principle of an action integral including the gauge invariant term(s) of the stress of the gauge potential belonging to a force field and the matter field interacting to this force. In the geometrical sense they correspond to a connection 1 - form on a fiber bundle and its curvature 2 - form. On the other hand, if ∇ is a connection with connection 1 - form A on a fiber bundle, then its covariant derivative can be shown by $\nabla = \mathrm{d} + A$. If ψ is a local section on a fiber bundle, as a physical field which interacts to a force field, then the kinetic term $\nabla\psi \wedge *\nabla\psi$ induces a term such that $A \wedge *A$. However, since a connection A on any fiber bundle doesn't homogeneously transform under a continuous or a local gauge transformation $g \in G$, i.e

$$A' = g^{-1}Ag + g^{-1}\mathrm{d}g, \tag{2.1}$$

the terms $A \wedge A$ and $A \wedge *A$ derived by the kinetic term don't become gauge invariant. In such a case, adding the mass term μ of the gauge field such that $\mu^2 A \wedge *A$ into the physical Lagrangian, as similar to the Higgs mechanism (Higgs, 1964), the gauge fields are acquired by a (spontaneously) symmetry breaking mechanism (Huang, 1992). To add a mass term into a gauge invariant Lagrangian isn't an easy work, because the mass term tries to vanish the gauge field. First trying was made by Proca Proca (1936). The best known example of the massive Abelian gauge fields is the Stückelberg action integral which describes a massive spin - 1 field as an $U(1)$ Maxwell field coupled to a real scalar field. The models gives a mass to the physical photon (Ruegg & Ruiz-Altaba (2004)). However, a massive gauge theory without the symmetry breaking and to escape the non invariant Lagrangian under a local gauge transformation, one may consider the flat connection notion, so that this connection doesn't induces a curvature and then it transforms as a tensorial field such that

$$A' = g^{-1}Ag. \tag{2.2}$$

Every connection are indeed flat at asymptotically, because the connection (or gauge potential) is a pure gauge, i.e

$$|x| \to \infty, \quad A \sim g^{-1}\mathrm{d}g, \quad F = \mathrm{dA} + A \wedge A = 0. \tag{2.3}$$

For example, the Newton and Columb potentials, i.e $\sim \frac{1}{|x|}$, present vanishing force fields at infinity. On the other hand, in the Einstein' s metric theory for the gravitational force the Universe is flat asymptotically, because the Riemann curvature of a spacetime manifold (that is the field stress of the gravitational force) vanish at infinity. Therefore, adding a flat connection 1 - form $A = g^{-1}\mathrm{d}g$ into a physical Lagrangian together with the mass term μ of a massive gauge field such that

$$\mu^2 A \wedge *A, \tag{2.4}$$

then this Lagrangian becomes invariant under a local gauge transformation.

A geometrical structure naturally including flat connection is the holomorphic vector bundles. The Poincarè lemma doesn't generally valid for the covariant exterior derivative acting on vector valued exterior forms. However a holomorphic vector bundle can be constructed if and only if the Poincarè lemma is satisfied by the covariant exterior derivative of the antiholomorphic part of a connection, as well as that all transition function on the bundle are holomorphic (details are below). In this context, the best known example of the flat connection in complex geometry is the Chern connection Kobayashi (1987) which is compatible by each hermitian metric and holomorphic structure. The flat connection concept in this thesis is in the sense of the differential geometry of the holomorphic bundles. In this context, the Yang - Mills type massive and massless gauge theories are interpreted in the geometrical frame of the holomorphic principal bundles over a complex 2 - manifold.

This thesis work is focused on the paper titled with *Massive And Massless Gauge Fields Formed by Flat Connections* (Şener et al., 2016). In chapter 1.1 some geometrical preliminaries belonging to the differential geometry of the differentiable manifolds and fiber bundles are presented. The geometrical concepts in the Yang -

Mills type gauge theories and some using of the flat connection notion in the gauge theories, i.e Chern Simons and BF theories. In chapter 3, firstly one presents the flat connection notion on a holomorphic bundle. Constructing the duality notions on 1 and 2 - forms over complex 2 manifold, a Lagrangian formalism including the mass matrix of the gauge field is given. Next one gives the topological bounds and some solutions to the Lie algebras $\mathfrak{sl}(2,\mathbb{C})$ for the massive gauge fields and $\mathfrak{so}(2,\mathbb{C})$ and $\mathfrak{u}(1)\otimes\mathbb{C}$ for the massless ones. Chapter 5 is the conclusion.

It is seen in this formalism that, although the connection is flat, the component $(1,1)$ of curvature of this connection appears because of the holomorphic structure. Thus it is possible to write a Lagrangian for a Yang - Mills theory including massive and massless gauge fields. However, the mass matrix of a massive gauge field on such a bundle isn't nilpotent and this field is generated by a noncommutative flat connection on the same bundle, then the structure group of this bundle is a non - Abelian complex Lie group. On the other hand, If gauge field is massless, then this is generated by a commutative flat connection, and so the structure group of bundle is an Abelian complex Lie group. Also one sees that second Chern number or topological charge is proportional to the total volume of the base manifold for each massless and massive gauge theories and Abelian (massless) gauge theories which is indeed the theories of Kähler potential on complex projective space $\mathbb{C}P^2$.

3. MATERIALS AND METHODS

We define a complex vector bundle E with the typical fiber space $\mathbb{C}^N$ and structure group G together with a differentiable projection map $[\pi : E \to M, G, \mathbb{C}^N]$ over a complex manifold M. Then the E - valued exterior forms live on the bundle $\Lambda_E^{(r,q)} = \mathcal{C}^\infty(E \otimes \Lambda^{(r,q)})$. Thus the bundle of E - valued K - forms decomposes as

$$\Lambda_E^{(K)} = \bigoplus_{(K=r+q)} \Lambda_E^{(r,q)}. \tag{3.1}$$

Here we will deal with the matter field defined on the adjoint bundle of a principal bundle, i.e Higgs field, monopole. So that, consider a complex manifold M and a complex Lie group G. Since a principal G - bundle is a fiber bundle whose typical fiber is a Lie group G, over a complex manifold one can construct a holomorphic principal bundle, so that the transition functions and the structure group G are holomorphic because a complex Lie group has a holomorphic group morphism together with its inverse. Therefore we define a bundle notion which we call "*holomorphic principal bundle*".

Definition 3.1. *(Holomorphic Principal Bundle) Let M be a complex manifold and G a complex Lie group with Lie algebra $\mathfrak{g}$. Consider a principal G - bundle over this manifold M by a projection map $[\pi : P \to M, G]$. If all transition functions and their inverses on this bundle are holomorphic, then we call the holomorphic principal bundle to this bundle.*

The adjoint representation of any principal G - bundle P over a base manifold M is shown by

$$\mathrm{ad}(P) = P \times \mathfrak{g}/G. \tag{3.2}$$

Thus the exterior forms on this bundle are also the Lie algebra - valued. Hence the bundle of the Lie algebra valued (K) - forms becomes

$$\Lambda_{\mathrm{ad}(P)}^{(K)} = \bigoplus_{(K=r+q)} \Lambda_{\mathrm{ad}(P)}^{(r,q)}, \tag{3.3}$$

where

$$\Lambda^{(r,q)}_{\mathrm{ad}(P)} = \mathcal{C}^{\infty}(\mathrm{ad}(P) \otimes \Lambda^{(r,q)}). \tag{3.4}$$

Hence a connection on the adjoint bundle $\mathrm{ad}(P)$ is defined by the map $\nabla : \Lambda^0_{\mathrm{ad}(P)} \to \Lambda^1_{\mathrm{ad}(P)}$. Considering a local section $\psi \in \Lambda^0(\mathrm{ad}(P))$ a gauge transformation $\psi \to g\psi$ for this section under gauge group $g \in G$ satisfies following covariant differentiation

$$\nabla\psi = \mathrm{d}\psi + A\psi, \tag{3.5}$$

where A is the Lie algebra valued connection 1 - form.

Let $\nabla : \Lambda^0_{\mathrm{ad}(P)} \to \Lambda^1_{\mathrm{ad}(P)}$ be connection on the adjoint bundle $\mathrm{ad}(P)$ of a principal G - bundle P over a complex manifold M, where G is any complex Lie group. The covariant derivative of this connection (shown by the same symbol) $\nabla = \mathrm{d} + A$ decomposes as follow

$$\begin{aligned} \nabla &= \nabla^{(1,0)} \oplus \nabla^{(0,1)}, &(3.6)\\ A &= A^{(1,0)} \oplus A^{(0,1)}, &(3.7) \end{aligned}$$

where A is the connection 1 - form and also

$$\nabla^{(1,0)} = \partial + A^{(1,0)} \qquad \nabla^{(0,1)} = \bar{\partial} + A^{(0,1)}. \tag{3.8}$$

Hereafter we will show by $'$ and $''$ the holomorphic and antiholomorphic parts of a 1 - form, respectively. On the other hand, the exterior covariant derivatives on the differentiable forms on this bundle given by the maps

$$\begin{aligned} \mathrm{d}^{\nabla} &: \Lambda^K_{\mathrm{ad}(P)} \to \Lambda^{K+1}_{\mathrm{ad}(P)}, &(3.9)\\ \partial^{\nabla} &: \Lambda^{(r,q)}_{\mathrm{ad}(P)} \to \Lambda^{(r+1,q)}_{\mathrm{ad}(P)}, &(3.10)\\ \bar{\partial}^{\nabla} &: \Lambda^{(r,q)}_{\mathrm{ad}(P)} \to \Lambda^{(r,q+1)}_{\mathrm{ad}(P)} &(3.11) \end{aligned}$$

are as follow

$$\begin{aligned} \mathrm{d}^{\nabla} &= \partial^{\nabla} \oplus \bar{\partial}^{\nabla}, &(3.12)\\ \partial^{\nabla} &= \partial + A'\wedge, &(3.13)\\ \bar{\partial}^{\nabla} &= \bar{\partial} + A''\wedge, &(3.14) \end{aligned}$$

where $K = r + q$. Then the curvature 2 - form of the connection ∇ on the bundle $\mathrm{ad}(P)$ is given by map $F : \Lambda^0_{\mathrm{ad}(P)} \to \Lambda^2_{\mathrm{ad}(P)}$ and it is written explicitly

$$F = \mathrm{d}^{\nabla} A = \mathrm{d}A + A \wedge A \in \Lambda^2_{\mathrm{ad}(P)} \tag{3.15}$$

and this curvature decomposes over the complex manifold M such that

$$F = F^{(2,0)} \oplus F^{(1,1)} \oplus F^{(0,2)} \tag{3.16}$$

where

$$\begin{aligned} &F^{(2,0)} = \partial^{\nabla} A', \quad F^{(0,2)} = \bar{\partial}^{\nabla} A'', \quad F^{(2,0)} = \overline{F^{(0,2)}}, \\ &F^{(1,1)} = \bar{\partial}^{\nabla} A' + \partial^{\nabla} A''. \end{aligned} \tag{3.17}$$

The parts $F^{(2,0)}$ and $F^{(0,2)}$ are called the curvatures of the holomorphic and antiholomorphic parts of a connection 1 - form A over a complex manifold.

