T.C.
AĞRI İBRAHİM ÇEÇEN ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI

Ali ŞİMŞEK

# EKSPONANSİYEL VE h-KONVEKS FONKSİYONLAR İÇİN RAİNA KESİRLİ İNTEGRAL OPERATÖRÜ YARDIMIYLA İNTEGRAL EŞİTSİZLİKLER

**YÜKSEK LİSANS TEZİ**

**TEZ YÖNETİCİSİ**

Prof. Dr. Ahmet Ocak AKDEMİR

**AĞRI – 2025**

**T.C.**
**AĞRI İBRAHİM ÇEÇEN ÜNİVERSİTESİ**
**LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ**
**MATEMATİK ANABİLİM DALI**

**Ali ŞİMŞEK**

# EKSPONANSİYEL VE h-KONVEKS FONKSİYONLAR İÇİN RAİNA KESİRLİ İNTEGRAL OPERATÖRÜ YARDIMIYLA İNTEGRAL EŞİTSİZLİKLER

**YÜKSEK LİSANS TEZİ**

**TEZ YÖNETİCİSİ**

**Prof. Dr. Ahmet Ocak AKDEMİR**
**JÜRİ ÜYELERİ**
**Doç. Dr. Abdullah ÇAĞMAN**
**Doç. Dr. Mustafa GÜRBÜZ**

**AĞRI-2025**

**EKSPONENSİYEL VE h-KONVEKS FONKSİYONLAR İÇİN RAINA KESİRLİ İNTEGRAL OPERATÖRÜ YARDIMIYLA İNTEGRAL EŞİTSİZLLİKLER**

Prof. Dr. Ahmet Ocak AKDEMİR danışmanlığında, Ali ŞİMŞEK tarafından hazırlanan bu çalışma, 25.08.2025 tarihinde aşağıda isimleri belirtilen jüri tarafından değerlendirilmiş olup Matematik Anabilim Dalı Analiz ve Fonksiyonlar Teorisi Bilim Dalında yüksek lisans olarak **oybirliği** ile kabul edilmiştir.

| | | |
| --- | --- | --- |
| Başkan : | Doç. Dr. Abdullah ÇAĞMAN | İmza: |
| Üye : | Prof. Dr. Ahmet Ocak AKDEMİR | İmza: |
| Üye : | Doç. Dr. Mustafa GÜRBÜZ | İmza: |

Yukarıdaki sonuç;

Enstitü Yönetim Kurulu …/…/… tarih ve . . . . / . . . . . . . . numaralı kararı ile onaylanmıştır.

**Prof. Dr. Mehmet TEYFUR**
**Enstitü Müdürü**

**Not:** Bu tezde kullanılan özgün ve başka kaynaklardan yapılan bildiriş, çizelge, şekil ve fotoğrafların kaynak olarak kullanımı, 5846 sayılı Fikir ve Sanat Eserleri Kanunundaki hükümlere tabidir.

| **Benzerlik Oranlarına Dair Beyannamedir** | | |
|---|---|---|
| **Öğrenci Adı Soyadı** | **Ali ŞİMŞEK** | |
| **Numarası** | **230601229** | |
| **Ana Bilim Dalı** | **Matematik** | |
| **Bilim Dalı** | **Analiz ve Fonksiyonlar Teorisi** | |
| **Programı** | | |
| **Danışman Unvanı Adı Soyadı** | **Prof. Dr. Ahmet Ocak AKDEMİR** | |
| **Tez Başlığı** | **EKSPONENSİYEL VE h-KONVEKS FONKSİYONLAR İÇİN RAİNA KESİRLİ İNTEGRAL OPERATÖRÜ YARDIMIYLA İNTEGRAL EŞİTSİZLLİKLER** | |
| **Bölümler** | **Turnitin Sonucu** | **En Yüksek Değer** |
| Giriş | %19 | % 25 |
| Kuramsal Temeller/Kavramsal Çerçeve | %18 | % 30 |
| Materyal ve Yöntem | %26 | % 30 |
| Araştırma Bulguları | %14 | % 20 |
| Sonuç/Tartışma/Öneri | %1 | % 20 |
| **TEZ (GENEL)** | %13 | % 25 |

**30.08.2025**

Yukarıda bilgileri yazılı tez çalışmasının benzerlik oranları Turnitin raporundan alındığı haliyle yazılmış olup tüm bilgilerin doğru olduğunu, aksi halde doğacak hukuki sorumlulukları kabul ve beyan ederiz.

Ali ŞİMŞEK Prof. Dr. Ahmet Ocak AKDEMİR

# İçindekiler

# ÖZET

## YÜKSEK LİSANS TEZİ

## EKSPONANSİYEL VE h-KONVEKS FONKSİYONLAR İÇİN RAİNA KESİRLİ İNTEGRAL OPERATÖRÜ YARDIMIYLA İNTEGRAL EŞİTSİZLİKLER

**Jüri: Prof. Dr. Ahmet Ocak AKDEMİR**

**Doç. Dr. Abdullah ÇAĞMAN**

**Doç. Dr. Mustafa GÜRBÜZ**

Bu tezde, literatürde tanımları mevcut Raina kesirli integral operatörleri incelenmiş ve $h$-konveks ve eksponansiyel konveks fonksiyon sınıfları için yeni integral eşitsizlikleri elde edilmiştir. Birinci bölüm giriş kısmında, konveks fonksiyonlar , eşitsizlik ve kesirli analiz teorisinin tarihinden , öneminden ve kullanım alanlarından kısaca bahsedilmiş olup, literatürde bulunan çalışmalar ile ilgili bilgiler verilmiştir. İkinci bölüm Kuramsal Temeller kısmında, konveks fonksiyonlar için temel tanım ve kavramlar yer almış, devamında konveks fonksiyon sınıfları arasındaki ilişkiye değinilmiştir. Raina kesirli integral operatörleri ile ilgili temel tanım ve kavramlar, bu kesirli integral operatörleri için önceden elde edilmiş çeşitli integral eşitsizlikleri ve bu eşitsizliklerin ispat yöntemleri üçüncü bölümde verilmiştir. Daha sonra bu bölümde $h$-konveks ve eksponansiyel konveks fonksiyon sınıfları için detaylı bir literatür bilgisi sunulmmuştur. Dördüncü bölümde ise $h$-konveks ve eksponansiyel konveks fonksiyon sınıfları için yeni integral eşitsizliklere ilişkin sonuçlar elde edilmiştir.



**2025, 71 sayfa**

**Anahtar Kelimeler**: Eşitsizlikler, Raina kesirli integral eşitsizlikleri, konveks fonksiyon, $h$-konveks fonksiyon, eksponansiyel-konveks fonksiyon

# ABSTRACT

## MASTER'S THESIS

## INTEGRAL INEQUALITIES VİA RAİNA FRACTİONAL INTEGRAL OPERATOR FOR EXPONENTIAL AND h-CONVEX FUNCTIONS

**Jury: Prof. Dr. Ahmet Ocak AKDEMİR**

**Assoc. Prof. Dr. Abdullah ÇAĞMAN**

**Assoc. Prof. Dr. Mustafa GÜRBÜZ**

In this thesis, the Raina fractional integral operators, which are defined in the literature, have been studied, and new integral inequalities have been obtained for the classes of $h$-convex and exponential convex functions. In the first chapter, convex functions, the history, importance and usage areas of the theory of inequality and fractional calculus are briefly mentioned and information about the studies in the literature is given. In the second chapter, the theoretical foundations, the basic definitions and concepts for convex functions are presented, followed by a discussion on the relationships between different convex function classes. The third chapter provides the basic definitions and concepts related to Raina fractional integral operators, various integral inequalities previously obtained for these fractional integral operators, and the proof methods of these inequalities. Additionally, a detailed literature review on $h$-convex and exponential convex function classes is presented in this chapter. In the fourth chapter, new integral inequalities for the $h$-convex and exponential convex function classes are obtained.

**2025, 71 pages**

**Key Words**: Inequalities, Raina fractional integral inequalities, convex function, $h$-convex function, exponential-convex function.

# Şekil Listesi

# TEŞEKKÜR

Yüksek Lisans Tezi olarak sunduğum bu çalışma Ağrı İbrahim Çeçen Üniversitesi Fen Fakültesi Matematik Bölümünde yapılmıştır.

Yüksek Lisans tez çalışmamın başlangıcından bu yana, bu alanda çalışmamı destekleyen, bütün enerjisi ve engin birikimiyle beni motive eden, çalışmalarımda eşsiz katkıları bulunan, bana bilimsel çalışma ve düşünme yeteneğini aşılayan saygıdeğer danışman hocam,

Sayın Prof. Dr. Ahmet Ocak Akdemir'e;

teşekkür ve şükranlarımı sunarım.

Öğrenim hayatım boyunca kendilerinden görmüş olduğum maddi ve manevi destekten dolayı aileme ve her zaman yanımda olan sevgili eşim Zehra Şimşek'e sonsuz teşekkürlerimi sunarım.

**Haziran 2025**

**Ali ŞİMŞEK**

# SİMGELER VE KISALTMALAR DİZİNİ

| | |
|---|---|
| $\mathbb{R}$ | : Reel Sayılar Kümesi |
| $\mathbb{R}_+$ | : Pozitif Reel Sayılar Kümesi |
| $I$ | : $\mathbb{R}$'de Bir Aralık |
| $I^0$ | : $I$'nın İçi |
| $L_1[a,b]$ | : $[a,b]$ Aralığında İntegrallenebilen Fonksiyonların Kümesi |
| $f''$ | : $f$ Fonksiyonunun İkinci Mertebeden Türevi |
| $SX(h,I)$ | : $h$ -Konveks Fonksiyonların Sınıfı |
| $SV(h,I)$ | : $h$ -Konkav Fonksiyonların Sınıfı |
| $K_s^2$ | : İkinci Anlamda $s$ -Konveks Fonksiyonların Sınıfı |
| $K_s^1$ | : Birinci Anlamda $s$ -Konveks Fonksiyonların Sınıfı |
| $C(I)$ | : Konveks Fonksiyonlar Sınıfı |
| $SX(h,I)$ | : $I$ Üzerinde $h$ -Konveks Fonksiyonlar Sınıfı |
| $SV(h,I)$ | : $I$ Üzerinde $h$ -Konkav Fonksiyonlar Sınıfı |
| $J_{\kappa+}^{\nu}f(\alpha)$ | : $\alpha$ mertebeden sağ taraflı RL Kesirli İntegral |
| $J_{\tau-}^{\nu}f(\alpha)$ | : $\alpha$ mertebeden sol taraflı RL Kesirli İntegral |
| ${}^{\xi_2}(I_\alpha \upsilon)(\iota)$ | : $\alpha$ mertebeden sol uyumlu kesirli integral |
| ${}^{\xi_2}(I_\alpha \upsilon)(\iota)$ | : $\alpha$ mertebeden sağ uyumlu kesirli integral |
| $(\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma}\phi)(x)$ | : Sol taraflı genelleştirilmiş Raina kesirli integral operatörü |
| $(\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma}\phi)(x)$ | : Sağ taraflı genelleştirilmiş Raina kesirli integral operatörü |

# 1. GİRİŞ

**Kesirli Analiz, Eşitsizlikler ve Konveks Fonksiyonların Tarihsel Süreci, Uygulamaları ve Önemi**

Kesirli analiz, eşitsizlikler ve konveks fonksiyonlar, her biri modelleme, tahmin ve optimizasyon için benzersiz araçlar sunan modern matematik biliminde üç önemli sütunu temsil eder.

