DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NEATNESS AND EXTENSIONS OF HOMOMORPHISMS

by
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İZMİR

NEATNESS AND EXTENSIONS OF HOMOMORPHISMS

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Ph.D. THESIS EXAMINATION RESULT FORM

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NEATNESS AND EXTENSIONS OF HOMOMORPHISMS

ABSTRACT

This thesis consists of two main parts. The first part is concerned with neat submodules over commutative rings and the second part is about neat homomorphisms and extensions of homomorphisms over an arbitrary ring. Let R be a commutative ring with identity. A short exact sequence $\mathbb{E}$ of R-modules is said to be neat (respectively, $\mathcal{P}$-pure) if the sequence $\mathrm{Hom}_R(S, \mathbb{E})$ (respectively, $S \otimes_R \mathbb{E}$) is exact for every simple R-module S. We generalized Fuchs' characterization of integral domains where neatness and $\mathcal{P}$-purity coincide to commutative rings. We show that every maximal ideal of R is finitely generated and projective if and only if R has projective socle and neatness and $\mathcal{P}$-purity coincide. We prove that neatness and $\mathcal{P}$-purity coincide if and only if every maximal ideal of R is finitely generated and the unique maximal ideal P_P of the local ring R_P is principal for every maximal ideal P of R. This result is proved firstly over commutative local rings and then using localization over any commutative ring. The Auslander-Bridger transpose of simple modules is used for the passage between proper classes of short exact sequences of modules that are projectively generated and these that are flatly generated by a set of finitely presented modules. The second part of this thesis presents our work with N. Er in a detailed form; we investigate the following properties which are motivated by our discussion on neatness: A ring R is said to satisfy the property (P) if all nonzero R-module epimorphisms with cyclic domains are neat; it is said to satify (Q) if every non-injective R-module is a test for injectivity by neatness. A large class of rings including local perfect rings and those that satisfy (Q) share a common property: any two noninjective modules admit a nonzero homomorphism between them (property (T)). All three classes of rings are characterized and various examples are provided.

Keywords: Neat submodule, $\mathcal{P}$-pure submodule, the Auslander-Bridger transpose, proper class, rings with projective socle, injective R-module, test for injectivity by neatness, injective hull, neat homomorphism, uniserial module, V-ring, semi-Artinian ring, SI-ring, Morita theory

DÜZENLİLİK VE HOMOMORFİZMALARIN GENİŞLEMELERİ

ÖZ

Bu tez iki ana bölümden oluşmaktadır. İlk bölüm değişmeli halkalar üzerinde düzenli altmodüllerle ilgilidir ve ikinci bölüm herhangi bir halka üzerinde düzenli homomorfizmaları ve homomorfizmaların genişlemeleriyle ilgilidir. R birimli ve değişmeli bir halka olsun. R-modüllerin kısa bir tam dizisi $\mathbb{E}$'ye düzenli (sırasıyla, $\mathcal{P}$-saf) denir eğer her basit R-modül S için $\mathrm{Hom}_R(S, \mathbb{E})$ (sırasıyla, $S \otimes_R \mathbb{E}$) dizisi tam dizi ise. Fuchs'un düzenlilik ve $\mathcal{P}$-saflığın çakıştığı tamlık bölgesi karakterizasyonunu değişmeli halkalara genelleştirdik. Şunu gösterdik: R'nin her maksimal ideali sonlu üretilmiş ve projektiftir ancak ve ancak R projektif bir kaideye sahiptir ve düzenlilik ve $\mathcal{P}$-saflık çakışır. Düzenlilik ve $\mathcal{P}$-saflığın çakışmasının ancak ve ancak şu durumda olduğunu kanıtladık, R'nin her maksimal ideali P sonlu üretilmiştir ve her maksimal ideal P için lokal halka R_P'nin tek maksimal ideali P_P'nin bir esas idealidir. Bu sonuç ilk olarak değişmeli lokal halkalarda ve sonra lokalizasyon kullanılarak herhangi bir değişmeli halka üzerinde ispat edildi. Basit modüllerin Auslander-Bridger devriği, sonlu temsilli modüllerden oluşan bir küme tarafından projektif olarak üretilen kısa tam diziler öz sınıfı ile düz olarak üretilen kısa tam diziler öz sınıfı arasında geçiş için kullanılır. Bu tezin ikinci bölümü N. Er ile olan çalışmamızı ayrıntılı bir biçimde sunmaktadır; düzenlilik üzerine tartışmamızın motive ettiği şu özellikleri araştırıyoruz: Bir R halkasına, eğer tanım kümeleri devirli olan ve sıfır olmayan tüm örten R-modül homomorfizmaları düzenli ise (P) özelliğini sağlıyor denir; R-halkası her injektif olmayan modül düzenlilik yardımıyla injektiflik için bir test oluyorsa (Q) özelliğini sağlıyor denir. (Q)'yu sağlayan yerel mükemmel halkaları da içeren büyük bir halka sınıfı şu ortak özelliği paylaşır: iki injektif olmayan modül arasında sıfır olmayan bir homomorfizma vardır ((T) özelliği). Bu üç sınıf halka da karakterize edilmiş ve çeşitli örnekler verilmiştir.

Anahtar kelimeler: Düzenli altmodül, $\mathcal{P}$-saf altmodül, Auslander-Bridger devriği, öz sınıf, kaidesi projektif olan halka, injektif R-modül, düzenlilik vasıtasıyla injektiflik için test, injektif bürüm, düzenli homomorfizma, uniserial module, V-halka, yarı-Artin halka, SI-halka, Morita teori

CONTENTS

Page

Ph.D. THESIS EXAMINATION RESULT FORM ii

ACKNOWLEDGEMENTS iii

ABSTRACT iv

ÖZ v

CHAPTER ONE – INTRODUCTION 1

1.1 Neat Submodules over Commutative Rings 2

1.2 Neat Homomorphisms and Extensions of Homomorphisms 6

CHAPTER TWO – NEAT SUBMODULES OVER COMMUTATIVE RINGS 9

2.1 Auslander-Bridger Transpose of Modules over Commutative Rings 9

2.2 When do neatness and $\mathcal{P}$-purity coincide? 12

2.3 Finitely Generated and Projective Maximal Ideals 25

CHAPTER THREE – NEAT HOMOMORPHISMS AND EXTENSIONS OF HOMOMORPHISMS 35

3.1 Rings Over Which Nonzero Epimorphisms With Cyclic Domains Are Neat 35

3.2 Rings over which Every Non-injective is t.i.n. 39

3.3 Rings Admitting Nonzero Homomorphisms Between Any Two Non-Injective Modules 47

CHAPTER FOUR – CONCLUSION 60

CHAPTER ONE
INTRODUCTION

All rings considered in this text are associative with an identity element. Unless otherwise stated, all modules are right unitary modules. This thesis consists of two main parts. The first main part is Chapter 2 Neat Submodules over Commutative Rings and the second main part is Chapter 3 Neat Homomorphisms and Extensions of Homomorphisms.

In the first main part we deal with some problems which are related with neat submodules over commutative rings. Although mentioned concepts have been defined for an arbitrary ring, the main results in this part are for commutative rings. We use some standard localization tools, projective covers and the Auslander-Bridger transpose of finitely presented modules which is an essential tool in the representation theory of Artin algebras. Let R be a commutative ring. A short exact sequence $\mathbb{E}$ of R-modules is said to be neat (respectively, $\mathcal{P}$-pure) if the sequence $\mathrm{Hom}_R(S, \mathbb{E})$ (respectively, $S \otimes_R \mathbb{E}$) is exact for every simple R-module S. The characterization of the integral domains over which neatness and $\mathcal{P}$-purity coincide has been given by László Fuchs in Fuchs (2012): they are the integral domains where every maximal ideal is projective (and so also finitely generated). We show that for a commutative ring R, every maximal ideal of R is finitely generated and projective if and only if R has projective socle and neatness and $\mathcal{P}$-purity coincide. For a commutative ring R, we prove that neatness and $\mathcal{P}$-purity coincide if and only if every maximal ideal of R is finitely generated and the unique maximal ideal P_P of the local ring R_P is a principal ideal for every maximal ideal P of R. This result is proved firstly over commutative local rings and then using localization over any commutative ring. The Auslander-Bridger transpose of simple modules is used in proving these equivalences because it is an essential tool for the passage between proper classes of short exact sequences of modules that are projectively generated and these that are flatly generated by a set of finitely presented modules. For a commutative ring R, we prove that neatness and $\mathcal{P}$-purity coincide if and only if every simple R-module S is finitely presented and an Auslander-Bridger transpose of S is projectively equivalent to S. See Section 1.1.

The second main part of this thesis presents our work with N. Er in a detailed form; we investigate the following properties which are motivated by our discussion on neatness: A ring R is said to satisfy the property (P) if all nonzero R-module epimorphisms with cyclic domains are neat; it is said to satify (Q) if every non-injective R-module is a test for injectivity by neatness. A large class of rings including local perfect rings and those that satisfy (Q) share a common property: any two noninjective modules admit a nonzero homomorphism between them (property (T)). We characterized all three classes of rings with these properties by using some well known ring characterizations and various examples are provided. See Section 1.2.

The symbols Rad(M), $Z(M)$, Soc(M), and $E(M)$ will respectively denote the Jacobson radical, the singular submodule, the socle and the injective hull of a module M. For the composition length of M, we will use the symbol $cl(M)$. We will use $J(R)$ (or just J when there is no ambiguity) exclusively for the Jacobson radical of a ring R. For $a \in M$, ann$_R(a)$ denotes the annihilator of a.

For all other terminology not defined here, the reader may refer to Lam (1999), Anderson & Fuller (1992), Osborne (2012), Cartan & Eilenberg (1956), Facchini (2013), Wisbauer (1991), Clark et al. (2006) and Goodearl & Warfield Jr (2004). In particular for localization in commutative rings, we use Atiyah & Macdonald (1969), Eisenbud (2013), Kunz (1985) and Kemper (2011).

1.1 Neat Submodules over Commutative Rings

A subgroup A of an abelian group B is said to be a *neat subgroup* if $A \cap pB = pA$ for all prime numbers p. There are several reasonable ways to generalize the concept of neat subgroups of abelian groups introduced in Honda (1956) to modules and a natural question is when these generalizations are equivalent. For an account of this problem, see Fuchs (2012) and Crivei (2014). We will deal with this problem using the terminology and results of proper classes of short exact sequences of modules; see Alizade & Mermut (2015) for a survey of related results and its Appendix Section A for the terminology and notation for proper classes. Following the notation as in Stenström (967a) and Sklyarenko (1978), for a given class $\mathcal{M}$ of R-modules, we define the following proper classes of short exact sequences of R-modules:

(i) $\pi^{-1}(\mathcal{M})$, the *proper class projectively generated by* $\mathcal{M}$, consists of all short exact sequences

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.1}$$

of R-modules such that the sequence $\mathrm{Hom}_R(M, \mathbb{E})$, that is,

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C) \longrightarrow 0$$

is exact for every $M \in \mathcal{M}$.

(ii) $\tau^{-1}(\mathcal{M})$, the *proper class flatly generated by* $\mathcal{M}$, consists of all short exact sequences $\mathbb{E}$ of R-modules as in (1.1) above such that the sequence

$$M \otimes_R \mathbb{E}: \qquad 0 \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0$$

is exact for every $M \in \mathcal{M}$.

In this part, we are interested in neatness and $\mathcal{P}$-purity defined by the following proper classes:

$${}_R\mathcal{N}eat = \pi^{-1}(\{\text{all simple } R\text{-modules}\}) = \pi^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R\})$$

and

$${}_R\mathcal{P}\text{-}\mathcal{P}ure = \tau^{-1}(\{\text{all simple } R\text{-modules}\}) = \tau^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R\}).$$

Note that for a short exact sequence $\mathbb{E}$ as in (1.1) and for a maximal ideal P of R, $(R/P) \otimes_R \mathbb{E}$ is an exact sequence if and only if $A' \cap PB = PA'$ where A' is the image of the monomorphism $f : A \to B$ in $\mathbb{E}$ (by (Sklyarenko, 1978, Lemma 6.1)). In the noncommutative case, we have defined $\mathcal{P}$-purity by taking the ideal P in the set $\mathcal{P} = \{\text{all left primitive ideals of } R\}$, see Mermut et al. (2009). In the commutative case,

$$\mathcal{P} = \{\text{all maximal ideals of } R\}.$$

In Fuchs (2012), Fuchs calls a short exact sequence $\mathbb{E}$ of modules *coneat* if it is $\mathcal{P}$-pure but we do not use this terminology since we have already used coneat for another dual concept to neat in Büyükaşık et al. (2010).

Fuchs proved in Fuchs (2012) that for an integral domain R, ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ if and only if every maximal ideal of R is projective and so necessarily finitely generated. Our problem is to characterize the commutative rings R such that ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$. In the last Section 2.3, we prove that all maximal ideals of a commutative ring R are finitely generated and projective if and only if R has projective socle and ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$.

As observed by Crivei in (Crivei, 2014, Theorem 2.1 and Remark 2.2), the condition that every maximal ideal of R is finitely generated and projective, is not a necessary condition for ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$. As shown there, if every maximal ideal of R is principal, then ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ but each maximal ideal need not be projective. For example, for a prime number p and positive integer $n \geq 2$, the unique maximal ideal $p\mathbb{Z}/p^n\mathbb{Z}$ of the commutative local Artinian principal ideal ring $R = \mathbb{Z}/p^n\mathbb{Z}$ is not a projective R-module but ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ holds.

Indeed, for a commutative local ring R with a unique maximal ideal $\mathfrak{m}$, we show in Corollary 2.2.5 that ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ if and only if $\mathfrak{m}$ is a principal ideal. Note that by a *commutative local ring*, we just mean a nonzero commutative ring with a unique maximal ideal; we do not assume that it is also Noetherian. If the unique maximal ideal $\mathfrak{m}$ of a commutative local ring R is projective, then it is necessarily a principal ideal as is well-known (Proposition 2.2.9). Over any commutative ring, we firstly understand in Lemma 2.2.6, how neatness and $\mathcal{P}$-purity behave under localization and then we obtain our main result in Theorem 2.2.7: for a commutative ring R, ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ if and only if every maximal ideal $\mathfrak{m}$ of R is finitely generated and the maximal ideal $\mathfrak{m}_{\mathfrak{m}}$ of the local ring $R_{\mathfrak{m}}$ is a principal ideal. It yields Fuch's result for integral domains as a corollary.

We use a tool from representation theory of Artin algebras: the Auslander-Bridger transpose of a finitely presented R-module M. We summarize its properties over a commutative ring that we use in Section 2.1. For a finitely presented R-module M, an Auslander-Bridger transpose $\mathrm{Tr}(M)$ is also a finitely presented R-module and for a short exact sequence $\mathbb{E}$ of R-modules,

$$\text{the sequence } \mathrm{Hom}_R(M,\mathbb{E}) \text{ is exact} \quad \Longleftrightarrow \quad \text{the sequence } \mathrm{Tr}(M)\otimes_R\mathbb{E} \text{ is exact}$$

by (Sklyarenko, 1978, Corollary 5.1) or see (Fuchs, 2012, §4) where free presentations of finitely presented modules are used when taking an Auslander-Bridger transpose which is called an associated finitely presented module there (and in (Sklyarenko, 1978, § 5 and § 8)). This gives the below result that we use mainly for our problem:

Theorem 1.1.1. *(Sklyarenko, 1978, Theorem 8.3) If $\mathcal{M}$ is a set of finitely presented R-modules, where R is a commutative ring, then we have:*

(1) $\pi^{-1}(\{M \mid M \in \mathcal{M}\}) = \tau^{-1}(\{\mathrm{Tr}(M) \mid M \in \mathcal{M}\})$.

(2) $\tau^{-1}(\{M \mid M \in \mathcal{M}\}) = \pi^{-1}(\{\mathrm{Tr}(M) \mid M \in \mathcal{M}\})$.

Turning back to the proper classes we deal with, if R is a commutative ring such that every maximal ideal is *finitely generated*, then by Theorem 1.1.1,

$${}_R\mathcal{N}eat = \pi^{-1}(\{S \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\}) = \tau^{-1}(\{\mathrm{Tr}(S) \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\}$$

and

$${}_R\mathcal{P}\text{-}\mathcal{P}ure = \tau^{-1}(\{S \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\}) = \pi^{-1}(\{\mathrm{Tr}(S) \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\}.$$

So in this case, to characterize when we have ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$, we investigate $\mathrm{Tr}(S)$ for a finitely presented simple R-module S. We prove in Theorem 2.2.7 that for a commutative ring R, ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ if and only if every simple R-module S is finitely presented (that is, every maximal ideal of R is finitely generated) and Tr(S) is projectively equivalent to S which means Tr$(S) \oplus P \cong S \oplus Q$ for some (finitely generated) projective modules P and Q.

1.2 Neat Homomorphisms and Extensions of Homomorphisms

A homomorphism $f : M \to N$ of modules is called neat if given any proper submodule H of a module G and any homomorphism $\sigma : H \to M$, the homomorphism σ has a proper extension in G whenever $f\sigma$ has a proper extension in G, that is, a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\sigma"'] & G' \arrow[r] \arrow[dr, dashed, "\tau"'] & G \\ M \arrow[rr, "f"'] & & N \end{tikzcd}$$

with $H \subsetneqq G' \subseteq G$, always guarantees the existence of a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\sigma"'] & G'' \arrow[r] \arrow[dl, dashed] & G \\ M \arrow[rr, "f"] & & N \end{tikzcd}$$

with $H \subsetneqq G'' \subseteq G$. Some characterizations of neat homomorphisms were given by Theorem 1.2 in Bowe (1972). We use the following characteriztation: A homomorphism $f : M \to N$ of modules is a neat homomorphism of modules if and only if there are no proper extensions of f in the injective hull E(M) of M.

In Osofsky (1964), it was shown that a ring all of whose cyclic modules are injective is semisimple Artinian. Also in Osofsky (1968b), that proof is greatly simplified. After this result, the following question type is raised: what is the structure of rings with the cyclic modules (or proper cyclic modules) having a fixed property? There are a lot of publications about this question, see for example Osofsky (1968a), Ahsan (1973), Goel & Jain (1978), Smith (1979), Jain & Saleh (1987), van Huynh & Dan (1989), Tuganbaev (2011), Aydoğdu et al. (2012), Er et al. (2016). With this motivation, we deal with the rings R where all nonzero R-module epimorphisms with cyclic domains are neat in Section 3.1. We call this property (P). To deal with such type of questions have been suggested by Noyan Er, and we proved that a ring R satisfies the property (P) if and only if either R is semisimple Artinian or the injective hull of every cyclic R-module is uniserial.

Recall that an R-module M is said to be injective provided that the functor $\text{Hom}_R(-, M)$ is exact. Of course M is injective if and only if $\text{Ext}_R(N, M) = 0$ for all R-modules N. For any ring R, it is well known that every R-module is a submodule of an injective R-module. So a possible strategy to investigate all R-modules is to describe all injective R-modules and for each injective R-module I, all its submodules. There is a huge list of studies on the injectivity of modules and rings with different perspectives, a considerable amount of which involves notions derived from relative injectivity, i.e, injectivity of a module with respect to a fixed module inspired by Baer's Criterion.

In the literature of injectives, there are different "test modules for injectivity" definitions and a prototype question is: What are the rings over which all modules are injective or "test module for injectivity by a property"? With these motivation we deal with the rings R which has property (Q), that is, every non-injective module is test for injectivity by neatness. An R-module M is called test for injectivity by neatness (t.i.n.) if all nonzero maps $N \to M$ being neat implies that N is injective. To deal with such type of questions suggested by Noyan Er, we proved that for a ring R, the following are equivalent:

(i) R satisfies the property (Q).

