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ECOLOGICAL SYSTEM: STABILITY AND BIFURCATION

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Doctor of Philosophy

Supervisor

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I hereby declare that all information/data presented in this graduation project has been obtained in full accordance with academic rules and ethical conduct. I also declare all unoriginal materials and conclusions have been cited in the text and all references mentioned in the Reference List have been cited in the text, and vice versa as required by the abovementioned rules and conduct.

Sabah Ali RAHI

PREFACE

In the beginning, I can hardly express my gratitude to " Allah " and I shall always remember his generosity for giving me a hand and a great amount of help to complete this work. My deepest appreciation and thank go first to my supervisor " Asst. Prof. Dr. Sefer Kurnaz ", I highly appreciate his kindness, advice, and valuable suggestions throughout the entire period of my project. I owe my thanks and gratitude to my respected teacher “Prof. Dr. Raid Kamel Naji” in the Department of Mathematics/College of Science/University of Baghdad, who stood with me in an honorable way throughout work, many thanks for your cooperation to present this work to the fullest Finally, I would like to express my great thanks and appreciation to my husband, my kids, my parents, and all members of my "beloved family" for their patience, cooperation, and their continuous support to complete this work.

ABSTRACT

ECOLOGICAL SYSTEM: STABILITY AND BIFURCATION

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A three-dimensional prey-predator model is suggested to explain the interplay between prey and two age stages of the predator. The fear function is modeled in the prey population. It has also been hypothesized that the prey is strong enough to show anti-predator behavior and has the ability to kill the predator as a natural response to resistance to predation. The traditional Holling's disk Eq. (Holling type II) is modified to explain the prey consumption to involve an additional food (AF) for the predator, taking into consideration the handling time, and the search time for the prey. The existence, uniqueness, and boundedness of the Sol. of the proposed model are investigated. The local stability analyses are carried out after deciding all viable equilibrium points (EPs). The Lyapunov method is also used to investigate the global stability analyses for this model. The probability of local bifurcation (LB), along with Hopf bifurcation (HB), is studied. Moreover, to complete our study and confirm our analytical results, Numerical simulations are performed to investigate the global dynamics of the model and determine the set of control parameters.

Keywords: Stage Structure, Anti-Predator, Fear, Stability, Bifurcation.

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ABBREVIATIONS

P-P : Prey-Predator.

Sol : Solution.

Sys : System

ODE : Ordinary Differential Equation.

Eq. : Equation

CEP : Coexistence Equilibrium Point.

VEP : Vanishing Equilibrium Point.

PYFEP : Prey-Free Equilibrium Point.

PDFEP : Predator-Free Equilibrium Point.

SNB : Saddle-Node Bifurcation.

TB : Transcritical Bifurcation.

PB : Pitchfork Bifurcation.

AS : Asymptotically Stable.

LAS : Local Asymptotically Stable.

AS : Asymptotically Stable.

LAS : Local Asymptotically Stable.

GAS : Global Asymptotically Stable.

JM : Jacobian Matrix.

EVU : Eigenvalue.

EVC : Eigenvector.

LF : Lyapunov Function.

ASYM : Asymptotically.

AF : Additional Food

1 INTRODUCTION AND MATHEMATICAL PRELIMINARIES AND TOOLS

1.1 INTRODUCTION

Due to its global existence and significance, the behavior of ecosystems (ecological Syss) dynamic comprising of P-P interaction or mutualism interaction or competition interaction has always been and will remain one of the dominant and major subjects in both theoretical ecology and applied mathematics. These problems may seem computationally simple at first glance, but they are, in reality, it is often very difficult and complicated. Many researchers have proposed and investigated ecological models that include a wide range of interaction kinds in a single model. The majority of these models are defined by a set of nonlinear differential Eqs, which can exhibit a variety of behaviors that are not feasible in linear Syss. The 2-species continuous-time models that are characterized by a 2D autonomous nonlinear Sys. can only show two main styles approach either equilibrium or limit cycle. Furthermore, Three (or more) species continuous-time models, these models that can be characterized by a three (or more)-dimensional autonomous nonlinear Sys., have been suggested to contain patterns that are more complex than those found in two species models, such as chaos and quasi-periodic.

Undoubtedly, the construction of the classic Lotka-Volterra model [2], [3] is a milestone in studies dealing with ecological Syss, especially the P-P interaction. The past nine decades witnessed a remarkable development in mathematical modeling through the introduction of several complex Syss based on the Lotka-Voltera models to include more than two interacting species, taking into consideration the important biological factors and injecting them into these models.

The primary goal of this study is to highlight many critical methods in biological environmental management that contribute significantly to the dynamical behavior of P-P models, such as the impacts of additional food (AF), fear, stage structure, and anti-predator.

The biologists often classify the animal as prey or predator and take into account that direct killing is the only distinguishing feature of predator Syss (due to its being easily noticed in

nature), but in fact, the ecological role of individuals is frequently unknown. In other words, There are a variety of anti-predator responses that species exhibit when they feel threatened(perceive fear), including changing their environments and foraging habits, as well as becoming more observant. Simultaneously, the others undergo psychological transformations. For example, Prey is more likely to relocate from a risky natural environment to one of low risk if they feel threatened. These movements could result in significant energy dissipation. Particularly when the new low-fear environment is not always suitable for prey, [4]. While this behavior may aid in survival, it may also have a negative effect on offspring in the long term owing to the stoppage of foraging because of hiding, see [4],[5],[6],[7]. Additionally, certain prey possesses an anti-predator defense mechanism, and there are several examples in nature, particularly when they coexist together. This means that the attacking adult predator may be killed by the prey, see [8],[4],[7],[9],[10],[11],[12].

On the other hand, Biologists and researchers should pay attention to the issue of supplying a predator with an alternate source of food in order to maintain ecological stability and prevent predation of certain rare or valuable species of prey by giving their predators an alternative source of nutrition. Some researchers have been suggested that giving predators additional or alternative sources of food may be used to manage and eradicate prey through predators, but this doesn't always help in the fight against biological pests since it depends on the purpose of this food whether it's an alternative or supplement to prey.[7],[2],[13],[14],[11],[12],[15],

Recently, Numerous research and publications have been published on the impacts of stage structure, anti-predator, fear, additional resource, refuge, disease, harvesting, and other significant factors have been addressed, see [7],[16],[17],[18],[19],[20],[21],[22],[23],[24],[25]. Ying Huang,[6] studied " a two-dimension P-P Sys. involving fear impact, Allee impact, and prey refuge, demonstrated that raising the prey refuge parameter or Allee impact complicates the Sys.'s dynamical behavior; the Allee impact or the fear impact may contribute to a reduction in predator species around positive equilibrium while the prey density is unaffected". Samanta,[12] considered "the three-dimension stage-structured P-P model involving cannibalizing in the predator across aquatic environments. Naturally, the adult predators capture their own

juvenile predators. To minimize such intraspecific catching, substitute food is provided to the adult predators as complimentary food mentioned that this model is helpful especially in the fishery Sys.". Panja,[5] studied "a three-dimension P-P Sys. among adult prey, juvenile prey, and predator; a stage-structure on the prey population is considered. It is supposed that adult prey very vigorous to exhibit an anti-predator behavior and kill the predator".

1.1.1 Problem Statement

In light of these researches, An ecological model inspired by reality based on the modifications of the most recent P-P models has been presented. It includes several significant biological factors such as fear, stage structure in the predator, AF for the predator, anti-predation, and developed form of Holling's disk Eq. The predation system is simulated in a new way by including these mentioned important factors. A theoretical and numerical investigation has been carried out to better comprehend the impact of these real biological relevant factors on their systems' dynamical behavior in the real world.

The suggested model may be seen in an ecosystem with lions or tigers as predators and buffaloes or zebras as prey. It is generally known that these predators live in age stages and that supplying food and hunting for prey is often one of the adult's jobs in order to provide food for their juvenile, while the zebra or buffalo displays a natural response to predation and might even murder the adult predator. Additionally, it is well established that supplying AF rather than prey reduces predation rates owing to predator satiety, which contributes greatly if the objective is to preserve specific creatures or species. [26].

1.1.2 The Aim of The Thesis

The purpose of the thesis is to structure and analyze a novel ecological model that incorporates a variety of biological factors in order to better understand how these real biological factors affect the dynamical behavior of their Syss in the real world. The suggested model can be mathematically represented as sets of first-order nonlinear ODEs. Additionally, is to

i. Study the existence, uniqueness, and boundedness of the suggested model's Sol. .

ii. Establish the existence conditions of all possible EPs.

iii. Then to perform the local stability analyses Also, to conduct analyses of global stability using the Lyapunov method for this model.

iv. Examine the incidence of local bifurcations (LBs), taking into account.

v. Detect the presence of periodic dynamics caused by the appearance of Hopf bifurcation (HB), in addition to the existence of complex dynamics.

vi. Finally, discuss the numerical simulations for the proposed model to complete our study and confirm our theoretical results by studying the global dynamics of the models and specifying the set of key parameters in the proposed model.

1.1.3 The Contribution of The Thesis

The thesis contribution can be summarized as follows:

i. The new and realistic model is represented by the Sys. (3.1) is suggested based on the development of the most recent related P-P models, to investigate the role of fear with anti-predator property on a stage structure P-P Sys., in which the age stages are imposed in the predator population.
ii. Sys. (3.1) is formulated by a new set of non-linear differential Eqs.
iii. Sys. (3.1) includes significant biological factors (fear, predator stage structure, AF for the predator, anti-predation, and developed form of Holling's disk Eq. functional response).
iv. Those factors have been suggested in a novel model to describe the predation system in reality.
v. The functional response (FR) has been considered as a modified Holling's disk Eq. **(**Holling type II) FR which takes into consideration the handling time, the search time for the prey, and the additional source of food for the predator. Where the traditional Holling's disk Eq. is defined by:

$$f(X) = \frac{aX}{b + X} \tag{1.1}$$

where: a is the maximum attack rate, b is the half-saturation level.

In which an individual predator's consumption rate decelerates as prey density increases until it becomes fixed at the level of satiation.

And another form of Holling's disk Eq. which takes into consideration the handling time, the search time for the prey is defined by:

$$f(A) = \frac{vA}{1 + uvA} \tag{1.2}$$

where: v is the searching time, u is the handling time.

These two forms of Holling's disk Eq. (1.1) and (1.2) have been modified to the form (1.3) which takes into consideration the handling time, the search time for the prey, and the additional source of food for the predator, as shown below:

$$f(X, A) = \frac{a(X + vA)}{c + uvA + X} \tag{1.3}$$

vi. Prey intrinsic growth rate $r > 0$ in the first Eq. of Sys. (3.1) is constructed based on the fear function when the prey community grows exponentially when the predator is not available.

The fear function has been chosen from among several forms, such as:

I. $F(n, Z) = \frac{1}{1+nZ}$ II. $F(n, Z) = \frac{1}{1+n_1 Z+ n_2 Z^2}$

III. $F(n, Z) = e^{-nz}$ IV. $F(n, Z) = \frac{e^{-nz}}{1+\beta \sin(nz)}$, where $0 < \beta < 1$

Here we consider the fear function $F(n, Z)$ as follows:

$$F(n, Z) = \frac{1}{1 + nZ} \tag{1.4}$$

Where n is a level of fear and Z is the predator,

Hence the first Eq. of the Sys. (3.1) becomes:

$$\frac{dX}{dT} = \frac{rX}{1+nZ} - bX^2 - \frac{aXZ}{c+uvA+X} \tag{1.5}$$

The properties of the general fear function are:

a. "$F(0,Z) = 1,\ F(n,0) = 1$,"

b. "$\lim_{n\to+\infty} F(n,Z) = 0,\quad \lim_{Z\to+\infty} F(n,Z) = 0$,"

c. "$\frac{\partial F(n,Z)}{\partial n} < 0, \qquad \frac{\partial F(n,Z)}{\partial Z} < 0$"

vii. Furthermore, it is supposed that the prey is powerful enough and may kill the adult predator as a natural defense against predation.

1.1.4 The Validation of System (3.1)

The validity and acceptability of this proposed model (mathematically and biologically) will be achieved if we can establish the following conditions:

i. The existence, uniqueness, and boundedness of the Sol. of the Sys (3.1).
ii. The existence conditions of all possible EPs, especially the positive (coexistence) equilibrium point (CEP), which guarantee that all assumed species (X, Y, and Z) exist.
iii. Numerical simulation to confirm the obtained analytical results.

1.1.5 Thesis Organization

This thesis is organized into six chapters:

Chapter two: In this chapter, a literary review is presented to give a historical introduction to the related works the ecological Syss, prey and predator interactions, mathematical modeling and its developments, and the environmental factors that have been injected into mathematical modeling to study the interactions of living organisms and their relationship with each other and with their environment.

Chapter three: in this chapter, the interplay between prey and two predator age stages is described using a three-dimensional P-P model. The fear function is modeled on the prey population. It has also been hypothesized that the prey is strong enough to exhibit anti-predator behavior and has the ability to kill the predator as a natural response to resistance to predation. The traditional Holling's disk Eq. (Holling type II) is modified to explain the prey consumption to involve an AF for the predator, taking into consideration the handling time, and the search time for the prey. The existence, uniqueness, and boundedness of the Sol. of the proposed model are investigated. The local stability analyses are carried out after deciding all viable EPs. The Lyapunov method is also used to investigate the global stability analyses for this model.

Chapter four: In this chapter, the occurrence of LBs (pitchfork bifurcation(PB), transcritical bifurcation (TB), and saddle-node bifurcation (SNB)) of all EPs are carried out. Further investigation of the conditions which guarantee the occurrence of the HB around the positive EP is discussed.

Chapter five: In this chapter, the global dynamics of the model are carried out numerically, and the conclusions and discussion of our obtained results are also included.

Chapter six: In this chapter, conclusions and discussion are presented as well as all advice and suggestions for future studies.

1.2 MATHEMATICAL PRELIMINARIES AND TOOLS

In this section, all the analytical mathematical tools, concepts, definitions, and theories that we utilized analytically in this thesis will be covered.

Definition (1.1) [27] **:** A function $g: R^n \to R^n$ is said to be Lipschitz on $\mathcal{B}$ where $\mathcal{L}$ is a Lipschitz positive constant if $\|g(x) - g(y)\| \leq \mathcal{L}\|x - y\|$; $\forall x, y \in \mathcal{B}$, for some $\mathcal{B}$ in R^n. Further, g is called globally Lipschitz, if it is Lipschitz on R^n , however, g is called locally Lipschitz, if it is Lipschitz on every bounded subset of R^n but is not globally Lipschitz.

Lemma (1.2) (Gronwall Lemma):

Consider a function $x(t)$ satisfies the following differential inequality $x' \leq ax + b; x(0) = x_0$ with a, b are constants. Then for all $t \geq 0$, we have:

$$x(t) \leq x_0 e^{at} + \frac{b}{a}(e^{at} - 1); \quad a \neq 0 \text{ and } x(t) \leq x_0 + bt; \quad a = 0. \text{ [28]}$$

1.2.1 Descartes Rule of Sings:

Let

$$P(x) = m_0 x^n + m_1 x^{n-1} + \cdots + m_n = 0,$$

be a polynomial function with real and non-zero coefficients $m_0, m_1, \ldots, m_n$, with descending power terms of x. Thus,

i. The positive real roots' number of $P(x) = 0$ is either equal to the No. of the changes of sign in P(x) coefficients or smaller than the No. of the changes of the sign by a positive even integer.

ii. The negative real roots' number of $P(x) = 0$ either equals the No. of sign changes in $P(-x)$, coefficients or smaller than the No. of the changes of the sign by a positive even integer. [29]

1.2.2 Local Stability Analysis:

Suppose the n-dimensional 1st order autono. ODEs system:

$$\left.\begin{array}{l} \dot{x_1} = g_1(x_1, \ldots, x_n) \\ \dot{x_2} = g_2(x_1, \ldots, x_n) \\ \quad \vdots \\ \dot{x_n} = g_n(x_1, \ldots, x_n) \end{array}\right\} \Longrightarrow \frac{dx}{dt} = \dot{x} = g(x), \qquad x \in R^n . \tag{1.6}$$

Where $g \in C^1(D, R^n)$ is the non-linear function with $D \subseteq R^n$ is the domain of g. We give some fundamental concepts that are required to investigate the stability of the Sys. (1.6) in the following:

Definition (1.3) [27]**:** if $f(\hat{x}) = 0$ then a point $\hat{x} \in R^n$ is termed an equilibrium point (EP) (critical point, steady-state Sol., etc.) of the Various kinds of attractors of the Sys.(3.2) with typical values of (1.6).

Definition (1.4) [27]**:** An EP $\hat{x}$ of the Sys. (1.6) is called:

I. Stable: if $\forall\ \epsilon > 0$, $\exists\ \delta(\epsilon) > 0$ s.t., $\forall\ x_0$ for which $\| x_0 - \hat{x} \| < \delta(\epsilon)$, then the Sol. $x(t)$ of the Sys. (1.6) satisfies $\| x(t) - \hat{x} \| < \epsilon\ \forall t \geq 0$.

II. Asymptotically stable (AS): if it's stable, and $\exists \gamma > 0$ (γ constant) s.t. if $\| x_0 - \hat{x} \| < \gamma$ then $\| x(t) - \hat{x} \| \to 0$ as $t \to \infty$,

III. Unstable: otherwise.

Definition (1.5) [27]: An EP $\hat{x}$ of the Sys. (1.6) is called globally asymptotically stable (GAS) if it's AS and the domain $D_{\hat{x}}$ of AS where,

$$D_{\hat{x}} = \{x_0 \in R^n : \lim_{t\to\infty} \| x(t) - \hat{x} \| = 0\} = Int.R^n ,$$

that is (every Sol. $x(t)$ of the Sys. (1.6) possesses the property $\| x(t) - \hat{x} \| \to 0$ as $t \to \infty$).

Definition (1.6) [27]: The basin of attraction of an EP $\hat{x}$ of the Sys. (1.6), indicated by $B(\hat{x})$ represents the set of all point u such that the Sol. of Eq. initiating at u will be subsequently approached to the point $\hat{x}$, that is,

$B(\hat{x}) = \{u \in R^n : \lim_{t\to\infty} \| x(t,u) - \hat{x} \| = 0\}$, here $x(t,u)$ represents the Sol. of Eq. (1.6) started at u. Further, $B(\hat{x})$ is equal to $Int.R^n$ if and only if $\hat{x}$ is GAS. Therefore, as a result, if the EP $\hat{x}$ is GAS then it's locally asymptotically stable (LAS), But the opposite is not true.

Now, in order to explain the application of the linearization technique on the Sys. (1.6), consider:

$$x(t) = \hat{x} + \varphi(t), \tag{1.7}$$

here $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))^T$ represents a small perturbation of the original Sol. Then depending on both Sys. (1.6) and Eq. (1.7), we get:

$$\frac{d\varphi}{dt} = f\left(\hat{x} + \varphi(t)\right). \tag{1.8}$$

Note that, the other EP of this Eq. which corresponds to $\hat{x}$ of the Sys. (1.6) is $\varphi(t) = 0$. Now applying Taylor expansion we obtain:

$$f(\hat{x} + \varphi) = f(\hat{x}) + Df(\hat{x})\varphi + R(\varphi), \tag{1.9}$$

where $R(\varphi)$ represents the remainder terms and $Df(\hat{x})$ represents the Jacobian matrix (JM) (or Variational matrix) of f evaluated at $\hat{x}$, in general, is written as:

$$Df(\hat{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=\hat{x}}$$

Since $\hat{x}$ is an EP, so $f(\hat{x}) = 0$. As a result, the Sys. (1.9) devolves into:

$$\frac{dw}{dt} = Aw + R(w), \tag{1.10}$$

Here, $A = Df(\hat{x})$

Accordingly, close to the origin Sys. (1.10) can be written as:

$$\frac{dw}{dt} = Aw, \tag{1.11}$$

which is a linearized system to Sys. (1.6). Therefore, the EP $w(t) = 0,$ and hence $\hat{x}$, is AS provided that the (EVUs) of $A = Df(\hat{x})$ are in the left-plane as shown in the next theorem.

