Effects of Budget and Competition on Search Engine Advertising

by

Beste Karnibat

A Dissertation Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Industrial Engineering

KOÇ ÜNİVERSİTESİ

September 2023

Effects of Budget and Competition on Search Engine Advertising

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

Beste Karnibat

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee Members:

Prof. Dr. Fikri Karaesmen (Advisor)

Assoc. Prof. Dr. Evrim Didem Güneş

Asst. Prof. Burak Gökgür

Date:

I dedicate my thesis to everyone, who supported me throughout this process, patiently stood by me and believed in me.

ABSTRACT

Effects of Budget and Competition on Search Engine Advertising
Beste Karnibat
Master of Science in Industrial Engineering
September 2023

One of the significant challenges marketers face nowadays is attempting to optimize their advertising strategies under a given budget constraint. As dominant bidding environments such as Google, Yahoo, etc., allow publishers to allocate positions in online advertising, budget constraints influence bidding strategy by inducing advertisers to deplete the budgets of higher-ranked competitors in order to gain premier positions. This strategic consideration has an impact on the profits of advertisers, especially when the competition is intense. Determining equilibrium strategies is vital for the advertiser and all parties involved in the problem. With the obtained bidding strategy, each advertiser is expected to allocate their budget effectively and attract customers with measurable returns.

This thesis studies the above problem in terms of finding optimal strategies for search engine advertising with budget constraints and competitors' interactions. Unlike similar studies in the literature, instead of solving an optimization problem with a single decision-maker, a game theory problem specified in the Nash equilibrium will be considered with all the competitors in the environment. In this respect, the paper is expected to fill the gap in the literature while making a practical impact on the search engine advertising market.

ÖZETÇE

Arama Motoru Reklam Yatırımlarında Bütçe ve Rekabetin Etkileri
Beste Karnibat
Endüstri Mühendisliği, Yüksek Lisans
Eylül 2023

Pazarlamacıların günümüzde karşılaştığı önemli zorluklardan biri, belirli bir bütçe kısıtı altında reklam stratejilerini optimize etmeye çalışmaktır. Google, Yahoo gibi önde gelen teklif ortamları, yayıncılara çevrimiçi reklamcılıkta pozisyonları tahsis etme olanağı sağladığından, bütçe kısıtlamaları, reklam verenlerin üst sıradaki rakiplerin bütçelerini tüketerek öncelikli pozisyonlar elde etme stratejisini etkiler. Bu stratejik düşünme, özellikle rekabetin yoğun olduğu durumlarda reklam verenlerin karlarını etkiler. Denge stratejilerini belirlemek, reklam veren ve probleme dahil olan tüm taraflar için hayati öneme sahiptir. Elde edilen teklif stratejisi ile her reklam verenin bütçesini etkili bir şekilde tahsis etmesi ve ölçülebilir getirilerle müşterileri çekmesi beklenmektedir.

Bu çalışma, bütçe kısıtlamaları ve rakiplerin etkileşimleri ile arama motoru reklamcılığı için optimal stratejileri bulma konusundaki yukarıdaki problemi inceler. Literatürdeki benzer çalışmalardan farklı olarak, tek bir karar verici ile bir optimizasyon problemi çözmek yerine, tüm rakiplerin çevrede olduğu Nash dengesi olarak belirlenen bir oyun teorisi problemi dikkate alınacaktır. Bu bakımdan, bu çalışmanın literatürdeki boşluğu doldurması ve arama motoru reklamcılık pazarında pratik bir etki yapması beklenmektedir.

ACKNOWLEDGMENTS

Before diving into the details, there are many people who deserve acknowledgment and gratitude, for without them, neither I nor this work would exist.

First and foremost, I would like to extend my most profound appreciation to my esteemed professors, Fikri Karaesmen and Pelin Canbolat. Your unwavering support and invaluable guidance throughout this research journey have been instrumental in shaping this thesis. Your expertise and patience have enriched my understanding of the subject matter and inspired me to strive for excellence. I am truly grateful for your mentorship and the countless hours you dedicated to providing feedback, discussing ideas, and challenging me to think critically.

I would like to thank the staff and faculty of Koç University, whose efforts behind the scenes ensured a conducive learning environment. Your dedication to providing resources, facilities, and opportunities for growth has been instrumental in shaping my academic experience.

Last but not least, I would like to express my deepest gratitude to my family for their unwavering love, encouragement, and belief in me. Your constant support, understanding, and sacrifices have been the driving force behind my accomplishments. I am forever grateful for your presence in my life.

To all those mentioned and those inadvertently left out, please accept my sincerest appreciation for your contributions. This thesis stands as a testament to the collaborative spirit and collective effort that played a significant role in its completion.

Thank you all for being an integral part of this remarkable journey.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABBREVIATIONS

CPC	Cost per Click
CPM	Cost per Impression
CTR	Click Through Rate
FPA	First Price Auction
ROI	Return on Investment
SERP	Search Engine Results Page
VPC	Value per Click

Chapter 1

INTRODUCTION

Search engine advertising is a form of internet advertising that allows retailers to utilize result pages such as Google, Yahoo, and Bing. These ads, also known as keyword ads, appear on the same page as organic search results, and advertisers only get charged when a user clicks on the ad that takes them to the advertiser's website. The retailer's e-commerce manager determines these ads according to various features, such as the words chosen by the individual and the advertiser's marketing strategy. Search engine advertising is now a major online revenue and has become even more important with the increase in online shopping due to the pandemic. According to Statista (2022) estimates, ad spending in Turkey in the Search Advertising segment is projected to reach \$100.40 m in 2022. The average ad spending per internet user is projected to amount to \$1.47 in 2022 [52]. In global comparison, the ad spending in the USA is projected to reach \$58.02 bn, and the share of ad revenue generated via search is expected to be %42.2 in 2022 [53].

Search engine advertising offers a very high return on investment for millions of advertisers compared to traditional marketing and also an important source of income for search engines. It is for this reason that billions of auctions are held every day in real-time to determine the position of the advertiser's keyword on the page and how it will be priced as a result. Many of these auctions operate with first price auction and similar mechanisms. In the case of Google auctions, each advertiser sets a maximum amount they are willing to pay per click, or the bid price. This advertiser-determined value is the most dominant factor influencing bid strategies. In addition, the quality of ads, ad rank, the context of the ad, and the expected impact from the ad assets and other ad formats determine the ad's ranking on the results page. This ranking is a critical factor influencing click-through rates or the frequency with which the page is viewed. The relationship between ranking and ad position is not trivial and received some attention from researchers. Kucukaydin et al., Brooks (2004a, 2004b), Regelson and Fain (2006), and Feng et al. (2007) studies that individuals are more likely to click on the first ad they see.

The advertiser's budget represents the second most crucial factor affecting bid strategies [1]. Abrams also states that budget constraints significantly influence most advertisers' bid decisions, especially those of small and medium-sized businesses. A key operational issue is the effective allocation of a limited advertising budget across advertising channels. Decisions regarding budget allocation on keywords can have adverse or favorable effects on advertisers, competitors, and search engine ad revenues. Much of the literature on budget constraints in search engine advertising has examined how a given budget should be divided among several keywords to maximize the total number of clicks without considering the competition in the relevant industry or assuming it is a specific, constant factor. Thus, decision interactions between competitors are often ignored. Lu et al. (2015) demonstrate that the budget constraint significantly influences the competition between two advertisers competing for ad positions.

Özlük and Cholette (2007) define the total revenue from advertisements to a certain search engine for a specific product as a function of the bid strategy. They establish the equations that characterize the strategy that maximizes the total revenue. Feldman et al. (2007) address the problem of optimally utilizing the advertising budget allocated to search engine words within a deterministic auction framework. They propose random strategies that yield relatively high click-through rates, albeit not optimal for the advertiser. Muthukrishnan et al. (2010) investigate an algorithmically efficient random version of this problem. Zhang and Feng (2011) and Yao and Mela (2011) consider the problem's dynamic variants and attempt to understand how the proposals change over time.

Cholette et al. (2012) examine the model proposed by Özlük and Cholette (2007) for the case where the position of the advertisement on the page is a random variable. Selçuk and Özlük (2013) add the targeted visibility constraint to this random model.

Kucukaydin et al. (2020) demonstrate that the level of competition and revenue potential are key factors in determining the bid price put forward for a keyword; with all other parameters being equal, a more challenging competitive environment or a higher income expectation will push the bid prices up. In these articles, competition is modeled with a given, fixed parameter for each advertiser.

The primary aim of the proposed thesis is to examine search engine advertising, which is becoming increasingly important in marketing, from the advertiser's perspective, considering budget constraints and interactions between competitors. For this purpose, research will be conducted on how each advertiser should allocate their budget to maximize their potential profit. This research will involve a detailed analysis of CPC bidding strategies, considering each advertiser's distinct parameters and limitations. The analysis will be conducted through the application of mathematical modeling, game theory, and numerical examples with computation. The objective is to pinpoint the conditions that lead to the discovery of a Nash equilibrium by maximizing profit. This involves identifying the optimal bidding strategies for each advertiser that, when adopted by all advertisers, result in no advertiser being able to increase their profit by unilaterally changing their strategy. The impact of changes in the budget and competitive environment on these strategies will be examined in depth. While doing this, unlike similar studies in the literature, instead of solving an optimization problem with a single decision-maker, a game theory problem that considers all the competitors in the environment will be handled, and the Nash equilibrium will be investigated. Determining equilibrium strategies is important for the advertiser and all parties involved in the problem. With the right bid strategy, each advertiser gets the most benefit from the budget allocated for their business.

In contrast, search engine managers who understand the advertisers' strategies can make design changes in line with their corporate goals. Similarly, it is possible for the competent authorities, who analyze the attitudes in the market correctly, to make the necessary market regulations. In this respect, the project is expected to fill the gap in the literature while making a practical impact on the search engine advertising market.

Chapter 2

LITERATURE REVIEW

Search engine advertising is a type of online advertising that involves placing sponsored ads on search engine results pages (SERPs). Advertisers bid on keywords that are relevant to their products or services, and the winning bids determine the placement of the ads on the SERPs. During the last decade, there has been a growing interest in analytical models of search-based advertising in academia due to the substantial increase in the number of both internet users and online service providers. In this literature review, we will focus on the advertiser perspective and the role of game theory in understanding the behavior of advertisers in search engine advertising auctions.

The early work on search advertising primarily concerns difficulties relating to search engines (Edelman et al., 2007; Edelman & Schwarz, 2010; Varian, 2007). Edelman et al. (2007) analyze the behavior of advertisers in search engine advertising auctions, focusing on both single and multiple keywords. In their paper, they employ a game-theoretic model to analyze the behavior of advertisers in sponsored search auctions. They define the "Envy-Free Nash equilibrium" of the generalized second-price auction and show that truthful bidding may not be the best strategy. They find that advertisers have an incentive to bid higher in auctions when the number of dominant advertisers is small and when the search engine has a high market share. Varian (2007) also analyzes the behavior of advertisers in search engine advertising auctions, focusing on multiple keywords. He proposes a symmetric Nash equilibrium of a search advertising auction to explain the behavior of advertisers in sponsored search auctions. He finds that advertisers with a higher VPC are more likely to bid higher in auctions, while advertisers with a lower VPC are more likely to bid lower. Researchers, including Edelman and Schwarz (2010), also explore the design of the search advertising auction. They look into the impact of the minimum bid required to compete in the search advertising auction. Others examine the financial consequences of auction design from a search engine perspective. Chen et al. (2010) analyze the behavior of advertisers in search engine advertising auctions, focusing on single keywords. They use a game-theoretic model to analyze the relationship between search engines and advertisers in the context of sponsored search auctions. They find that advertisers have an incentive to bid higher in auctions when they have a higher VPC and when the number of dominant advertisers is small. Overall, the literature suggests that game theory can be a valuable tool for understanding the behavior of advertisers in search engine advertising auctions, regardless of whether they are bidding on a single keyword or multiple keywords. Advertisers seek to find a Nash equilibrium, where their bid prices are in balance with the bid prices of other advertisers, in order to maximize their profits. Further research is needed to explore the implications of these findings for search engine advertising strategies and policies.

Additionally, some researchers consider the advertiser perspective such as optimal bids [24], keyword choices [49], search engine optimization [3], and cross-media strategy of search advertising with traditional media [28]. Finding an ideal position based on the bid price and the landing page quality is crucial for advertisers in search advertising since the position affects both the number of clicks received and the money paid to the search engine per click. Advertisers can generally improve their ranking by increasing their bids in the advertising auction in order to get more consumer clicks. However, increasing a bid to acquire a higher position may not be the best option for some advertisers because a higher position means a higher CPC, which can significantly raise total search advertising expenditures [17] and result in difficulties allocating budget constraints [49]. As a result, experts have looked into various factors concerning advertisement positions [42].

Many studies, including this one, have primarily focused on a single keyword framework, even though advertisers often bid on multiple keywords in their search advertising campaigns. The complexity of managing multiple keywords can present additional challenges, which is why this area may not have received as much attention. However, it's important to acknowledge that this is a significant aspect of search advertising and could be a promising avenue for future academic exploration. Furthermore, while several empirical works explored multiple keywords [23] and [48] they focused on various keywords of a single brand or retailer.

Several game-theory-based articles in the literature use a prior known list of individual bid prices as the basis for their models. For example, Zhao and Nagurney (2008) present a network equilibrium framework that allows for the calculation of both the equilibrium online advertising budget and the advertising expenditures across many websites. Their approach looks at the dilemma of several companies competing for ad space on multiple websites.

Lu, Zhu, and Dukes (2015) present a game-theory-based study that considers budget constraints. By studying a simultaneous game with complete information played by two advertisers competing for two ad spaces, the authors show that budget constraints significantly impact equilibrium bidding. Every advertiser is requested to set a daily budget. If a higher-ranked advertiser's budget is depleted before the end of the day, he or she is removed from the listing, and a lower-ranked advertiser takes his or her place. The research reveals that an advertiser's bid is highly dependent on the competition's budget. An advertiser can intentionally bid lower than the opponent to quickly deplete the competitor's budget. The influence of budget limits on the publisher's revenue is another consequence of budget constraints. Suppose a higher-ranked advertiser's budget exceeds a specific threshold. In that case, the lower-ranked advertiser loses the opportunity to go to first place, thus decreasing their bid, reducing the publisher's overall revenue.

The methodology adopted in this study is an extension of the research conducted by Kucukaydin et al. (2020), that creates a tractable analytical model that examines the competitive nature of search-based advertising. However, this thesis differs from Kucukaydin et al. (2020) in terms of research focus and modeling approach. Kucukaydin et al.'s (2020) objective is to understand how to bid on a group of keywords in an effort to maximize potential revenue generation while respecting budgetary restrictions. They model the situation as a stochastically driven nonlinear optimization problem. By contrast, this thesis introduces a competition parameter modeled with a beta random variable, focusing on how advertisers' budget constraints affect their profits under Nash equilibrium and competition. Küçükaydın (2020) utilizes the exhaucanust competition parameter in their study. Most other studies, in comparison, may not consider this specific parameter or its implications on budgetary constraints in competitive environments.

The rest of the thesis is organized as follows: Chapter 3 gives background information on the first price auction by explaining the impacting factors of search engine advertising, which are frequently used terms throughout the article, and describes the ad bidding model & its formulation. Chapter 4 presents the model of a single keyword in a homogeneous market with two advertisers with the impact of budgets on advertiser profits. Chapter 5 extends the discussion to a heterogeneous market, again with a single keyword and two advertisers. This chapter offers quantitative examples to support the theoretical concepts presented. Finally, Chapter 6 concludes the thesis by summarizing the key findings and paves the way for future research.

Chapter 3

SEARCH ENGINE ADVERTISING DECISIONS: AUCTIONS AND MODEL FORMULATION

This chapter focuses on the search engine advertising decisions: first price auctions and model formulation. Our aim is to provide a clear understanding of how first price auction system works and the key factors that affect its results. Understanding these dynamics is essential as it supports the strategic decisions made by advertisers when bidding on keywords. This analysis aims to show the effects of parameters that we could not consider in this thesis, specifically within the context of the FPA model. Thereafter, we will present the ad bidding model and it's formulation that encapsulates budget, bid price, impression, CTR and clicks. This formulation will serve as a practical framework for understanding and predicting bidding behavior in first-price auctions, specifically in the context of a Nash equilibrium.

The chapter is organized as follows. Section 1 provides an overview of the FPA. Section 2 discusses the key factors that affect the FPA. Section 3 presents the ad bidding model & it's formulation.

3.1 First Price Auction

First-price auction is a key part of auction theory and have broad applications across various fields, from traditional auctions of art and antiques to contemporary digital advertising platforms. In these auctions, multiple bidders compete for one item by submitting a confidential bid. The advertiser with the highest bid wins, paying the exact amount they bid to secure their advertisement's position on a search engine. This differs from second-price auctions, where the highest bidder also wins, but they pay the amount of the second-highest bid. This distinction introduces a strategic element to the bidding process, requiring bidders to consider not only their own valuation of the ad space but also the potential bids of others.

In a first-price auction, bidders are often motivated to bid slightly below their true valuation. This is because the winning bidder's profit is the difference between their valuation and their bid. Therefore, by bidding less than their true valuation, bidders can potentially increase their profit. However, this strategy is not without its risks. Bidders must strike a balance between increasing their potential profit and the risk of losing the auction by bidding too low.

In contrast, in second-price auctions, the strategic considerations are less complex. The dominant strategy for each bidder is to bid their true valuation, eliminating the need for strategic bidding. This difference in strategic complexity between first-price and second-price auctions can influence the choice of auction format in different contexts.

Determining the optimal bid in a first-price auction is a complex task. It depends on the bidder's valuation of the item, their beliefs about other bidders' valuations, and their risk tolerance. In a homogeneous first-price auction, where all bidders share the same valuation distribution, the Nash equilibrium bidding strategy suggests bidding less than one's valuation. The exact bid under this strategy depends on the number of bidders and the distribution of valuations.