(ii) R is a right V-ring and for any two non-injective modules A and B $\text{Hom}_R(A, B) \neq 0$.

(iii) $R \cong \Lambda' \times \Lambda$ such that Λ' is semisimple Artinian and Λ is a right V-ring with homogeneous right socle modulo which Λ is semisimple Artinian.

After the characterization of rings with the property (Q), my advisor Engin Mermut asked that "What are the rings over which $\mathrm{Hom}(A, B) \neq 0$ for all non-injective right modules A and B?". We call this property (T). Using the characterization of rings with the property (Q), it is proved that rings with property (T), see Theorem 3.3.1, are precisely those rings

(i) Λ is a right V-ring with homogeneous right socle and $\frac{\Lambda}{\mathrm{Soc}(\Lambda_\Lambda)}$ is semisimple Artinian (i.e. R is a ring satisfying (Q)),

(ii) Λ is Morita equivalent to a ring in the form $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$, where D is a division ring,

(iii) Λ is a right max, right semi-Artinian (nonzero) with singular right socle and a unique non-injective simple right module S such that $S_{\frac{\Lambda}{J(\Lambda)}}$ is injective and the following hold:

(a) $J(\Lambda)$ is small in C_Λ, where $\frac{C}{J(\Lambda)}$ is the S-component of the right socle of $\frac{\Lambda}{J(\Lambda)}$, and

(b) The ring $\frac{\Lambda}{C}$ is semisimple Artinian and orthogonal to S as a Λ-module.

CHAPTER TWO
NEAT SUBMODULES OVER COMMUTATIVE RINGS

We started to deal with some problems which are solved in this chapter in my M.Sc. thesis, for more details you can see Türkoğlu (2013). For solving these problems, we used an essential tool Auslander-Bridger transpose of modules. In the next section we will give the definiton of this tool and some important properties of it. Moreover for completeness of this thesis we note some results with its proofs in Section 2.3 from my M.Sc. thesis.

2.1 Auslander-Bridger Transpose of Modules over Commutative Rings

Let M be a *finitely presented* R-module, where R is a commutative ring. Take a *presentation* of M, that is, take an exact sequence

$$\gamma : \qquad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0 \tag{2.1}$$

where P_0 and P_1 are *finitely generated projective* modules. Apply the contravariant left exact functor $(-)^* = \mathrm{Hom}_R(-, R)$ (from the category of R-modules to itself since R is commutative) to f and complete it on the right with the canonical epimorphism σ onto the R-module

$$\mathrm{Tr}_\gamma(M) = \mathrm{Coker}(f^*) = P_1^* / \mathrm{Im}(f^*)$$

to obtain the following exact sequence that we denote by γ^*:

$$\gamma^* : \qquad P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \mathrm{Tr}_\gamma(M) \longrightarrow 0,$$

Since P_0^* and P_1^* are also finitely generated projective R-modules, γ^* is a presentation for the finitely presented R-module $\mathrm{Tr}_\gamma(M)$ which is called the *Auslander-Bridger transpose of the finitely presented R-module M with respect to the presentation γ*. See Auslander & Bridger (1969) and (Auslander et al., 1995, §IV.1) where the main investigation is on Artin algebras on which every finitely presented module has a minimal projective presentation. Over a semiperfect ring R, taking the

Auslander-Bridger transpose with respect to a minimal presentation, gives a duality between the classes of the finitely presented left and right modules without nonzero projective direct summands; see (Anderson & Fuller, 1992, pp. 354-358). Remember that a presentation γ of M as in (2.1) is called a *minimal presentation* of M if both $g : P_0 \to M$ and $f' : P_1 \to \mathrm{Ker}(g)$ that is obtained by restricting the codomain of f to $\mathrm{Im}(f) = \mathrm{Ker}(g)$ are projective covers, that is, $\mathrm{Ker}(g) \ll P_0$ and $\mathrm{Ker}(f) \ll P_1$. The notation $N \ll M$ means that N is a *small submodule* of M, that is, for every submodule X of M, $N + X = M$ implies $X = M$. For projective covers of modules, see (Anderson & Fuller, 1992, p. 199–201).

Two modules A and B are said to be *projectively equivalent*, denoted by $A \approx B$, if there exist projective modules P and Q such that $A \oplus P \cong B \oplus Q$; if A and B are finitely generated, then the projective modules P and Q can be choosen to be finitely generated by (Hilton & Stammbach, 1997, Proof of Theorem 10.4).

Observe that projectively equivalent modules have the same projective dimension. We denote the projective dimension of a module X by $\mathrm{pd}(X)$.

If we have another presentation δ for the finitely presented module M, then $\mathrm{Tr}_\gamma(M)$ and $\mathrm{Tr}_\delta(M)$ are projectively equivalent, that is, $\mathrm{Tr}_\gamma(M) \oplus P \cong \mathrm{Tr}_\delta(M) \oplus Q$ for some projective R-modules P and Q; since $\mathrm{Tr}_\gamma(M)$ and $\mathrm{Tr}_\delta(M)$ are finitely generated, P and Q can be choosen to be finitely generated.

We just write $\mathrm{Tr}(M)$ for *an Auslander-Bridger transpose of a finitely presented R-module M with respect to a presentation of M* keeping in mind that this is unique up to projective equivalence. Observe that

$$M \text{ is a projective } R\text{-module} \quad \Longleftrightarrow \quad \mathrm{Tr}(M) \approx 0 \quad \Longleftrightarrow \quad \mathrm{Tr}(M) \text{ is projective.}$$

We always have $\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \cong M$ which may also been seen by Proposition 2.1.1 below, simply because of the fact that the transpose of the transpose of a matrix is itself. If we drop the subscript for the dependent presentations γ^* and γ in the isomorphism

$\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \cong M$, then we can only say that $\mathrm{Tr}(\mathrm{Tr}(M))$ is projectively equivalent to M.

For each $m \times n$ matrix $A = [a_{ij}]_{i,j=1}^n$ over a commutative ring R, where $m, n \in \mathbb{Z}^+$, define the R-module homomorhism $\mu_A : R^n \to R^m$ for all $(x_1, x_2, \ldots, x_n) \in R^n$ by

$$\mu_A((x_1, x_2, \ldots, x_n)) = (y_1, y_2, \ldots, y_m), \text{ where } y_i = \sum_{k=1}^{n} a_{ik} x_k \quad \text{for } i = 1, 2, \ldots, m,$$

that is, as in usual linear algebra, $\mu_A(x) = Ax$ for all $x \in R^n$ if elements of R^n and R^m are written as column vectors and A is its matrix representation in the standard bases. We will also denote μ_A just by the matrix A also. The name transpose for the Auslander-Bridger transpose comes from the following observation:

Proposition 2.1.1. *Let M be a finitely presented module over a commutative ring R with a free presentation*

$$\delta : \qquad R^n \xrightarrow{\mu_A} R^m \longrightarrow M \longrightarrow 0,$$

where $A = [a_{ij}]_{i,j=1}^n$ is an $m \times n$ matrix over R, $m, n \in \mathbb{Z}^+$ and $\mu_A : R^n \to R^m$ is the R-module homomorhism represented by the matrix A as given above. Then $\mathrm{Tr}_\delta(M) \cong \mathrm{Coker}(\mu_{A^\mathsf{T}})$ *via the below isomorphism ψ, where A^T is the transpose of the matrix A and $\mu_{A^\mathsf{T}} : R^m \to R^n$ is the R-module homomorhism represented by the matrix A^T because we can construct the following commutative diagram with exact rows:*

$$\begin{tikzcd}
(R^m)^* \arrow[r, "\mu_A^*"] \arrow[d, "\cong"] & (R^n)^* \arrow[r] \arrow[d, "\cong"] & \mathrm{Tr}_\delta(M) \arrow[r] \arrow[d, dashed, "\psi"] & 0 \\
R^m \arrow[r, "\mu_{A^\mathsf{T}}"] & R^n \arrow[r] & \mathrm{Coker}(\mu_{A^\mathsf{T}}) \arrow[r] & 0
\end{tikzcd}$$

Here we have the natural isomorphisms $(R^m)^* \to R^m$ and $(R^n)^* \to R^n$ which induce the R-module homomorhism $\psi : \mathrm{Tr}_\delta(M) \to \mathrm{Coker}(\mu_{A^\mathsf{T}})$ and by the 5-Lemma (when the diagram is extended to the right by zeros), we obtain that ψ is an isomorphism.

Note also the following properties, see (Maşiek, 2000, Remarks (2) and (5) before Proposition 5); rings are assumed to be commutative Noetherian there but the following hold without the Noetherian assumption:

Proposition 2.1.2. *If $A_1,\ A_2,\ ...,A_n$ are finitely presented R-modules, where R is a commutative ring and $n \in \mathbb{Z}^+$, then* $\mathrm{Tr}\left(\overset{n}{\underset{i=1}{\oplus}} A_i\right) \approx \overset{n}{\underset{i=1}{\oplus}} \mathrm{Tr}(A_i)$.

Proposition 2.1.3. *If M is a finitely presented R-module, where R is a commutative ring, and $\mathfrak{m}$ is a maximal ideal of R, then $(\mathrm{Tr}(M))_{\mathfrak{m}} \approx \mathrm{Tr}(M_{\mathfrak{m}})$ as $R_{\mathfrak{m}}$-modules, where the right side is an Auslander-Bridger transpose of the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ with respect to a presentation.*

2.2 When do neatness and $\mathcal{P}$-purity coincide?

Quasi-injective modules have the exchange property, see (Fuchs & Salce, 2001, §I.9 and §IX.8) or (Fuchs, 1969, Theorem 3). Since semisimple modules are quasi-injective, they also have the exchange property and we use this frequently below as noted in the following definition.

Definition 2.2.1. An R-module M is said to have *exchange property* if, for any internal direct sum decomposition $A = M' \oplus N = \underset{i\in I}{\oplus} A_i$ of an arbitrary R-module A, where $M' \cong M$, A_i, N are submodules of A and I is an index set, there always exist submodules B_i of A_i for every $i \in I$ such that $A = M' \oplus \left(\underset{i\in I}{\oplus} B_i\right)$. In this case, B_i must be a direct summand of A_i for every $i \in I$, say $A_i = B_i \oplus B_i'$ for some submodule B_i' of A_i, and we obtain: $M' \cong \underset{i\in I}{\oplus} B_i'$ and $N \cong \underset{i\in I}{\oplus} B_i$.

For completeness, note the following elementary properties of projectively generated and flatly generated proper classes of short exact sequences of modules that we use frequently in our investigation. The below first three properties follow just by the definition of projectively generated and flatly generated proper classes and the fact that for a projective module X, the functors $\mathrm{Hom}_R(X, -)$ and $X \otimes_R -$ are exact functors. For a proper class $\mathcal{A}$ of short exact sequences of R-modules, an R-module X is said to be *$\mathcal{A}$-projective* if the sequence $\mathrm{Hom}_R(X, \mathbb{E})$ is exact for every short exact sequence $\mathbb{E} \in \mathcal{A}$.

Theorem 2.2.2. *(Sklyarenko, 1978, §1, 2 and 8) Let $\mathcal{M}$ be a set of R-modules, where R is a commutative ring.*

(1) Every $M \in \mathcal{M}$ is $\pi^{-1}(\mathcal{M})$-projective.

(2) In the generating set $\mathcal{M}$, we can replace any module by a module projectively equivalent to it without changing the proper class. More precisely, if $\mathcal{M}' = \{M' \mid M \in \mathcal{M}\}$ where for each $M \in \mathcal{M}$, the module M' is projectively equivalent to M, then

$$\pi^{-1}(\mathcal{M}') = \pi^{-1}(\mathcal{M}) \qquad \textit{and} \qquad \tau^{-1}(\mathcal{M}') = \tau^{-1}(\mathcal{M}).$$

(3) In the generating set $\mathcal{M}$, we can take out the projective ones without changing the proper class. More precisely, we have

$$\pi^{-1}(\{M \mid M \in \mathcal{M} \textit{ and } M \textit{ is not projective}\}) = \pi^{-1}(\mathcal{M})$$

and

$$\tau^{-1}(\{M \mid M \in \mathcal{M} \textit{ and } M \textit{ is not projective}\}) = \tau^{-1}(\mathcal{M}).$$

(4) (Sklyarenko, 1978, Proposition 2.1) If a module A is $\pi^{-1}(\mathcal{M})$-projective, then A is isomorphic to a direct summand of $F \oplus (\bigoplus_{j\in J} M_j)$ where F is a free R-module and $M_j \in \mathcal{M}$ for all $j \in J$ for some index set J.

Using the above property (4) for the proper class

$${}_R\mathcal{N}eat = \pi^{-1}(\{S \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \text{ is a maximal ideal of } R\}),$$

that is projectively generated by a set of simple modules, we obtain that a ${}_R\mathcal{N}eat$-projective module N is isomorphic to a direct summand of a direct sum of a free R-module and a semisimple module. As noted in (Fuchs, 2012, Theorem 2.6-(ii)) (without proof), such a direct summand is then a direct sum of a projective module and a semisimple module. This is proved using the exchange property (Definition 2.2.1) of semisimple modules (see the argument in the begining of the proof of (1)⇒(2) of Theorem 2.2.3).

Theorem 2.2.3. *Let R be a commutative local ring with the unique maximal ideal*

$\mathfrak{m} \neq 0$. *Suppose* $\mathfrak{m}$ *is a finitely generated ideal and* k *is the smallest number of generators for* $\mathfrak{m}$*. Fix some* $a_1, \ldots, a_k \in \mathfrak{m}$ *such that* $\mathfrak{m} = Ra_1 + \ldots + Ra_k$ *and take the following minimal free presentation of the simple* R*-module* $S = R/\mathfrak{m}$ *which is not projective:*

$$\gamma : R^k \xrightarrow{f} R \xrightarrow{\pi} S = R/\mathfrak{m} \longrightarrow 0,$$

where π *is the canonical epimorphism and* $f(x_1, \ldots, x_k) = a_1x_1 + \ldots + a_kx_k$ *for all* $(x_1, \ldots, x_k) \in R^k$*, that is,* f *is represented by the row matrix* $[a_1 \quad \cdots \quad a_k]$*. Then the following are equivalent:*

(1) ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$*.*

(2) $\mathrm{Tr}_\gamma(S) \cong S$*.*

(3) $\mathfrak{m}$ *is a principal ideal.*

(4) $\mathrm{Tr}_\gamma(S)$ *is projectively equivalent to* S*.*

(5) $\mathrm{Tr}(S)$ *is projectively equivalent to* S*.*

Firstly, we will prove the following lemma which also gives the $(2) \Rightarrow (3)$ part of the theorem:

Lemma 2.2.4. *Under the hypothesis of Theorem 2.2.3, we have:*

(1) The exact sequence, where ρ *is the canonical epimorphism,*

$$\gamma^* : R^* \xrightarrow{f^*} (R^k)^* \xrightarrow{\rho} \mathrm{Coker}(f^*) = \mathrm{Tr}_\gamma(S) \longrightarrow 0$$

is also a minimal projective presentation of $\mathrm{Tr}_\gamma(S)$ *and moreover* $\mathrm{Tr}_\gamma(S) \neq 0$ *has no nonzero projective direct summand.*

(2) If $\mathrm{Tr}_\gamma(S) \cong S^j$ *for some* $j \in \mathbb{Z}^+$*, then* $j = k$*. In particular, if* $\mathrm{Tr}_\gamma(S) \cong S$*, then* $k = 1$ *and so* $\mathfrak{m}$ *is a principal ideal.*

Proof. Part (1) follows from (Anderson & Fuller, 1992, Theorem 32.13) since local rings are semiperfect and the non-projective simple module S clearly has no nonzero projective direct summand. $\mathrm{Tr}_\gamma(S) \neq 0$ because otherwise we would have

$S \cong \mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(S)) = \mathrm{Tr}_{\gamma^*}(0)$ will be projectively equivalent to 0, that is, S would be projective which is not by our hypothesis. Furthermore, by Proposition 2.1.1, we can construct the following commutative diagram with exact rows using the natural isomorphisms $R^* \cong R$ and $(R^k)^* \cong R^k$ and completing it with the isomorphism ψ, where $\bar{f}$ is represented by the column matrix $[a_1 \quad \cdots \quad a_k]^\intercal$ and σ is the canonical epimorphism:

$$\begin{tikzcd}
R^* \arrow[r, "f^*"] \arrow[d, "\cong"] & (R^k)^* \arrow[r] \arrow[d, "\cong"] & \mathrm{Tr}_\gamma(S) \arrow[r] \arrow[d, dashed, "\psi"] & 0 \\
R \arrow[r, "\bar{f}=[a_1 \cdots a_k]^\intercal"] & R^k \arrow[r, "\sigma"] & \mathrm{Coker}(\bar{f}) \arrow[r] & 0
\end{tikzcd}$$

So $\mathrm{Coker}(f^*) = \mathrm{Tr}_\gamma(S) \cong \mathrm{Coker}(\bar{f})$. Since σ is a projective cover and $\mathrm{Tr}_\gamma(S) \cong S^j$, we have that the composition

$$R^k \xrightarrow{\psi^{-1}\sigma} \mathrm{Tr}_\gamma(S) \xrightarrow{\cong} S^j$$

is a projective cover of S^j. Since $R \xrightarrow{\pi} S$ is a projective cover of S and every finite direct sum of projective covers is also a projective cover (by (Anderson & Fuller, 1992, Lemma 27.2)), we obtain that $\bigoplus_{i=1}^{j} \pi : R^j \to S^j$ is also projective cover of S^j. Hence we must have $R^k \cong R^j$ since projective covers of a module are unique up to isomorphism (by (Anderson & Fuller, 1992, Lemma 17.17)). Then $k = j$ because commutative rings have the invariant basis number property (by (Passman, 1991, Lemma 2.6)). $\square$

Proof of Theorem 2.2.3. The simple module $S = R/\mathfrak{m}$ is not projective since the unique largest proper ideal $\mathfrak{m} \neq 0$ of the local ring R is not a direct summand of R. The presentation γ is a minimal free presentation of $S = R/\mathfrak{m}$ because $\mathrm{Ker}\,\pi = \mathfrak{m} \ll R$ as R is local and so $\mathfrak{m}^k \ll R^k$ (by (Anderson & Fuller, 1992, 5.20)). It suffices then to show $\mathrm{Ker}\, f \subseteq \mathfrak{m}^k$. This holds if $k = 1$ since $\mathrm{Ker}\, f$ is a proper submodule of R in this case as $f \neq 0$, and $\mathfrak{m} \neq 0$ is the largest proper ideal of the local ring R. Suppose $k > 1$. If there exists $x = (r_1, r_2, \ldots, r_k) \in \mathrm{Ker}(f) \setminus \mathfrak{m}^k$, then without loss of generality say $r_1 \notin \mathfrak{m}$, so r_1 is a unit since R is a local ring. Thus there exists $r \in R$ such that $rr_1 = 1$. Clearly $rf(x) = 0$, that is, $a_1 = -rr_2a_2 - rr_3a_3 - \ldots rr_ka_k$. But this contradicts with k being the smallest number of generators for $\mathfrak{m}$.

$(4) \iff (5)$: Clear since $\mathrm{Tr}_\gamma(M)$ and $\mathrm{Tr}(M)$ are projectively equivalent for any finitely presented module M.

$(3) \Rightarrow (2)$: Since $\mathfrak{m}$ is a principal ideal, we have $k = 1$ in the minimal presentation γ of S and so $\mathrm{Tr}_\gamma(S) \cong S$ by Proposition 2.1.1 since the transpose of the 1×1 matrix $[a_1]$ representing f is again $[a_1]$.

$(2) \Rightarrow (3)$: Follows by Lemma 2.2.4-(2).

$(2) \Rightarrow (4)$: Clear.