Theorem (1.7)[30]: The EP $\hat{x}$ of (1.6) is called:

I. LAS, if all EVUs of A in the linearized Sys. (1.11) exists on the left of the complex plane.

II. Unstable, if any of the EVUs are in the right of the complex plane.

Definition (1.8)[31]**:** Suppose $\hat{x}$ is an EP of the Sys. (1.6), then $\hat{x}$ is called a hyperbolic EP If all EVUs of $Df(\hat{x})$ have no a zero real part. And it's a non-hyperbolic EP otherwise.

Thus according to the above definition, the EP $\hat{x}$ of the Sys. (1.6) can be classified as follows:

i. Stable point (sink):

A hyperbolic EP of a vector field described by the Sys. (1.6) is called a stable node or sink if all EVUs of the JM $J = DF(\hat{x})$ possess negative real parts.

ii. Unstable point (source):

If all EVUs of J possess real parts which are positive, then the EP is called an Unstable node or source.

iii. Saddle point:

If some of the EVUs of J possess real parts negative and the other possesses real parts positive, then the EP is called a Saddle point**.**

iv. Linear center point:

If all EVUs of J are non-zero and pure imaginary, then the non-hyperbolic EP is called a linear center point.

Now, we need to state the following fundamental theorem of dynamic Syss. Thus .the relation between the asymptotic behavior of the Sol. of (1.6) near the EP $\hat{x}$ and that of the approximate linear Sys. (1.11) is described by this theorem.

Theorem (1.9) (Hartman-Grobman theorem):

Let $\hat{x}$ is a hyperbolic EP of the Sys. (1.6), then there is a homeomorphism Ң specified in some neighborhood N of $\hat{x}$ in R^n locally transferring orbits of the non-linear flux $x(t)$ of the Sys. (1.6) to the orbits of the linear flux $e^{Df(\hat{x})t}$ of the Sys. (1.6). The homeomorphism can preserve the orbits sense, and additionally has the option of preserving parameterization across time.[32]

As a result, the structure of the orbit near a hyperbolic EP of the Sys. (1.6) is qualitatively identical to the structure of the orbit given by the corresponding linearized dynamical Sys. (1.11). Consequently, the local stability of a hyperbolic EP $\hat{x}$ of the non-linear Sys. (1.6) depends on the real parts sign of the EVUs of $Df(\hat{x})$ in the linearized Sys. (1.11). Therefore, in the following two subsections, we give the most often used criterion for determining the sign of the EVUs.

1.2.3 Trace-Determinate Stability Criterion:

Assume that the planar system: [33]

$$\frac{dx(t)}{dt} = Ax(t) \ , \tag{1.12}$$

where A is the 2×2 matrix with real constant elements given by:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Then the auxiliary Eq. for A is given by:

$$P(\ \lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} \tag{1.13}$$

The trace of the matrix A represented by $T = a_{11} + a_{22}$ and the determinant of matrix A represented by $D = a_{11}a_{22} - a_{12}a_{21}$ So the auxiliary Eq. of A may be rephrased as:

$$\lambda^2 - T\lambda + D = 0 . \tag{1.14}$$

Note that the auxiliary Eq. of *A* depends only on *D* and *T*, so the EVUs of *A* also depend on *D* and *T*. Then the EVUs of *A* can be written by:

$$\lambda_i = \frac{T \pm \sqrt{T^2 - 4D}}{2} \quad , \; i=1, 2, \; \Delta = \sqrt{T^2 - 4D} \tag{1.15}$$

From Eq. (1.15) we see that the EVUs of *A* are:

i. Complex if: $\Delta < 0$,

ii. Repeated if: $\Delta = 0$,

iii. Real and distinct if: $\Delta > 0$.

The behavior of the Sys. (1.12) in the different cases for the EVUs will be discussed as follows:

Case I: If $\Delta > 0$, we have the following subcases:

Subcase i: If $D > 0$ and $T > 0$ in this subcase the Sys. has two positive EVUs, so the origin is an unstable node.

Subcase ii: If $D < 0$ and ($T > 0$ or $T < 0$) in this subcase the Sys. has one negative and one positive EVU, so the origin is a saddle point.

Subcase iii: If $D = 0$ and $T > 0$ in this subcase the Sys. has one zero and one positive EVU, so the origin is unstable.

Subcase iv: If $D > 0$ and $T < 0$ in this subcase the Sys. has two negative EVUs, so the origin is stable.

Subcase v: If $T < 0$ and $D = 0$, in this subcase, the Sys. has one zero and one negative EVU, so the origin is stable but not AS.

Case II: If $\Delta < 0$, the EVUs will be complex and their real part is $T/2$ hence we have the following subcases:

Subcase i: If $T < 0$ in this subcase, the EVUs possess real parts, negative so the origin is an AS focus.

Subcase ii: If $T > 0$ in this subcase the EVUs possess real parts positive, so the origin is an unstable focus.

Subcase iii: If $T = 0$ in this subcase the origin is a center**.**

Case III: If $\Delta = 0$ i, the Sys. possesses double real EVU $\lambda = T/2$, so we have the following subcases:

Subcase i: If $T < 0$, in this subcase the two EVUs are negative, so the origin is stable.

Subcase ii: If $T > 0$, in this subcase the two EVUs are positive, so the origin is unstable.

Subcase iii: If $T = 0$, in this subcase the two EVUs are equal to zero.

1.2.4 Routh-Hurwit Criteri:

Consider the auxiliary Eq. of $Df(\hat{x})$ which is defined by: [34]

$$P_n(\lambda) = \lambda^n + b_1\lambda^{n-1} + \cdots\cdots + b_n = 0 \tag{1.16}$$

Suppose:

$$D_1 = b_1, \quad D_2 = \begin{vmatrix} b_1 & b_3 \\ 1 & b_2 \end{vmatrix}, \ldots, \quad D_k = \begin{vmatrix} b_1 & b_3 & b_5 & \cdots & b_{2k-1} \\ 1 & b_2 & b_4 & \cdots & b_{2k-2} \\ 0 & b_1 & b_3 & \cdots & b_{2k-3} \\ 0 & 1 & b_2 & \cdots & b_{2k-4} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & b_k \end{vmatrix}$$

where $b_i = 0$ if $i > n.$ Then the polynomial $P_n(\lambda)$ possesses roots with negative real parts iff $D_k > 0$ $\forall$ *k= 1,2,….,n.*

When *n=2,* (1.16) will be $P_2(\lambda) = \lambda^2 + b_1\lambda + b_2 = 0,$ and hence

$$D_1 = b_1\,,\ D_2 = \begin{vmatrix} b_1 & 0 \\ 1 & b_2 \end{vmatrix} = b_1 b_2$$

Then if $n = 2$ the conditions (necessary and sufficient) for the polynomial to possess negative real parts roots: $b_1 > 0\,,\ b_2 > 0.$

And when *n=3,* (1.16) will be $P_3(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0,$ and hence

$$D_1 = b_1\,,\ D_2 = \begin{vmatrix} b_1 & b_3 \\ 1 & b_2 \end{vmatrix} = b_1 b_2 - b_3,\ D_3 = \begin{vmatrix} b_1 & b_3 & 0 \\ 1 & b_2 & 0 \\ 0 & b_1 & b_3 \end{vmatrix} = b_3(b_1 b_2 - b_3)$$

So if *n=3,* the conditions (necessary and sufficient) for the polynomial to possess negative real parts roots: $b_1 > 0\,,\ b_3 > 0$ and $b_1 b_2 - b_3 > 0.$

As a result, using the Routh – Hurwitz criteria yields a set of restrictions on the coefficients $b_1, b_2, \ldots, b_n$ that are necessary and sufficient to guarantee that all roots of the auxiliary Eq. (EVUs of the JM $A = Df(\hat{x})$) appear in the left half of the complex plane. As a consequence, if the Routh-Hurwitz criteria are concurrently met, the Sys. (1.6) is AS at $\hat{x}$. On the other hand, any breach of any of the requirements indicates an unstable point.

1.2.5 Lyapunov Method:

The stability and instability of the EPs (particularly of the non-hyperbolic EPs) of a nonlinear Sys. can be determined using the Lyapunov method. This method can also be used to investigate the EP stability of a linear Sys. It also works in both finite and infinite dimensions. And the method's core principle is as follows: Assume that, $\hat{x}$ is an EP of the Sys. (1.6), then finding the neighborhood U of $\hat{x}$ where orbits beginning in U remain throughout the positive time in U, is sufficient to determine if $\hat{x}$ is stable or not. This requirement would be met if we could demonstrate the orbit of the Sys. (1.6) is either

pointing inward toward $\hat{x}$ or tangent to the boundary of U, and this scenario holds even when U is shrunk down onto the EP $\hat{x}$. [27]

It's worth noting that the Lyapunov method is used to describe this situation. As a result, the general theorem for EP stability, which clarifies these concepts, is as follows:

Theorem (1.10) (Lyapunov stability theorem)[31]:

Suppose that $\hat{x}$ is an EP of the Sys. (1.6) and let $V: U \to R$ be a C^1 a real-valued function defined on some neighborhood $U (U \subseteq R^n)$ of $\hat{x}$ such that:

i. $V(x) > 0$ and $V(\hat{x})=0$ $\forall\, x \neq \hat{x}\,;\ x \in U$ (which means V is a positive definite function).

ii. $\hat{x}$ is stable, if $\frac{dV(x)}{dt} \leq 0$ in $U - \{\hat{x}\}$.

iii. Furthermore, $\hat{x}$ is AS, if $\frac{dV(x)}{dt} < 0$ in $U - \{\hat{x}\}$.

In addition, the function V given above is called the Lyapunov function (LF) (weak LF) iff the conditions (i) and (ii) hold, and it is called strictly LF (strong LF) iff conditions (i) and (iii) hold. Moreover, $\hat{x}$ is said to be GAS, if U is selected to be all of the domain R^n, and if conditions (i) and (iii) hold.

1.2.6 Local Bifurcation (LB):

suppose a general Sys. of ODEs depending on a single parameter α: [35]

$$\frac{dx}{dt} = f(x, \mu) \tag{1.17}$$

where $\frac{dx}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \dots, \frac{dx_n}{dt}\right)^T$ $= f = (f_1, f_2, \dots, f_n)^T$, and $\alpha \in R$ is a parameter. We shall assume throughout this thesis that $f \in \mathrm{C}^1$ $(E \times I)$ where $I \subset R$ is an interval and E is an open set in $R^n, n \in N$. $Df(x, \mu)$ denoted the Jacobian matrix (JM) and $f_\mu\ (x, \mu)$ denote the vector of the partial derivatives of the components of f w.r.t the parameter μ.

Definition (1.11) [36]: “The word bifurcation is a qualitative (topological) change of the nature of the Sol. under the Sys.'s parameter variation”.

Definition (1.12)[36] : “A value μ_0 is called a bifurcation value of Eq. (1.17) if in any neighborhood of μ_0 there is a value corresponding to a topologically non-equivalent member of the family given by Eq. (1.17).”

Theorem (1.13) (Sotomayor theorem)[35]

Assume that there is an EP $\hat{x}$ for Sys. (1.17) and $\mu = \mu_0$ that is $f(\hat{x}, \mu_0) = 0$. Suppose that the $n \times n$ matrix $J = Df(\hat{x}, \mu_0)$ possess a simple EVU $\lambda = 0$ with eigenvector (EVC) V and that the $n \times n$ matrix J^T possess an e EVC Ψ corresponding to the EVU $\lambda = 0$. Furthermore, suppose that J has k EVUs with negative real parts and $(n - k - 1)$ EVUs with positive real parts and that if the following conditions hold:

$$\Psi^T f_\mu(\hat{x}, \mu_0) \neq 0,\ \Psi^T[D^2 f(\hat{x}, \mu_0)(V, V)] \neq 0 \tag{1.18}$$

Then the Sys. (1.17) experiences an SNB at the EP $\hat{x}$ as τ passes through $\mu = \mu_0$. However, if the conditions (1.18) are changed to

$$\left.\begin{array}{l} \Psi^T f_\mu(\hat{x}, \mu_0) = 0, \Psi^T\left[D f_\mu(\hat{x}, \mu_0) V\right] \neq 0 \\ \Psi^T[D^2 f(\hat{x}, \mu_0)(V, V)] \neq 0 \end{array}\right\}. \tag{1.19}$$

Then the Sys. (1.17) experiences a TB at the EP $\hat{x}$ as μ passes through $\mu = \mu_0$. Finally, if the conditions given by (1.18) are changed to

$$\left.\begin{array}{l} \Psi^T f_\mu(\hat{x}, \mu_0) = 0,\ \Psi^T\left[D f_\mu(\hat{x}, \mu_0) V\right] \neq 0 \\ \Psi^T[D^2 f(\hat{x}, \mu_0)(V, V)] = 0,\ \Psi^T[D^3 f(\hat{x}, \mu_0)(V, V, V)] \neq 0 \end{array}\right\}. \tag{1.20}$$

Then the Sys. (1.17) experiences a PB at the EP $\hat{x}$ as μ passes through the $\mu = \mu_0$.

1.2.7 Hopf Bifurcation (HB):

In the following the conditions, which gradients that a simple HB for a dynamical Sys. (1.17) occur are presented.

Definition (1.14)[37]: Let $f(\hat{x}, \mu) = 0$ for all $\mu \in R$. A value $\mu = \tilde{\mu}$ is said to be a Hopf bifurcation value for a steady Sol. $\hat{x}$ of eq. (1.17) if the JM $Df(\hat{x}, \tilde{\mu})$ possesses a simple pair of purely imaginary EVUs and no other EVUs with zero real parts, while the phase portrait of eq.(1.17) will change as μ passes through $\tilde{\mu}$, where a periodic orbit is created in the neighborhood of $\tilde{\mu}$.

Theorem (1.15) [38]: Consider the Sys. (1.17) with an EP $\hat{x}$ Then, if there is a parameter $\tilde{\mu} \in I$ with $I \subset R$ and $f \in C^{\alpha}(R^n \times R, R^n)$ for some $\alpha \geq n$ with $f(\hat{x}, \mu) = 0 \ \forall \mu \in R$, so that:

I. The JM $J(\hat{x})$ of the Sys. (1.17) at the EP $\hat{x}$ has a simple pair of complex EVUs, say $\lambda(\mu) = \varepsilon_1(\mu) \mp i\varepsilon_2(\mu)$

, s.t. they become purely imaginary at $= \tilde{\mu}$, while all the other EVUs remain negative and real

II. $\frac{d\varepsilon_1(\mu)}{d\mu}\Big|_{\mu=\tilde{\mu}} \neq 0$, (the transversality condition).

Then the Sys. (4.1) has an HB at $\tilde{\mu}$.

Now, according to the above theorem, the classic simple HB criteria for $n = 3$ is in terms of EVUs properties as shown below [36].

Consider the characteristic Eq. of Sys. (1.17) given by:

$$P_3(\gamma) = \gamma^3 + A_1\gamma^2 + A_2\gamma + A_3 = 0, \tag{1.21}$$

here

$A_1(\mu) = -tr(J(\hat{x}))$, $C_2(\tau) = M(J(\hat{x}))$ and $A_3(\mu) = -\det(J(\hat{x}))$ with $M(J(\hat{x}))$ represents the sum of the determinants of the principal minors of order two of $J(\hat{x})$.

Clearly the first condition of theorem (1.15) holds if and only if $A_i > 0$; $i = 1,2$ and $\Delta = A_1A_2 - A_3 = 0$ (that means that $A_3 = A_1A_2$). So, the characteristic Eq. becomes:

$$P_3(\gamma) = (\gamma + A_1)(\gamma^2 + A_2) = 0 \tag{1.22}$$

Clearly, the roots of eq. (1.22) are

$$\gamma_{1,2} = \pm i\sqrt{A_2} \text{ and } \gamma_3 = -A_1$$

Now, to check the transversality condition of HB, Substituting $\gamma(\mu) = \xi_1(\mu) \mp i\xi_2(\mu)$ into eq. (1.21), computing its derivative w.r.t the bifurcation parameter μ , ${P_3}'\big(\gamma(\mu)\big) = 0$, and then comparing the two sides of the resulting Eq. and then equating their imaginary and real parts, we get:

$$\left.\begin{array}{l} \Psi(\mu){\xi_1}'(\mu) - \Phi(\mu){\xi_2}'(\mu) = -\Theta(\mu) \\ \Phi(\mu){\xi_1}'(\mu) + \Psi(\mu){\xi_2}'(\mu) = -\Gamma(\mu) \end{array}\right\}, \tag{1.23}$$

here

$$\left.\begin{array}{l} \Psi(\mu) = 3(\xi_1(\mu))^2 + 2A_1(\mu)\xi_1(\mu) + A_2(\mu) - 3\big(\xi_2(\mu)\big)^2 \\ \Phi(\mu) = 6\xi_1(\mu)\xi_2(\mu) + 2A_1(\mu)\xi_2(\mu) \\ \Theta(\mu) = \big(\xi_1(\mu)\big)^2{A_1}'(\mu) + {A_2}'(\mu)\xi_1(\mu) + {A_3}'(\mu) - {A_1}'(\mu)\big(\xi_2(\mu)\big)^2 \\ \Gamma(\mu) = 2\xi_1(\mu)\xi_2(\mu){A_1}'(\mu) + {A_2}'(\mu)\xi_2(\mu) \end{array}\right\}. \tag{1.24}$$

Using Cramer's rule for the unknowns ${\xi_1}'(\mu)$ and ${\xi_2}'(\mu)$ to solve the linear Sys. (1.23), we get:

$${\xi_1}'(\tau) = -\frac{\Theta(\mu)\Psi(\mu)+\Gamma(\mu)\Phi(\mu)}{\big(\Psi(\mu)\big)^2+\big(\Phi(\mu)\big)^2} \quad \text{and} \quad {\xi_2}'(\mu) = \frac{-\Gamma(\mu)\Psi(\mu)+\Theta(\mu)\Phi(\mu)}{\big(\Psi(\mu)\big)^2+\big(\Phi(\mu)\big)^2}$$

Hence the transversality condition not being zero if and only if

$$\Theta(\tilde{\mu})\Psi(\tilde{\mu}) + \Gamma(\tilde{\mu})\Phi(\tilde{\mu}) \neq 0 \tag{1.25}$$

2. LITERATURE REVIEW

The initial pattern of the P-P model was presented by the renowned Lotka in 1925 [2] and Volterra in 1926[39] who started from a classic and uncomplicated hypothesis.

$$\acute{x}(t) = f(x) - g(x,y)y$$

$$\acute{y}(t) = eg(x,y)y - \alpha\,(y)y \tag{2.1}$$

Where: x, y: the densities of the prey and predator at time t respectively.

$f(x)$: is a continuous function that represents the prey's growth rate in the absence of a predator.

e: is the conversion rate parameter of eaten prey into new predator

$\alpha(y)$: is a continuous function that represents the predator's natural death rate in the absence of its prey.

$g(x,y)$: is a continuous function that represents the FR in the Eq. of the prey, which is also known as a numerical response in the Eq. of the predator."

More complex but realistic predator and prey models have been created by ecologists and mathematicians. Berryman [40] in 1992, stated that the interaction dynamic between the prey and the predator took a long time and continued to be one of the most significant and key issues in mathematical ecology, technological sciences, society, and nature, particularly in biology and ecology study.[34],[41],[42],[43],[44].

Numerous researchers who work in these fields are putting a lot of effort into improving and generalizing the pattern formation of P-P models to a more involved and realistic model, and incorporating this advantage into a variety of applications that are interested in the existing nature of P-P models global asymptotic stability.