Kesirli analiz, tam sayı olmayan mertebeden integrallere ve türevlere izin vererek klasik hesabı genişletir. Bu genelleme yalnızca teorik bir merak değil; viskoelastik malzemeler ve anormal difüzyon süreçleri gibi hafıza ve kalıtım etkilerine sahip gerçek dünya sistemlerini modellemede çok önemlidir. Kesirli türevlerin yerel olmayan doğası, tarihsel etkileri yakalar ve bunları mühendislik, kontrol sistemleri ve biyolojik modellemede güçlü kılar.

Eşitsizlikler, matematiksel analizde sınırları, yaklaşımları ve yakınsamayı belirlemek için temeldir. Cauchy-Schwarz, Hölder ve Minkowski eşitsizlikleri gibi klasik eşitsizlikler, fonksiyonel analizin, olasılığın ve sayısal yöntemlerin omurgasını oluşturur. Hataları kontrol etmek, algoritmaları optimize etmek ve teorik sonuçları kanıtlamak için titiz araçlar sağlarlar ve bu da onları hem saf hem de uygulamalı matematikte vazgeçilmez kılar.

Grafikteki iki nokta arasındaki herhangi bir doğru parçasının grafiğin kendisinin üzerinde yer aldığı özelliğiyle tanımlanan konveks fonksiyonlar, optimizasyon ve ekonomi teorisinde merkezi bir rol oynar. Zarif yapıları, yerel minimumların küresel olmasını sağlayarak karmaşık optimizasyon problemlerini büyük ölçüde basitleştirir. Konvekslik, makine öğreniminden finans gibi birçok uygulamada doğal olarak ortaya çıkar ve burada analiz ve algoritma tasarımını kolaylaştırır.

Bu üç kavram birlikte, matematiksel genellemelerin ve soyut özelliklerin derin bir pratik öneme sahip olabileceğini gösterir. Hem teorik olarak sağlam hem de gerçek dünyadaki zorluklara uygulanabilir modeller oluşturmamızı sağlar ve matematiksel yeniliğin zamansız değerini vurgular.

**Tarihsel Arka Plan**

Kesirli hesaplamanın (veya kesirli analizin) yolculuğu üç yüzyıl önce başladı. Önemli an, 1695'te Marquis de L'Hôpital'in Gottfried Wilhelm Leibniz'e $\frac{1}{2}$. mertebesindeki bir türevin anlamının ne olacağını sormasıyla geldi. Leibniz'in cevabı, böyle bir fikrin matematiksel meşruiyetini kabul etti, ancak bunun resmileştirilmesi zaman aldı. 19. yüzyılda Joseph Liouville, Augustin-Louis Cauchy ve Bernhard Riemann önemli adımlar attı. Liouville, günümüzde Liouville kesirli türevi olarak bilinen şeyi tanıttı, Riemann ise Riemann-Liouville kesirli integralini tanımladı. Daha sonra, Grünwald-Letnikov yaklaşımı alternatif, ayrıksı bir şekilde ilham veren bir formülasyon sundu. Erken tarihinin büyük bir kısmında, kesirli hesaplama, net uygulamaları olmayan matematiksel bir merak konusuydu. Ancak, 20. yüzyılın ortalarından sonlarına doğru, mühendisler ve fizikçiler, kesirli türevlerin, hafıza ve kalıtımsal özelliklere sahip gerçek dünya sistemlerini (viskoelastik malzemeler ve anormal difüzyon süreçleri gibi) zarif bir şekilde modelleyebileceğini fark ettiler. Bu canlanma, yeni kesirli operatörlerin (örneğin, Caputo, Atangana-Baleanu ve Prabhakar operatörleri) alanı zenginleştirmesiyle devam etti.

Eşitsizliklerin tarihi antik matematiğe dayanır. İlk örnekler Öklid'in "Öğeler"inde bulunabilir. Zamanla eşitsizlikler, 18. ve 19. yüzyıllarda Joseph-Louis Lagrange, Cauchy ve Chebyshev gibi matematikçilerin çalışmalarıyla yönlendirilen zengin bir alana dönüştü. Cauchy-Schwarz eşitsizliği, Hölder eşitsizliği, Minkowski eşitsizliği ve Chebyshev eşitsizliği temel araçlar haline geldi. 20. yüzyılda eşitsizliklerin incelenmesi fonksiyonel analiz ve olasılık teorisine doğru genişledi. Jensen eşitsizliği gibi yeni biçimler, konveks fonksiyonlar ve eşitsizlikler arasındaki bağlantıyı vurgularken, daha uzmanlaşmış eşitsizlikler bilgi teorisi, harmonik analiz ve operatör teorisinde ortaya çıktı.

Konveks Fonksiyonlar konveksligin geometrik kavramı antiktir ve kökleri Yunan matematiğine dayanır, ancak resmi analitik çalışması daha sonra geldi. Konveks fonksiyonun (epifografisi konveks bir küme olan bir fonksiyon) modern tanımı 19. yüzyılın sonlarında resmileştirildi. Önemli gelişmeler arasında beklentiler ve ortalamalarda konveksliğin rolünü vurgulayan Jensen eşitsizliği (1906) yer alır. 20. yüzyılda konvekslik, Harold Kuhn ve Albert Tucker gibi matematikçilerin yönlendirdiği optimizasyon, dualite teorisi ve ekonominin merkezi haline geldi. Karush-Kuhn-Tucker (KKT) koşulları üzerine çalışmaları modern kısıtlı optimizasyonun temelini oluşturmaktadır.

**Kullanım ve Uygulamalar**

Mühendislik ve Fizik , Kesirli analiz, hafıza, kalıtımsal etkiler ve yerel olmayan davranışlar sergileyen sistemlerin modellenmesinde parlar.
Örnekler şunlardır:

Viskoelastisite: Kesirli diferansiyel denklemler, gerilim-şekil değiştirme ilişkilerini klasik modellerden daha doğru bir şekilde tanımlar.
Anormal difüzyon: Klasik Brown hareketinden sapan süreçler, kesirli zaman türevleriyle doğal olarak modellenir.
Kontrol sistemleri: Kesirli mertebeden denetleyiciler, bazı karmaşık sistemlerde daha iyi sağlamlık ve performans sağlar.

Eşitsizlikler mühendislikte temeldir: Sayısal yöntemlerde hata sınırları ve yakınsama garantileri sağlar. Kararlılık analizinde görünür (örneğin, Lyapunov'un doğrudan yöntemi genellikle eşitsizlik tahminlerini kullanır). Filtre tasarımı ve sinyal işleme için kritiktir. Konveks fonksiyonlar, mühendislik tasarımı, ağ akışı ve güç sistemlerindeki optimizasyon problemlerinin temelini oluşturur.Konveks optimizasyon, sistem güvenilirliği ve maliyet etkinliği için kritik olan küresel olarak en iyi çözümleri garanti eder. Veri Bilimi ve Makine Öğrenimi Konveks ve eşitsizlikler modern makine öğreniminin omurgasını oluşturur: Birçok kayıp fonksiyonu (örneğin, lojistik kayıp, menteşe kaybı) dışbükeydir ve verimli optimizasyona izin verir. L1 ve L2 normları gibi düzenleyiciler dışbükeydir ve seyrekliği veya düzgünlüğü teşvik eder. Hölder ve Jensen gibi eşitsizlikler genelleme ve yakınsama konusunda teorik garantiler sağlar. Kesirli analizin yeni uygulamaları vardır:

Özellik mühendisliği: kesirli türevler ince sinyal özelliklerini vurgulayabilir.
Derin öğrenme: araştırmacılar sinir ağı esnekliğini artırmak için kesirli aktivasyon fonksiyonlarını ve operatörlerini araştırır.
Ekonomi ve Finans: Konveks fonksiyonlar azalan getirileri, riskten kaçınmayı ve faydayı modeller: Fayda fonksiyonları genellikle konkavdır (konveksin negatifi), azalan marjinal faydayı yansıtır. Üretim ve maliyet fonksiyonları genellikle dışbükeylik veya içbükeylik sergiler ve fiyatlandırmayı ve kaynak tahsisini yönlendirir. Aritmetik-geometrik ortalama eşitsizliği gibi eşitsizlikler, ekonomik modellerde teorik sınırların oluşturulmasına yardımcı olur. Ekonomide daha az geleneksel olmasına rağmen kesirli analiz, uzun hafızalı finansal süreçlerin ve oynaklığın modellenmesinde ortaya çıkmaya başlamıştır.
Saf Matematik ve Teorik İlerlemeler: Kesirli analiz, klasik sonuçları genişleterek şunlarla bağlantı kurar: İntegral dönüşümler (Laplace, Mellin, Mittag-Leffler fonksiyonları). Özel fonksiyonlar ve diferansiyel denklemler. Eşitsizlikler şunları mümkün kılar: Fonksiyon uzayı yerleştirmelerinin oluşturulması. Serilerin ve integrallerin yakınsamasının kanıtlanması. Konveks analiz, şunların temelini oluşturur: Fonksiyonel analiz (örneğin, Banach uzaylarındaki

konveks kümeler). Kısmi diferansiyel denklemlerin çözümünde kritik olan varyasyonel yöntemler. Önem ve Teorik İçgörüler Bu konular sadece teorem koleksiyonları değildir; derin yapısal gerçekleri ortaya çıkarırlar: Kesirli analiz, tam sayı düzenindeki hesaplamayı genelleştirir ve yerel olmayan ve belleğe bağlı dinamikleri doğal olarak yakalayan esnek araçlar sunar. Eşitsizlikler, nesneler arasındaki ilişkileri nicelleştirir ve genellikle ayrık ve sürekli matematiği birbirine bağlar. Konveks fonksiyonlar, yerel minimumların küresel olmasını sağlayarak karmaşık optimizasyon problemlerini basitleştirir. Teorik derinlikleri yeni çalışma dallarına ilham vermiştir: Genelleştirilmiş konvekslik (invex, preinvex fonksiyonlar) daha geniş fonksiyon sınıflarını araştırır. Yeni kesirli operatörler (örn. Caputo-Fabrizio, Prabhakar) farklı bellek çekirdeklerini yakalar. Keskin eşitsizlikler, harmonik analizi ve bilgi teorisini ilerletir. Modern Etki ve Disiplinlerarası Erişim Kontrol ve Robotik Kesirli denetleyiciler, mekanik sistemlerdeki ve robotik dinamiklerdeki belirsizlikleri tam sayı mertebeli denetleyicilerden daha iyi ele alır. Biyomedikal Mühendisliği Kesirli modeller, ilaç dağıtım dinamikleri ve doku viskoelastisitesi gibi karmaşık biyolojik sistemleri tanımlar. Görüntü ve Sinyal İşleme Konveks optimizasyon yöntemleri, görüntü yeniden yapılandırma, gürültü giderme ve özellik çıkarma işlemlerini destekler. Eşitsizlikler, hata analizini ve kararlılığı yönlendirir. Çevre Bilimi Yeraltı suyundaki ve atmosferik dağılımdaki anormal taşıma süreçleri, doğal olarak kesirli diferansiyel denklemlerle modellenir. Yapay Zeka Konveks analiz, verimli algoritma tasarımına olanak tanır. Yeni araştırma, kesirli operatörleri grafik sinir ağları ve takviyeli öğrenmeyle birleştirir. Sonuç Kesirli analiz, eşitsizlikler ve dışbükey fonksiyonlar, matematiğin soyutlama yoluyla nasıl evrildiğini ve gerçek dünyadaki zorluklarda nasıl yankı bulduğunu örneklendirir. Meraktan vazgeçilmez araçlara uzanan tarihi yolculukları, matematiksel düşüncenin yüzyılları ve disiplinleri birbirine bağlama gücünü gösterir. Günümüzde mühendislik, veri bilimi, ekonomi ve fizikteki ilerlemelere rehberlik ederken, teorik keşfe ilham vermeye devam ediyorlar. Veri ve karmaşıklık tarafından giderek daha fazla yönlendirilen bir dünyada, bu kavramlar her zamankinden daha hayati önem taşıyor; kesin modeller, zarif çözümler ve evrenimizi şekillendiren kalıpların daha derin bir anlayışını sağlıyor.