$(4) \Rightarrow (1)$: The (up to isomorphism) unique simple R-module S is finitely presented by the hypothesis. Since $\mathrm{Tr}_\gamma(S)$ is projectively equivalent to S, we obtain by Theorem 1.1.1 and Theorem 2.2.2-(2) that

$$_R\mathcal{N}eat = \pi^{-1}(\{S\}) = \tau^{-1}(\{\mathrm{Tr}_\gamma(S)\}) = \tau^{-1}(\{S\}) = {}_R\mathcal{P}\text{-}\mathcal{P}ure.$$

$(1) \Rightarrow (2)$: Since $_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$, we obtain by Theorem 1.1.1 that

$$_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure = \tau^{-1}(\{S\}) = \pi^{-1}(\{\mathrm{Tr}_\gamma(S)\}).$$

So $\mathrm{Tr}_\gamma(S)$ is a $_R\mathcal{N}eat$-projective module by Theorem 2.2.2-(1). By (Fuchs, 2012, Theorem 2.6-(ii)), a $_R\mathcal{N}eat$-projective module is a direct sum of a projective module and a semisimple module. This is proved using the exchange property of semisimple modules as follows. Since $\mathrm{Tr}_\gamma(S)$ is $_R\mathcal{N}eat$-projective and $_R\mathcal{N}eat = \pi^{-1}(\{S\})$ by definition, we have by Theorem 2.2.2-(4) that $\mathrm{Tr}_\gamma(S)$ is isomorphic to a direct summand A of the module $(\bigoplus_{\alpha\in I} R_\alpha) \oplus (\bigoplus_{j\in J} S_j)$ (internal direct sum) where $R_\alpha \cong R$ and $S_j \cong S$ for all $\alpha \in I$ and $j \in J$ for some index sets I and J. Since $A \cong \mathrm{Tr}_\gamma(S)$ is finitely generated, we can assume that the index sets I and J are finite sets, say $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ for some $m, n \in \mathbb{Z}^+$. So

$A \oplus B = (\overset{m}{\underset{i=1}{\oplus}} R_i) \oplus (\overset{n}{\underset{t=1}{\oplus}} S_t)$ for some submodule B. Since semisimple modules have the exchange property (see Definition 2.2.1), there exist some submodules A_1, A_1', B_1, B_1' such that

$$A\oplus B = (\overset{m}{\underset{i=1}{\oplus}} R_i)\oplus(\overset{n}{\underset{t=1}{\oplus}} S_t) = A_1\oplus B_1\oplus(\overset{n}{\underset{t=1}{\oplus}} S_t) \quad \text{and} \quad A_1\oplus A_1' = A, \quad B_1\oplus B_1' = B.$$

Thus $A_1\oplus B_1 \cong \overset{m}{\underset{i=1}{\oplus}} R_i \cong R^m$ is a free R-module and $A_1'\oplus B_1' \cong \overset{n}{\underset{t=1}{\oplus}} S_t$ is semisimple. So A_1 is projective and A_1' is semisimple. Thus $\mathrm{Tr}_\gamma(S) \cong A = A_1\oplus A_1'$ is also a direct sum of a finitely generated projective module and a finitely generated semisimple module. But $A \cong \mathrm{Tr}_\gamma(S) \neq 0$ has no nonzero projective direct summand by Lemma 2.2.4-(1), so the projective direct summand A_1 of A must be zero and then $0 \neq \mathrm{Tr}_\gamma(S) \cong A_1'$ is semisimple over the local ring R over which S is the (up to isomorphism) unique simple module. Then $\mathrm{Tr}_\gamma(S) \cong S^j$ for some positive integer $j \leq n$ and by Lemma 2.2.4-(2), we obtain $j = k$. Call this isomorphism $\phi : \mathrm{Tr}_\gamma(S) \to S^k$. Now use the commutative diagram in the proof of Lemma 2.2.4. We have that $\pi : R \to S$ is a projective cover of S, $f' : R^k \to \mathrm{Ker}\,\pi$ that is obtained by restricting the codomain of f to $\mathrm{Im}\, f = \mathrm{Ker}\,\pi$ is a projective cover of $\mathrm{Ker}\,\pi$, $\psi^{-1}\sigma : R^k \to \mathrm{Tr}_\gamma(S)$ is a projective cover of $\mathrm{Tr}_\gamma(S)$ and $\bar{f}' : R \to \mathrm{Ker}\,\sigma$ obtained by restricting the codomain of $\bar{f}$ to $\mathrm{Im}(\bar{f}) = \mathrm{Ker}\,\sigma$ is a projective cover of $\mathrm{Ker}\,\sigma$. Since $\phi\psi^{-1}\sigma$ and $\overset{k}{\underset{i=1}{\oplus}} \pi$ are projective covers of S^k, by uniqueness of projective covers, there exists an isomorphism $h : R^k \to R^k$ such that $(\overset{k}{\underset{i=1}{\oplus}} \pi)h = \phi\psi^{-1}\sigma$. So we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\,\sigma \arrow[r] \arrow[d, "g=h_{|\mathrm{Ker}(\sigma)}"] & R^k \arrow[r, "\psi^{-1}\sigma"] \arrow[d, "h"] & \mathrm{Tr}_\gamma(S) \arrow[r, "\phi", "\cong"'] & S^k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \overset{k}{\underset{i=1}{\oplus}} \mathrm{Ker}\,\pi \arrow[r] & R^k \arrow[rr, "\overset{k}{\underset{i=1}{\oplus}} \pi"] & & S^k \arrow[r] & 0
\end{tikzcd}$$

Then by the 5-Lemma (when we extend by zeros to the left), we obtain that g is also an isomorphism, that is, $\mathrm{Ker}\,\sigma \cong \overset{k}{\underset{i=1}{\oplus}} \mathrm{Ker}\,\pi$. We also have that $\bar{f}' : R \to \mathrm{Ker}\,\sigma$ and $\overset{k}{\underset{i=1}{\oplus}} f' : (R^k)^k \to \overset{k}{\underset{i=1}{\oplus}} \mathrm{Ker}\,\pi$ are projective covers. Hence again by the uniqueness of projective covers up to isomorphism, we obtain $R \cong R^{k^2}$ and then $k^2 = 1$ since commutative rings have the invariant basis number property. Thus $k = 1$ and so

$\mathrm{Tr}_\gamma(S) \cong S$. $\square$

If R is a field, then every R-module is projective and flat and so the conditions (1) and (5) hold in Theorem 2.2.3. So we obtain including the $\mathfrak{m} = 0$ case for Theorem 2.2.3, the following:

Corollary 2.2.5. *For a commutative local ring R with the unique maximal ideal $\mathfrak{m}$ that is finitely generated, the following are equivalent:*

(1) ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$.

(2) $\mathrm{Tr}(S)$ *is projectively equivalent to* $S = R/\mathfrak{m}$.

(3) $\mathfrak{m}$ *is a principal ideal.*

To obtain the characterization for ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ over any commutative ring R, we firstly understand how this behaves under localization:

Lemma 2.2.6. *If every maximal ideal of a commutative ring R is finitely generated, then for a short exact sequence*

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules, we have the following, where for every maximal ideal $\mathfrak{m}$ of R,

$$\mathbb{E}_\mathfrak{m}: \qquad 0 \longrightarrow A_\mathfrak{m} \xrightarrow{f_\mathfrak{m}} B_\mathfrak{m} \xrightarrow{g_\mathfrak{m}} C_\mathfrak{m} \longrightarrow 0$$

is the localization of $\mathbb{E}$ at $\mathfrak{m}$.

(1) $\mathbb{E} \in_R \mathcal{N}eat \iff \mathbb{E}_\mathfrak{m} \in_{R_\mathfrak{m}} \mathcal{N}eat$ *for every maximal ideal $\mathfrak{m}$ of R.*

(2) $\mathbb{E} \in_R \mathcal{P}\text{-}\mathcal{P}ure \iff \mathbb{E}_\mathfrak{m} \in_{R_\mathfrak{m}} \mathcal{P}\text{-}\mathcal{P}ure$ *for every maximal ideal $\mathfrak{m}$ of R.*

Proof. (1) $(\Rightarrow)$: Suppose $\mathbb{E} \in {}_R\mathcal{N}eat$, that is,

$$0 \longrightarrow \mathrm{Hom}_R(S, A) \longrightarrow \mathrm{Hom}_R(S, B) \longrightarrow \mathrm{Hom}_R(S, C) \longrightarrow 0$$

is exact for every simple R-module S. Then since localization is an exact functor (by (Atiyah & Macdonald, 1969, Proposition 3.3)), we have that

$$0 \longrightarrow (\mathrm{Hom}_R(S,A))_{\mathfrak{m}} \longrightarrow (\mathrm{Hom}_R(S,B))_{\mathfrak{m}} \longrightarrow (\mathrm{Hom}_R(S,C))_{\mathfrak{m}} \longrightarrow 0$$

is exact for every simple R-module S and for every maximal ideal $\mathfrak{m}$ of R. Since all simple R-modules are finitely presented by our hypothesis, we have $(\mathrm{Hom}_R(S,X))_{\mathfrak{m}} \cong \mathrm{Hom}_{R_{\mathfrak{m}}}(S_{\mathfrak{m}}, X_{\mathfrak{m}})$ for every R-module X by (Eisenbud, 2013, Proposition 2.10). So we obtain that

$$0 \longrightarrow \mathrm{Hom}_{R_{\mathfrak{m}}}(S_{\mathfrak{m}}, A_{\mathfrak{m}}) \longrightarrow \mathrm{Hom}_{R_{\mathfrak{m}}}(S_{\mathfrak{m}}, B_{\mathfrak{m}}) \longrightarrow \mathrm{Hom}_{R_{\mathfrak{m}}}(S_{\mathfrak{m}}, C_{\mathfrak{m}}) \longrightarrow 0$$

is exact for every simple R-module S and for every maximal ideal $\mathfrak{m}$ of R. In particular for $S = R/\mathfrak{m}$, we obtain that the sequence

$$0 \to \mathrm{Hom}_{R_{\mathfrak{m}}}((R/\mathfrak{m})_{\mathfrak{m}}, A_{\mathfrak{m}}) \to \mathrm{Hom}_{R_{\mathfrak{m}}}((R/\mathfrak{m})_{\mathfrak{m}}, B_{\mathfrak{m}}) \to \mathrm{Hom}_{R_{\mathfrak{m}}}((R/\mathfrak{m})_{\mathfrak{m}}, C_{\mathfrak{m}}) \to 0$$

is exact for every maximal ideal $\mathfrak{m}$ of R where $(R/\mathfrak{m})_{\mathfrak{m}} = R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$ is the (up to isomorphism) unique simple $R_{\mathfrak{m}}$-module. Then $\mathbb{E}_{\mathfrak{m}} \in {}_{R_{\mathfrak{m}}}\mathcal{N}eat$ for every maximal ideal $\mathfrak{m}$ of R.

(1) ($\Leftarrow$) : Suppose $\mathbb{E}_{\mathfrak{m}} \in_{R_{\mathfrak{m}}}\mathcal{N}eat$ for every maximal ideal $\mathfrak{m}$ of R, that is,

$$0 \to \mathrm{Hom}_{R_{\mathfrak{m}}}((R/\mathfrak{m})_{\mathfrak{m}}, A_{\mathfrak{m}}) \to \mathrm{Hom}_{R_{\mathfrak{m}}}((R/\mathfrak{m})_{\mathfrak{m}}, B_{\mathfrak{m}}) \to \mathrm{Hom}_{R_{\mathfrak{m}}}((R/\mathfrak{m})_{\mathfrak{m}}, C_{\mathfrak{m}}) \to 0$$

is exact for every maximal ideal $\mathfrak{m}$ of R. Then, since $(R/\mathfrak{m}')_{\mathfrak{m}} = 0$ if $\mathfrak{m}' \neq \mathfrak{m}$ are maximal ideals of R, we have

$$0 \longrightarrow \mathrm{Hom}_{R_{\mathfrak{m}}}(S_{\mathfrak{m}}, A_{\mathfrak{m}}) \longrightarrow \mathrm{Hom}_{R_{\mathfrak{m}}}(S_{\mathfrak{m}}, B_{\mathfrak{m}}) \longrightarrow \mathrm{Hom}_{R_{\mathfrak{m}}}(S_{\mathfrak{m}}, C_{\mathfrak{m}}) \longrightarrow 0$$

is exact for every simple R-module S and for every maximal ideal $\mathfrak{m}$ of R. Again using

(Eisenbud, 2013, Proposition 2.10), we obtain that

$$0 \longrightarrow (\mathrm{Hom}_R(S,A))_{\mathfrak{m}} \longrightarrow (\mathrm{Hom}_R(S,B))_{\mathfrak{m}} \longrightarrow (\mathrm{Hom}_R(S,C))_{\mathfrak{m}} \longrightarrow 0$$

is exact for every simple R-module S and for every maximal ideal $\mathfrak{m}$ of R. Then since exactness is a local property (by (Kunz, 1985, Corollary IV.1.5)), we obtain that

$$0 \longrightarrow \mathrm{Hom}_R(S,A) \longrightarrow \mathrm{Hom}_R(S,B) \longrightarrow \mathrm{Hom}_R(S,C) \longrightarrow 0$$

is exact for every simple R-module S, that is, $\mathbb{E} \in {}_R\mathcal{N}eat$.

(2) $(\Rightarrow)$: Suppose $\mathbb{E} \in_R \mathcal{P}\text{-}\mathcal{P}ure$, that is,

$$0 \longrightarrow S \underset{R}{\otimes} A \longrightarrow S \underset{R}{\otimes} B \longrightarrow S \underset{R}{\otimes} C \longrightarrow 0$$

is exact for every simple R-module S. Then since localization is an exact functor (by (Atiyah & Macdonald, 1969, Proposition 3.3)), we have that

$$0 \longrightarrow (S \underset{R}{\otimes} A)_{\mathfrak{m}} \longrightarrow (S \underset{R}{\otimes} B)_{\mathfrak{m}} \longrightarrow (S \underset{R}{\otimes} C)_{\mathfrak{m}} \longrightarrow 0$$

is exact for every simple R-module S and for every maximal ideal $\mathfrak{m}$ of R. Since localization commutes with the tensor product (by (Atiyah & Macdonald, 1969, Proposition 3.7)), the sequence

$$0 \longrightarrow S_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} A_{\mathfrak{m}} \longrightarrow S_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} B_{\mathfrak{m}} \longrightarrow S_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} C_{\mathfrak{m}} \longrightarrow 0$$

is exact for every simple R-module S and for every maximal ideal $\mathfrak{m}$ of R. In particular, for $S = R/\mathfrak{m}$, we obtain that

$$0 \longrightarrow (R/\mathfrak{m})_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} A_{\mathfrak{m}} \longrightarrow (R/\mathfrak{m})_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} B_{\mathfrak{m}} \longrightarrow (R/\mathfrak{m})_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} C_{\mathfrak{m}} \longrightarrow 0$$

is exact, that is, $\mathbb{E}_{\mathfrak{m}} \in_{R_{\mathfrak{m}}} \mathcal{P}\text{-}\mathcal{P}ure$ for every maximal ideal $\mathfrak{m}$ of R.

(2) $(\Leftarrow)$: Suppose $\mathbb{E}_{\mathfrak{m}} \in_{R_{\mathfrak{m}}} \mathcal{P}\text{-}\mathcal{P}ure$ for every maximal ideal $\mathfrak{m}$ of R, that is,

$$0 \longrightarrow (R/\mathfrak{m})_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} A_{\mathfrak{m}} \longrightarrow (R/\mathfrak{m})_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} B_{\mathfrak{m}} \longrightarrow (R/\mathfrak{m})_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} C_{\mathfrak{m}} \longrightarrow 0$$

is exact for every maximal ideal $\mathfrak{m}$ of R. Since $(R/\mathfrak{m}')_{\mathfrak{m}} = 0$ if $\mathfrak{m}' \neq \mathfrak{m}$ are maximal ideals of R, we obtain that

$$0 \longrightarrow S_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} A_{\mathfrak{m}} \longrightarrow S_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} B_{\mathfrak{m}} \longrightarrow S_{\mathfrak{m}} \underset{R_{\mathfrak{m}}}{\otimes} C_{\mathfrak{m}} \longrightarrow 0$$

is exact for every simple R-module S and for every maximal ideal $\mathfrak{m}$ of R. Again using the fact that localization commutes with the tensor product, we obtain that the sequence

$$0 \longrightarrow (S \underset{R}{\otimes} A)_{\mathfrak{m}} \longrightarrow (S \underset{R}{\otimes} B)_{\mathfrak{m}} \longrightarrow (S \underset{R}{\otimes} C)_{\mathfrak{m}} \longrightarrow 0$$

is exact for every simple R-module S and for every maximal ideal $\mathfrak{m}$ of R. Since exactness is a local property (by (Kunz, 1985, Corollary IV.1.5)), it follows that the sequence

$$0 \longrightarrow S \underset{R}{\otimes} A \longrightarrow S \underset{R}{\otimes} B \longrightarrow S \underset{R}{\otimes} C \longrightarrow 0$$

is exact for every simple R-module S, that is, $\mathbb{E} \in {}_R\mathcal{P}\text{-}\mathcal{P}ure$. □

Now we can prove our main result. Just note also the following terminology. We call an ideal I of a commutative ring R to be *locally principal* if $I_{\mathfrak{m}}$ is a principal ideal of the local ring $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.

Theorem 2.2.7. *For a commutative ring R, the following are equivalent:*

(1) ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$.

(2) Every maximal ideal $\mathfrak{m}$ *of* R *is finitely generated and* $\mathrm{Tr}(S)$ *is projectively equivalent to* S *for every simple* R*-module* S.

(3) Every maximal ideal $\mathfrak{m}$ *of* R *is finitely generated and* ${}_{R_{\mathfrak{m}}}\mathcal{N}eat = {}_{R_{\mathfrak{m}}}\mathcal{P}\text{-}\mathcal{P}ure$.

(4) Every maximal ideal $\mathfrak{m}$ *of* R *is finitely generated and the maximal ideal* $\mathfrak{m}_{\mathfrak{m}}$ *of the local ring* $R_{\mathfrak{m}}$ *is a principal ideal.*

(5) Every maximal ideal $\mathfrak{m}$ *of* R *is finitely generated and locally principal.*

Proof. (3) $\iff$ (4) follows from Corollary 2.2.5 for the local ring $R_{\mathfrak{m}}$.

(4) $\iff$ (5) is clear since for maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of R, $\mathfrak{m}' \neq \mathfrak{m}$ implies that $\mathfrak{m}_{\mathfrak{m}'} = 0$ is a principal ideal.

(3) $\Rightarrow$ (1) follows by Lemma 2.2.6.

(2) $\Rightarrow$ (4): By Proposition 2.1.3, Tr$(S_{\mathfrak{m}}) \approx (\text{Tr}(S))_{\mathfrak{m}}$ and this is projectively equivalent to $S_{\mathfrak{m}}$ since Tr$(S) \approx S$ by hypothesis (2), where $S = R/\mathfrak{m}$ and $\mathfrak{m}$ is any maximal ideal of R. Then $S_{\mathfrak{m}} = R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$ and by Corollary 2.2.5, Tr$(S_{\mathfrak{m}}) \approx S_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-modules implies that the unique maximal ideal $\mathfrak{m}_{\mathfrak{m}}$ of the local ring $R_{\mathfrak{m}}$ is a principal ideal.