3.1 Flat Connection

We will consider the adjoint bundle of a holomorphic principal bundle, because the curvature of the antiholomorphic part of the connection vanishes and then this part of the connection becomes flat. On the other hand, since $F^{(2,0)} = \overline{F^{(0,2)}}$, the curvature of the holomorphic part vanishes as well as the $F^{(0,2)}$.

In the sense of the complex geometry, a well known example of the flat connection is Chern connection which is compatible to each hermitian metric and holomorphic structure. Let E be a complex vector bundle over a complex manifold M and let $\Psi, \Psi' \in \Lambda^0(E)$ be two local sections which are compatible by Hermitian structure H. A connection D which is compatible by the Hermitian structure preserves this structure as follow

$$\mathrm{d}H(\Psi, \Psi') = H(D\Psi, \Psi') + H(\Psi, D\Psi') \tag{3.18}$$

or makes parallel it (Kobayashi, 1987) such that

$$DH = 0. \tag{3.19}$$

The decomposition of D can be shown as

$$D = D^{(1,0)} + D^{(0,1)} \tag{3.20}$$

and they obey

$$D^{(1,0)} \circ D^{(1,0)} = 0, \quad and \tag{3.21}$$

$$D^{(0,1)} \circ D^{(0,1)} = 0, \tag{3.22}$$

and so the components $(2, 0)$ and $(0, 2)$ of curvature vanish, but, since it will be

$$D \circ D = D^{(1,0)} \circ D^{(0,1)} + D^{(0,1)} \circ D^{(1,0)} = D^{(1,1)}, \tag{3.23}$$

the curvature of Chern connection consists only of the component $(1, 1)$ of the curvature.

Let $\mathrm{ad}(P)$ be the adjoint bundle of a holomorphic principal bundle over a complex manifold M and ∇ a connection with connection 1 - form A on this bundle. This connection decomposes over this manifold such that $\nabla = \nabla' \oplus \nabla''$ and $A = A' \oplus A''$, respectively. Since the connections on fiber bundles aren't generally tensor fields, they have no homogeneous transformation rules under a continuous transformations, i.e gauge transformations, that is, if A is a connection 1 - form and $g \in G$ is a (continuous) gauge transformation, then

$$A \to A' = g^{-1}Ag + g^{-1}\mathrm{d}g. \tag{3.24}$$

However, all elements of a complex Lie groups are holomorphic. Hence, if G is a complex Lie group, then we have

$$\bar{\partial} g = 0, \quad g \in G, \tag{3.25}$$

and so the holomorphic and antiholomorphic parts of a connection 1 - form A on a holomorphic bundle transform as

$$A' \rightarrow g^{-1}A'g + g^{-1}\partial g, \tag{3.26}$$

$$A'' \rightarrow g^{-1}A''g \tag{3.27}$$

Here one easily see that the antiholomorphic part A'' appears a tensorial field and it becomes a flat connection.

On the other hand, the Poincarè Lemma for the ordered exterior derivative d over a smooth manifold isn't valid for the exterior covariant derivative d^{∇} of a connection ∇ on a vector or principal bundle. Therefore, if $\mathrm{d}^{\nabla} = \mathrm{d} + A$ is an exterior covariant derivative, then it is generally

$$\mathrm{d}^{\nabla} \circ \mathrm{d}^{\nabla} \neq 0. \tag{3.28}$$

On the other hand, If

$$\mathrm{d}^{\nabla} \circ \mathrm{d}^{\nabla} = 0, \tag{3.29}$$

then the connection ∇ is flat, so that, if ψ is a local section, then

$$\mathrm{d}^{\nabla} \circ \mathrm{d}^{\nabla}\psi = (\mathrm{d}A + A \wedge A)\psi = F\psi = 0, \tag{3.30}$$

and so vanishing curvature $F = \mathrm{d} + A \wedge A = 0$ is generated by flat connection given as

$$A = g^{-1}\mathrm{d}g, \quad g \in G. \tag{3.31}$$

This relates also in Maurer - Cartan 1 - form defined by

$$\theta = g^{-1}\mathrm{d}g \tag{3.32}$$

of a structure group satisfying Maurer - Cartan equation

$$\mathrm{d}\theta + \theta \wedge \theta = 0 \tag{3.33}$$

for invariant 1 - form fields.

Definition 3.2. *(Maurer - Cartan* 1 *- Form On Complex Lie Groups) Let* $G \subset GL(N, \mathbb{C})$ *be a complex Lie group with Lie algebra* $\mathfrak{g}$*. Also* ∂ *and* $\bar{\partial}$ *be the holomorphic and antiholomorphic exterior derivatives over a complex manifold. Since* $\bar{\partial} g = 0$ *for an element* $g \in G$ *of a complex Lie group G, the Maurer - Cartan* 1 *- form of such a group becomes only*

$$\theta = g^{-1}\partial g \in \Lambda^{(1,0)}(\mathfrak{g}). \tag{3.34}$$

However, a flat connection in the complex sense is a natural features of the holomorphic bundles. Therefore, if ψ is a local section on a holomorphic bundle, then the exterior covariant derivative $\bar{\partial}^{\nabla}$ must satisfies

$$\bar{\partial}^{\nabla} \circ \bar{\partial}^{\nabla}\psi = (\bar{\partial} A'' + A'' \wedge A'')\psi = F^{(0,2)}\psi = 0. \tag{3.35}$$

This means that the curvature of the antiholomorphic part of a connection vanishes because of the holomorphic structure requiring the Poincarè lemma $\bar{\partial}^{\nabla} \circ \bar{\partial}^{\nabla} = 0$, and so the antiholomorphic part of the exterior covariant derivative becomes

$$\bar{\partial}^{\nabla} = \bar{\partial}, \tag{3.36}$$

this is also known as Dolbeult derivative. The curvature $F^{(0,2)}$ of the antiholomorphic part of the connection ∇ then becomes

$$F^{(0,2)} = \bar{\partial}^{\nabla} A'' = \bar{\partial} A'' + A'' \wedge A'' = 0, \tag{3.37}$$

Hence, since $F^{(2,0)} = \overline{F^{(0,2)}}$, the curvature of the holomorphic part A' of the connection also vanishes

$$F^{(2,0)} = \partial^{\nabla} A' = \partial A' + A' \wedge A' = 0, \tag{3.38}$$

but the part $F^{(1,1)}$ doesn't vanish

$$\begin{aligned} \Omega &= \bar{\partial}^{\nabla} A' + \partial^{\nabla} A'' \\ &= \partial A'' + \bar{\partial} A' + A' \wedge A'' + A'' \wedge A' \neq 0. \end{aligned} \tag{3.39}$$

Hereafter we will show by Ω the part $F^{(1,1)}$ of the curvature. Therefore, for $F^{(2,0)} = 0$, the holomorphic part A' of the connection is the Maurer - Cartan 1 - form given in eq. (3.34) of the complex Lie group G

$$A' = \theta = g^{-1}\partial g \in \Lambda^1(\mathfrak{g}), \qquad g \in G. \tag{3.40}$$

On the other hand, for $F^{(0,2)} = 0$ we present a pure gauge potential in physical sense (that is it is at the same time flat connection) such that

$$A'' = \phi^{-1}\bar{\partial}\phi, \tag{3.41}$$

where $\phi \in GL(N, \mathbb{C})$. Since $\bar{\partial}g = 0$, we get

$$\bar{\partial}A' = 0 \tag{3.42}$$

and also locally we have

$$A' \wedge A'' + A'' \wedge A' = 0. \tag{3.43}$$

Hence curvature on a holomorphic bundle consists only of part $(1, 1)$.

Definition 3.3. *(Curvature Of Flat Connection On Adjoint Bundle Of Holomorphic Principal Bundle) Let* $\mathrm{ad}(P)$ *be the adjoint bundle of a holomorphic principal* G *- bundle* P *with connection* 1 *- form* $A = A' + A''$ *and* $\phi \in G$*. Then the total curvature of this connection is*

$$F = \Omega = \partial A'' = \partial(\phi^{-1}\bar{\partial}\phi). \tag{3.44}$$

3.2 Action Principle

In geometrical sense a Lagrangian is n - form if base manifold is of real n - dimension. Thus an action integral is scalar valued differential n - form. On the other hand, in the context of the dynamics of a gauge field, a connection 1 - form induces a current $n-1$ - form. Therefore, over a complex 2 - dimension the current form of a physical field is a 3 form and it decomposes as

$$\Lambda^3 \;=\; \Lambda^{(3,0)} \oplus \Lambda^{(2,1)} \oplus \Lambda^{(1,2)} \oplus \Lambda^{(0,3)}. \tag{3.45}$$

However, considering a duality map $* : \Lambda^{(p,q)} \to \Lambda^{(2-p,2-q)}$ over the complex 2 - dimension, some meaningless situations appear such that

$$*\Lambda^{(3,0)} \to \Lambda^{(-1,2)}, \quad *\Lambda^{(0,3)} \to \Lambda^{(2,-1)}. \tag{3.46}$$

Then the parts $\Lambda^{(3,0)}$ and $\Lambda^{(0,3)}$ must vanish. Thus, for 3 - forms over a complex manifold $M \cong \mathbb{C}^2$ one writes only

$$\Lambda^3 = \Lambda^{(2,1)} \oplus \Lambda^{(1,2)}. \tag{3.47}$$

As will be seen that each gauge field Lagrangian and matter Lagrangian are $(2,2)$ - forms and the current forms of matter Lagrangian consist of the components $(2,1)$ and $(1,2)$ of exterior 3 - form over real 4 - dimension. Similarly, considering the decomposition for the exterior 4 - forms

$$\Lambda^4 = \Lambda^{(4,0)} \oplus \Lambda^{(3,1)} \oplus \Lambda^{(2,2)} \oplus \Lambda^{(1,3)} \oplus \Lambda^{(0,4)}, \tag{3.48}$$

there doesn't exist any geometrical object on the parts $\Lambda^{(4,0)}$, $\Lambda^{(0,4)}$, $\Lambda^{(3,1)}$ and $\Lambda^{(1,3)}$, but there exists only on the part $\Lambda^{(2,2)}$ because of

$$*\Lambda^{(4,0)} \to \Lambda^{(-2,2)}, \quad *\Lambda^{(0,4)} \to \Lambda^{(2,-2)}, \tag{3.49}$$

$$*\Lambda^{(3,1)} \to \Lambda^{(-1,1)}, \quad *\Lambda^{(1,3)} \to \Lambda^{(1,-1)}. \tag{3.50}$$

Thus, we write the decomposition of 4 - forms over complex 2 - dimension as

$$\Lambda^4 = \Lambda^{(2,2)}. \tag{3.51}$$

In a result, the part $\Lambda^{(2,0)}$ and $\Lambda^{(0,2)}$ of the bundle Λ^2 of the 2 - forms over complex 2 - dimension are the bundles of the exact forms on the parts $\Lambda^{(1,0)}$ and $\Lambda^{(0,1)}$ with respect to the exterior derivatives ∂ and $\bar{\partial}$, respectively. At the same way, the parts $\Lambda^{(2,1)}$ and

$\Lambda^{(1,2)}$ of Λ^3 are the bundles of closed forms. Therefore, 3 and 4 - forms over complex 2 - dimension decompose only as $\Lambda^3 = \Lambda^{(2,1)} \oplus \Lambda^{(1,2)}$ and $\Lambda^4 = \Lambda^{(2,2)}$.