# 2. KURAMSAL TEMELLER

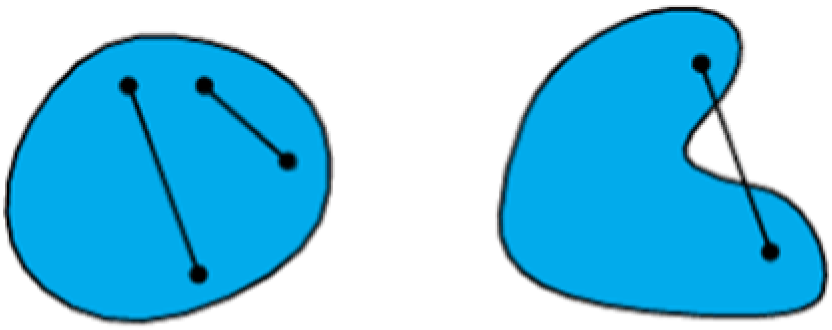
## 2.1. Genel Kavramlar

Bu bölümde, çalışmamın temel taşı olan bazı kavramlar verilecektir.

**Tanım 2.1** ***(Konveks Küme):*** *L bir lineer uzay $A \subseteq L$ ve $\eta$,$\theta \in A$ keyfi olmak üzere*

$$B = \{z \in: z = t\eta + (1-t)\theta, \qquad 0 \leq t \leq 1\} \subseteq A$$

*İse $A$ kümesine konveks küme denir.Eğer $z \in B$ ise z=t$\eta$+(1-t)$\theta$ eşitliğindeki $\eta$ ve $\theta$' nın katsayıları için t+(1-t)=1 bağıntısı her zaman doğrudur.Bu sebeple konveks küme tanımındaki t,1-t yerine t+k=1 şartını sağlayan ve negatif olmayan t,k reel sayıları alınabilir. Geometrik olarak $B$ kümesi uç noktaları x ve y olan bir doğru parçasıdır.Bu durumda sezgisel olarak konveks küme , boş olmayan ve herhangi iki noktasını birleştiren doğru parçasını ihtiva eden kümedir (Bayraktar, 2000).*



**Şekil 2.1. Konveks küme ve Konveks olmayan küme**

**Tanım 2.2** ***(Konveks Fonksiyon):*** *"$I$ $\mathbb{R}$'de bir aralık $f$: $I \to \mathbb{R}$ bir fonksiyon olmak üzere her $\eta$, $\theta \in I$ ve t $\in$ [0,1] için,*

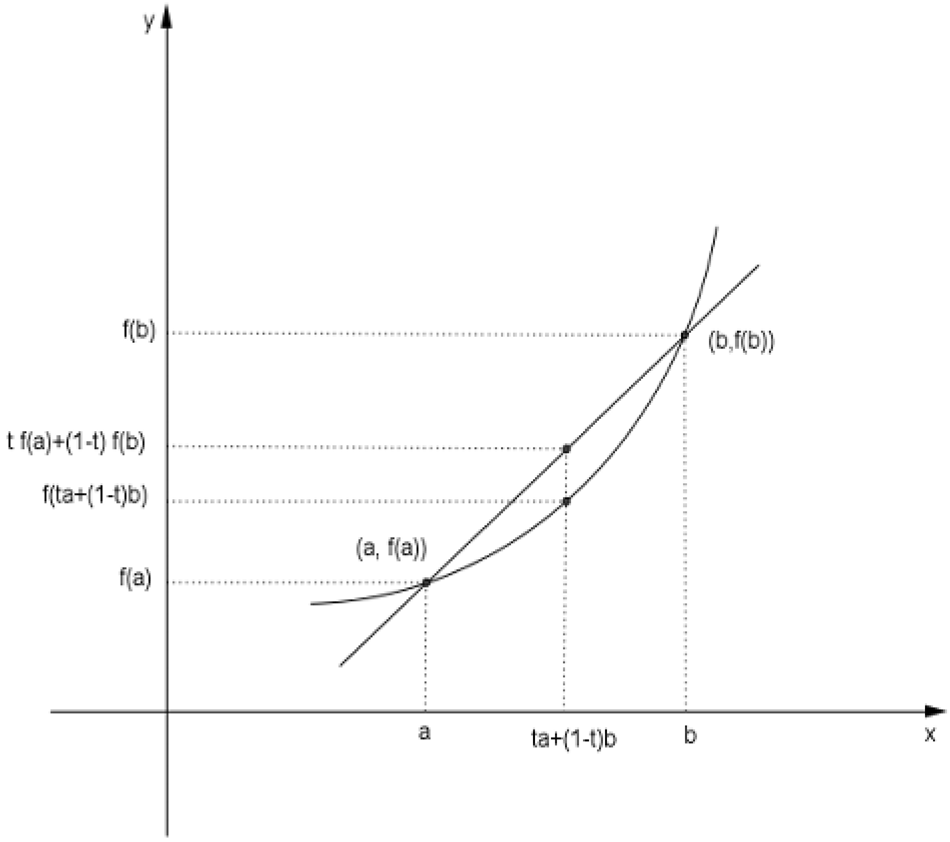
$$f(t\eta + (1-t)\theta) \leq tf(\eta) + (1-t)f(\theta)$$

*şartını sağlayan f fonksiyonuna konveks fonksiyon denir"(Pečarić and Tong, 1992).*

*Eğer $\iota_1 \in$ (0,1) aralığında alınırsa bu durumda*

$$f(\iota_1\xi_1 + (1-\iota_1)\xi_2) < \iota_1 f(\xi_1) + (1-\iota_1)f(\xi_2)$$

*olur.Bu f fonksiyonuna da stritly konveks fonksiyon denir. -f konveks (strictly konveks) ise bu taktirde f'ye konkav (strictly konkav) denir.*




**Şekil 2.2. Aralık üzerinde konvkes fonksiyon**

**Tanım 2.3** ***(h-Konveks Fonksiyon):***"$h$*:* $J \subseteq \mathbb{R} \to \mathbb{R}$ *pozitif bir fonksiyon olsun.Her* $\xi_1$,$\xi_2 \in I$, $\nu \in$ *(0,1) için*

$$f(\nu\xi_1 + (1-\nu)\xi_2) \leq h(\nu)f(\xi_1) + h(1-\nu)f(\xi_2)$$

*şartını sağlayan negatif olmayan* $f$*:*$I \subseteq \mathbb{R} \to \mathbb{R}$ *fonksiyonuna* $h$*-konveks fonksiyon veya SX(*$h$*,*$I$*) sınıfina aittir denir" (Varošanec, 2007).*

$h(\theta) = \theta$ alınırsa, tüm negatif olmayan konveks fonksiyonlar $SX(h, I)$ sınıfina aittir; $h(\theta) = \frac{1}{\theta}$ alınırsa, $SX(h, I) = Q(I)$; $h(\theta) = 1$ alınırsa, $SX(h, I) \supseteq P(I)$ ve $h(\theta) = \theta^s, s \in (0, 1)$ alınırsa $SX(h, I) \supseteq K_s^2$ olacağı açıktır.

**Tanım 2.4** ***(Birinci Anlamda s-Konveks Fonksiyon):*** *"$\mathbb{R}_+$=[0,$\infty$),f $\mathbb{R}_+ \to \mathbb{R}$ ve 0<s≤1 olsun.$\nu^s$+$\mu^s$=1 olmak üzere her $u$,$\upsilon \in \mathbb{R}_+$ ve her $\nu$,$\mu \geq 0$ için*

$$f(\nu u + \beta\upsilon) \leq \nu^s f(u) + \mu^s f(\upsilon)$$

*eşitsizliği sağlanıyorsa f fonksiyonuna birinci anlamda s-konveks fonksiyon denir"(Hudzik and Maligranda, 1994).*

*Bu fonksiyonların sınıfı $K_s^1$ ile gösterilir (Hudzik and Maligranda, 1994)*

**Tanım 2.5** ***(İkinci Anlamda s-Konveks Fonksiyon):*** *"$\mathbb{R}_+$= [0,$\infty$),f $\mathbb{R}_+ \to \mathbb{R}$ ve 0<s≤1 olsun.$\nu$+$\mu$=1 olmak üzere her $u$,$\upsilon \in \mathbb{R}_+$ ve her $\nu$,$\mu \geq 0$ için*

$$f(\nu u + \mu\upsilon) \leq \nu^s f(u) + \mu^s f(\upsilon)$$

*eşitsizliği sağlanıyorsa f fonksiyonuna birinci anlamda $s$-konveks fonksiyon denir. İkinci anlamda $s$-konveks fonksiyonların sınıfı $K_s^2$ ile gösterilir"(Breckner, 1978).Yukarıda verilen her iki $s$-konvekslik tanımı $s$=1 için bilinen konveksliğe dönüşür.*

*Bu fonksiyonların sınıfı $K_s^2$ ile gösterilir (Breckner, 1978)*

**Tanım 2.6** ***(Eksponansiyel Konveks Fonksiyon):*** *$f : K \subseteq \mathbb{R} \to \mathbb{R}$ fonksiyonu her $\eta$,$\theta \in I$ ,$\iota_1 \in$ [0,1] ve $\nu \in \mathbb{R}$ için*

$$f((1-\iota_1)\eta + \iota_1\theta) \leq (1-\iota_1)\frac{f(\eta)}{e^{\nu\eta}} + \iota_1\frac{f(\theta)}{e^{\nu\theta}}$$

*eşitsizliğini sağlıyorsa eksponansiyel konveks fonksiyondur (Awan and et al, 2018).*

**Tanım 2.7** ***(Beta Fonksiyonu):***

$$\beta(\iota_1, \kappa_1) = \int_0^1 \xi_1^{\iota_1 - 1}(1-\iota_1)^{\kappa_1 - 1} d\xi_1, \qquad \iota_1, \kappa_1 > 0$$

*şeklindedir. Bu eşitlik Euler tipli Beta fonksiyonu ya da birinci eşit Euler integrali olarak adlandırılır. Beta fonksiyonu,*

*(1)* $\beta(\iota_1, \kappa_1)$=$\beta(\kappa_1, \iota_1)$

*(2)* $\beta(\iota_1 + 1, \kappa_1)$=$\frac{\iota_1}{\iota_1+1}\beta(\iota_1, \kappa_1)$

*özelliklerini sağlar (Srivastava and Choi, 2012).*

**Tanım 2.8** ***(Tam Olmayan Beta Fonksiyonu):*** $0 \leq x < 1$ *olmak üzere tam olmayan beta fonksiyonu,*

$$B_x(v, \mu) = \int_0^x t^{v-1}(1-t)^{\mu-1} dt$$

*şeklinde ifade edilir. Ayrıca, $B_x(\mu,v)$ ve $B_{1-x}(v,\mu)$ fonksiyonları birlikte ele alındığında*

$$B_x(\mu,v)+B_{1-x}(v,\mu)=$$
$$\int_0^x t^{\mu-1}(1-t)^{v-1}dt+\int_0^{1-x} t^{v-1}(1-t)^{\mu-1}dt=\int_0^x t^{\mu-1}(1-t)^{v-1}dt+\int_x^1 u^{\mu-1}(1-u)^{v-1}du$$

$$B_x(\mu,v)+B_{1-x}(v,\mu)=\int_0^1 t^{\mu-1}(1-t)^{v-1}dt=B(v,\mu)$$

*sonucuna ulaşılır. Bu eşitlik, tam olmayan beta fonksiyonu ile beta fonksiyonu arasındaki ilişkiyi verir (Gradshteyn and Ryzhik, 1994).*

