IMPACT OF ADDITIONAL DONORS IN LUNG EXCHANGE

A Master's Thesis

by
DUYGU SİLİ

Department of
Economics
İhsan Doğramacı Bilkent University
Ankara
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DUYGU SİLİ

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist.Prof. Dr. Kemal Yıldız

Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Nuh Aygün Dalkıran

Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assistant.Prof.Dr.Ethem Akyol

Examining Committee Member

Approval of the Graduate School of Economics and Social Sciences

---

Prof. Dr. Erdal Erel

Director

ABSTRACT

IMPACT OF ADDITIONAL DONORS IN LUNG EXCHANGE

SİLİ, DUYGU

M.A., Department of Economics

Supervisor: Assist.Prof.Dr Kemal Yıldız

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In this thesis, we mainly focus on lung exchange. There are significant differences between lung and kidney exchanges. While in kidney exchange patients come with a single donor, lung exchange often requires two donors. Incompatible patient-donor-donor triples can exchange donor lungs with other triples.We consider a model in which some patients bring three donors; one is compatible and two are incompatible with themselves. We design a two stage mechanism to show that bringing additional donor to the exchange pool will improve efficiency of the matching system through increasing number of patients that can be matched. Additionally, we provide a necessary condition that shows when allowing additional donors can also reduce the number of necessary transplant teams to run the exchange sequence, and therefore increase,what we call, implementational efficiency.

Keywords: : Market design, Matching, Lung exchange, Implementational efficiency.

ÖZET

EK BAĞIŞÇILARIN AKCİĞER TAKASINA ETKİSİ

SİLİ, DUYGU

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Assist.Prof.Dr. Kemal Yıldız

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Bu tez çalışmamızda, temelde akciğer takasına odaklanılmaktadır. Akciğer ve böbrek takasları arasında önemli farklılıklar bulunmaktadır. Böbrek takasına hastalar tek bir verici ile katılırken, akciğer takasında iki verici gerekmektedir. Birbiriyle uyuşmaz hasta-verici-verici üçlülerindeki hastalar sahip oldukları verici akciğerlerini diğer üçlülerin verici akciğerleri ile takas edebilirler. Burada bazı hastaların bir tanesi kendileriyle uyumlu ve diğer iki tanesi kendileriyle uyumsuz olan üç tane verici getirebildiği bir model düşünülmektedir.Eşleştirilebilen hasta sayısının artması yoluyla takas havuzuna ek verici getirmenin eşleştirme sisteminin refah verimliliğinin geliştirebileceğini göstermek için iki aşamalı bir mekanizma tasarlanmıştır. Ayrıca, ek vericilere izin verildiği taktirde hem takas sıralamasını çalıştırmak için gerekli nakil ekibi sayısının azalabileceğini hemde bu yüzden uygulama verimliliği olarak adllandırılan kavramın artacağını gösteren gerekli bir koşul verilmiştir.

Anahtar Kelimeler: : Pazar tasarımı, Eşleştirme, Akciğer takası,Uygulama verimliliği.

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CHAPTER 1

INTRODUCTION

Most serious forms of kidney, lung and liver disease can be treated by transplantation. Unfortunately, due to a substantial shortfall of deceased donor organs, transplantation will decrease. Therefore tissue/organ donations from living donors become an important source of organ transplantation. A live donor is a relative or friend of the patient who is willing to donate only if his/her friend gets an organ. However, a patient is often unable to receive the organ of his/her willing healty donor because of incompatibilities. Such incompatibilities can be overcome by exchange(of donors) between patients with incompatible donors. These exchanges become nationwide in the last decade with bringing optimization and market design techniques into kidney exchange. Even though Starnes, Barr, and Cohen [1990] brought living donor lobar lung transplantation two decades ago and it has been especially widespread in JapanSato et al[2014], introduction of living donor lobar lung exchange has not been done until 2014. Ergin, Sonmez, Unver [2014] develop a lung exchange model and introduce optimal lung exchange mechanisms under various logistical constraints. Firstly, we shortly describe the lung exchange model of ESU[2014].

A healty human has five lung lobes:

- Three lobes in the right lung and two lobes in the left lung.
- In a living lobar lung transplantation two donors each donate a lower lobe to the patient to replace patient's dysfunctional lungs
- Each donor must be blood type compatible with the patient and have similar size.

As in the case of kidney exchange , all operations in a lung exchange will be carried out simultaneously. However lung exchange differs from kidney exchange in two key ways:

1. presence of two donors
2. size compatibility

A simplified model is considered with only blood type compatibility. Each patient is defined as a triple of blood types(one for the patient and two for her incompatible donors). Let $\mathcal{B} = \{O, A, B, AB\}$ be the set of blood types with generic elements $X, Y, Z \in \mathcal{B}$. Let $\trianglerighteq$ be the partial order on blood types defined by $X \trianglerighteq Y$ if and only if blood type X can donate to blood type Y. Patient and her donors can be denoted in the form of blood types $X - Y - Z \in \mathcal{B}^3$, where X is the blood type of the patient, and Y and Z are the blood types of the donors.
A lung exchange pool is a vector of nonnegative integers

$$\mathcal{E} = \left\{n\left(X - Y - Z\right) : X - Y - Z \in \mathcal{B}^3\right\}.$$

The number $n\left(X - Y - Z\right)$ denotes the number of patients of type $X-Y-Z$. $X - Y - Z$ and $X - Z - Y$ represents the same type and compatible pairs do not participate in exchange. ESU[2014] denote these properties suct that:

1. $n\left(X - Y - Z\right) = n\left(X - Z - Y\right)$ for all $X - Y - Z \in \mathcal{B}^3$.

2. $n(X - Y - Z) = 0$ for all $X - Y - Z \in \mathcal{B}^3$ such that $Y \trianglerighteq X$ and $Z \trianglerighteq X$.

In this paper, we consider a model in which each patient can provide an additional donor. In other words, some patients bring three donors; one is compatible and two are incompatible with himself. Firstly, we show that bringing extra donors to the exchange pool will improve welfare efficiency of the matching system through increasing number of patients that can be matched. Secondly, we prove that using additional donors in the exchange pool will increase implementational efficiency (reduction of maximum exchange size).We introduce a two stage mechanism such that in the first stage, the mechanism chooses two of patient's three donors as to reach the maximum number of transplantations and in the second stage, optimal lung exchange mechanism introduced by ESU[2014] is applied. To see that bringing additional donors can increase the efficiency let us consider the following example.