Then we must consider $(1,1)$ and $(2,2)$ - forms over complex manifold $M \cong \mathbb{C}^2$ in order to construct a Lagrangian formalism. Besides, the curvature of a connection on the adjoint bundle of a holomorphic principal bundle consist only of part $(1,1)$. In this context, over complex 2 - dimension this form induces a quadratic term as a $(2,2)$ - form, that Yang - Mills invariance. Therefore we request to the duality notions of 1 and 2 forms.

In the sense of (anti-) self dual gauge theories the duality notion is generally given by 2 - forms over real 4 - dimension, so that this duality notion is the Hodge sense for the half dimension of a real even dimension. However, for the exterior forms in more higher or less degree than 2, we cannot directly mention from the Hodge duality notion. Since our connection is $A'' \in \Lambda^{(0,1)}_{\mathrm{ad}(P)}$ and curvature is $\Omega \in \Lambda^{(1,1)}_{\mathrm{ad}(P)}$, we define a duality notion in the Hodge sense for 2 - forms and a different notion for 1 - forms.

Definition 3.4. *(Hodge Duality Over Complex 2 - Dimension) Let $\omega \in \Lambda^{(1,1)}$ be a $(1,1)$ - form over complex 2 - dimension and take a linear map as follows*

$$* : \Lambda^{(1,1)}(\mathbb{C}^2) \to \Lambda^{(1,1)}(\mathbb{C}^2). \tag{3.52}$$

Then we can construct a duality equation under this duality map such that

$$\omega = \rho * \omega \in \Lambda^{(1,1)}, \qquad \rho = \{\mathrm{Eigen}(*)\}. \tag{3.53}$$

Thus the quadratic term of the Lie algebra $\mathfrak{g}$ valued - curvature 2 - form $\omega \in \Lambda^{(1,1)}(\mathbb{C}^2, \mathfrak{g})$ becomes proportional to its the second characteristic polynomial over $\mathbb{C}^2$:

$$\mathrm{Tr}[\omega \wedge *\omega] = \rho \mathrm{Tr}[\omega \wedge \omega] \in \Lambda^{(2,2)}. \tag{3.54}$$

Secondly, we define another duality notion for 1 - forms. Since the wedge product of any 1 - form by its own vanishes, we need any geometrical object valued 1 - form, i.e Lie algebra or matrix.

Definition 3.5. *(Φ - Duality of $(0,1)$ - Forms Over Complex 2 - Dimension) Let $\xi \in \Lambda^{(0,1)}(\mathfrak{g})$ be a $\mathfrak{g}$ - valued $(0,1)$ - form and $\Phi \in \Lambda^{(2,0)}(M)$ a globally defined and covariant constant auxiliary form over a complex 2 manifold M, $\mathrm{d}\Phi = 0$ and $\nabla\Phi = 0$. Let*

$$*_{\Phi} : \Lambda^{(0,1)}(\mathbb{C}^2, \mathfrak{g}) \to \Lambda^{(0,1)}(\mathbb{C}^2, \mathfrak{g}) \tag{3.55}$$

be a linear map acting on bundle $\Lambda^{(0,1)}(\mathbb{C}^2, \mathfrak{g})$. Then we can write a duality equation under this map as follow

$$*_{\Phi}\xi = \epsilon(\xi \wedge \Phi), \tag{3.56}$$

where ϵ is a constant.

Considering these duality notions we define a Lagrangian $(2,2)$ - form of the massive gauge fields as follow.

Definition 3.6. *(Lagrangian $(2,2)$ Over Complex 2 - Dimension) Let $\mathrm{ad}(P)$ be the adjoint bundle of a holomorphic principal bundle over a complex 2 manifold $M \cong \mathbb{C}^2$. Given the antiholomorphic part A'' of a connection on this bundle and its curvature $\Omega = \partial A''$. Therefore, for a local section $\psi \in \Lambda^0_{\mathrm{ad}(P)}$, satisfying $\bar{\partial}\psi \neq 0$, under a linear map*

$$\mathcal{L} : \Lambda^{(0,1)}_{\mathrm{ad}(P)} \times \Lambda^{(1,1)}_{\mathrm{ad}(P)} \to \Lambda^{(2,2)}, \tag{3.57}$$

a Lagrangian $(2,2)$ - form $\mathcal{L}(A, \Omega, \psi, \bar{\partial}\psi)$ is given by

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}[\Omega \wedge *\Omega] + \mathrm{g}_1\mathrm{Tr}[\mu^2 A'' \wedge *_{\Phi}A''] + \mathrm{g}_2\mathrm{Tr}[\mu A'' \wedge *_{\Phi}\bar{\partial}\psi] \\ +\mathrm{Tr}[\bar{\partial}\psi \wedge *_{\Phi}\bar{\partial}\psi] - \mathcal{V}(\psi),\end{aligned} \tag{3.58}$$

where g_1, g_2 are some constants, $\mu \in GL(N, \mathbb{C})$ mass matrix belonging to the gauge field ψ and $\mathcal{V}(\psi) \in \Lambda^{(2,2)}$ potential $(2,2)$ - form for gauge field ψ.

Therefore using the quadratic terms given in eqs. (3.54) and (3.56), the Lagrangian of the massive gauge fields becomes

$$\begin{aligned}\mathcal{L} = \rho\mathrm{Tr}[\Omega \wedge \Omega] + \epsilon\mathrm{g}_1\mathrm{Tr}[\mu^2 A'' \wedge A'' \wedge \Phi] + \mathrm{g}_2\epsilon\mathrm{Tr}[\mu A'' \wedge \bar{\partial}\psi \wedge \Phi] \\ +\epsilon\mathrm{Tr}[\bar{\partial}\psi \wedge \bar{\partial}\psi \wedge \Phi] - \mathcal{V}(\psi).\end{aligned} \tag{3.59}$$

The field and wave equations of this Lagrangian with respect to the A'' and ψ are obtained as, respectively,

$$\begin{aligned} g_1\mu^2 A'' + g_2\mu\bar{\partial}\psi &= 0, & (3.60)\\ \frac{\delta\mathcal{V}(\psi)}{\delta\psi} - g_2\epsilon\mu A''\wedge A''\wedge\Phi &= 0. & (3.61)\end{aligned}$$

where we used relation $F^{(0,2)} = \bar{\partial}A'' + A''\wedge A'' = 0$. At the limit $\mu\to 0$, the wave equation reads

$$\textbf{??}\frac{\delta\mathcal{V}(\psi)}{\delta\psi} = 0 \tag{3.62}$$

and the solution(s)

$$\psi_0 = \left\{\psi\in\Lambda^0_{\mathrm{ad}(P)}|\frac{\delta\mathcal{V}(\psi)}{\delta\psi} = 0\right\} \tag{3.63}$$

creates the ground level(s) for the massless gauge fields. Thus the Lagrangian belonging to massless gauge fields is written as

$$\mathcal{L} = \mathrm{Tr}[\Omega\wedge *\Omega] + \mathrm{Tr}[\bar{\partial}\psi\wedge *_\Phi\bar{\partial}\psi] - \mathcal{V}(\psi). \tag{3.64}$$

On the other hand, If we write the field equation (3.60) as

$$\mu^2(g_1 A'' + g_2\mu^{-1}\bar{\partial}\psi) = 0, \tag{3.65}$$

then this equation has a meaning either the square of mass matrix vanish or not. If $\mu^2\neq 0$, then it requires

$$g_1 A'' + g_2\mu^{-1}\bar{\partial}\psi = 0. \tag{3.66}$$

If we derive the equation (3.66) by $\bar{\partial}$, then we have

$$g_1\bar{\partial}A'' = 0. \tag{3.67}$$

Therefore, either $g_1 = 0$ or $\bar{\partial}A'' = 0$. Suppose that $g_1 = 0$. Then the Lagrangian of the massive gauge field is

$$\mathcal{L} = \rho\mathrm{Tr}[\Omega\wedge\Omega] + \epsilon g_2\mathrm{Tr}[\mu A''\wedge\bar{\partial}\psi\wedge\Phi] - \mathcal{V}(\psi) \tag{3.68}$$

and, since it will be $\mathrm{g}_2 \neq 0$, this Lagrangian reads a field equation with respect to the A'' as follow

$$\mu\bar{\partial}\psi = \bar{\partial}(\mu\psi) = 0. \tag{3.69}$$

If $\det[\mu] = 0$, that is μ is a degenerate matrix, then μ^{-1} is undefined and in this case field eq. (3.65) gives $\mu^2 = 0$ or $A'' = 0$. As a result, if the mass matrix μ is a nilpotent, then the part A'' of the connection becomes unmeaningful, i.e $A'' = 0$. Therefore, the mass matrix cannot be nilpotent. On the other hand, in eq. (3.69) ψ is either a matrix - valued holomorphic 0 - form or a closed 0 - form. Then it is written

$$\mu\psi = \mathrm{C}, \qquad \mathrm{C} \in \Lambda^0(GL(N, \mathbb{C})) \tag{3.70}$$

and this equation is expanded as $\mu^a_c\psi^c_b = \mathrm{C}^a_b$. Therefore the massive gauge field ψ is obtained as the algebraic solution of this linear equation:

$$\psi = \mu^{-1}\mathrm{C}. \tag{3.71}$$

On the other hand, if $\bar{\partial}A'' = 0$, then $\bar{\partial}A'' = A'' \wedge A'' = 0$, namely connection A'' is commutative and the structure group of the bundle is Abelian. Therefore, the cases of $\mathrm{g}_1 = 0$ and the commutative connection A'' on the bundle with Abelian structure group are equivalent. Hence, in each case the wave equation (3.61) leads to the massless limit, and so the gauge fields have ground level(s)

$$\mu \to 0, \qquad \psi_0 = \left\{ \psi \in \Lambda^0_{\mathrm{ad}(P)} | \frac{\delta\mathcal{V}(\psi)}{\delta\psi} = 0 \right\}. \tag{3.72}$$

This means that if a connection on the adjoint bundle of a holomorphic principal bundle is commutative flat, then this case presents a massless gauge field. Therefore, as a result, a connection forming the massless gauge fields on the adjoint bundle of a holomorphic principal bundle is commutative and the structure group of this bundle is Abelian. However, if this connection is noncommutative, that is the structure group is non - Abelian, then the gauge fields are massive.