**Tanım 2.9** ***(Gama Fonksiyonu):***

$$\Gamma(\mu)=\int_0^\infty e^{-\iota_1}t^{\mu-1}d\iota_1.$$

*şeklinde tanımlıdır (Samko et al., 1993).*

**Tanım 2.10** ***(k-Gamma Fonksiyon):*** *k > 0 için k-Gama fonksiyonu şu şekilde verilebilir:*

$$\Gamma_k(x)=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},$$

*x bir karmaşık sayıdır ve R(x) > 0 dır (Kokologiannaki and Krasniqi, 2013).*

**Teorem 2.1** ***(Üçgen Eşitsizliği):*** *Herhangi $\delta$ , $\eta$ reel sayıları için*

$$|\delta+\eta\leq|\delta|+|\eta|,$$
$$||\delta|-|\eta||\leq|\delta-\eta|,$$
$$||\delta|-|\eta||\leq|\delta+\eta|,$$

*ve tümevarım metoduyla*

$$|\delta+...+\delta_z|\leq|\delta|+...+|\delta_z|$$

*eşitsizliği gerçeklenir (Mitrinović et al., 1993).*

**Teorem 2.2** ***(Üçgen Eşitsizliğinin İntegral Versiyonu):*** *$\xi$ , $[\delta,\eta]$ aralığında sürekli gerçek değerli fonksiyon olsun.*

$$\left|\int_\delta^\eta \xi(z)dz\right|\leq\int_\delta^\eta|\xi(z)|dz \qquad (\delta<\eta)$$

*eşitsizliği gerçeklenir (Mitrinović et al., 1993).*

**Teorem 2.3** ***(İntegraller için Hölder Eşitsizliği):*** *$p > 1$ ve $\frac{1}{p}+\frac{1}{q}$=1 olsun. $\xi_1$ ve $\xi_2$ , $[\rho,\sigma]$ aralığında tanımlı reel fonksiyonlar $|\xi_1|^p$ ve $|\xi_2|^q$ , $[\rho,\sigma]$ aralığında integrallenebilir fonksiyonlar ise ;*

$$\int_{\rho}^{\sigma} |\xi_1(\delta)\xi_2(\delta)|d\delta \leq \left(\int_{\rho}^{\sigma} |\xi_1(\delta)|^p d\delta\right)^{\frac{1}{p}} \left(\int_{\rho}^{\sigma} |\xi_2(\delta)|^q d\delta\right)^{\frac{1}{q}}$$

*eşitsizliği gerçeklenir (Mitrinović et al., 1993).*

Ayrıca ; Power mean eşitsizliği , Hölder eşitsizliğinin bir sonucudur ve şu şekilde ifade edilir :

**Sonuç 2.1** ***(Power Mean Eşitsizliği):*** *$q \geq 1$ olsun. $\xi_1$ ve $\xi_2$ , $[\rho,\sigma]$ aralığında tanımlı reel fonksiyonlar $|\xi_1|$ ve $|\xi_2|^q$ , [$\rho$ , $\sigma$] aralığında integrallenebilir fonksiyonlar ise*

$$\int_{\rho}^{\sigma} |\xi_1(\delta)\xi_2(\delta)|d\delta \leq \left(\int_{\rho}^{\sigma} |\xi_1(\delta)|d\delta\right)^{1-\frac{1}{q}} \left(\int_{\rho}^{\sigma} |\delta_1(\delta)||\xi_2(\delta)|^q d\delta\right)^{\frac{1}{q}}$$

*eşitsizliği gerçeklenir (Mitrinović et al., 1993).*

**Teorem 2.4** ***(Young Eşitsizliği):*** *$\theta$,k $>$ 0 için [0,k] üzerinde tanımlı reel değerli , sürekli ve artan fonksiyon olsun. $\theta(0)$=0 , $\rho \in$ [0,k] , ve $\sigma \in$ [0,$\theta(k)$] için Young eşitsizliği aşağıdaki gibidir :*

$$\int_{o}^{\rho} \theta(x) + \int_{0}^{\sigma} \theta^{-1}(x)dx \geq \rho\sigma.$$

*Burada $\theta^{-1}$ , $\theta$ fonksiyounun tersidir. Eğer $\theta(\sigma)$=$\theta(\rho)$ ise eşitsizlik eşitliğe dönüşür.$\theta$ kuvvet fonksiyonu olan $\theta(\omega)$=$\omega^{p-1}$:*

$$\frac{\rho^p}{p} + \left(\frac{p-1}{p}\right)\sigma^{p/p-1} \geq \rho\sigma,$$

*benzer şekilde ;*

$$\frac{\rho^p}{p} + \frac{\sigma^q}{q} \geq \rho\sigma,$$

*dir. Burada $\rho$,$\sigma \geq$ 0 , $p > 1$ ve $\frac{1}{p}+\frac{1}{q}$=1 dir (Mitrinović et al., 2013).*

**Teorem 2.5** ***(Mittag-Leffler Fonksiyonu):***

$$E_\alpha(\iota) = \sum_{k=0}^{\infty} \frac{\iota^\kappa}{\Gamma(a\kappa+1)} \qquad (Re(\alpha) > 0, \iota \in \mathbb{C})$$

*şeklinde tanımlanan bu fonksiyon ismini ünlü Gösta Mittag-Leffler den almnıştır. Üstel fonksiyonun McLaurin seri açılımı göz önüne alındığında , $Re(\alpha) > 0$ olmak üzere üstel fonksiyonun bir genellemesidir ve a=1 olduğunda üstel fonksiyondur. Yukarıda verilen fonksiyon aynı zamanda tek parametreli Mittag-Leffler olarak adlandırılır (Gorenflo et al., 2020).*

**Tanım 2.11** ***(Riemann-Liouville Kesirli İntegral Operatörü):*** *$I \in L_1\left[\kappa_1, \tau\right]$ olmak şartıyla $J^{\nu}_{\kappa_1+}f$ ve $J^{\nu}_{\tau-}f$ Riemann-Liouville kesirli integrallerinin $\xi_1 > 0$ ve $\nu \geq 0$ için tanımları sırasıyla*

$$J^{\nu}_{\kappa+}f = \frac{1}{\Gamma(\nu)}\int_{\kappa_1}^{\omega_1}(\omega_1 - \iota_1)^{\nu-1}f(\iota_1)d\iota_1, \qquad \omega_1 > \kappa_1$$

*ve*

$$J^{\nu}_{\tau-}f = \frac{1}{\Gamma(\nu)}\int_{\omega_1}^{\tau}(t - \omega_1)^{\nu-1}f(\iota_1)d\iota_1, \qquad \omega_1 < \tau$$

*dir. Burada $\Gamma(\nu)$=$\int_0^{\infty} e^{-\iota_1}u^{\nu-1}du$*

$$J^{0}_{\kappa_1+}f(\omega_1) = f(\omega_1)$$

*ve*

$$J^{0}_{\tau-}f(\omega_1) = f(\omega_1)$$

*eşitlikleri geçerlidir (Kilbas et al., 2006).*

**Tanım 2.12** ***(Uyumlu Kesirli İntegral Operatörü):*** *$\alpha \in (z, z+1]$ , n= 0,1,2 ,... , $\beta$= α-z ; $\xi_1$,$\xi_2 \in \mathbb{R}$ ve $\xi_1 < \xi_2$ ve f $\in$ L$[\xi_1, \xi_2]$ olsun. $\alpha > 0$ için $\alpha$ mertebeden sol uyumlu kesirli integral :*

$$(I^{\xi_1}_{\alpha}\upsilon)(\iota) = \frac{1}{z!}\int_a^{\iota}(\iota - \tau)^z(\tau - \xi_1)^{\beta-1}f(\tau)d\tau. \qquad \iota > \xi_1$$

*şeklinde ve $\alpha$. mertebeden sağ uyumlu kesirli integral*

$${}^{\xi_2}(I_{\alpha}\upsilon)(\iota) = \frac{1}{z!}\int_a^{\iota}(\tau - \iota)^z(\xi_2 - \tau)^{\beta-1}f(\tau)d\tau. \qquad \iota < \xi_2$$

*eşitlikleri geçerlidir (Khalil et al., 2014).*

**Tanım 2.13** ***(Raina Kesirli İntegral Operatörü):***

$$\mathcal{J}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),..}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k + \lambda)}x^k$$

*$\rho$,$\lambda > 0$ ve $|x| < \mathbb{R}$ ve $\phi(\tau)$ integrallenebilir bir fonksiyon olacak şekilde $\sigma_1$=($\sigma_1(1)$,...,$\sigma_1(k)$,...) pozitif reeel sayıların sınırlı bir dizisidir. Sol taraflı genelleştirilmiş kesirli integral operatörü :*

$$(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\phi)(x) = \int_a^x (x-\tau)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(x-\tau)^{\rho}]\phi(\tau)d\tau, \qquad (x > a > 0)$$

*Sağ taraflı genelleştirilmiş kesirli integral operatörü :*

$$(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\phi)(x) = \int_x^b (\tau-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(\tau-x)^{\rho}]\phi(\tau)d\tau, \qquad (0 < x < b),$$

*şeklinde tanımlanmıştır. (Raina, 1987).*

# 3. MATERYAL VE YÖNTEM

## 3.1. Farklı Türden Bazı Konveks Fonksiyonlar İçin İntegral Eşitsizlikler

**Teorem 3.1** ***(Hermite-Hadamard Eşitsizliği):*** *$f : I \subset \mathbb{R} \to \mathbb{R}$ reel sayıların $I$ aralığında tanımlanmış konveks fonksiyon ve $\xi_1$, $\xi_2 \in I$ $\xi_1 < \xi_2$ olsun. Aşağıdaki eşitsizlik konveks fonksiyonlar için Hermite-Hadamard eşitsizliği olarak ifade edilir (Pečarić and Tong, 1992).*

$$f\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} f(z)dz \leq \frac{f(\xi_1)+f(\xi_2)}{2}$$

**Teorem 3.2** ***(Ostrowski Eşitsizliği):*** *$\xi_1$ , $\xi_2$ , $\zeta$ , $\xi \in \mathbb{T}$ , $\xi_1 < \xi_2$ ve $\kappa_1 : [\xi_1, \xi_2] \to \mathbb{R}$ tanımlı türevlenebilir bir fonksiyon olsun. Bu taktirde*

$$\left|\kappa_1(\xi) - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} \kappa_1^{\theta}(\zeta)\Delta\zeta\right| \leq \frac{M}{\xi_2-\xi_1}(h_2(\zeta,\xi_1)+h_2(\zeta,\xi_2))$$

*eşitsizliği vardır.$M$=$sup_{\xi_1<\xi<\xi_2}\left|\kappa_1^{\Delta}(\xi)\right|$ dir (Bohner and Matthews, 2008).*

**Teorem 3.3** ***(Simpson Eşitsizliği):*** *$f$ : $I$ = $[\kappa,\tau]$ $\subset \mathbb{R} \to \mathbb{R}$ bir fonksiyon olsun. $||f^{(4)}||_\infty < \infty$ olmak üzere f fonksiyonu $I^0$ üzerinde 4. mertebeden sürekli türevlenebilir fonksiyon olsun. Bu taktirde , aşağıdaki eşitsizlik sağlanır:*

$$\left|\frac{1}{3}\left[\frac{f(\kappa)+f(\tau)}{2}+2f\left(\frac{\kappa+\tau}{2}\right)\right]-\frac{1}{\tau-\kappa}\int_{\kappa}^{\tau} f(\xi)d\xi\right| \leq \frac{1}{2880}||f^4||_\infty(\tau-\kappa)^4.$$