However, many real-world auction scenarios are not homogeneous, and bidders may have different valuation distributions or varying amounts of information. The optimal bidding strategy can be more intricate in these heterogeneous auctions. This complexity is further amplified by the potential for collusion among bidders, which can significantly alter the dynamics of the auction.

These complexities make first-price auction a rich and active area of research in auction theory. Search engines often opt for first-price auctions for several reasons. One advantage is their simplicity and ease of understanding, encouraging advertiser participation. Additionally, first-price auctions can potentially generate higher revenue for the search engine, as advertisers may be willing to bid more in a first-price auction than in a second-price auction. Theoretical models of bidding behavior in first-price auctions continue to evolve, with ongoing research seeking to better understand and predict the strategies employed by bidders.

This thesis discusses first-price auctions due to their common use in search engine advertising platforms. Since many search engines employ first-price auctions to allocate advertising space, understanding the underlying dynamics of this auction type is relevant to the subject matter. The examination of first-price auctions lays the groundwork for the calculations and analyses that follow in the thesis. By exploring the strategic considerations and complexities of first-price auctions, we provide context that aligns with online advertising practices. This alignment helps to ensure that the insights and applications presented in the thesis are applicable to the field of digital advertising.

3.2 Impacting factors on Search Advertising

Search advertising, a critical component of the digital marketing ecosystem, is a multidimensional process influenced by various parameters. A search advertising campaign's efficacy and success depend not only on the advertiser's bid, which, while indisputably important, is only one piece of a larger, more complex picture. The advertisers bid, a critical determinant of the ad's placement and visibility, is influenced by the competitive landscape of advertisers. However, because this study primarily focuses on advertisers bidding behavior and competition among them, we will not dig into the specifics of this parameter here.

The ad's genuine success is determined by the harmonious interaction of various crucial parameters, each of which carries a special significance and contributes to the ad's overall impact. One of the parameters is the number of clicks the ad receives, which indicates the user's interest and engagement. The number of times an ad is viewed, known as impressions, serves as a testament to its visibility and the extent of its reach within the digital ecosystem. Its potential impact is increased through keyword selection, guaranteeing the ad's accessibility to the most fitting user. The strategic positioning of the ad on the search engine results page is a critical factor that can significantly affect the click-through rate and, consequently, the campaign's effectiveness. Further, the CTR directly influences the campaign's engagement and ROI. Lastly, the final parameter we will mention in the scope of this research is the keyword selection enables advertisers to target the most pertinent users and foster engagement.

These parameters, collectively functioning, determine the course and outcome of the search advertising campaign. They influence its reach, engagement, visibility, and, ultimately, its ability to deliver a substantial ROI. Hence, the success of a search advertising campaign is not solely determined by the competitive bid, but rather it is the result of a harmonious blend of these numerous impactful factors.

In the context of this thesis, the exploration of various parameters influencing search advertising campaigns serves to acknowledge the multifaceted nature of digital marketing. This thesis primarily focuses on the bidding behavior and competition among advertisers. However, it's crucial to understand that these behaviors exist within a larger framework. While this thesis doesn't delve into every detail, the parameters discussed in this section provide a comprehensive view of the real-life complexities that shape search advertising. We establish a foundational understanding that informs the subsequent analyses by describing these parameters. This acknowledgment contextualizes the model presented in this thesis, offering a perspective on the various factors that may contribute to the success of a search advertising campaign without claiming to capture all the complexities of the real-world scenario.

3.2.1 Keyword Research and Selection

Choosing keywords holds significant importance in search engine advertising and demands thoughtful deliberation. This process involves the selection of suitable search terms that potential customers are anticipated to employ when looking for goods or services similar to those offered by the advertiser. Jansen et al. (2011) emphasize the importance of understanding user behavior as they interact with search engine results, which can aid in selecting relevant keywords [23].

The selected keywords should ideally have a high search volume, indicating that users frequently include in their search queries. A study by Zhang and Feng (2011) suggests that search engine marketing can provide better ROI when high-volume keywords are used [59]. However, the competitiveness of the keywords, or keyword difficulty, should also be considered. A paper by Feng et al. (2007) highlights the importance of considering the competitiveness of keywords in the generation process for search engine advertising [16].

The strategic selection of keywords is a fundamental aspect of search advertising that extends beyond the ad copy, the main text of an ad. This selection directly influences the optimization of the landing page, a significant component of the campaign. The landing page's content should align with the chosen keywords to effectively meet the user's search intent. This alignment not only validates the relevance of the advertisement but also enhances the user experience.

Moreover, the clear and concise presentation of the product or service being offered, in line with the keyword strategy, strengthens the effectiveness of the search advertising campaign. Search engines use a metric known as the quality score to assess the quality and applicability of selected keywords and PPC advertisements. This score impacts the ad's position on SERP. As Thurow (2007) and Zhou et al. (2008) indicate, to achieve a high rank on search engines, landing pages must contain keyword phrases that relate to the ad, further emphasizing the importance of strategic keyword selection in search advertising [61].

In summary, the strategic selection and optimization of keywords constitute vital elements in the landscape of search engine advertising. While this thesis has not delved into the domain of keyword optimization, it is imperative to acknowledge this aspect as a significant factor affecting search engine advertising. Numerous studies by other authors have dedicated attention to this subject, emphasizing its role in influencing user behavior, ad visibility, and overall effectiveness. Although outside the scope of this thesis, the recognition of keyword optimization and its impact on search advertising enriches the understanding of the multifaceted dynamics at play. It serves as a testament to the complexity of the field and highlights potential avenues for future research and exploration. The exclusion of this dimension from the present thesis does not diminish its importance but rather underscores the vast and varied nature of the subject matter, providing opportunities for inclusion in subsequent studies.

3.2.2 Position

An ad's position in SERP can significantly impact its effectiveness and the ROI for the advertiser. From both the competitor's and the advertiser's perspective, a higher ad position can lead to increased visibility and clicks, potentially resulting in a larger market share or a higher ROI [35].

The impact of ad position on CTR is well-documented in the literature. For instance, a study by Richardson et al. (2007) observes that the likelihood of an ad being clicked on drops significantly with its position [46]. This aligns with a study reported by Search Engine Journal, ads positioned first in search results experience a significant drop in average CTR after the top position. Specifically, the second and third positions observed a 15% and 11% click-through rate, respectively [25].

Furthermore, Ghose and Yang (2009) highlights that the final rank of an advertisement has a more significant effect on the CTR and cost per click than the current bid [17]. This suggests that advertisers are willing to pay higher fees to secure a higher ad position, as it can lead to more visibility and clicks.

However, a higher ad position does not necessarily result in more conversions, such as sales or leads, for the advertiser. Goldfarb and Tucker (2011) emphasizes that the profitability of an advertisement is not affected by ad position [19]. This suggests that a higher ad position can lead to more visibility and clicks, but it does not necessarily translate into more conversions.

On the other hand, a study by Su and Zhang (2010) shows that the position of an ad in search results can directly impact the ad's visibility and effectiveness [54]. This suggests that a higher ad position can lead to more conversions and, ultimately, a higher return on investment for the advertiser.

For instance, Google's data reveals that ads in the top position receive nearly three times more clicks than ads in the second position and almost six times more than ads in the third position, highlighting the importance of ad position in generating impressions and illustrating the potential impact of competition on ad performance [21].

Similarly, a study by Bing Ads shows that ads appearing at the top of the page receive twice as many clicks as those in the middle of the page and three times as many clicks as those at the bottom of the page. This further emphasizes the importance of ad position and its direct impact on the number of impressions an ad receives [4].

In conclusion, as we state with references, an ad's position in search results plays an integral role, directly influencing the ad's visibility, effectiveness, and, ultimately, the ROI for the advertiser. Within the context of this thesis, the significance of ad position has been mathematically represented through a Beta distribution in Section 3 Equation (3.3.3), where it is modeled as an X_{jk} random variable. Advertisers' willingness to pay higher fees to secure a more favorable position, leading to increased visibility, clicks, and conversions, underscores the multifaceted relationship between ad position and performance. Consistent with references from various studies, this section aims to provide a perspective that can improve our understanding of ad positioning dynamics in the complex landscape of search engine advertising.

3.2.3 Impression, CTR & Clicks

Three key metrics play a fundamental role in determining the success of an advertising campaign: impressions, clicks, and CTR. These metrics, individually and collectively, provide valuable insights into the performance of an ad, and their optimization is crucial for maximizing the ROI for advertisers.

Impressions, which denote the number of times an ad has been displayed to users, serve as a fundamental measure of an ad's visibility and potential reach. The frequency of impressions not only indicates the reach of an advertising campaign but also reflects the effectiveness of ad targeting strategies. Various factors, including the competition for ad space, can influence impressions. For instance, if a large number of advertisers target the same keywords, the CPM for those keywords is likely to be higher. This can make it more difficult for individual advertisers to secure a high position in the search results, which can in turn affect the number of impressions they receive [17].

Clicks, on the other hand, represent the number of times users have interacted with an ad. The number of clicks an ad receives is a direct measure of user engagement and is often used as a primary signal for the usefulness of ads to users. A study by Anil et al. (2022) emphasizes the importance of ad clicks in CPC advertising systems, where click rate expectations feed directly into value estimation [2].

The CTR, which is the ratio of users who click on an ad to the number of total users who view the ad (impressions), is a key performance indicator in search advertising. CTR is a measure of the relevance and effectiveness of an ad and is used by search engines to determine the quality and relevance of keywords and PPC advertisements. A high CTR indicates a successful ad campaign, as it suggests that a high percentage of users who see the ad, find it relevant and choose to click on it. A study by Zhang et al. (2021) highlights the importance of deep learning models for CTR estimation tasks, emphasizing the need for models that can learn the interactions between each type of auxiliary data and the target ad, to emphasize more important hidden information, and to fuse heterogeneous data in a unified framework [58].

In the context of this thesis, impressions, clicks, and CTR are essential parameters that lay the groundwork for further studies. These parameters are integral to the bidding process and can significantly influence the Nash equilibrium bid price and profit, as will be explored in Chapter 5. This section aims to underscore the importance of these parameters in search advertising and highlight their role in shaping an advertiser's bidding strategy. This understanding will provide a more comprehensive perspective on the bidding process and enrich the subsequent discussions in this thesis.

3.3 The Ad Bidding Model and Its Formulation

Consider an advertiser planning a search-based advertising campaign, where the task involves determining bid prices for a preselected list of keywords. The bid price set for each keyword plays a crucial role in determining the ad´s position on the SERP. The position of the advertisement on the SERP, combined with the number of impressions, collectively impacts the total number of clicks the ad receives.

The search engine provider charges the advertiser for each of these clicks, each of which also holds the potential to generate revenue. The influence diagram in Figure (3.1) first developed by Cholette et al. (2007) shows the decisions and the uncertainties faced by an advertiser who needs to choose the keywords to invest in and to determine how much to invest in each of the selected keywords given the available budget.

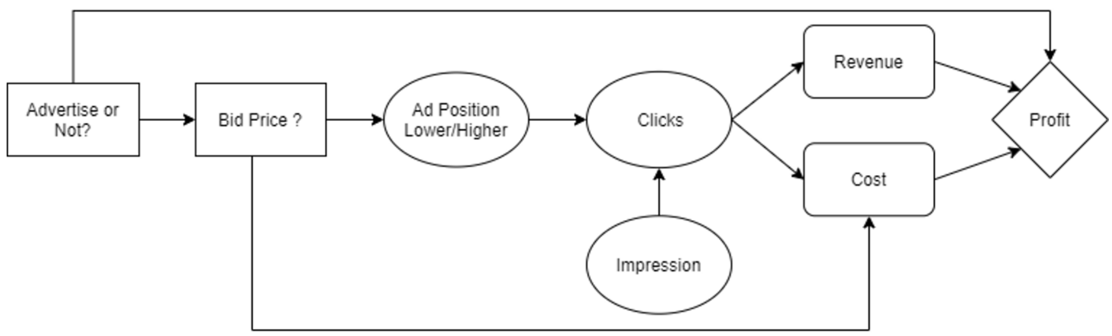

Figure 3.1: Influence Diagram Highlighting Key Components and Interactions in the Bidding Process

Search-based advertising employs a system known as keyword auctions, where advertisers place bids on specific keywords to have their advertisements displayed when those keywords are queried on a search engine such as Google. The most commonly known and used auction type, as discussed in Section (3.1), is the First Price Auction, a mechanism used by Google.

In a search-based advertising auction, advertisers bid on keywords that they want their ads to appear for, when a user conducts a search using those keywords. The auction is typically a real-time process, where the highest bidder's ad is shown at the top of the search results for that keyword. Then players are allocated slots according to the decreasing order of their bids. The advertiser is then charged the amount of their bid for each click on their ad. Some bidders do not get a slot, so do not make any payments. The process allows the search engine to maximize revenue by auctioning off the ad space to the highest bidder, while also providing relevant ads to users.

According to Figure (3.1), the advertiser's bid for a keyword affects the ad's position on the results page when searched by the user, the cost per click, the advertiser's resulting revenue and expenditure. The position of the ad on the page and the number of users that the search engine shows the given ad determine how often the ad is clicked, which, in turn, determines the revenue from the ad. CTR affects not only ad revenue, but also its cost, because for each click, the advertiser pays the search engine's calculated price for the given ad. The main innovation proposed by Cholette et al. (2012) is the treatment of the ad position as a random variable with parameters consisting of the bid price and a competition-level parameter representing the intensity of the competition faced by the advertiser [8].

Table 3.1: Notation

K	number of keywords
J	number of advertisers
r_{jk}	expected revenue per click for advertiser j for keyword k
i_{jk}	number of daily impressions associated with advertiser j and keyword k
c_{jk}	maximum CTR of advertiser j for keyword k at the top position
X_{jk}	random ad position of advertiser j for keyword k
m	parameter of the relationship between Beta random variable and ad position
b_{jk}	bid of advertiser j for keyword k
d_j	total daily budget of advertiser j

In our extension of the model developed by Cholette et al. (2012), we explicitly formulate competition rather than representing it through a single fixed parameter. As shown in the Table (3.1), our general model considers K keywords and J competing advertisers interested in the keywords being auctioned. We denote r_{jk} as the expected revenue of advertiser j for each click following a search for keyword k, b_{jk} as the bid price associated with keyword k and advertiser j, d_j as the total daily budget for advertiser j. To maintain simplicity and without compromising the generality of the thesis, we proceed with the assumption that all bids and budgets are denominated in the same currency, Turkish Lira (TL). This assumption ensures that monetary units are consistent across all calculations and strategic considerations made by the advertisers. Moreover, i_{jk} denotes the number of daily impressions associated with keyword k and advertiser j and c_{jk} as the maximum click-through rate when the ad of advertiser j is displayed at the top position after a search for keyword k. The maximum click-through rate of advertiser j's ad is then calculated as the product $i_{jk}c_{jk}$ of the number of impressions and the maximum click-through rate at the top position. The realized number of clicks for the ad of advertiser j is proportional to the ad's position on the results page. More precisely, we assume this relationship is as follows where m is a positive scalar and X_{jk} is a Beta random variable.

The daily clicks an ad receives for a keyword is also influenced by the ad's position and can be considered a random variable. We can denote this variable as Z_{jk}, and assume it follows a specific functional form.

$$Z_{jk} = i_{jk}c_{jk}\left(1 - X_{jk}\right)^m \tag{3.3.1}$$

Beta distribution is a very flexible distribution and has an increasing usage in search engine advertising literature. The detailed reason why Beta distribution is good fit for this particular model is explained in Cholette et al. (2012). The Beta density is one of the most flexible densities for modeling an event constrained by two endpoints; 0 referring the top of the page and 1 referring the bottom of the page. A continuous density will be an approximation as the number of sponsored ads on a page is a discrete number that varies, currently Google has 4 ad slots available at the top of the results and remaining in the bottom of the results page, but typically this number ranges between 8 to 12 for such slots [8]. For this problem, the parameters of the distribution of X_{jk} are the bid b_{jk} of advertiser j for keyword k and the competition level represented as the sum of the other advertisers' bids for this keyword are assumed to be continuous. The random variable X_{jk} represents the position of advertiser j's ad on the results page. Its probability density function is

$$f_{jk}(x) = \frac{\Gamma(\sum_{j'} b_{j'k})}{\Gamma(\sum_{j' \neq j} b_{j'k})\Gamma(b_{jk})} x^{\sum_{j' \neq j} b_{j'k} - 1}(1 - x)^{b_{jk}-1} \tag{3.3.2}$$

for $x \in (0, 1)$, where $\Gamma(z) = \int_0^\infty y^{z-1}e^{-y}dy$ is the Gamma function. Small values of this random variable represent a higher, so more favorable position on the page whereas large values indicate a lower, so less favorable position; therefore, the advertiser prefers the density to be concentrated around small values. The expected position of advertiser j's ad on the results page of keyword k is obtained as

$$\mathrm{E}[(1 - X_{jk})^m] = \frac{\sum_{j' \neq j} b_{j'k}}{\sum_{j'} b_{j'k}}, \tag{3.3.3}$$

which increases as competition for keyword k increases, placing the ad closer to the bottom of the page and decreases as the bid b_{jk} placed by advertiser j, placing the ad closer to the top of the page. The expected number of clicks is found by taking the expectation of (3.3.1), leading to

$$\mathrm{E}[Z_{jk}] = i_{jk}c_{jk}\mathrm{E}[(1 - X_{jk})^m]. \tag{3.3.4}$$

For the purpose of analytical tractability within this thesis, we will sometimes adopt the assumption that $m = 1$. Under this assumption, equation (3.3.4) simplifies to equation (3.3.5).

$$\mathrm{E}[Z_{jk}] = i_{jk}c_{jk}\,(1 - \mathrm{E}[X_{jk}])^m = \frac{i_{jk}c_{jk}b_{jk}}{\sum_{j'} b_{j'k}}. \tag{3.3.5}$$

The expected revenue of advertiser j from this ad is

$$r_{jk}\mathrm{E}[Z_{jk}] = \frac{r_{jk}i_{jk}c_{jk}b_{jk}}{\sum_{j'} b_{j'k}}, \tag{3.3.6}$$

so the expected total revenue from the bidding strategy $(b_{jk})_{k=1}^{K}$ is

$$\sum_{k=1}^{K} r_{jk}\mathrm{E}[Z_{jk}] = \sum_{k=1}^{K} \frac{r_{jk}i_{jk}c_{jk}b_{jk}}{\sum_{j'} b_{j'k}}. \tag{3.3.7}$$

The expected total cost of this bidding strategy is calculated as

$$\sum_{k=1}^{K} b_{jk}\mathrm{E}[Z_{jk}] = \sum_{k=1}^{K} \frac{b_{jk}i_{jk}c_{jk}b_{jk}}{\sum_{j'} b_{j'k}}. \tag{3.3.8}$$

The bidding strategy (b_{jk}) is considered feasible for advertiser j if it meets the budget constraint, i.e.,

$$\sum_{k=1}^{K} b_{jk}\mathrm{E}[Z_{jk}] \leq d_j. \tag{3.3.9}$$

It is worth noting that when determining the bidding strategy, the advertiser does not know the number of clicks the ad will receive, and since the realized cost will be proportional to this uncertain number, it will be random (unless the bidding strategy is null or maybe if there is no competition).