(1) $\Rightarrow$ (2): Suppose ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$. Then every maximal ideal $\mathfrak{m}$ of R is finitely generated by (Fuchs, 2012, Lemma 2.4), so all simple R-modules are finitely presented. Remember that $\mathcal{P}$ is the set of all maximal ideals of R. We obtain that

$$\begin{aligned}
\pi^{-1}(\{S \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\}) &= {}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure \\
&= \tau^{-1}(\{S \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\}) \\
&= \pi^{-1}(\{\text{Tr}(S) \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\})
\end{aligned}$$

where the last equality holds by Theorem 1.1.1. This then implies that Tr$_\beta(S)$ is ${}_R\mathcal{N}eat$-projective for any projective presentation β of any simple R-module S by Theorem 2.2.2-(1). Now our claim is to show that Tr$(S) \approx S$ for any simple R-module S. It suffices to show that Tr$(S) \approx S$ for any non-projective simple R-module S because Tr$(S) \approx S$ does automatically hold if S is a projective simple R-module since in this case both Tr(S) and S are projective.

Let S be a non-projective simple R-module and let β be a projective presentation of it. By the same argument as in the beginning of the proof of ((1) $\Rightarrow$ (2)) part of Theorem 2.2.3, the ${}_R\mathcal{N}eat$-projective module Tr$_\beta(S) = S_1 \oplus S_2 \oplus \ldots \oplus S_j \oplus A$ for some $j \in \mathbb{Z}^+$, where S_1, S_2,…,S_j are simple submodules and A is a finitely generated projective submodule. Using the projective presentation β^* of $\operatorname{Tr}_\beta(S)$, we have that $S \cong \operatorname{Tr}_{\beta^*}\operatorname{Tr}_\beta(S) = \operatorname{Tr}_{\beta^*}(S_1 \oplus S_2 \oplus \ldots \oplus S_j \oplus A) \approx \operatorname{Tr}(S_1 \oplus S_2 \oplus \ldots \oplus S_j \oplus A) \approx \operatorname{Tr}(S_1) \oplus \operatorname{Tr}(S_2) \oplus \ldots \oplus \operatorname{Tr}(S_j) \oplus \operatorname{Tr}(A)$ by Proposition 2.1.2. The module $\operatorname{Tr}(A)$ is projective since A is projective. Then $S \oplus P \cong \operatorname{Tr}(S_1) \oplus \operatorname{Tr}(S_2) \oplus \ldots \oplus \operatorname{Tr}(S_j) \oplus Q$ for some (finitely generated) projective R-modules P and Q. Hence $S \oplus P = M_1 \oplus M_2 \oplus \ldots \oplus M_j \oplus \tilde{Q}$ where the sum on the right is an internal direct sum of submodules of $S \oplus P$ where $\tilde{Q} \cong Q$ and $M_i \cong \operatorname{Tr}(S_i)$ for all $i \in \{1, 2, \ldots, j\}$. Since (semi)simple modules have the exchange property (see Definition 2.2.1), there exist submodules A_1, A'_1, A_2, A'_2, …, A_j, A'_j, $\tilde{Q}'$, $\tilde{Q}''$ of $S \oplus P$ such that

$$S \oplus P = S \oplus A_1 \oplus A_2 \oplus \ldots \oplus A_j \oplus \tilde{Q}', \qquad \tilde{Q}' \oplus \tilde{Q}'' = \tilde{Q} \quad \text{and} \quad A_i \oplus A'_i = M_i \cong \operatorname{Tr}(S_i)$$

for all $i \in \{1, 2, \ldots, j\}$, and so we have

$$P \cong A_1 \oplus A_2 \oplus \ldots \oplus A_j \oplus \tilde{Q}' \quad \text{and} \quad S \cong A'_1 \oplus A'_2 \oplus \ldots \oplus A'_j \oplus \tilde{Q}''$$

The first isomorphism gives us that each A_i is projective. The modules $\tilde{Q}'$ and $\tilde{Q}''$ are also projective as $\tilde{Q}' \oplus \tilde{Q}'' = \tilde{Q} \cong Q$ is projective. Since the simple module S is indecomposable and non-projective, the second isomorphism implies that for some $i_0 \in \{1, 2, \ldots, j\}$, the module A'_{i_0} is isomorphic to the simple module S. Then $S \cong A'_{i_0}$ and $\operatorname{Tr}(S_{i_0}) \cong A_{i_0} \oplus A'_{i_0}$ implies $\operatorname{Tr}(S) \approx \operatorname{Tr}(A'_{i_0}) \approx \operatorname{Tr}(A'_{i_0}) \oplus \operatorname{Tr}(A_{i_0})$ since $\operatorname{Tr}(A_{i_0})$ is a projective module as A_{i_0} is projective, and by Proposition 2.1.2, $\operatorname{Tr}(A'_{i_0}) \oplus \operatorname{Tr}(A_{i_0}) \approx \operatorname{Tr}(A'_{i_0} \oplus A_{i_0}) \cong \operatorname{Tr}(\operatorname{Tr}(S_{i_0})) \approx S_{i_0}$. As a result, $\operatorname{Tr}(S) \approx S_{i_0}$. Suppose for the contrary that $S \ncong S_{i_0}$, say $S \cong R/\mathfrak{m}$ and $S_{i_0} \cong R/\mathfrak{m}_0$ for some different maximal ideals $\mathfrak{m}$ and $\mathfrak{m}_0$ of R. Since $\operatorname{Tr}(S) \approx S_{i_0}$, using Proposition 2.1.3, we obtain $\operatorname{Tr}(S_\mathfrak{m}) \approx (\operatorname{Tr}(S))_\mathfrak{m} \approx (S_{i_0})_\mathfrak{m} = 0$ where the last equality follows since $\mathfrak{m} \neq \mathfrak{m}_0$. Then $S_\mathfrak{m} \approx \operatorname{Tr}\operatorname{Tr}(S_\mathfrak{m}) \approx \operatorname{Tr}(0) \approx 0$. So $S_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module. For every maximal ideal $\tilde{\mathfrak{m}} \neq \mathfrak{m}$, we have that $S_{\tilde{\mathfrak{m}}}$ is zero and so a projective $R_{\tilde{\mathfrak{m}}}$-module. Thus the finitely presented simple module S is a locally projective R-module which then implies that S is a projective R-module since being projective is a local property for finitely presented modules by (Kunz, 1985, Corollary IV.3.6). But this contradicts the non-projectivity of S. Thus we must have $S \cong S_{i_0}$ and so $\operatorname{Tr}(S) \approx S$

as required. $\square$

The motivating result for our problem was Fuchs' characterization of N-domains, that is, the integral domains R where ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$. We can now obtain this because in an integral domain R, a nonzero finitely generated ideal I is invertible if and only if I is locally principal; see (Gabelli, 2013, Proposition 2) or (Gilmer, 1992, Corollary 7.5) or (Kemper, 2011, Proof of Proposition 14.6).

Corollary 2.2.8. *(Fuchs, 2012, Theorem 5.2)* Fuchs' characterization of N-domains. *An integral domain R is an N-domain if and only if all the maximal ideals of R are (finitely generated) projective modules, i.e. every nonzero maximal ideal of R is invertible.*

Proof. Clear if R is a field. Suppose R is not a field, that is, 0 is not a maximal ideal of R.

($\Rightarrow$): By Theorem 2.2.7, every maximal ideal $\mathfrak{m}$ of R is nonzero (R is not a field), finitely generated and locally principal; so it is invertible as noted before the corollary, and hence it is finitely generated and projective.

($\Leftarrow$): Let $\mathfrak{m}$ be a maximal ideal of R. Then it is projective by hypothesis. It is nonzero as R is not a field. Thus it is invertible and so finitely generated since R is an integral domain. Since $\mathfrak{m}$ is projective, its localization $\mathfrak{m}_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module by (Kunz, 1985, Corollary IV.3.6). Then $\mathfrak{m}_{\mathfrak{m}}$ is a principal ideal of the local ring $R_{\mathfrak{m}}$ by the well-known below Proposition 2.2.9 whose proof we include for completeness. Then by Theorem 2.2.7, we obtain ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$, that is, the integral domain R is an N-domain. $\square$

Proposition 2.2.9. *Let R be a commutative local ring with the unique maximal ideal $\mathfrak{m}$. If $\mathfrak{m}$ is projective, then $\mathfrak{m}$ is a principal ideal, and if $\mathfrak{m} \neq 0$, it is generated by a regular element, that is, an element that is not a zerodivisor.*

Proof. Projective modules are free over a local ring by Kaplansky's Theorem (see (Passman, 1991, Theorem 10.8)). A basis element a in the free R-module $\mathfrak{m} \neq 0$ is regular since the natural map $R \to Ra$ sending each $r \in R$ to ra must be an isomorphism. If $\mathfrak{m} \neq 0$ and a basis for $\mathfrak{m}$ contains two distinct elements a and b, then $ab \in Ra \cap Rb = 0$ which yields a contradiction with the regularity of a and b. Thus a basis for $\mathfrak{m} \neq 0$ must consist of only one element, that is, $\mathfrak{m} = Ra$ for some regular element a of R. $\square$

The 'if' part of Corollary 2.2.8 holds over any commutative ring and we will investigate this some more in the next section:

Theorem 2.2.10. *If R is a* commutative *ring such that every maximal ideal of R is finitely generated and projective, then* Tr(S) *is projectively equivalent to S for every simple R-module S and ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. Since it is finitely generated and projective by hypothesis, so is its localization $\mathfrak{m}_\mathfrak{m}$. Then by Proposition 2.2.9, the unique maximal ideal $\mathfrak{m}_\mathfrak{m}$ of the local ring $R_\mathfrak{m}$ is a principal ideal. So ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ and Tr(S) is projectively equivalent to S for every simple R-module S by Theorem 2.2.7. $\square$

In Fuchs (2012), this result was proved over integral domains using free presentations for the Auslander-Bridger transpose of finitely presented simple modules. In the next section, we will use projective presentations of the simple modules when every maximal ideal is finitely generated and projective. See Theorem 2.3.6 and Proposition 2.3.7.

2.3 Finitely Generated and Projective Maximal Ideals

Note firstly the properties of the Auslander-Bridger transpose related to modules of projective dimension ≤ 1 that we will use in this section.

In the notation of Section 2.1 or Theorem 2.3.2 below, for a finitely presented R-module M, where R is a commutative ring, with a presentation γ as below and with the presentation γ^* of $\mathrm{Tr}_\gamma(M)$, we have that $\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) = \mathrm{Coker}(f^{**})$ is defined by the exact sequence

$$\gamma^{**}: \qquad P_1^{**} \xrightarrow{f^{**}} P_0^{**} \xrightarrow{\sigma'} \mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \longrightarrow 0,$$

where σ' is the canonical epimorphism. On the other hand, applying the functor $(-)^*$ to the exact sequence γ^*, we obtain the following exact sequence:

$$0 \longrightarrow (\mathrm{Tr}_\gamma(M))^* \xrightarrow{\sigma^*} P_1^{**} \xrightarrow{f^{**}} P_0^{**}$$

Since we have natural isomorphisms $P \cong P^{**}$ for every finitely generated projective R-module P, we obtain $(\mathrm{Tr}_\gamma(M))^* \cong \mathrm{Im}(\sigma^*) = \mathrm{Ker}(f^{**}) \cong \mathrm{Ker}(f)$. This proves the following which also shows some part of Theorem 2.3.2-(2):

Proposition 2.3.1. *((Hügel & Bazzoni, 2010, Lemma 6.1-(2))) For a finitely presented R-module M, where R is a commutative ring,* $\mathrm{pd}(M) \leq 1$ *if and only if there exists a presentation*

$$\gamma : \qquad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

of M such that (f is monic and) $(\mathrm{Tr}_\gamma(M))^* = 0$.

Theorem 2.3.2. *((Sklyarenko, 1978, Proposition 5.1, Remarks 5.1 and 5.2)) Let M be a* finitely presented *R-module, where R is a commutative ring, and let γ be a presentation of M:*

$$\gamma : \qquad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

(1) *For every R-module N, there is a monomorphism* $\mathrm{Ext}^1_R(M, N) \to \mathrm{Tr}_\gamma(M) \otimes_R N$ *and an epimorphism* $\mathrm{Hom}_R(\mathrm{Tr}_\gamma(M), N) \to \mathrm{Tor}^R_1(M, N)$ *of R-modules. Both are natural in N.*

(2) *If* $\mathrm{pd}(M) \leq 1$, *then the map $f : P_1 \to P_0$ in the above presentation γ can be taken to be a monomorphism and in this case the monomorphism and epimorphism in part (1) become isomorphisms. Moreover by taking $N = R$, we obtain*

$$\mathrm{Tr}_\gamma(M) \cong \mathrm{Ext}^1_R(M, R) \qquad \textit{and} \qquad (\mathrm{Tr}_\gamma(M))^* = \mathrm{Hom}_R(\mathrm{Tr}_\gamma(M), R) = 0$$

for the presentation γ of M where the map $f : P_1 \to P_0$ is a monomorphism.

(3) *If* $\mathrm{pd}(M) \leq 1$, *then* $\mathrm{Tr}(M)$ *is projectively equivalent to* $\mathrm{Ext}^1_R(M, R)$.

(4) *If* $M^* = \mathrm{Hom}_R(M,R) = 0$*, then in the presentation*

$$\gamma^* : \qquad P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \mathrm{Tr}_\gamma(M) \longrightarrow 0,$$

we necessarily have that $f^* : P_0^* \to P_1^*$ *is a monomorphism and so* $\mathrm{pd}(\mathrm{Tr}_\gamma(M)) \leq 1$ *which implies*

$$M \cong \mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \cong \mathrm{Ext}_R^1(\mathrm{Tr}_\gamma(M), R).$$

(5) *If* M *is* not *projective, then* $\mathrm{Tr}_\gamma(M) \neq 0$.

Proof. (1), (2) and (3) follows from (Sklyarenko, 1978, Proposition 5.1, Remarks 5.1 and 5.2). The first part of (4) is obtained by applying the left exact functor $(-)^* = \mathrm{Hom}_R(-,R)$ to γ to obtain the exact sequence

$$0 \longrightarrow M^* \xrightarrow{g^*} P_0^* \xrightarrow{f^*} P_1^*$$

Since $M^* = 0$ by hpothesis, this gives us that f^* is monic and so the presentation γ^* implies that $\mathrm{pd}(\mathrm{Tr}_\gamma(M)) \leq 1$. Then the isomorphism $\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \cong \mathrm{Ext}_R^1(\mathrm{Tr}_\gamma(M), R)$ for the finitely presented R-module $\mathrm{Tr}_\gamma(M)$ follows from (2). We already know that $M \cong \mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M))$, see Section 2.1. (5) is easily seen since if $\mathrm{Tr}_\gamma(M) = 0$, then $M \cong \mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) = \mathrm{Tr}_{\gamma^*}(0)$ is projectively equivalent to 0, and so M is projective. □

In (Hügel & Bazzoni, 2010, Lemma 6.1-(1)), it is stated that for a finitely presented R-module U, $\mathrm{pd}(\mathrm{Tr}(U)) \leq 1$ if and only if $U^* = 0$. For the 'only if' part, we need to assume that U has no nonzero projective direct summands:

Theorem 2.3.3. *(Türkoğlu, 2013, Theorem 4.4.1) Let* M *be a finitely presented* R*-module, where* R *is a commutative ring.*

(1) *If* $M^* = 0$*, then* $\mathrm{pd}(\mathrm{Tr}(M)) \leq 1$.

(2) *If* $\mathrm{pd}(\mathrm{Tr}(M)) \leq 1$*, then* M^* *is projective and finitely generated.*

(3) If pd(Tr(M)) $\leq$ 1 and M *has* no *nonzero projective direct summands, then* $M^* = 0$.

Proof. (1) is just Theorem 2.3.2-(4). See the proof of (Hügel & Bazzoni, 2010, Lemma 6.1-(1)). Suppose Tr(M) has projective dimension at most 1. Let γ be a presentation of M:

$$\gamma : \qquad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

We have the following exact sequence

$$0 \longrightarrow M^* \xrightarrow{g^*} P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \mathrm{Tr}_\gamma(M) \longrightarrow 0$$

and so pd(Tr(M)) ≤ 1 implies that M^* is projective. Indeed $M^* \cong \mathrm{Im}(g^*) = \mathrm{Ker}(f^*)$ is a direct summand of P_0^* (since $\mathrm{Im}(f^*)$ is projective as pd(Tr(M)) $\leq$ 1). Since P_0^* is finitely generated, so is its direct summand. By (Maşiek, 2000, Proposition 5) (which holds over any commutative ring without assuming it is Noetherian), we have the following exact sequence:

$$0 \longrightarrow \mathrm{Ext}_R^1(\mathrm{Tr}(M), R) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow \mathrm{Ext}_R^2(\mathrm{Tr}(M), R) \longrightarrow 0,$$

where $\sigma_M : M \to M^{**}$ is the natural map into the double dual. The last term $\mathrm{Ext}_R^2(\mathrm{Tr}(M), R) = 0$ since pd(Tr(M)) $\leq$ 1. Since M^* is finitely generated and projective, M^{**} is also projective and so the exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^1(\mathrm{Tr}(M), R) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow 0,$$

splits which gives $M \cong M^{**} \oplus \mathrm{Ext}_R^1(\mathrm{Tr}(M), R)$. If we assume that M has no nonzero projective direct summands, the projective direct summand $M^{**} = 0$ must hold. This then gives $M^* \cong M^{***} = 0^* = 0$, that is, $M^* = 0$. □

Since the only nonzero direct summand of a simple module is itself, we obtain:

Corollary 2.3.4. *(Türkoğlu, 2013, Corollary 4.4.2) If S is a finitely presented simple*

R-module that is not *projective, where R is a commutative ring, then*

$$\mathrm{pd}(\mathrm{Tr}(S)) \leq 1 \qquad \Leftrightarrow \qquad S^* = 0.$$

This corollary does not hold if S is projective: for a finitely generated nonzero projective module M, we have $\mathrm{Tr}(M) \approx 0$ and so $\mathrm{pd}(\mathrm{Tr}(M)) = 0 \leq 1$ but $M^* \neq 0$ because otherwise $M \cong M^{**} = 0^* = 0$ would hold. Thus:

Proposition 2.3.5. *(Türkoğlu, 2013, Proposition 4.4.3) Let R be a commutative ring. The following are equivalent for a finitely presented simple R-module S for which S^* is homogeneous semisimple with every simple submodule isomorphic to S:*

(1) $\mathrm{pd}(\mathrm{Tr}(S)) \leq 1$.

(2) $S^* = 0$ *or* S *is projective.*

(3) S^* *is projective (and finitely generated).*

Proof. Equivalence of (1) and (2) follows from the above corollary. (2) implies (3) clearly.

(3)⇒(2): The simple module $S \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. So $S^* = \mathrm{Hom}_R(S, R)$ is annihilated by $\mathfrak{m}$ also and thus S^* is homogeneous semisimple with every simple submodule isomorphic to $S \cong R/\mathfrak{m}$. Thus $S^* = \bigoplus_{i \in I} S_i$ for some simple submodules $S_i \cong S$ of S^* for each $i \in I$, where I is some indexing set. So if S^* is projective, there are two cases: either $S^* = 0$ (index set $I = \emptyset$) or $S^* \neq 0$. In the second case, the projective module S^* has a direct summand isomorphic to S which must then be also projective. □

Note that for a commutative ring R and a simple R-module S, $S^* = 0$ if and only if R has no simple submodule isomorphic to S. So $\mathrm{Soc}\, R = 0$ if and only if $S^* = 0$ for every simple R-module S. Commutative rings with zero socle are in some sense near to domains and examples of such rings are given in Example 2.3.9. Indeed, it suffices for some results below to assume a bit less as in the above proposition and such rings in the noncommutative case have been named left PS-rings in Nicholson & Watters (1988): rings with projective left socle. We say that a commutative ring R is a *PS-ring* if Soc R is projective and that is equivalent to the following: $S^* = 0$ or S is projective for every simple R-module S. See (Nicholson & Watters, 1988, Theorem 2.4).