*(Pečarić and Tong, 1992)*

**Teorem 3.4** *$I$ $\mathbb{R}$'de bir aralık $\eta$ , $\theta \in I$ ve $\eta < \theta$ olmak üzere $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ konveks bir fonksiyon olsun.Bu taktirde*

$$\psi\left(\frac{\eta+\theta}{2}\right) \leq \frac{q}{\theta-\eta}\int_{\eta}^{\theta} \psi(z)dz \leq \frac{\psi(\eta)+\psi(\theta)}{2}$$

*eşitsizlik gerçeklenir. Bu eşitsizlik konveks fonksiyonlar için literatürde Hermite-Hadamard eşitsizliği olarak bilinmektedir. (Pečarić and Tong, 1992).*

**Teorem 3.5** *$p > 1$ ve $\frac{1}{p}+\frac{1}{q}$=1 olsun. Eğer $\psi$ ve $\upsilon$ , $[\eta,\theta]$ aralığında tanımlı reel fonksiyon $|\psi|^p$ ve $|\upsilon|^q$ , $[\eta,\theta]$ integrallenebilen fonksiyon ise*

$$\begin{aligned}
&\int_\eta^\theta |\psi(z)\upsilon(z)|dz \\
&\leq \frac{1}{\theta-\eta}\Bigg|\left(\int_\eta^\theta(\theta-z)|\psi(z)|^p dz\right)^{\frac{1}{p}}\left(\int_\eta^\theta(\theta-z)|\upsilon(z)|^q dz\right)^{\frac{1}{q}} \\
&+\left(\int_\eta^\theta(z-\eta)|\psi(z)|^p dz\right)^{\frac{1}{p}}\left(\int_\eta^\theta(z-\theta)|\upsilon(z)|^q dz\right)^{\frac{1}{q}} \\
&\leq \left(\int_\eta^\theta |\psi(z)|^p dz\right)\left(\int_\eta^\theta |\upsilon(z)|^q dz\right)^{\frac{1}{q}}\Bigg|
\end{aligned}$$

*eşitsizliği geçerlidir (Akdemir et al., 2022).*

**Not 3.1** *Her negatif olmayan konveks fonksiyon aynı zamanda üstel tipli konveks fonksiyondur (Kadakal et al., 2021).*

**Not 3.2** *Her üstel tipli konveks fonksiyon $h(\delta)$=$e^{\delta-1}$ için bir $h$ konveks fonksiyondur (Kadakal and İşcan, 2020)*

**Teorem 3.6** *Eğer $\psi$:$\zeta\to\eta$ konveks bir fonksiyon $\upsilon$:$\eta\to\mathbb{R}$ üstel tipli azalmayan konveks bir fonksiyon ise $\upsilon o\psi$:$\zeta\to\mathbb{R}$ üstel tipli konveks bir fonksiyondur (Kadakal and İşcan, 2020).*

**Lemma 3.1** *$\psi$ : $\zeta^o\subset\mathbb{R}\to\mathbb{R}$ fonksiyonu $\zeta^o$ üzerinde diferansiyellenebilen bir dönüşüm olsun. $\eta$ , $\theta\in\zeta^o$ ve $\eta<\theta$ olmak üzere $\psi'\in L[\eta,\theta]$ ise*

$$\frac{\psi(\eta)+\psi(\theta)}{2}-\frac{1}{\theta-\eta}\int_\eta^\theta\psi(z)dz=\frac{\eta-\theta}{2}\int_0^1(1-2\delta)\psi'(\delta\eta+(1-\delta)\theta)d\delta$$

*eşitliği geçerlidir (Dragomir and Agarwal, 1998).*

**Teorem 3.7** *$\psi$ : $\zeta\to\mathbb{R}$ fonksiyonu $\zeta^o$ üzerinde diferansiyellenebilen bir fonksiyon $\eta$ , $\theta\in\zeta^o$ ve $\eta<\theta$ olsun. $\psi'\in L[\eta,\theta]$ olmak üzere eğer $|\psi'|$ , $[\eta,\theta]$ olmak üzere eğer $|\psi'|$ , $[\eta,\theta]$ üstel tipli konveks bir fonksiyon ise*

$$\begin{aligned}
&\left|\frac{\psi(\eta)+\psi(\theta)}{2}-\frac{1}{\theta-\eta}\int_\eta^\theta\psi(z)dz\right| \\
&\leq(\theta-\eta)\left(4\sqrt{e}-e-\frac{7}{2}\right)\zeta(|\psi'(\eta)|,|\psi'(\theta)|)
\end{aligned}$$

*eşitsizliği elde edilir. Burada $\zeta(\nu,\mu)$ , $\mu$ ile $\nu$ nun aritmetik ortalamasıdır (Kadakal and İşcan, 2020).*

**Lemma 3.2** *$\psi : \zeta \subset \mathbb{R} \to \mathbb{R}$ fonksiyonu $\zeta^o$ üzerinde diferansiyellenebilen bir dönüşüm olsun. $\eta$ , $\theta \in \zeta$ ve $\eta < \theta$ olmak üzere $\psi' \in L[\eta, \theta]$ ise*

$$\psi(z) - \frac{1}{\theta - \eta}\int_{\eta}^{\theta} \psi(\xi)d\xi = (\theta - \eta)\int_{0}^{1} p(\delta)\psi'(\delta\eta + (1-\delta)\theta)d\delta$$

*eşitliği geçerlidir. Her $\delta \in [0,1]$ için ,*

$$p(\delta) = \begin{cases} \delta, & \delta \in \left[0, \frac{\theta - z}{\theta - \eta}\right] \\ \delta - 1, & \delta \in \left(\frac{\theta - z}{\theta - \eta}, 1\right] \end{cases}$$

*dir. Burada $z \in [\eta, \theta]$ dir (Alomari and Darus, 2010).*

**Teorem 3.8**

$$\begin{aligned} &\left|\psi(z) - \frac{1}{\theta - \eta}\int_{\eta}^{\theta} \psi(\omega)d\omega\right| \\ &\leq \frac{1}{(\theta-\eta)^{\frac{1}{p}}}\left\{(\theta - z)^{1+\frac{1}{p}}\left(\left(e^{\frac{\theta - z}{\theta - \eta}} - \frac{\theta - z}{\theta - \eta} - 1\right)|\psi'(\theta)|^q\right.\right. \\ &+ \left.\left(e - e^{\frac{z-\eta}{\theta-\eta}} - \frac{\theta - z}{\theta - \eta}\right)|\psi'(\eta)|^q\right)^{\frac{1}{q}} + (z-\eta)^{\frac{p+1}{p}} \\ &\times \left.\left(\left(e - e^{\frac{\theta - z}{\theta - \eta}} - \frac{z - \eta}{\theta - \eta}\right)|\psi'(\eta)|^q + \left(e\frac{z-\eta}{\theta-\eta} - 1\right)|\psi'(\theta)|^q\right)^{\frac{1}{q}}\right\} \end{aligned}$$

*eşitsizşiği geçerlidir.Burada $z \in [\eta, \theta]$ , $p > 1$ ve $\frac{1}{p} + \frac{1}{q}$=1 dir (Yıldız and Yergöz, 2023).*

**Teorem 3.9** *$\psi : \zeta \subset [0, \infty) \to \mathbb{R}$ tanımlı $\zeta^o$ üzerinde diferansiyellenebilen bir dönüşüm olmak üzere $\eta$ , $\theta \in \zeta$ , $\eta$ ve $\psi' \in L\ [\eta, \theta]$ olsun. Eğer $|\psi'|^q$ , $[\eta, \theta]$ üzerinde üstel tipli konveks* *bir fonksiyon ise*

$$
\begin{aligned}
&\left|\psi(z)-\frac{1}{\theta-\eta}\int_{\eta}^{\theta}\psi(\omega)d\omega\right| \\
&\leq \frac{(\theta-\eta)^2}{\theta-z}\left\{\left(\frac{1}{(p+1)(p+2)}\left(\frac{\theta-z}{\theta-\eta}\right)^{p+2}\right)^{\frac{1}{p}}\right. \\
&\times\left(\left(e^{\frac{\theta-z}{\theta-\eta}}-\frac{(\theta-z)^2}{2(\theta-\eta)^2}+\frac{\eta-2\theta+z}{\theta-\eta}\right)|\psi'(\eta)|^q\right. \\
&\left.+\left(e^{\frac{(z-\eta)}{(\theta-\eta)}}-\frac{(\theta-z)^2}{2(\theta-\eta)^2}-\frac{e(z-\eta)}{\theta-\eta}\right)|\psi'(\theta)|^q\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{(p+2)}\left(\frac{\theta-z}{\theta-\eta}\right)^{p+2}\right)^{\frac{1}{2}} \\
&\times\left(\left(1-\frac{(\theta-z)^2}{2(\theta-\eta)^2}-\frac{(z-\eta)e^{\frac{\theta-z}{\theta-\eta}}}{\theta-\eta}\right)|\psi'(\eta)|^q\right. \\
&\left.\left.+\left(e-e^{\frac{(\theta-z)^2}{2(\theta-\eta)^2}}+\frac{(\eta-2\theta+z)e^{\frac{z-\eta}{\theta-\eta}}}{\theta-\eta}\right)|\psi'(\theta)|^q\right)^{\frac{1}{q}}\right\} \\
&+\frac{(\theta-\eta)^2}{z-\eta}\left\{\left(\frac{1}{p+2}\left(\frac{z-\eta}{\theta-\eta}\right)^{p+2}\right)^{\frac{1}{q}}\right. \\
&\times\left(\left(e-e^{\frac{\theta-z}{\theta-\eta}}-\frac{(z-\eta)\left(e^{\frac{\theta-z}{\theta-\eta}}+1\right)}{\theta-\eta}-\frac{(z-\eta)(\eta-2\theta+z)}{2(\theta-\eta)^2}\right)|\psi'(\eta)|^q\right. \\
&\left.+\left(\frac{1}{2}-\frac{(\theta-z)e^{\frac{z-\eta}{\theta-\eta}}}{(\theta-\eta)}-\frac{(\theta-z)(\theta-2\eta+z)}{2(\theta-\eta)^2}\right)|\psi'(\theta)|^q\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{(p+1)(p+2)}\left(\frac{z-\eta}{\theta-\eta}\right)^{p+2}\right)^{\frac{1}{p}} \\
&\times\left(\left(e^{\frac{\theta-z}{\theta-\eta}}-\frac{(\theta-z)^2}{2(\theta-\eta)^2}-\frac{(\theta-z)(e-1)}{\theta-\eta}-\frac{1}{2}\right)|\psi'(\eta)|^q\right. \\
&\left.\left.+\left(e^{\frac{z-\eta}{\theta-\eta}}-\frac{1}{2}-\frac{(\theta+z)^2+4\eta(\eta-\theta-z)}{2(\theta-\eta)^2}\right)|\psi'(\theta)|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

*eşitsizliği geçerlidir. Burada $z\in[\eta,\theta]$ ve $p>1$ için $\frac{1}{p}+\frac{1}{q}$=1 dir (Yıldız and Yergöz, 2023).*

**Lemma 3.3** *f : $[a,b]\subseteq\mathbb{R}^{+}\cup\{0\}\to\mathbb{R}$ , (a,b) aralığında differansiyellenebilir bir fonksiyon ve $a<b$ olsun . $|f'|\in[a,b]$ ve $|f'|$ s-konveks fonksiyon ise $0<\alpha\leq 1$ , $s\in(0,1]$ ve $a<x<$*

*b olmak üzere*

$$
\begin{aligned}
& \frac{(x-a)[(a+x)f(x)-xf(a)]+(b-x)[(x+b)f(x)-xf(b)]}{(b-a)^2} \\
& -\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-a)^{\alpha-1}}\mathcal{J}_{x^-}^{\alpha}f(a)+\frac{b}{(b-x)^{\alpha-1}}\mathcal{J}_{x^+}^{\alpha}f(b)\right) \\
\leq \quad & \frac{(x-a)^2}{(b-a)^2}\left[\left(\frac{a}{\alpha+s+1}+\frac{x}{s+1}\right)f'(x)+\left(\frac{a\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}+\frac{x}{s+1}\right)f'(a)\right] \\
& +\frac{(b-x)^2}{(b-a)^2}\left[\left(\frac{b\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}+\frac{x}{s+1}\right)f'(b)+\left(\frac{b}{\alpha+s+1}+\frac{x}{s+1}\right)f'(x)\right]
\end{aligned}
$$