The decision problem faced by advertiser $j = 1, \ldots, J$ given the other advertisers' strategies can be formulated as an optimization problem where the decision variables are the bids b_{jk} for keywords $k = 1, \ldots, K$. This decision should be aligned with the objectives of the advertiser (e.g., maximizing the expected revenue, the expected profit, the expected utility) and with the budget constraint.

Kucukaydin et al. (2020) characterize the bidding strategy that maximizes the expected total revenue for a single advertiser with the parameter $m = 1$ by using Karush-Kuhn-Tucker conditions. They assume a fixed value for the competition level [31]. The objective of this thesis is to incorporate competition by modeling the influence of competitors' bidding strategies on an advertiser's cost, revenue, and profit, a topic that will be analyzed in the following chapters. By analyzing the game-theoretic model we develop, we aim to propose efficient ways to obtain the equilibrium strategies for advertisers and focus on understanding the effects of the competition and of the budgetary constraints on the equilibrium. To this end, we will start by considering the special case with one keyword and identical advertisers, then the case with one keyword and different advertisers, and then as a future study move on to the study of multiple keywords.

Chapter 4

HOMOGENEOUS MARKET

A unique and complex dynamic in search engine advertising arises as advertisers compete for visibility and clicks. This chapter explores the strategy of homogeneous advertisers - a market where two advertisers compete for a single keyword. Homogeneity in this context implies that advertisers share identical parameters. With only one keyword and identical advertisers, subscripts in the parameters can be dropped. Accordingly, r represents the expected revenue of an advertiser for each click following a search for the specified keyword, i is the number of daily impressions, and c is the maximum click-through rate when the ad is displayed at the top position after a search for the keyword. The assumption of equal revenue per click for each advertiser may be valid if the advertisers in the market sell an equivalent product (or products) and a potential user visiting each advertiser's site is equally likely to make a purchase. It is also assumed that the advertisers have an equal budget d allocated to this keyword.

This assumption of homogeneity simplifies the market dynamics and allows us to focus on the strategic interactions between the advertisers. The exploration begins with the objective of revenue maximization. Each advertiser, within their budget constraints, seeks to optimize their bidding strategy to maximize the revenue generated from the clicks on their ads. However, the pursuit of revenue is just one part of the equation. The ultimate goal of any business endeavor is profit maximization. Therefore, the analysis extends to profit maximization, where not only the revenue but also the cost associated with each click are considered.

A game-theoretic approach is employed to unravel the strategic considerations and interactions between the advertisers. The bidding process is modeled as a game and seeks to find the Nash equilibrium - a state of the game where no player can benefit from unilaterally changing their strategy, given the strategies of the other players. This equilibrium will provide insights into the optimal bidding strategies for advertisers in this homogeneous market.

In the following sections, a deeper dive into these aspects will shed light on the complex dynamics of search engine advertising from the perspective of homogeneous advertisers. This exploration aims to contribute to understanding online advertising strategies and provide practical insights for advertisers navigating this competitive landscape.

4.1 Revenue Maximization

The pursuit of revenue maximization forms an initial step of search engine advertising strategy. Within their budget constraints, advertisers aim to optimize their bidding strategy to maximize the revenue generated from the clicks on their ads. This process involves a complex interplay of factors, including keyword selection, bid amount, and ad placement.

In this thesis, the mathematical representation for revenue maximization is articulated as follows:

$$\begin{aligned}
\max \quad & ric\frac{b_j}{\sum b_j} \\
\text{s.t} \quad & \frac{{b_j}^2 ic}{\sum b_j} \leq d, \\
& b_j \geq 0 \\
& j \in \{1, 2\}
\end{aligned}$$

The presented model aims to describe an optimization problem faced by advertisers in the context of search engine advertising. The model captures the essence of the trade-off that advertisers face: They aim to maximize their revenue while staying within their budget and ensuring their bid amounts are non-negative. The term r represents the revenue generated per click, i denotes the number of impressions, and c is the click-through rate. The fraction $\frac{b_j}{\sum b_j}$ represents the advertiser's share of the total bid, which is used to determine the ad position. The first constraint $\frac{{b_j}^2 ic}{\sum b_j} \leq d$ ensures that the advertiser does not exceed their budget d. The second constraint $b_j \geq 0$ ensures that each advertiser's bid amount b_j is non-negative, which is a practical requirement in any bidding scenario.

In the context of a homogeneous market, where all advertisers possess the same parameters, the problem of revenue maximization takes on a unique dimension. The following sections will explore this aspect, studying how advertisers can navigate this competitive landscape to maximize their revenue.

To illustrate our approach and results, let's examine the two advertiser situation by adapting the general form given in Equation (3.3.6) and (3.3.8). In this case, the expected revenue of each advertiser is

$$r\mathrm{E}[Z] = ric\mathrm{E}[1 - X_j]^m, \tag{4.1.1}$$

and the expected cost is

$$b_j\mathrm{E}[Z] = b_jic\mathrm{E}[1 - X_j]^m. \tag{4.1.2}$$

As discussed in Section 3, the bid of advertiser $j = 1, 2$, denoted as b_j, and the ad positions X_1 and X_2 follow specific Beta distributions based on the bids. The interplay between the bids of the advertisers and their respective ad positions is further elaborated in that section.

For the sake of analytical tractability, the case $m = 1$ is examined and the expected revenue of advertiser $j = 1, 2$, by adapting the general form given in Equation (3.3.5) to the specific case of two advertisers. The expected revenue for each advertiser is then given by:

$$ric(1 - \mathrm{E}[X_j]) = \frac{ricb_j}{b_1 + b_2} \tag{4.1.3}$$

The expected cost for each advertiser under budget constraint is then given by:

$$b_jic(1 - \mathrm{E}[X_j]) = \frac{icb_j^{\,2}}{b_1 + b_2} \tag{4.1.4}$$

We begin by considering the objective of advertisers to maximize revenue, subject to their respective budget constraints. Prior to identifying the Nash equilibrium, where the strategy profile corresponds to optimal bids (b_1^*, b_2^*), we first seek to determine the best response of advertiser 1 given a specific bid from advertiser 2. This involves identifying the optimal bid price for advertiser 1 b_1^* that maximizes their payoff, given advertiser 2's bid b_2. To facilitate our discussion and ensure clarity, we employ several notations associated with the bidding strategies of advertisers. Table 4.1 provides a comprehensive list of these notations and their respective descriptions, which will be consistently used throughout this thesis.

Table 4.1: Notations used for Best Response Function

$BR_j(b_{j'})$	Best response function for advertiser j in response to the bid price of another advertiser
$R(b_j, b_{j'})$	Revenue of advertiser j when another advertiser bids $b_{j'}$
$P(b_j, b_{j'})$	Profit of advertiser j when another advertiser bids $b_{j'}$
$f_n(b_j)$	the function that gives the bid price for advertiser j when a specific condition n is met
$\overline{b_j}$	Maximum allowable bid for advertiser j in response to another advertiser's bid
${b_j}^*$	Optimal bid for advertiser j in response to another advertiser's bid

In the context of a two-player strategic game, the determination of a player's best response is pivotal to understanding the strategic dynamics at play. The best response essentially captures a player's optimal strategy given the choices of the other player.

Before delving into the proposition, it is assumed that the parameters revenue r, impression i, and click-through rate c are all greater than 0. These assumptions are crucial for the validity of the subsequent propositions and proofs.

In the following proposition, the best response functions $BR_1(b_2)$ and $BR_2(b_1)$ represent the optimal bidding strategies for advertiser 1 and 2, given the bid of the other advertiser. They are derived from the advertisers' objective to maximize their revenue, subject to their budget constraints and the strategic considerations of the bidding game.

Proposition 4.1.1 *The best response functions for advertisers 1 & 2, contingent upon the bid prices set by their respective competitors, are as follows:*

$$BR_1(b_2) = \begin{cases} b_1 = ric - b_2 & \textit{if budget constraint is not binding} \\ \frac{d+\sqrt{d}\cdot\sqrt{d+4b_2ci}}{2ci} & \textit{if budget constraint is binding} \end{cases}$$

$$BR_2(b_1) = \begin{cases} b_2 = ric - b_1 & \textit{if budget constraint is not binding} \\ \frac{d+\sqrt{d}\cdot\sqrt{d+4b_1ci}}{2ci} & \textit{if budget constraint is binding} \end{cases}$$

Proof Let $R(b_1) = \frac{ricb_1}{b_1+b_2}$ represent the revenue function for advertiser 1. Taking the derivative with respect to b_1, we have:

$$\frac{dR(b_1)}{db_1} = \frac{ricb_2}{(b_1+b_2)^2}$$

This derivative is positive when $b_2 > 0$, indicating that the revenue function is increasing in b_1 when $b_2 > 0$. Taking the second derivative with respect to b_1, we have:

$$\frac{d}{db_1}\left(\frac{dR(b_1)}{db_1}\right) = -\frac{2ricb_2}{(b_1+b_2)^3} < 0$$

The second derivative is negative, indicating the function is concave down. This means that although the function is increasing, it is doing so in a decreasing rate. The concavity is more pronounced as b_1 increases as b_2 is fixed.

Thus, we checked that

$$\frac{dR(b_1)}{db_1} > 0$$

therefore $R(b_1)$ is an increasing function.

Let's now check the left hand side of the budget constraint (let $\beta = ic$).

$$g(b_1) = \frac{b_1^2\beta}{(b_1+b_2)}$$

$$\frac{dg(b_1)}{db_1} = \frac{2b_1\beta(b_1+b_2) - b_1^2\beta}{(b_1+b_2)^2} = \frac{b_1\beta + 2b_1\beta b_2}{(b_1+b_2)^2}$$

therefore the consumed budget $g(b_1)$ is also increasing in b_1.

Since $R(b_1)$ is increasing in b_1 and $g(b_1)$ is also increasing in b_1, the constraint must be binding.

$$\frac{b_1^2\beta}{(b_1+b_2)} = d$$

Solving $b_1^2\beta - d(b_1+b_2) = 0$ we obtain as the only positive solution:

$$b^* = \frac{d + \sqrt{d}\sqrt{d+4bci}}{2ci}$$

To obtain the symmetric NE we must have:

$$b_{NE} = \frac{d + \sqrt{d}\sqrt{d+4b_{NE}ci}}{2ci}$$

Solving this equation, we get:

$$b_{NE} = \frac{2d}{\beta}$$

Note that when $b_2 = \frac{2d}{\beta}$,

$$BR(b_2) = \frac{d + \sqrt{d}\sqrt{9d}}{2\beta} = \frac{d+3d}{2\beta} = \frac{2d}{\beta}$$

which verifies the Nash Equilibrium. ■

These best response functions capture the strategic considerations of the advertisers in the bidding game. Given the bid of the other advertiser, each advertiser chooses their bid to maximize their revenue, subject to their budget constraint.

For advertiser 1, the process begins by considering each potential strategy that advertiser 2 might adopt. For every such strategy of advertiser 2, we scrutinize the set of possible strategies available to advertiser 1 to identify which one yields the highest payoff. The outcome of this analysis provides us with advertiser 1's best response function, which maps each of advertiser 2's strategies to the strategy that maximizes advertiser 1's payoff. Conversely, for advertiser 2, the analytical process mirrors that of advertiser 1 but with roles reversed.

Through this rigorous analysis, we obtain a comprehensive understanding of the strategic interdependencies between the two players. These best response functions serve as foundational building blocks in the subsequent search for the game's Nash equilibria, where the strategies of the two players converge in mutual optimality. A Nash equilibrium is a state of the game where no player can improve their outcome by unilaterally changing their strategy, given the other players' strategies.

In our case, a Nash equilibrium is represented by a strategy profile (b_1^*, b_2^*), where b_1^* and b_2^* are the best response bids of advertisers 1 and 2. Specifically, b_1^* is the bid that maximizes the revenue of advertiser 1, given that advertiser 2 bids b_2^*, and vice versa. This means that, at the Nash equilibrium, each advertiser's bid is the best response to the other advertiser's bid.

Proposition 4.1.2 *The revenue-maximizing bid of each advertiser in the unique symmetric Nash equilibrium of the homogeneous case with one keyword and two identical advertisers is*

$$b^* = \frac{2d}{ic},$$

the expected revenue of each advertiser is $\frac{ric}{2}$ *and the expected cost is found by exhausting entire budget for a single keyword.*

Proof Starting with the equation that represents the budget constraint for advertiser 1:

$$\frac{\beta b_1^2}{b_1 + b_2} = d$$

Multiplying both sides by $b_1 + b_2$, we get:

$$\beta b_1^2 = d(b_1 + b_2)$$

Given that the advertisers are homogeneous, their bidding strategies and constraints are identical. Thus, each b_j can be represented as b. Substituting these values into the equation, we have:

$$\beta b^2 = d(2b)$$

Solving for b, we find:

$$b = \frac{2d}{\beta}$$

This equation provides the optimal bid for an advertiser given their budget constraint d and the coefficient β. This result can be obtained by using the KKT conditions. Please see Appendix A.

■

If each advertiser bids the same amount, the expected ad position for each advertiser is 0.5 and the expected number of clicks is $\mathrm{E}[Z] = \frac{1}{2}ic$. The expected ad expense for each advertiser bidding b^* is $b^*\mathrm{E}[Z] = (\frac{2d}{ic})(\frac{1}{2}ic) = d$, meaning that each advertiser bids the maximum possible without violating the budget constraint modelled as a bound on expected expenses. This is an expected result, since there is one keyword and the objective is set as maximizing the expected revenue. Each advertiser allocates all available budget to the keyword under consideration in order to maximize the expected position on the page. This would not necessarily hold if the expenses are included in the objective function.

The proposition is extended to the homogeneous market with j identical advertisers in Proposition 4.1.3.

Table 4.2: Experimental test cases for 2 homogeneous advertisers

Case	Definition	(i_1, i_2)	(c_1, c_2)	(r_1, r_2)	(d_1, d_2)
Case 1	Effect on Bid Price by changing Budget	(10,10)	(5,5)	(3,3)	-
Case 2	Effect on Bid Price by changing Revenue	(10,10)	(5,5)	-	(50,50)
Case 3	Effect on Bid Price by changing CTR	(10,10)	-	(3,3)	(50,50)
Case 4	Effect on Bid Price by changing Impression	-	(5,5)	(3,3)	(50,50)

In our numerical tests we consider relationships of the parameters with the bid price as 4 different cases which defined in Table 4.2. In Case 1, we vary the total budget ranges between 10 and 100 with a step size of 10, and we compute the bid

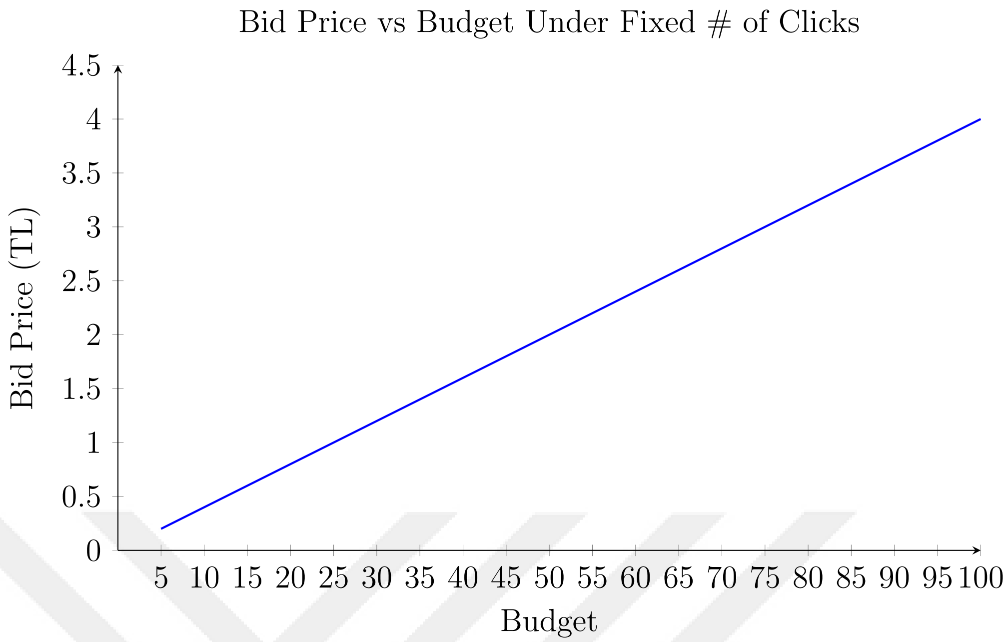

Figure 4.1: Under fixed number of clicks, the bid price of a single keyword increases as the budget increases.

prices of the homogeneous advertiser. Figure 4.1 visually represents the relationship between an advertiser's budget and the bid price for a single keyword. The slope of this line, given by $\frac{2}{\beta}$, signifies the rate at which the bid price changes with respect to the budget. Such a relationship underscores the aggressive nature of the bidding strategy. As advertisers allocate more funds to their advertising campaigns, they are willing to bid higher amounts for keywords to secure prime ad placements. This trend is intuitive, as a higher budget often signifies a greater willingness to invest in advertising, leading to increased competition and, consequently, higher bid prices.