Example 1.0.1. Consider an exchange pool with

- 3 blood type B patients and 3 blood type B donors
- 3 blood type A patients and 7 blood type A donors, and
- 2 blood type O donors. Let us define μ as our matching contains:

1. $B - A - A$
2. $A - O - B$
3. $B - O - A$
4. $B - A - A$
5. $A - B - A - B$
6. $A - B - A$

Since a 3- way lung exchange involves nine simultaneous operations and a 2-way lung exchange involves six operations, In this example fifteen transplants can result from the one possible 3-way exchange and one 2- way exchange :

- $(B-A-A, A-B-A, A-B-A)$ and
- $(B-A-A, A-O-B)$.

Only $B-O-A$ triple stays unmatched.

- When three way exchange is obtained, Step 1 of Sequential Matching Procedure of Two and Three Way Exchange is used: two types from the dotted end and one type from non-dotted end.

- When two way exchange is obtained, Step 2 of Sequential matching procedure is used: Match the maximum number of $A-O-B$ types of remaining $B-A-A$ and $B-B-A$ types.

If any one of $A-B-A$ triples brings extra donor of type B and behaves like type $A-B-B$, we can achieve the three possible two way exchanges:

1. $(B-A-A, A-B-B)$
2. $(B-A-A, A-O-B)$
3. $(B-O-A, A-B-A)$

When we implement the outcome of one 3- way exchange + one 2- way exchange with three 2- way exchange, firstly we match $(B-A-A, A-B-B)$ then match remaining $B-A-A$ and $A-B-A$ triples with $A-O-B$ and $B-O-A$ by following sequential procedure for two way exchange.

Before including extra donor in our matching, we could obtain one 3-way exchange, one possible 2-way exchange and only five number of patient-donor pairs could be matched. After bringing extra donor in the exchange model, we can obtain three number of 2- way exchanges and six number of patient-donor pairs can be matched.

As you have seen in the above example, the extra donors can be used by the mechanism in such a way that improve efficiency of everyone in the system. In this paper, our mechanism is a two stage mechanism. We find the details of first stage through seperating it into three different parts. We will show that impact of extra donors changes regarding whether it is two way exchange, two and three way exchange or unrestricted size exchange. When only two way exchange is allowed, number of patients that can be matched will increase under some constraints on the number of extra donors. When two and three way exchange are allowed, bringing extra donors to the exchange pool would increase number of patients that can be matched differently according to supply and demand balance of A and B blood type donors. Finally, when unrestricted size exchange is allowed, we show that extra donors will increase the upper bound to the number of triple that can be matched. Moreover extra donors increase number of donors compatible with B blood type patients that can be supplied by A blood type patients and increase number of donors compatible with A blood type patients that can be supplied by B blood type patients. We obtain a significant theorem by combining these observations. In this theorem, we show that under at most 6 way exchange how many patients can be matched by using extra donors in the mechanism. The final part of the paper gives us two significant results.The first one is optimal number of patients that can be matched under extra donor condition through this theorem and the second one is to follow a necessary condition in order to increase implementational efficiency. When supply and demand equality is satisfied, number of additional A and B blood type donor used in exchange determine how the maximum number of exchange size reduces.

1.1 Literature Review

Medical doctor F.T Rapaport[1986]proposed paired kidney exchange.Roth, Sonmez and Unver [2004,2005,2007] introduced optimization and market design techniques to kidney exchange. The two main sources of kidneys for transplantation are deceased donor kidney and live donations from family and friends. In addition to direct exchange between incompatible pairs, another form of exchange is indirect exchange Ross and Woodle[2003]. In this kind of exchange, the patient of the incompatible pair receives an upgrade in the deceased donor priority list in exchange for donor's kidney. Kidney Exchange with Good Samaritan Donor is the closest study to our model introduced by Sonmez and Unver[2006]. Good Samaritan donor model differs from our model considerably. In good samaritan model, there is a deceased donor priority list and an exchange is among an altruistic living donor(good samaritan donor), two incompatible patient-donor pairs and a patient with highest priority on a deceased donor priority list. In our model an exchange is between patients having at least one incompatible donor.In contrast to Good Samaritan model, there is no deceased priority list and in our model, patients bring three donors instead of two have flexibility to behave two different types. Good Samaritan donor gives a gift of life to a stranger and facilitates two patients and one patient on the waitlist to be matched in return for nothing. In contrast in our model, patients with extra donors can be matched through implementing our mechanism even though they are unmatched before bringing extra donors. In addition the mechanism improves welfare efficiency of the system through increasing the number of patients that can be matched in the exchange pool and increases implementational efficiency through reducing exchange size without changing the number of patients that can be matched in the exchange pool.

CHAPTER 2

TWO WAY LUNG EXCHANGE IN ESU[2014] AND EFFICIENT MECHANISM WITH EXTRA DONORS

2.1 Two Way Exchange in ESU[2014]

We analyze the efficient mechanism with extra donor through using the following lemma and sequential matching procedure of ESU[2014]. So let us describe them.

The following lemma of ESU shows that two patients can participate in two way lung exchange if their donors can be partitioned such that two donors can donate to first patient and the remaining two donors can donate to the second patient.

Lemma 2.1.1 (1)**.** In any given exchange pool $\mathcal{E}$,the only types that could be part of two way exchange are $A-Y-B$ and $B-Y-A$ where $Y \in \{A, B, O\}$.

Proof. Given in ESU[2014] □

There are six type of triples in two way exchange. Every A blood type patient has at least one B blood type donor and every B blood type patient has at least one A blood type donor. Therefore, B blood type patients can

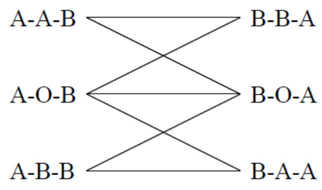

Figure 2.1: Possible Two way Exchange

only take part in an two way exchange with A blood type patients and vice versa.In two way exchange $A-B-A$ and $B-A-B$ types must exchange exactly one donor ; the $A-B-B$ and $B-A-A$ types must exchange both donors; and the $A-O-B$ and $B-O-A$ types might exchange one or two donors. Since O blood type donor can be given both A and B blood type patients. ESU[2014] show that following matching procedure maximized the number of transplants through two way exchanges.

Sequential Matching Procedure for Two way Exchanges:

Step 1:Match the maximum number of $A-A-B$ and $B-B-A$ types.

Match the maximum number of $A-B-B$ and $B-A-A$ types.

Step 2: Match the maximum number of $A-O-B$ types with any subset of remaining $B-A-A$ and $B-B-A$ types.

Match maximum number of $B-O-A$ types with any subset of remaining $A-B-A$ and $A-B-B$ types.

Step 3: Match the maximum number of remaining $A-O-B$ and $B-O-A$ types.

In the next theorem, ESU[2014] show the optimality of this procedure and qualify the maximum number of transplants through two way exchange.