*eşitliği geçerlidir. (Gao et al., 2017).*

**Teorem 3.10** *f :* $[a,b] \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *, (a,b) aralığında differansiyellenebilir bir fonksiyon ve a < b olsun .* $|f'|^q \in$ *[a,b] ve* $|f'|^q$ *s-konveks fonksiyon ise* $0 < \alpha \leq 1$ *, a < x < b , q > 1 ,* $\frac{1}{p}+\frac{1}{q}$*=1 ve s* $\in (0,1]$ *olmak üzere*

$$
\begin{aligned}
& \left|\frac{(x-a)[(a+x)f(x)-xf(a)]+(b-x)[(x+b)f(x)-xf(b)]}{(b-a)^2}\right. \\
& \left.-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-a)^{\alpha-1}}\mathcal{J}_{x^-}^{\alpha}f(a)+\frac{b}{(b-x)^{\alpha-1}}\mathcal{J}_{x^+}^{\alpha}f(b)\right)\right| \\
& \leq \frac{(x-a)^2}{(b-a)^2}\left(\frac{2^{p-1}a^p}{p\alpha+1}+2^{p-1}x^p\right)^{\frac{1}{q}}\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}} \\
& +\frac{(b-x)^2}{(b-a)^2}1\left(\frac{2^{p-1}b^p}{p\alpha+1}+2^{p-1}x^p\right)^{\frac{1}{q}}\left(\frac{|f'(b)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}
\end{aligned}
$$

*eşitsizliği geçerlidir. (Gao et al., 2017)*

**Teorem 3.11** *f :* $[a,b] \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *, (a,b) aralığında differansiyellenebilir bir fonksiyon ve a < b olsun .* $|f'|^q \in$ *[a,b] ve* $|f'|^q$ *s-konveks fonksiyon ise 0 <* $\alpha \leq$ *1 , a < x < b , q > 1 ve s* $\in (0,1]$ *olmak üzere*

$$
\begin{aligned}
& \left|\frac{(x-a)[(a+x)f(x)-xf(a)]+(b-x)[(x+b)f(x)-xf(b)]}{(b-a)^2}\right. \\
& \left.-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-a)^{\alpha-1}}\mathcal{J}_{x^-}^{\alpha}f(a)+\frac{b}{(b-x)^{\alpha-1}}\mathcal{J}_{x^+}^{\alpha}f(b)\right)\right| \\
& \leq \frac{(x-a)^2}{(b-a)^2}\left(\frac{a}{\alpha+1}+x\right)^{1-\frac{1}{q}}\psi(\alpha,s,x) \\
& +\frac{(b-x)^2}{(b-a)^2}\left(\frac{b}{\alpha+1}+x\right)^{1-\frac{1}{q}}\omega(\alpha,s,x),
\end{aligned}
$$

*eşitsizliği geçerlidir. Burada*

$$\begin{aligned}
\psi(\alpha,s,x) &= \left(\left(\frac{a}{\alpha+s+1}+\frac{x}{s+1}\right)|f'(x)|^q\right.\\
&\quad \left.+\left(\frac{a\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}+\frac{x}{s+1}\right)|f'(a)|^q\right)^{\frac{1}{q}}\\
\omega(\alpha,s,x) &= \left(\left(\frac{a\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}+\frac{x}{s+1}\right)|f'(b)|^q\right.\\
&\quad \left.+\left(\frac{b}{\alpha+s+1}+\frac{x}{s+1}|f'(x)|^q\right)\right)^{\frac{1}{q}}
\end{aligned}$$

*şeklindedir. (Gao et al., 2017).*

**Teorem 3.12** *$f : [a,b] \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ , $(a,b)$ aralığında iki kez differansiyellenebilir bir fonksiyon ve $a < b$ olsun . $|f''|^q \in [a,b]$ ve $|f''|^q$ s-konveks fonksiyon ise $0 < \alpha \leq 1$ , $a < x < b$ ve $s \in (0,1]$ olmak üzere*

$$\begin{aligned}
&\left|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(b)}{b-x}+\frac{(b(\alpha+1)+x)f(x)-xf(a)}{x-a}\right]\right.\\
&\qquad \left.-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(b-x)^{\alpha+1}}\mathcal{J}^{\alpha}_{x^+}f(b)+\frac{b}{(x-a)^{\alpha+1}}\mathcal{J}^{\alpha}_{x^-}f(a)\right)\right|\\
&\qquad\qquad \leq \frac{(b-x)}{(b-a)^2}\left[\left(\frac{x}{\alpha+s+2}+\frac{b}{s+2}\right)|f''(x)|\right.\\
&\qquad\qquad\quad \left.+\left(\frac{x\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(\alpha+s+3)}+\frac{b}{(s+1)(s+2)}\right)|f''(b)|\right]\\
&\qquad\qquad\quad +\frac{(x-a)}{(b-a)^2}\left[\left(\frac{x\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(\alpha+s+3)}+\frac{b}{(s+1)(s+2)}\right)|f''(b)|\right.\\
&\qquad\qquad\quad \left.+\left(\frac{b}{\alpha+s+2}+\frac{x}{s+2}\right)|f''(x)|\right]
\end{aligned}$$

*eşitsizliği geçerlidir (Gao et al., 2017).*

**Teorem 3.13** *$\upsilon \to SX(h,I)$, $\xi_1$, $\xi_2 \in I$, $\xi_1 < \xi_2$ ve $\upsilon \in L_1\,[\xi_1,\xi_2]$ olsun. h-konveks fonksiyolar için ,*

$$\begin{aligned}
\upsilon(\xi_1+\xi_2) &\leq \frac{h(\frac{1}{2}\Gamma(\alpha+1))}{\Gamma(n+1)\Gamma(\alpha-n)(b-n)^{\alpha}}[I^a_{\alpha}\,(\upsilon(\xi_2))+{}^{b}I_{\alpha}\,(\upsilon(\xi_1))]\\
&\leq \frac{h\left(\frac{1}{2}\right)}{\Gamma(n+1)\Gamma(\alpha-n)}[\upsilon(\xi_1)+\upsilon(\xi_2)]\\
&\quad \times\int_0^1 \iota_1^n(1-\iota_1)^{\alpha-n-1}[h(\iota_1)+h(1-\iota_1)]d\iota_1
\end{aligned}$$

*uyumlu kesirli integral eşitsizliği sağlanır. (Uçar and Korkmaz, 2019).*

**Teorem 3.14** *$F \in SX(H_1, I)$ , $G \in SX(H_2, I)$ , $m, n \in I$ , $m < n$ ve $FG \in L_1\,[m,n]$ ve $H_1 H_2$*

$\in L_1\ [0,1]$ *ise*

$$\frac{1}{n-m}\int_m^n F(\theta)G(\theta)d\theta$$
$$\leq M(m,n)\int_0^1 H_1(t)H_2(t)dt + N(m,n)\int_0^1 H_1(t)H_2(1-t)dt$$

*dir. Burada*

$$M(m,n) = F(m)G(m) + F(n)G(n) ve N(m,n) = F(m)G(n) + F(n)G(m)$$

*dir.*

*$F \in SX(H_1, I)$, $G \in SX(H_2, I)$, $m, n \in I$, $m < n$ ve $FG \in L_1[m,n]$ ve $H_1 H_2 \in L_1([0,1])$ ise*

$$\frac{1}{2H_1(\frac{1}{2})h_2(\frac{1}{2})}F\left(\frac{m+n}{2}\right)G\left(\frac{m+n}{2}\right) - \frac{1}{n-m}\int_m^n F(\theta)G(\theta)d\theta$$
$$\leq M(m,n)\int_0^1 H_1(t)H_2(t)dt + N(m,n)\int_0^1 H_1(t)H_2(1-t)dt$$

*dir (Uçar and Korkmaz, 2019).*

**Lemma 3.4** *$\eta_1$, $\eta_2 \in I^0$ ve $\eta_1 < \eta_2$ olduğunda $\psi : I^0 \subseteq \mathbb{R} \to \mathbb{R}$, $I$ üzerinde diferansiyellenebilir bir fonksiyon olsun. Eğer $\psi' \in \mathcal{L}\ [\eta_1, \eta_2]$ ise aşağıdaki eşitsizlik geçerlidir.*

$$-\frac{\psi(\eta_2) + \psi(\eta_2 + \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}(\eta_1 - \eta_2))}{2} +$$
$$\frac{1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}(\eta_1 - \eta_2)}\int_{\eta_2}^{\eta_2 + \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}(\eta_1 - \eta_2)} \psi(x)dx$$
$$= \frac{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}(\eta_1 - \eta_2)}{2}\int_0^1 (1-\lambda_1)\psi'(\eta_2 + \lambda_1 \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}(\eta_1 - \eta_2))d\lambda_1$$

*eşitsizliği geçerlidir. (Tariq et al., 2023).*

**Teorem 3.15**

$$\Bigg| -\frac{\psi(\eta_2) + \psi(\eta_2 + \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}(\eta_1 - \eta_2))}{2}$$
$$+ \frac{1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}(\eta_1 - \eta_2)}\int_{\eta_2}^{\eta_2 + \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}(\eta_1 - \eta_2)} \psi(x)dx \Bigg|$$
$$\leq \frac{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}(\eta_1 - \eta_2)}{2}\left(\Omega_1(z)|\psi'(\eta_2)| + m\Omega_2(z)\|\psi'(\frac{\eta_2}{m})|\right),$$

*$\lambda_1 \in [0,1]$, $(\alpha, m) \in (0,1] \times (0,1]$ ve $z \in [0,1]$ olduğunda aşağıdaki eşitlikler geçerlidir:*

$$\Omega_1(z) = \int_0^1 |1 - 2\lambda_1|(1 - (z\lambda_1)^\alpha)d\lambda_1$$
$$= \frac{2\alpha^2 + 6\alpha + 4 - 2.2^{-\alpha+1}z^\alpha - 4\alpha z^\alpha}{4(\alpha+1)(\alpha+2)},$$

$$\Omega_2(z) = \int_0^1 |1 - 2\lambda_1|(1 - (z(1-\lambda_1))^\alpha)d\lambda_1.$$

*(Tariq et al., 2023).*

**Lemma 3.5** *$\lambda_1,\rho_1$ >0, $\omega_1 \in \mathbb{R}$ ve $\sigma_1$ ,negatif olmayan gerçek sayıların bir dizisidir. $\xi_1 < \xi_2$ ve $\lambda_1 > 0$ olmak üzere $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ , $(\xi_1, \xi_2)$ aralığında bir fonksiyondur. Eğer $\upsilon$' $\in \mathcal{L}$ $[\xi_1, \xi_2]$ ise aşağıdaki eşitsizlik geçerlidir.*

$$
\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}\\
&-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq \frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right]\\
&\times|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\\
&-\int_0^1(\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(\zeta)^{\rho_1}]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta
\end{aligned}
$$

*(Hernández and Vivas-Cortez, 2019).*

***İspat*** *Lemmanın sağ tarafindaki integrallere kısmı integrasyon uygulanırsa*

$$
\begin{aligned}
&\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\psi'(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta\\
&-\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\psi'(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta\\
&=(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\left.\frac{\psi(\zeta\xi_1+(1-\zeta)\xi_2)}{\xi_1-\xi_2}\right|_0^1\\
&-\frac{1}{\xi_2-\xi_1}\int_0^1(1-\zeta)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\iota_1)^{\rho_1}]\\
&\times\psi(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta\\
&+\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\left.\frac{\psi(\zeta\xi_1+(1-\zeta)\xi_2)}{\xi_2-\xi_1}\right|_0^1\\
&-\frac{1}{\xi_2-\xi_1}\int_0^1\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\psi(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta.
\end{aligned}
$$