In Case 2, the relationship between bid prices and a set budget contrasts markedly with that observed in Case 1. As depicted in Figure 4.2, an initial surge in clicks corresponds to a pronounced drop in bid prices, likely due to saturation effects. However, as clicks continue to accumulate, this price decline becomes more gradual, hinting at the diminishing influence of additional clicks. This trend underscores the importance of budget allocation and optimization. Advertisers need to strike a balance between securing a high number of clicks and ensuring that the bid price does

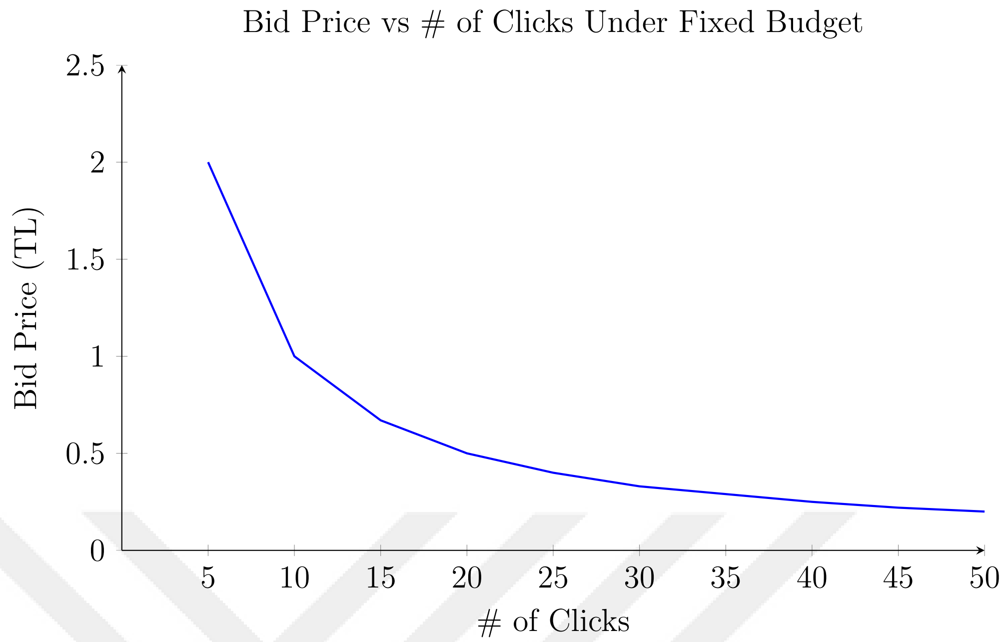

Figure 4.2: Under a fixed budget price, the bid price of a single keyword decreases as the number of clicks increases.

not drop to a level where the ad's visibility or positioning is compromised.

In Case 3, as showcased in Figure 4.3, we explore the relationship between the number of clicks and the revenue an advertiser generates. The graph presents a consistent linear increase in revenue as the number of clicks rises. Given that the revenue parameter r is constant, this uniform growth naturally leads to a direct proportionality between clicks and total revenue. However, beyond the mathematical explanation, this steady progression underscores an interesting point about consumer behavior. If each click, from the first to the fiftieth, contributes equally to the revenue, suggests that every potential customer values the brand consistently. This could happen due to the brand's strong and uniform appeal across its audience.

In Case 4, as illustrated in Figure 4.4, we delve into the relationship between the number of impressions and the bid prices set by advertisers. At first glance, the graph reveals an exponential decline in bid prices as impressions increase. The rapid decrease in bid prices with increasing impressions could be indicative of a market saturation effect. As the number of impressions grows, the incremental value or

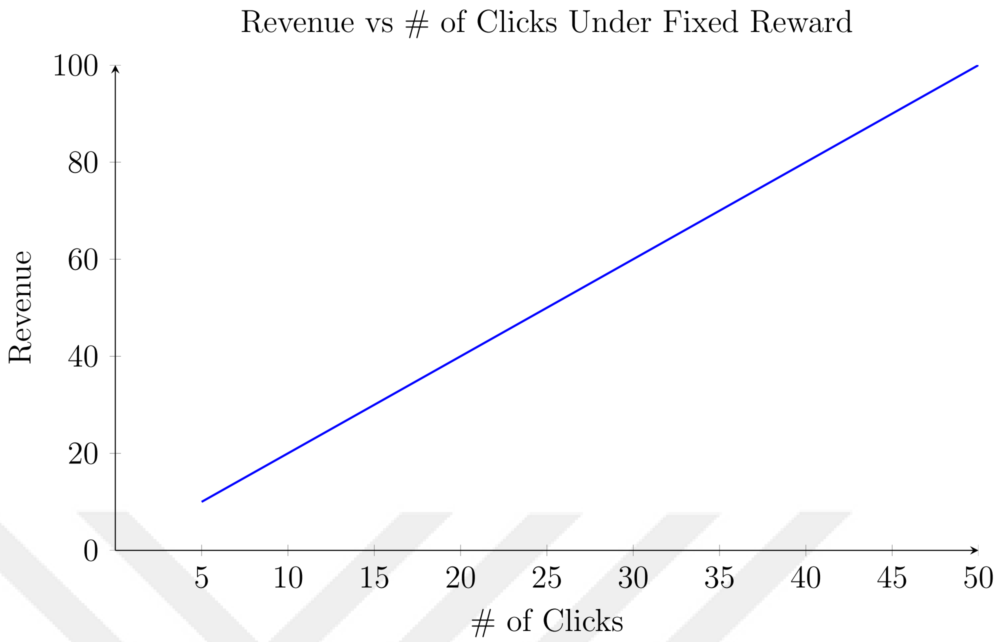

Figure 4.3: Under the fixed reward from a single click, the revenue of a single keyword increases as the number of clicks increases.

impact of each additional impression diminishes. This means that advertisers might be less willing to pay a premium for additional visibility once a certain threshold of impressions is reached.

One may think that similarly to the strategy profile (b^*, b^*), any profile where the two advertisers bid equally will lead to an equilibrium, since the expected revenue will be the same. However, for any other feasible strategy profile, fixing the other's bid, each advertiser can improve the revenue by increasing its own bid slightly. For example, think of the extreme situation where both advertiser bid a minuscule amount, very close to zero. Each will then have the equal expected revenue $0.5ric$ (same as in the equilibrium described in the previous paragraph). However, given that advertiser 2 bids a very small amount, advertiser 1 can raise its bid by any amount without violating the budget constraint, improve the expected ad position and increase the expected revenue. Hence any other symmetric strategy profile is not a Nash equilibrium. In other words, what we obtain above is the unique symmetric Nash equilibrium.

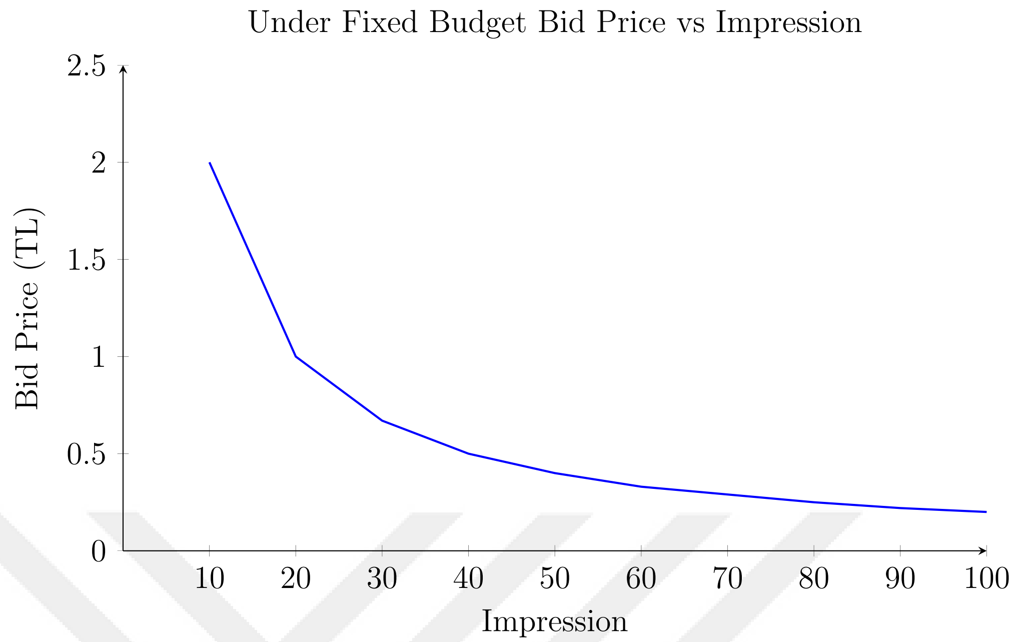

Figure 4.4: Under the fixed reward from a single click, the revenue of a single keyword increases as the number of clicks increases.

As in the case with two advertisers, we use the conditions proposed by Kucukaydin et al. (2020), and replace $b_j = b$ to restrict attention to symmetric equilibrium. With j advertisers bidding the same amount, the expected ad position is found as $1/J$, and the expected number of clicks is ic/J. For the equal bid b^*, the expected ad expense becomes d, meaning that as expected, when there is only one keyword, each advertiser allocates all the budget to it in order to maximize the expected revenue.

Here, it is useful to underline that in presence of multiple competitors, we used to the sum of their bids to represent the level of competition for the ad position. The parameters of the Beta random variable representing the ad position are the advertisers' competitors' total bid and the advertiser's own bid.

Proposition 4.1.3 *The revenue-maximizing bid of each advertiser in the unique symmetric Nash equilibrium of the homogeneous case with one keyword and j identical advertisers is*

$$b^* = \frac{jd}{ic},$$

expected revenue of each advertiser is $\frac{ric}{j}$ and the expected cost equates to the budget allocated for j advertisers. It's important to note that the budget for these advertisers is fully utilized with respect to the expected number of clicks.

Proof The proof of Proposition 4.1.2 can be extended to the case involving j advertisers. Starting with the equation that represents the budget constraint for advertiser 1:

$$\frac{\beta b_1^2}{b_1 + b_2 + \ldots + b_j} = d$$

Multiplying both sides by $b_1 + b_2 + \ldots + b_j$, we get:

$$\beta b_1^2 = d(b_1 + b_2 + \ldots + b_j)$$

Given that the advertisers are homogeneous, their bidding strategies and constraints are identical. Thus, each b_j can be represented as b. Substituting these values into the equation, we have:

$$\beta b^2 = d(jb)$$

Solving for b, we find:

$$b = \frac{jd}{\beta}$$

This equation provides the optimal bid for an advertiser given their budget constraint d and the coefficient β. This result can be obtained by using the KKT conditions. Please see Appendix B.

■

Some Remarks on the case with $m \geq 2$

In Chapter 3, Equation (3.3.5) was derived under the assumption of $m = 1$, a condition that facilitated analytical tractability for the case of two homogeneous advertisers. This assumption served as the foundation for the previous calculations. As part of our ongoing analysis, we seek to explore the effects of increasing m on the calculations. Specifically, we extend the assumption to $m = 2$, considering a scenario with two homogeneous advertisers. By doing so, we aim to evaluate whether and how the increment in m affects the results. Under this expanded framework, the expected number of clicks can be calculated as follows:

$$\mathrm{E}[(1-X_j)^2] = (1 - 2\mathrm{E}[X_j] + \mathrm{E}[X_j^2], \tag{4.1.5}$$

$$b_j ic\left(1 - \frac{2b_{j'}}{b_{j'}+b_j} + \frac{b_{j'}b_j}{(b_{j'}+b_j)^2(b_{j'}+b_j+1)} + \left(\frac{b_{j'}}{b_{j'}+b_j}\right)^2\right) \tag{4.1.6}$$

When $b_{j'} = b_j = b$

$$bic\left(1 - 2\frac{b}{2b} + \frac{b^2}{4b^2+(2b+1)} + \left(\frac{b}{2b}\right)^2\right) \tag{4.1.7}$$

$$bic\left(\frac{1}{4(2b+1)} + \frac{1}{4}\right) = d \tag{4.1.8}$$

$$bic\left(\frac{1}{(2b+1)} + 1\right) = 4d \tag{4.1.9}$$

$$\frac{bic(2b+2)}{(2b+1)} = 4d \tag{4.1.10}$$

$$bic(2b+2) = 4d(2b+1) \tag{4.1.11}$$

$$\frac{(b+1)}{(2+\frac{1}{b})} = \frac{2d}{ic} \tag{4.1.12}$$

whenever $m >= 1$, expected revenue is increasing in b_j given $b_{j'}$.

When considering situations with $m = 2$, higher order moments of the Beta distribution must be used in addition to the first order moment in order to derive an expression for $E[b_j]$, making the problem analytically intractable.

The random ad position is expressed as Beta r.v. X with parameters (a, b) if we want to bid b. The expected value $E[(1-X)^m]$ for $m \geq 1$ is a property of the Beta, which will be connected to the expected number of clicks per time given the ad location $X = x$. This expectation can be calculated using the gamma function $\Gamma(.)$ as follows:

$$\mathrm{E}[(1-X_j)^m] = \frac{\Gamma(a+b)\Gamma(b+m)}{\Gamma(b)\Gamma(a+b+m)}, \tag{4.1.13}$$

from Cholette et. al. (2012).

$\Gamma(b+m) = \int_0^\infty e^{-t}t^{x-1}dt$

Proposition 4.1.4 *$E[(1-X_j)^m]$ is increasing in b for any $m \geq 1$.*

Proof We begin by establishing the relationship $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$. Differentiating the function $e^{-t}t^x$ with respect to t, we obtain:

$$\frac{d}{dt}e^{-t}t^x = -e^{-t}t^x + xe^{-t}t^{x-1}.$$

Integrating both sides from 0 to ∞, we have:

$$\begin{aligned} e^{-t}t^x\Big|_0^\infty &= -\int_0^\infty e^{-t}t^x dt + x\int_0^\infty e^{-t}t^{x-1}dt \\ &= -\Gamma(x+1) + x\Gamma(x). \end{aligned}$$

From which it follows that:

$$\Gamma(x+1) = x\Gamma(x).$$

For the base case $m = 1$, we have:

$$\mathrm{E}[(1-X_j)^1] = \frac{\Gamma(a+b)\Gamma(b+1)}{\Gamma(b)\Gamma(a+b+1)} = \frac{b}{a+b},$$

which is an increasing function of b.

Assuming, as our induction hypothesis, that $\mathrm{E}[(1-X_j)^{m-1}]$ is increasing in b for some $m-1 \geq 1$, we consider:

$$\mathrm{E}[(1-X_j)^m] = \frac{\Gamma(a+b)\Gamma(b+m)}{\Gamma(b)\Gamma(a+b+m)} = \frac{b+m-1}{a+b+m-1}\mathrm{E}[(1-X_j)^{m-1}].$$

The first term on the right-hand side, $\frac{b+m-1}{a+b+m-1}$, is increasing in b. This can be verified by differentiating with respect to b to get $\frac{a}{(a+b+m-1)^2}$, which is positive. Thus, the product of two positive increasing functions is also positive and increasing.

■

4.2 Profit Maximization

The objective of profit maximization is an essential consideration that exceeds the straightforward task of maximizing revenue. While revenue maximization focuses on generating the highest possible income from ad clicks, profit maximization considers the costs associated with each click and seeks to optimize the net gain. This shift in focus from revenue to profit is a more comprehensive approach to advertising strategy, as it considers the advertising campaign's income and expenditure aspects.

Several studies have explored profit maximization in the context of search engine advertising. For instance, Kambhampati (2008) discusses the optimal ad ranking for profit maximization, where individual vendors bid for certain keywords. The study emphasizes the importance of ad positioning and its impact on the campaign's profitability [27].

In this section, we extend the analysis of our game-theoretic model to the case of profit maximization. We consider the same homogeneous market of two advertisers competing for a single keyword. However, instead of merely maximizing revenue, advertisers now seek to maximize their profit, considering the cost per click. This adds another layer of complexity to the strategic interactions between the advertisers, as they must now consider not only the potential revenue from their ads but also the costs associated with them.

The optimization problem is given as follows,

$$
\begin{aligned}
\max \quad & (r - b_j)ic\frac{b_j}{\sum b_j} \\
\text{s.t} \quad & \frac{{b_j}^2 ic}{\sum b_j} \leq d, \\
& b_j \geq 0 \\
& j \in \{1, 2\}
\end{aligned}
$$

In contrast to revenue maximization, which emphasizes gross earnings, profit maximization becomes a focal point, necessitating advertisers to strike a balance between increasing revenues and managing costs. Achieving this balance is crucial for maintaining steady growth and a competitive edge in the marketplace. This problem aims to maximize the profit for an advertiser, denoted by j. The objective function $(r-b_j)ic\frac{b_j}{\sum b_j}$ represents the profit, where r is the revenue per click, b_j is the bid price of advertiser j, i is the number of impressions, and c is the CTR. The term $\sum b_j$ represents the sum of all bid prices in the auction. The constraint $\frac{{b_j}^2 ic}{\sum b_j} \leq d$ ensures that the advertiser's spending does not exceed their budget d. Finally, the non-negativity constraint $b_j \geq 0$ ensures that the bid price cannot be negative. The goal is to find the optimal bid price b_j that maximizes this profit while adhering to the budget constraint and ensuring a non-negative bid.