Moreover, several articles and papers have looked into the dynamical behavior of prey-predator models, resulting in a significant expansion in the structures of P-P models to include multi-species; numerous types of food chains and food webs, various FRs, stage-structure, fear, refuge, AF, harvesting, and numerous types of diseases...etc.

Recently, Pal et al. [45] investigated the hydra impacts in the models of stable food chains, which has been a prominent study topic in the last decade. The six-dimensional model is defined by:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{\propto_1 x\, y_1}{h_1 + x} - q_0 e_0 x$$

$$\frac{dy_1}{dt} = \frac{\beta_1 \propto_1 x\, y_1}{h_1 + x} - \frac{\propto_2 y_1\, y_2}{h_2 + y_1} - \mathrm{m}_1 y_1 - q_1 e_1 y_1$$

$$\frac{dy_2}{dt} = \frac{\beta_2 \propto_2 y_1\, y_2}{h_2 + y_1} - \frac{\propto_3 y_2 y_3}{h_3 + y_3} - \mathrm{m}_2 y_2 - q_2 e_2 y_2$$

$$\frac{dy_3}{dt} = \frac{\beta_3 \propto_3 y_2\, y_3}{h_2 + y_2} - \frac{\propto_4 y_3 y_4}{h_4 + y_4} - \mathrm{m}_3 y_3 - q_3 e_3 y_3 \qquad (2.2)$$

$$\frac{dy_4}{dt} = \frac{\beta_4 \propto_4 y_3\, y_4}{h_4 + y_4} - \frac{\propto_5 y_4 y_5}{h_5 + y_4} - \mathrm{m}_4 y_4 - q_4 e_4 y_4$$

$$\frac{dy_5}{dt} = \frac{\beta_5 \propto_5 y_4\, y_5}{h_4 + y_4} - q_5 e_5 y_5,$$

Noting that the effect of hydra, which is one of the contradictory discoveries in applied and theoretical ecology indicates the notion that raising a population's death rate improves its stock, and the major goal is to present a food chain dynamical Sys. model with a stable state condition and to calculate the change in the stock of targeted species as the death rate increases, where this food chain has just one prey (x) and five specialized food levels for predators (y_1, y_2, y_3, y_4 and y_5). And they came out with results that are that " the per capita rate of growth of every predator food level is independent on its density. The P-P model that has such a characteristic for predator growth is known as a 'pure predator Sys.'" . Sen et al. [46] studied " the complex dynamics and the intraguild predation of a three-species P-P model. They defined "Omnivores" as those whose feeding does not depend on one level in the food chain and they have taken on an important role in ecology lately. they introduced a special type of omnivores that is intraguild predation, which is found in a wide range of biological Sys.s, it includes a prey (resource) (N_1), an intermediate predator (N_2) that consumes only prey and a higher predator (intraguild predator) (N_3) which consumes both intermediate predator and prey.The model is defined by:

$$\frac{dN_1}{dt} = N_1(r_1 - a_{11}N_1 - a_{12}N_2 - a_{13}N_3)$$

$$\frac{dN_2}{dt} = N_2\left(-r_2 + a_{21}N_1 - a_{22}N_2 - \frac{a_{23}N_3}{1+\beta N_2}\right) \quad (2.3)$$

$$\frac{dN_3}{dt} = N_3\left(-r_3 + a_{31}N_1 - a_{33}N_3 + \frac{a_{32}N_2}{1+\beta N_2}\right),$$

They proposed a saturated FR to explain the interaction between intraguild prey and intraguild predator. Other interspecific competitions are represented as a linear function. Because limitless consumption is not necessarily a realistic assumption to explain grazing behavior, so they looked at Holling's disk function to better understand its effect on the dynamical Sys. And they have also taken into account intraspecific competition for intraguild and intermediate predators since individuals are forced to struggle among themselves owing to a scarcity of resources, which negatively affects their growth. Huang et al. [47] studied "the stability and the HB across a delayed P-P Sys. involving disease in the predator population. The model is defined by:

$$\frac{dP}{dt} = r\left(1 - \frac{P(t)}{K}\right)P(t) - \frac{nP(t)I(t)}{M+P(t)} - mP(t)S(t)$$

$$\frac{dS}{dt} = \frac{\beta\, nP(t)I(t)}{M+P(t)} - \alpha I(t)S(t) + a\,S(t) - d_1\,S(t) + \eta I(t), \quad (2.4)$$

$$\frac{dI}{dt} = \frac{\beta nP(t)I(t)}{M+P(t)} + \alpha I(t-\tau)S(t-\tau) - d_1 I(t) - \eta I(t),$$

they have developed a novel delayed SIS epidemiological P-P Sys. based on the hypothesis that "the disease is only transmitted within the predator species and that different kinds of predators respond differently, the prey (P) is consumed by the infected predator (I) according to Holling's disk function and consumed by the susceptible predator (S) following the mass action law. Noting that trying to deal with an epidemiological P-P model is essential for understanding the dynamic features of population models and the current research indicates that introducing disease into the predator's group can disrupt the stability of P-P ecosystems".

Furthermore, many factors influence predation, and the density of predator and prey are unavoidable features of all cases of P-P where the major interest is population effects. The responses to these two factors will represent the key parts of predation since they are universal. Two types of responses are significant because the total number of prey killed by predators represents the No. of predators present and the No. of prey consumed per predator [48], [49],[50]. Solomon [51] has effectively introduced two concepts to describe the dual nature of predation; they are the FR (which concerns prey consumption) and the numerical response (which concerns predator density) "As a result, the fundamental components obtained by the two universal factors can be regarded as the predator's FRs to the density of prey and predator, and the numerical response. As the numerical response is a result of the impact of the consumption of food on reproduction, emigration, immigration, and mortality. This means the FRs (which are measured in food consumption terms), are of great importance. So it makes sense to focus on FRs and study their impact on prey and predator Syss extensively.

The relationship between prey (pests) and predator (natural enemies) is well known, the standard models included a distinctive description of the method of feeding the predators on prey (feeding rate), which is referred to as the FR, to structure the change in the rate of predation as a function of prey density using a set of differential Eqs. The method of imposing and selecting the FR depends on the nature of the ecosystems. Due to the complexity of the natural environment, which includes different levels of human intervention, many FRs have been proposed to study the dynamics of interaction between predator and prey such as:

The Holling type I FR [52],[53], which is in the form $f_1(X) = \begin{cases} \frac{bx}{a}, & 0 < x < a \\ b & x > a \end{cases}$.

The Holling's disk function (Holling type II) FR [54], which is in the form $f_2(x) = \frac{rx}{a+bx}$.

The Holling type III FR [55], which is in the form $f_3(x) = \frac{rx^2}{a+bx^2}$., ,

The Ivlev FR [56],[57], which is in the form $f_4(x) = r(1 - e^{-\rho x})$.

The Beddington-DeAngelis FR [58],[59], which is in the form $f_5(x, y) = \frac{rx}{a+bx+cy}$.

The ratio-dependent FR [4],[60], which is in the form $f_6\left(\frac{x}{y}\right) = \frac{\frac{\alpha x}{y}}{\mu+\frac{x}{y}} = \frac{\alpha x}{\mu y+x}$.

The Watt FR [61],[62], which is in the form $f_7(x)= e^{\frac{-\rho x}{\alpha y}}$

The Hassell-Varley FR [62],[63], which is in the form $f_8(x) = \frac{\alpha x}{\mu y^{\gamma}+x}$.

The Square-Root FR [64], [65], which is in the form $f_9(x) = \frac{\alpha\sqrt{x}}{1+\beta\sqrt{x}}$.

The Monod-Haldance FR [66],[67], which is in the form $f_3(x)=\frac{rx}{a+x^2}$, etc.

Ali and Majeed [68] investigated the four-dimension stage-structured P-P interaction dynamics which involve different types of FRs. The model is defined by:

$$\dot{X_1} = \alpha\, x_2\left(1-\frac{X_2}{k}\right) - \frac{\beta_1\,(1-\eta)\,X_1}{b+X_1}\,Y_1 - s\,X_1 - d_1\,X_1$$

$$\dot{X_2} = s\,X_1 - \beta_2\,(1-\eta)\,X_2\,Y_1 \; - d_2\,X_2 \qquad (2.5)$$

$$\dot{Y_1} = \frac{e_1\beta_1\,(1-\eta)\;X_1}{b+X_1}\,Y_1 + e_2\beta_2\,(1-\eta)\,X_2\,Y_1 - \beta_3\,Y_1\,Y_2 - d_2\,Y_1$$

$$\dot{Y_2} = e_3\beta_3\,Y_1\,Y_2 - d_4\,Y_2\,.$$

Two FRs have been imposed in this model, where the type II FR was considered, to explain the interplay between the intermediate predator (Y_1) and juvenile prey (X_1) characterized by lack of resources, whereas the Lotka-Volterra FR is imposed in two separate places, the first describes the unlimited resources and the linear relationship between intermediate predator (Y_1) and adult prey (X_2). Second, describes the interaction of intermediate predator (Y_1) and top predator (Y_2) according to a food chain. Mortuja [69] et al. introduced a dynamic of a P-P model with square root FR and nonlinear harvesting of prey. The model is defined by:

$$\dot{x(t)} = x(1-x) - \frac{\beta x}{x+a} - \frac{y\sqrt{x}}{1+\mu\sqrt{x}}$$

$$\dot{y(t)} = \frac{e\sqrt{x}y}{1+\mu\sqrt{x}} - dy, \qquad (2.6)$$

They introduced the dynamic of a P-P model with square root FR and nonlinear harvesting of prey, this work aims to study and analyze the dynamics of a P-P model that takes into account the square root FRs for prey group behavior along with the nonlinear harvesting of prey populations. The square root FR and nonlinear harvesting of prey are included in the model when herd behavior is present.

Common methods for modeling the population growth in the P-P Sys. have usually included certain FRs, and this type of modeling is only applicable over a certain ecoSys. under specific environmental conditions. Liu et al. [62] tried to bridge the gap by presenting a study different from previous studies and proposing a prey and predator Sys. that includes a generalized impulsive FR $f(x,y)$ for the first time instead of using a specific FR. The model is defined by: (2.7)"

"
$$\left.\begin{aligned} \dot{x(t)} &= \left(a - \frac{a}{k}x(t)\right)x(t) - f(x,y)x(t)y(t) \\ \dot{y(t)} &= (kf(x,y)x(t) - D)y(t) \end{aligned}\right\}, t \neq (n+i-1)T, t \neq nT$$

$$\left.\begin{aligned} \Delta x(t) &= -\beta_1 x(t) \\ \Delta y(t) &= -\beta_2 y(t) \end{aligned}\right\}, \quad t = (n+i-1)T \qquad (2.7)$$

$$\left.\begin{aligned} \Delta x(t) &= 0 \\ \Delta y(t) &= \gamma \end{aligned}\right\}, t = nT \quad \text{“}$$

This study improves modeling for integrated pest control particularly when a predator (natural enemy) (y) stocking and prey (pest) (x) harvesting occurs impulsively and at separate times. The theoretical and numerical results obtained using the generalized impulsive FR came in complete agreement with the results obtained in previous studies and were validated using Holling type II (Holling's disk Eq.) as an example.

On the other hand, Living organisms interact in different ways with each other and with their natural environment, almost all organisms in the real world have an immature and mature age stage, and species with various stages may behave differently. Ecologically, the individuals of mature and immature organisms are completely different, Thus their niches and position in their local food web vary, and they may also inhabit different food webs and habitats such that caterpillars and butterflies; aquatic caterpillars with terrestrial amphibians; malaria-carrying mosquitoes; seedlings and Adult trees and accordingly, the way they obtain food and the way they defend themselves when facing dangers differ in order to sustain survival. It is the responsibility of adults to obtain food and provide it for their young, as is prevalent in most living organisms, and it is been known the population grows at a constant rate until it reaches a fixed age distribution, where the proportion of the population in each age group remains approximately the same annually. Often, it does not make sense for a single species to be modeled as a single community but is better treated as an integrated society with all its age groups in which the population's individuals are classified or staged.[70], [71],[72],[73],[74],[75]. Zhang et al. [70] investigated a stage-structure P-P periodic Sys. with a general FR and two predators. The model is defined by:

$$\frac{dx_1}{dt} = a_1(t)x_2(t) - a_2(t)x_1(t) - d_1(t){x_1}^2(t) - \frac{s_1(t){x_1}^{r_1}(t)}{b_1(t) + h_1(t){x_1}^{r_2}(t)}y_1(t)$$

$$\frac{dx_2}{dt} = c_1(t)x_1(t) - c_2(t){x_2}^2(t) - \frac{s_2(t){x_2}^{r_1}(t)}{b_2(t) + h_2(t){x_2}^{r_2}(t)}y_2(t)$$

$$\frac{dy_1}{dt} = \left(\frac{e_1(t){x_1}^{r_1}(t)}{b_1(t) + h_1(t){x_1}^{r_2}(t)} - m_1(t) - p_1(t)y_1(t)\right)y_1(t) \qquad (2.8)$$

$$\frac{dy_2}{dt} = \left(\frac{e_2(t){x_2}^{r_1}(t)}{b_2(t) + h_2(t){x_2}^{r_2}(t)} - m_2(t) - p_2(t)y_2(t)\right)y_2(t),$$

where prey species consist of immature prey (x_1) and mature prey (x_2) and the two predators (y_1) and (y_2) consume the immature prey (x_1) and mature prey (x_2) respectively. Kaushik and Banerjee [76] studied a P-P model involving counterattacking behavior and age stages in the predator population. The model is defined by:

$$\acute{x} = r\,x - (a_1 y_J + a_2 y_A)x$$

$$\acute{y_J} = \alpha\, a_2 y_A x - a_3 y_J x - \mu y_J - \beta y_J \tag{2.9}$$

$$\acute{y_A} = \beta y_J - m y_A - d {y_A}^2 ,$$

such that the juvenile predators (y_J)can be attacked by the prey (x), due to their small size , and since they are still in the predation learning stage, they may be attacked by the prey as a means of self-defense and finally get killed. While adult predators (y_A) may professionally overcome prey techniques intended to defend themselves from predation. Mortoja et al. [73] structured and analyzed a four-dimensions P-P model based on stage structure. They take into account stage structure in both the predator and prey populations. The model is defined by:

$$\acute{x} = r\,y - (a_1 + d_1)x - \frac{s_1\,x}{b_1 + x} w$$

$$\acute{y} = a_1 x - c_1\,y^2 - \frac{s_2\,y}{b_2 + b_3 y^2} w - d_2 y$$

$$\acute{z} = \frac{e_1 s_1\,x}{b_1 + x} w + \frac{e_2 s_2\,y}{b_2 + b_3 y^2} w - (m - d_2)z \tag{2.10}$$

$$\acute{w} = mz - d_2 w\,.$$

As the predator species is split into two age stages: immature predator (z) and mature predator (w), and the prey species are likewise split into age stages: immature prey (x) and mature prey (y) .Only adult predator is considered to be capable of predation, and they feed on both immature and mature prey. They also assumed that the immature prey growth rate depends entirely on the mature prey, in other words, that the immature prey cannot reproduce.

Furthermore, some theoretical and experimental work has investigated the impacts of prey's fear of predators and drawn different conclusions. [4],[6], [7], [77],[78],[79],[80]. Mukherjee [81] investigated the mechanism of fear in the P-P Sys. that involves a rival for the prey by considering an environment in which prey and predator coexist, the prey has a

rival, and the predator (y) preys on one of the two rival species (x_1) and (x_2). The model is defined by:

$$\dot{x_1} = x_1\left(\frac{r_1}{1+ky}\right) - a_{11}x_1 - a_{12}x_2 - \beta y,$$

$$\dot{x_2} = x_{21}(r_2 - a_{21}x_1 - a_{22}x_2), \tag{2.11}$$

$$\dot{y} = y(c\beta x_1 - d),$$

The researcher emphasized that fear of predators may influence prey behavior and decrease reproduction, according to field research on vertebrates. Huang.[6] investigated the fear impact and the Allee impact in a two-dimensional P-P Sys. with a refuge in the prey. The model is defined by:

$$\frac{du}{dt} = ru\left(1-\frac{u}{k}\right)(u-\emptyset_0)\frac{1}{1+f_0 v} - \propto (1-\eta)uv,$$

$$\frac{dv}{dt} = a \propto (1-\eta)uv - \mu_0 v, \tag{2.12}$$

indicating that the fear has no effect on the density of the prey (u) but can result in a reduction in predator density (v) at coexistence equilibrium. Zhu et al.[77] analyzed the fear impact on the Lotka–Volterra P-P model when the predator has another supplementary food. The model is defined by:

$$\dot{u} = \frac{r_0 u}{1+k} - du - au^2 - puv,$$

$$\dot{v} = cpuv + mv - d_1 v^2, \tag{2.13}$$

According to this study, "the fear factor is one of the most significant factors that contribute to the extinction of prey species (u). It should be noted that such a discovery is significantly different from the previously known findings".

Moreover, in the P-P Sys., predators can affect the prey population in two different ways, either by direct killing or by instilling fear of predation, which eventually forces the prey to display different types of anti-predator reactions against the threat of predation, the researchers discovered that prey's fear of predators produces defenses against the predator.

Preisser and Bolnick [82] studied the multiple faces of fear through a comparative study between the effects and pathways of non-consumptive predator impacts on prey species, they present a basic Lotka-Volterra P-P model to provide a guiding tool for making a distinction between the demographic influences of consuming (consumptive influences) and anti-predator defenses (non-consumptive influences), as well as the various means whereby anti-predator defenses can reduce the growth rate of the prey population, Noting that in addition to killing and direct consuming, predators can decrease prey population by compelling prey to adopt costly defense methods.

Prey grouping is widely known to be an anti-predator behavior in several prey species, Some research has also shown how group size response to counter predation risks, Despite the advantage of prey grouping in the advanced detection of predators, but sometimes the large groups may be more easily detected by the predators. It is generally recognized that enhancing protection against predators is costly and causes a reduction in forage intake of prey and that this choice will be preferred by individuals who are looking for an optimal balance between the benefits and costs of risk reduction. As a result, many preys (such as ungulates) use ” many eyes” effects by gathering into big groups to decrease the individual defensive costs and increase survivability by "safety in numbers" effects. For example, Ungulates have a variety of anti-predator responses, but they may be divided into two categories: reducing the chances of being detected by predators and improving the chance of surviving after being detected by a predator by raising their group size or group defense. [83]. Sasmal and Takeuchi [84] used FR of the Monod-Haldane type to investigate the dynamics of a P-P model that includes: (i) The effect of fear in reducing the growth rate of prey due to the predator's presence; and (ii) prey group defense against predation. The model is defined by:

$$\dot{x} = \frac{rx}{1+k\alpha y} - d_1 x - d_2 x^2 - \frac{\beta xy}{a+b\alpha x+x^2},$$

$$\dot{y} = \frac{c\beta xy}{a + b\alpha x + x^2} - my \tag{2.14}$$

Furthermore, they have used the sensitivity of the predator-taxis to correlate these factors, when prey (x) spends the same amount of energy or time for defense and foraging. If the prey expends more energy or time for defense, this effort may lead to reduced reproduction

The results of the mathematical and numerical study showed that, for predator survival, a certain threshold density must be greater than the sensitivity of the predator-taxis . They have performed a sensitivity analysis of the Sols. of the model w.r.t. the three significant parameters model, namely, the sensitivity of predator-taxi, fear level, and predator tolerance limit, noticed that the dynamics of the model are greatly affected by the tolerance limit perturbation of the predator (y). Initially, the prey is positively affected due to the sensitivity of the predator-taxi as the killing rate decreases; but in the long run, it affects both Sols. negatively because it decreases the prey growth rate, affecting the fitness of both species.