*elde edilir ve ispat tamamlanmış olur.*

**Teorem 3.16** *$\lambda_1,\rho_1$ >0, $\omega_1 \in \mathbb{R}$ ve $\sigma_1$ , negatif olmayan gerçek sayıların bir dizisidir. $\xi_1 < \xi_2$ ve $\lambda_1 > 0$ olmak üzere $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ , $(\xi_1, \xi_2)$ aralığında bir fonksiyondur. Eğer $\upsilon$'*

$\in \mathcal{L}\,[\xi_1,\xi_2]$ *ise aşağıdaki eşitsizlik geçerlidir.*

$$\begin{aligned}&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\\&\times\left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\Big|\\&\le(\xi_2-\xi_1)\frac{\mathcal{F}^{\sigma_1'}_{\rho_1,\lambda_1+2}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\frac{\upsilon'(\xi_1)+\upsilon'(\xi_2)}{2},\end{aligned}$$

*ve*

$$\sigma_1'=(\kappa_1)=\sigma_1(\kappa_1)\left(1-\frac{1}{2^{\rho_1\kappa_1+\lambda_1}}\right).$$

**Lemma 3.6** *$\xi_1<\xi_2$ olmak üzere $\upsilon:[\xi_1,\xi_2]\to\mathbb{R}$, $(\xi_1,\xi_2)$ aralığında bir fonksiyondur. Eğer $\upsilon$' $\in\mathcal{L}\,[\xi_1,\xi_2]$ ise aşağıdaki eşitsizlik geçerlidir.*

$$\begin{aligned}&\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4k(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\&\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\\&=\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\&\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)d\zeta\end{aligned}$$

*Burada*

$$\begin{aligned}&\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)\\&=[\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]\\&-[\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]\\&-[\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]\\&+[\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-(1-\zeta)\xi_1))^{\rho_1}].\end{aligned}$$

*ile verilen $\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}:[0,1]\to\mathbb{R}$ şeklinde tanımlı bir fonksiyondur. (Tunç et al., 2020)*

***İspat***

$$\begin{aligned}&\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+,\omega_1}\Upsilon(\xi_2)=\xi_2-\xi_1\int_0^1\frac{\psi'((1-\zeta)\xi_1)+\zeta\xi_2}{[\psi(\xi_2)-\psi((1-\zeta)\xi_1)+\zeta\xi_2)]^{1-\frac{\lambda_1}{\kappa_1}}}\\&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1}[\omega_1(\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2))^{\rho_1}]\Upsilon(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta.\end{aligned}$$

*Kısmı integrasyon uygulanınca*

$$\begin{aligned}&\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+,\omega_1}\Upsilon(\xi_2)\\&=\kappa_1[\psi(\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\Upsilon(\xi_1)\\&+(\xi_2-\xi_1)\kappa_1\int_0^1[\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2)]^{\frac{\lambda_1}{\kappa_1}}\\&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1}[\omega_1(\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2))^{\rho_1}]\Upsilon'(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta.\end{aligned}$$

*Benzer şekilde*

$$\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-,\omega_1}\Upsilon(\xi_2)=\int_0^1\frac{(\xi_2-\xi_1)\psi'((1-\zeta)\xi_1+\zeta\xi_2)}{[\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1)]^{1-\frac{\lambda_1}{\kappa_1}}}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1}[\omega_1(\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1))^{\rho_1}]\Upsilon(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta.$$

*Kısmı integrasyon uygulanırsa*

$$\begin{aligned}
&\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-,\omega_1}\Upsilon(\xi_2)\\
&=\kappa_1[\psi(\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\Upsilon(\xi_2)\\
&-(\xi_2-\xi_1)\kappa_1\int_0^1[\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1))^{\rho_1}]\Upsilon'(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta.
\end{aligned}$$

*üstteki eşitsizlikler ve ilişkiyi kullanrak .....$F(z)=f(z)+f(\xi_1+\xi_2-z)$*

$$\begin{aligned}
&\frac{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}{\xi_2-\xi_1}\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}\\
&\times\frac{-1}{4k(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\\
&=\int_0^1[\psi((1-\zeta)\xi_1)+\zeta\xi_2]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&\times\Upsilon'(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta\\
&-\int_0^1[\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2))^{\rho_1}]\\
&\times\Upsilon'(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta.
\end{aligned}$$

*elde edilir. Diğer taraftan $F'(z)=f'(z)+f'(\xi_1+\xi_2-z)$*

$\Upsilon'(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta=f'(\zeta\xi_2+(1-\zeta)\xi_1)-f'(\zeta\xi_1+(1-\zeta)\xi_2)$

*buradan*

$$\int_0^1 [\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$\times\Upsilon'(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta$$
$$=\int_0^1 [\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$\times\upsilon'(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta$$
$$-\int_0^1 [\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$\times\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta$$
$$=\int_0^1 [\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$\times\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta$$
$$-\int_0^1 [\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi((1-\zeta)\xi_1+\zeta\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$\times\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta$$

*ve*

$$\int_0^1 [\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2))^{\rho_1}]$$
$$\times\Upsilon'(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta$$
$$=\int_0^1 [\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2))^{\rho_1}]$$
$$\times\upsilon'(\zeta\xi_2+(1-\zeta)\xi_1)d\zeta$$
$$-\int_0^1 [\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2))^{\rho_1}]$$
$$\times\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta$$
$$=\int_0^1 [\psi(\xi_2)-\psi((1-\zeta)\xi_2+\zeta\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi((1-\zeta)\xi_2+\zeta\xi_1))^{\rho_1}]$$
$$\times\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta$$
$$-\int_0^1 [\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi((1-\zeta)\xi_1+\zeta\xi_2))^{\rho_1}]$$
$$\times\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)d\zeta.$$

*sonuca ulaşılır ve ispat tamamlanmış olur.*

**Teorem 3.17**

$$\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{4k(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$
$$\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]$$
$$\leq \frac{I^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}}{4(\xi_2-\xi_1)(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}[|\upsilon'(\xi_1)|+|\upsilon'(\xi_2)|].$$

$$I^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g} = L^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\xi_2,\xi_2)+L^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\xi_1,\xi_2)-L^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\xi_2,\xi_1)-L^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\xi_1,\xi_1)$$

*ve $L^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\eta,\theta)$ şu şekilde tanımlanır:*

$$L^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\eta,\theta) = \int_{\xi_1}^{\frac{\xi_1+\xi_2}{2}} |\eta-u||\psi(\xi_2-\psi(u)|^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1|\psi(\xi_2)-\psi(\xi_1)|^{\rho_1}]du$$
$$-\int_{\frac{\xi_1+\xi_2}{2}}^{\xi_2} |\eta-u||\psi(\xi_2-\psi(u)|^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1|\psi(\xi_2)-\psi(\xi_1)|^{\rho_1}]du.$$
$$(HernndezandVivas-Cortez,\ 2019).$$

# 4. ARAŞTIRMA BULGULARI

Bu bölümde; literatürde tanımları mevcut Raina kesirli integral operatörü kullanılarak h- ve eksponansiyel konveks fonksiyon sınıfları için yeni integral eşitsizlikleri üzerine çalışmalar yapılacaktır.

## 4.1. h- ve Eksponansiyel Konveks Fonksiyonlar ve Yeni İntegral Eşitsizlikler

**Teorem 4.1** *$\lambda_1$,$\rho_1$ >0, $\omega_1 \in \mathbb{R}$ ve $\sigma_1$ ,negatif olmayan gerçek sayıların bir dizisidir. $\xi_1 < \xi_2$ ve $\lambda_1 > 0$ olmak üzere $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ , $(\xi_1, \xi_2)$ aralığında bir fonksiyondur. Eğer $\upsilon$' $\in \mathcal{L}[\xi_1, \xi_2]$ ve $|\upsilon'|^q$ h-konveks fonksiyon ise Raina kesirli integral operatörü için aşağıdaki eşitsizlik geçerlidir*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]} \\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2) + (\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg| \\
&\leq \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]\left[\frac{\psi(\xi_2)+\psi(\xi_1)}{\xi_2-\xi_1}\right] \\
&-\frac{1}{\xi_2-\xi_1}\left[\int_0^1 (1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right. \\
&\times(h(\zeta)|\psi(\xi_1)| + h(1-\zeta)|\psi(\xi_2)|)d\zeta \\
&\left.+\int_0^1 \zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}](h(\zeta)|\psi(\xi_1)| + h(1-\zeta)|\psi(\xi_2)|)d\zeta\right].
\end{aligned}$$

***İspat*** *Lemma 3.5 ve mutlak değer yardımıyla*

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}\right.$$
$$-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}$$
$$\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|$$
$$\leq \frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}$$
$$\times\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta$$
$$-\int_0^1(\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(\zeta)^{\rho_1}]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta$$

*yazılır. h-konveks fonksiyon tanımı dikkate alınarak ve gerekli düzenlemeler yapılarak*

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right.$$
$$\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|$$
$$\leq \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]\left[\frac{\psi(\xi_2)+\psi(\xi_1)}{\xi_2-\xi_1}\right]$$
$$-\frac{1}{\xi_2-\xi_1}\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right.$$
$$\times(h(\zeta)|\psi(\xi_1)|+h(1-\zeta)|\psi(\xi_2)|)d\zeta$$
$$\left.+\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}](h(\zeta)|\psi(\xi_1)|+h(1-\zeta)|\psi(\xi_2)|d\zeta\right]$$

*sonuca ulaşılır ve ispat tamamlanmış olur.*

**Teorem 4.2** $\lambda_1$ *,* $\rho_1 > 0$*,* $\omega_1 \in \mathbb{R}$ *ve* $\sigma_1$ *, negatif olmayan gerçek sayıların bir dizisidir.* $\xi_1 < \xi_2$ *ve* $\lambda_1 > 0$ *olmak üzere* $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ *,* $(\xi_1, \xi_2)$ *aralığında bir fonksiyondur.Eğer* $\upsilon$*’* $\in \mathcal{L}\ [\xi_1, \xi_2]$ *ve* $|\upsilon'|^q$ *h-konveks fonksiyon ise Raina kesirli integral operatörü için aşağıdaki*

*eşitsizlik geçerlidir*

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right.$$
$$\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|$$
$$\leq\frac{\psi(\xi_2)-\psi(\xi_1)}{2}$$
$$-\frac{1}{\xi_2-\xi_1}\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^1|h(\zeta)\psi(\xi_1)|^q+|h(1-\zeta)\psi(\xi_2)|^q d\zeta\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}$$
$$\left.\times\left(\int_0^1|h(\zeta)\psi(\xi_1)|^q+|h(1-\zeta)\psi(\xi_2)|^q d\zeta\right)^{\frac{1}{q}}\right].$$

***İspat*** *Lemma 3.5 ve mutlak değer yardımıyla*

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}\right.$$
$$-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}$$
$$\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|$$
$$\leq\frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}$$
$$\times\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta$$
$$-\int_0^1(\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(\zeta)^{\rho_1}]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta$$

*yazılır. Hölder eşitsizliği uygulanırsa*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right.\\
&\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|\\
&\leq\frac{\psi(\xi_2)-\psi(\xi_1)}{2}\\
&-\frac{1}{\xi_2-\xi_1}\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\right.\\
&\times\left(\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\\
&\left.\times\left(\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta\right)^{\frac{1}{q}}\right].
\end{aligned}$$