To shed light on our approach and results, we examine the case of two advertisers. In this framework, the expected profit for each advertiser is articulated as the difference between expected revenue and anticipated cost. Referring back to the revenue maximization section, we adapt Equations (4.1.1) - (4.1.2) to cater to the specifics of profit maximization. The profit function becomes:

$$(r - b_j)\mathrm{E}[Z_k] = (r - b_j)ic\mathrm{E}[1 - X_j]^m, \tag{4.2.1}$$

The expected cost remains consistent with the revenue maximization scenario, as our budget constraint is invariant to the objective shift. The expected cost is given by:

$$b_j\mathrm{E}[Z_k] = b_j ic\mathrm{E}[1 - X_j]^m. \tag{4.2.2}$$

For analytical simplicity, we focus on the case where $m = 1$. Adapting the general form presented in Equation (3.3.5) to the two-advertiser scenario, the expected profit for each advertiser becomes:

$$ric(1 - \mathrm{E}[X_j]) - b_j ic(1 - \mathrm{E}[X_j]) = \frac{ricb_j - icb_j{}^2}{b_1 + b_2}. \tag{4.2.3}$$

We now focus on advertisers' primary objective: maximizing profit while adhering to their individual budget constraints. Our analysis of this objective will be twofold. First, we will examine the unconstrained case, focusing on advertiser 1's best response to a predetermined bid b_2 from advertiser 2. This will involve identifying the optimal bid b_1^* that maximizes advertiser 1's profit, given advertiser 2's strategy, and subsequently characterizing the Nash equilibrium in terms of bid prices (b_1^*, b_2^*). Second, we will delve into the constrained case, analyzing how budget limitations impact both the best response and the Nash equilibrium of the bidding strategies. Finally, we will synthesize these findings to provide a comprehensive understanding of profit-maximizing bidding strategies under both unconstrained and constrained scenarios. For clarity and consistency, we will employ the notations associated with advertisers' bidding strategies as outlined in Table 4.1, which provides a detailed overview of these notations and their implications.

Unconstrained Profit Maximization

Before delving into the complexities of constrained optimization, we first consider the simpler scenario of unconstrained optimization. In this context, advertisers aim to maximize their profits without any budgetary constraints. The focus here is to identify the bid prices that yield the highest profit, taking into account the mathematical properties of the maximization function. For a more detailed understanding, we will refer to Equation (3.3.5). Specifically, we will investigate the case where $m = 1$, under the assumption that the market is a duopoly. Here, b_j represents the bid of each advertiser, where j indexes the advertisers and can take the values 1 or 2.

In the context of an unconstrained optimization problem, the bid price that maximizes revenue can be determined by setting the first derivative of the maximization function to zero. Given the strictly concave nature of the function, as evidenced by its positive second derivative, the optimal bid price can be calculated as the following proposition.

Proposition 4.2.1 *The best response functions for advertisers 1 & 2, contingent upon the bid prices set by their respective competitors, are as follows:*

$$BR_1(b_2) = \begin{cases} b_1^u = \sqrt{b_2 r_1 + {b_2}^2} - b_2 \text{ if budget constraint is not binding} \end{cases}$$

$$BR_2(b_1) = \begin{cases} b_2^u = \sqrt{b_1 r_2 + {b_1}^2} - b_1 \text{ if budget constraint is not binding} \end{cases}$$

Proof For this problem, the random ad position appears in both the expected revenue and expected cost functions is obtained from (3.3.3) as $\frac{b_j}{\sum b_j}$. The expected revenue and expected cost terms are respectively given as $\frac{ricb_j}{b_1+b_2}$, and $\frac{icb_j^2}{b_1+b_2}$ so that, profit becomes , which is $(r-b_1)ic\frac{b_1}{b_1+b_2}$ a strictly concave function with the first and second derivatives,

$$f'(b_1) = \frac{\partial}{\partial b_1}\left(\frac{ic(r_1-b_1)b_1}{b_1+b_2}\right) = \frac{-ic({b_1}^2+2b_2b_1-b_2r_1)}{(b_1+b_2)^2} = 0,$$

$$f''(b_1) = \frac{\partial}{\partial b_1}\left[\frac{-ic({b_1}^2+2b_2b_1-b_2r_1)}{(b_1+b_2)^2}\right] = -\frac{2icb_2(r_1+b_2)}{(b_1+b_2)^3} < 0.$$

Since $f(b_1)$ is strictly concave, its global maximizer is found by solving $f'(b_1) = 0$ which gives $b_1^u = \sqrt{b_2r_1 + {b_2}^2} - b_2$ where $b_2(b_2 + r_1) \geq 0$.

■

Having established the best response functions for both advertisers in Proposition 4.2.1, we now turn our focus to the question of profit maximization in an unconstrained setting. While the best response functions provide insights into each advertiser's optimal strategy given the other's bid, it is also important to understand what these strategies yield in terms of overall profitability. This leads us to the next proposition, where we derive the unconstrained profit-maximizing bid price and its associated expected profit and cost for each advertiser.

Proposition 4.2.2 *The unconstrained profit maximizing bid price is as follows:*

$$b_u^* = \frac{r}{3} \tag{4.2.4}$$

The expected profit for each advertiser is,

$$\frac{ric}{3} \tag{4.2.5}$$

and the expected cost is,

$$\frac{ric}{6} \tag{4.2.6}$$

Proof Given the above proof or Proposition 4.2.1, when we assume that the advertisers are homogeneous, the optimal unconstraint bid b^u simplifies to:

$$b^u = \frac{sqrt(b^2 + br)}{2}$$

From which we can derive:

$$b^2 = \frac{b^2 + br}{4}$$

Leading to:

$$b_u^* = \frac{r}{3}$$

■

By substituting this unconstrained bid price into the profit function, we can determine the expected profit for each advertiser in terms of the given constants. Additionally, this substitution allows us to compute the expected cost associated with this bid. This analytical approach provides a comprehensive understanding of the advertiser's potential returns and expenditures, emphasizing the importance of the bid price in shaping the overall profitability in an unconstrained scenario.

In the context of profit maximization under an unconstrained scenario, determining the optimal bid price solely depends on the revenue generated per click. This is a unique characteristic of the unconstrained case, further amplified by the homogeneity of the advertisers.

The profit and the expected cost for each advertiser become constant values, directly proportional to the number of advertisers. This highlights a fundamental shift in the strategic considerations of advertisers, emphasizing the importance of individual revenue and cost structures in shaping optimal bidding strategies in the pursuit of profit maximization.

As a result, the optimal bid price for two homogeneous advertisers that maximizes profit is a constant $\frac{r}{3}$. This bid price remains invariant, irrespective of any increase in the budget. The rationale behind this constancy is rooted in the principle of profit maximization. In an environment devoid of budgetary constraints, advertisers naturally gravitate towards this optimal bid price to maximize their returns. Thus, even as the budget increases, advertisers have no incentive to bid higher than $\frac{r}{3}$, as it already represents the profit-maximizing bid in the unconstrained setting.

Proposition 4.2.3 *The findings in Proposition 4.2.2 can be generalized to j identical advertisers, the bid that maximizes unconstrained profit for each advertiser is given by:*

$$b_u^* = \frac{(j-1)r}{(2j-1)} \tag{4.2.7}$$

The expected profit for each advertiser is,

$$\frac{ric}{(2j-1)} \tag{4.2.8}$$

and the expected cost is,

$$\frac{(j-1)ric}{(2j^2-j)} \tag{4.2.9}$$

Proof The first step is to take the derivative of the objective function with j advertisers, which gives us:

$$\frac{d}{db_1} = \frac{ic(b_1^2 + 2b_1(b_2 + b_3 + ... + b_n) - r(b_2 + b_3 + ... + b_j)}{(b_1 + b_2 + b_3 + ... + b_j)^2} = 0 \tag{4.2.10}$$

Let's simplify this equation by setting $x = b_2 + b_3 + ... + b_j$, which leads to:

$$b_1^2 + 2b_1 x - rx = 0 \tag{4.2.11}$$

Solving this equation for b_1 yields the optimal bid price for advertiser 1:

$$b_1 = \sqrt{x(x+r)} - x \tag{4.2.12}$$

In the case of homogeneous advertisers, all bids are equal, i.e., $b_1 = b_2 = ... = b_j$. This simplifies the equation for b_1 to (4.2.7). By substituting this optimal bid price into the profit function, we obtain the expected profit for each advertiser, as shown in Equation (4.2.8):

$$\mathrm{E}[P(b)] = (r - \frac{(j-1)r}{(2j-1)})ic\frac{\frac{(j-1)r}{(2j-1)}}{j\frac{(j-1)r}{(2j-1)}} \tag{4.2.13}$$

Finally, the cost function can be articulated using the subsequent equation:

$$\frac{(\frac{(j-1)r}{(2j-1)})^2 ic}{j\frac{(j-1)r}{(2j-1)}} \tag{4.2.14}$$

Simplifying this equation gives us the condition shown in Equation (4.2.9).

■

In the scenario of a single keyword and j identical advertisers, the bid that maximizes unconstrained profit is calculated as $(j-1)$ times the revenue divided by $(2j-1)$. This calculation underscores the intricate relationship between the number of advertisers, their revenue, and the optimal bid price.

The expected profit for each advertiser is then determined by dividing the revenue times number of clicks by $2j-1$, reflecting the proportional distribution of profit among the advertisers.

The expected cost, on the other hand, is calculated as $(j-1)/j$ times the profit. This relationship between the expected cost and profit reveals a critical insight: the expected cost for each advertiser is directly proportional to their profit and inversely proportional to the number of advertisers.

This highlights the complex interplay between profit, cost, and the number of advertisers in shaping the optimal bidding strategies. It underscores the importance of understanding these dynamics in the pursuit of profit maximization in a competitive search engine advertising market with homogeneous advertisers.

Constrained Profit Maximization

Transitioning from the unconstrained profit maximization scenario, we now focus on the more pragmatic case of constrained profit maximization. Advertisers operate under various constraints in the real world, with the budget constraint being the most significant. The budget allocated for advertising is finite, thus imposing a limit on the bidding strategies of the advertisers. This constraint adds an additional layer of complexity to the problem, as advertisers must balance the goal of maximizing profit with the necessity of staying within their budget.

The constrained profit maximization problem necessitates a different analytical approach than the unconstrained case. While the unconstrained scenario allowed for the direct calculation of the optimal bid price, the constrained scenario requires the consideration of the budget limit in determining the optimal bid. This shift in the problem setting highlights the importance of understanding the interplay between revenue, cost, and budget in search engine advertising.

The following paragraphs will explore the constrained profit maximization problem in detail. We will derive the optimal bidding strategies under budget constraints. We will examine how these strategies differ from those in the unconstrained case. This analysis aims to provide valuable insights into the strategic considerations of advertisers operating under tight budget constraints, thereby contributing to a more comprehensive understanding of the dynamics of search engine advertising.

Proposition 4.2.4 *As previously demonstrated in the Proof of Proposition 4.1.1, the consumed budget $g(b_1)$ is increasing in b_1. Therefore,*

$$BR_1(b_2) = \begin{cases} b_1^c = \frac{d+\sqrt{d}\cdot\sqrt{d+4b_2ci}}{2ci} & \text{if budget constraint is binding} \end{cases}$$

$$BR_2(b_1) = \begin{cases} b_2^c = \frac{d+\sqrt{d}\cdot\sqrt{d+4b_1ci}}{2ci} & \text{if budget constraint is binding} \end{cases}$$

Proof As a preliminary note, it is important to clarify that while Proposition 4.1.1 focused on revenue maximization, the current proposition, Proposition 4.2.4, aims at profit maximization. Despite this difference in objectives, the nature of the budget constraint remains unchanged, and as established in the proof of Proposition 4.1.1, the consumed budget $g(b_1)$ is an increasing function of b_1.

■

Proposition 4.2.5 *In the context of a constrained optimization problem, the bid price that maximizes profit can be determined by using budget constraint function. Given the budget constraint d and the parameters b, c, and i, the optimal bid price can be calculated as:*

$$b^* = \frac{d + \sqrt{d} \cdot \sqrt{d + 4bci}}{2ci} \tag{4.2.15}$$

This bid price maximizes the advertiser's profit while ensuring that the budget constraint is not violated.

Proof To validate the equation presented in Proposition 4.2.5, we shall embark on a step-by-step derivation. We commence by considering the budget constraint for the two-advertiser scenario:

$$\frac{{b_j}^2 ic}{\sum b_j} \leq d_j$$

For simplicity and without loss of generality, let's equate the budget constraint to the budget:

$$\frac{{b_1}^2 ic}{b_1 + b_2} = d$$

Rearranging terms, we get:

$$ic{b_1}^2 = d(b_1 + b_2)$$

To simplify this equation further, substitute $ic = \beta$ which denotes the number of clicks and get:

$$\beta {b_1}^2 - db_1 - db_2 = 0$$

Given the quadratic nature of our equation in terms of b_1, we can solve for b_1 using the quadratic formula. Doing so, we arrive at the equation presented in Proposition 4.2.5.

The derived result can further be corroborated using the Karush-Kuhn-Tucker (KKT) conditions, which are necessary conditions for a solution in nonlinear programming to be optimal. A detailed derivation using the KKT conditions is provided in Appendix C.

Unified Analysis of Unconstrained and Constrained Cases

In the preceding sections, we have separately analyzed the scenarios of unconstrained and constrained profit maximization. While each case offers valuable insights, a comprehensive understanding necessitates a unified analysis that bridges these two. This part aims to achieve this by first presenting the best response functions that encapsulate both the unconstrained and constrained cases. Subsequently, we derive the optimal bid prices under these unified conditions and validate these results. We proceed as follows.

Proposition 4.2.6 *We present the best response function for the combined case as follows:*

$$BR_1(b_2) = \begin{cases} b_1 = \sqrt{b_2 r_1 + {b_2}^2} - b_2 \ \textit{if } d > \beta\sqrt{b_2 r_1 + {b_2}^2} - b_2 \\ b_1 = \frac{d+\sqrt{d}\cdot\sqrt{d+4b_2ci}}{2ci} \ \textit{otherwise} \end{cases}$$

$$BR_2(b_1) = \begin{cases} b_2 = \sqrt{b_2 r_1 + {b_2}^2} - b_2 \ \textit{if budget constraint is not binding} \\ b_2 = \frac{d+\sqrt{d}\cdot\sqrt{d+4b_1ci}}{2ci} \ \textit{if budget constraint is binding} \end{cases}$$

Proof The proofs for the individual cases—both unconstrained and constrained—have been rigorously established in the preceding sections. Readers are referred to Proposition 4.2.1 for the proof concerning the unconstrained case and to Proposition 4.2.4 for the proof related to the constrained case.

Let's now check when the budget constraint is binding

$$b_{1,u}^* = \sqrt{b_2 r_1 + {b_2}^2} - b_2$$

The constraint is not binding if $g(b_{1,u}) < d$

$$\frac{b_{1,u}^2\beta}{b_{1,u} + b_2} < d$$

$$b_{1,u}^2\beta - db_{1,u} - db_2 < 0$$

If $d > \beta\sqrt{b_2 r_1 + {b_2}^2} - b_2$ then constraint is not binding. We can now write the Proposition 4.2.6.

■

The decision-making process of a bidder, considering their budget constraint, can be visualized graphically, as shown in Figure 4.7. The x-axis represents different bid prices, while the y-axis represents the profit obtained from these bids. The plotted function, which is concave, represents the profit as a function of the bid price.

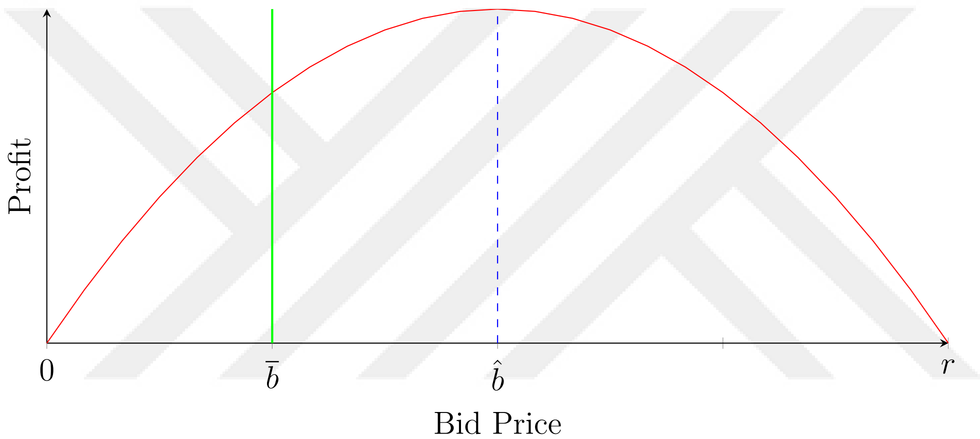

Figure 4.5: The budget constraint precludes bidding at the optimal level

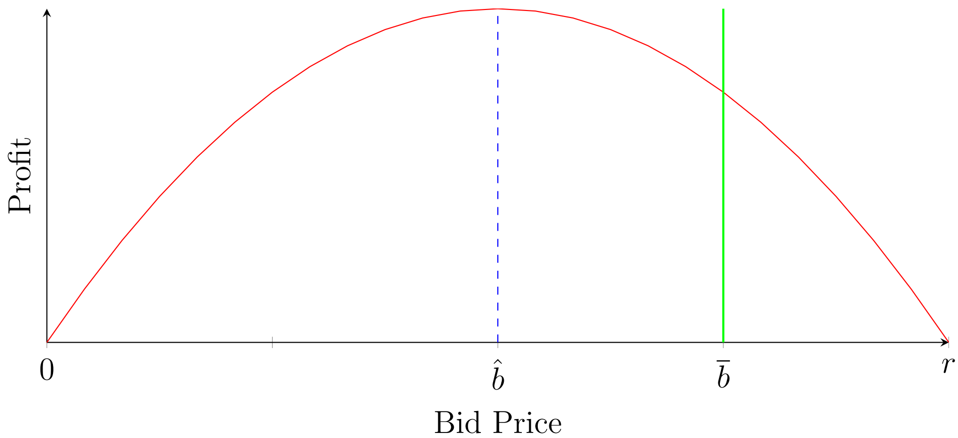

Figure 4.6: The budget constraint permits bidding at the optimal level

Figure 4.7: Profit as a function of the bid price, considering budget constraints

At the lowest end of the bid price spectrum, the bidder is unwilling or unable to submit a bid. As the bid price increases, the potential profit also increases, reaching a maximum at the optimal bid price, denoted by $\hat{b}$ on the x-axis. This bid price yields the highest possible profit for the bidder.