Theorem 2.1.2 (1)**.** Given a lung exchange problem , the sequential two way lung exchange algorithm maximizes the number of two way exchanges. The maximum number of transplants through two way exchange is $2\min\{N_1, N_2, N_3, N_4\}$

where :

$$
\begin{aligned}
N_1 &= n(A-A-B)+n(A-O-B)+n(A-B-B)\\
N_2 &= n(A-O-B)+n(A-B-B)+n(B-B-A)+n(B-O-A)\\
N_3 &= n(A-A-B)+n(A-O-B)+n(B-O-A)+n(B-A-A)\\
N_4 &= n(B-B-A)+n(B-O-A)+n(B-A-A).
\end{aligned}
$$

Proof. Given in ESU[2014] □

2.2 Efficient Mechanism With Extra Donors For Two Way Exchange

In this section, we extend the model of Sonmez, Unver and Ergin as to allow each patient to come up with three donors instead of two. However, for each patient only two of three donors can be used in the exchange process. We find out the efficient mechanism that would achieve maximal number of transplantations under different exchange feasibilities. First, we analyse how efficiency can be improved if any arbitrary number of patients enter the pool with extra donor for the case of 2-way exchange.

The mechanism that we come up with is a two stage mechanism which first finds out how many A type donor and how many B type donor should be selected from among the extra donors. Then the patient - donor profile reduces to that of Ergin, Sonmez, Unver. In particular, only $A-A-B$ and $B-A-B$ type triples must come with extra donors. In other cases, extra donors doesn't make any sense. A blood type patients can bring extra B blood type donors and chooses to behave like $A-B-A$ or $A-B-B$ type in the first stage of the mechanism and B blood type patient can bring extra A blood type donor and chooses to behave like $B-A-B$ or $B-A-A$ in the first stage of our mechanism. Since $A-O-B$ and $B-O-A$ can use their O blood type donor instead of B and A blood type donors. $A-O-B$ and $B-O-A$ triples don't benefit from extra donors. Procedure for two way exchange given by ESU[2014] is used to reach optimality in the second stage.

In the following lemma, we will show that bringing extra one more donor by any number of triples to the exchange system will increase the number of transplants through two way exchange when number of $B-A-B$ type triples our mechanism utilize as $B-A-A$ type triple is greater than number of $A-B-A$ type triples our mechanism utilize as $A-B-B$ type triple in the first stage. Let us define:

$e_a =$ number of $B-A-B$ type triples who brings extra A blood type donor.

$e_b =$ number of $A-B-A$ type triples who brings extra B blood type donor.

For the case of $e_b \geq e_a$, we get the similar result symmetrically.

Lemma 2.2.1. Given a lung exchange problem , the sequential two way lung exchange algorithm maximizes the number of two way exchanges. Suppose that number of extra donors satisfies $e_a \geq e_b$. The maximum number of transplants through two way exchange will increase iff

$$(e_a - e_b) \leq n\,(A-B-B) + (B-A-B) - n\,(A-B-A) - (B-A-A)\,.$$

Proof. We use theorem 2.1.2. Let us consider four cases:

- Case 1: We assume that $\min\{N_1, N_2, N_3, N_4\} = N_1$ is true and we assume that all extra donors are used. N_1 does not change. Since e_b number of $A-B-A$ type triple behave $A-B-B$, number of $A-A-B$ type triple decreases number of e_b and $A-B-B$ type triple increases number of e_b. Therefore there is no change totally. N_2 decreases number of $e_a - e_b$. Since number of $A-B-B$ type triple increases number of e_b and number of $B-B-A$ type triple decreases amount of e_a. The number of N_3 increases amount of $e_a - e_b$. Since number of $B-A-A$ type triple increases number of e_a and number of $B-B-A$ type triple decreases amount of e_b. Finally N_4 stays the same because of similar reason with N_1.

After bringing additional donor $\min\{N_1, N_2-(e_a-e_b), N_3+(e_a-e_b), N_4\} = N_1$ or $N_2-(e_a-e_b)$. If it is equal to $N_2-(e_a-e_b)$ it means that $N_2-(e_a-e_b) < N_1$. We don't want to decrease the number of transplant. Therefore this case must be eliminated.

- Case 2: if $\min\{N_1, N_2, N_3, N_4\} = N_2$ is true, after bringing additional donor the minimum of N_1, N_2, N_3, N_4 equals to $N_2-(e_a-e_b)$. So we also eliminate this case.

- Case 3: if $\min\{N_1, N_2, N_3, N_4\} = N_3$ is true, we will have three subcases:

  (1) $\min\{N_1, N_2-(e_a-e_b), N_3+(e_a-e_b), N_4\} = N_1$: number of patients can be matched increases amount of $2(N_1-N_3) > 0$.

  (2) $\min\{N_1, N_2-(e_a-e_b), N_3+(e_a-e_b), N_4\} = N_2-(e_a-e_b)$:number of patient can be matched increases amount of

  $2(N_2-(e_a-e_b)-N_3) > 0$ iff

  $$N_2 - N_3 = n(A-B-B) + n(B-A-B) - n(A-B-A) - n(B-A-A) > e_a - e_b.$$

  (3) $\min\{N_1, N_2-(e_a-e_b), N_3+(e_a-e_b), N_4\} = N_3+(e_a-e_b)$: number of patients can be matched increases amount of $2(e_a-e_b)$.

- Case 4: $\min\{N_1, N_2, N_3, N_4\} = N_4$: number of patients can be matched increases amount of $2(N_4 - N_3)$.

$\square$

CHAPTER 3

TWO AND THREE WAY LUNG EXCHANGE IN ESU[2014] AND EFFICIENT MECHANISM WITH EXTRA DONORS

3.1 Two and Three Way Lung Exchange In ESU[2014]

In this chapter, we will show that under particular conditions any random number of extra donors brought by $A-B-A$ or $B-A-B$ type will improve efficiency through 2-way and 3-way exchanges. When we obtain an optimal matching procedure for the first stage of our mechanism. We use the following lemma and sequential procedure for two and three way exchanges. In the following lemma, ESU [2014] explain that every K-way exchange must involve an A and a B blood type patient, but If $K \geq 0$, then it might also involve O blood type patients.

Lemma 3.1.1 (2)**.** Let $\mathcal{E}$ and $K \geq 2$ be given. Then the only types that could be part of a K way exchange are $O-Y-A, O-Y-B, A-Y-B$ and $B-Y-A$ where $Y \in \{O, A, B\}$. Furthermore, every K way exchange must involve an A and a B blood type patient.

Proof. Given in ESU[2014] □

There is an assumption about the types $O-O-A$ and $O-O-B$ for the remaining results on lung exchange.

Definition 3.1.2. Long Run Assumption: A lung exchange pool $\mathcal{E}$ satisfies the assumption if for every matching composed of arbitrary size exchanges, there remains at least one "unmatched" patient from each of two types $O-O-A$ and $O-O-B$.