Although most mathematical models investigated the dynamics by observing the anti-predator behavior only since it has both benefits and costs. Panday [85] studied the "dynamics of a stage-structured P-P model: cost and advantage of group defense that is fear-induced" by formulating a P-P model in which the prey population is divided into two age stages: immature (J) and mature (A), and a single predator (P). The model is defined by:

$$\dot{J} = \frac{rA}{1+f\alpha P} - d_1 J - mJ - s_1 J^2 + \frac{a_1 JP}{1+b_1 J}$$

$$\dot{A} = mJ - d_2 A - s_2 A^2 + \frac{a_2 AP}{1 + b_2 A + \alpha b_3 A^2} \tag{2.15}$$

$$\dot{P} = \frac{ea_1 JP}{1 + b_1 J} + \frac{ea_2 AP}{1 + b_2 A + \propto b_3 A^2} - d_3 P$$

Assuming that mature prey only use group defense once they are threatened by predators. Although group defense has a positive role in reducing the predation of mature prey, at the same time it has a negative effect as it reduces their reproductive ability.

Finally, Traditional P-P interaction models are frequently centered on an organism's principal purpose within a food chain (e.g., predator and prey). Now, what happens if the predators are unable to reach and consume the prey? .In this case, predators may stray from their typical diet and show an interest in alternative prey or/and more food to counteract the lack of resources and ensure their survival. An alternative diet is emerging, on the surface, to help keep predators in the Sys. When the density of preferred

prey decreases, predators usually move to alternative foods. AF represents a key component of most predators' diets (e.g., coccinellid). despite receiving less attention in the scientific literature than basal prey, these foods have a significant impact on the life cycles of several predator species, P-P Syss that include AF indicate that prey will enhance the predator density and have an impact on the intended prey. The alternative food source might be a different prey species residing in the same or another environment, or it might be a supply of food or non-reproducing prey. Apparent competition is a term given to the relationship between prey and AF when the density of equilibrium prey is reduced due to alternative food. [86]. In P-P models, The provision of appropriate supplementary food (food that is not prey) can have a great influence on the Sys. dynamics. Many scientists have been interested in the repercussions of giving AF to the predator, as well as the related impacts on predator and prey dynamics and its value in biocontrol (e.g pest management and species conservation). Several researchers, experimentalists, and biologists have presented a large number of studies on the effects of providing the predator with AF and its impact on the dynamic Syss in recent years. When the favored prey is scarce, nearly all predators will seek to switch to alternative prey or they may even turn to a herbivorous diet or scavenging if possible. The lynx–caribou–hare interaction in Newfoundland is a good example of a P-P model with alternative food.[87]. Some researchers investigated the qualitative characteristics of the P-P model when AF is provided to the predator and this research provides insight into potential management options that include manipulating the quality and level of supplemental food supply to predators, in favor of biocontrol. [88]. This study came out with results consistent with the conclusions presented in a recent study introduced by Wade et al. [89] that dealt with the impact of "artificial food sprays on conservational biological control", which emphasizes that the efficacy of biological control is highly dependent on the quality and quantity of food sprays. Bai and Li [90]studied the dynamics of a three -dimension predator-prey system where the population divided into two age stages: immat ure prey(x_1) and mature prey (x_2) with refuge (m) for the mature prey and AF (A^*) for the predator (y). The model is defined by:

$$\frac{dx_1}{dt} = \alpha\, x_2(t) - a\, x_1(t) - bx_1(t)$$

$$\frac{dx_2}{dt} = bx_1(t) - cx_2(t) - d\, {x_2}^2(t) - \frac{e_1\, s_1\,(1-m)x_2(t)}{\beta + e_1\, h_1 x_2(t) + e_2\, h_2 A^*}\, y(t) \qquad (2.16)$$

$$\frac{dy}{dt} = \frac{s_2\,[e_1\,(1-m)x_2(t-\tau) + e_2\, A^*]}{\beta + e_1\, h_1 x_2(t-\tau) + e_2\, h_2 A^*}\, y(t-\tau) - \, py(t),$$

They see that, for a CEP to be stable, the refuge must be constrained by a value that relies on the quality and quantity of AF. The findings in this research give a valuable foundation for understanding the effects of AF and refuge. As a result, AF and refuge can be used as population controls when studying predator- preymodels. Batabyal and Jana.[91] analyzed the importance of AF on the dynamics of a spatiotemporal Sys. caused by mutually interacting predators in a herd of prey. they presented a P-P model in both temporal-spatial and temporal contexts, using Beddington-DeAngelis FR in which the predator (P) is provided with AF (A)in addition to its primary food in order to enhance the predator's absolute growth and control the prey population (N), which in turn forms a herd that is square-shaped as a form of collective defense. In this situation, alternative food, in addition to primary food, plays an important role in the predator's survival. As a result, alternative food is included in the FR. Therefore, the model becomes:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right) - \frac{\alpha P\sqrt{N}}{\beta + \gamma\mu A + \varepsilon\sqrt{x}}$$

$$\frac{dP}{dt} = \frac{\delta\left(\sqrt{N} + \mu A\right)P}{\beta + \gamma\mu A + \varepsilon\sqrt{x}} - dP, \qquad (2.17)$$

They came out with important conclusions that the preys form a herd to defend themselves from predators, which is the most common type of group defense. So in the absence of herd behavior then predators may easily hunt them. This study deals with herd-shaped prey, thus the benefit of AF mixed with the primary food is very great for feeding predators to hunt the prey in the herds. the study has equated pests (that are extremely damaging to crops) to prey herds. Chhetri et al. [92] presented a quantitative and qualitative study into the impact of gestation delay and the effect of AF in P-P Syss using

Holling type III FR .This mathematical study is a simulation of reality as some mammals such as seals, bears, and pine martins, show gestation delay, or a delay in embryo implantation when the environment is not suitable to bear and raise the offspring. This perplexing reproductive approach has the potential to increase survival chances. The authors perform a mathematical simulation for such a scenario to show the gestation delay upon prey consumption by using Holling type III FR. The predator (P) is provided with numerous quantities (β) and qualities (α) of AF in order to preserve the Sys.. they investigated the Sys.'s stability at the EPs where the bifurcation parameter is a gestation delay (time delay). The model is defined by:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right) - \frac{s(1-c)N^2P}{a^2 + (1-c)N^2 + \alpha\beta^2}$$

$$\frac{dP}{dt} = \frac{e((1-c)N^2(t-\tau) + \beta^2)P(t-\tau)}{a^2 + (1-c)N^2(t-\tau) + \alpha\beta^2} - dP \qquad (2.18)$$

It is noted that population size fluctuations can be decreased by providing a large amount of AF or by enhancing the habitat complexity strength, In other words, providing large amounts of AF serves in the interest of the prey and predator in a stable coexisting and will conserve both of them (Sys.) from extinction. According to the findings, the Sys.'s stability varies w.r.t delay. The bifurcation study revealed that the bifurcation value (critical value) is affected by both the quantity and quality of the AF Furthermore, it was discovered that with larger values of habitat complexity, population fluctuations may be eliminated.

A real ecological model named Sys. (3.1) is constructed as a result of these studies is based on the most recent P-P model developments. It includes several significant biological factors such as fear, stage structure in the predator, AF for the predator, anti-predation, and developed form of Holling's disk Eq.

3. THE MODELING OF THE ROLE OF FEAR AND ANTIPREDATOR PROPERTIES ON A STAGE STRUCTURE PREY-PREDATOR SYSTEM

3.1 INTRODUCTION:

Mathematical modeling of P-P Sys. is a critical area that piques mathematicians' interest in simulating reality by converting real issues to mathematical Eqs., solving mathematical Eqs., then making conclusions in terms of the real world [93]. The studying of real-world issues in ecology, epidemiology, and eco-epidemiology are important because they affect human lives. Clearly, these issues have a direct impact on human survival and natural resources. To gain a deeper understanding of these real-world problems, a wide range of mathematical models in ecology, epidemiology, and later eco-epidemiology has been proposed and analyzed over the last years. These models included a variety of factors (such as stage structure, fear function, different types of FR, anti-predator, refuge, disease, switching, and delay). [8],[94],[4],[5]. The majority of proposed models are stated as ODEs Syss. Additionally, their dynamics are investigated using dynamical Sys theory and bifurcation analysis.

The most critical strategies in biological environmental control that have a significant impact on the dynamical behavior of P-P models are the FR selection, the impact of stage structure, fear, the quality of AF,..etc.

The stage and age-based dynamics of the P-P Sys. have been extensively studied and explored using mathematical models [95]. Many mammals' evolution and dynamics are heavily influenced by their age. The rate of growth, survival and reproduction is almost entirely determined by developmental stage or age, and it has been observed that many species' life histories are divided into at least two stages, juvenile and adult, and in the first stage, species rarely interact with other species or reproducing, instead it is raised by their adult parents[95] [96],[85].

At present, mathematical modeling has been developed by many pieces of research based on the study of the effect of fear, and those researches found that the prey's fear of predators generates anti-predator defenses and decreases prey reproduction rates. Several

studies propose the influence of the predation rule on the prey's direct intake of predators; nevertheless , there is a direct confirmation that the fear from a predator is just as essential as direct consumption [97]. Organisms showed different types of anti-predator responses against the threat of predation, and foraging patterns, habitat use changes, alertness, and physiological changes are only a few examples [4],[97],[98].

Moreover, providing AF to the predator has been extensively researched, in the P-P interaction dynamics because of its extensive applications in biocontrol and species preservation. Research and studies in this area have come out with different results, as some studies have shown that providing AF could raise the target predator and reduce the attack rate of prey biomass . Other studies have demonstrated that more food resources can make predators more powerful or raise predator reproductive rates, leading to the extinction of prey populations [14] . On the other hand, Some researchers have observed that the quality and quantity of supplemental food are of great importance in the dynamical Sys. They observed that high-quality food led to an increase in the predation rate, while low-quality food may help to keep the target prey alive [99],[100].

Consequently, in this chapter, a three-species food chain across a stage-structure P-P Sys. with fear effect and anti-predator behavior is structured and investigated by using a 3D autonomous nonlinear Sys. The consumption of the prey is described by a Modified Holling's disk Eq. which involves the additional source of food for the predator. The impact of fear, the anti-predator behavior, and the AF for the predator on the dynamical behavior of the Sys. is analyzed theoretically.

3.2 MATHEMATICAL MODEL FORMULATION

"A P-P model consisting of prey and two ages of predators (juvenile and adult) has been proposed and analyzed in this section. We supposed that $X(T)$ represents the density of the prey population at time T, while the variables $Y(T)$, and $Z(T)$ represent the density of the two compartments of the predator's population juvenile predator, and adult predator at time T respectively.

Now, the following hypotheses are considered in order to structure and describe the dynamics of such a Sys.:

i. The traditional Holling's disk Eq. is modified to explain the prey consumption to involve the constant biomass of AF.

ii. Prey intrinsic growth rate r>0 , is constructed based on the fear function when the prey community grows exponentially when the predator is not available.

iii. Furthermore, it has also been hypothesized that the prey is strong enough to show anti-predator behavior and has the ability to kill the predator as a natural response to resistance to predation.

Under the aforementioned assumptions, the following differential equations set may be used to simulate such real-world Sy's. dynamics.". [26]

$$\begin{aligned} \frac{dX}{dT} &= \frac{rX}{1+nZ} - bX^2 - \frac{aXZ}{c+uvA+X}, \\ \frac{dY}{dT} &= \frac{ea(X+vA)Z}{c+uvA+X} - (p+d_1)Y, \\ \frac{dZ}{dT} &= pY - qXZ - d_2Z, \end{aligned} \tag{3.1}$$

With $X(0) \geq 0, Y(0) \geq 0$, and $Z(0) \geq 0$ (nonnegative initial conditions)

And Table 3.1 shows the biological definitions of the Sys. (3.1) parameters.

Table 3.1: Descriptions of model parameters

Parameter	Definition
r	The intrinsic growth rate of the prey.
b	Intraspecific competition.
n	The intensity of the level of fear.
a	The highest predation rate of prey by an adult predator.
c	The half-saturation value of the predator
u	The ratio of the handling time of the predator per unit quantity of the additional food and prey respectively.
v	The ratio of the constants is dependent on the rate of movement of the predator in searching for additional food and prey respectively.
e	The conversion rate.
p	The maturation rate of the predator.
q	The reduction rate of the predator by the anti-predator property of the prey.
d_1, d_2	The death rates of the juvenile and adult predators respectively.
A	The biomass of additional food.

And the block diagram of this model is shown in Fig. (3.1).

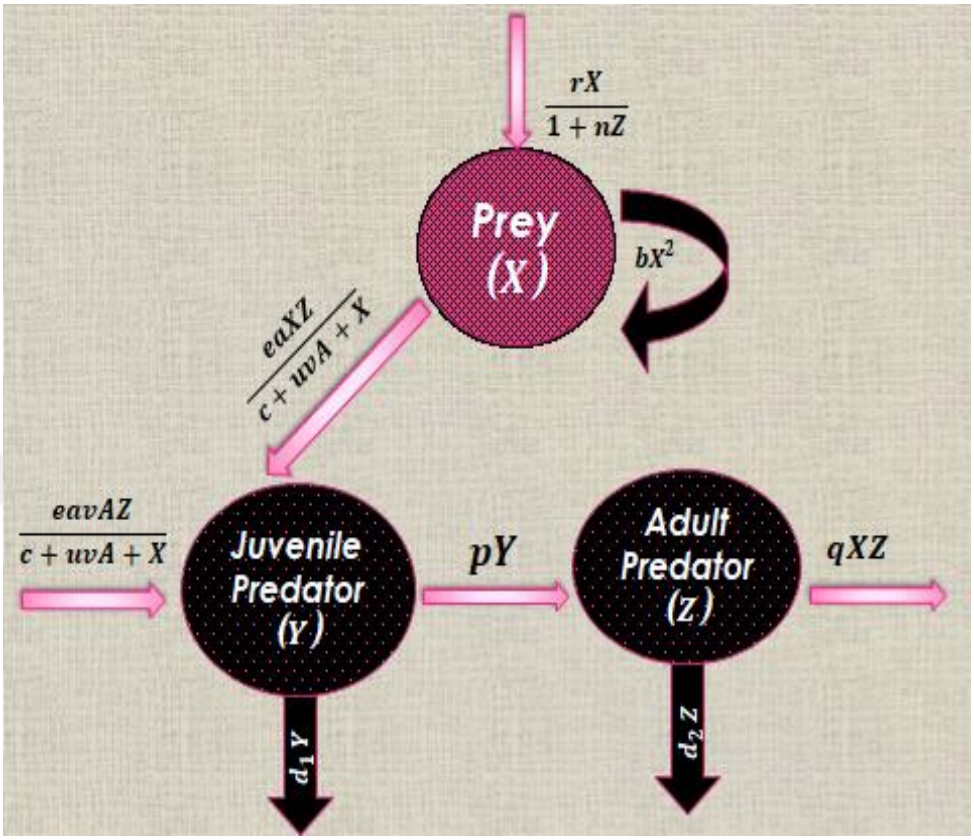

Figure 3.1: The block diagram of Sys. (3.1).

To simplify the preceding Sys. of Eqs., and study it more generally, we will eliminate all units from it and replace them with the determined constants and scaling variables set:

$$t = rT, \qquad x = \frac{b}{r}X, \qquad y = \frac{b}{r}Y, \qquad z = \frac{ab}{r^2}Z,$$

$$w_1 = \frac{n\,r^2}{ab}, \qquad w_2 = \frac{cb}{r}, \qquad w_3 = \frac{buvA}{r}, \qquad w_4 = \frac{bvA}{r},$$

$$w_5 = \frac{p + d_1}{r}, \qquad w_6 = \frac{ap}{r^2}, \qquad w_7 = \frac{q}{b}, \qquad w_8 = \frac{d_2}{r}$$

Thus the following dimensionless Sys. is obtained:

$$\left.\begin{aligned}\frac{dx}{dt} &= x\left[\frac{1}{1+w_1 z} - x - \frac{z}{w_2+w_3+x}\right] = xf_1(x,y,z)\\ \frac{dy}{dt} &= \frac{e(w_4+x)z}{w_2+w_3+x} - w_5 y = f_2(x,y,z),\\ \frac{dz}{dt} &= w_6 y - w_7 xz - w_8 z = f_3(x,y,z).\end{aligned}\right\} \quad (3.2)$$

Obviously, the foregoing selection of dimensional constants and variables results in a reduction of the parameters' No. from 14 in Sys. (3.1) to 9 in Sys. (3.2). On the following positive 3D octant, interaction functions of the Sys (and their partial derivatives) are continuous.

$$R_+^3 = \{(x,y,z) \in R^3 : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0\}.$$

Hence, these functions are Lipschitzian on R_+^3 , and then the Sys. (3.2) Sol. exists uniquely. Furthermore, the following theorem establishes the uniformly bounded to the Sys. (3.2) Sol.s, Which ensures that the Sol. converges to an attractor.

Theorem 3.1: All the Sys. (3.2) Sol.s, which are started in R_+^3, are uniformly bounded in the region $\omega = \left\{(x,y,z) \in R^3 : 0 < x \leq 1, x+y+z < \frac{2}{\rho}\right\}$, under the following sufficient condition.

$$w_8 - \frac{ew_4}{w_2+w_3} > 0, \quad (3.1)$$

where $\rho = \min\left\{1, (w_5 - w_6), \left(w_8 - \frac{ew_4}{w_2+w_3}\right)\right\}$.

Proof: By the first Eq. of the Sys. (3.2), it is noticed that:

$$\frac{dx}{dt} \leq x\left[\frac{1}{1+w_1 z} - x\right] \leq x(1-x).$$

Therefore, checking that $x\,(t) \leq 1\; for\; t \to \infty$. is very easy.

Now, let $Ḥ(t) = x(t) + y(t) + z(t)$,

then by using the biological facts that $0 < e < 1,\ w_5 - w_6 > 0$, with condition (3.1) it is obtained that:

$$\frac{dḤ}{dt} \leq 2x - x - (w_5 - w_6)y - \left(w_8 - \frac{ew_4}{w_2+w_3}\right)z \leq 2 - \rho Ḥ.$$

Now, after solving the above differential inequality is for the initial value $Ḥ\,(0) = Ḥ_0$, it is found that:

$$Ḥ(t) \leq \frac{2}{\rho} + \left(Ḥ_0 - \frac{2}{\rho}\right)\, e^{-\rho t}\ ,\ \text{and then}\ \lim_{t\to\infty} Ḥ(t) \leq \frac{2}{\rho}.$$

Hence, all the Sys. (3.2) Sols are uniformly bounded in the region ω.