In Theorem 2.2.10, we have seen that if every maximal ideal of a commutative ring R is finitely generated and projective, then Tr(S) is projectively equivalent to S for every simple R-module S. In the next theorem, we show that if γ is the below natural projective presentation of the simple module S in this case, then $\mathrm{Tr}_\gamma(S)$ is isomorphic to S if S is not projective. This also generalizes the result (Angeleri Hügel & Sánchez, 2011, Corollary 6.18) for simple modules over Dedekind domains.

Theorem 2.3.6. *(Türkoğlu, 2013, Theorem 4.4.4) Let R be a commutative ring and $\mathfrak{m}$ be a* finitely generated *maximal ideal of R that is* projective*. Take the following projective presentation of the simple module $S = R/\mathfrak{m}$, where f is the inclusion monomorphism and g is the natural epimorphism:*

$$\gamma : \qquad \mathfrak{m} \xrightarrow{f} R \xrightarrow{g} S \longrightarrow 0$$

(1) If S is projective, then $S^* \neq 0$ and $\mathrm{Tr}_\gamma(S) = 0$.

(2) If S is not *projective, then $S^* = 0$ and* $\mathrm{Tr}_\gamma(S) \cong \mathrm{Ext}^1_R(S, R) \cong S$.

(3) S is projective if and only if $S^* \neq 0$.

Proof. For the finitely presented simple module S, pd$(S) \leq 1$ by hypothesis. So we obtain $\mathrm{Tr}_\gamma(S) \cong \mathrm{Ext}^1_R(S, R)$ by Theorem 2.3.2-(2). Since $\mathrm{Ext}^1_R(S, R)$ is annihilated by $\mathfrak{m}$ (as S is annihilated by $\mathfrak{m}$), it must be a homogeneous semisimple module with every simple submodule isomorphic to S. Since $\mathrm{Tr}_\gamma(S)$ is finitely generated, $\mathrm{Ext}^1_R(S, R)$ must be a finite direct sum of copies of S, that is, $\mathrm{Tr}_\gamma(S) \cong \mathrm{Ext}^1_R(S, R) \cong \bigoplus_{i=1}^m S$ for some $m \in \mathbb{Z}^+ \cup \{0\}$.

If S is projective, then $\mathrm{Ext}^1_R(S, R) = 0$ and $S^* \neq 0$ (because otherwise for the finitely generated nonzero projective module S, $S \cong S^{**} = 0^* = 0$ would hold); this proves (1).

To prove (2), assume that S is not projective. By Theorem 2.3.2-(5), $\mathrm{Tr}_\gamma(S) \neq 0$. So the m in $\mathrm{Tr}_\gamma(S) \cong \mathrm{Ext}^1_R(S,R) \cong \bigoplus_{i=1}^m S$ must be positive. Since $\mathrm{Tr}_\gamma(S) \cong \bigoplus_{i=1}^m S$ and $\mathrm{pd}(S) \leq 1$ by hypothesis, we have $\mathrm{pd}(\mathrm{Tr}_\gamma(S)) \leq 1$. Then by Corollary 2.3.4, $S^* = 0$. So we can use Theorem 2.3.2-(4) to obtain that $S \cong \mathrm{Ext}^1_R(\mathrm{Tr}_\gamma(S), R)$. Thus

$$S \cong \mathrm{Ext}^1_R(\mathrm{Tr}_\gamma(S), R) \cong \mathrm{Ext}^1_R\left(\bigoplus_{i=1}^m S, R\right) \cong \bigoplus_{i=1}^m \mathrm{Ext}^1_R(S,R) \cong \bigoplus_{i=1}^m \left[\bigoplus_{j=1}^m S\right].$$

This then implies that $m = 1$ by the results for the structure of semisimple modules.

(3) follows just by (1) and (2). □

Nunke shows that $I^{-1}/R \cong R/I$ if I is a nonzero ideal of a Dedekind domain R; see (Nunke, 1959, Lemma 4.4). For an invertible maximal ideal $\mathfrak{m}$ of a commutative ring R, we show next that $\mathfrak{m}^{-1}/R \cong R/\mathfrak{m}$ where the invertibility is in the total quotient ring of R, that is, the localization of R with respect to the set of all regular elements of R, and $\mathfrak{m}^{-1}$ consists of all q in the total quotient ring of R such that $q\mathfrak{m} \subseteq R$.

Proposition 2.3.7. *(Türkoğlu, 2013, Proposition 4.4.5) If R is a commutative ring and $\mathfrak{m}$ is a maximal ideal of R that is invertible in the total ring of quotients of R, then for the simple module $S = R/\mathfrak{m}$ and for the projective presentation*

$$\gamma: \qquad \mathfrak{m} \xrightarrow{f} R \xrightarrow{g} S \longrightarrow 0$$

of S, where f is the inclusion monomorphism and g is the natural epimorphism, we have

$$\mathrm{Tr}_\gamma(S) \cong \mathfrak{m}^{-1}/R \cong S = R/\mathfrak{m}.$$

Proof. By (Lam, 1999, Theorem 2.17), $\mathfrak{m}$ must be projective and it contains a regular element. So $S^* = 0$ by the below Example 2.3.9-(2) and $\mathrm{pd}(S) \leq 1$. Then by Theorem 2.3.6, $\mathrm{Tr}_\gamma(S) \cong S = R/\mathfrak{m}$. The following exact sequence defines $\mathrm{Tr}_\gamma(S)$:

$$\mathrm{Hom}_R(R,R) \xrightarrow{f^*} \mathrm{Hom}_R(\mathfrak{m},R) \xrightarrow{\sigma} \mathrm{Tr}_\gamma(S) = \mathrm{Hom}_R(\mathfrak{m},R)/\operatorname{Im}(f^*) \longrightarrow 0, \quad (2.2)$$

where σ is the canonical epimorphism. By (Lam, 1999, Theorem 2.14), we have $\mathfrak{m}^{-1} \cong \mathrm{Hom}_R(\mathfrak{m}, R)$ by the isomorphism $\beta : \mathfrak{m}^{-1} \to \mathrm{Hom}(\mathfrak{m}, R)$ given for each $q \in \mathfrak{m}^{-1}$ by $\beta(q)(p) = pq$ for every $p \in \mathfrak{m}$. Observe that under this isomorphism R goes onto $\mathrm{Im}(f^*)$. So $\mathrm{Tr}_\gamma(S) = \mathrm{Hom}_R(\mathfrak{m}, R)/\mathrm{Im}(f^*) \cong \mathfrak{m}^{-1}/R$. □

As noted in the introduction, for a commutative ring R, the condition ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ does not imply that every maximal ideal of R is finitely generated and projective. This will hold only if R is a PS-ring:

Theorem 2.3.8. *For a commutative ring R, the following are equivalent:*

(1) R is a PS-ring and ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$.

(2) Every maximal ideal of R is finitely generated and projective.

(3) R is a PS-ring, every maximal ideal of R is finitely generated and $\mathrm{Tr}(S)$ is projectively equivalent to S for every simple R-module S.

Proof. (3) ⇒ (1) follows from Theorem 2.2.7.

(1) ⇒ (2): ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ implies that every maximal ideal is finitely generated by (Fuchs, 2012, Lemma 2.4), so every simple R-module is finitely presented. Also $S^* = 0$ or S is projective for every simple R-module S since $\mathrm{Soc}(R)$ is projective by hypothesis. Then by Proposition 2.3.5, we obtain $\mathrm{pd}(\mathrm{Tr}(S)) \leq 1$ for every simple R-module S. Let $\mathcal{P} = \{\mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal of } R\}$. Let U be a simple R-module. Then U is ${}_R\mathcal{N}eat$-projective since ${}_R\mathcal{N}eat = \pi^{-1}(\{S \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\})$. On the other hand, our hypothesis ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ implies by Theorem 1.1.1 that

$$\tau^{-1}(\{S \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\}) = \pi^{-1}(\{\mathrm{Tr}(S) \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\}),$$

and so by Theorem 2.2.2-(4), the ${}_R\mathcal{N}eat$-projective module U is isomorphic to a direct summand of a free R-module and modules in $\{\mathrm{Tr}(S) \mid S = R/\mathfrak{m} \text{ and } \mathfrak{m} \in \mathcal{P}\}$ all of which have projective dimension ≤ 1 since $\mathrm{pd}(\mathrm{Tr}(S)) \leq 1$ for every simple R-module

S. Thus we obtain that U also has projective dimension ≤ 1 for every simple R-module U, that is, every maximal ideal of R is projective.

$(2) \Rightarrow (3)$: By hypothesis, every simple R-module is finitely presented. Our claim is to show that R is a PS-ring, that is, Soc(R) is projective. Let $\mathfrak{m}$ be a maximal ideal of R, which is finitely generated and projective by hypothesis, and take the simple R-module $S = R/\mathfrak{m}$. If it is not projective, then by Theorem 2.3.6-(2), we obtain $S^* = 0$. Thus either S is projective or $S^* = 0$ for every simple R-module S, which implies that R is a PS-ring. By Theorem 2.2.10, Tr(S) is projectively equivalent to S for every simple R-module S, or use Theorem 2.3.6-(1 and 2) to obtain this. $\square$

Example 2.3.9. Examples of commutative rings R such that $S^* = 0$ for every simple R-module S, that is, Soc$(R) = 0$, are given below:

(1) Commutative domains are clearly among such rings.

(2) Commutative rings in which every maximal ideal contains a regular element are also among such rings. Let R be such a ring. For the simple module $S = R/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of R, if $f : S \to R$ is in $S^* = \text{Hom}_R(S, R)$, then either Im$(f) = 0$ (and so $f = 0$) or Im$(f) \cong S$ is simple. If Im$(f) \cong S$, then Im$(f) = Ra \subseteq R$ for some $a \in R$ such that $\mathfrak{m}$ is the annihilator of a. So $\mathfrak{m}a = 0$ and $a \neq 0$ (because $Ra \cong R/\mathfrak{m} = S \neq 0$). But by our hypothesis, every maximal ideal of R contains a regular element, say $b \in \mathfrak{m}$ is a regular element. Then $\mathfrak{m}a = 0$ implies that $ba = 0$ contradicting regularity of b since $a \neq 0$. This contradiction shows that Im$(f) \cong S$ is not possible. So for every $f \in S^*$, we must have $f = 0$. That is, $S^* = 0$ for every simple R-module S.

(3) A finite product of commutative domains are commutative rings such that every maximal ideal contains a regular element. It suffices to prove this for $R \times S$ where R and S are commutative rings such that every maximal ideal contains a regular element. Since ideals of $R \times S$ are necessarily of the form $I \times J$ for some ideal I of R and some ideal J of S, the maximal ideals of $R \times S$ are of the form $\mathfrak{m} \times S$ or $R \times Q$ where $\mathfrak{m}$ is a maximal ideal of R and Q is a maximal ideal of S. Since $\mathfrak{m}$ contains a regular element a in R, the element $(a, 1) \in \mathfrak{m} \times S$ is a regular element of $R \times S$. Similary $R \times Q$ also contains a regular element.

(4) A finite product of commutative rings each of which satisfies the property that every maximal ideal contains a regular element is also a commutative ring such that every maximal ideal contains a regular element. This is what we proved in (3) above.

(5) Patrick F. Smith has showed that for any nontrivial commutative ring R, the polynomial ring $R[x]$ and the formal power series ring $R[[x]]$ are rings in which every maximal ideal contains a regular element. Let R be any nontrivial commutative ring (that is, $1 \neq 0$). Let $\mathfrak{m}$ be a maximal ideal of the polynomial ring $S = R[x]$. If $x \in \mathfrak{m}$, then $\mathfrak{m}$ contains the regular element x. Suppose that $x \notin \mathfrak{m}$. Then Sx is an ideal of S different from $\mathfrak{m}$ and hence $S = Sx + \mathfrak{m}$. There exists a polynomial $f(x) \in S$ such that $1 - f(x)x \in \mathfrak{m}$. Observe that $1 - f(x)x$ is a regular element of S for every polynomial $f(x) \in S$. Because if $(1 - f(x)x)g(x) = 0$ for some nonzero $g(x) \in S$, then by taking a to be the nonzero coefficient of $g(x)$ corresponding to the lowest possible power of x, we obtain $1a = 0$, a contradiction. Similar proof can be adapted to show that every maximal ideal of the formal power series ring $R[[x]]$ contains a regular element. In (Nicholson & Watters, 1988, Theorem 3.1), it is shown that $R[x]$ and $R[[x]]$ are PS-rings if R is a PS-ring. As the above argument shows this always holds (even in the noncommutative case) for a nontrivial commutative ring R.

Remark 2.3.10. By Example 2.3.9-(2), if R is a commutative ring such that every maximal ideal contains a regular element, then $\mathrm{Soc}(R) = 0$. Conversely, if R is a commutative Noetherian local ring such that $\mathrm{Soc}(R) = 0$, then the unique maximal ideal of R contains a regular element by for example (Northcott, 1960, §9.4, Proposition 6*).

CHAPTER THREE
NEAT HOMOMORPHISMS AND EXTENSIONS OF HOMOMORPHISMS

This chapter consists of the extended version of Er & Türkoğlu (2018). Let us start with giving the definition of neat homomorphisms and a characterization of them which is given in Enochs (1971) and Bowe (1972).

A homomorphism $f : M \to N$ of modules is called neat if given any proper submodule H of a module G and any homomorphism $\sigma : H \to M$, the homomorphism σ has a proper extension in G whenever $f\sigma$ has a proper extension in G, that is, a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\sigma"'] & G' \arrow[r] \arrow[dr, dashed, "\tau"] & G \\ M \arrow[rr, "f"'] & & N \end{tikzcd}$$

with $H \subsetneq G' \subseteq G$, always guarantees the existence of a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\sigma"'] & G'' \arrow[r] \arrow[dl, dashed] & G \\ M \arrow[rr, "f"] & & N \end{tikzcd}$$

with $H \subsetneq G'' \subseteq G$. Some characterizations of neat homomorphisms were given by Theorem 1.2 in Bowe (1972). We use the following characterization : a homomorphism $f : M \to N$ of modules is a neat homomorphism if and only if there are no proper extensions of f in the injective hull $\mathrm{E}(M)$ of M.

In the next section, we have characterized the rings where all nonzero R-module epimorphisms with cyclic domains are neat.

3.1 Rings Over Which Nonzero Epimorphisms With Cyclic Domains Are Neat

In Osofsky (1964), it was shown that a ring all of whose cyclic modules are injective is semisimple Artinian. Also in Osofsky (1968b), that proof is greatly simplified. After this result, the following question type is raised: what is the structure of rings with the cyclic modules (or proper cyclic modules) having a fixed property? There are a lot of publications about this question, see for example Osofsky (1968a), Ahsan (1973), Goel & Jain (1978), Smith (1979), Jain & Saleh (1987), van Huynh & Dan (1989), Tuganbaev (2011), Aydoğdu et al. (2012), Er et al. (2016). With this motivation, we deal with the rings R where all nonzero R-module epimorphisms with cyclic domains are neat. We call this property (P). To deal with such type of questions have been suggested by Noyan Er, and we proved that the following Theorem.

Theorem 3.1.1. *The following conditions are equivalent for a ring R:*

(1) R satisfies the property (P), that is, every nonzero R module epimorphism with cyclic domain is neat.

(2) Either R is semisimple Artinian or the injective hull of any cyclic R-module is uniserial.

The proof of this theorem will be divided into propositions.

Proposition 3.1.2. *If a ring R satisfies (P), then any cyclic R-module A is either above or below each submodule of its injective hull $E(A)$.*

Proof. Let R be a ring which satisfies (P). Let A be a nonzero cyclic R-module and $B \subseteq E(A)$ be any submodule. Assume that A and B could be mutually exclusive. The obvious (nonzero) epimorphism $A \to \frac{A}{A \cap B}$ would then properly extend to the composition $A + B \to \frac{A+B}{A \cap B} \to \frac{A}{A \cap B}$, where the former map is the canonical epimorphism and the latter is the projection $\frac{A}{A \cap B} \oplus \frac{B}{A \cap B} \to \frac{A}{A \cap B}$, contradicting the assumption of the property (P). This proves that a cyclic module A is either above or below each submodule B of its injective hull $E(A)$. □

Proposition 3.1.3. *If a ring R satisfies (P), then any injective hull of nonzero non-injective cyclic R-module is indecomposable (uniform).*

Proof. Let R be a ring which satisfies (P). Let A be a nonzero non-injective cyclic R-module. Suppose for a moment $E(A) = X \oplus Y$. Then one of X and Y would have to be zero, since, otherwise, one of them would have to be mutually exclusive with

A, contradicting by the 3.1.2. So if A is nonzero non-injective cyclic, then $E(A)$ is indecomposable, hence uniform. $\square$

Proposition 3.1.4. *If a ring R satisfies (P), then any injective hull of nonzero non-injective cyclic R-module is uniserial.*

Proof. Let R be a ring which satisfies (P). Let A be a nonzero non-injective cyclic R-module. It is clear that injective hull $E(A)$ of A is uniform by Proposition 3.1.3. Let B and C are any two nonzero cyclic submodules of $E(A)$, then $E(B) = E(C) = E(A)$, since $E(A)$ is uniform, so that again by Proposition 3.1.2 one of B and C contains the other, showing that $E(A)$ is uniserial. $\square$

Proposition 3.1.5. *If a ring R satisfies (P), then R is a right q.f.d. ring, that is, every cyclic right R-module has finite Goldie dimension (or uniform dimension).*

Proof. Let R be a ring which satisfies (P). Let A be a cyclic R-module. A is injective or A is uniform by Proposition 3.1.3. It is clear that injective R-modules and uniform R-modules are extending. Hence every cyclic R-module is an extending module, furthermore every cyclic R-module is a completely extending module. Then by a well known result due to Osofsky & Smith (1991, Theorem 1), R is a right q.f.d. $\square$

Proposition 3.1.6. *If a ring R satisfies (P), then R is semiperfect and direct sum of uniform right ideals, say $R = \overset{n}{\underset{i=1}{\oplus}} e_i R$ where $\{e_i : i = 1, 2, ..., n\}$ is a complete set of primitive orthogonal idempotents.*

Proof. Let R be a ring which satisfies (P). It is clear that R is cyclic R-module so R is right uniserial by Proposition 3.1.4 or right self-injective ring with finite uniform dimension by Proposition 3.1.5. In any case R is semiperfect and direct sum of uniform right ideals, say $R = \overset{n}{\underset{i=1}{\oplus}} e_i R$ where $\{e_i : i = 1, 2, ..., n\}$ is a complete set of primitive orthogonal idempotents. $\square$

Proposition 3.1.7. *If a ring R satisfies (P), then either R semisimple Artinian or the injective hull of any cyclic module is uniserial.*

Proof. Let R be a ring which satisfies (P). Suppose for a moment there is a cyclic module with non-uniserial injective hull. Then that cyclic module must be injective by Proposition 3.1.4 and so by assumption non-uniserial as it is own injective hull. This implies that R_R is non-uniserial, otherwise every cyclic module is uniserial. R is semiperfect and direct sum of uniform right ideals by Proposition 3.1.6, say $R = \overset{n}{\underset{i=1}{\oplus}} e_i R$ where $\{e_i : i = 1, 2, ..., n\}$ is a complete set of primitive orthogonal idempotents. If $n = 1$, then R is injective hull of every cyclic submodule because R is self-injective and uniform. Furthermore by Proposition 3.1.2 R is uniserial. This contradicts with the non-uniserial property of R. Thus we must have $n \geq 2$. Now for $i \neq j \in \{1, 2, ..., n\}$ and any right ideal A properly contained in $e_j R$ the module $\frac{e_i R \oplus e_j R}{A} \cong e_i R \oplus \frac{e_j R}{A}$ being non-uniserial and cyclic, is injective by Proposition 3.1.4. Thus, every factor of $e_j R$ is injective. Since each $e_i R$ is a local uniform module and $e_i J(R)$ is a sum of factors of $e_k R$ ($k \in \{1, 2, ..., n\}$), by the preceeding argument $e_i J(R)$ must be zero. Hence from the simplicity of $\frac{e_i R}{e_i J}$ we have $e_i R$ is simple implies that R is semisimple Artinian. This completes the proof. □

Proposition 3.1.8. *If either R is semisimple Artinian or injective hull of any cyclic module is uniserial, then R satisfies the property (P).*

Proof. Let R be a semisimple Artinian ring. It is clear by the well known result in Osofsky (1964) every cyclic R-module M is injective. Hence for any cyclic R-module M has no proper extension in $E(M)$. This proves that if R is semisimple Artinian ring, then every nonzero R module epimorphism with cyclic domain is neat, that is, R satisfies (P). It suffices to prove that if injective hull of any cyclic R-module M is uniserial, then any nonzero R module epimorphism with cyclic domain is neat. Let M be a cyclic R-module and its injective hull $E(M)$ be a uniserial R-module. Suppose for the contrary that there exist a proper submodule A of M such that $\pi : M \to M/A$ is not neat. This means that π has a proper extension $\tilde{\pi}$ in $E(M)$ where $\tilde{\pi} : M' \to M/A$, where $M \leq M'$. It is clear that Ker$(\tilde{\pi}) \subseteq M$ or $M \subseteq$ Ker$(\tilde{\pi})$ since $E(M)$ is uniserial. Furthermore Ker$(\tilde{\pi}) \subseteq M$ since $\tilde{\pi}_{|_M}$ is onto and Ker$(\tilde{\pi}) \cap M =$ Ker$(\pi) = A$. We have $\tilde{\pi}(M') = \tilde{\pi}(M) = M/A$ by our assumption. If we take inverse image of both side, then $M' + \text{Ker}(\tilde{\pi}) = M + \text{Ker}(\tilde{\pi})$. Hence by the above observation $M' = M$. This contradicts with $\tilde{\pi}$ is a proper extension of π in $E(M)$. This contradiction completes the proof. □

Proof of Theorem 3.1.1 By Propositions 3.1.7 and 3.1.8.