Here we will use the $\mathfrak{sl}(2,\mathbb{C})$ (for the massive gauge fields), $\mathfrak{so}(2,\mathbb{C})$ and $\mathfrak{u}(1)\otimes\mathbb{C}$ (for the massless gauge fields) - valued exterior forms. Since the Lie algebras $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(2,\mathbb{C})$ are traceless, we can write

$$\Omega = \begin{pmatrix} f_1 & f_2 \\ f_3 & -f_1 \end{pmatrix}, \qquad f_A = f_{(A)\mu\bar{\nu}} dz^{\mu} \wedge dz^{\bar{\nu}} \in \Lambda^{(1,1)} \tag{3.73}$$

$$A'' = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & -\gamma_1 \end{pmatrix}, \qquad \gamma_A = \gamma_{(A)\bar{\mu}} dz^{\bar{\mu}} \in \Lambda^{(0,1)}. \tag{3.74}$$

On the other hand, since $A'' = \phi^{-1}\bar{\partial}\phi$, the matrix $\phi \in GL(2,\mathbb{C})$ is as follow

$$\phi = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix}, \ \phi_A \in \mathcal{C}^{\infty}(M) \tag{3.75}$$

Then the connection matrix we find as

$$A'' = \frac{1}{\det[\phi]} \begin{pmatrix} \phi_4\bar{\partial}\phi_1 - \phi_2\bar{\partial}\phi_3 & \phi_4\bar{\partial}\phi_2 - \phi_2\bar{\partial}\phi_4 \\ -\phi_3\bar{\partial}\phi_1 + \phi_1\bar{\partial}\phi_3 & -\phi_3\bar{\partial}\phi_2 + \phi_1\bar{\partial}\phi_4 \end{pmatrix}. \tag{3.76}$$

To be the $\mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{so}(2,\mathbb{C})$ valued of this matrix, it must be

$$\phi_4\bar{\partial}\phi_1 - \phi_2\bar{\partial}\phi_3 - \phi_3\bar{\partial}\phi_2 + \phi_1\bar{\partial}\phi_4 = 0. \tag{3.77}$$

Indeed this is

$$\bar{\partial}(\phi_1\phi_4 - \phi_2\phi_3) = \bar{\partial}(\det[\phi]) = 0. \tag{3.78}$$

Then either $\det[\phi] = 1$ or ϕ is a holomorphic matrix. If it is holomorphic, then the connection vanishes. Hence it should not be holomorphic. Then

$$\det[\phi] = 1. \tag{3.79}$$

If we set as

$$\phi_4 = \frac{1 + \phi_2\phi_3}{\phi_1}. \tag{3.80}$$

then the connection matrix we rewrite as

$$A'' = \begin{pmatrix} \left(\frac{1+\phi_2\phi_3}{\phi_1}\right)\bar{\partial}\phi_1 - \phi_3\bar{\partial}\phi_2 & \phi_2^2\bar{\partial}\left(\frac{1+\phi_2\phi_3}{\phi_1\phi_2}\right) \\ \phi_1^2\bar{\partial}(\frac{\phi_3}{\phi_1}) & \phi_3\bar{\partial}\phi_2 - \left(\frac{1+\phi_2\phi_3}{\phi_1}\right)\bar{\partial}\phi_1 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & -\gamma_1 \end{pmatrix} \tag{3.81}$$

Thus its curvature is

$$\Omega = \partial A'' = \begin{pmatrix} f_1 & f_2 \\ f_3 & -f_1 \end{pmatrix} = \begin{pmatrix} \partial\gamma_1 & \partial\gamma_2 \\ \partial\gamma_3 & -\partial\gamma_1 \end{pmatrix}. \tag{3.82}$$

Therefore we can constructe a massive gauge theory using $\mathfrak{sl}(2,\mathbb{C})$ and massless one by $\mathfrak{so}(2,\mathbb{C})$. Hence the Lagrangian for the massive gauge fields which are invariant under the group $SL(2,\mathbb{C})$ is

$$\begin{aligned}\mathcal{L}_{Massive} = \mathrm{Tr}[\Omega\wedge *\Omega] + \mathrm{g}_1\mathrm{Tr}[\mu^2 A''\wedge *_\Phi A''] + \mathrm{g}_2\mathrm{Tr}[\mu A''\wedge *_\Phi \bar{\partial}\psi] \\ +\mathrm{Tr}[\bar{\partial}\psi\wedge *_\Phi\bar{\partial}\psi] - \mathcal{V}(\psi),\end{aligned} \tag{3.83}$$

where $\psi\in\mathfrak{sl}(2,\mathbb{C})$, $A''\in\Lambda^{(0,1)}(\mathfrak{sl}(2,\mathbb{C}))$ and $\Omega\in\Lambda^{(1,1)}(\mathfrak{sl}(2,\mathbb{C}))$.

On the other hand, we know that $\mathfrak{so}(2)\cong\mathfrak{u}(1)$ and $\mathfrak{so}(2,\mathbb{C}) = \mathfrak{so}(2)\otimes_{\mathbb{R}}\mathbb{C}$. Therefore, we can write a morphism such that $\mathfrak{so}(2,\mathbb{C})\cong\mathfrak{u}(1)\otimes_{\mathbb{R}}\mathbb{C}$. Here we show as

$$\mathfrak{u}(1,\mathbb{C}) = \mathfrak{u}(1)\otimes_{\mathbb{R}}\mathbb{C}. \tag{3.84}$$

Thus the Lagrangian for massless gauge fields which is invariant under group the $SO(2,\mathbb{C})$ or $U(1)\otimes_{\mathbb{R}}\mathbb{C}$ is

$$\mathcal{L}_{Massless} = \mathrm{Tr}[\Omega\wedge *\Omega] + \mathrm{Tr}[\bar{\partial}\psi\wedge *_\Phi\bar{\partial}\psi] - \mathcal{V}(\psi), \tag{3.85}$$

where

$$\psi\in\mathfrak{so}(2,\mathbb{C}),\quad A''\in\Lambda^{(0,1)}(\mathfrak{so}(2,\mathbb{C}))\ \ and\ \ \Omega\in\Lambda^{(1,1)}(\mathfrak{so}(2,\mathbb{C})),$$

or

$$\psi\in\mathfrak{so}(\mathfrak{u}(1,\mathbb{C})),\quad A''\in\Lambda^{(0,1)}(\mathfrak{u}(1,\mathbb{C}))\ \ and\ \ \Omega\in\Lambda^{(1,1)}(\mathfrak{u}(1,\mathbb{C})).$$

3.3 Topological Bounds

Topologically, an action integral of a gauge field has a boundary determined by the characteristic classes of the bundle. In real 4 - dimension it is

$$\int_M \mathrm{Tr}[F\wedge *F] \geq \int_M \mathrm{Tr}[F\wedge F], \tag{3.86}$$

where term $\int_M \mathrm{Tr}[F \wedge F]$ is related to the Chern classes of the bundle. Since we have complex 2 - dimension, then we consider only the first and the second Chern classes for the curvature 2 - form. Thus we write

$$\begin{aligned} \mathrm{ch}_1 &= \frac{i}{2\pi}\mathrm{Tr}[\Omega], & (3.87)\\ \mathrm{ch}_2 &= \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 (\mathrm{Tr}[\Omega] \wedge \mathrm{Tr}[\Omega] - \mathrm{Tr}[\Omega \wedge \Omega]). & (3.88)\end{aligned}$$

Here $\mathrm{Tr}[\mathfrak{sl}(2,\mathbb{C})] = 0$ and $\mathrm{Tr}[\mathfrak{so}(2,\mathbb{C})] = 0$. Then the first classes vanishes for $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(2,\mathbb{C})$ - valued curvatures, that is, if $\Omega \in \Lambda^{(1,1)}(\mathfrak{so}(2,\mathbb{C}))$ or $\Omega \in \Lambda^{(1,1)}(\mathfrak{sl}(2,\mathbb{C}))$, then

$$\mathrm{ch}_1 = \frac{i}{2\pi}\mathrm{Tr}[\Omega] = 0. \tag{3.89}$$

Thus topological bound is only controlled by the second Chern class

$$\mathrm{ch}_2 = \frac{1}{8\pi^2}\mathrm{Tr}[\Omega \wedge \Omega]. \tag{3.90}$$

On the other hand, the Chern class for $\Omega \in \Lambda^{(1,1)}(\mathfrak{u}(1,\mathbb{C}))$ doesn't vanish the first Chern class,

$$\mathrm{ch}_1 = \frac{i}{2\pi}\mathrm{Tr}[\Omega] \neq 0, \tag{3.91}$$

but the second one is undefined. Then we can write a more general topological bound over complex 2 - dimension

$$\int_M \mathrm{Tr}[\Omega \wedge *\Omega] \geq \int_M \mathrm{ch}_2 \quad for \quad \mathfrak{so}(2,\mathbb{C}) \quad and \quad \mathfrak{sl}(2,\mathbb{C}) \tag{3.92}$$

or

$$\int_M \mathrm{Tr}[\Omega \wedge *\Omega] \geq \int_M \mathrm{ch}_1 \wedge \mathrm{ch}_1 \quad for \quad \mathfrak{u}(1,\mathbb{C}). \tag{3.93}$$

3.4 Massless (Abelian) Gauge Theory

The Lagrangian for the massless gauge fields which is invariant under group $SO(2,\mathbb{C})$ or $U(1,\mathbb{C})$ is

$$\mathcal{L}_{Massless} = \mathrm{Tr}[\Omega \wedge *\Omega] + \mathrm{Tr}[\bar{\partial}^{\nabla}\psi \wedge *_{\Phi}\bar{\partial}^{\nabla}\psi] - \mathcal{V}(\psi), \tag{3.94}$$

where

$$\psi \in \mathfrak{so}(2,\mathbb{C}),\ A'' \in \Lambda^{(0,1)}(\mathfrak{so}(2,\mathbb{C})),\ \Omega \in \Lambda^{(1,1)}(\mathfrak{so}(2,\mathbb{C})),$$

or

$$\psi \in \mathfrak{u}(1,\mathbb{C}),\ A'' \in \Lambda^{(0,1)}(\mathfrak{u}(1,\mathbb{C})),\ \Omega \in \Lambda^{(1,1)}(\mathfrak{u}(1,\mathbb{C})).$$