3.3 EXISTENCE OF EQUILIBRIUM POINTS

This section discusses the presence of all possible EPs for the Sys. (3.2). It is worth noting that Sys. (3.2) has exactly four EPs, they are detailed below.

i. The VEP $E_0 = (\,0, 0, 0\,)$, exists always.

ii. The PDFEP that is denoted by $E_1 = (1, 0, 0)$, exists always.

iii. The PYFEP that is denoted by $E_2 = (0, \bar{y}, \bar{z})$, exists (uniquely) in the $yz{-}plane$ if there is a positive Sol to the following set of Eqs. :

$$\frac{ew_4 z}{w_2 + w_3} - w_5 y = 0 \qquad (3.2\text{ a})$$

$$w_6 y - w_8 z = 0 \qquad (3.2\text{ b})$$

By substituting Eq. (3.2 b) in Eq. (3.2 a), we obtain that:

$$\bar{y} = \frac{w_8}{w_6}\,\bar{z}, \qquad \text{and} \quad \bar{z} > 0,$$

provided that the following necessary and sufficient condition holds:

$$\frac{e\, w_4}{w_2 + w_3} = \frac{w_5\, w_8}{w_6}. \tag{3.2 c}$$

i. Finally, the CEP of the Sys. (3.2) referred to as $E_3 = (x^*, y^*, z^*)$, exists (uniquely) in the interior R_+^3 if there is a positive Sol. to the following set of Eqs. :

$$x\left[\frac{1}{1 + w_1 z} - x - \frac{z}{w_2 + w_3 + x}\right] = 0 \tag{3.3 a}$$

$$\frac{e(w_4 + x)z}{w_2 + w_3 + x} - w_5 y = 0 \tag{3.3 b}$$

$$w_6 y - w_7 xz - w_8 z = 0 \tag{3.3 c}$$

From Eq. (3.3 b) we obtain that:

$$y^* = \frac{e(w_4 + x^*)\, z^*}{w_5(w_2 + w_3 + x^*)} \tag{3.3 d}$$

Substituting Eq. (3.3 d) in Eq. (3.3 c), we obtain that:

$$x^* = \frac{-\delta_1}{2\delta_2} + \frac{\sqrt{{\delta_1}^2 - 4\delta_0\delta_2}}{2\delta_2}, \tag{3.3 e}$$

where: $\delta_2 = w_5 w_7 > 0,$

$$\delta_1 = w_5 w_8 + w_5 w_7 (w_2 + w_3) - \mathrm{e}\, w_6,$$

$$\delta_0 = w_5 w_8 (w_2 + w_3) - \mathrm{e} w_4 w_6.$$

And from Eq. (3.3 a), we obtain that:

$$z^* = \frac{-\mu_1}{2\mu_2} + \frac{\sqrt{{\mu_1}^2 - 4\mu_0\mu_2}}{2\mu_2} \tag{3.3f}$$

where:

$$\mu_2 = w_1 > 0, \mu_1 = 1 + w_1 x^*(w_2 + w_3 + x^*), \quad \mu_0 = (w_2 + w_3 + x^*)(x^* - 1).$$

Clearly, Eq. $(3.3\,e)$ has a unique positive root, denoted by x^* if $\delta_0 < 0$, while Eq. $(3.3f)$ has a unique positive root, denoted by z^* if $\mu_0 < 0$.

Therefore, the CEP $E_3 = (x^*, y^*, z^*)$, exists (uniquely) if the following sufficient conditions satisfy:

$$\frac{w_5\ w_8\,(w_2+w_3\,)}{\mathrm{e}\,w_6} < w_4 \tag{3.3 g}$$

$$x^*\ < 1 \tag{3.3 h}$$

3.4 LOCAL STABILITY ANALYSIS

This section discusses the conditions that guarantee the LAS for all of the aforementioned EPs of Sys. (3.2); those conditions are found utilizing the Linearization method. The simple calculation indicates that general JM for Sys. (3.2) is as follows:

$$J = \begin{bmatrix} x\left(\frac{z}{Q^2}-1\right)+f_1 & 0 & -x\left(\frac{w_1}{L^2}+\frac{1}{Q}\right) \\ \frac{e(w_2+w_3-w_4)z}{Q^2} & -w_5 & \frac{e(x+w_4)}{Q} \\ -w_7 z & w_6 & -(w_7 x+w_8) \end{bmatrix}, \tag{3.4}$$

where: $Q = w_2 + w_3 + x,\ \ L = 1 + w_1 z$ and $f_1 = \frac{1}{1+w_1 z} - x - \frac{z}{w_2+w_3+x}$.

a. The local stability analysis at E_0

The JM of the Sys. (3.2) at the vanishing EP, $E_0 = (0, 0, 0)$ can be written as:

$$J_0 = J(E_0) = \begin{bmatrix} 1 & 0 & 0) \\ 0 & -w_5 & \frac{e\,w_4}{w_2+w_3} \\ 0 & w_6 & -w_8 \end{bmatrix}, \tag{3.5 a}$$

Then the characteristic Eq. of J_0 is given by:

$$\lambda^2 - T_0\lambda + D_0 = 0, \tag{3.5 b}$$

where $T_0 = -(w_5 + w_8) < 0$, and $D_0 = w_5 w_8 - \frac{e w_4 w_6}{w_2+w_3}$.

Clearly, J_0 has three EVUs given by:

$$\lambda_{0x} = 1 > 0, \quad \lambda_{0y} = \frac{T_0}{2} - \frac{\sqrt{{T_0}^2 - 4D_0}}{2} \text{ and } \quad \lambda_{0z} = \frac{T_0}{2} + \frac{\sqrt{{T_0}^2 - 4D_0}}{2}, \tag{3.5 c}$$

Accordingly, E_0 is an unstable point.

b. The local stability analysis at E_1

The JM of the Sys. (3.2) at the VEP, $E_1 = (1, 0, 0)$ can be written as:

$$J_1 = J(E_1) = \begin{bmatrix} -1 & 0 & -(w_1 + \frac{1}{\hat{Q}}) \\ 0 & -w_5 & \frac{e(1+w_4)}{\hat{Q}} \\ 0 & w_6 & -(w_7 + w_8) \end{bmatrix} \tag{3.6 a}$$

where $\hat{Q} = w_2 + w_3 + 1$.

Then the characteristic Eq. of J_1 is given by:

$$(-1 - \lambda)(\lambda^2 - T_1\lambda + D_1) = 0, \tag{3.6 b}$$

where $T_1 = -(w_5 + w_7 + w_8) < 0$, and $D_1 = w_5(w_7 + w_8) - \frac{e(1+w_4)\, w_6}{\hat{Q}}$.

Therefore the EVUs are given by

$$\lambda_{1x} = -1 < 0, \ \lambda_{1y} = \frac{T_1}{2} + \frac{\sqrt{{T_1}^2 - 4D_1}}{2}, \text{ and } \ \lambda_{1z} = \frac{T_1}{2} - \frac{\sqrt{{T_1}^2 - 4D_1}}{2}, \tag{3.6 c}$$

Then the PDFEP, E_1 is LAS under the following condition.

$$\frac{w_5(w_7 + w_8)}{w_6} > \frac{e(1 + w_4)}{\hat{Q}}. \tag{3.6 d}$$

c. The local stability analysis at E_2

The JM of the Sys. (3.2) at the PYFEP, $E_2 = (0, \bar{y}, \bar{z})$ is given by:

$$J_2 = J(E_2) = \begin{bmatrix} \frac{1}{\bar{\bar{L}}} - \frac{\bar{z}}{w_2+w_3} & 0 & 0 \\ \frac{e(w_2+w_3-w_4)\bar{z}}{(w_2+w_3)^2} & -w_5 & \frac{ew_4}{w_2+w_3} \\ -w_7\,\bar{z} & w_6 & -w_8 \end{bmatrix}, \qquad (3.7a)$$

where $\bar{L} = 1 + w_1\bar{z}$

Then the characteristic Eq. of J_1 is given by:

$$\left(\frac{1}{\bar{\bar{L}}} - \frac{\bar{z}}{w_2+w_3} - \lambda\right)(\lambda^2 - T_2\lambda + D_2) = 0, \qquad (3.7\ b)$$

where $T_2 = -(w_5 + w_8) < 0$, and $D_2 = w_5w_8 - \frac{ew_4\,w_6}{w_2+w_3}$.

Therefore, according to the existing condition (3.2 c), the EVUs are given by:

$$\lambda_{2x} = \frac{1}{\bar{L}} - \frac{\bar{z}}{w_2+w_3}, \qquad \lambda_{2y} = -(w_5 + w_8), and \qquad \lambda_{2z} = 0 \qquad (3.7\ c)$$

Accordingly, Because we have gotten a zero EVU, so the PYFEP is a non-hyperbolic point. Therefore, the Lyapunov method is used to study this point stability.

d. The local stability analysis at E_3

Finally, The JM of the Sys. (3.2) at the CEP, $E_3 = (x^*, y^*, z^*)$ can be written as:

$$J_3 = [\,a_{ij}\,]_{3\times3}, \qquad (3.8\ a)$$

Where:

$$a_{11} = -x^*\left[1 - \frac{z^*}{(Q^*)^2}\right] \quad , \quad a_{12} = 0, \quad a_{13} = -x^*\left[\frac{w_1}{(L^*)^2} + \frac{1}{Q^*}\right] < 0,$$

$$a_{21} = \frac{e(w_2+w_3-w_4)z^*}{(Q^*)^2}, \quad a_{22} = -w_5 < 0, \quad a_{23} = \frac{e(x^*+w_4)}{Q^*} > 0,$$

$$a_{31} = -w_7z^* < 0, \quad a_{32} = w_6 > 0, \quad a_{33} = -(w_7x^* + w_8) < 0.$$

with $Q^* = w_2 + w_3 + x^*$, and $L^* = 1 + w_1z^*$.

Then the characteristic Eq. of J_3 is given by:

$$\lambda^3 + A_1\,\lambda^2 + A_2\,\lambda + A_3 = 0, \tag{3.8 b}$$

Where:

$$A_1 = -(a_{11} + a_{22} + a_{33}),$$

$$A_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{31}a_{13},$$

$$A_3 = a_{11}[a_{23}a_{32} - a_{22}a_{33}] + a_{13}\,[a_{31}\,a_{22} - a_{21}a_{32}],$$

with, $\Delta = A_1A_2 - A_3$

$$= -(a_{22} + a_{33})[a_{22}a_{33} - a_{23}a_{32}] - a_{11}a_{22}(a_{11} + a_{22} + a_{33})$$
$$-(a_{11} + a_{33})[a_{11}a_{33} - a_{13}a_{31}] + [a_{13}a_{21}\,a_{32} - a_{11}a_{22}a_{33}].$$

As a result, the theorem below establishes that the CEP E_3 of the Sys. (3.2) is a LAS.

Theorem 3.2: The CEP is LAS in the *int.* R_+^3 under the following sufficient conditions:

$$z^* < Q^{*2}, \tag{3.9 a}$$

$$w_2 + w_3 > w_4, \tag{3.9 b}$$

$$\frac{ew_6(x^* + w_4)}{Q^*(w_7x^* + w_8)} < w_5 < \frac{ew_6(w_2 + w_3 - w_4)}{w_7Q^{*2}}, \tag{3.9 c}$$

$$\max\{S_1, S_2\} < 1, \tag{3.9 d}$$

Where:

$$S_1 = z^*\left[\frac{w_7Q^*L^{*2} + (w_7x^* + w_8)L^{*2} + w_1w_7Q^{*2}}{L^{*2}Q^{*2}(w_7x^* + w_8)}\right],$$

$$and \quad S_2 = \frac{(w_1Q^* + (L^*)^2)ew_6(w_2 + w_3 - w_4)z^*}{w_5\,Q^*(L^*)^2[(Q^*)^2 - z^*](w_7x^* + w_8)}.$$

Proof: As per the requirements of Ruoth–Hurwitz criterion, Eq. (3.8 b) possesses three EVUs that possess real negative parts if:

$A_1 > 0$, $A_3 > 0$, and $\Delta = A_1A_2 - A_3 > 0$,

Direct computation shows that the given sufficient conditions guarantee the fulfillment of these requirements.

Therefore all J_3 EVUs possess real negative parts, so the CEP. is LAS. and the proof is now complete.

3.5 GLOBAL STABILITY ANALYSIS

The Lyapunov method is used to specify the attraction's basin for the EPs., of the Sys. (3.2), which are non-hyperbolic or LAS, in this section, as demonstrated in the theorems below. Recall that, the basin of attraction of an EP represents a GAS manifold of it.

Theorem 3.3: Suppose that the PDFEP, $E_1 = (1,0,0)$ of the Sys. (3.2) is LAS in the R_+^3, then it is a GAS in the R_+^3 provided that:

$$e < \min\left\{\frac{w_8(w_2+w_3)}{1+w_4}, \frac{w_7}{w_1}\right\} \tag{3.10}$$

Proof: Consider the following LF:

$$V_1(x, y, z) = a_1[x - 1 - \ln x] + a_2 y + a_3 z,$$

where $a_i; i = 1,2,3$ are positive constants to be determined. Clearly, $V_1: R_+^3 \to R$ is a C^1 positive definite function so that $V_1(1,\ 0,\ 0) = 0$ and $V_1(x, y, z) > 0$ for all $\{(x,\ y,\ z) \in R_+^3: x > 0, y \geq 0, z \geq 0\ with\ (x, y,\ z) \neq (1,\ 0,\ 0)\}$.

$$\frac{dV_1}{dt} = a_1\frac{(x-1)}{x}\frac{dx}{dt} + a_2\frac{dy}{dt} + a_3\frac{dz}{dt}$$

Now by taking the time derivative of V_1 and substituting the values of the $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ from the Sys. (3.2) in the previous Eq. and then performing some algebraic manipulation, gives that:

$$\frac{dV_1}{dt} \leq -a_1(x-1)^2 - \left[a_3 w_8 - \frac{a_1 + e a_2 w_4}{w_2 + w_3}\right] z - (a_3 w_7 - a_1 w_1)xz$$
$$-(a_1 - e a_2)\frac{xz}{w_2 + w_3 + x} - (a_2 w_5 - a_3 w_6)y.$$

So by choosing $a_1 = 1$ and $a_2 = a_3 = \frac{1}{e}$, which are positive constant, it is obtained that

$$\frac{dV_1}{dt} \leq -(x-1)^2 - \left[\frac{w_8}{e} - \frac{1+w_4}{w_2+w_3}\right] z - \left(\frac{w_7}{e} - w_1\right) xz - \frac{1}{e}(w_5 - w_6)y,$$

Clearly, $\frac{dV_1}{dt}$ is negative definite under the condition (3.10), and the biological facts $w_5 - w_6 > 0$ Hence E_1 is GAS on R^3_+, and the proof is now complete.

Theorem 3.4: The EP $E_2 = (0, \bar{y}, \bar{z})$ of the Sys. (3.2) has the basin of attraction $\mathcal{B} \subset \mathrm{R}^3_+$ that satisfies the following sufficient conditions:

$$z < \frac{w_4 \bar{z}}{\bar{Q}}, \tag{3.11 a}$$

$$w_7 \bar{z} < \frac{1+e\bar{y}}{w_2+w_3+1} \tag{3.11 b}$$

$$\left(\frac{ew_4}{Q} + w_6\right)^2 < 4w_5 w_8 \tag{3.11 c}$$

$$\frac{\mu^2}{4} < \left[\sqrt{w_5}\,(y-\bar{y}) - \sqrt{w_8}\,(z-\bar{z})\right]^2 \tag{3.11 d}$$

where $\mu = 1 + \frac{e\, w_4 \bar{y}\, \bar{z}}{\bar{Q}^2}$, $\bar{Q} = w_2 + w_3$, and Q is given above.

Proof: Consider the following LF for E_2:

$$V_2(x, y, z) = x + \frac{(y-\bar{y})^2}{2} + \frac{(z-\bar{z})^2}{2}.$$

Now by taking the time derivative of V_2 and doing some algebraic simplification gives that:

$$\begin{aligned}\frac{dV_2}{dt} = {} & \frac{\mathrm{x}}{1+w_1 z} - x^2 - \frac{xz}{Q} + \frac{e\, xyz}{Q} - \frac{e\, x\bar{y}z}{Q} \\ & - \frac{ew_4\, xy\bar{z}}{Q\bar{Q}} + \frac{ew_4\, x\bar{y}\bar{z}}{Q\bar{Q}} - w_7 xz^2 + w_7 xz\bar{z} \\ & - w_5(y-\bar{y})^2 + \left[\frac{ew_4}{Q} + w_6\right](y-\bar{y})(z-\bar{z}) - w_8(z-\bar{z})^2.\end{aligned}$$

Therefore, due to the given sufficient conditions, the above Eq. can be written as:

$$\frac{dV_2}{dt} \leq \left[1 + \frac{e\, w_4 \bar{y}\, \bar{z}}{\bar{Q}^2}\right] x - x^2 - \left[\frac{1 + e\bar{y}}{Q} - w_7 \bar{z}\right] xz - \frac{exy}{Q}\left[\frac{w_4 \bar{z}}{\bar{Q}} - z\right]$$
$$-\left[\sqrt{w_5}\,(y - \bar{y}) - \sqrt{w_8}\,(z - \bar{z})\right]^2.$$

Then, by using the bound of the logistic term with the above conditions (3.11 a) - (3.11 c), it is obtained that:

$$\frac{dV_2}{dt} \leq \frac{\mu^2}{4} - \left[\sqrt{w_5}\,(y - \bar{y}) - \sqrt{w_8}\,(z - \bar{z})\right]^2.$$

Clearly, $\frac{dV_2}{dt}$ is negative definite, under the sufficient condition (3.11 d), for any initial point in the interior of $\mathcal{B}$. Hence V_2 is strictly LF in the interior $\mathcal{B}$, thus $\mathcal{B}$ is GAS manifold for E_2. Theorem 3.5: Suppose that the CEP. E_3 is LAS Then it is GAS in the R_+^3 interior under the following sufficient conditions.

$$S_{11} > 0, \tag{3.12 a}$$

$$(S_{12})^2 < S_{11}S_{22}, \tag{3.12 b}$$

$$(S_{13})^2 < S_{11}S_{33}, \tag{3.12 c}$$

$$(S_{23})^2 < S_{22}S_{33}, \tag{3.12 d}$$

*w*here:

$$S_{11} = 1 - \frac{z^*}{Q\,Q^*}, \qquad S_{22} = w_5,$$

$$S_{33} = (w_7 x + w_8), \qquad S_{12} = \frac{ez^*(w_2 + w_3 - w_4)}{QQ^*},$$

$$S_{13} = \left[\frac{w_1}{L\,L^*} + \frac{1}{Q} + w_7 z^*\right], \qquad S_{23} = \frac{ew_4}{Q} + \frac{e(w_2 + w_3)x}{Q\;Q^*} + w_6.$$

while Q, L, Q^*, and L^* are given above.

Proof: Consider the following LF:

$$V_3(x, y, z) = \left(x - x^* - x^* \ln\frac{x}{x^*}\right) + \frac{(y - y^*)^2}{2} + \frac{(z - z^*)^2}{2}$$

By using similar arguments as above give

$$\frac{dV_3}{dt} = -S_{11}(x-x^*)^2 + S_{12}(x-x^*)(y-y^*) - S_{22}(y-y^*)^2$$
$$-S_{13}(x-x^*)(z-z^*) + S_{23}(y-y^*)(z-z^*) - S_{33}(z-z^*)^2$$

Therefore, due to the given sufficient conditions, the above Eq. can be written as:

$$\frac{dV_3}{dt} \leq -\frac{1}{2}\left[\sqrt{S_{11}}\,(x-x^*) - \sqrt{S_{22}}\,(y-y^*)\right]^2 - \frac{1}{2}\left[\sqrt{S_{11}}\,(x-x^*) + \sqrt{S_{33}}\,(z-z^*)\right]^2$$
$$-\frac{1}{2}\left[\sqrt{S_{22}}\,(y-y^*) - \sqrt{S_{33}}\,(z-z^*)\right]^2$$

Clearly, $\frac{dV_3}{dt}$ is negative definite for any initial point in the interior of R_+^3. Hence V_3 is strictly LF, thus E_3 is GAS.

4. BIFURCATION ANALYSIS

4.1 INTRODUCTION

In linguistic terms, bifurcation refers to a branching process in which a point or area forks and divides into multiple pieces or branches. Is widely used to characterize any circumstance or phenomenon in which the qualitative, topological portrayal of the topic we're studying changes when one or more of the parameters on which the subject depends are a little bit changed. Curves or surfaces, functions or maps, real or complex, differential or integral Eqs., and vector fields are only a few examples of possible topics. The objective of this talk will be a dynamic Sys. with a differential Eqs. formula. When someone formulates a Sys. with Eqs. of motion to a mathematical or physical model, that kind of dynamical Sys. is developed frequently in the sciences. The phase space (or portion) of the Sys. is where these Eqs. are set, In a differential Eq. case, a point p in the phase space represents all possible situations for the Sys., and a Sol. with initial condition $p_{\circ}$ defines a curve going through p in the phase space. The phase portrait is the global description of these curves that correspond to all phase space points. This portrait depicts the dynamics in a global qualitative sense, and it is affected by any parameters in the boundary conditions or Eqs. of motion. Once any of these factors is changed, the phase portrait may deform somewhat while its qualitative aspects (i.e., topological) do not change, Alternatively, the dynamics may well be drastically altered, resulting in a qualitative alteration in the phase portrait, such as the emergence or absence of equilibria, periodic orbits, or more complicated properties, for example, unexpected attractors [101].