3.2 Rings over which Every Non-injective is t.i.n.

Recall that an R-module M is said to be injective provided that the functor $\text{Hom}_R(-, M)$ is exact. Of course M is injective if and only if $\text{Ext}_R(N, M) = 0$ for all R-modules N. For any ring R, it is well known that every R-module is a submodule of an injective R-module. So a possible strategy to investigate all R-modules is to describe all injective R-modules and for each injective R-module, I, all its submodules. There is a huge list of studies on the injectivity of modules and rings with different perspectives, considerable amount of which involves notions derived from relative injectivity, i.e, injectivity of a module with respect to a fixed module inspired by Baer's Criterion.

In the literature of injectives, there are different "test modules for injectivity" definitions and a prototype question is: What are the rings over which all modules are injective or "test module for injectivity by a property"? Also we have introduced neat homomorphisms in Section 3.1. With these motivation, we deal with the rings R which has property (Q), that is, every non-injective module is t.i.n. An R-module M is called test for injectivity by neatness (t.i.n.) if all nonzero maps $N \to M$ being neat implies that N is injective. To deal with such type of questions suggested by Noyan Er, we proved that the following Theorem:

Before giving the theorem recall that a right V-ring is one whose simple right modules are injective, equivalently, every right module has zero Jacobson radical (see Cozzens & Faith (1975)).

Theorem 3.2.1. *The following conditions are equivalent for a ring R:*

(1) R satisfies the property (Q), that is, every non-injective R-module is t.i.n.

(2) R *is a right* V*-ring and for any two non-injective modules* A *and* B $\mathrm{Hom}_R(A, B) \neq 0$*.*

(3) $R \cong \Lambda' \times \Lambda$ *such that* Λ' *is semisimple Artinian and* Λ *is a right* V*-ring with* $\mathrm{Soc}(\Lambda_\Lambda)$ *homogeneous and* $\frac{\Lambda}{\mathrm{Soc}(\Lambda_\Lambda)}$ *is semisimple Artinian.*

Let us pointing out that every ring R has a t.i.n. R-module.

Lemma 3.2.2. *Every ring* R *has an injective t.i.n.* R*-module.*

Proof. It is clear that for being t.i.n. of an injective module E, we must guarantee that there exists a nonzero homomorphism from A to E. This is point out that an injective cogenerator. Hence let E be an injective cogenerator. Let every element of $\mathrm{Hom}_R(A, E) \neq 0$ is a neat homomorphism. Then it is clear that A is injective, otherwise $A \neq E(A)$ and for any element $f \in \mathrm{Hom}_R(A, E) \neq 0$ extends to $\tilde{f} : E(A) \to E$. But this contradicts with the neatness of f. $\square$

For example $E = E(M)$ where M is a direct sum of cyclic right modules one from each isomorphism class.

Recall that a ring R is said to be semi-Artinian if it is semi-Artinian as a right R-module, i.e., every nonzero cyclic module has nonzero socle. There are alternative characterizations of semi-Artinian rings (see for example Tuganbaev (2012), Stenström (1975)). Let us note that some well-known results and its proofs with the following lemmas.

Lemma 3.2.3. *If* $R/\mathrm{Soc}(R_R)$ *is a semisimple Artinian ring, then* R *is a semi-Artinian ring.*

Proof. Let M be a nonzero R-module. Take any cyclic N in M. Let $N \cong R/A$ for some $A \leq R$. We have two cases $\mathrm{Soc}(R_R) \subseteq A$ or $\mathrm{Soc}(R_R) \subsetneq A$. If $\mathrm{Soc}(R_R) \subseteq A$, then $R/A \cong \frac{R/\mathrm{Soc}(R_R)}{A/\mathrm{Soc}(R_R)}$. R/A is a semisimple $R/\mathrm{Soc}(R_R)$-module by this isomorphism, so as semisimple R-module. If $\mathrm{Soc}(R_R) \subsetneq A$, then there exists a simple $S \subsetneq A$ then the submodule $\frac{S \oplus A}{A} \cong S$. In both case $\mathrm{Soc}(N) \neq 0$, hence

$\mathrm{Soc}(M) \neq 0$, that is, R is a semi-Artinian ring see the Proposition 2.5 of Stenström (1975). □

Lemma 3.2.4. *If R satisying for any two non-injective modules A and B, $\mathrm{Hom}_R(A, B) \neq 0$ and R has a ring decomposition $R = \Lambda' \times \Lambda$ such that Λ' is semisimple Artinian, then Λ satisying for any two non-injective Λ-modules A and B, $\mathrm{Hom}_\Lambda(A, B) \neq 0$.*

Proof. Let R satisying for any two non-injective modules A and B, $\mathrm{Hom}_R(A, B) \neq 0$ and R has a ring decomposition $R = \Lambda' \times \Lambda$ such that Λ' is semisimple Artinian. Let A, B non-injective Λ-modules implies A, B non-injective R-modules. Otherwise A is injective R-module implies A must be injective Λ-module since Λ' semisimple Artinian. So there exists a nonzero R-module $f : A_\Lambda \to B_\Lambda$ by our assumption. But clearly f is also nonzero Λ-module homomorphism. Hence Λ satisfying for any two non-injective Λ-modules A and B, $\mathrm{Hom}_\Lambda(A, B) \neq 0$. □

Before the following lemma recall that a right SI-ring is a ring over which every singular right R-module is injective. Such rings were characterized by Theorem 3.11 in Goodearl (1972), that is, R is a right SI-ring if and only if there is a ring decomposition $R = K \times R_1 \times R_2 \times ... \times R_n$ such that $K/\mathrm{Soc}(K_K)$ is a semisimple Artinian ring and each R_i is Morita-equivalent to a right SI-domain.

Lemma 3.2.5. *If a ring R is a right V-ring such that $R/\mathrm{Soc}(R_R)$ semisimple Artinian, then R is a right SI-ring.*

Proof. Let R be a right V-ring such that $R/\mathrm{Soc}(R_R)$ semisimple Artinian. Let I be right ideal of R where $\mathrm{Soc}(R_R) \cap I = 0$. It is cleat that $I \cong \frac{I \oplus \mathrm{Soc}(R_R)}{\mathrm{Soc}(R_R)}$, that is, I is a semisimple $R/\mathrm{Soc}(R_R)$ right ideal, so as I semisimple R-ideal. This implies $I \subseteq \mathrm{Soc}(R_R)$, so I must be 0. Hence $\mathrm{Soc}(R_R) \leq_e R$. Every simple right ideal of R is injective by the right V property of R. Furthermore every simple right ideal is projective, so nonsingular. Hence $\mathrm{Soc}(R_R)$ is nonsingular right ideal of R. As a consequence R_R is also nonsingular since $\mathrm{Soc}(R_R) \leq_e R$ and $\mathrm{Soc}(R_R)$ nonsingular. Let I be any essential right ideal of R. It is clear that $\mathrm{Soc}(R_R) \subseteq I$ and R/I can be seen homomorphic image of $R/\operatorname{Soc}(R_R)$ by constructed the following map $R/\operatorname{Soc}(R_R) \to R/I$, that is, R/I semisimple R-module. Hence by the Proposition 3.1 of Goodearl (1972) R is a right SI-ring. $\square$

Proposition 3.2.6. *A ring R has the property (Q) if and only if for any two non-injective right R-modules A and B, there exists a proper essential extension A' of A and a homomorphism $f : A' \to B$ of modules such that $A \subsetneq \operatorname{Ker}(f)$. In this case in particular,* $\operatorname{Hom}_R(A, B) \neq 0$.

Proof. Let R has property (Q), that is, every right R-module is injective or t.i.n. Take any non-injective right R-module which are namely A, B. By our assumption A and B are t.i.n. Since A is non-injective and B is t.i.n. there exists a nonzero $f' : A \to B$ which is not neat. Otherwise every nonzero map from A to B is neat implies A is injective since B is t.i.n. Since f' is not neat there exists $f : A' \to B$ where $A \leq_e A' \leq E(A)$ such that $f'_{|A} = f$. So $A \subsetneq \operatorname{Ker}(f)$, otherwise $f(A) = f'(A) = 0$ but this contradicts with f' is nonzero.

For the converse take any right R-module M which is not injective. Our aim is to show that M is t.i.n. Take any right R-module A. Suppose every nonzero element $\operatorname{Hom}_R(A, M) \neq 0$ is neat. If A is not injective, then by our assumption there exists $f : A' \to M$ where $A \leq_e A'$ and $A \subsetneq \operatorname{Ker}(f)$. It is clear that $f_{|A}$ nonzero and non-neat homomorphism but this contradicts with every nonzero element of $\operatorname{Hom}_R(A, M)$ is neat. Hence A must be injective. $\square$

Lemma 3.2.7. *If R has a ring decomposition $R = \Lambda' \times \Lambda$ such that Λ' is semisimple Artinian and Λ has the property (Q), then R has the property (Q).*

Proof. Let R has a ring decomposition $R = \Lambda' \times \Lambda$ such that Λ' is semisimple Artinian and Λ has the property (Q). Let A and B are any two non-injective R-modules. $A\Lambda$ and $B\Lambda$ are also nonzero non-injective Λ modules where $A = A\Lambda' \oplus A\Lambda$ and $B = B\Lambda' \oplus B\Lambda$. Otherwise by the semisimplicity of Λ', A and B must be injective R-modules. By our assumption and the Proposition 3.2.6 there exists a proper essential extension A' of $A\Lambda$ as a Λ-module and a Λ-module homomorphism $f : A' \to B\Lambda$ such that $A\Lambda \subsetneq \operatorname{Ker}(f)$. It is clear that there exists a proper essential extension A'' of A as a R-module and a R-module homomorphism $\tilde{f} : A'' \to B$ such that $A \subsetneq \text{Ker}(\tilde{f})$. Hence R satisfies the property (Q). $\square$

Lemma 3.2.8. *A ring satisfying the property (Q) is a right V-ring.*

Proof. Let R has the property (Q). Take any simple right R-module, say S. Suppose for a moment S is not injective, by the property (Q) and Proposition 3.2.6 there would be a proper essential extension S' of S and a homomorphism $f : S' \to S$ such that $S \subsetneq \text{Ker}(f)$. If $\text{Ker}(f) \neq 0$, then $S \cap \text{Ker}(f) \neq 0$ so $S \subseteq \text{Ker}(f)$. But this contradicts with $S \subsetneq \text{Ker}(f)$. This contradiction shows that $\text{Ker}(f) = 0$, so f is one-to-one. It is clear that $S' = S$ since $f(S') = S = f(S)$. But this contradicts with the S' is a proper essential extension of S. This contradiction shows that S is injective right R-module. Hence R is a right V-ring. $\square$

Lemma 3.2.9. *If R is a right Noetherian, right V-ring and for any two non-injective modules A and B,* $\text{Hom}_R(A, B) \neq 0$*, then R is semisimple Artinian.*

Proof. Let R be a right Noetherian, right V-ring and for any two non-injective modules A and B $\text{Hom}_R(A, B) \neq 0$. Let M be a non-semisimple right R-module. Splitting of $\text{Soc}(M)$ as a direct summand, that is, $M = \text{Soc}(M) \oplus M'$. Take any cyclic submodule M'' of M', that is, M'' is finetely generated and Noetherian since R is a Noetherian ring. So M'' can not be contain infinite direct sum. If M'' is not uniform, then there exists a nonzero submodule which is not essential in M'', say M_1, that is, there exists a nonzero $M_2 \leq M''$ such that $M_1 \cap M_2 = 0$ and $M_1 \oplus M_2 \leq M''$. If M_1 and M_2 is not uniform, then we can continue in this manner. But M'' has no infinite direct sum of submodules, so there must be a uniform submodule U of M''. Take any cyclic submodule N of U, that is, N is uniform and non-injective since U is uniform. Again by Noetherianity of R, we can pick an element $a \in N$ with maximal annihilator in R among these nonzero elements of N. Set $A = aR$. Then A has proper essential submodule C since A is also out of $\text{Soc}(M)$, that is, A is not semisimple. Since both A and C are non-injective we can find a nonzero homomorphism $f : A \to C$ by our assumption. If $\text{Ker}(f)$ were nonzero, we would have $A/\text{Ker}(f) \cong R/I$. But $\text{ann}(a) \lneq I$. This implies that C, and hence A, contained a nonzero element with strictly larger annihilator than that of a, this contradicts with maximality of $\text{ann}(a)$. So f is one-to-one and thus A properly contains a copy of itself. Then we can find in $E(A)$ an infinite chain $... \subset A_{-2} \subset A_{-1} \subset A_0 \subset A_1 \subset A_2...$ of copies A_k of A.

Set $B = \bigcup_{k\in\mathbb{Z}} A_k$. Now if $B \neq E(A)$, then B is not injective, otherwise $A \leq B \lneq E(A)$. So that, there is a nonzero homomorphism. $g : B \to A$ by the assumption. If $\text{Ker}(g) \neq 0$, then there must be some $k \in \mathbb{Z}$ such that $A_k \subsetneq \text{Ker}(g)$. Since $g_{|A_k} : A_k \to A$ and $g_{|A_k}(A_k) \cong \frac{A_k}{\text{Ker}(g)\cap A_k} \cong R/I$ for some proper right ideal I which contains $\text{ann}(a)$ properly. But this is also contradicts with the maximality of $\text{ann}(a)$. Hence g is an embedding and so A contains infinitely increasing chain. But this is also contradicts with the Noetherianity of A. Hence $E(A) = B$.

Next take any nonzero proper submodule D of $E(A)$. It is clear that D is non-injective by the uniformity of B. Then $\text{Hom}_R(D, A) \neq 0$ by our assumption, so we can find a nonzero homomorphism $h : D \to A$ and this extends to a homomorphism $h' : E(A) \to E(A)$. If we have furthermore $\text{Ker}(h) \neq 0$, then $\text{Ker}(h') \neq 0$ as well. Thus $h' \neq 0$ and $E(A) = B$ there would be some $k \in \mathbb{Z}$ such that $A \cap \text{Ker}(h') \neq 0$ and $A_k \subsetneq \text{Ker}(h')$. So that $h'_{|A_k}(A_k) \cong \frac{A_k}{\text{Ker}(h')\cap A_k} \cong R/I$ for some proper right ideal I which contains $\text{ann}(a)$ properly. Since by cyclicity of A_k, $h'(A_k) \subseteq A_t \cong A$ for some $t \in \mathbb{Z}$. This situation yields a contradiction. Therefore we must have $\text{Ker}(h) = 0$, whence D embed in A and is thus finitely generated from Noetherianity of A. Thus every submodule of $E(A)$ is finitely generated. Now since R is right V-ring by the Theorem 6.2. of Jain et al. (2012) every R module contains a maximal submodule. In particular $E(A)$ contains a maximal submodule, say T, otherwise $\text{Rad}(E(A)) = E(A)$. Then T is finitely generated by the above argument. Hence $T \leq A_i$ for some $i \in \mathbb{Z}$. But then $A_{i+1} = E(A)$, this contradicts with A is non-injective. This contradiction shows that Mod-R can not have a nonzero non-semisimple module, where R is semisimple Artinian ring. □

Definition 3.2.10. A pair of R-modules A and B are called orthogonal if A does not contain any nonzero submodule isomorphic to a submodule of B and B does not contain any nonzero submodule isomorphic to a submodule of A.

It is clear that A and B are semisimple R-modules and orthogonal if and only if $\mathrm{Hom}_R(A, B) = 0$ and $\mathrm{Hom}_R(B, A) = 0$.

Lemma 3.2.11. *If a ring R satisfying for any two non-injective modules A and B, $\mathrm{Hom}_R(A, B) \neq 0$, then R has decomposition $R = \Lambda' \times \Lambda$ such that Λ' is semisimple Artinian and Λ is either zero ring or an indecomposable ring with homogeneous socle.*

Proof. Let R be ring and R satisfying for any two non-injective modules A and B, $\mathrm{Hom}_R(A, B) \neq 0$. In any pair of orthogonal semisimple modules A and B at least one of them must be injective by our assumption, otherwise $\mathrm{Hom}_R(A, B) \neq 0$ or $\mathrm{Hom}_R(B, A) \neq 0$, so A, B can not be orthogonal. $\mathrm{Soc}(R_R)$ can not have infinitely many nonzero homogeneous components since R is finitely generated. Furthermore $\mathrm{Soc}(R_R)$ can have at most one non-injective homogeneous components and each injective homogeneous components must be finitely generated and splits into direct sum. Splitting all nonzero homogeneous components (note that this does not mean we are splitting all injective simple right ideals of R). We obtain a direct sum $R = \Lambda' \times \Lambda$ where Λ' is semisimple Artinian and the right socle of Λ is homogeneous (possibly zero).

Now for the ring Λ it is clear that Λ satisfying for any two non-injective Λ-modules A and B, $\mathrm{Hom}_\Lambda(A, B) \neq 0$ by the Lemma 3.2.4 and Λ has no splitting socle part by the above argument. Furthermore suppose that there is a nontrivial ring direct sum $\Lambda = \Lambda_1 \oplus \Lambda_2$. Since any Λ_1-module is orthogonal to any Λ_2-module (as Λ-modules). One of the categories Mod-Λ_1 and Mod-Λ_2 consists of semisimple modules. Otherwise both Mod-Λ_1 and Mod-Λ_2 contains a non-injective Λ_1 and Λ_2 module respectively, so they are also Λ-module non-injectives. But by our assumtion there must be nonzero Λ-module homomorphism which is not possible by the orthogonality. Hence without loss of generality Mod-Λ_2 modules must be injective. This contradicts with Λ has no splitting socle part. This contradiction shows that Λ is indecomposable. $\square$

Before the following proposition note that you can see Lam (1999, Ch. 7) for Morita Theory.