The profit then decreases as the bid price continues to increase, eventually dropping to zero at the revenue limit, denoted by r on the x-axis. If the bidder submits a bid equal to the revenue r, their profit would be zero because the cost of the bid would exactly match the revenue.

In Figure 4.7, the budget constraint is visually represented by the green line, which sets the upper limit for feasible bid prices. This line indicates that while the budget allows bids up to this point, it cannot be exceeded. If the cost associated with the optimal bid price, $\hat{b}$, surpasses the available budget, the bidder must adjust their bid to match the budget, as shown in Figure 4.5. However, this adjusted bid might not be the one that maximizes their profit, highlighting the significant impact of budget constraints on bidding decisions.

When the budget is sufficient enough to accommodate the optimal bid price $\hat{b}$ or even a higher amount, the bidder will naturally opt for $\hat{b}$ to maximize profit. This is visually represented in Figure 4.6 by the portion of the function between $\overline{b}$ and $\hat{b}$, which indicates the range of bid prices a bidder can afford when the budget constraint is relaxed.

The blue dashed line in Figure 4.7, originating from $\hat{b}$ on the x-axis and extending to the function's peak, underscores the optimal bid price, clearly demonstrating that this bid yields the highest profit.

Mathematically, the budget constraint is expressed as:

$$b_1^2 ic = d_1(b_1 + b_2)$$

The determination of the optimal bid price, b^*, is based on the following conditions:

- In the case where $\overline{b} > \hat{b}$, as depicted in Figure 4.6, the optimal bid price is $b^* = \hat{b}$.

- When $\overline{b} < \hat{b}$ and the budget d is sufficient to cover the cost of bidding at $\hat{b}$, the optimal bid remains $b^* = \hat{b}$.

- However, if $\overline{b} < \hat{b}$ and the budget d inadequate to cover the cost associated with $\hat{b}$, as illustrated in Figure 4.5, the optimal bid shifts to $b^* = \overline{b}$.

The present analysis constitutes a basic model, and the actual decision-making process could be subject to a multitude of other factors not encapsulated within this framework.

We can now find the first candidate for N.E (both budgets are larger than a threshold) using symmetry:

$$b_{N.E.} = \sqrt{b_{N.E.}r + {b_{N.E.}}^2} - b_{N.E.}$$

$$b_{N.E.} = \frac{r}{3}$$

For this to be feasible the budget constraint must not be binding:

$$\frac{(\frac{r}{3})^2\beta}{2\frac{r}{3}} < d$$

$$\frac{r\beta}{6} < d$$

To find the second type of N.E. we assume that the budget constraint is binding and solve $\frac{d+\sqrt{d}\cdot\sqrt{d+4bci}}{2ci}$. This yields:

$$b_{N.E.} = \frac{2d}{\beta}$$

Unlike in the unconstrained case, determining the optimal bid price under a budget constraint presents a distinct challenge. In the constraint case, which is shown in Proposition 4.2.2, the optimal bid remains constant. Under budgetary constraints, the bid price starts to increase, but only up to a certain point. This relationship is visualized in the Figure 4.8.

The bid price is variable and is governed by a square root function. This suggests that as the budget increases, the bid price does rise, but at a diminishing rate, reflecting the principle of diminishing returns commonly observed in advertising. Interestingly, the factor of 2β in the denominator, representing the total

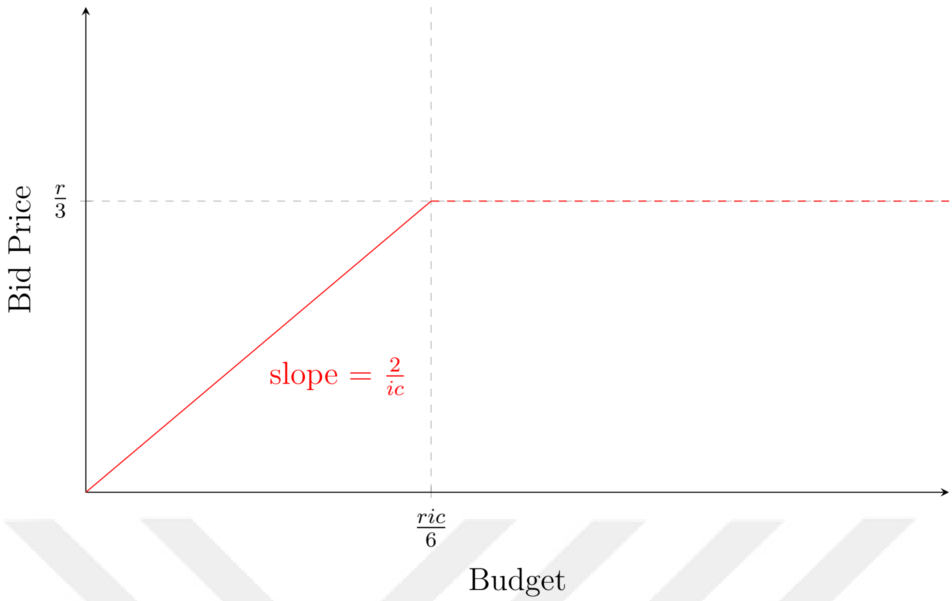

Figure 4.8: The Budget's Effect on Bid Price in the Unified Analysis of Unconstrained and Constrained Profit Maximization

potential number of clicks, indicates that the optimal bid price is inversely related to the number of clicks an ad receives. When faced with a high potential for clicks, advertisers can achieve their profit maximization objectives even with a lower bid, allowing them to conserve their budget for other lucrative opportunities.

Therefore,

$$(b^*_{N.E.}, b^*_{N.E.}) = \begin{cases} \frac{r}{3} \text{ if } d > \frac{r\beta}{6} \\ \frac{2d}{\beta} \text{ otherwise} \end{cases}$$

Chapter 5

HETEROGENEOUS MARKET

In this section, we relax the assumption that all advertisers are identical. Instead, we consider a more general scenario where two distinct advertisers, each potentially differing in their parameters. Given the diversity of advertisers and the presence of only one keyword, we use subscripts in the parameters to denote the distinctions between advertisers.

In a diverse advertising market, these differences in parameters among advertisers can derive from unique marketing strategies, budget allocations, target audience types, and website designs. For instance, some advertisers may focus on creating engaging ads to attract more clicks, leading to variations in click-through rates. Budget differences are also common, with advertisers allocating different amounts to specific keywords based on their financial resources and strategic objectives. Furthermore, the target audience can vary among advertisers, affecting both click-through and conversion rates, as different audience groups exhibit distinct behaviors and preferences. Lastly, the design of an advertiser's website can greatly impact user experience and, consequently, the click-through and conversion rates. A well-designed, user-friendly website can lead to higher rates than a poorly designed one. These factors together contribute to the heterogeneity of parameters in an advertising market.

5.1 Mathematical Analysis of Heterogenous Advertisers

The above heterogeneity introduces additional complexity to the bidding game. The best response function analysis becomes more nuanced in such an advertising environment. Within this landscape, it becomes evident that the approach from Chapter 4 needs to be revised. Given their unique parameters and constraints, the optimal bid that maximizes the return for one advertiser is likely differ from the optimal strategy for another. This underscores the importance of individualized bid strategies and the challenges inherent in navigating a multifaceted advertising ecosystem.

To determine the best response for each advertiser, we need to consider their individual characteristics. For instance, an advertiser with a larger budget may be able to afford higher bids, while an advertiser with a higher valuation for the ad space may be willing to bid more aggressively. Given a specific bid from one advertiser, the best response of the other advertiser is the bid that maximizes their payoff, taking into account their own characteristics and the bid of the other advertiser. This involves identifying the bid price that yields the highest payoff for each advertiser, given the other's bid. In this complex interplay, determining the optimal bid is not merely about outbidding competitors. It is about determining the intricate balance between one's own characteristics and the prevailing bids from competitors.

Building upon the foundations established in Chapter 3's 'The Ad Bidding Model and Its Formulation' and Chapter 4's 'Profit Maximization,' the parameters i_1, i_2, c_1, c_2, and d_1, d_2 are introduced to represent the distinct characteristics of advertisers 1 and 2, respectively. These parameters encapsulate a range of factors, including the number of impressions, CTR, and budgets. For a detailed breakdown and understanding of these parameters, please refer to Table 3.1 in Chapter 3.

The best response functions, denoted as $BR_1(b_2)$ and $BR_2(b_1)$, are formulated to encapsulate the optimal bidding strategies for advertisers 1 and 2, respectively. These functions are predicated on the bid of the other advertiser and the unique characteristics of each advertiser. They are derived from the objective of each advertiser to optimize their individual profit, while adhering to their budget constraints and considering the strategic dynamics of the bidding game. For a detailed exploration of the derivation and application of these best response functions in a homogeneous setting, please refer to Chapter 4.

Proposition 5.1.1 *The best response functions for heterogeneous advertisers, denoted as $BR_1(b_2)$ and $BR_2(b_1)$, can be defined as follows:*

$$BR_1(b_2) = \begin{cases} b_1 = -\frac{1}{3}\frac{b_1c_1i_1}{b_1+b_2} + \frac{c_1i_1(-b_1+r_1)}{3(b_1+b_2)} & \textit{if budget constraint is not binding} \\ b_1 = \frac{d_1+\sqrt{d_1}\sqrt{d_1+4b_2c_1i_1}}{2c_1i_1} & \textit{if budget constraint is binding} \end{cases}$$

$$BR_2(b_1) = \begin{cases} b_2 = -\frac{1}{3}\frac{b_2c_2i_2}{b_1+b_2} + \frac{c_2i_2(-b_2+r_2)}{3(b_1+b_2)} & \textit{if budget constraint is not binding} \\ b_2 = \frac{d_2+\sqrt{d_2}\sqrt{d_2+4b_1c_2i_2}}{2c_2i_2} & \textit{if budget constraint is binding} \end{cases}$$

In Proposition 5.1.1, we delve into the best response functions for heterogeneous advertisers. Given the diversity among advertisers, the best response functions, represented as $BR_1(b_2)$ and $BR_2(b_1)$, become inherently more intricate. This complexity arises due to the unique characteristics and constraints each advertiser brings to the table.

Proof If the budget constraint is not binding the first derivative of the heterogeneous profit maximizing function with respect to b_j is:

$$\frac{d}{db_j}\left((r_j - b_j)i_jc_j\frac{b_j}{b_1+b_2}\right) = -\frac{1}{3}\frac{b_jc_ji_j}{b_1+b_2} + \frac{c_ji_j(-b_j+r_j)}{3(b_1+b_2)} \tag{5.1.1}$$

The second derivative of the function with respect to b_j is:

$$\frac{d^2}{db_j^2}\left((r_j - b_j)i_jc_j\frac{b_j}{b_1+b_2}\right) = -\frac{2c_ji_j}{9(b_1+b_2)} \tag{5.1.2}$$

Given that c_j, i_j, and b_1+b_2 are all positive (assuming positive costs, impressions, and bids), the entire expression is negative. This means that the second derivative is negative for all values of b_j in its domain.

Since $f''(b_j) < 0$ for all b_j, the function $f(b_j)$ is concave with respect to b_j over its entire domain. This implies that any local maximum found by setting the first derivative to zero will be a global maximum for the function.

If the budget constraint is binding:

$$\frac{{b_j}^2 i_jc_j}{\sum b_j} \le d_j$$

For simplicity and without loss of generality, let's equate the budget constraint to the budget:

$$\frac{{b_1}^2 i_1c_1}{b_1+b_2} = d_1$$

Rearranging terms, we get:

$$i_1 c_1 {b_1}^2 = d_1(b_1 + b_2)$$

To simplify this equation further, substitute $i_j c_j = \beta_j$ which denotes the number of clicks and get:

$$\beta_1 {b_1}^2 - d_1 b_1 - d_1 b_2 = 0$$

Given the quadratic nature of our equation in terms of b_1, we can solve for b_1 using the quadratic formula.

■

In this heterogeneous setting, the Nash equilibrium corresponds to the strategy profile (b_1^*, b_2^*) where each advertiser's bid is the best response to the other's bid, given their individual characteristics. This equilibrium reflects the strategic interactions between the heterogeneous advertisers in the competitive bidding environment.

In a homogeneous environment, where all advertisers have similar valuations, a unique equilibrium value can be computed directly. However, in a heterogeneous environment, where advertisers' valuations differ, the task of finding an equilibrium value becomes significantly more complex.

Proposition 5.1.2 *The optimal bid price b_1 for advertiser 1 in a heterogeneous setting, taking into account various factors such as budget constraint d_1, click-through rate c_1, and impressions i_1. The equation aims to capture the complexities of a competitive bidding environment where advertisers have different valuations and constraints.*

$$b_j^* = \frac{d_j + \sqrt{d_j}\sqrt{d_j + 4b_{j'}{}_{c_j i_j}}}{2c_j i_j} \tag{5.1.3}$$

Proof To navigate this complexity, we begin with a two-advertiser scenario and utilize the conditions outlined by Kucukaydin et al. (2020) to characterize the Nash equilibrium. Specifically, the optimal response b_1 of advertiser 1 to a fixed bid b_2 of advertiser 2 is determined by taking the first derivative of budget constraint. Given the concavity and the binding nature of the budget, this differentiation is pivotal. The detailed computation is presented in the proof of Proposition 5.1.1.

■

The computational complexity of this problem escalates substantially when transitioning to a heterogeneous environment, where each advertiser is characterized by distinct variables. This complexity is encapsulated in the equation (5.1.3), which outline the optimal bidding strategies in such a context. These variables could encompass their valuations for different outcomes, such as click-through rates, impressions, conversions, allocated budgets, risk tolerance, and other factors. Moreover, the dynamic nature of search engine advertising, where bids can fluctuate in real-time, introduces an additional layer of complexity. These factors make it difficult to find a unique equilibrium value that satisfies all advertisers.

The challenge lies in the fact that the Nash equilibrium is a function of all these variables. To find the equilibrium, we need to solve a system of equations that includes an equation for each advertiser. As the number of advertisers (and therefore the number of variables and equations) increases, the problem becomes more computationally intensive. This is known as the curse of dimensionality.

Moreover, the problem is further complicated by the fact that these equations are likely to be nonlinear and may include interaction terms, reflecting the fact that an advertiser's optimal strategy depends not only on their own variables but also on the strategies of their competitors. Solving such a system of equations directly is often infeasible, particularly in a real-time bidding environment where decisions need to be made quickly. Therefore, alternative methods need to be explored. These could include approximation techniques, heuristic methods, or machine learning algorithms that can handle high-dimensional, nonlinear problems more efficiently.

To address this challenge, a numerical algorithm has been employed that provides a practical approach to finding the Nash equilibrium in a heterogeneous environment. This algorithm, which is detailed in Appendix D as a pseudocode, handles the high-dimensional, nonlinear nature of the problem by using a discrete grid and exhaustively checking whether each cell in the grid is a N.E. or not. The pseudocode provides a step-by-step procedure for calculating the optimal bidding strategies for both advertisers, taking into account their individual parameters and the strategies of their competitors.

The development of this algorithm offers a valuable perspective in understanding the intricate dynamics of search engine advertising in a heterogeneous environment. While it may not fully capture the complexities of real-world scenarios, it serves as a useful theoretical tool for advertisers. In the sections that follow, numerical examples will be presented to demonstrate the algorithm's application. These examples will specifically focus on illustrating the impact of various parameters on bid prices and profits within the constraints of the model's assumptions.

5.2 Numerical Experiments

In the preceding chapters, we delved deep into the theoretical construct of advertising strategies for homogeneous and heterogeneous advertisers. However, the complexity of real-world conditions and the computational intensity of the problem constrained us from extensively exploring the heterogeneous case.

Here, we will pivot from the more theoretical aspects of our research, instead venturing into a hands-on, empirical approach. This chapter presents a numerical experiment designed to calculate the Nash equilibrium. Our numerical experiment includes an algorithm that can compute the Nash equilibrium of the game, given specific parameters regarding the advertisers and the keyword. While this experiment focuses on a pair of homogeneous and heterogenous advertisers for a singular keyword, the findings from this analysis will serve as an essential stepping-stone for future research in this field. It will pave the way for more sophisticated analyses, potentially tackling the computationally intense heterogeneous case with multiple advertisers.

This chapter also underscores the significance of computational methodologies in bridging the gap between theoretical economic constructs and their practical application, especially in the context of modern digital advertising. To facilitate a comprehensive understanding of the variables involved in our analysis, we provide a detailed explanation of each in Table 5.1.