By using "Long Run Assumption", They can construct a new matching $\mu^{'}$ from μ by replacing every $O-A-A$ and $O-B-A$ type taking part in an exchange by an unmatched $O-O-A$ and every $O-B-B$ type taking part in an exchange by an unmatched $O-O-B$. Size exchanges, number of transplants does not change and only O blood type patients matched under $\mu^{'}$ belong to the triples of types $O-O-A$ and $O-O-B$. The numbers of $O-O-B$ and $O-O-A$ participants in the market is non binding. ESU[2014] describes a consistent matching consists of two and three way exchanges.

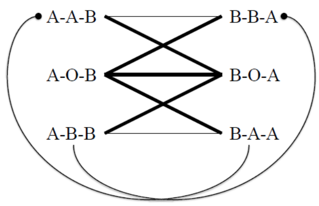

Figure 3.1: A Subset of Two and Three way Exchange

1 A regular (non-bold/no dotted end) edge between two types represents a two way exchange involving those two types.

2 A bold edge between two types represents a three way exchange involving those two types and a $O-O-A$ or $O-O-B$ type.

3 An edge with a dotted end represents three way exchange involving two types from the dotted end and one type from the non-dotted end.

ESU[2014] show that the following procedure maximizes number of transplants through two and three way exchange.

Sequential Matching Procedure for Two and Three Way Exchanges:

Step 1:Carry out the 2 and 3 way exchanges in Lemma 3 among $A-A-B, A-B-B, B-B-A$ and $B-A-A$ types to maximize the number of transplants subject to following constraints(*):

(1) Leave at least a total of $\min\left\{n\left(A-A-B\right)+n\left(A-B-B\right), n\left(B-O-A\right)\right\}$ $A-A-B$ and $A-B-B$ types unmatched.

(2) Leave at least a total of $\min\left\{n\left(B-B-A\right)+n\left(B-A-A\right), n\left(A-O-B\right)\right\}$ $B-B-A$ and $B-A-A$ types unmatched.

Step 2: Carry out the maximum number of 3-way exchanges in Lemma 3 involving $A-O-B$ types and the remaining $B-A-A$ or $B-B-A$ types. Carry out the maximum number of 3-way exchanges in Lemma 3 involving $B-O-A$ types and the remaining $A-A-B$ or $A-B-B$ types.

Step 3: Carry out the maximum number of 3-way exchanges in Lemma 3 involving the remaining $B-O-A$ or $A-O-B$ types.

3.2 Efficient Mechanism With Extra Donors For Two and Three Way Exchange

We will show that If a particular condition holds then extra donors would increase the efficiency of matching through two and three way exchanges. To formulate that condition, we would use the following numbers K_A and K_B

defined by ESU[2014]:

$$\begin{aligned} K_A &:= n(A-O-B) - n(B-B-A) - n(B-A-A) \\ K_B &:= n(B-O-A) - n(A-A-B) - n(A-B-B) \end{aligned}$$

Lemma 3.2.1. Suppose that the lung exchange pool $\mathcal{E}$ satisfies long run assumption and $\max\{K_A, K_B\} < 0$ then number of patients can be matched increases if any arbitrary number of patient enter the pool with extra donor.

Proof. Firstly, we will show why in the case of $\max\{K_A, K_B\} \geq 0$ there will be no change in a matching.
Suppose without loss of generality that $K_B \leq K_A$. Then, $K_A = \max\{K_A, K_B\} \geq 0$. This implies, by the definition of K_A that $n(A-O-B) \geq n(B-A-B) + (B-A-A)$. Therefore, all $B-A-B$ and $B-A-A$ types participate in three way exchanges with $A-O-B$ types in Step 2 of the sequential matching procedure for two and three way exchange. The number of $B-O-A$ types that are not matched in Step 2 is given by:
$n(B-O-A) - \min\{n(A-A-B) + n(A-B-B), n(B-O-A)\}$

$$\begin{aligned} &= \max\{n(B-O-A) - n(A-A-B) - n(A-B-B), 0\} \\ &= \max\{K_A, 0\} \\ &\leq K_A = n(A-O-B) - n(B-B-A) - n(B-A-A). \end{aligned}$$

As a result the number of $B-O-A$ types that are not matched in Step 2 is less than or equal to the number of $A-O-B$ types that are not matched in Step 2. Therefore, all $B-O-A$ types participate in three way exchange in Step 2 and Step 3. In this case, all $A-A-B$,$A-B-B$,$B-B-A$ and $B-A-A$ type triples are matched with $B-O-A$ and $A-O-B$ type triples. Since $B-A-B$ and $B-A-A$ types can participate in three way exchanges with $A-O-B$ types;$A-A-B$ and $A-B-B$ types can participate in three way exchanges with $B-O-A$ types, the behavior of $A-A-B-B$ and the behavior of $B-A-B-A$ change nothing. Therefore bringing extra donors to the pool doesn't improve efficiency of the matching.

if $\max\{K_A, K_B\} < 0$ is satisfied, the constraint in Step 1 becomes equivalent to :

1. Leave at least a total $n(B-O-A)$ of $A-A-B$ and $A-B-B$ types unmatched.

2. Leave at least a total $n(A-O-B)$ of $B-A-B$ and $B-A-A$ types unmatched.

Hence all $A-O-B$ types take part in three way exchange with $B-B-A$ and $B-A-A$ types, and all $B-O-A$ types take part in three way exchange with $A-B-A$ and $A-B-B$ types. This implies that $n(B-B-A)+n(B-A-A)-n(A-O-B)$ number of $B-B-A$ and $B-A-A$ types remain unmatched and $n(A-B-A)+n(A-B-B)-n(B-O-A)$ number of $A-B-A$ and $A-B-B$ types remain unmatched. Therefore the behavior of patients that enter with extra donor to the pool is matter and extra donor will improve efficiency through controlling the supply and demand balance of A and B blood type patients. □

In the following lemma, we will show that If particular assumptions hold,bringing extra donors to the exchange pool would increase number of patients that can be matched differently according to supply and demand balance of A and B blood type donors.

Lemma 3.2.2. Suppose that the lung exchange pool $\mathcal{E}$ satisfies long run assumption and $\max\{K_A, K_B\} < 0$.

1. If supply of B blood type donor is greater than demand of it and demand of A blood type donor is greater than supply of it then number of patients

increases at least amount of $e_a \leq n(A-B-A) - n(B-A-B) + 2n(A-B-B) - 2n(B-A-A)$ iff

$$n(B-A-B) > n(A-B-B) - n(B-A-A) + \frac{n(A-B-A)}{2}$$

and number of unmatched patients before additional donors is greater than and equal to e_a.