This Lagrangian reads the field and wave equations with respect to the A'' and ψ, respectively, as follow

$$\bar{\partial}^{\nabla} * A'' + \psi * \bar{\partial}^{\nabla}\psi = 0, \tag{3.95}$$

$$\frac{\delta\mathcal{V}(\psi)}{\delta\psi} + \bar{\partial}^{\nabla}(*\bar{\partial}^{\nabla}\psi) = 0. \tag{3.96}$$

Since $\bar{\partial}^{\nabla} = \bar{\partial}$, because of the holomorphic structure and eq. (3.56), one gets

$$\bar{\partial}^{\nabla}(*\bar{\partial}^{\nabla}\psi) = 0, \tag{3.97}$$

and so these equation are rewritten as

$$\bar{\partial} * A'' + \psi * \bar{\partial}\psi = 0, \tag{3.98}$$

$$\frac{\delta\mathcal{V}(\psi)}{\delta\psi} = 0. \tag{3.99}$$

Since we have complex 2 - dimension, then we consider only the first and the second Chern classes. Thus we write

$$\mathrm{ch}_1 = \frac{i}{2\pi}\mathrm{Tr}[\Omega], \tag{3.100}$$

$$\mathrm{ch}_2 = \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 (\mathrm{Tr}[\Omega] \wedge \mathrm{Tr}[\Omega] - \mathrm{Tr}[\Omega \wedge \Omega]). \tag{3.101}$$

Here $\mathrm{Tr}[\mathfrak{so}(2,\mathbb{C})] = 0$. Then the first Chern class for $\mathfrak{so}(2,\mathbb{C})$ - valued curvatures vanishes

$$\mathrm{ch}_1 = \frac{i}{2\pi}\mathrm{Tr}[\Omega] = 0. \tag{3.102}$$

Thus the topological bound is controlled by the second Chern class

$$\mathrm{ch}_2 = \frac{1}{8\pi^2}\mathrm{Tr}[\Omega \wedge \Omega]. \tag{3.103}$$

On the other hand, for $\mathfrak{u}(1, \mathbb{C})$ the first Chern class doesn't vanish

$$\mathrm{ch}_1 = \frac{i}{2\pi}\mathrm{Tr}[\Omega] \neq 0, \tag{3.104}$$

however the second one is undefined. Therefore the action integral of the massless gauge fields is bounded such that

$$\begin{aligned} \int_M \big(\mathrm{Tr}[\Omega \wedge *\Omega] + \mathrm{gTr}[\bar{\partial}\psi \wedge *_\Phi \bar{\partial}\psi] - \mathcal{V}(\psi)\big) \;\; &\geq \;\; \int_M \mathrm{ch}_2, \quad for \;\; \mathfrak{so}(2, \mathbb{C}) \;\; or \\ &\geq \;\; \int_M \mathrm{ch}_1 \wedge \mathrm{ch}_1, \quad for \;\; \mathfrak{u}(1, \mathbb{C}). \end{aligned} \tag{3.105}$$

3.4.1 $\mathfrak{so}(2, \mathbb{C})$ - Valued Gauge Theory

For Lie algebra $\mathfrak{so}(2, \mathbb{C})$ we have

$$\mathrm{ch}_2 = \frac{1}{8\pi^2}\mathrm{Tr}[\Omega \wedge \Omega]. \tag{3.106}$$

Therefore, since the term $\mathrm{Tr}[\Omega \wedge \Omega]$ associates to the second Chern class ch_2, then we write following topological bound

$$\frac{1}{(1 - 8\pi^2)} \int_M \big(\mathrm{gTr}[\bar{\partial}\psi \wedge *_\Phi \bar{\partial}\psi] - \mathcal{V}(\psi)\big) \geq \int_M \mathrm{ch}_2. \tag{3.107}$$

Considering a fixed connection and a fixed gauge field (ψ_0), then we have

$$\frac{1}{(1 - 8\pi^2)} \int_M \mathcal{V}(\psi_0) \leq \int_M \mathrm{ch}_2. \tag{3.108}$$

Since the generator of the Lie algebra $\mathfrak{so}(2, \mathbb{C})$ is

$$\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.109}$$

then the $\mathfrak{so}(2,\mathbb{C})$ valued connection $(0,1)$ - form and its curvature $(1,1)$ - form are written as

$$A'' = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \ \gamma = \gamma_{\bar{\mu}} dz^{\bar{\mu}} \in \Lambda^{(0,1)}, \tag{3.110}$$

$$\Omega = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}, \ f = f_{\mu\bar{\nu}} dz^{\mu} \wedge dz^{\bar{\nu}} \in \Lambda^{(1,1)}. \tag{3.111}$$

Considering eq. (3.81) for the $\mathfrak{so}(2,\mathbb{C})$ - valued connection, then it must be

$$\left(\frac{1+\phi_2\phi_3}{\phi_1}\right)\bar{\partial}\phi_1 = \phi_3\bar{\partial}\phi_2, \tag{3.112}$$

$$\phi_1^2\bar{\partial}(\frac{\phi_3}{\phi_1}) = -\phi_2^2\bar{\partial}\left(\frac{1+\phi_2\phi_3}{\phi_1\phi_2}\right) \tag{3.113}$$

If one rewrites the first condition as

$$\left(\frac{1+\phi_2\phi_3}{\phi_1\phi_3}\right)\bar{\partial}\phi_1 = \bar{\partial}\phi_2, \tag{3.114}$$

then the wedge derivatives of two side become

$$\bar{\partial}\left(\frac{1+\phi_2\phi_3}{\phi_1\phi_3}\right) \wedge \bar{\partial}\phi_1 = 0. \tag{3.115}$$

Therefore, if the wedge product of two 1 - forms vanihes, then these forms are the same. Then one gets

$$\phi_3 = \frac{1}{\phi_1^2 - \phi_2}. \tag{3.116}$$

On the other hand, from the second condition one finds

$$\bar{\partial}\bar{\partial}(\frac{\phi_3}{\phi_1}) = -\frac{\phi_2}{\phi_1}\bar{\partial}(\frac{\phi_2}{\phi_1}) \wedge \bar{\partial}\left(\frac{1+\phi_2\phi_3}{\phi_1\phi_2}\right). \tag{3.117}$$

From here one gets

$$\phi_3 = \phi_2^2 - \phi_2 - 1, \qquad \phi_1^2 = \phi_2 + \frac{1}{\phi_2^2 - \phi_2 - 1}. \tag{3.118}$$

Then, defining a new smooth function such that

$$\tilde{\phi}^2 = \left(\frac{\phi_3}{\phi_1}\right)^2 = \frac{\phi_2^2 - \phi_2 - 1}{\phi_2} \tag{3.119}$$

one writes

$$\left(\frac{\phi_3}{\phi_1}\right)^2 = \frac{\phi_2^2 - \phi_2 - 1}{\phi_2} \tag{3.120}$$

Therefore the element of the connection matrix is obtained as

$$\gamma = -\bar{\partial}(\frac{\phi_3}{\phi_1}) = -\bar{\partial}\tilde{\phi}. \tag{3.121}$$

The components of the curvature matrix of this connection becomes

$$f = \partial\gamma = -\partial\bar{\partial}\tilde{\phi}. \tag{3.122}$$

Thus we have

$$\begin{aligned} \mathrm{ch}_2 &= \frac{1}{8\pi^2}\mathrm{Tr}[\Omega \wedge \Omega] = -\frac{2}{8\pi^2} f \wedge f \\ &= -\frac{1}{4\pi^2}\langle \partial\bar{\partial}\tilde{\phi}, \partial\bar{\partial}\tilde{\phi}\rangle \mathrm{dVol}, \end{aligned} \tag{3.123}$$

where

$$\partial\bar{\partial}\tilde{\phi} \wedge \partial\bar{\partial}\tilde{\phi} = \langle \partial\bar{\partial}\tilde{\phi}, \partial\bar{\partial}\tilde{\phi}\rangle \mathrm{dVol} \tag{3.124}$$

On the other hand, we can write the potential $(2,2)$ - form $\mathcal{V}(\psi)$ such that

$$\mathcal{V}(\psi_0) = V(\psi_0)\mathrm{dVol}. \tag{3.125}$$

Considering the classic field theories, the potential function are generally given for a field ψ by

$$V(\psi) \sim \mathrm{Tr}[\psi^2] + (\mathrm{Tr}[\psi^2])^2 + \mathrm{Tr}[\psi^4] + \cdots . \tag{3.126}$$

Thus we can write the $\mathfrak{so}(2, \mathbb{C})$ - valued gauge field as

$$\psi = \begin{pmatrix} 0 & \psi_1 \\ -\psi_1 & 0 \end{pmatrix}, \tag{3.127}$$

where $\psi_1 \in \mathcal{C}^\infty(M)$. Therefore we get

$$\mathrm{Tr}[\psi^2] = -2\psi_1^2 \tag{3.128}$$

and the potential function is rewritten such that

$$\begin{aligned} V(\psi) &\sim \mathrm{Tr}[\psi^2] + (\mathrm{Tr}[\psi^2])^2 + \mathrm{Tr}[\psi^4] + \cdots \\ &= -2a\psi_1^2 + b\psi_1^4, \end{aligned} \tag{3.129}$$

where a, b are constants. Therefore the fixed values ψ_0 via wave equation given in eq. (3.61),

$$\psi_0 = \left\{ \psi \in \Lambda^0_{\mathrm{ad}(P)} | \frac{\delta \mathcal{V}(\psi)}{\delta \psi} = 0 \right\}, \tag{3.130}$$

are obtained from relation

$$\frac{\delta \mathcal{V}(\psi)}{\delta \psi} = 4\psi_1(-a + b\psi_1^2) = 0 \tag{3.131}$$

as non - zero

$$\psi_0 = \pm \sqrt{\frac{a}{b}}. \tag{3.132}$$

In result the value of the potential function at fixed value ψ_0 becomes

$$V(\psi_0) = -\frac{a^2}{b} = -a\psi_0^2, \tag{3.133}$$

Then we get

$$\int_M \left(\frac{-a\psi_0^2}{(1 - 8\pi^2)} + \frac{1}{4\pi^2} \langle \partial\bar{\partial}\tilde{\phi}, \partial\bar{\partial}\tilde{\phi} \rangle \right) \mathrm{dVol} \leq 0, \tag{3.134}$$

and so the massless gauge theory of the $\mathfrak{so}(2, \mathbb{C})$ - valued flat connection over complex 2 manifold is given by field equation

$$\langle \partial\bar{\partial}\tilde{\phi}, \partial\bar{\partial}\tilde{\phi} \rangle = \frac{4a\pi^2\psi_0^2}{(1 - 8\pi^2)}. \tag{3.135}$$