Proposition 3.2.12. *If R is a right V-ring and for any two non-injective modules A and B $\operatorname{Hom}_R(A, B) \neq 0$ then there is a ring decomposition $R = \Lambda' \times \Lambda$ such that Λ' is semisimple Artinian and Λ is a right V-ring with $\operatorname{Soc}(\Lambda_\Lambda)$ homogeneous and $\frac{\Lambda}{\operatorname{Soc}(\Lambda_\Lambda)}$ is semisimple Artinian.*

Proof. By Lemma 3.2.11 and Lemma 3.2.4 we can assume, without loss of generality, that R indecomposable ring with homogeneous right socle, and furthermore, R is not semisimple Artinian. So by the Lemma 3.2.9 R is not right Noetherian. Now our claim is to show that R is right semi-Artinian. It is clear that there exists a non-injective semisimple right R-module, say B, since R is right V-ring and not right Noetherian. Take any nonzero R-module L. Let K be a complement of $\operatorname{Soc}(L)$ in L. If K is nonzero, then it has a proper essential submodule K' since $K \cap \operatorname{Soc}(L) = 0$. Since K' is non-injective, so $\operatorname{Hom}_R(B, K') \neq 0$ by our assumption. Take any $f \in \operatorname{Hom}_R(B, K') \neq 0$, it is clear that $f(B)$ is semisimple but this contradicts with $K \cap \operatorname{Soc}(L) = 0$. This argument shows that R is right semi-Artinian.

Now we want to show that R is right SI-ring. If R is not right SI-ring, then it has a non-injective singular module, so that by assumption every non-injective module contains a nonzero singular submodule, and every nonsingular module is injective, so semisimple. Let T be a complement of $Z_2(R_R)$ in R_R. Then T is semisimple and injective. And since $Z_2(R_R)$ is a complement of T as well. We have $R = T \oplus Z_2(R_R)$. Clearly this is a ring direct sum where T is semisimple, so that $T = 0$ and $R = Z_2(R_R)$ since R is an indecomposable and non-semisimple Artinian ring. By the semi-Artinity of R, $\operatorname{Soc}(R_R) \cap Z_2(R_R) \neq 0$. But since R is also right V-ring this means that R contains a singular simple ideal which is then projective which is not possible. Therefore R is right SI-ring. This implies by the Theorem 3.11 of Goodearl (1972) that $R \cong R_1 \times R_2 \times ... \times R_n \times R'$ where R_i Morita equivalent to PCI domain (SI-domain) and $R'/\operatorname{Soc}(R'_{R'})$ is semisimple Artinian. By indecomposability of R, it is either Morita equivalent to PCI domain or $R/\operatorname{Soc}(R_R)$ semisimple Artinian. If R is Morita equivalent to a PCI domain, then it would be a right Noetherian which is not possible. Therefore $R/\operatorname{Soc}(R_R)$ semisimple Artinian. This completes the proof. $\square$

Proposition 3.2.13. *If R is a ring with a ring decomposition $R = \Lambda' \oplus \Lambda$ such that Λ' is semisimple Artinian and Λ is a right V-ring with* $\mathrm{Soc}(\Lambda_\Lambda)$ *homogeneous and* $\frac{\Lambda}{\mathrm{Soc}(\Lambda_\Lambda)}$ *semisimple Artinian, then R satisfies the property (Q).*

Proof. By the Lemma 3.2.7 it suffices to prove result for a right V-ring R with $\mathrm{Soc}(R_R)$ homogeneous and $\frac{R}{\mathrm{Soc}(R_R)}$ semisimple Artinian; so we will assume R is as such and first see that R has no pair of orthogonal semisimple modules. By the Lemma 3.2.5 R is right SI-ring. See also the proof of the Lemma 3.2.5 our assumption imply that $\mathrm{Soc}(R_R) \leq_e R_R$. Choose any simple right ideal, say S, of R. If N is semisimple module orthogonal to S, then for any nonzero element $n \in N$ $nR \cong R/\operatorname{ann}(n)$ where $\mathrm{Soc}(R_R) \subseteq \operatorname{ann}(n) \leq_e R_R$ by the orthogonality of S and N. So $R/\operatorname{ann}(n)$ is singular, so injective. Hence $N = \sum\limits_{n \in N} nR = \bigoplus\limits_{n \in N} nR$ is injective. Therefore N is non-injective semisimple implies N is not orthogonal to S.

Next take any two non-injective modules A and B. By the Lemma 3.2.3 R is semi-Artinian ring so neither $\mathrm{Soc}(A)$ nor $\mathrm{Soc}(B)$ is injective since they are essential in A and B respectively by the semi-Artinity of R. So that by the first paragraph and the right V-ring assumption A and B contains submodule V_1 and V_2 such that $V_1 \cong S \cong V_2$. Since R is right V-ring we can construct a map f from $E(A)$ to B, that is, $E(A) = V_1 \oplus D \to V_1 \cong V_2 \subseteq B$ where A is not contained in $\mathrm{Ker}(f)$. Now the result follows by Proposition 3.2.6. □

Now Proof of the Theorem 3.2.1 follows from Propositions 3.2.12, 3.2.13, 3.2.6 and the Lemma 3.2.8.

3.3 Rings Admitting Nonzero Homomorphisms Between Any Two Non-Injective Modules

After the characterization of rings with the property (Q), my advisor Engin Mermut asked that "What are the rings over which $\mathrm{Hom}(A, B) \neq 0$ for all non-injective right modules A and B?". We call this property (T). Recall that ring R is said to be a right max ring if every nonzero right module has a maximal submodule (see Faith (1976)). Making use of Theorem 3.2.1 we can now characterize rings satisfying the property

(T) by the following Theorem.

Theorem 3.3.1. *Let R be a ring, R satisfies the property (T) if and only if there is a ring decomposition $R = \Lambda' \oplus \Lambda$ such that Λ' is a semisimple Artinian ring and Λ is a ring with homogeneous right socle that satisfies one of the following conditions:*

(i) *Λ is a right V-ring with homogeneous right socle and $\frac{\Lambda}{\mathrm{Soc}(\Lambda_\Lambda)}$ is semisimple Artinian (i.e. R is a ring satisfying (Q)),*

(ii) *Λ is Morita equivalent to a ring in the form $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$, where D is a division ring,*

(iii) *Λ is a right max, right semi-Artinian (nonzero) with singular right socle and a unique non-injective simple right module S such that $S_{\frac{\Lambda}{J(\Lambda)}}$ is injective and the following hold:*

(a) *$J(\Lambda)$ is small in C_Λ, where $\frac{C}{J(\Lambda)}$ is the S-component of the right socle of $\frac{\Lambda}{J(\Lambda)}$, and*

(b) *The ring $\frac{\Lambda}{C}$ is semisimple Artinian and orthogonal to S as a Λ-module.*

In case (iii), $\frac{\Lambda}{J(\Lambda)}$ is a ring with property (Q), and if Λ is a right Artinian, then $\Lambda_\Lambda \cong (e\Lambda)^n$ for some local idempotent $e \in \Lambda$ and some $n \in \mathbb{N}$.

It may be interesting to observe that a moderate condition such as perfectness would encompass several of the conditions in Theorem 3.3.1 (iii), namely Λ being max and semi-Artinian, the semisimplicity of the ring $\frac{\Lambda}{C}$ and the injectivity of $S_{\frac{\Lambda}{J(\Lambda)}}$; however, the ring of Theorem 3.3.1 (iii) does not even have to be semilocal (see Example 3.3.6 $(iii)(b)$). Moreover, all the three conditions in the theorem are mutually exclusive as well as viable, a fact that is also shown in several parts of Example 3.3.6. Once again we will need to divide the proof into pieces.

Lemma 3.3.2. *Let R be a right Artinian ring satisfying (T) with* $\mathrm{Soc}(R_R)$ *homogeneous and singular. Then $R_R \cong (eR)^n$ for some local idempotent e of R and some $n \in \mathbb{N}$.*

Proof. There is a non-injective simple right R-module since $\mathrm{Soc}(R_R)$ is singular and this non-injective right R-module is up to isomorphism unique by the property (T), say S. It is clear that $\mathrm{Soc}(R_R)$ is an S-component since $\mathrm{Soc}(R_R)$ is homogeneous and singular. Now let $\{e_i : i = 1, 2, ..., n\}$ be a complete set of primitive idempotents of R. Note that at least one of the e_iR must have image isomorphic to S. All non-injective e_iR must be isomorphic to each other since $\frac{e_iR}{e_iJ(R)} \cong S$ by the (T) condition. Let e_kR be an injective one. e_kR is not a semisimple Artinian since e_kR indecomposable and injective. Then e_kR has proper essential (non-injective) submodule so by the property (T) has an S copy and more $E(S) \cong e_kR$. Hence all injective e_kR must be isomorphic to $E(S)$. Now we will prove that e_iR are either all injective or all non-injective. Else we would have $i \neq k$ such that e_iR is injective and e_kR non-injective. Then $\frac{e_kR}{e_kJ(R)} \cong S$ and $\mathrm{Hom}(e_iR, S) = 0$, otherwise $e_iR \cong e_kR$. By the locality of e_iR, this also implies that there is no nonzero homomorphism from any factor of e_iR into S. Hence, all nonzero factors of e_iR are injective, again by (T). Then, each such factor is either simple injective or isomorphic to $E(S)$, as S is the only non-injective simple. Since e_iR is of finite length at least 2, this forces $cl(e_iR) = 2$. Let V be a copy of S in e_kR. The embedding $V \to e_iR$ extend to some map $f : e_kR \to e_iR$, by injectivity of latter ($e_iR \cong E(S)$). Since $V \subsetneq \mathrm{Ker}(f)$ and both ideals V and $\mathrm{Ker}(f)$ in $e_kJ(R)$, $f(e_kR)$ contains $\mathrm{Soc}(e_iR)$ properly, so that f must be an epimorphism since $cl(e_iR) = 2$, then $\mathrm{Ker}(f) = 0$ since e_iR projective and $\mathrm{Ker}(f)$ small in e_kR. This contradict the fact that e_kR is not injective. This concludes the proof. □

Lemma 3.3.3. *If R has the property (T), then R/I has the property (T) for any ideal I.*

Proof. Let A and B be are non-injective R/I-modules. Then it is clear that A and B are also non-injective R-modules. There exists a nonzero R-homomorphism $f : A \to B$ by the property (T) of R. But this homomorphism f is also can be seen as a nonzero R/I-homomorphism in a natural way since B is an R/I module and so annhilated by I. □

Proposition 3.3.4. *Let R be a ring with property (T). Then R has the decomposition described in Theorem 3.3.1.*

Proof. A right V-ring with property (T) satisfies (Q); so, the first case follows from Theorem 3.2.1. Thus we will assume, throughout the rest of the proof, that R is not a right V-ring. By this assumption and Lemma 3.2.11 there is a ring decomposition $R = \Lambda' \oplus \Lambda$ such that Λ' semisimple Artinian and Λ is an indecomposable non-right-V-ring with homogeneous right socle satisfying property (T). Then there exists a unique non-injective simple right Λ-module S. If A is a non-semisimple module, it has a proper essential submodule A', which cannot be injective, so that $\mathrm{Hom}(S, A') \neq 0$ implying that A contains a copy of S. Therefore Λ is right semi-Artinian and $\mathrm{Soc}(\Lambda_\Lambda)$ is homogeneous S-component.

Case 1: First assume that Λ is a right SI-ring and as above. Then S is neccessarily nonsingular whence projective. Since every non-injective right Λ-module has a factor isomorphic to S, by projectivity of S, every non-injective contains a copy of S as a direct summand. In particular, we must have $cl(E(S)) = 2$ since S is maximal submodule of $E(S)$. Since a semisimple module that is orthogonal to S must be injective, it is easy to see that the right Noetherianity of Λ depends on whether $E(S)$ is $\sum$-injective or not. Hence if Λ were not right Noetherian $E(S)^{\mathbb{N}}$ would be non-injective, thereby containing a copy of S as a direct summand (in a finite sub-sum of $E(S)^{\mathbb{N}}$), contradicting the non-injectivity of S. Thus, Λ is right Noetherian as well as right semi-Artinian, whence it is right Artinian. Now take any indecomposable module B. If B is non-injective, then it contains a copy of S as a direct summand, whence $B \cong S$. So assume that B is furthermore injective. If B is not simple, then it is not semisimple because of indecomposobility, whence it contains a proper essential (hence non-injective) submodule B', which in turn has contain a copy of S. But then B will contain a copy of $E(S)$, implying that $E(S) \cong B$ by the indecomposability of B. This argument shows that every indecomposable right module either simple which is isomorphic to S or isomorphic to $E(S)$, thus having composition length at most two. In particular $\Lambda_\Lambda = \bigoplus_{i=1}^{n} e_i\Lambda$ where $e_i\Lambda \cong S$ or $e_i\Lambda \cong E(S)$ (with at least one copy of each in appearing in the decomposition, because S is projective and Λ is not semisimple Artinian). Now it follows that such a ring Morita equivalent to $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ fo a division ring D (see Dung et al. (1994, 12.21 and 13.5 $(a) \iff (e)$)). This establishes case (ii).

Case 2: Next assume Λ is not right SI and as above of the Case 1. Then there is a non-injective singular right Λ-module G. Since $\mathrm{Soc}(\Lambda_\Lambda)$, which is now proper and essential in Λ_Λ, is an S-component, $\mathrm{Hom}(G, S) \neq 0$, so that Λ_Λ, whence every right Λ-module, contains its singular submodule essentially and note that $\mathrm{Soc}(\Lambda_\Lambda) \leq J(\Lambda)$. Since the property (T) passes over factor rings by the Lemma 3.3.3, $\bar{\Lambda} = \frac{\Lambda}{J(\Lambda)}$ satisfies (T). Then $\bar{\Lambda}$ is one of the following rings: a ring of Theorem 3.2.1 or a ring that decomposes into semisimple Artinian ring and a ring is as in $Case$ 1 above or satisfies the part of $Case$ 2 that has thus far been proved. The last situation is not possible $J(\bar{\Lambda})$ would then need to contain a simple singular right ideal; however, it is zero. The second one would force $\bar{\Lambda}$ to be semisimple Artinian because in this situation $\bar{\Lambda}$ is a non-right V and right SI with the property (T) implies $\bar{\Lambda}$ is Artinian with $J(\bar{\Lambda}) = 0$, contradicting the fact that $\bar{\Lambda}$ non-right V. Thus the only viable case is the first one, where $\bar{\Lambda}$ is a ring satisfying the property (Q) (note that $\bar{\Lambda}$ is right SI). In particular, it is a right V-ring.

In summary of the last discussion, there is now decomposition $\bar{\Lambda} = \Gamma' \oplus \Gamma$ such that Γ' is semisimple Artinian and Γ is a right V-ring with homogeneous right socle $\frac{\Gamma}{\mathrm{Soc}(\Gamma)}$ is semisimple Artinian.

First assume that $\bar{\Lambda}$ is not semisimple Artinian. Then $\mathrm{Soc}(\Gamma)$ can not be injective (as a module either one of the rings Λ and $\bar{\Lambda}$) since Λ injectivity implies $\bar{\Lambda}$ injectivity and $\bar{\Lambda}$ semi-Artinian so $\mathrm{Soc}(\Gamma) = \Gamma$ implies $\bar{\Lambda}$ is semisimple Artinian, whence it is the S-component, say $\frac{C}{J(\Lambda)}$, of $\mathrm{Soc}(\bar{\Lambda})$ and Γ', being orthogonal to S is $\sum$-injective (as well as semisimple) as a right Λ-module, whence every Γ'- module is an injective Λ-module since they are isomorphic to a quotient free Γ'-module so semisimple and orthogonal to S. Furthermore, $\frac{\Gamma}{\mathrm{Soc}(\Gamma)}$ is a singular $\bar{\Lambda}$-module since $\bar{\Lambda}$ right semi-Artinian, whereas S is not, thus it is orthogonal to S and thus it is $\sum$-injective as Λ-module, as well. From this argument it follows that $\frac{\Lambda}{C}$-modules are semisimple and injective as Λ-modules,

$\frac{\Lambda}{C} \cong \Gamma' \oplus \frac{\Gamma}{\mathrm{Soc}(\Gamma_\Gamma)}$ is orthogonal to S.

Now, we will see that $J = J(\Lambda)$ is small in the Λ-module C: Assume the contrary, namely that $J + D = C$ for some right ideal D of Λ such that $J \subsetneq D$. Put $M = \frac{\Lambda}{D}$. Then $\mathrm{Rad}(M)$ contains $\frac{J+D}{D} = \frac{C}{D} \subseteq \mathrm{Rad}(M)$ since the canonical epimorphism from Λ to $\frac{\Lambda}{D}$ maps J to $\frac{J+D}{D}$ in $\mathrm{Rad}(\frac{\Lambda}{D} = \mathrm{Rad}(M))$ and since $U = \frac{M}{C/D}$ is then a right module over the right V-ring $\frac{\Lambda}{J(\Lambda)}$ since $J \subseteq C$ so annihilates U, then $\mathrm{Rad}(U) = 0$ so $\mathrm{Rad}(M) \subseteq \frac{C}{D}$. So, we have $\mathrm{Rad}(M) = \frac{C}{D}$. Now, if M_Λ is not injective it must have a factor isomorphic to S, whence so will $\frac{M}{\mathrm{Rad}(M)} = \frac{M}{C/D} \cong \Lambda/C$. However, $\frac{\Lambda}{C} \cong \Gamma' \oplus \frac{\Gamma}{\mathrm{Soc}(\Gamma_\Gamma)}$, which is a semisimple module orthogonal to S. This contradicts the above assertion about M having factor isomorphic to S. Thus M must be injective. Recall that by our assumption $\bar{\Lambda}$ is not semisimple Artinian, whence Λ cannot be Noetherian, otherwise $\bar{\Lambda}$ Noetherian and semi-Artinian, so Artinian with radical zero, that is, semisimple Artinian. This means, by the same argument as in the second paragraph of this proof, that $E(S)$ is not $\sum$-injective. Thus, $E(S)^{\mathbb{N}}$ has a factor isomorphic to S. This implies that $E(S)$ has a factor isomorphic to S. Now, $\frac{C}{D} = \frac{J+D}{D} \cong \frac{J}{J \cap D}$, being a small module which is nonzero by our assumption, is non-injective, so that it contains a copy of S. This means that M contains a copy of $E(S)$. Since that copy of $E(S)$ is then a direct summand of M and $E(S)$ has a factor isomorphic to S, then so does M, once again a contradiction since M is orthogonal to S. Therefore, $J = J(\Lambda)$ is small in the Λ-module C, as desired.

Next assume that $\bar{\Lambda}$ is semisimple Artinian (note that, in this case too, the S-component $\bar{\Lambda}$ cannot be zero). The arguments above showing the smallness of J in C works as same when Λ is not right Noetherian; else, if it is right Noetherian, then Λ is right Artinian, in which case Lemma 3.3.2 applies to yield that $\Lambda_\Lambda \cong (e\Lambda)^n$ for a local idempotent e of Λ, so that $S \cong \frac{e\Lambda}{eJ(\Lambda)}$, whence $C = \Lambda$. Since $C = \Lambda$ in this case, it follows once again that J is small in C. The orthogonality of $\frac{\Lambda}{C}$ to S is immediate in this case.