Bifurcation theory's methodology and results are necessary to understand nonlinear dynamical Syss, and then this theory applies to any subject of nonlinear Syss found in the real world.

Furthermore, the bifurcations are divided into two classes parts: local and global bifurcation; . Bifurcations from EPs that happen in the neighborhood of a specific point are referred to as local bifurcations (LB) (e.g. transcritical (TB), saddle-node (SNB), and pitchfork (PB)). This limitation ignores a big region on global bifurcations in which a few qualitative changes in the phase portrait happen that aren't picked up by viewing near a specific point, an introduction to this part of the topic is provided by Wiggins (1988) [102].

On the other hand, A HB is also defined as the appearance or disappearance of a periodic path all over a local change in the stability qualities at a stable point. We can more accurately say that it is a type of local bifurcation in which a stable point is destabilized when two complex conjugates EVUs intersect the imaginary axis of the complex plane [103]. According to general logical assumptions regarding the dynamical Syss, a cycle of short-range limits split from a stable point. Another name for the HB is, a Poincare–Andronov–Hopf bifurcation.

In addition, the roots of the bifurcation theory topic are classical mathematical roots, that can be seen in the work presented by Euler (1744), Twenty years ago, Arnold (1972) presented the characterization, while Poincare introduces the modern development of the subject together with the differential Eq. theory [101].

This theory has recently undergone a huge evolution, with new ideas and methods used and incorporated into the theory of dynamic Syss. Song et al. [104] studied Bifurcation analysis which includes the presence of HB, direction of the bifurcation, and the bifurcating periodic Sol.'s stability in a P-P Sys. including a Predator harvesting in the Michaelis–Menten type with diffusion term by the normal form theory and method of the center manifold. Naji and Majeed [72] examined The possibility of LB occurring near all the EPs of the P-P Sys. with refuge and stage structure in the population of the prey. Majeed and Rahi [105] presented all the conditions that guarantee the occurrence of all kinds of LB (such as SNB, TB, and PB) around each EP as well as an HB near the CEP of an ecological mathematical Sys. consisting of a food chain prey-predator model involving two different types of FRs including a prey refuge. Ibarra et al. [106] studied the bifurcation analysis of the Bazykin prey-predator model using ratio-dependent FR and intraspecific competition within the predator community, they demonstrated that the model has a wide variety of bifurcations, such as SNBs, HBs, Bogdanov-Takens bifurcations, and homoclinic bifurcations. Shang, et al. [1] discussed the bifurcation analysis in a Leslie-type prey-predator model and increasing FR in which the dynamical behaviors of this Sys. are investigated. when the FR for both prey and predator is increasing. Shown that the Sys. undergoes HB and Bautin bifurcation. The complex Sys. is also used to obtain the normal form of Bautin bifurcation. Aldebert et al.[107] discussed the bifurcation analysis of a three-dimensional prey-predator model with inconclusive formulation, in which bifurcation

analysis is used to investigate the problem of the parameter sensitivity caused by a change in formulation of the model. According to a three-dimensional analysis, the majority of the changes that occur as a result of the formulation change are related to a codimension-three Bogdanov—Takens degenerated bifurcation.

4.2 LOCAL BIFURCATION ANALYSIS

This section discusses the effect of parameter value changes on the Sys. (3.2) dynamics around EPs. Non-hyperbolic EP of the Sys. (3.2) is necessary but not sufficient for bifurcation to take place. Sotomayor's theorem for LB can be applied by selecting a certain parameter value that causes the EP to switch from hyperbolic mode into a non-hyperbolic mode when the bifurcation occurs. To make the notations easier to study, the Sys. (3.2) may now be expressed as a vector:

$$\frac{dX}{dt} = F(X)\,, \text{with } X = (x\,, y, z)^T, F = (F_1, F_2, F_3)^T = (xf_1, f_2, f_3)^T,$$

where $f_i; i = 1,2,3$ represents the functions in the R.H.S of the Sys. (3.2). Moreover, straightforward computation on the general JM shows that $\forall\, V \neq 0;\ V = (v_1, v_2, v_3)^{\mathrm{T}}$, the following second and third directional derivatives are obtained:

$$D^2F_\mu(X\,, \mu)(V, V) = \begin{bmatrix} 2\left[\left(\frac{(w_2+w_3)z}{Q^3} - 1\right){v_1}^2 - \left(\frac{w_2+w_3}{Q^2} + \frac{w_1}{L^2}\right)v_1v_3 + \left(\frac{{w_1}^2\,x}{L^2}\right){v_3}^2\right] \\ 2\,\frac{e(w_2+w_3-w_4)}{Q^2}\,v_1\left(v_3 - \frac{zv_1}{Q}\right) \\ -2\,w_7v_1v_3 \end{bmatrix} \tag{4.1}$$

And

$$D^3F_\mu(X\,, \mu)(V, V, V) = \begin{bmatrix} -6\left[\frac{(w_2+w_3)}{Q^3}\,v_1^2\left(\frac{z}{Q}v_1 - v_3\right) + \frac{{w_1}^2}{L^3}{v_3}^2\left(\frac{w_1x}{L}v_3 - v_1\right)\right] \\ \frac{6\,e(w_2+w_3-w_4)}{Q^3}\,v_1^2\left(\frac{z}{Q}v_1 - v_3\right) \\ 0 \end{bmatrix} \tag{4.2}$$

Now, the possibility of bifurcation occurring around each EP is discussed in the following theorems.

Theorem 4.1: As the parameter w_5 passes through the value $w_5 \equiv \widehat{w}_5$, where

$$\widehat{w}_5 = \frac{ew_6(1+w_4)}{\widehat{\mathrm{Q}}(w_7+w_8)}. \tag{4.3 a}$$

Then near the PDFEP, $E_1 = (\,1,0,0\,)$ the Sys. (3.2) has:

a. No SNB

b. A TB if the following condition holds

$$\mathrm{e}(\mathrm{w}_2 + \mathrm{w}_3)w_6 \neq \mathrm{e}\mathrm{w}_4 w_6 + \widehat{w}_5 \mathrm{w}_7 \widehat{\mathrm{Q}}^2. \tag{4.3 b}$$

c. A PB if the condition (4.3 b) is violated and the following condition holds.

$$(w_2 + w_3) \neq w_4. \tag{4.3 c}$$

Proof: It's very easy to check that, when the parameter w_5 goes through the value $w_5 \equiv \widehat{w}_5$, the JM is given by Eq. (3.6 a) of the Sys. (3.2) at the EP, E_1 has zero EVU (say $\hat{\lambda} = 0$), then the EP E_1 becomes a non-hyperbolic point and the JM (J_1) when $w_5 = \widehat{w}_5$ becomes:

$$\hat{J}_1 = J_1(E_1, \widehat{w}_5) = \begin{bmatrix} -1 & 0 & -\left(w_1 + \frac{1}{\hat{Q}}\right) \\ 0 & -\widehat{w}_5 & \frac{e(1+w_4)}{\hat{Q}} \\ 0 & w_6 & -(w_7 + w_8) \end{bmatrix}.$$

Let $V^{[1]} = \left(v_1^{[1]}, v_2^{[1]}, v_3^{[1]}\right)^T$ be the EVC associated with the EVU $\hat{\lambda} = 0$ of $\hat{J}_1$.

Then $V^{[1]}$ can be written as $V^{[1]} = \left(-P_1 v_3^{[1]}, P_2 v_3^{[1]}, v_3^{[1]}\right)^T$, where $v_3^{[1]}$ be any nonzero real number, while $P_1 = w_1 + \frac{1}{\hat{Q}}$ and $P_2 = \frac{w_7+w_8}{w_6}$.

Let $\Psi^{[1]} = \left(\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}\right)^T$ be the EVC associated with the EVU $\hat{\lambda} = 0$ of the matrix ${\hat{J}_1}^T$.

Then $\Psi^{[1]}$ can be written as $\Psi^{[1]} = \left(0, \frac{w_6}{\widehat{w}_5}\psi_3^{[1]}, \psi_3^{[1]}\right)^T$, where $\psi_3^{[1]}$ is any nonzero real numbers.

Now, since

$$\frac{\partial F}{\partial w_5} = F_{w_5}(X\,, w_5) = \left(\frac{\partial F_1}{\partial w_5}, \frac{\partial F_2}{\partial w_5}, \frac{\partial F_3}{\partial w_5}\right)^T = (0, -y\,, 0)^T.$$

Hence, $F_{w_5}(E_1\,, \widehat{w}_5) = (0\,, 0\,, 0\,)^T$, and then $\left(\Psi^{[1]}\right)^T F_{w_5}(E_1\,, \widehat{w}_5) = 0.$

Therefore, the first condition of TB is satisfied, while the SNB cannot occur according to Sotomayor's theorem. Now, since we have

$$DF_{w_5}(X\,, w_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $DF_{w_5}(X\,, w_5)$ is the derivative of $F_{w_5}(X\,, w_5)$ w.r.t $X = (x\,, y\,, z\,)^T$.

So it's very easy to show that:

$$\left(\Psi^{[1]}\right)^T \left[DF_{w_5}(E_1\,, \widehat{w}_5)V^{[1]}\right] = -\frac{w_6}{\widehat{w}_5} P_2 v_3^{[1]} \psi_3^{[1]} \neq 0.$$

Now, by substituting $E_1, \widehat{w}_5$ and $V^{[1]}$ in Eq. (4.1), it is obtained:

$$D^2F(E_1\,, \widehat{w}_5)\left(V^{[1]}\,, V^{[1]}\right) = \begin{bmatrix} 2\left[P_1\left(\dfrac{w_2 + w_3}{\widehat{Q}^2} + w_1\right) + {w_1}^2 - {P_1}^2\right]\left(v_3^{[1]}\right)^2 \\ -2\,\dfrac{e(w_2 + w_3 - w_4)}{\widehat{Q}^2} P_1 \left(v_3^{[1]}\right)^2 \\ 2\,w_7 P_1 \left(v_3^{[1]}\right)^2 \end{bmatrix}.$$

Hence, under the condition (4.3 b) it is obtained that:

$$\left(\Psi^{[1]}\right)^T D^2F(E_1\,, \widehat{w}_5)\left(V^{[1]}\,, V^{[1]}\right) = -2P_1\left[\frac{e(w_2 + w_3 - w_4)w_6}{\widehat{Q}^2 \widehat{w}_5} - w_7\right]\psi_3^{[1]}\left(v_3^{[1]}\right)^2 \neq 0.$$

Thus, as per Sotomayor's theorem, at the predator−free EP E_1, Sys. (3.2) possess a TB, where the parameter $w_5 = \widehat{w}_5$.

Now, by violating condition (4.3 b) and substituting $E_1, \widehat{w}_5$ and $V^{[1]}$ in Eq. (4.2), it is obtained:

$$D^3F(E_1,\widehat{w}_5)\left(V^{[1]},V^{[1]},V^{[1]}\right)=\begin{bmatrix}-6\left[w_1{}^2(w_1+P_1)-\dfrac{(w_2+w_3)}{\hat{Q}^3}P_1{}^2\right]\left(v_3^{[1]}\right)^3\\ \dfrac{-6\,e(w_2+w_3-w_4)}{\hat{Q}^3}P_1{}^2\left(v_3^{[1]}\right)^3\\ 0\end{bmatrix}.$$

Hence, under the condition (4.3 b) it is obtained that:

$$\left(\Psi^{[1]}\right)^T D^3F(E_1,\widehat{w}_5)\left(V^{[1]},V^{[1]},V^{[1]}\right)=-\frac{6\,e\,w_6(w_2+w_3-w_4)}{\widehat{w}_5\,\hat{Q}^3}P_1{}^2\psi_3^{[1]}\left(v_3^{[1]}\right)^3\neq 0.$$

Thus, as per Sotomayor's theorem, at the PDFEP, E_1 with the parameter $w_5=\widehat{w}_5$, Sys. (3.2) possess a PB .

It is easy to verify that, the EVC which associated with the zero EVUs of the JM at the PYFEP, J_2, is written as $V^{[2]}=\left(0,v_2^{[2]},\frac{w_5}{e}v_2^{[2]}\right)^T$. Hence, Sys. (3.2) does not undergo any kind of bifurcation at the PYFEP E_2 with any parameter, due to the obtained result that $D^2F_\mu(X,\mu)(V,V)=0$.

Theorem4.2: Suppose that the conditions $(3.9\,a)$-$(3.9\,b)$ along with the following set of conditions are satisfied.

$$\max\{S_1,S_3\}<1, \tag{4.4 a}$$

$$M_1\neq M_2, \tag{4.4 b}$$

where S_1 is given in Eq. $(3.9\,d)$, $S_3=z^*\left[\frac{(w_1Q^*+(L^*)^2)(w_2+w_3-w_4)+(x^*+w_4)L^{*2}}{L^{*2}Q^{*2}(x^*+w_4)}\right]$, and $M_i: i=1,2$ are given in the proof.

Then as the parameter w_6 passes through the value $w_6{}^*$, where $w_6{}^*=\frac{a_{22}(a_{11}a_{33}-a_{13}a_{31})}{a_{11}a_{23}-a_{13}a_{21}}$, the Sys.(3.2) near the CEP $E_3=(x^*,y^*,z^*)$ possess an SNB, but neither TB nor PB can occur.

Proof: It is well known that the characteristic Eq. given by Eq. (3.8 b) has a zero root if and only if $A_3=0$, and then E_3 becomes a non-hyperbolic EP.

Clearly, the JM of the Sys.(3.2) at the CEP E_3 with the parameter $w_6 = {w_6}^*$ has a zero EVU say ($\lambda^* = 0$) and it becomes

$J_3^* = J_3(w_6 = {w_6}^*) = \left[a_{ij}^*\right]_{3\times3}$, where $a_{ij}^* = a_{ij}$ for all $i, j = 1, 2, 3$ with $a_{32}^* = {w_6}^*$.

Straightforward computation shows that the bifurcation value ${w_6}^* = \frac{a_{22}(a_{11}a_{33} - a_{13}a_{31})}{a_{11}a_{23} - a_{13}a_{21}} > 0$, where the term $(a_{11}a_{33} - a_{13}a_{31}) > 0$, while $(a_{11}a_{23} - a_{13}a_{21}) < 0$ under the condition (4.4 a).

Let $V^{[3]} = \left(v_1^{[3]}, v_2^{[3]}, v_3^{[3]}\right)^T$ be the EVC associated with the EVU $\lambda^* = 0$. Therefore, by direct computation, we can show that $V^{[3]} = \left(v_1^{[3]}, L_1 v_1^{[3]}, L_2 v_1^{[3]}\right)^T$, where $v_1^{[3]}$ be any nonzero real number with $L_1 = \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{13}a_{22}} < 0$ and $L_2 = -\frac{a_{11}}{a_{13}} < 0$.

Let $\Psi^{[3]} = \left(\psi_1^{[3]}, \psi_2^{[3]}, \psi_3^{[3]}\right)^T$ be the EVC associated with the EVU $\lambda^* = 0$ of the matrix J_3^{*T}.

Then after simple computation, it is obtained that $\Psi^{[3]} = \left(L_3\psi_2^{[3]}, \psi_2^{[3]}, L_4\psi_2^{[3]}\right)^T$, where $\psi_2^{[4]}$ be any nonzero real number with $L_3 = \frac{z^*[(\mathrm{e}w_4{w_6}^* + \mathrm{w}_7 w_5(Q^*)^2) - \mathrm{e}(\mathrm{w}_2 + \mathrm{w}_3){w_6}^*]}{x^*\left[1 - \frac{z^*}{(Q^*)^2}\right]{w_6}^*(Q^*)^2}$ and $L_4 = \frac{w_5}{{w_6}^*} > 0$.

Now, consider:

$\frac{\partial F}{\partial w_6} = F_{w_6}(X, w_6) = \left(\frac{\partial F_1}{\partial w_6}, \frac{\partial F_2}{\partial w_6}, \frac{\partial F_3}{\partial w_6}\right)^T = (0, 0, y)^T$.

So, $F_{w_6}(E_3, {w_6}^*) = (0, 0, y^*)^T$ and hence $\left(\Psi^{[3]}\right)^T F_{w_6}(E_3, {w_6}^*) = L_4 y^* \psi_2^{[3]} \neq 0$.

Therefore, the first condition of an SNB is satisfied. While neither a TB nor a PB can occur at E_3 according to Sotomayor's theorem.

Now, by substituting $E_3, {w_6}^*$ and $V^{[3]}$ in Eq. (4.1), we get:

$$D^2F(E_3, {w_6}^*)\left(V^{[3]}, V^{[3]}\right) = \begin{bmatrix} 2(K_1 - K_2)\left(v_1^{[3]}\right)^2 \\ 2\frac{e(w_2+w_3-w_4)}{(Q^*)^2}(L_2 - \frac{z^*}{Q^*})\left(v_1^{[3]}\right)^2 \\ -2\, w_7 L_2\left(v_1^{[3]}\right)^2 \end{bmatrix},$$

where $K_1 = \frac{(w_2+w_3)z^*}{(Q^*)^3} + \frac{(w_1)^2\,(L^*)^2 x^*}{(L^*)^3}$ and

$$K_2 = 1 + L_2(\frac{(w_2+w_3)}{(Q^*)^2} + \frac{w_1}{(L^*)^2}.$$

Hence, under the condition (4.4 b) it is obtained that:

$$\left(\Psi^{[3]}\right)^T D^2F(E_3, {w_6}^*)\left(V^{[3]}, V^{[3]}\right) = 2[M_1 - M_2]\psi_2^{[3]}\left(v_1^{[3]}\right)^2 \neq 0.$$

where $M_1 = K_1 L_3 + \frac{e}{(Q^*)^2}\left[(w_2 + w_3)L_2 + \frac{w_4\, z^*}{Q^*}\right]$,

$$M_2 = K_2 L_3 + w_7 L_2 L_4 + \frac{e}{(Q^*)^2}\left[\frac{(w_2+w_3)z^*}{Q^*} + L_2 w_4\right].$$

Thus, Sys.(3.2) has an SNB at E_3 by Sotomayor's theorem and the proof is now complete.

4.3 HOPF BIFURCATION ANALYSIS

The probability of HB occurring near CEP (3.2) is examined in this section, as proven by the next theorem .

Theorem 4.3: Suppose that the conditions $(3.9\,a) - (3.9\,c)$ along with $S_1 < 1$ and the following sufficient condition is satisfied:

$$(a_{11} + a_{33})(a_{13}a_{31} - a_{11}a_{33}) < -a_{32}(a_{33}a_{23} + a_{13}a_{21}), \qquad (4.5\,a)$$

Then the Sys.(3.2) undergoes an HP close to the CEP, $E_3$when the parameter w_5 goes through this value ${w_5}^* = \frac{1}{2\,N_1}\left(N_2 + \sqrt{(N_2)^2 - 4N_1\,N_3}\right)$, where $N_i; i = 1,2,3$ are given in the proof.

Proof: According to Eq. (3.8 b), it is easy to verify that $\Delta = A_1A_2 - A_3$ can be written as the following

$$\Delta = A_1A_2 - A_3 = N_1(w_5)^2 - N_2w_5 + N_3, \qquad (4.5\ b)$$

where $N_1 = -(a_{11} + a_{33}) > 0$,

$N_2 = -(a_{11} + a_{33})^2 + a_{32}a_{23}$,

$N_3 = (a_{11} + a_{33})(a_{13}a_{31} - a_{11}a_{33}) + a_{33}a_{23}a_{32} + a_{32}a_{13}a_{21}$.