Table 5.1: Explanation of Variables

Variable	**Description**
$r(A_1)$	Revenue per click accrued by advertiser one
$i(A_1)$	Number of impressions received by advertiser one
$c(A_1)$	CTR achieved by advertiser one
$d(A_1)$	Budgetary constraint imposed on advertiser one
$P^*(A_j)$	Profit obtained by advertiser j when advertisers are in Nash equilibrium
$b^*(A_j)$	Optimal bid price for advertiser j when advertisers are in Nash equilibrium
$\Delta b(A_j)$	Change in the bid price for advertiser j
$\Delta P(A_j)$	Change in the profit for advertiser j

Lastly, the development of our numerical model will be outlined, discuss the theoretical underpinnings of the algorithm used, and present some initial findings from our numerical experiment. Through this process, we hope to provide deeper insights into the intricate dynamics of advertiser behavior in search engine advertising.

5.2.1 Impact of Revenue Parameter r on Bid Prices and Profits

In the context of ad auctions, the revenue parameter r represents the potential revenue an advertiser can gain from a user's engagement with an advertisement, such as by clicking on the ad. The value of r is crucial as it shapes the bidding behavior of advertisers. A larger r value signals greater potential returns from a user's engagement with the advertisement, which motivates higher bidding by advertisers. Higher bids can lead to more prominent ad placements, increasing the likelihood of user engagement and, thereby the potential revenue. As a result, changes in r can substantially influence the Nash equilibrium outcomes regarding bid prices and profits. This study aims to explore these dynamics systematically. Specifically, we investigate the impact of variations in r on the equilibrium bid prices and profits for both advertisers in the ad auction, thereby providing valuable insights into the strategic interactions in this competitive setting.

In the given profit function, the term $(r_j - b_j)ic\frac{b_j}{\sum b_j}$ signifies the product of the potential revenue per click and the expected number of clicks. Here, r_j is the potential revenue per click, and the remainder of the term calculates the expected number of clicks based on the bid and position. Consequently, an increase in r_j would directly proportion the profit, given that all other variables remain constant. This highlights the crucial role of r_j in shaping the profitability of advertisers.

However, r_j also appears in the term $(r_j - b_j)$, representing the net revenue per click after deducting the bid price. Therefore, while increasing r would increase the profit, the effect might be offset if the advertiser also needs to increase their bid price to maintain the same number of clicks. In a highly competitive environment, a significant increase in r_j might compel other advertisers also to increase their bids, which could reduce the overall profitability despite the higher potential revenue per click. Moreover, advertisers typically have a budget constraint for their ad campaigns. If increasing r_j leads to higher bid prices, advertisers might hit their budget limit faster, which can limit the number of ads they can place and thus their total profit.

In the Nash equilibrium, both advertisers choose their bid prices to maximize their profit, taking into account the other advertiser's bid price. The Nash equilibrium bid prices are therefore influenced by both the profit function and the bid prices of the other advertiser.

When r_j changes, it affects the profit function and, therefore, the optimal bid prices for both advertisers. However, the effect is not straightforward because the advertisers' bid prices also depend on the other advertiser's bid price, and a change in r might lead both advertisers to adjust their bid prices.

Above Table 5.2 shows, as advertiser 1's r increases, advertiser 1's profits at the Nash equilibrium increase, as expected. However, advertiser 2's profits decrease initially, then stabilize. The initial decrease in advertiser 2's profits is due to the increased competitiveness of advertiser 1 as their r value increases. However, after advertiser 1's bid price reaches 1.2, any further increases in advertiser 1's r value do not affect their bid price or advertiser 2's profits. The bid prices at the Nash equilibrium for both advertisers also show interesting dynamics.

$r(A_1)$	$P^*(A_1)$	$P^*(A_2)$	$b^*(A_1)$	$b^*(A_2)$	$\Delta b(A_1)$	$\Delta b(A_2)$	$\Delta P(A_1)$	$\Delta P(A_2)$
1	56.70	56.70	0.3	0.3	NaN	NaN	NaN	NaN
2	136.08	38.88	0.6	0.4	100.00	33.33	140.00	-31.43
3	237.60	32.40	0.8	0.4	33.33	0.00	74.60	-16.67
4	347.68	29.91	0.9	0.4	12.50	0.00	46.33	-7.69
5	463.32	25.92	1.1	0.4	22.22	0.00	33.26	-13.33
6	583.20	24.30	1.2	0.4	9.09	0.00	25.87	-6.25
7	704.70	24.30	1.2	0.4	0.00	0.00	20.83	0.00
8	826.20	24.30	1.2	0.4	0.00	0.00	17.24	0.00
9	947.70	24.30	1.2	0.4	0.00	0.00	14.71	0.00
10	1069.20	24.30	1.2	0.4	0.00	0.00	12.82	0.00

Table 5.2: Change in Bid Price and Profit for advertiser 2 as r for advertiser 1 Increases

Advertiser 1's bid price increases with their r value, as expected. However, advertiser 2's bid price initially increases, then decreases and stabilizes. The initial increase is due to the increased competitiveness of advertiser 1, which forces advertiser 2 to also bid higher. However, as advertiser 1's r value continues to increase, advertiser 2 realizes that they cannot compete and reduces their bid price.

The percentage change in advertiser 1's bid price and profit shows a similar trend to the previous case, where both advertisers' r values were increased simultaneously. The percentage change initially increases rapidly, then slows down, showing the effect of the law of diminishing returns. However, the percentage change in advertiser 2's bid price and profit shows a different trend. The bid price initially increases, then decreases and stabilizes. The profit initially decreases, then stabilizes. This demonstrates that increasing advertiser 1's r value has a negative impact on advertiser 2's competitiveness and profits, but this impact is limited and does not gets lower once advertiser 1's bid price reaches a certain level.

This analysis shows that increasing one advertiser's r value while keeping the other's constant can lead to an imbalance in the game. The advertiser with the higher r value will be more competitive and make higher profits, while the other advertiser will be less competitive and make lower profits. However, there is a limit to this imbalance, as the less competitive advertiser can reduce their bid price to a level where any further increases in the competitive advertiser's r value do not affect their profits.

To summarize, the parameter r represents the value of winning the auction, and increasing this value for one advertiser can give them a competitive advantage. However, the other advertiser can adapt their strategy to limit the impact on their profits. This shows the complex dynamics and strategic interactions in this game.

5.2.2 Influence of Impression Parameter i on Bidding Strategies

The impression parameter, denoted as i, quantifies the estimated number of exposures an advertiser's ad will receive. It is one of the determinants in shaping the competitive landscape of online advertising auctions, as it directly impacts the potential reach and, therefore, the possible returns from winning the auction. This tendency for increased exposure can, in turn, influence the bidding strategies deployed by advertisers. In this section, we delve into the intricate dynamics that unfold as a result of variations in the i parameter. We examine its implications on the Nash equilibrium — the system where no advertiser can unilaterally increase their profit by altering their bid — and, consequently, on the resultant profits accrued by the advertisers.

The effects of changing the impression value on the Nash equilibrium bid prices can be complex and depend on the specific values of the other parameters. In general, an increase in the impression value might decrease the bid prices if the increased exposure leads to a lower need for high bid prices to achieve the same profit level. However, the actual changes in the bid prices depend on the specific form and parameters of the profit function, and a detailed analysis would require a deep understanding of the underlying economics of the advertising market.

In the analysis we performed, as advertiser 1's i value increases, the correspond-

$i(A_1)$	$P^*(A_1)$	$P^*(A_2)$	$b^*(A_1)$	$b^*(A_2)$	$\Delta b(A_1)$	$\Delta b(A_2)$	$\Delta P(A_1)$	$\Delta P(A_2)$
1	15.30	15.30	1.6	1.6	NaN	NaN	NaN	
2	30.60	15.30	1.6	1.6	-5.88	-5.88	106.06	
3	45.90	15.30	1.6	1.6	-5.88	-5.88	54.55	
...	...	...	...	...	...	...	...	
18	267.30	14.85	1.7	1.7	0.00	0.00	5.88	
19	290.70	15.30	1.6	1.6	-5.88	-5.88	8.75	
20	306.00	15.30	1.6	1.6	-5.88	-5.88	8.45	

Table 5.3: Change in Bid Price and Profit for advertiser 2 as i for advertiser 1 Increases

ing profit at the Nash equilibrium increases as well, which is expected given the direct relationship between i and the profit function. On the other hand, advertiser 2's profit remains relatively constant. This suggests that changes in advertiser 1's i value primarily affect advertiser 1's profit, while having minimal impact on advertiser 2's profit. This can be attributed to the fact that advertiser 2's i value remains constant, and thus its profit function is not directly affected by changes in advertiser 1's i value.

The Nash equilibrium bid prices also reveal interesting dynamics. As advertiser 1's i value increases, neither advertiser 1's nor advertiser 2's bid price changes.

This analysis shows that increasing advertiser 1's i value while keeping advertiser 2's constant can lead to significant changes in advertiser 1's bidding strategy and profit, while having minimal impact on advertiser 2. Advertiser 1's increased i value enhances its competitiveness, leading to higher profits. However, the impact on advertiser 2 is limited due to its constant i value and its ability to adapt its bidding strategy in response to advertiser 1's behavior.

5.2.3 Role of Click Through Rate Parameter c in Determining Profits

The CTR c represents the rate at which clicks convert into profitable actions, such as purchases. An increase in c makes each click more valuable, influencing the bids advertisers are willing to submit. In this section, we investigate how variations in c impact the Nash equilibrium bid prices and profits.

In the context of this bidding scenario, the click-through rate c indicates the ad's effectiveness – a higher c value means that a higher proportion of impressions will result in clicks, leading to higher potential revenue. The click-through rate effectively scales the number of impressions i into the number of clicks, which are then converted into revenue by the revenue per click r and the bid price b.

In the profit function, c directly multiplies the term $b_2^2 i$, meaning that an increase in c, all else being equal, will result in an increase in the profit. However, it is essential to note that in the context of a bidding scenario, all else is not equal – an increase in the click-through rate for one advertiser will also affect the denominator of the fraction, $b_1 + b_2$, which may lead to changes in the Nash equilibrium and the optimal bid prices.

From the Nash equilibrium perspective, each advertiser chooses their bid price to maximize their profit, considering the other advertiser's bid price. An increase in c for one advertiser makes their ad more effective, which can lead to changes in the optimal bid prices for both advertisers.

Above Table 5.4 shows, as advertiser 1's c value increases, the bid price at Nash equilibrium initially stays constant but then starts decreasing after $c = 9$. This suggests that advertiser 1 perceives a higher value for each impression as c increases, thus increasing their willingness to bid higher. However, as c continues to increase beyond a certain threshold, advertiser 1 starts decreasing their bid price, which could be a strategic move to maintain profit levels.

Interestingly, advertiser 2's bid price at Nash equilibrium decreases as advertiser 1's c value increases. This suggests that advertiser 2 is becoming less competitive as advertiser 1's c value increases, leading them to bid lower.

Advertiser 1's profit at Nash equilibrium increases as their c value increases,

$c(A_1)$	$P^*(A_1)$	$P^*(A_2)$	$b^*(A_1)$	$b^*(A_2)$	$\Delta b(A_1)$	$\Delta b(A_2)$	$\Delta P(A_1)$	$\Delta P(A_2)$
1	29.70	267.30	1.7	1.7	NaN	NaN	NaN	NaN
2	59.40	267.30	1.7	1.7	0.00	0.00	100.00	0.00
3	89.10	267.30	1.7	1.7	0.00	0.00	50.00	0.00
4	118.80	267.30	1.7	1.7	0.00	0.00	33.33	0.00
5	148.50	267.30	1.7	1.7	0.00	0.00	25.00	0.00
6	178.20	267.30	1.7	1.7	0.00	0.00	20.00	0.00
7	207.90	267.30	1.7	1.7	0.00	0.00	16.67	0.00
8	237.60	267.30	1.7	1.7	0.00	0.00	14.29	0.00
9	267.30	267.30	1.7	1.7	0.00	0.00	12.50	0.00
10	296.73	275.40	1.6	1.7	-5.88	0.00	11.01	3.03
11	335.32	284.28	1.5	1.6	-6.25	-5.88	13.01	3.23
12	362.88	293.76	1.4	1.6	-6.67	0.00	8.22	3.33
13	388.12	303.89	1.3	1.6	-7.14	0.00	6.95	3.45
14	425.60	315.00	1.2	1.5	-7.69	-6.25	9.66	3.66
14	417.97	303.89	1.3	1.6	8.33	6.67	-1.79	-3.53
15	456.00	315.00	1.2	1.5	-7.69	-6.25	9.10	3.66
16	475.20	327.12	1.1	1.5	-8.33	0.00	4.21	3.85
17	504.90	327.12	1.1	1.5	0.00	0.00	6.25	0.00
18	518.40	340.20	1.0	1.5	-9.09	0.00	2.67	4.00
19	547.20	340.20	1.0	1.5	0.00	0.00	5.56	0.00
20	576.00	340.20	1.0	1.5	0.00	0.00	5.26	0.00

Table 5.4: Change in Bid Price and Profit for advertiser 2 as c for advertiser 1 Increases

reflecting the increased value they perceive from each impression. However, the rate of increase in profit declines as c continues to increase, suggesting that the marginal benefit from each additional unit of c is decreasing.

Advertiser 2's profit at Nash equilibrium shows an increasing trend as advertiser

1's c value increases. This could be due to advertiser 2's strategic bidding response lower as advertiser 1's c increases, thereby maintaining their profit levels.

In conclusion, changes in the click-through rate c can significantly alter the dynamics of the bidding game, affecting both the bid prices and profits at Nash equilibrium. Understanding these dynamics can provide valuable insights for advertisers in deciding their bidding strategies.

5.2.4 Effects of Budget Constraint d on Nash Equilibrium

The budget parameter d in our ad auction model delineates the financial capacity of an advertiser to participate in the bidding process. By systematically altering d for advertiser 1, while maintaining advertiser 2's budget constant, we investigate the influence of budget variations on the Nash equilibrium bidding behavior and subsequent profit outcomes for both advertisers.

$d(A_1)$	$P^*(A_1)$	$P^*(A_2)$	$b^*(A_1)$	$b^*(A_2)$	$\Delta b(A_1)$	$\Delta b(A_2)$	$\Delta P(A_1)$	$\Delta P(A_2)$
100	268.70	303.89	1.3	1.6	0.0	0.0	0.0	0.0
110	272.16	293.76	1.4	1.6	7.69	0.0	1.29	-3.33
120	274.35	284.28	1.5	1.6	7.14	0.0	0.81	-3.23
130	267.05	275.40	1.6	1.7	6.67	6.25	-2.66	-3.13
140	267.30	267.30	1.7	1.7	6.25	0.0	0.09	-2.94
150	267.30	267.30	1.7	1.7	0.0	0.0	0.0	0.0
160	267.30	267.30	1.7	1.7	0.0	0.0	0.0	0.0
170	267.30	267.30	1.7	1.7	0.0	0.0	0.0	0.0
180	267.30	267.30	1.7	1.7	0.0	0.0	0.0	0.0
190	267.30	267.30	1.7	1.7	0.0	0.0	0.0	0.0
200	267.30	267.30	1.7	1.7	0.0	0.0	0.0	0.0

Table 5.5: Change in Bid Price and Profit for advertiser 2 as d for advertiser 1 Increases

The analysis presented in Table 5.5 investigates the repercussions of varying the budget for advertiser 1, while maintaining a constant budget for advertiser 2. As the budget allocated to advertiser 1 increases, there is a corresponding rise in the bid price at the Nash equilibrium for advertiser 1. This continues until the budget reaches a certain threshold (130 in this instance). The rationale behind this can be attributed to the increased financial flexibility afforded by a larger budget, enabling advertiser 1 to place higher bids and thereby increasing their probability of winning the auction.

However, an increase in the bid price precipitates a corresponding decrease in the profit levels for both advertiser 1 and advertiser 2. As the bid from advertiser 1 ascends, a greater fraction of the overall cost is allocated to advertiser 1, leading to a subsequent reduction in their realized profits.

Interestingly, upon surpassing a certain budget threshold (130 in this case), further budget increments fail to impact either the bid price or the profits. The bid price at the Nash equilibrium is essentially a function of the intersection of the strategies adopted by advertiser 1 and 2. Hence, once the budget is sufficiently large to no longer restrict advertiser 1's bidding strategy, any subsequent increases do not alter the equilibrium dynamics.

In conclusion, while a larger budget affords an advertiser the ability to place higher bids and potentially secure a larger number of auction wins, this also leads to a contraction in profits due to the higher proportion of the total cost the advertiser has to cover. Once the budget crosses a certain threshold that allows the advertiser's bidding strategy to be unrestricted, further budget increases cease to influence the bidding behavior or the profit levels.

Chapter 6

CONCLUSION AND FUTURE WORK

The results of this thesis have provided an understanding of the dynamics that govern the bidding strategies of advertisers competing for the same keyword within a fixed budget. This thesis has systematically investigated key parameters—revenue r, impression i, click-through rate c, and budget d—and their significant influence on the Nash equilibrium and bid prices.

One of the primary contributions of this research is the exploration of the profit maximization case under competition, shedding light on alternative equilibrium bidding strategies. This thesis made an attempt to address heterogeneous cases, aiming to enhance our understanding of the underlying dynamics. The numerıcal analysis developed to identify Nash equilibria demonstrates a practical orientation for the research, albeit under the assumption of perfect information of all parameters for both advertisers. While many studies touch upon similar themes, this thesis places emphasis on the best response functions of advertisers and presents the Nash equilibrium bid prices in a manner that aims to be clear and useful for practitioners. The key finding is that competition significantly impacts bidding behavior and therefore should not be ignored.

However, the model presented in this thesis does not fully encapsulate the complexities of real-world advertising auctions. As Even-dar et al. (2009) highlighted, optimizing bids to maximize returns in broad-match ad auctions presents a challenging problem due to the broad-match feature. Furthermore, Li et al. (2016) discussed that the dynamic nature of real-world auctions adds another layer of complexity that the model needs to account for. Moreover, external factors such as market trends and user behavior, which were not considered in this study, could significantly influence bidding strategies and outcomes.