Therefore the topological charge, that is the second Chern number, determined by the second Chern class ch_2 is

$$Q = \mathrm{c}_2 = \int_M \mathrm{ch}_2 = -\frac{a\psi_0^2}{(1 - 8\pi^2)} \int_M \mathrm{dVol}. \tag{3.136}$$

Then the fixed value ψ_0 of the gauge field depends on the topological charge. Since the integral of the dVol over the base manifold is total volume measure of the base manifold, we may show it by

$$Volume = \int_M \mathrm{dVol}, \tag{3.137}$$

then

$$\psi_0^2 = -\frac{Q(1-8\pi^2)}{a \times Volume}, \tag{3.138}$$

and so the field equation then reads

$$\langle \partial\bar{\partial}\tilde{\phi}, \partial\bar{\partial}\tilde{\phi} \rangle = \frac{4a\pi^2 Q}{Volume}. \tag{3.139}$$

3.4.2 $\mathfrak{u}(1,\mathbb{C})$ - Valued Gauge Theory

Since the second Chern class for Lie algebra $\mathfrak{u}(1,\mathbb{C})$ is undefined, the term $\mathrm{Tr}[\Omega \wedge \Omega]$ vanishes and we have only a topological invariant over complex 2 - dimension as follow

$$\mathrm{ch}_1 \wedge \mathrm{ch}_1 = -\frac{1}{4\pi^2}\mathrm{Tr}[\Omega] \wedge \mathrm{Tr}[\Omega]. \tag{3.140}$$

Thus the Lagrangian of the massless gauge fields is given by

$$\mathcal{L}_{Massless} = \mathrm{Tr}[\bar{\partial}\psi \wedge *_\Phi \bar{\partial}\psi] - \mathcal{V}(\psi) \tag{3.141}$$

and this Lagrangian reads a topological bound such that

$$\int_M \left(\mathrm{gTr}[\bar{\partial}\psi \wedge *_\Phi \bar{\partial}\psi] - \mathcal{V}(\psi) \right) \geq \int_M \mathrm{ch}_1 \wedge \mathrm{ch}_1. \tag{3.142}$$

Considering an fixed gauge field ψ_0, then

$$\int_M \mathcal{V}(\psi_0) \leq \int_M \mathrm{ch}_1 \wedge \mathrm{ch}_1. \tag{3.143}$$

Since $\phi \in U(1,\mathbb{C})$ is a smooth scalar, we have

$$\begin{aligned} A'' &= \phi^{-1}\bar{\partial}\phi = \bar{\partial}\ln|\phi|, && (3.144) \\ \Omega &= \partial A'' = \partial\bar{\partial}\ln|\phi|, && (3.145) \end{aligned}$$

so,

$$\begin{aligned} \mathrm{ch}_1 \wedge \mathrm{ch}_1 &= -\frac{1}{4\pi^2}\mathrm{Tr}[\Omega] \wedge \mathrm{Tr}[\Omega] \\ &= -\frac{1}{4\pi^2}\langle \partial\bar{\partial}\ln|\phi|, \partial\bar{\partial}\ln|\phi|\rangle \mathrm{dVol}, && (3.146) \end{aligned}$$

where

$$\partial\bar{\partial}\ln|\phi| \wedge \partial\bar{\partial}\ln|\phi| = \langle \partial\bar{\partial}\ln|\phi|, \partial\bar{\partial}\ln|\phi|\rangle \mathrm{dVol}. \tag{3.147}$$

Therefore the topological charge is determined by the wedge square of the first Chern class ch_1 by

$$Q = \int_M \mathrm{ch}_1 \wedge \mathrm{ch}_1 = -\frac{1}{4\pi^2}\int_M \langle \partial\bar{\partial}\ln|\phi|, \partial\bar{\partial}\ln|\phi|\rangle \mathrm{dVol}. \tag{3.148}$$

Since $\psi \in \mathfrak{u}(1,\mathbb{C})$, we get

$$\mathrm{Tr}[\psi^2] = \psi^2 \tag{3.149}$$

and for the potential $(2,2)$ - form $\mathcal{V}(\psi)$ given such that

$$\mathcal{V}(\psi_0) = V(\psi_0)\mathrm{dVol}. \tag{3.150}$$

The potential function is rewritten such that

$$\begin{aligned} V(\psi) &\sim \mathrm{Tr}[\psi^2] + (\mathrm{Tr}[\psi^2])^4 + \mathrm{Tr}[\psi^2] + \cdots \\ &= 2a\psi^2 + b\psi^4, && (3.151) \end{aligned}$$

where a, b are constants. Therefore the fixed values ψ_0 of the gauge field ψ given by

$$\psi_0 = \left\{ \psi \in \Lambda^0_{\mathrm{ad}(P)} | \frac{\delta\mathcal{V}(\psi)}{\delta\psi} = 0 \right\}, \tag{3.152}$$

are obtained from relation

$$\frac{\delta \mathcal{V}(\psi)}{\delta \psi} = 4\psi(a + b\psi^2) = 0 \tag{3.153}$$

as non - zero

$$\psi_0 = \pm\sqrt{-\frac{a}{b}}. \tag{3.154}$$

In result the potential function becomes

$$V(\psi_0) = -\frac{a^2}{b} = -a\psi_0^2 \tag{3.155}$$

and we have

$$\int_M \left(-a\psi_0^2 + \frac{1}{4\pi^2}\langle \partial\bar{\partial}\ln|\phi|, \partial\bar{\partial}\ln|\phi|\rangle\right) \mathrm{dVol} \leq 0. \tag{3.156}$$

Then the topological charge is written as

$$Q = \int_M \mathrm{ch}_1 \wedge \mathrm{ch}_1 = -a\psi_0^2 \int_M \mathrm{dVol}, \tag{3.157}$$

and so we have

$$\psi_0^2 = \frac{Q}{a \times Volume}, \tag{3.158}$$

that is the fixed value ψ_0 depends on the topological charge. Thus, the massless gauge theory of the $\mathfrak{u}(1,\mathbb{C})$ - valued flat connection over complex 2 manifold is given by field equation

$$\langle \partial\bar{\partial}\ln|\phi|, \partial\bar{\partial}\ln|\phi|\rangle = -\frac{Q}{Volume}. \tag{3.159}$$

3.4.3 Massless Gauge Theories Associated To $\mathbb{C}P^2$

We obtained some geometrical notions associated to the topological invariants of the $\mathfrak{so}(2,\mathbb{C})$ and $\mathfrak{u}(1,\mathbb{C})$ - valued curvatures. Common point at that of the $\mathfrak{so}(2,\mathbb{C})$

and $\mathfrak{u}(1,\mathbb{C})$ is that each curvature (component) is proportional to a term such that $\partial\bar{\partial}\phi$, where $\phi \in \mathcal{C}^{\infty}(M)$. The components of $\mathfrak{so}(2,\mathbb{C})$ and $\mathfrak{u}(1,\mathbb{C})$ - valued curvatures given in eqs. (3.122) and (3.144) we can summarize as follow

$$f = \begin{cases} -\partial\bar{\partial}\tilde{\phi} & if\ \mathfrak{so}(2,\mathbb{C}) - valued, \\ \partial\bar{\partial}\ln|\phi| & if\ \mathfrak{u}(1,\mathbb{C}) - valued \end{cases} \tag{3.160}$$

Therefore, generally a term such that $\partial\bar{\partial}\phi$ for a smooth scalar indeed indicates a Kähler potential and its fundamental 2 - form. Then we can say that each of these curvatures behaves as a Kähler form.

Let M be a Kähler manifold of complex n - dimension and

$$ds^2 = \mathrm{g}_{\mu\bar{\nu}} dz^{\mu} \otimes dz^{\bar{\nu}} \tag{3.161}$$

hermitian metric over this manifold. Therefore, the Kähler 2 - form is

$$K = \mathrm{g}_{\mu\bar{\nu}} dz^{\mu} \wedge dz^{\bar{\nu}} = \frac{i}{2}\partial\bar{\partial}\phi, \tag{3.162}$$

where we call to the ϕ Kähler potential. If

$$\mathrm{dVol} = dz^1 \wedge dz^{\bar{2}} \wedge \cdots \wedge dz^n \wedge dz^{\bar{n}} \tag{3.163}$$

is volume element over the manifold M, then following (Moroianu, 2007) the Kähler 2 - form satisfies following relation

$$K^{(n)} = \underbrace{K \wedge \cdots \wedge K}_{n\ times} = \frac{2^n n!}{(1+|z|^2)^{n+1}}\mathrm{dVol} = 2^n n! \det[\mathrm{g}_{\mu\bar{\nu}}]\mathrm{dVol}, \tag{3.164}$$

where

$$|z|^2 = \sum_{\mu=1}^{n} |z^{\mu}|^2, \tag{3.165}$$

so metric $\mathrm{g}_{\mu\bar{\nu}}$ has following determinant

$$\det[\mathrm{g}_{\mu\bar{\nu}}] = \frac{1}{(1+|z|^2)^{n+1}} \tag{3.166}$$

This is indeed the determinant of the metric (called also Fubini - Study metric) over the complex projective space $\mathbb{C}P^n$ which is naturally Kähler manifold:

$$g_{\mu\bar{\nu}} = \frac{\delta_{\mu\nu}(1+|z|^2) - z^{\bar{\mu}} z^{\nu}}{(1+|z|^2)^2}. \tag{3.167}$$

Since we have complex 2 - dimension, we choose $\mathbb{C}P^2$ as a base manifold M. Therefore, from eq. (3.162) the Kähler potential ϕ is given as

$$\phi = \ln(1+|z|^2) \tag{3.168}$$

Since any complex projective space $\mathbb{C}P^n$ can be regarded as a quotient of the sphere S^{2n+1} over $\mathbb{C}^n$, that is

$$\mathbb{C}P^n = S^{2n+1}/S^1, \tag{3.169}$$

we say that

$$\mathbb{C}P^2 = S^5/S^1, \tag{3.170}$$

and so locally we can use the coordinates of sphere S^5. Then, we write the volume element of S^5 such that

$$\mathrm{dVol} = r^4 \sin^3\theta_1 \sin^2\theta_2 \sin\theta_3\, dr\, d\theta_1\, d\theta_2\, d\theta_3\, d\theta_4, \tag{3.171}$$

where $r \in [0,\infty)$, $\theta_1, \theta_2, \theta_3 \in [0,\pi)$ and $\theta_4 \in [0,2\pi)$. Thus,

$$|z|^2 = r^2 \tag{3.172}$$

If sphere S^5 has unit radius, then its total volume is

$$\begin{aligned} Volume = \int_{S^5} \mathrm{dVol} &= \int_0^1 r^4 dr \int_0^{\pi} \sin^3\theta_1 d\theta_1 \int_0^{\pi} \sin^2\theta_2 d\theta_2 \int_0^{\pi} \sin\theta_3 d\theta_3 \int_0^{4\pi} d\theta_4 \\ &= \frac{8\pi^2}{15} \end{aligned} \tag{3.173}$$