As for Λ being a right max ring, take any nonzero right Λ-module M and note that if M is not injective or if M contains an injective simple, then M must have a maximal submodule since by the property (T) or injectivity of simple. So assume that M is an injective module that does not contain an injective simple module. Then M contains a copy of $E(S)$ by semi-Artinianity and our assumptions. As discussed in a previous paragraph, Λ is Noetherian (so Artinian) or $E(S)^{\mathbb{N}}$ is non-injective and thus has a factor isomorphic to S; in either case M has a maximal submodule since Artinian rings are max ring and $E(S)$ has a factor isomorphic to S, so has M. $\square$

Now, we prove the converse.

Proposition 3.3.5. *If R has a decomposition as is described in Theorem 3.3.1, then R satisfies (T).*

Proof. If R has decomposition as in Theorem 3.3.1 (i), then R satisfies (T) by Theorem 3.2.1.

Next assume that R has a decomposition as in Theorem 3.3.1 (ii). Since the property (T) is clearly a Morita invariant and the ring direct sum of a semisimple Artinian ring and a ring with property (T) satisfies (T), in order to show that R has (T), it suffices to identify R with the ring $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ and show that the latter has (T): In this case R Artinian serial and right SI-ring with exactly one non-injective simple right R-module $S = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ up to isomorphism. Assume that A is non-injective right module. Since R is Artinian serial, A decomposes into cyclic uniserial each of which is singular or isomorphic to one of the modules S and $E(S)(\cong \begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix})$. Since A is non-injective, one of those cyclics must be S. Thus A has a submodule and a factor isomorphic to S. This implies that R satisfies (T).

Following the same discussion as in the preceding paragraph, assume that R satisfies the same properties as the ring Λ in Theorem 3.3.1 (iii) does, S is the unique non-injective simple right R-module and C is the right ideal of R such that $\frac{C}{J(R)}$ is the S-component of the right socle of $\frac{R}{J(R)}$. Since $\mathrm{Soc}(R_R)$ must be singular by our assumptions, it must be an S-component, and thus homogeneous. We will now see that every right $\frac{R}{C}$-module is injective as an R-module; to that end we will first show that every nonzero cyclic subfactor of the R-module C contains a copy of S: Assume that $x \in C$ and Y is a proper submodule of xR. Note that by the assumptions that S is the only non-injective simple right R-module and R is the right semi-Artinian and the fact that any (sub)factor of J is a small module (so that it cannot contain a nonzero injective submodule), we have that for any ordinal α, $\mathrm{Soc}_{\alpha}(J)$ must be an S-component (possibly zero), and the same goes for any subfactor of J. So, we can assume without loss of generality that xR is not contained in J. Now, if $xR \cap J \subseteq Y$ then $\frac{xR}{Y}$ is isomorphic to $\frac{xR+J}{Y+J}$, which in turn is isomorphic to a factor of $\frac{C}{J}$, which is an S-component (over any ring possibly considered here), yielding the desired conclusion. Else, assume that Y does not contain $xR \cap J$. Then $\frac{(xR\cap J)+Y}{Y}$, being isomorphic to a nonzero subfactor of J, will contain a copy of S by the argument above. This establishes our claim that nonzero cyclic subfactors of C contain copies of S. Next, assume that K is a right $\frac{R}{C}$-module, let A be a right ideal of R and $f : A \to K$ be any (R-linear) homomorphism. By our assumption of orthogonality of semisimple $\frac{R}{C}$ to S, $K \cong \frac{\oplus R/C}{Y}$ cannot contain a copy of S. Thus $\mathrm{Ker}(f)$ must contain $A \cap C$ otherwise $x \in C$ and $f(x) \neq 0$ implies $f_{|xR}(xR) \cong \frac{xR}{\mathrm{ker}(f_{|xR})} \subseteq K$, so K contains a copy of S by what we shown above. Then, the map $f' : A + C \to K$ defined by the rule $f'(a + c) = f(a)$ (for any $a \in A$ and $c \in C$) is well defined, it extends f and $\mathrm{ker}(f') = \mathrm{ker}(f) + C$. Furthermore, f' induces an obvious homomorphism $\bar{f}' : \frac{A+C}{\mathrm{ker}(f')} \to K$. Since the R-module $\frac{R}{C}$ is semisimple, then so is $\frac{R}{\mathrm{ker}(f')}$, so that there is a decomposition $\frac{R}{\mathrm{ker}(f')} = \frac{A+C}{\mathrm{ker}(f')} \oplus \frac{Z}{\mathrm{ker}(f')}$. Let $\mu : R \to \frac{R}{\mathrm{ker}(f')}$ and $\pi : \frac{A+C}{\mathrm{ker}(f')} \oplus \frac{Z}{\mathrm{ker}(f')} \to \frac{A+C}{\mathrm{ker}(f')}$ be obvious projections. Now, it is easy to check that the map $\bar{f}'\pi\mu$ extends f to a member of $\mathrm{Hom}(R, K)$, proving our initial claim that K is injective as a right R-module.

Next, let M be a non-injective right R-module. Since all right $\frac{R}{C}$-modules are injective as R-modules by the preceding paragraph, we must have $MC \neq 0$ otherwise M is $\frac{R}{C}$-module, so injective. Now, for each $a \in M$, aJ is a small submodule of aC since J is small in C by our assumption. Thus, for each $a \in M$, $aJ \subseteq \mathrm{Rad}(MC)$, whence, by the right max ring assumption, we have $MJ \subseteq \mathrm{Rad}(MC) \subsetneq MC$.

Viewing $\frac{M}{MJ}$ as a module over the ring $\frac{R}{J}$, this means that $\frac{M}{MJ}\frac{C}{J}$ is nonzero, so that $\frac{M}{MJ}$ contains a copy of S, say V. As S injective as a module over the ring $\frac{R}{J}$, the module V is a direct summand of $\frac{M}{MJ}$. It follows that S is an epimorphic image of the module $\frac{M}{MJ}$, whence of M as well.

Next, we will see that every non-injective module N contains a copy of S: If $NJ \neq 0$, then there is some $b \in N$ with $bJ \neq 0$. Recall that bJ is a small module, hence it cannot contain an injective simple submodule. By semi-Artinianity, bJ must then contain a copy of the only non-injective simple S. Else, assume $NJ = 0$. By the above argument, $NC \neq 0$ since N is non-injective. Then N is an $\frac{R}{J}$-module with $N\frac{C}{J} \neq 0$. Thus there must be a copy of S in N.

Now, we have seen that every non-injective right R-module both contains a copy of S and has an image isomorphic to it. This proves R has (T), as desired. $\square$

Proof of Theorem 3.3.1 By Proposition 3.3.4 and 3.3.5 and Lemma 3.3.2.

Example 3.3.6. All three cases of Theorem 3.3.1 are clear that mutually exclusive (and in fact disjoint); and show that each one is viable:

(i) (Cozzens & Faith (1975, Example 5.14)) Let V_F be an infinite dimensional vector space over a field F, $L = End(V_F)$ be the ring of linear transformations (from V into V), S be the ideal of L consisting of elements of L of finite rank, and R be its subring generated by S and the scalar transformations in L. Then $S = \mathrm{Soc}(R_R)$ and $\frac{R}{S}$ is a simple right R-module. Furthermore, R is a right but not left V-ring. For any two nonzero idempotents e and f in R, $eRf \neq 0$, so that $\mathrm{Soc}(R)$ is homogeneous. Hence, R is a ring satisfying (Q), and thus it is a ring as in Theorem 3.3.1 (i). Note that since R is not a left V-ring but it is a semiprime ring, this example also shows, by Theorem 3.3.1, that none of the properties (Q) and (T) is left-right symmetric. It shows, furthermore, that a ring with (Q) is not necessarily fully saturated on any side, because R is neither simple von Neumann regular nor Artinian (see Trlifaj (1996)).

(ii) Any hereditary Artinian serial ring R with $J(R)^2 = 0$ and homogeneous right socle is a ring of Theorem 3.3.1 (ii). A non-semisimple such ring is fully saturated and has (T) but not (Q).

(iii) The following are examples of a ring of Theorem 3.3.1 (iii):

(a) Any non-semisimple local and perfect ring satisfies Theorem 3.3.1 (iii), but not (Q). Take, for example, the ring $\frac{\mathbb{Z}}{p^n\mathbb{Z}}$, where p is a prime and $n > 1$. For an example of this case with infinitely generated socle, consider, for a division ring D, the subring of $\begin{pmatrix} D & D^{(\mathbb{N})} \\ 0 & D \end{pmatrix}$ consisting of matrices $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where $a \in D, b \in D^{(\mathbb{N})}$.

(b) Let R be any non-semisimple ring with homogeneous right socle satisfying (Q) (e.g. the ring in (i) above) and L be the subring of the ring $\begin{pmatrix} R & \mathrm{Soc}(R_R) \\ 0 & R \end{pmatrix}$ comprised of matrices $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where $a \in R$ and $b \in \mathrm{Soc}(R_R)$. We will see that L is an example of Theorem 3.3.1 (iii) which is not semilocal: Let $I = \begin{pmatrix} 0 & \mathrm{Soc}(R_R) \\ 0 & 0 \end{pmatrix}$. Since I is the kernel of the ring epimorphism $\phi : L \to R$ with $\phi(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}) = a$ and since R is a right V-ring, so that $J(R) = 0$, we have $J(L) \subseteq I$. Now assume $I + B = L$ for some right ideal B of L. Then there is a sum $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} r & b \\ 0 & r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where the former element belongs to I and the latter to B. Thus, $r = 1$ and $b = -x$. Then, $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in B$, proving that I is a small right ideal of L, whence $I = J(L)$. Next, we claim that $\mathrm{Soc}(L_L) = J(L) = I$ and it is a singular right L-module. Since I is a semisimple right ideal of L with $I^2 = 0$, it suffices to show that I is an essential right ideal of L: Assume that $0 \neq \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in L$, and that $a \neq 0$ to avoid triviality. Since $Z(R_R) = 0$ and $\mathrm{Soc}(R_R)$ is essential in R_R, there is some $c \in \mathrm{Soc}(R_R)$

such that $ac \neq 0$. Then, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ac \\ 0 & 0 \end{pmatrix}$ is a nonzero element of I, establishing our claim that $\mathrm{Soc}(L_L) = J(L) = I$. The homogeneity of $\mathrm{Soc}(L_L)$ follows from that of $\mathrm{Soc}(R_R)$ easily. For any simple right ideal S of R, the right ideal $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ is isomorphic to $\frac{A}{J(L)}$ via the (L-linear) epimorphism $A \to \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ defined by the rule $\begin{pmatrix} a & x \\ 0 & a \end{pmatrix} \to \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, where $A = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a \in S, b \in \mathrm{Soc}(R_R)\}$, implying that $Z \subseteq C$, where $\frac{C}{J(L)}$ is the $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$-component of $\mathrm{Soc}(\frac{L}{J(L)})$ and $Z = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a, b \in \mathrm{Soc}(R_R)\}$. Because of the ring isomorphism $\frac{L}{J(L)} \cong R$ effected by the ϕ above, the essentiality of $\mathrm{Soc}(R_R)$ in R_R implies that of $\frac{Z}{J(L)}$ in $\frac{L}{J(L)}$, yielding that $\frac{Z}{J(L)} = \mathrm{Soc}(\frac{L}{J(L)}) = \frac{C}{J(L)}$, whence $Z = C$. Now, we will see that L is a right max ring: Take any nonzero right L-module M. If $MJ(L) = M$, then since $J(L) = \mathrm{Soc}(L_L)$, M is semisimple and thus it has a maximal submodule, which contradicts the initial assumption; so, we must have $MJ(L) \neq M$, so that $\frac{M}{MJ(L)}$ is a nonzero module over $\frac{L}{J(L)} \cong R$, which is a right V-ring, whence it has a maximal submodule, and so does M, proving the claim. Next, $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$, being a singular right ideal, is a non-injective simple module. Let V be any simple right L-module not isomorphic to $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$. If V_L is not injective, we can choose some $m \in E(V_L) - V$. As mL is orthogonal to $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ (so that $mJ(L) = 0$), we must have $C \subseteq \mathrm{ann}_L(m)$, whence mL is semisimple (because mL is isomorphic to a factor of the L-module $\frac{L}{C}$, which, as a ring, is isomorphic to the semisimple Artinian ring $\frac{R}{\mathrm{Soc}(R_R)}$), forcing $V = mL$, a contradiction. Therefore V_L is injective, and thus

$\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ is the unique non-injective simple right L-module up to isomorphism. Moreover, $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ is injective as a module over the right V-ring $\frac{L}{J(L)}$. Now we will show that $J(L)$ is small in C: Assume that $J(L) + D = C$ for some right ideal D of L. Let $a \in \text{Soc}(R_R)$ be such that aR is simple. Clearly, D must contain an element of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, for some $b \in \text{Soc}(R_R)$. As aR is a direct summand of R_R, there is some $r \in R$ such that ar is an idempotent with $ara = a$. Then $ra \in \text{Soc}(R_R)$, so that $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} . \begin{pmatrix} 0 & ra \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in D$, showing that $\text{Soc}(L_L) = J(L) \subseteq D = C$. Therefore, $J(L)$ is small in C. Finally, L is clearly right semi-Artinian since $\frac{L}{\text{Soc}(L_L)}$ is a such, and $\frac{L}{C}$ is a semisimple Artinian ring that isorthogonal to $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ as an L-module, because $\frac{L}{C}.C = 0$, whereas $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.C \neq 0$. Now, it follows by Theorem 3.3.1 that the ring L satisfies (T), and it is not semilocal. Finally, note that L does not satisfy (Q).

The following example shows that the rather unfamiliar condition in Theorem 3.3.1 (iii) that J is small in C is indispensable, as it is independent of the rest of the conditions.

Example 3.3.7. There is an example of a ring satisfying all conditions of Theorem 3.3.1 (iii) except J being small in C: Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$, $e = \begin{pmatrix} \bar{1} & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & \bar{1} \end{pmatrix}$. It is routine to verify the following facts: $\text{Soc}(eR) = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix} \cong \text{Soc}(fR) = \begin{pmatrix} 0 & 0 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$. In particular, $\text{Soc}(R_R) = J$ (the Jacobson radical). If $\frac{eR}{eJ}$ where isomorphic to $\frac{fR}{fJ}$, then eR and fR would be isomorphic to each other. However, $eReR \neq 0$, whereas $fReR = 0$. This, along with the fact that $eRf \neq 0$, implies that $\text{Soc}(eR) \cong \frac{fR}{fJ}$. Calling this module S, we now have $\frac{R}{J} \cong S \oplus \frac{eR}{eJ}$, and that $C = fR + eJ = fR + J$, where $\frac{C}{J}$ is the S-component of the right socle of $\frac{R}{J}$. Thus, J is not small in C. The only simple module other than S is $V = \frac{eR}{eJ}$ and it is orthogonal to the modules eR and fR, both of which are of length 2, implying that V is injective relative to both eR and fR, whence it is injective. Thus, S is the only non-injective simple right R-module. Finally, $\frac{R}{C}$, which is isomorphic to $\frac{eR}{eJ}$ as an R-module, is a semisimple Artinian ring that is orthogonal to S as an R-module. Now, it follows easily that R satisfies all conditions in Theorem 3.3.1 (iii) except the condition that J is small in C.

The following example and Example 3.3.6 (i) together establish the fact that none of the classes of fully saturated rings and rings satisfying (Q) contains the other.

Example 3.3.8. A non-semisimple serial quasi-Frobenius ring is fully saturated but it does not satisfy (Q).

Remark 3.3.9. In Alizade et al. (2014), the authors consider a type of ring motivated by the prototypical question of non-injectives being some sort of test modules for injectivity, which is shown to coincide with fully saturated rings except when R is von Neumann regular $(*)$, and our treatment of (Q) in Section 2 bears, on some occasions, a formal similarity to theirs of the said type of ring. The ring treated there is one over which, for any pair A and B of non-injective modules, there exists some $f : B \to A$ that cannot be extended to any map $E(B) \to A$ (see Alizade et al. (2014, Proposition 2 (iii))). The property that defines (Q), however, stipulates what appears to be a somewhat opposite condition, namely that there be some $f : B \to A$ that can be extended nontrivially inside $E(B)$. On the other hand, both rings satisfy (T). In order to complete the picture, we note that none of the two classes of rings mentioned above contains the other (see $(*)$ above and Examples 3.3.6 (i) and 3.3.8).

CHAPTER FOUR
CONCLUSION

In this thesis, our first problem is to characterize the commutative rings R such that ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$. Our motivating result is Fuchs' complete answer for integral domains in Fuchs (2012): they are exactly the integral domain whose maximal ideals are projective and so necessarily finitely generated (invertible ideals if they are nonzero). We first tried to understand how much this can be extended. In Section 2.3, we prove that all maximal ideals of a commutative ring R are finitely generated and projective if and only if R has projective socle and ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$. For a commutative local ring R with the unique maximal ideal $\mathfrak{m}$, we show that ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ if and only if $\mathfrak{m}$ is a principal ideal. Note also that if the unique maximal ideal $\mathfrak{m}$ of a commutative local ring R is projective, then it is necessarily a principal ideal as is well-known (Proposition 2.2.9). To see what happens over any commutative ring, we firstly understand in Lemma 2.2.6, how neatness and $\mathcal{P}$-purity behave under localization and then we obtain our main result in Theorem 2.2.7: for a commutative ring R, ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ if and only if every maximal ideal $\mathfrak{m}$ of R is finitely generated and the maximal ideal $\mathfrak{m}_\mathfrak{m}$ of the local ring $R_\mathfrak{m}$ is a principal ideal.

The rest of this thesis consists of the extended version of Er & Türkoğlu (2018). In that paper, we deal with some problems which are related with neat homomorphisms: a homomorphism $f : M \to N$ of modules is called a neat homomorphism if and only if there are no proper extensions of f in the injective hull $\mathrm{E}(M)$ of M. Firstly we deal with the rings R where all nonzero R-module epimorphisms with cyclic domains are neat in Section 3.1. We call this property (P). To deal with such type of questions have been suggested by Noyan Er, and we proved that a ring R satisfies the property (P) if and only if either R is semisimple Artinian or the injective hull of every cyclic R-module is uniserial. Then we deal with the rings R which has property (Q), that is, every non-injective module is test for injectivity by neatness. An R-module M is called test for injectivity by neatness (t.i.n.) if all nonzero maps $N \to M$ being neat implies that N is injective. We proved that for a ring R, the following are equivalent:

(i) R satisfies the property (Q).

(ii) R is a right V-ring and for any two non-injective modules A and B $\mathrm{Hom}_R(A, B) \neq 0$.

(iii) $R \cong \mathit{\Lambda}' \times \mathit{\Lambda}$ such that $\mathit{\Lambda}'$ is semisimple Artinian and $\mathit{\Lambda}$ is a right V-ring with homogeneous right socle modulo which $\mathit{\Lambda}$ is semisimple Artinian.

After the characterization of rings with the property (Q), my advisor Engin Mermut asked that "What are the rings over which $\mathrm{Hom}(A, B) \neq 0$ for all non-injective right modules A and B?". We call this property (T). Using the characterization of rings with the property (Q), it is proved that rings with property (T) are precisely those rings

(i) Λ is a right V-ring with homogeneous right socle and $\frac{\Lambda}{\mathrm{Soc}(\Lambda_\Lambda)}$ is semisimple Artinian (i.e. R is a ring satisfying (Q)),

(ii) Λ is Morita equivalent to a ring in the form $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$, where D is a division ring,

(iii) Λ is a right max, right semi-Artinian (nonzero) with singular right socle and a unique non-injective simple right module S such that $S_{\frac{\Lambda}{J(\Lambda)}}$ is injective and the following hold:

(a) $J(\Lambda)$ is small in C_Λ, where $\frac{C}{J(\Lambda)}$ is the S-component of the right socle of $\frac{\Lambda}{J(\Lambda)}$, and

(b) The ring $\frac{\Lambda}{C}$ is semisimple Artinian and orthogonal to S as a Λ-module.

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