For future research, broadening the model to account for these additional complexities would be beneficial. Goel and Munagala (2009) propose using hybrid auctions, which could provide a more realistic representation of real-world auctions. Furthermore, the application of reinforcement learning, as proposed by Zhao et al. (2018), could be a promising approach to handling the complex dynamic environment of sponsored search auctions.

In addition, investigating the impacts of other external factors, such as market trends and user behavior, could yield even deeper insights into the bidding strategies of advertisers. This could be informed by the work of Even-dar et al. (2007) and Feldman and Muthukrishnan (2008), who discuss the role of these factors in sponsored search advertising.

Another potential avenue for future research could be to extend the model to consider multiple keywords and a larger number of advertisers. This would allow for a more comprehensive understanding of the dynamics of keyword bidding in advertising auctions. Furthermore, the use of machine learning algorithms could be explored to handle high-dimensional, nonlinear problems more efficiently, as suggested by the research from WordStream [35].

This research establishes a fundamental understanding of the key dynamics involved in keyword bidding within advertising auctions. The insights provided here are expected to inform more effective bidding strategies for advertisers and stimulate further research in this fascinating area, building upon the work of Goel and Munagala (2009), Even-dar et al. (2009), Li et al. (2016), Feldman and Muthukrishnan (2008), and Zhao et al. (2018).

BIBLIOGRAPHY

[1] Z. Abrams, O. Mendelevitch, and J. Tomlin. Optimal delivery of sponsored search advertisements subject to budget constraints. In *Proceedings of the 8th ACM conference on Electronic commerce*, pages 272–278, 2007.

[2] R. Anil, S. Gadanho, D. Huang, N. Jacob, Z. Li, D. Lin, et al. On the factory floor: Ml engineering for industrial-scale ads recommendation models. *arXiv preprint arXiv:2209.05310*, 2022.

[3] M. R. Baye, B. De los Santos, and M. R. Wildenbeest. Search engine optimization: what drives organic traffic to retail sites? *Journal of Economics & Management Strategy*, 25(1):6–31, 2016.

[4] Bing. The impact of ad position on performance. `https://advertise.bingads.microsoft.com/en-us/insights/the-impact-of-ad-position-on-performance`, 2019.

[5] N. Brooks. The atlas rank report: How search engine rank impacts traffic. *Atlas Institute Digital Marketing Insights*, 2004.

[6] N. Brooks. The atlas rank report ii: How search engine rank conversions. *Atlas Institute Digital Marketing Insights*, 2004.

[7] N Brooks. How search engine rank impacts traffic. *The Atlas Rank Report i*, 2004.

[8] S. Cholette, Ö. Özlük, and M. Parlar. Optimal keyword bids in search-based advertising with stochastic advertisement positions. *Journal of Optimization Theory and Applications*, 152(1):225–244, 2012.

[9] B. Edelman and M. Schwarz. Optimal auction design in a multi-unit environment: The case of sponsored search auctions. Unpublished manuscript, Harvard Business School, 2007.

[10] B. Edelman and M. Schwarz. Optimal auction design and equilibrium selection in sponsored search auctions. *American Economic Review*, 100(2):597–602, 2010.

[11] E. Even-Dar, V. S. Mirrokni, and Y. Mansour. Bid optimization for broad match ad auctions. In *Proceedings of the 18th international conference on World wide web*, pages 231–240. ACM, 2009.

[12] J. Feldman and S. Muthukrishnan. Algorithmic game theory for sponsored search. In *Proceedings of the 1st international workshop on Data mining and audience intelligence for advertising*, pages 1–6. ACM, 2008.

[13] J. Feldman and S. Muthukrishnan. Algorithmic methods for sponsored search advertising. In *Performance Modeling and Engineering*, pages 91–122. Springer US, 2008.

[14] J. Feldman and S. Muthukrishnan. Algorithmic marketing. *arXiv preprint arXiv:1402.3172*, 2014.

[15] J. Feldman, S. Muthukrishnan, M. Pal, and C. Stein. Budget optimization in search-based advertising auctions. In *Proceedings of the 9th ACM Conference on Electronic Commerce*, pages 40–49, 2007.

[16] J. Feng, H. K. Bhargava, and D. M. Pennock. Implementing sponsored search in web search engines: Computational evaluation of alternative mechanisms. In *International Conference on Web Information Systems Engineering*, pages 653–661. Springer, Berlin, Heidelberg, 2007.

[17] A. Ghose and S. Yang. An empirical analysis of search engine advertising:

Sponsored search in electronic markets. *Management Science*, 55(10):1605–1622, 2009.

[18] G. Goel and K. Munagala. Hybrid keyword search auctions. In *Proceedings of the 18th international conference on World wide web*, pages 221–230. ACM, 2009.

[19] A. Goldfarb and C. Tucker. Location, location, location: An analysis of profitability of position in online advertising markets. *Journal of Marketing Research*, 48(6):1057–1073, 2011.

[20] Google. Understanding ad position and ad rank. `https://support.google.com/google-ads/answer/1722122?hl=en`, 2011.

[21] Google. The value of google result positioning. `https://www.google.com/ads/innovations/position.html`, 2019.

[22] D. Hoy, K. Jain, and C. A. Wilkens. A dynamic axiomatic approach to first-price auctions. *arXiv preprint arXiv:1304.7718*, 2013.

[23] B. J. Jansen and S. Schuster. Bidding on the buying funnel for sponsored search and keyword advertising. *Journal of Electronic Commerce Research*, 12(1):1, 2011.

[24] K. Jerath et al. A "position paradox" in sponsored search auctions. *Marketing Science*, 30(4):612–627, 2011.

[25] Search Engine Journal. How google's first page dominates clicks, 2020. Accessed: [August 2023].

[26] J. H. Kagel, R. M. Harstad, and D. Levin. Information impact and allocation rules in auctions with affiliated private values: A laboratory study. *Econometrica: Journal of the Econometric Society*, pages 1275–1304, 1987.

[27] S. Kambhampati. Optimal ad ranking for profit maximization. 2008.

[28] A. J. Kim, S. Jang, and H. S. Shin. How should retail advertisers manage multiple keywords in paid search advertising? *Journal of Business Research*, 130:539–551, 2021.

[29] M. Kionka, O. Mußhoff, M. Ritter, J. Uhlemann, and M. Odening. Optimal reserve prices for land auctions in eastern germany. *Applied Economics Letters*, pages 1–5, 2023.

[30] B. Kitts and B. Leblanc. Optimal bidding on keyword auctions. *Electronic Markets*, 14(3):186–201, 2004.

[31] H. Küçükaydin, B. Selçuk, and Ö. Özlük. Optimal keyword bidding in search-based advertising with budget constraint and stochastic ad position. *Journal of the Operational Research Society*, 71(4):566–578, 2020.

[32] S. Lahaie. An analysis of alternative slot auction designs for sponsored search. pages 218–227, 2006.

[33] B. Leblanc and B. Kitts. Optimal bidding on keyword auctions. *Electronic Markets*, 14(3):186–201, 2004.

[34] D. Levin and J. L. Smith. Equilibrium in auctions with entry. *The American Economic Review*, pages 585–599, 1994.

[35] A. Li. Attribution strategies and return on keyword investment in paid search advertising. *Marketing Science*, 35(3):399–416, 2016.

[36] S. Li, J. Ni, and X. Zhang. Keyword bidding: to be rational or aggressive? In *Proceedings of the 21st ACM international conference on Information and knowledge management*, pages 1247–1256. ACM, 2012.

[37] D. Liu, J. Chen, and A. B. Whinston. A dynamic axiomatic approach to first-price auctions. *Management Science*, 59(9):2113–2130, 2013.

[38] S. Lu, Y. Zhu, and A. Dukes. Position auctions with budget constraints: Implications for advertisers and publishers. *Marketing Science*, 34(6):897–905, 2015.

[39] A. Mehta, A. Saberi, U. Vazirani, and V. Vazirani. Adwords and generalized online matching. *Journal of the ACM (JACM)*, 54(5):22, 2007.

[40] S. Muthukrishnan. Ad exchanges: Research issues. In *Proceedings of the 5th international workshop on Internet and network economics*, pages 1–12. Springer, 2009.

[41] S. Muthukrishnan, M. Pal, and Z. Svitkina. Stochastic models for budget optimization in search-based advertising. *Algorithmica*, 58(4):1022–1044, 2010.

[42] S. Narayanan and K. Kalyanam. Position effects in search advertising and their moderators: A regression discontinuity approach. *Marketing Science*, 34(3):388–407, 2015.

[43] Ö. Ozluk and S. Cholette. Allocating expenditures across keywords in search advertising. *Journal of Revenue and Pricing Management*, 6(4):347–356, 2007.

[44] G. Pin and P. Key. Stochastic user equilibrium in search-based advertising auctions. *Operations Research*, 61(1):133–148, 2013.

[45] M. Regelson and D. Fain. Predicting click-through rate using keyword clusters. In *Proceedings of the Second Workshop on Sponsored Search Auctions*, volume 9623, pages 1–6, 2006.

[46] M. Richardson, E. Dominowska, and R. Ragno. Predicting clicks: estimating the click-through rate for new ads. In *Proceedings of the 16th international conference on World Wide Web*, pages 521–530, 2007.

[47] M. Russell and D. Skillicorn. The economics of the cloud. *Journal of Virtualization*, 2(1):7–18, 2010.

[48] O. Rutz, M. Trusov, and R.E. Bucklin. Modeling indirect effects of paid search advertising: Which keywords lead to more future visits? *Marketing Science*, 30(4):646–665, 2011.

[49] A. Sayedi, K. Jerath, and K. Srinivasan. Competitive poaching in sponsored search advertising and its strategic impact on traditional advertising. *Marketing Science*, 33(4):586–608, 2014.

[50] B. Selçuk and Ö. Özlük. Optimal keyword bidding in search-based advertising with target exposure levels. *European Journal of Operational Research*, 226(1):163–172, 2013.

[51] V. L. Smith. Incentives and behavior in english, dutch and sealed-bid auctions. *Economic Inquiry*, 18(1):1–22, 1980.

[52] Statista. Search advertising - turkey, n.d. Retrieved December 10, 2022.

[53] Statista. Search advertising - united states, n.d. Retrieved December 10, 2022.

[54] C. Su and Z. Zhang. Brand positioning strategy using search engine marketing. *MIS Quarterly*, 34(2):261–279, 2010.

[55] Author Unknown. Mediated collusion. Technical report, MIT Department of Economics, 2023.

[56] H. R. Varian. Position auctions. *International Journal of industrial Organization*, 25(6):1163–1178, 2007.

[57] Q. H. Vuong. Econometrics of first-price auctions. *Econometrica: Journal of the Econometric Society*, pages 953–980, 1993.

[58] Weinan Zhang, Jiarui Qin, Wei Guo, Ruiming Tang, and Xiuqiang He. Deep learning for click-through rate estimation. *arXiv preprint arXiv:2104.10584*, 2021.

[59] Xiaoquan Zhang and Juan Feng. Cyclical bid adjustments in search-engine advertising. *Management Science*, 57(9):1703–1719, 2011.

[60] D. Zhao, X. Zhang, W. Ding, and Y. Chen. Deep reinforcement learning for sponsored search real-time bidding. In *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, pages 1021–1030. ACM, 2018.

[61] D. Zhou, C. Perlich, P. Melville, S. Ma, B. Moorthy, and F. Provost. Predicting ad click-through rates via feature-based fully bayesian sparse learning. In *Proceedings of the 14th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 1079–1088. ACM, 2008.

Appendix A

Proof Based on the results obtained by Kucukaydin et al.(2020), the optimal response b_1^* of advertiser 1 to a fixed bid b_2 of advertiser 2 satisfies the equation

$$\frac{b_1}{b_1+b_2} = 1 - \sqrt{\frac{\lambda_1 b_2}{\lambda_1 b_2 + r}}, \tag{A.0.1}$$

where λ_1 satisfies

$$icb_2\left(\frac{2\lambda_1 b_2 + r}{\sqrt{\lambda_1 b_2}\sqrt{\lambda_1 b_2 + r}} - 2\right) = d. \tag{A.0.2}$$

Similarly, the optimal response b_2 of advertiser 2 to a fixed bid b_1 of advertiser 1 satisfies the equations

$$\frac{b_2}{b_1+b_2} = 1 - \sqrt{\frac{\lambda_2 b_1}{\lambda_2 b_1 + r}} \tag{A.0.3}$$

$$icb_1\left(\frac{2\lambda_2 b_1 + r}{\sqrt{\lambda_2 b_1}\sqrt{\lambda_2 b_1 + r}} - 2\right) = d. \tag{A.0.4}$$

Advertisers are expected to bid the same amount if they are identical, so we focus on symmetric Nash equilibrium where $b_1 = b_2 = b$. By replacing this equality in (A.0.1) -(A.0.4), we reduce the equations characterizing the Nash equilibrium to the following:

$$\sqrt{\frac{\lambda b}{\lambda b + r}} = \frac{1}{2}, \tag{A.0.5}$$

where λ is obtained from the equation

$$icb\left(\frac{2\lambda b + r}{\sqrt{\lambda b}\sqrt{\lambda b + r}} - 2\right) = d. \tag{A.0.6}$$

The first equation leads to $\lambda b = r/3$, and substituting this into the second gives

$$b^* = \frac{2d}{ic}. \tag{A.0.7}$$

■

Appendix B

Proof The proof in Appendix A can be extended to the case involving j advertisers. In this context, we consider the bid prices up to b_j, as delineated between equations A.0.1 to A.0.4. Given that advertisers are anticipated to bid an identical amount, our analysis centers on the symmetric Nash Equilibrium, wherein all bids are equalized, i.e., $b_1 = b_2 = \ldots = b_j = b$. This consideration leads us to the following mathematical expressions:

$$\sqrt{\frac{\lambda b}{\lambda b + r}} = \frac{1}{j}, \tag{B.0.1}$$

where λ is derived from the following equation:

$$icb\left(\frac{j\lambda b + r}{\sqrt{\lambda b}\sqrt{\lambda b + r}} - j\right) = d. \tag{B.0.2}$$

The solution to the first equation yields $\lambda b = r/3$, and the subsequent substitution into the second equation culminates in:

$$b^* = \frac{jd}{ic}. \tag{B.0.3}$$

These equations collectively establish the framework for understanding the bid dynamics within the context of j advertisers operating under a symmetric Nash Equilibrium.

■

Appendix C

Proof We begin by simplifying the profit maximization function, where we substitute $i \cdot c$ with β and $\frac{b_1}{b_1+b_2}$ with y:

$$\begin{aligned} \max \quad & (r-b_1)\beta y \\ \text{s.t} \quad & \beta b_2 \frac{y^2}{1-y} \leq d_j, \\ & 0 \leq y \leq 1 \end{aligned}$$

To solve this constrained optimization problem, we apply the Karush-Kuhn-Tucker (KKT) conditions. The Lagrangian of the problem is:

$$L(y) = (r-b_1)\beta y - \lambda(\beta b_2 \frac{y^2}{1-y} - d_j) \tag{C.0.1}$$

Setting the derivative of the Lagrangian with respect to y equal to zero, we obtain:

$$\frac{\partial L}{\partial y} = \beta(r-b_1) - \lambda\left(-\frac{\beta b_2(y-2)y}{(y-1)^2}\right) = 0$$

Since the objective is to maximize profit, the term $\beta(r-b_1)$ cannot be zero. Therefore, the other term must be zero:

$$\lambda\left(\beta b_2 \frac{y^2}{1-y} - d\right) = 0$$

Solving for y, we get:

$$\frac{y^2}{1-y} = \frac{d}{\beta b_2}$$

Let $a = \frac{d}{\beta b_2}$. Substituting this into the equation, we get a quadratic equation:

$$y^2 + ay - a = 0$$

Solving this quadratic equation for y, we find the optimal y:

$$y^* = \frac{-a + \sqrt{a(a+4)}}{2}$$

Substituting $a = \frac{d}{\beta b_2}$ back into the equation, we find the optimal bid price:

$$b^* = \frac{-d + \sqrt{d} \cdot \sqrt{d + 4bci}}{ci}$$

■

Appendix D

This pseudocode is written in the structured basic style. It represents the key steps and logic of the code used. The goal of the code is to find the Nash equilibrium for the given bidding game.

```
BEGIN
  DECLARE player1_bid, player2_bid AS ARRAYS OF REAL
  SET player1_bid = [0.1, 0.2, ..., 10.0] (increments of 0.1)
  SET player2_bid = [0.1, 0.2, ..., 10.0] (increments of 0.1)
  DECLARE i1, i2, c1, c2, B1, B2, r1, r2 AS REAL
  SET i1 = 18, i2 = 18, c1 = 9, c2 = 9, r1 = 5, r2 = 5, d1 = 150, d2 = 150
  DECLARE profits AS 2D ARRAY OF REAL
  FOR EACH bid1 IN player1_bid
    FOR EACH bid2 IN player2_bid
      SET profits[bid1, bid2] = calculateProfit(bid1, bid2)
    END FOR
  END FOR
  PRINT profits
END

PROCEDURE calculate_profit(player1_bid, player2_bid)
  CREATE player1_bid and player2_bid arrays
  CREATE empty profits array
  FOR EACH bid1 IN player1_bid DO
    FOR EACH bid2 IN player2_bid DO
      CALCULATE profits for bid1, bid2 using the given formulas
      ADD profits to the profits array
```

```
    END FOR
  END FOR
  CONVERT profits array to a DataFrame
  CREATE a pivot table from the DataFrame
  CREATE lists for bids and profits
  FOR EACH cell in the pivot table DO
    EXTRACT profits for first player in current column and second player in curre
    IF profit of current cell is max in column for player 1
       AND max in row for player 2 THEN
      PRINT Nash Equilibrium, Profit for player 1 and 2
    END IF
  END FOR
END PROCEDURE
```