2. If supply of A blood type donor is greater than demand of it and demand of B blood type donor is greater than supply of it then number of patients increases at least amount of $e_b \leq n(B-B-A) - n(A-A-B) + 2n(B-A-A) - 2n(A-B-B)$ iff

$$n(A-B-A) > n(B-A-A) - n(A-B-B) + \frac{n(B-B-A)}{2}$$

and number of unmatched patients before additional donors is greater than and equal to e_b.

Proof. **Case 1:** If supply of B blood type donor is greater and equal to the demand of B blood type donor, it implies that demand of A blood type donor is greater and equal to the supply of A blood type donor. Since we constitute a feasible matching, the supply of donors in a feasible matching μ that are compatible with A blood type patients should be at least as large as 2 multiples of the number of A blood type patients and a similar statement holds for B blood type patients. ESU[2014] shows in the proof of lemma 5 that $d_A[\mu] + d_O[\mu] \geq 2p_A[\mu]$ and $d_B[\mu] + d_O[\mu] \geq 2p_B[\mu]$. The first inequality implies that

$$n(A-O-B)+2n(B-O-A)+2n(B-A-A)+n(A-B-A)+n(B-A-B)$$
$$\geq 2n(A-O-B) + 2n(A-B-B) + 2n(A-B-A)$$

By our assumption we know that $\max\{K_A, K_B\} < 0$. Therefore all B-O-A and A-O-B type triples are already matched. We analyse supply demand balance of A-B-B, A-B-A,B-A-B,B-A-A type triples. We get from the inequality above

$$\begin{aligned} n(A-A-B)+2n(A-B-B)+n(B-A-B) &\geq 2n(B-A-B)+ \\ &2n(B-A-A) \\ n(A-A-B)+2n(A-B-B) &\geq 2n(B-A-A)+ \\ &n(B-A-B) \end{aligned}$$

- If supply of A blood type donor $\leq$ demand of A blood type donor:

$$\begin{aligned} n(A-A-B)+2n(B-A-A)+n(B-A-B) &\leq 2n(A-A-B)+ \\ &2n(A-B-B) \\ n(B-A-B)+2n(B-A-A) &\leq 2n(A-B-B)+ \\ &n(A-B-A) \end{aligned}$$

Therefore we can say, supply of B $\geq$ demand of B blood type donor $\Leftrightarrow$ demand of A blood type donor $\geq$ supply of it. When e_a number of $B-A-B$ types behave like $B-A-A$ then difference between supply of B and demand of B becomes

$$n(A-B-A)+2n(A-B-B)-2n(B-A-A)-2e_a-n(B-A-B)+e_a \geq 0.$$

We obtain

$$e_a \leq n(A-B-A)-n(B-A-B)+2n(A-B-B)-2n(B-A-A)*$$

Morever we know that number of additional A donors is at most $n(B-A-B)$.

Therefore we get

$$e_a = n\,(A - B - A) - n\,(B - A - B) + 2n\,(A - B - B) -$$
$$2n\,(B - A - A) \leq n\,(B - A - B)$$

However, under equality of $(A - B - A) - n\,(B - A - B) + 2n\,(A - B - B) - 2n\,(B - A - A) = n\,(B - A - B)$, number of patients that can be matched will not increase at least number of e_a. Therefore we must have

$$\begin{aligned}
n\,(A - B - A) - n\,(B - A - B) + 2n\,(A - B - B) - & \\
2n\,(B - A - A) &< n\,(B - A - B) \\
n\,(A - B - A) + 2n\,(A - B - B) - & \\
2n\,(B - A - A) &< 2n\,(B - A - B) \\
\frac{n\,(A - B - A)}{2} + n\,(A - B - B) - n\,(B - A - A) &< n\,(B - A - B)
\end{aligned}$$

Under this inequality, we can ensure that number of patients that can be matched increases at least amount of e_a.

- we can not consider the case where supply of both A and B blood type donors are greater than demand of them. Since both of them can not be possible.

Case 2: If supply of A blood type donor is greater and equal to the demand of A blood type donor that is

$$\begin{aligned}
n\,(A - A - B) + 2n\,(B - A - A) + n\,(B - A - B) &\geq 2n\,(A - A - B) + 2n\,(A - B - B) \\
n\,(B - A - B) + 2n\,(B - A - A) &\geq 2n\,(A - B - B) + n\,(A - B - A)
\end{aligned}$$

- This implies that supply of B blood type donor $\leq$ demand of B blood type donor:

$$\begin{aligned} n(B-A-B)+2n(A-B-B)+n(A-B-A) &\leq 2n(B-A-B)+2n(B-A-A) \\ n(A-B-A)+2n(A-B-B) &\leq 2n(B-A-A)+n(B-A-B) \end{aligned}$$

When e_b number of $A-B-A$ types behave like $A-B-B$ then difference between supply of A and demand of A becomes

$$n(B-A-B)+2n(B-A-A)-2n(A-B-B)-2e_b+n(A-B-A)+e_b \geq 0.$$

We obtain

$$e_b \leq n(B-B-A)-n(A-A-B)+2n(B-A-A)-2n(A-B-B).$$

Morever we know that number of additional B donors is at most $n(A-B-A)$. Therefore we get

$$\begin{aligned} e_b &= n(B-A-B)-n(A-A-B)+2n(B-A-A)-2n(A-B-B) \\ &\leq n(A-B-A) \end{aligned}$$

However, under equality of $(B-B-A)-n(A-A-B)+2n(B-A-A)-2n(A-B-B)=n(A-A-B)$, number of patients that can be matched will not increase at least number of e_b. Therefore we must have

$$\begin{aligned} n(B-B-A)-n(A-A-B)+2n(B-A-A)-2n(A-B-B) &< n(A-A-B) \\ n(B-B-A)+2n(B-A-A)-2n(A-B-B) &< 2n(A-A-B) \\ \frac{n(B-B-A)}{2}+n(B-A-A)-n(A-B-B) &< n(A-A-B) \end{aligned}$$

Under this inequality, we can ensure that number of patients that can be matched increases at least amount of e_b.

Case 3: Both Demand of A blood type donor is greater and equal to the supply of A blood type donor and demand of B blood type donor is greater and equal to the supply of B blood type donor can not be possible again

□

CHAPTER 4

UNRESTRICTED EXCHANGE SIZE ESU[2014] AND EFFICIENT MECHANISM WITH EXTRA DONORS

4.1 Unrestricted Exchange Size ESU[2014]

In this chapter, firstly we aim to find an upper bound to the number of patients that can be matched in a matching only consisting of triples with A blood type patients and B blood type patients through unrestricted size exchange with extra donors. Secondly, we use this upper bound in order to find the number of patients matched in an optimal matching. For achieving these, we use the following results of ESU[2014].