Clearly, N_3 is negative under the given sufficient condition $(4.5\ a)$, and hence Eq. $(4.5\ b)$ has a unique positive root given by ${w_5}^*$. Therefore, for $w_5 = {w_5}^*$, we have that $A_1A_2 = A_3$. Now by substituting $A_1A_2 = A_3$ in the characteristic given by Eq. (3.8 b), , and then simplifies the obtained result shows that the JM $J_3(E_3, {w_5}^*)$ has the following EVUs:

$$\lambda_1 = -A_1 < 0,\ \lambda_{2,3} = \pm i\sqrt{A_2},$$

where $A_i; i = 1,2,3$ are given in Eq. (3.8 b).

Now in the neighborhood of ${w_5}^*$, the above complex roots can be written as $\lambda_{2,3} = \xi_1 \pm i\xi_2$, where $\xi_1({w_5}^*) = 0$, that means the first condition of the necessary and sufficient conditions for HP to occur is satisfied at $w_5 = {w_5}^*$.

Therefore, the proof will be done if we can verify the transversality condition of the HP $\xi_1{}'(w_5) = \left.\frac{d\xi_1}{dw_5}\right|_{w_5={w_5}^*} \neq 0$.

So, by substitute the complex EVU, say $\lambda_2 = \xi_1 + i\xi_2$, in the Eq. $(4.5\ b)$ and then calculating the derivative w.r.t the bifurcation parameter w_5, comparing the two sides of the resulting Eq. and equating their real and imaginary parts, gives that:

$$\begin{aligned} \Psi(w_5)\xi_1{}'(w_5) - \Phi(w_5)\xi_2{}'(w_5) &= -\Theta(w_5) \\ \Phi(w_5)\xi_1{}'(w_5) + \Psi(w_5)\xi_2{}'(w_5) &= -\Gamma(w_5)' \end{aligned} \qquad (4.5\ c)$$

here $\Psi(w_5) = 3(\xi_1(w_5))^2 + 2A_1(w_5)\xi_1(w_5) + A_2(w_5) - 3(\xi_2(w_5))^2$

$\Phi(w_5) = 6\xi_1(w_5)\xi_2(w_5) + 2A_1(w_5)\xi_2(w_5)$

$\Theta(w_5) = \big(\xi_1(w_5)\big)^2 A_1{}'(w_5) + A_2{}'(w_5)\xi_1(w_5) + A_3{}'(w_5) - A_1{}'(w_5)\big(\xi_2(w_5)\big)^2$

$\Gamma(w_5) = 2\xi_1(w_5)\xi_2(w_5)A_1{}'(w_5) + A_2{}'(w_5)\xi_2(w_5)$

Now using Cramer's rule for the unknown $\xi_1{}'(w_5)$ to solve the linear Sys. $(4.5\ c)$, it is obtained that

$$\xi_1{}'(w_5) = -\frac{\Theta(w_5)\Psi(w_5)+\Gamma(w_5)\Phi(w_5)}{(\Psi(w_5))^2+(\Phi(w_5))^2}. \qquad (4.5\ d)$$

Therefore, the second necessary and sufficient condition of HP will be satisfied if and only if

$$\Theta(w_5{}^*)\Psi(w_5{}^*) + \Gamma(w_5{}^*)\Phi(w_5{}^*) \neq 0.$$

Note that for $w_5 = w_5{}^*$ we have:

$$\Psi(w_5{}^*) = -2A_2(w_5{}^*),$$

$$\Gamma(w_5{}^*) = -(a_{11} + a_{33})\sqrt{A_2(w_5{}^*)},$$

$$\Phi(w_5{}^*) = 2A_1(w_5{}^*)\sqrt{A_2(w_5{}^*)},$$

$$\Theta(w_5{}^*) = -\frac{1}{2}\left[(a_{11} + a_{33})^2 + a_{23}a_{32} - \sqrt{(N_2)^2 - 4N_1\,N_3}\right] \quad .$$

Substitution into Eq. $(4.5\ d)$ gives:

$$\Theta(w_5{}^*)\Psi(w_5{}^*) + \Gamma(w_5{}^*)\Phi(w_5{}^*) = 2A_2(w_5{}^*)\sqrt{(N_2)^2 - 4N_1\,N_3} > 0$$

Hence, the 2nd condition of the HP is satisfied. So, Sys. (3.2) undergoes an HP close to the CEP. E_3 at the parameter $w_5 = w_5{}^*$.

5. NUMERICAL SIMULATION

5.1 INTRODUCTION

After a computer program constructs a mathematical model of a physical system, a numerical simulation is a computational process that takes place on a computer. Because the mathematical models of most nonlinear Syss are too difficult to yield analytical solutions, numerical simulations are required to study their behavior. Sols.

In this chapter, we used Runge Kutta method with the predictor-corrector method in order to numerically confirm the results, we obtained in the previous chapters. turbo C++ is used for programming and MATLAB R2020b is used for plotting and then we discuss our obtained results.

5.2 NUMERICAL SIMULATION

The global dynamics of the Sys. (3.2) have been studied by analyzing the influence of varying parameter values of parameters on the dynamics of Sys. (3.2) through a numerical simulation using several sets of parameters and several sets of initial points. The study's secondary goal is to corroborate the stability and bifurcation findings we've already achieved. Numerically, based on initial values gained by the sixth-order Rung-Kutta, Sys. (3.2) is solved using the predictor/corrector method in four steps, and then the resulting trajectories are shown using Matlab R2020b as a 2D of time series and 3D phase plot.

For the following set of mathematically and biologically acceptable parameters, with the six distinct initial points, the Sys. (3.2) contains a GAS of the CEP $E_3 = (0.39, 0.17, 0.63)$, which confirms our stability findings of the presence of a GAS of the CEP for specific conditions. see figure (5.1):

$$\begin{aligned} &" \ w_1 = 0.10\,, \quad w_2 = 0.50, \quad w_3 = 0.25, \\ &\quad w_4 = 0.10, \quad w_5 = 0.54, \quad w_6 = 0.50\,, \\ &\quad w_7 = 0.10, \quad w_8 = 0.10, \quad e = 0.35." \end{aligned} \tag{5.1}$$

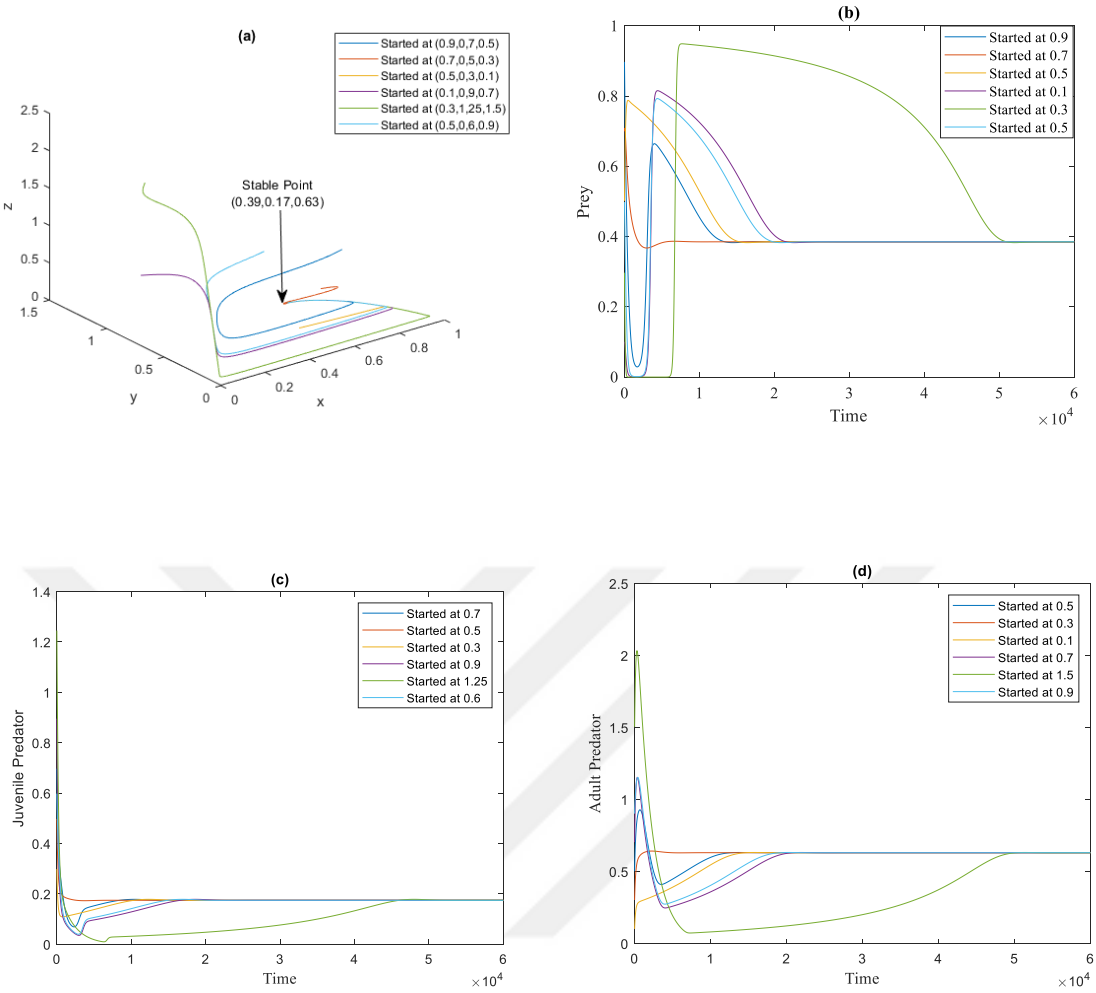

Figure 5.1: GAS of CEP (a) Sys. (3.2) 3D Phase plot for distinct initial points. (b) Prey Trajectories vs. t. (c) The juvenile predator's trajectories vs. t (d) The adult predator's trajectories vs. t.

Now, the impact of changing the parameter set's values on the Sys. (3.2) dynamical behavior is studied. Sys. (3.2) is numerically solved for the data given in (5.1) by changing one parameter at a time. It is noted that for the parameters set given by (5.1) with $w_1 = 0,15,30$, the Sys. (3.2) Sol. approaching asymptotically (ASYM) the CEPs given by $E_3 = (0.39,0.19.0.70)$, $E_3 = (0.39,0.02.0.08)$, and $E_3 = (0.39,0.01.0.04)$ respectively, see figure (5.2). Clearly, increasing the value of w_1 the predator population, in both the compartments juvenile and adult, decreases due to the hiding of the prey, while the population of the prey doesn't affect.

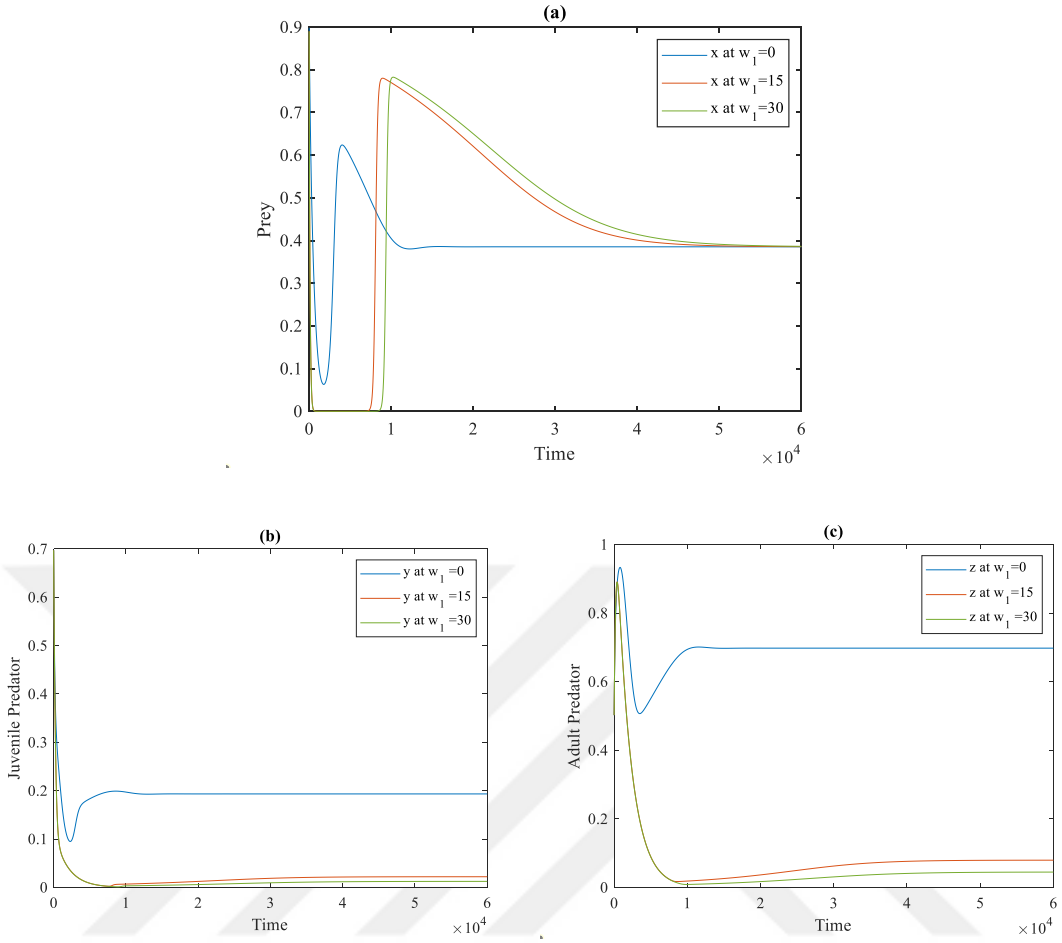

Figure 5.2: Sys. (3.2) trajectory vs. t for the data in (5.1) with distinct values of w_1 vs. t. (a) The prey's trajectories vs. t. (b) The juvenile predator's trajectories vs. t. (c) The adult predator's trajectories vs. t.

For the parameters set given in (5.1) with the parameter w_2 in the range $w_2 < 0.08$, it's observed that the Sys.(3.2) Sol. approaching ASYM to the PYFEP given by $E_2 = (0, \bar{y}, \bar{z})$ as shown in figure (5.3a). However, when $0.08 \leq w_2 < 0.41$, the Sys.(3.2) approaching ASYM to periodic dynamics in the interior R_+^3 as shown in figure (5.3b). Further increasing it in the range $0.41 \leq w_2 < 0.58$, the Sys.(3.2) approaching the CEP E_3 as shown in figure (5.3c) for the typical value $w_2 = 0.5$. Finally, for $w_2 \geq 0.58$, the Sys. (3.2) faces extinction in the species of the predator and the Sol. approaching ASYM to the PDFEP given by $E_1 = (1,0,0)$, as shown in figure (5.3d).

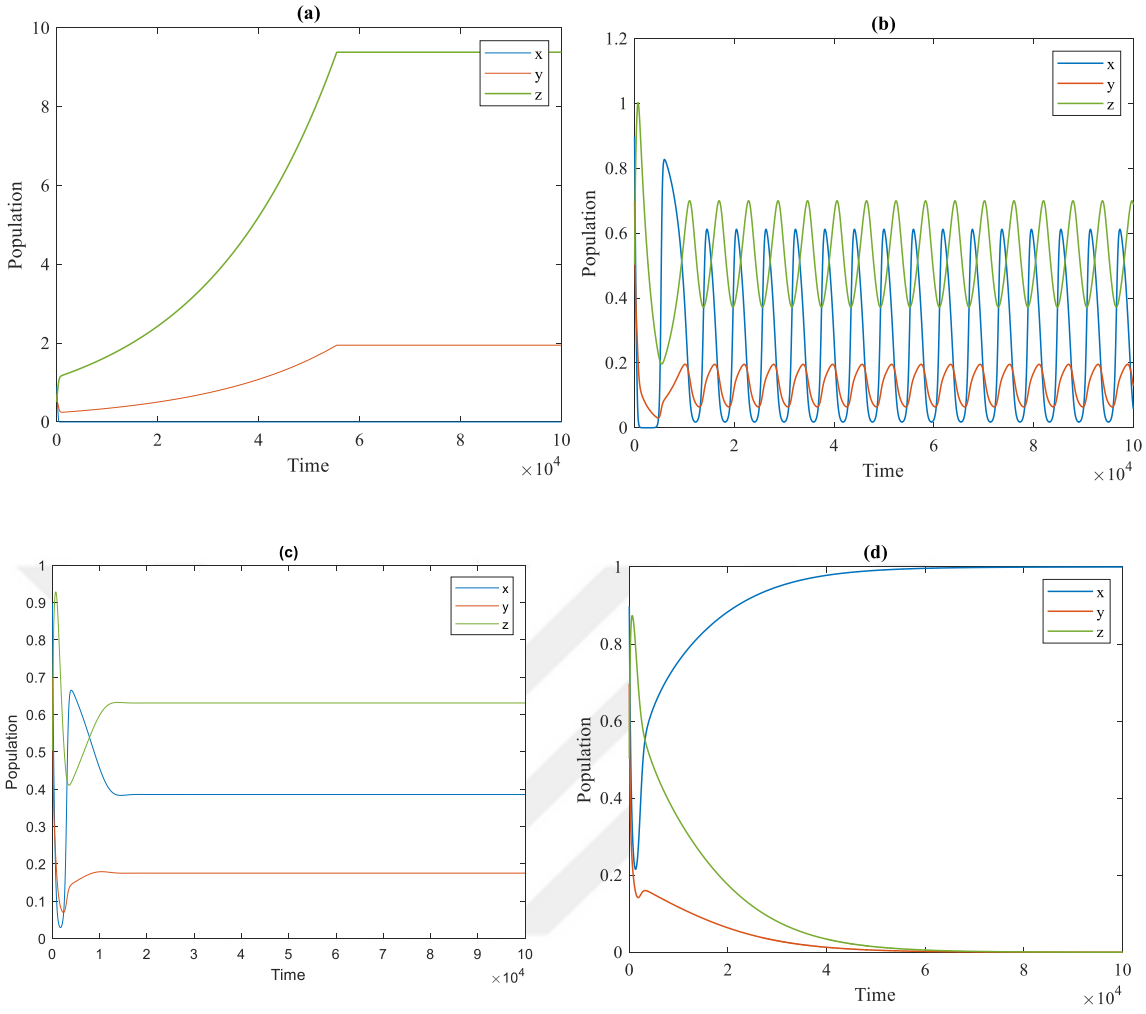

Figure 5.3: Various kinds of attractors of the Sys. (3.2) with typical values of w_2 for the data in (5.1). (a) The Sys. approaching to $E_2 = (0, 1.95, 9.38)$ for $w_2 = 0.06$, (b) Periodic attractor in R_+^3 for $w_2 = 0.32$. (c) The Sys. approaching CEP $E_3 = (0.39, 0.17.0.63)$ for $w_2 = 0.5$, (d) The Sys. approaching to $E_1 = (0, 1, 1)$ for $w_2 = 0.65$.

Now, varying the parameter w_3 in the range $w_3 < 0.16$ with the other parameters set to the data (5.1), it is noticed that the Sys. (3.2) Sol. is approaching ASYM to periodic dynamics in the interior R_+^3, see figure (5.4a). However, when $0.16 \leq w_3 < 0.33$, the Sol. approaching ASYM the CEP E_3, as shown in figure (5.4b) for the typical value $w_3 = 0.25$. Finally, when $w_3 \geq 0.33$, the Sys. (3.2) faces extinction in the population of predators and the Sol. approaching ASYM to the PDFEP given by $E_1 = (1,0,0)$, as shown in figure (5.4c).