Therefore, from eq. (3.164) we write for $\mathbb{C}P^2$

$$\begin{aligned}
\int_{\mathbb{C}P^2} K^{(2)} &= 2^2 2! \int_{S^5} \frac{1}{(1+r^2)^3} \mathrm{dVol} \\
&= 16 \int_0^1 \frac{r^4}{(1+r^2)^3} dr \int_0^{\pi} \sin^3\theta_1 d\theta_1 \int_0^{\pi} \sin^2\theta_2 d\theta_2 \int_0^{\pi} \sin\theta_3 d\theta_3 \int_0^{2\pi} d\theta_4 \\
&= \frac{16\pi^2}{3}\left(\frac{3\pi}{4} - 2\right) \qquad (3.174)
\end{aligned}$$

From eq. (3.162) then we get

$$\int_{\mathbb{C}P^2} K^{(2)} = \left(\frac{i}{2}\right)^2 \int_{\mathbb{C}P^2} \partial\bar{\partial}\phi \wedge \partial\bar{\partial}\phi \qquad (3.175)$$

Therefore, the topological charges $Q = \int_{\mathbb{C}P^2} \mathrm{ch}_2$ and $Q = \int_{\mathbb{C}P^2} \mathrm{ch}_1 \wedge \mathrm{ch}_1$ are proportional to the result of this integral:

$$Q \sim \int_{\mathbb{C}P^2} K^{(2)}. \qquad (3.176)$$

$\mathfrak{so}(2,\mathbb{C})$ *- Valued Theory On* $\mathbb{C}P^2$

Rewrite the component of the $\mathfrak{so}(2,\mathbb{C})$ - valued curvature 2 - form $\Omega = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}$ for an smooth function $\tilde{\phi} \in \mathcal{C}^\infty(M)$ over manifold $M = \mathbb{C}P^2$ as

$$f = -\partial\bar{\partial}\tilde{\phi} \qquad (3.177)$$

together with the second Chern class

$$\mathrm{ch}_2 = \frac{1}{8\pi^2}\mathrm{Tr}[\Omega \wedge \Omega] = -\frac{1}{4\pi^2}\partial\bar{\partial}\tilde{\phi} \wedge \partial\bar{\partial}\tilde{\phi}. \qquad (3.178)$$

If the smooth function $\tilde{\phi} \in \mathcal{C}^\infty(\mathbb{C}P^2)$ is a Kähler potential, then from eq. (3.162) we write

$$\partial\bar{\partial}\tilde{\phi} = -2iK = -2ig_{\mu\bar{\nu}} dz^\mu \wedge dz^{\bar{\nu}} \qquad (3.179)$$

and from eq. (3.164) the second Chern class is written such that

$$\mathrm{ch}_2 = \frac{1}{\pi^2}K^{(2)} = \frac{16}{\pi^2}\det[\mathrm{g}_{\mu\bar{\nu}}]\mathrm{dVol}, \tag{3.180}$$

For $\mathbb{C}P^2 = S^5/S^1$, because of $|z|^2 = r^2$, from eq. (3.168) the Kähler potential is written

$$\tilde{\phi} = \ln(1+r^2) \qquad \det[\mathrm{g}_{\mu\bar{\nu}}] = \frac{1}{(1+r^2)^3}. \tag{3.181}$$

Thus, using eq. (3.174) as well as this approach, the topological charge is obtained as

$$Q = \mathrm{c}_2 = \int_{S^5} \mathrm{ch}_2 = \frac{16\pi^2}{3}\left(\frac{3\pi}{4} - 2\right). \tag{3.182}$$

and from eq. (3.138) the fixed value of the gauge field becomes

$$\psi_0^2 = \frac{Q(1-8\pi^2)}{Volume} = 10\left(\frac{3\pi}{4} - 2\right)(1-8\pi^2). \tag{3.183}$$

$\mathfrak{u}(1,\mathbb{C})$ - *Valued Theory On* $\mathbb{C}P^2$

As similar to the $\mathfrak{so}(2,\mathbb{C})$ valued theory, the $\mathfrak{u}(1,\mathbb{C})$ - valued curvature 2 - form is

$$\Omega = \partial\bar{\partial}\ln\phi \tag{3.184}$$

together with the wedge square of the first Chern class

$$\mathrm{ch}_1 \wedge \mathrm{ch}_1 = -\frac{1}{4\pi^2}\mathrm{Tr}[\Omega] \wedge \mathrm{Tr}[\Omega] = -\frac{1}{4\pi^2}\partial\bar{\partial}\ln\phi \wedge \partial\bar{\partial}\ln\phi. \tag{3.185}$$

If the $\ln\phi \in \mathcal{C}^\infty(\mathbb{C}P^2)$ is a Kähler potential, then from eq. (3.162) we write

$$\partial\bar{\partial}\ln\phi = -2iK = -2i\mathrm{g}_{\mu\bar{\nu}}dz^\mu \wedge dz^{\bar{\nu}}. \tag{3.186}$$

and from eq. (3.164), the square of the first Chern class is also written such that

$$\mathrm{ch}_1 \wedge \mathrm{ch}_1 = \frac{1}{\pi^2}K^{(2)} = \frac{16}{\pi^2}\det[\mathrm{g}_{\mu\bar{\nu}}]\mathrm{dVol}, \tag{3.187}$$

Then, for $\mathbb{C}P^2 = S^5/S^1$, from eq. (3.168) Kähler potential is written

$$\phi = 1 + r^2. \tag{3.188}$$

Use eq. (3.174), topological charge is written as

$$Q = \int_{S^5} \mathrm{ch}_1 \wedge \mathrm{ch}_1 = \frac{16\pi^2}{3}\left(\frac{3\pi}{4} - 2\right). \tag{3.189}$$

Then from eq. (3.158) the fixed value of the gauge field becomes

$$\psi_0^2 = \frac{Q(1-8\pi^2)}{Volume} = 10\left(\frac{3\pi}{4} - 2\right)(1-8\pi^2). \tag{3.190}$$

3.5 Massive (Non - Abelian) Gauge Theory

For $\Omega \in \Lambda^{(1,1)}(\mathfrak{sl}(2,\mathbb{C}))$, because of $\mathrm{ch}_1 = 0$, the action integral of the massless gauge fields is bounded such that

$$\int_M \big(\mathrm{Tr}[\Omega \wedge \Omega] + \mathrm{g}_1 \mathrm{Tr}[\mu^2 A'' \wedge *_\Phi A''] + \mathrm{g}_2 \mathrm{Tr}[\mu A'' \wedge *_\Phi \bar{\partial}\psi] \\ + \mathrm{Tr}[\bar{\partial}\psi \wedge *_\Phi \bar{\partial}\psi] - \mathcal{V}(\psi)\big) \geq \int_M \mathrm{ch}_2, \tag{3.191}$$

where

$$\mathrm{ch}_2 = \frac{1}{8\pi^2}\mathrm{Tr}[\Omega \wedge \Omega]. \tag{3.192}$$

Therefore, since the term $\mathrm{Tr}[\Omega \wedge \Omega]$ in this integral is the second Chern class ch_2, then we another topological bound

$$\frac{1}{(1-8\pi^2)} \int_M \big(\mathrm{g}_1 \mathrm{Tr}[\mu^2 A'' \wedge *_\Phi A''] + \mathrm{g}_2 \mathrm{Tr}[\mu A'' \wedge *_\Phi \bar{\partial}\psi] \\ + \mathrm{Tr}[\bar{\partial}\psi \wedge *_\Phi \bar{\partial}\psi] - \mathcal{V}(\psi)\big) \geq \int_M \mathrm{ch}_2. \tag{3.193}$$

3.5.1 $\mathfrak{sl}(2,\mathbb{C})$ - Valued Gauge Theory

The generator of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ are

$$\theta_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.194}$$

Then an $\mathfrak{sl}(2,\mathbb{C})$ valued connection $(0,1)$ - form and its curvature $(1,1)$ - form are written as

$$\Omega = \begin{pmatrix} f_1 & f_2 \\ f_3 & -f_1 \end{pmatrix}, \quad f = f_{(A)\mu\bar{\nu}} dz^{\mu} \wedge dz^{\bar{\nu}} \in \Lambda^{(1,1)} \tag{3.195}$$

$$A'' = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & -\gamma_1 \end{pmatrix}, \quad \gamma = \gamma_{(A)\bar{\mu}} dz^{\bar{\mu}} \in \Lambda^{(1,1)}. \tag{3.196}$$

On the other hand, the gauge field ψ and the mass matrix are given by

$$\psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_3 & -\psi_1 \end{pmatrix}, \quad \psi \in \mathcal{C}^{\infty}(M) \tag{3.197}$$

$$\mu = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & -\mu_1 \end{pmatrix}, \tag{3.198}$$

where μ_1, μ_2, μ_3 are constants. Considering the connection matrix given in eq. (3.81), we write

$$A'' = \begin{pmatrix} (\frac{1+\phi_2\phi_3}{\phi_2})\bar{\partial}\phi_2 - \phi_3\bar{\partial}\phi_2 & \phi_2^2\bar{\partial}(\frac{1+\phi_2\phi_3}{\phi_2\phi_2}) \\ \phi_1^2\bar{\partial}(\frac{\phi_3}{\phi_2}) & \phi_3\bar{\partial}\phi_2 - (\frac{1+\phi_2\phi_3}{\phi_2})\bar{\partial}\phi_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & -\gamma_1 \end{pmatrix}. \tag{3.199}$$

Its curvature is then written as

$$\Omega = \partial A'' = \begin{pmatrix} f_1 & f_2 \\ f_3 & -f_1 \end{pmatrix} = \begin{pmatrix} \partial\gamma_1 & \partial\gamma_2 \\ \partial\gamma_3 & -\partial\gamma_1 \end{pmatrix}. \tag{3.200}$$

On the other hand,

$$\begin{aligned} \mu^2 &= (\mu_1^2 + \mu_2\mu_3) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= -\det[\mu]\mathbb{I}, \end{aligned} \tag{3.201}$$

and so

$$\begin{aligned}\mathrm{Tr}[\mu^2 A'' \wedge A'' \wedge \Phi] &= -4\det[\mu]\gamma_1 \wedge \gamma_1 \wedge \Phi = 0, &(3.202)\\ \mathrm{Tr}[\bar{\partial}\psi \wedge \bar{\partial}\psi \wedge \Phi] &= 2\bar{\partial}\psi_1 \wedge \bar{\partial}\psi_1 \wedge \Phi = 0. &(3.203)\end{aligned}$$