For a given exchange pool $\mathcal{E}$, ESU[2014] refer to an exchange pool $\mathcal{K} \leq \mathcal{E}$ as a sub pool of $\mathcal{E}$. Given a sub pool $\mathcal{K}$

- $d_X[\mathcal{K}]$ be the number of X blood type donors in matching $\mathcal{K}$
- $p_X[\mathcal{K}]$ be the number of X blood type patients in $\mathcal{K}$.

ESU[2014] denote $\mathbb{E}_{\mathbb{X}}$ for any $X \in \{A, B\}$, the triple types of that the patient's blood type is X:

$$\begin{aligned}\mathbb{E}_{\mathbb{A}} &:= \{A - B - A, A - O - B, A - B - B\} \text{ and} \\ \mathbb{E}_{\mathbb{B}} &:= \{B - B - A, B - O - A, B - A - A\}.\end{aligned}$$

Let $\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}} \leq \mathcal{E}$ be the sub pool with only essential type triples.

ESU[2014] define two non- negative numbers for triples in $\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}$. These tell us the minimum ($\underline{s}_A$) and maximum ($\overline{s}_A$) number of donors compatible with B blood type patients that can be supplied by A blood type patients:

$$\underline{s}_A := n(A - O - B) + n(A - B - A) + n(A - B - B)$$

$$\overline{s}_A := 2n(A - O - B) + n(A - B - A) + n(A - B - B).$$

Here $\underline{s}_A$ assumes that all A-O-B type triples are treated like A-B-A types and hence, the O blood type donor can be utilized internally.Hence , each A-O-B type requires one donor from outside, so does each A-B-A triple. In calculation of $\overline{s}_A$ we treat A-O-B type like A-B-B's. Therefore, each of them requires 2 donors from outside instead of 1. Symmetrically, we define $\underline{s}_B$ and $\overline{s}_B$. Observe that

$$\overline{s}_A - \underline{s}_A = n(A - O - B),$$

$$\overline{s}_B - \underline{s}_B = n(B - O - A).$$

Lemma 4.1.1. Upper Bound Lemma Consider the sub-pool $\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}$. Then $\overline{m}$, defined below is an upper bound to the number of triples that can be matched in a matching only consisting of triples in $\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}$:

$$\overline{m} := \overline{m_A} + \overline{m_B} \quad \text{where}$$

$$\overline{m_A} := min\left\{p_A\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right], \left\lfloor \frac{d_A\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right] + d_O\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]}{2} \right\rfloor, \overline{s_B}\right\} \text{and}$$

$$\overline{m_B} := min\left\{p_B\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right], \left\lfloor \frac{d_B\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right] + d_O\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]}{2} \right\rfloor, \overline{s_A}\right\}$$

Proof. Given in ESU[2014]. □

Proposition 4.1.2. Consider $\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}$ i.e, the sub-pool with types only from $\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}$. Procedure 4, Group and Match, matches the number of $A-O-B$ and $B-O-A$ type triples possible in any matching within $\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}$; and theses numbers are $\min\{n(A-O-B), \overline{s_B}\}$ and $\min\{n(B-O-A), \overline{s_A}\}$,respectively.

Proof. Given in ESU[2014]. □

Theorem 4.1.3. Suppose that the lung exchange pool $\mathcal{E}$ satisfies the long run assumption and all size of exchanges are allowed. Then Procedure 5, the sequential matching procedure without size constraints, finds an optimal matching.Morever, none of exchanges in this matching are larger than 6-way. The number of patients matched in an optimal matching is given by

$$\overline{m} - \mathbb{1} + \min\{n(A-O-B), \overline{s_B}\} + \min\{n(B-O-A), \overline{s_A}\}$$

where $\mathbb{1} \in \{0,1\}$ and $\overline{s_X}$ for $X \in \{A, B\}$ and $\overline{m}$ are defined as above.

Proof. Given in ESU[2014]. □

4.2 Efficient Mechanism With Extra Donors in Unrestricted Exchange Size

In the next lemma, we try to find out how an upper bound to the number of triples that can be matched in a matching only consisting of triples in $\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}$: $\overline{m} := \overline{m_A} + \overline{m_B}$ changes when random number of patients bring additional donors into the sub-pool $\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}$.

Lemma 4.2.1. Consider the sub-pool $\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}$. Suppose that number of patients comes with additional B blood type donor e_b is less than and equal to number of patients comes with additional A blood type donor e_a. Upper bound to the number of triples that can be matched $\overline{m}$ will increase amount of e_a iff $p_A - p_B \geq \lfloor \frac{n(B-O-A)-n(A-O-B)}{2} \rfloor$ when

$$p_B \leq \lfloor \frac{d_B \left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}\right] + d_O \left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}\right]}{2} \rfloor \quad \leq \quad \overline{s_A} + \lfloor \frac{e_a + e_b}{2} \rfloor \tag{4.1}$$

$$\overline{s_B} + e_a \quad \leq \quad p_A \tag{4.2}$$

$$\overline{s_B} + \lfloor \frac{e_a + e_b}{2} \rfloor \quad \leq \quad \lfloor \frac{d_A \left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}\right] + d_O \left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}\right]}{2} \rfloor. \tag{4.3}$$

are satisfied.

Proof. Let us observe the impact of extra donors on $\overline{m_A}$ and $\overline{m_B}$.

According to definition of $p_A = n\,(A - O - B) + (A - B - A) + (A - B - B)$, after extra donors number of A blood type patients becomes $p_A^{'} = n\,(A - O - B) + (A - B - A) - e_b + (A - B - B) + e_b = p_A$.

$$\lfloor\frac{d_A+d_O}{2}\rfloor' = n\,(B-A-A)+e_a+n\,(B-O-A)+$$
$$\lfloor\frac{n\,(A-B-A)-e_b+n\,(A-O-B)+n\,(B-A-B)-e_a}{2}\rfloor$$
$$=\lfloor\frac{d_A+d_O}{2}\rfloor+\lfloor\frac{e_a-e_b}{2}\rfloor.$$

$$\overline{s_B}' = 2n\,(B-O-A)+(B-A-B)-e_a+2n\,(B-A-A)+2e_a$$
$$=\overline{s_B}+e_a$$

According to definition of $p_B = n\,(B-O-A)+(B-A-B)+(B-A-A)$, after extra donors number of A blood type patients becomes $p_B^{'} = n\,(B-O-A)+ (B-A-B)-e_a+(B-A-A)+e_a = p_B$.

$$\lfloor\frac{d_B+d_O}{2}\rfloor' = n\,(A-B-B)+e_b+n\,(A-O-B)+$$
$$\lfloor\frac{n\,(B-A-B)-e_a+n\,(B-O-A)+n\,(A-B-A)-e_b}{2}\rfloor$$
$$=\lfloor\frac{d_B+d_O}{2}\rfloor-\lfloor\frac{e_a-e_b}{2}\rfloor.$$

$$\overline{s_A}' = 2n\,(A-O-B)+(A-B-A)-e_b+2n\,(A-B-B)+2e_b$$
$$=\overline{s_A}+e_b$$

If both $\overline{m_A}=\overline{s_B}+e_a$ and $\overline{m_B}=\overline{s_A}+e_b$ holds at the same time after being used extra donors, we get both $p_A > \overline{s_B}+e_a > p_B$ and $p_B > \overline{s_A}+e_b > p_A$..