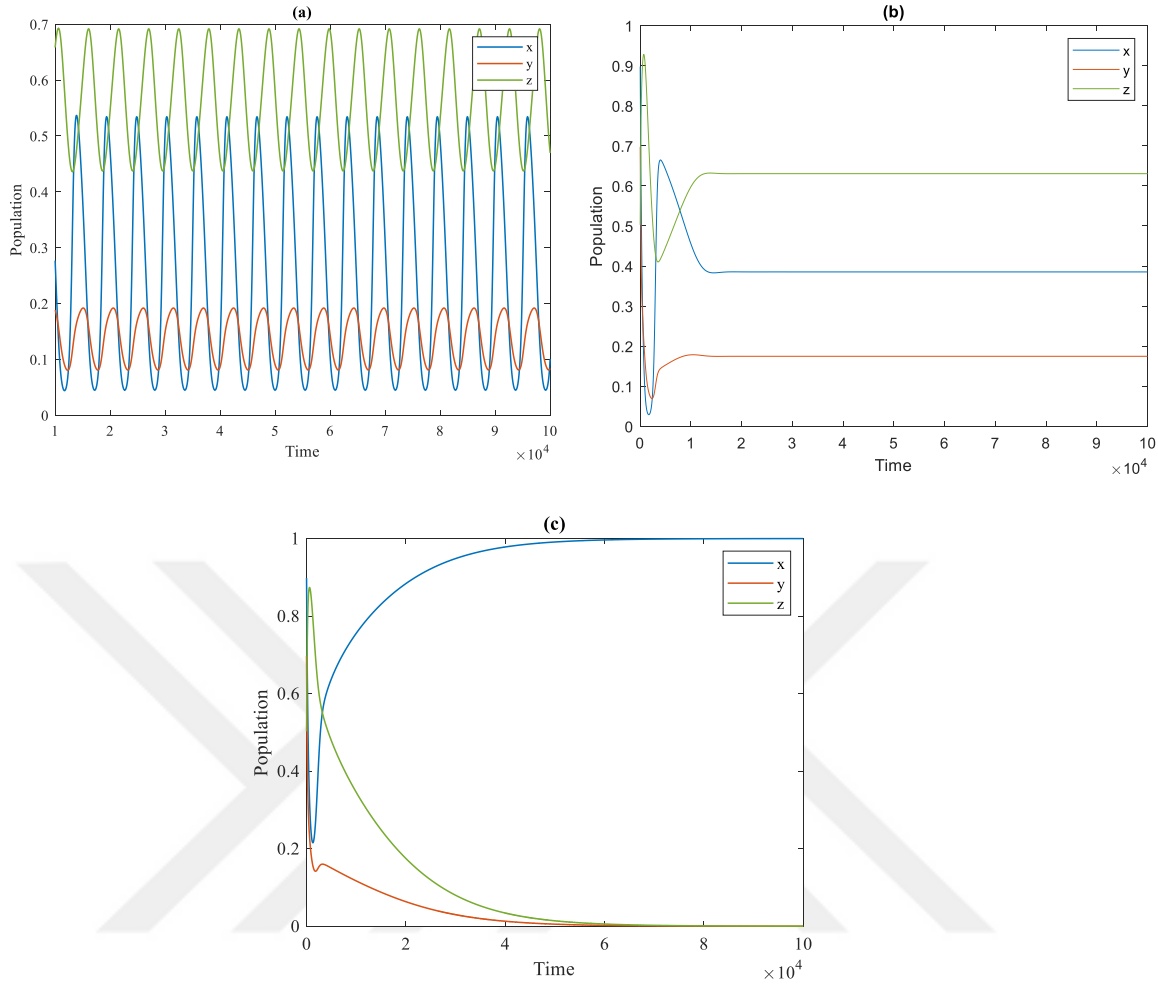

Figure 5.4: For $w_3 = 0.1$, (b) The Sys. approaching CEP $E_3 = (0, 39, 0.17, 0.63)$ for $w_3 = 0.25$, (c) The Sys. Various kinds of attractors of the Sys. (3.2) with typical values of w_3 for the data in (5.1). (a) Periodic attractor in R_+^3 approaching to $E_1 = (1,0,0)$ for $w_3 = 0.4$.

Moreover, for the parameters set given by (5.1) with varying the parameter w_4 in the range $w_4 < 0.06$, it's observed that the Sol. of the Sys.(3.2) approaching ASYM to the EP given by $E_1 = (1,0,0)$, see figure (5.5a), However, when $0.06 \leq w_4 < 0.16$, the Sol. of the Sys.(3.2) approaches to the CEP, as shown in figure (5.5b) for the typical value $w_4 = 0.1$. Further increasing this parameter in the range $0.16 \leq w_4 < 0.24$ leads to destabilizing the Sys.(3.2), and the trajectory is approaching ASYM to periodic dynamics in the interior R_+^3, see figure (5.5c). Finally, increasing the value of w_4 in the range $w_4 \geq 0.24$ leads to extinction in prey and the Sys. (3.2) trajectory is approaching ASYM to the PYFEP $E_2 = (0, \bar{y}, \bar{z})$, as shown in figure (5.5d).

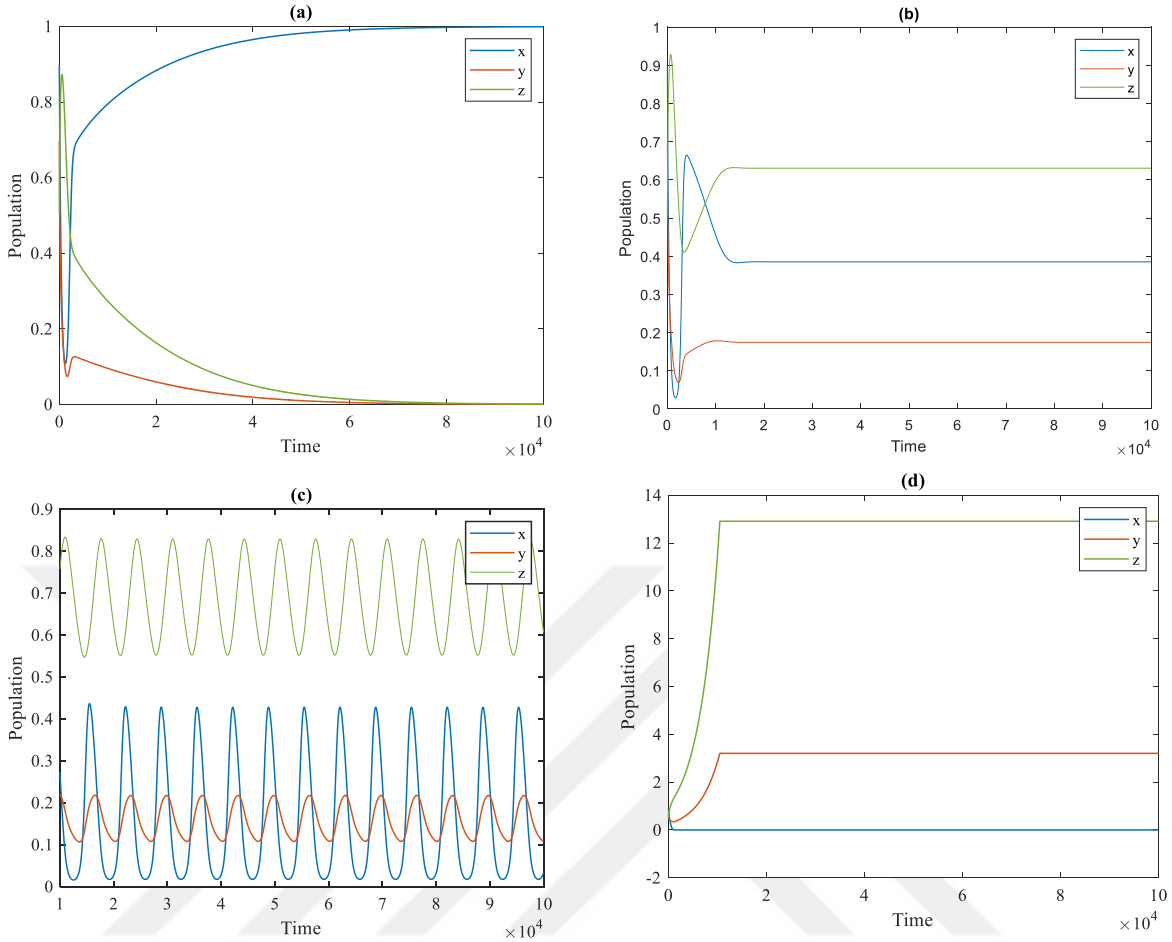

Figure 5.5: Various kinds of attractors of the Sys. (3.2) with typical values of w_4 for the data in (5.1). (a) The Sys. approaching to $E_1 = (1,0,0)$ for $w_4 = 0.03$, (b) The Sys. approaching CEP $E_3 = (0.39,0.17,0.63)$ for $w_4 = 0.10$, (c) Periodic attractor in R_+^3 for $w_4 = 0.18$, (d) The Sys. approaches to E_2= (0, 3.20, 12.91) for $w_4 = 0.3$.

For the parameters set given by (5.1) with the parameter w_6 in the range $w_6 < 0.48$, it's observed that the Sys.(3.2) Sol. is approaching ASYM to the PDFEP, $E_1 = (1,0,0)$, as shown in figure (5.6a), However, when $0.48 \leq w_6 < 0.54$, the Sol. is approaching the CEP, as shown in figure (5.6b), for the typical value $w_6 = 0.5$.

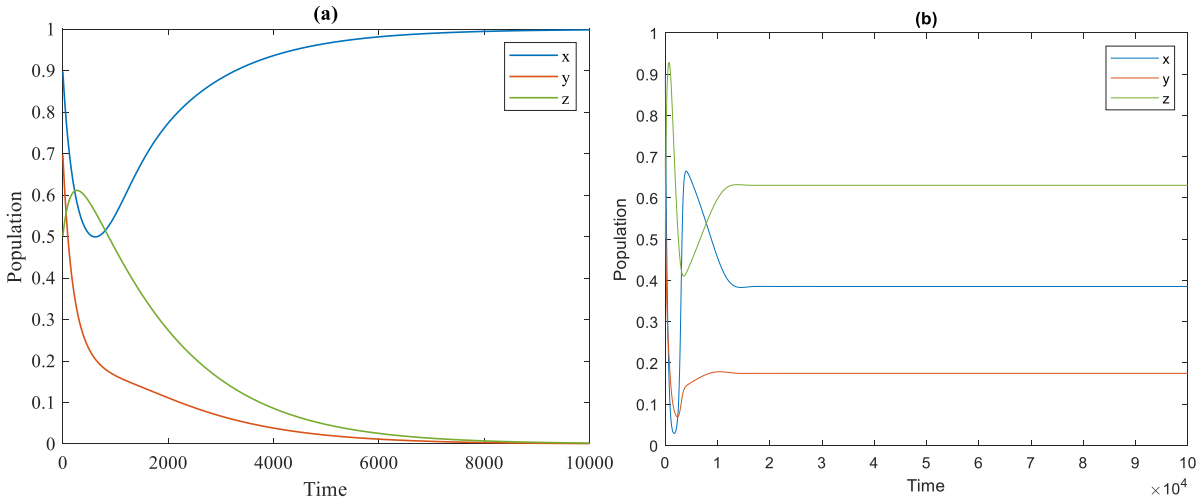

Figure 5.6: The Sys. (3.2) trajectory for the data in (5.1). (a) The Sys. approaching to $E_1 = (1,0,0)$ for $w_6 = 0.3$. (b) The Sys. approaching CEP $E_3 = (0,39,0.17,0.63)$ for $w_6 = 0.5$.

Now, for the parameters set given by (5.1) with the parameter w_7 in the range $w_7 < 0.12$, it's observed that the Sys.(3.2) Sol. is approaching ASYM to the CEP, as shown in figure (5.7a), for the typical value $w_7 = 0.1$. However, increasing the value of the parameter w_7 in the range $0.12 \leq w_7 < 1$, leads to extinction in the population of the predator and the Sys.(3.2) trajectory is approaching ASYM to the PDFEP, $E_1 = (1,0,0)$, as shown in figure (5.7b).

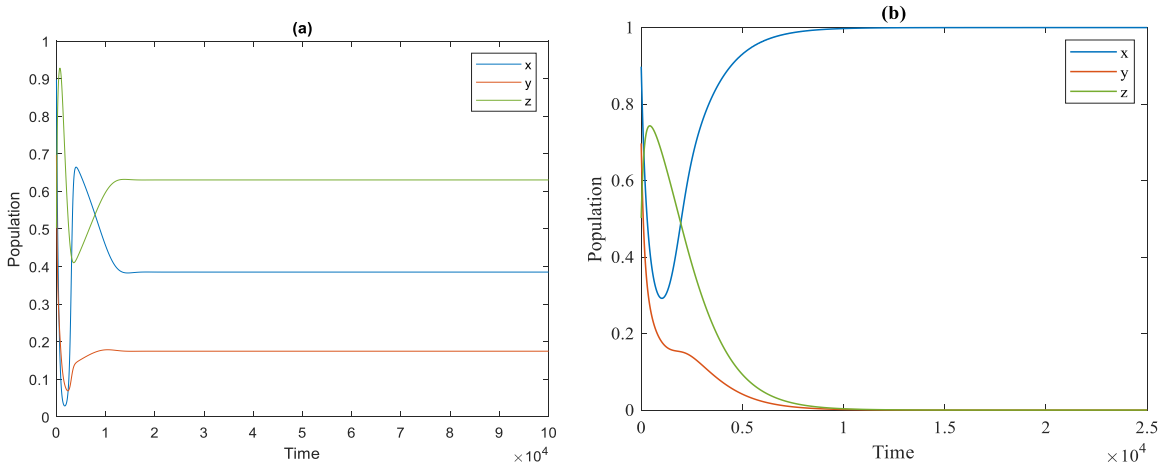

Figure 5.7: The Sys. (3.2) trajectory for the data in (5.1). (a) The Sys. approaching to CEP $E_3 = (0,39,0.17,0.63)$ for $w_7 = 0.1$. (b) The Sys. approaching to $E_1 = (1,0,0)$ for $w_7 = 0.2$.

Finally, the impact of changing the other parameters on the dynamical behavior of the Sys. (3.2) is studied, with the remaining parameters fixed as in Eq. (5.1), and identical findings are obtained as with the aforementioned parameters, therefore they have been summarized in the following table (5.1).

Table 5.1: The dynamical behavior of the Sys. (3.2) as the parameters vary.

Parameter	**Range**	**Behavior**	**Figure**
w_5	$w_5 < 0.24$	E_2 is AS.	Similar behavior to that of w_2
	$0.24 \leq w_5 < 0.48$	Periodic attractor in R_+^3	
	$0.48 \leq w_5 < 0.57$	E_3 is AS.	
	$0.57 \leq w_5 < 1$	E_1 is AS.	
w_8	$w_8 < 0.09$	Periodic attractor in R_+^3	Similar behavior to that of w_3
	$0.09 \leq w_8 < 0.11$	E_3 is AS.	
	$0.11 \leq w_8 < 1$	E_1 is AS.	
e	$0.05 \leq e < 0.34$	E_1 is AS.	Similar behavior to that of w_4
	$0.34 \leq e < 0.41$	E_3 is AS.	
	$0.41 \leq e < 0.82$	Periodic attractor in R_+^3	
	$0.82 \leq e < 1$	E_2 is AS.	

6. CONCLUSION AND FUTURE WORK

6.1 CONCLUSION AND DISCUSSION

In this thesis, a realistic and genuine ecological P-P model has been suggested and examined, including a variety of biological parameters to explain the relationship between prey and two predators (juvenile and adult). The fear function is modeled in the prey. It has also been hypothesized that the prey is strong enough to show anti-predator behavior and it has the ability to kill the predator as a natural response to resistance to predation. We have discussed the characteristics of the Sol. of the Sys. (3.2). The conditions for the existence of all possible EPs, particularly the CEP, have been examined. Analyses of local stability have been conducted. Additionally, global stability analyses utilizing the Lyapunov method have been performed on this model. A study of the LBs based on Sotomayor's theorem is performed. The presence of periodic dynamics as a result of HB incidence is explored and discussed. Additionally, to conclude our research of the proposed model, numerical simulations have been conducted to investigate the model's global dynamics, verify our analytical conclusions, and identify the set of control parameters. The numerical findings derived from Eq. (5.1) are summed up as follows:

i. For the Sys. (3.2), two forms of attractors are observed: stable point attractors and limit cycle attractors.

ii. Increasing the level of fear (w_1) doesn't have any effect on the prey population, while the predator population drops as a result of the prey's hiding, and the Sol. of the Sys. (3.2). and Sys. (3.2) Sol. continues to approach CEP asymptotically.

iii. Changing the half-saturation rate parameter (w_2) produces three specific bifurcation points, say τ_1, τ_2 and τ_3 , and the behavior of the Sol. of Sys. (3.2) can be listed as follows:

When $w_2 < \tau_1$: the Sol. approaches ASYM to the PYFEP E_2 (prey extinction).

When $\tau_1 \leq w_2 < \tau_2$: the Sol. approaches to periodic dynamics.

When $\tau_2 \leq w_2 < \tau_3$: the Sol. approaches ASYM to the CEP E_3 (stabilizing).

When $w_2 \geq \tau_3$: the Sol. approaches ASYM to the PDFEP E_1 (predators extinction).

The variation in the juvenile predator's death rate (w_5) produced comparable behavior to that of w_2.

iv. Changing the AF rate parameter (w_3), produces two specific bifurcation points, say ε_1 and ε_2, hence the behavior of the Sol. of Sys. (3.2) can be listed as follows:

When $w_3 < \varepsilon_1$: the Sol. approaches ASYM to periodic dynamics

When $\varepsilon_1 \leq w_3 < \varepsilon_2$: the Sol. approaches ASYM to ASYM to the CEP E_3 (stabilizing).

When $w_3 \geq \varepsilon_2$: the Sol. approaches ASYM to the PDFEP E_1 (predators extinction).

The variation in the adult predator's death rate (w_8) produced comparable behavior to that of w_3.

v. Changing the predator growth rate which depends on the AF parameter (w_4) produces three specific bifurcation points, say σ_1, σ_2 and σ_3 , hence the behavior of the Sol. of Sys. (3.2) can be listed as follows:

When $w_4 < \sigma_1$: the Sol. approaches ASYM to the PDFEP E_1 (predators extinction)

When $\sigma_1 \leq w_4 < \sigma_2$: the Sol. approaches ASYM to the CEP E_3.

When $\sigma_2 \leq w_4 < \sigma_3$: the Sol. approaches to periodic dynamics (destabilizing).

When $w_4 \geq \sigma_3$: the Sol. approaches ASYM to the PYFEP E_2 (prey extinction).

The variation in the juvenile predator's death rate (e) produced comparable behavior to that of w_4.

vi. Changing the predator maturation rate parameter (w_6), produces one specific bifurcation point, say δ, hence the behavior of the Sol. of Sys. (3.2) can be listed as follows:

When $w_6 < \delta$: the Sol. approaches ASYM to the PDFEP E_1 (predators extinction).

Otherwise, when $w_6 \geq \delta$: the Sol. approaches ASYM to the CEP E_3.

vii. Changing the anti-predator rate parameter (w_7), produces one specific bifurcation point, say μ, hence the behavior of the Sol. of Sys. (3.2) can be listed as follows:

when $w_7 \geq \mu$: the Sol. approaches ASYM to the PDFEP E_1 (predators extinction). Otherwise, When $w_7 < \mu$: the Sol. approaches ASYM to the CEP E_3.

As a result, Sys. (3.2) is very sensitive to changes in the values of the parameters and undergoes various forms of LBs and an HB.

6.2 FUTURE WORK

More models can be formulated by modifying the suggested ecological model by adding different biological factors or switching to another dimension or epidemiological model.

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