Also

$$\begin{aligned}\mathrm{Tr}[\mu A'' \wedge \bar{\partial}\psi \wedge \Phi] = \big[&\gamma_1 \wedge \bar{\partial}(\mu_3\psi_2 - \mu_2\psi_3) + \gamma_2 \wedge \bar{\partial}(\mu_1\psi_3 - \mu_3\psi_1)\\ &+\gamma_3 \wedge \bar{\partial}(\mu_2\psi_1 - \mu_1\psi_2)\big] \wedge \Phi \qquad (3.204)\end{aligned}$$

Here we can make a manipulation. Since we have a holomorphic bundle, then $F^{(0,2)} = \bar{\partial}^{\nabla} A'' = \bar{\partial} A'' = 0$. Thus we write the terms in parenthesis such that

$$\gamma \wedge \bar{\partial}(\mu_B\psi_C - \mu_C\psi_B) = \bar{\partial}[\gamma(\mu_B\psi_C - \mu_C\psi_B)]. \qquad (3.205)$$

Using Stokes theorem, if ∂M is the boundary of base manifold M, then we can set

$$\int_M \bar{\partial}[\gamma(\mu_B\psi_C - \mu_C\psi_B) \wedge \Phi] = \int_{\partial M} \gamma(\mu_B\psi_C - \mu_C\psi_B) \wedge \Phi = 0. \qquad (3.206)$$

Therefore we get

$$\int_M \mathrm{Tr}[\mu A'' \wedge \bar{\partial}\psi \wedge \Phi] = 0 \qquad (3.207)$$

On the other hand, the second Chern class ch_2 is written as

$$\begin{aligned}\mathrm{ch}_2 = \frac{1}{8\pi^2}\mathrm{Tr}[\Omega \wedge \Omega] &= \frac{1}{4\pi^2}(f_1 \wedge f_1 + f_2 \wedge f_3)\\ &= \frac{1}{4\pi^2}(\partial\gamma_1 \wedge \partial\gamma_1 + \partial\gamma_2 \wedge \partial\gamma_3). \qquad (3.208)\end{aligned}$$

Hence we have

$$\frac{1}{(1-8\pi^2)}\int_M \big(\mathrm{Tr}[\mu A'' \wedge \bar{\partial}\psi \wedge \Phi] - \mathcal{V}(\psi)\big) \geq \int_M \mathrm{ch}_2, \qquad (3.209)$$

or from eq. (3.207) the topological bound becomes

$$\int_M \left(\mathcal{V}(\psi_0) + \frac{(1-8\pi^2)}{4\pi^2}(\partial\gamma_1 \wedge \partial\gamma_1 + \partial\gamma_2 \wedge \partial\gamma_3)\right) \leq 0 \qquad (3.210)$$

Now define

$$\partial\gamma_1 \wedge \partial\gamma_1 + \partial\gamma_2 \wedge \partial\gamma_3 = \Sigma \mathrm{dVol} \qquad (3.211)$$

Then the components of gauge field ψ are constants and the potential form at a fixed ψ_0 can also be written as

$$\mathcal{V}(\psi_0) = V(\psi_0)\mathrm{dVol}. \tag{3.212}$$

Relation between the mass matrix μ and the gauge field ψ was given by eq. (3.71) such that

$$\psi = \mu^{-1}\mathrm{C}, \qquad \mathrm{C} \in \Lambda^0(GL(2,\mathbb{C})) \tag{3.213}$$

On the other hand, considering eq. (3.202), the wave equation given in (3.61) reads

$$\frac{\delta\mathcal{V}(\psi)}{\delta\psi} = 0. \tag{3.214}$$

Therefore the solution(s) of this variation is (are) given like in (3.72):

$$\psi_0 = \left\{ \psi \in \Lambda^0_{\mathrm{ad}(P)} | \frac{\delta\mathcal{V}(\psi)}{\delta\psi} = 0 \right\}. \tag{3.215}$$

Then, together with eq. (3.71), the potential form $\mathcal{V}(\psi_0)$ depends on the mass matrix:

$$\mathcal{V}(\psi_0) \equiv \mathcal{V}(\mu), \tag{3.216}$$

so we write

$$\mathcal{V}(\mu^2) = V(\mu^2)\mathrm{dVol}. \tag{3.217}$$

Considering the potential function for a field ψ mentioned above as

$$V(\psi) \sim \mathrm{Tr}[\psi^2] + (\mathrm{Tr}[\psi^2])^4 + \mathrm{Tr}[\psi^2] + \cdots, \tag{3.218}$$

then using eq. (3.201) we can write potential $V(\mu)$

$$V(\mu^2) \sim \mathrm{Tr}[\mu^2] + (\mathrm{Tr}[\mu^2])^2 + \mathrm{Tr}[\mu^2] + \cdots \tag{3.219}$$

$$= a\mathrm{det}[\mu] + b(\mathrm{det}[\mu])^2, \tag{3.220}$$

where one sees easily

$$\mu^4 = (\mathrm{det}[\mu])^2\mathbb{I}. \tag{3.221}$$

Therefore topological bound is rewritten as

$$\int_M \left(V(\mu^2) + \frac{(1-8\pi^2)}{4\pi^2}\Sigma \right) \mathrm{dVol} \leq 0. \tag{3.222}$$

We get then

$$\Sigma = -\frac{4\pi^2 V(\mu^2)}{(1-8\pi^2)} \tag{3.223}$$

Therefore the topological charge is

$$Q = \mathrm{c}_2 = \int_M \mathrm{ch}_2 = -\frac{4\pi^2 V(\mu^2)}{(1-8\pi^2)} \int_M \mathrm{dVol}. \tag{3.224}$$

Then the potential $V(\psi_0)$ at fixed ψ_0 depends on the topological charge. Since the total volume of the base manifold M is $Volume = \int_M \mathrm{dVol}$, then

$$V(\mu^2) = \frac{Q(1-8\pi^2)}{4\pi^2 \times Volume}, \tag{3.225}$$

and the field equation then reads

$$\Sigma = -\frac{Q}{Volume} \tag{3.226}$$

In result, the second Chern class ch_2 of the curvature $\Omega \in \Lambda^{(1,1)}(\mathfrak{sl}(2,\mathbb{C}))$ creates topologically massive gauge fields.

4. RESULTS AND DISCUSSIONS

There exists an important point here. Even though connection 1 - form isn't a gauge invariant, we can write following Lagrangian over real 4 - dimension, proved that the symmetry breaks,

$$\mathcal{L} = \rho \mathrm{Tr}[\Omega \wedge \Omega] + \mathrm{g}_1 \mu^2 \epsilon \mathrm{Tr}[A \wedge A \wedge \Phi] + \mathrm{g}_2 \epsilon \mathrm{Tr}[\mu A \wedge \mathrm{d}\psi \wedge \Phi] \\ + \epsilon \mathrm{Tr}[\mathrm{d}\psi \wedge \mathrm{d}\psi \wedge \Phi] - \mathcal{V}(\psi), \tag{4.1}$$

where the auxiliary form Φ is real 2 form. Since $\mathrm{d}^{\nabla}(*\Omega) = \mathrm{d}^{\nabla}\Omega = 0$ for the (anti-) self dual connection over 4 - dimension, then the action $S[A, \psi] = \int_M \mathcal{L}$ of this Lagrangian reads field equation with respect to the A such that

$$\mathrm{g}_1 \mu^2 A + \mathrm{g}_2 \mu \mathrm{d}\psi = 0. \tag{4.2}$$

Thus this equation gives either a massless gauge field, $\mu = 0$, or massive gauge field $\mu \neq 0$ with a flat connection which is invariant under a gauge transformation, i.e $A \to g^{-1} A g$. The humor of the flat connection ansatz given in this thesis appears precisely at this point.

As known that the flat connection vanishes its curvature, but, since the curvature of a connection over a complex vector bundle has three components via the decomposition of the complex structure, i.e $F = F^{(2,0)} \oplus F^{(1,1)} \oplus F^{(0,2)}$, even though the connection is flat, there exists a component of the curvature, i.e. $\Omega = F^{(1,1)}$. Namely, the holomorphic structure on a holomorphic bundle and its structure group occur naturally a flat connection vanishing the curvatures of holomorphic and antiholomorphic parts of connection, $F^{(2,0)} = F^{(0,2)}$, but the part $F^{(1,1)}$ of the curvature doesn't vanish.

On the other hand, in the physical meaning, to be flat the connection is corresponded by the mass matrix of gauge theory. So that, over complex 2 - manifold the flat connection A'' present a complex Lie group as the structure group of the bundle defined on it if the mass matrix μ isn't nilpotent, on contrast, if the mass matrix is nilpotent (in the power of 2), then unmeaningly connection appears, i.e $A'' = 0$.

5. CONCLUSIONS AND RECOMMENDATIONS

The number of the massive gauge fields is the dimension of subgroup of unbroken symmetry, (for example $SU(2)$ for Weinberg - Salam model $SU(2) \otimes U(1)$), however, if there don't exist unbroken part(s) of symmetry, then these fields are massless (Kibble, 1967). Therefore, if we consider a unique gauge group as the direct product of these groups, $G = G_{NA} \otimes G_A$, then total Lagrangian splits into massive and massless ones: $\mathcal{L} = \mathcal{L}_{Massive} + \mathcal{L}_{Massless}$, so that these parts must be invariant under groups $G_{NA} = SL(2, \mathbb{C})$, $G_A = SO(2, \mathbb{C})$ or $G_A = U(1) \otimes_{\mathbb{R}} \mathbb{C}$, respectively. Therefore the massless part is corresponded by long range forces (Streater, 1965), (Kibble, 1967), i.e gravitation and electromagnetism. Thus we can say that the group G_A may be the general gauge group for the long - range interactions (gauge fields).

In result, a massive gauge field is generated by a noncommutative flat connection on the same bundle with structure group $SL(2, \mathbb{C})$, however, the connection with $SO(2, \mathbb{C})$ or $U(1) \otimes_{\mathbb{R}} \mathbb{C}$ is commutative flat and the gauge field is massless and this predict the long - range gauge fields. As seen that second Chern class ch_2 of the curvature $\Omega \in \Lambda^{(1,1)}(\mathfrak{sl}(2, \mathbb{C}))$ creates topologically massive gauge fields. As seen from eqs. (3.136), (3.157) and (3.189) that the second Chern number or topological charge is proportional to the total volume of base manifold for each massless and massive gauge theories. On the other hand, the Abelian (massless) gauge theories are indeed theories of the Kähler potential over the complex projective space $\mathbb{C}P^2$.

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