This gives us contradiction. Therefore both of

$$\begin{aligned}
\overline{m_A} &:= min\left\{p_A\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right], \lfloor\frac{d_A\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]+d_O\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]}{2}\rfloor+\lfloor\frac{e_a-e_b}{2}\rfloor, \overline{s_B}+e_a\right\} \\
&= \overline{s_B}+e_a \\
\overline{m_B} &:= min\left\{p_B\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right], \lfloor\frac{d_B\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]+d_O\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]}{2}\rfloor-\lfloor\frac{e_a-e_b}{2}\rfloor, \overline{s_A}+e_b\right\} \\
&= \overline{s_A}+e_b
\end{aligned}$$

can not be true. When $\overline{m_A}=\overline{s_B}+e_a$ is satisfied , $\overline{m_B}=p_B$ or $\overline{m_B}=\lfloor\frac{d_B\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]+d_O\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]}{2}\rfloor-\lfloor\frac{e_a-e_b}{2}\rfloor$ must hold. However we don't want

$$\overline{m_B}=\lfloor\frac{d_B\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]+d_O\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]}{2}\rfloor-\lfloor\frac{e_a-e_b}{2}\rfloor$$

. Since $\overline{m}$ increases $\lfloor\frac{e_a+e_b}{2}\rfloor<e_a$. Therefore we obtain $p_B\leq\lfloor\frac{d_B\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]+d_O\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]}{2}\rfloor-\lfloor\frac{e_a-e_b}{2}\rfloor$ and $p_B\leq\overline{s_A}$. This implies that

$$p_B\leq\lfloor\frac{d_B\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]+d_O\left[\mathcal{E}_{\mathbb{E}_{\mathbb{A}}\cup\mathbb{E}_{\mathbb{B}}}\right]}{2}\rfloor\leq\overline{s_A}+\lfloor\frac{e_a+e_b}{2}\rfloor\ (1)$$

From solving (1),

$$\begin{aligned}
&n(B-A-B)+n(B-O-A)+ \\
&\qquad n(B-A-A)\leq\lfloor\frac{n(B-O-A)+n(B-A-B)+n(A-B-A)}{2}\rfloor \\
&\qquad\qquad +n(A-O-B)+n(A-B-B) \\
&\qquad\qquad \leq 2n(A-O-B)+\ n(A-B-A)+2n(A-B-B) \\
&\qquad\qquad +\lfloor\frac{e_a+e_b}{2}\rfloor
\end{aligned}$$

We obtain $p_A\geq\lfloor\frac{n(B-O-A)+n(B-A-B)-e_a+n(A-B-A)-e_b}{2}\rfloor$.
From using assumption (3), we get $p_B\leq\lfloor\frac{n(A-O-B)+n(B-A-B)-e_a+n(A-B-A)-e_b}{2}\rfloor$.

From assumption (2), we can say $p_B < \overline{s_B} + e_a \leq p_A$ and since

$$-p_B \geq -\lfloor \frac{n(A-O-B) + n(B-A-B) - e_a + n(A-B-A) - e_b}{2} \rfloor$$

$$p_A \geq \lfloor \frac{n(B-O-A) + n(B-A-B) - e_a + n(A-B-A) - e_b}{2} \rfloor$$

By adding both sides, we conclude that

$$p_A - p_B \geq \lfloor \frac{n(B-O-A) - n(A-O-B)}{2} \rfloor.$$

□

We have seen in the above lemma that s_B increases number of e_a and s_A increases number of e_b through bringing e_a random number of extra A blood type donors and e_b random number of extra B blood type donors to the exchange pool. Therefore in the following proposition,we show that maximum number of $A-O-B$ and $B-O-A$ type triples that can be matched within $\mathcal{E}_{\mathbb{E}_\mathbb{A} \cup \mathbb{E}_\mathbb{B}}$ would increase.

Proposition 4.2.2. Consider $\mathcal{E}_{\mathbb{E}_\mathbb{A} \cup \mathbb{E}_\mathbb{B}}$ i.e, the sub-pool with types only from $\mathbb{E}_\mathbb{A} \cup \mathbb{E}_\mathbb{B}$. If any random number of extra donors are used in any matching within $\mathcal{E}_{\mathbb{E}_\mathbb{A} \cup \mathbb{E}_\mathbb{B}}$, Procedure 4, Group and Match, matches the number of $A-O-B$ and $B-O-A$ type triples possible in any matching within $\mathcal{E}_{\mathbb{E}_\mathbb{A} \cup \mathbb{E}_\mathbb{B}}$; these numbers become $\min\{n(A-O-B), \overline{s_B} + e_a\}$ and $\min\{n(B-O-A), \overline{s_A} + e_b\}$ respectively.

Proof. The proof of the proposition is the same as with ESU[2014]. We look at the $\min\{n(A-O-B), \overline{s_B} + e_a\}$ instead of looking at $\min\{n(A-O-B), \overline{s_B}\}$ to find the number of $A-O-B$ type triples that can be matched.Similarly, we take the $\min\{n(B-O-A), \overline{s_A} + e_b\}$ instead of taking $\min\{n(B-O-A), \overline{s_A}\}$ to find the number of $B-O-A$ type triples that can be matched. □

By above lemma and proposition, we have shown that bringing extra donors improve welfare efficiency. In the lemma, we have found that upper bound for the triples consisting of A and B blood type patient would increase through using random number of additional donors.In the proposition, we have shown that number of $A-O-B$ and $B-O-A$ type triples would increase through using random number of additional donors. We combine these two results and we give the number of patients matched in an optimal matching in the following theorem.

Theorem 4.2.3. Suppose that the lung exchange pool $\mathcal{E}$ satisfies the long run assumption and all size of exchanges are allowed. Then Procedure 5, the sequential matching procedure without size constraints, finds an optimal matching. Morever, none of the exchanges in this matching are larger than 6-way. After bringing extra donors e_a and e_b where $e_a > e_b$ to the exchange pool $\mathcal{E}$, the number of patients matched in an optimal matching is equal to

$$\overline{m} + e_a - \mathbb{1} + \min\left\{n\left(A-O-B\right), \overline{s_B} + e_a\right\} + \min\left\{n\left(B-O-A\right), \overline{s_A} + e_b\right\}$$

where $\mathbb{1} \in \{0,1\}$ and $\overline{s_X}$ for $X \in \{A, B\}$ and $\overline{m}$ are defined as above.

Proof. Since upper bound for essential type triples increases amount of e_a by the above lemma. We write $\overline{m} + e_a - \mathbb{1}$ instead of $\overline{m} - \mathbb{1}$.
$\overline{s_B}$ increase amout of e_a and $\overline{s_A}$ increases amount of e_b through bringing additional donor.Therefore by using the above proposition for $A-O-B$ and $B-O-A$ we change the construction with respect to this. The general proof is the same with ESU[2014]. □

CHAPTER 5

IMPACT OF EXTRA DONORS TO IMPLEMENTATIONAL EFFICIENCY

Given a lung exchange problem, let m be the maximum exchange size under no size constraints. We increase implementational efficiency m number pertaining to a particular lung exchange problem goes down by having additional donor. For instance, if we can match 7 number of patients by $5 + 2$ way exchanges where m equals to 5, we can match them by $4+3$ way exchanges through using extra donors. While a 5 way exchange requires 15 simultaneous surgeries, a 4 way exchange requires 12 simultaneous surgeries. Therefore using extra donors increases implementational efficiency in a lung exchange. Usage of extra donors in order to increase implementational efficiency is only a necessary condition. It is not a sufficient condition. Before giving the following proposition, we define the lemma 5 of ESU[2014].

Lemma 5.0.4 (LEMMA 5 ESU [2014]). Suppose that $\mathcal{E}$ satisfies the long run assumption and μ is an optimal matching (without any exchange size constraints) within the essential type sub-pool $\mathcal{E}_{\mathbb{E}_{\mathbb{A}} \cup \mathbb{E}_{\mathbb{B}}}$. Suppose further that μ matches the maximum possible number of A-O-B and B-O-A type triples that can be matched in any matching.

- Then μ can be modified to obtain a matching ν such that $n\,(A-O-B)\,[\mu]+n\,(B-O-A)\,[\mu]$ many O-O-A and O-O-B type triples can be matched in addition to all triples matched by μ.

- Morever, ν is an optimal matching of $\mathcal{E}$ without any size constraints.

Proof. Given in ESU[2014] □

Proposition 5.0.5. Suppose that the lung exchange pool $\mathcal{E}$ satisfies the long run assumption and all size of exchanges are allowed. When supply of $X \in \{A, B, O\}$ blood type donor(d_X) equal to demand of it (equals to $2p_X$), if number of $B-A-B-A$ chosen by our mechanism to use as $B-A-A$ type is equal to number of $A-B-A-B$ chosen by our mechanism in order to use as $A-B-B$ type, exchange size decreases .

Proof. Supply demand relation of A,B,O blood type donors is based on the proof of lemma 5 of ESU[2014].According to ESU[2014], after adding O-O-A and O-O-B type triples to the essential types to obtain unrestricted exchange size, O blood type donors have to commit to O blood type donors. Therefore supply demand relation for A and B blood type patients in the sub pool will be $d_A = 2p_A$ and $d_B = 2p_B$. As a result, we can match all B and A blood type patients with B and A blood type donors within the sub pool respectively. Our two stage mechanism must choose number of B-A-B type triple behaving like B-A-A equals to number of A-B-A type triple behaving like A-B-B. In order to match all of patients in an unrestricted size exchange, supply of A,B,O blood type patients must be equal to demand of them. Suppose that number of A-B-A- B used in a lung exchange to increase implementational efficiency is greater than number of B-B-A- A used in a lung exchange. We obtain that $d_B > 2p_B$ and $d_A < 2p_A$. This ends up with unmatched patients. Therefore it is a necessary condition. □

It is not a sufficient condition. Even though our two stage mechanism designs the first stage wih respect to this condition, exchange size may not decrease and implemantational efficiency doesn't increase. Let us give an example to clarify this.

Example 5.0.6. Consider an exchange pool with triple types are :

1. A-O-B
2. B-O-A
3. B-O-A
4. O-O-B
5. O-O-B
6. O-O-B

Firstly, one of patients 1,2,3 need to be in the same exchange with one of patients 4,5,6. Then these six triples reduce to three type triples A-B-B, B-B-A, B-B-A and makes a three way exchange. As can be seen, There is a six way exchange and both number of triples comes with additional A blood type donors and B blood type donors equal to zero. However a six way exchange can not reduce to smaller way exchange. Therefore the condition in the above is only a necessary condition.

CHAPTER 6

CONCLUSION

To conclude, we model a two stage mechanism to show that bringing extra donors to the exchange pool increases both welfare efficiency and implementational efficiency. In the second chapter, by Lemma 2.2.1 we exhibit that maximum number of transplants through two way exchange will increase if and only if we put a restriction on the difference between additional donor A and additional donor B. Plus, in Lemma 3.2.1 we find out that under long run assumption and maximum of K_A and K_B is less than zero condition, number of patients can be matched increases if any arbitrary number of patient enter the pool with extra donor. Morever, we find out If supply of B blood type donor is greater than demand of it and demand of A blood type donor is greater than supply of it then number of patients increases at least amount of number of B-A-B type triples who brings extra A blood type donor and If supply of A blood type donor is greater than demand of it and demand of B blood type donor is greater than supply of it then number of patients increases at least amount of number of A-A-B type triples who brings extra B blood type donor within allowance of two and three way exchanges in Lemma 3.2.2. In addition we prove that when number of patients comes with additional B blood type donor is less than and equal to number of patients comes with additional A blood type donor , upper bound to the number of triples that can be matched will increase amount of number of patients with additional A blood type donors if and only if we obtain an lower bound to the difference between number of A blood type patients and number of B blood type patients through satisfying three main assumptions in Lemma 4.2.1. Furhermore in Proposition 4.2.2, we prove that number of A-O-B and B-O-A type triples that can be matched will increase. Since number of donors compatible with B blood type patients that can be supplied by A blood type patients and number of donors compatible with A blood type patients that can be supplied by B blood type patients increase. When long run assumption is satisfied and unrestricted exchange size is feasible,as a final result of this chapter in Theorem 4.2.3 we obtain number of patients matched in an optimal matching with extra donors. In the implementational efficiency chapter, we illustrate that number of A-A-B type triple brings additional B blood type donors must be equal to the number of B-B-A type triples brings additional A blood type donors in order to decrease maximal way of exchange size. Hence implementing transplantation will be more practical and struggle with surgical constraints easily.

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