*elde edilir.* $|\upsilon'|^q$ *fonksiyonunun h-konveks fonksiyon oluşu dikkate alınırsa*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right.\\
&\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|\\
&\leq\frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\right.\\
&\times\left(\int_0^1|h(\zeta)\psi(\xi_1)|^q+|h(1-\zeta)\psi(\xi_2)|^q d\zeta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\\
&\left.\times\left(\int_0^1|h(\zeta)\psi(\xi_1)|^q+|h(1-\zeta)\psi(\xi_2)|^q d\zeta\right)^{\frac{1}{q}}\right].
\end{aligned}$$

*sonucu elde edilir ve ispat tamamlanmış olur.*

**Teorem 4.3** $\lambda_1$ *,* $\rho_1>0$*,* $\omega_1\in\mathbb{R}$ *ve* $\sigma_1$ *, negatif olmayan gerçek sayıların bir dizisidir.* $\xi_1<\xi_2$ *ve* $\lambda_1>0$ *olmak üzere* $\upsilon:[\xi_1,\xi_2]\to\mathbb{R}$ *,* $(\xi_1,\xi_2)$ *aralığında bir fonksiyondur. Eğer* $\upsilon$' $\in\mathcal{L}\,[\xi_1,\xi_2]$ *ve* $|\upsilon'|^q$ *h-konveks fonksiyon ise Raina kesirli integral operatörü için aşağıdaki*

*eşitsizlik geçerlidir*

$$
\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq \frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^q|[h(\zeta)\psi(\xi_1)+h(1-\zeta)\psi(\xi_2)]|d\zeta\right)^{\frac{1}{q}}\right.\\
&\times\left(\int_0^1|[h(\zeta)\psi(\xi_1)+h(1-\zeta)\psi(\xi_2)]|d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^q|[h(\zeta)\psi(\xi_1)+h(1-\zeta)\psi(\xi_2)]|d\zeta\right)^{\frac{1}{q}}\\
&\left.\times\left(\int_0^1|[h(\zeta)\psi(\xi_1)+h(1-\zeta)\psi(\xi_2)]|d\zeta\right)^{1-\frac{1}{q}}\right].
\end{aligned}
$$

***İspat*** *Lemma 3.5 ve mutlak değer yardımıyla*

$$
\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}\\
&-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq\frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\\
&-\int_0^1(\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(\zeta)^{\rho_1}]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta
\end{aligned}
$$

*yazılır. Power mean eşitsizliği uygulanırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq\frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^q|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\right)^{\frac{1}{q}}\right.\\
&\times\left(\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^q|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\right)^{\frac{1}{q}}\\
&\left.\times\left(\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\right)^{1-\frac{1}{q}}\right].
\end{aligned}$$

*elde edilir.* $|\upsilon'|^q$ *fonksiyonunun h-konveks fonksiyon oluşu dikkate alınırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq\frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^q|[h(\zeta)\psi(\xi_1)+h(1-\zeta)\psi(\xi_2)]|d\zeta\right)^{\frac{1}{q}}\right.\\
&\times\left(\int_0^1|[h(\zeta)\psi(\xi_1)+h(1-\zeta)\psi(\xi_2)]|d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^q|[h(\zeta)\psi(\xi_1)+h(1-\zeta)\psi(\xi_2)]|d\zeta\right)^{\frac{1}{q}}\\
&\left.\times\left(\int_0^1|[h(\zeta)\psi(\xi_1)+h(1-\zeta)\psi(\xi_2)]|d\zeta\right)^{1-\frac{1}{q}}\right].
\end{aligned}$$

*sonucu elde edilir ve ispat tamamlanmış olur.*

**Teorem 4.4** $\lambda_1$ *,* $\rho_1>0$*,* $\omega_1\in\mathbb{R}$ *ve* $\sigma_1$ *, negatif olmayan gerçek sayıların bir dizisidir.* $\xi_1<\xi_2$ *ve* $\lambda_1>0$ *olmak üzere* $\upsilon:[\xi_1,\xi_2]\to\mathbb{R}$ *,* $(\xi_1,\xi_2)$ *aralığında bir fonksiyondur. Eğer* $\upsilon$*'* $\in\mathcal{L}\,[\xi_1,\xi_2]$ *ve* $|\upsilon'|^q$ *h-konveks fonksiyon ise Raina kesirli integral operatörü için aşağıdaki*

*eşitsizlik geçerlidir*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq \frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\Bigg[\Bigg(\int_0^1\frac{|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho}]|^p d\zeta}{p}\\
&+\int_0^1\frac{[|h(\zeta)\psi(\xi_1)|^q+|h(1-\zeta)\psi(\xi_2)|^q]d\zeta}{q}\Bigg)\\
&+\Bigg(\int_0^1\frac{|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta}{p}\\
&+\int_0^1\frac{[|h(\zeta)\psi(\xi_1)|)^q+|h(1-\zeta)\psi(\xi_2)|^q]d\zeta}{q}\Bigg).
\end{aligned}$$

***İspat*** *Lemma 3.5 ve mutlak değer yardımıyla*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}\\
&-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq\frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\\
&-\int_0^1(\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(\zeta)^{\rho_1}]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta
\end{aligned}$$

*yazılır. Young eşitsizliği uygulanırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq\frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\Bigg[\Bigg(\int_0^1\frac{|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta}{p}\\
&+\int_0^1\frac{(|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta}{q}\Bigg)\\
&+\Bigg(\int_0^1\frac{|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta}{p}\\
&+\int_0^1\frac{(|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q)d\zeta}{q}\Bigg).
\end{aligned}$$

*bulunur.* $|\upsilon'|^q$ *fonksiyonunun h-konveks fonksiyon oluşu dikkate alınırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq\frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\Bigg[\Bigg(\int_0^1\frac{|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho}]|^p d\zeta}{p}\\
&+\int_0^1\frac{[|h(\zeta)\psi(\xi_1)|^q+|h(1-\zeta)\psi(\xi_2)|^q]d\zeta}{q}\Bigg)\\
&+\Bigg(\int_0^1\frac{|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta}{p}\\
&+\int_0^1\frac{[|h(\zeta)\psi(\xi_1)|)^q+|h(1-\zeta)\psi(\xi_2)|^q]d\zeta}{q}\Bigg).
\end{aligned}$$

*elde edilir ve ispat tamamlanmış olur.*

**Sonuç 4.1** *Teorem 4.1 de h($\zeta$)=$\zeta$ alınırsa aşağıdaki eşitsizlik elde edilir:*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\\
&\times\left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\Bigg|\\
&\leq\frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times\Bigg[\frac{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho^1}]\psi(\xi_2)}{\xi_2-\xi_1}-\frac{1}{\xi_2-\xi_1}\\
&\times\int_0^1(1-\zeta)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\iota_1)^{\rho_1}]\\
&\times\left(|\zeta\psi(\xi_1)+(1-\zeta)\psi(\xi_2)|\right)d\zeta\\
&-\frac{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho^1}]\psi(\xi_1)}{\xi_2-\xi_1}-\frac{1}{\xi_2-\xi_1}\\
&\times\int_0^1\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\left(|\zeta\psi(\xi_1)+(1-\zeta)\psi(\xi_2)|\right)d\zeta\Bigg].
\end{aligned}$$

**Sonuç 4.2** *Teorem 4.1 de h($\zeta$)=$\zeta^s$ alınırsa aşağıdaki eşitsizlik elde edilir:*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times\left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\Bigg|\\
&\leq\frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times\Bigg[\frac{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho^1}]\psi(\xi_2)}{\xi_2-\xi_1}-\frac{1}{\xi_2-\xi_1}\\
&\times\int_0^1(1-\zeta)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\\
&\times\left(|\zeta^s\psi(\xi_1)+(1-\zeta)^s\psi(\xi_2)|\right)d\zeta\\
&-\frac{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho^1}]\psi(\xi_1)}{\xi_2-\xi_1}-\frac{1}{\xi_2-\xi_1}\\
&\times\int_0^1\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\left(|\zeta\psi(\xi_1)+(1-\zeta)\psi(\xi_2)|\right)d\zeta\Bigg].
\end{aligned}$$

**Teorem 4.5** *$\lambda_1$ , $\rho_1>0$, $\omega_1\in\mathbb{R}$ ve $\sigma_1$ , negatif olmayan gerçek sayıların bir dizisidir. $\xi_1<\xi_2$ ve $\lambda_1>0$ olmak üzere $\upsilon:[\xi_1,\xi_2]\rightarrow\mathbb{R}$ , $(\xi_1,\xi_2)$ aralığında bir fonksiyondur. Eğer $\upsilon'$ $\in\mathcal{L}\,[\xi_1,\xi_2]$ ve $|\upsilon'|^q$ eksponansiyel konveks fonksiyon ise Raina kesirli integral operatörü için*

*aşağıdaki eşitsizlik geçerlidir*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\right.\\
&\left.\times\left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\right|\\
&\leq \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]\left[\frac{\psi(\xi_2)+\psi(\xi_1)}{\xi_2-\xi_1}\right]\\
&-\frac{1}{\xi_2-\xi_1}\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right.\\
&\times\left(\frac{|\zeta\upsilon(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{|(1-\zeta)\upsilon(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\\
&\left.+\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\left(\frac{|\zeta\upsilon(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{|(1-\zeta)\upsilon(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right].
\end{aligned}$$

***İspat*** *Lemma 3.5 ve mutlak değer yardıumıyla*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}\right.\\
&-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\\
&\left.\times\left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\right|\\
&\leq\frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\\
&-\int_0^1(\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(\zeta)^{\rho_1}]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta.
\end{aligned}$$

*yazılır. Eksponansiyel konveks fonksiyon tanımından ve gerekli düzenlemeler yapılarak*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\right.\\
&\left.\times\left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\right|\\
&\leq \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]\left[\frac{\psi(\xi_2)+\psi(\xi_1)}{\xi_2-\xi_1}\right]\\
&-\frac{1}{\xi_2-\xi_1}\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right.\\
&\times\left(\left|\frac{\zeta\upsilon(\xi_1)}{e^{\omega_1\xi_1}}\right|+\left|\frac{(1-\zeta)\upsilon(\xi_2)}{e^{\omega_1\xi_2}}\right|\right)d\zeta\\
&\left.+\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\left(\left|\frac{\zeta\upsilon(\xi_1)}{e^{\omega_1\xi_1}}\right|+\left|\frac{(1-\zeta)\upsilon(\xi_2)}{e^{\omega_1\xi_2}}\right)\right|d\zeta\right].
\end{aligned}$$

*sonuca ulaşılır ve ispat tamamlanmış olur.*

**Teorem 4.6** *$\lambda_1$ , $\rho_1$ >0, $\omega_1 \in \mathbb{R}$ ve $\sigma_1$ , negatif olmayan gerçek sayıların bir dizisidir. $\xi_1 < \xi_2$ ve $\lambda_1 > 0$ olmak üzere $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ , $(\xi_1, \xi_2)$ aralığında bir fonksiyondur.Eğer $\upsilon$' $\in \mathcal{L}$ $[\xi_1, \xi_2]$ ve $|\upsilon'|^q$ eksponansiyel konveks fonksiyon ise Raina kesirli integral operatörü için aşağıdaki eşitsizlik geçerlidir.*

$$
\begin{aligned}
&\bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]} \\
&\times \left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2) + (\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\bigg| \\
&\le \frac{\psi(\xi_2)-\psi(\xi_1)}{2} \\
&-\frac{1}{\xi_2-\xi_1}\left[\left(\int_0^1 |(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\right. \\
&\times \left(\left(\int_0^1 \frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}\right) d\zeta\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 |\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}} \\
&\left.\times \left(\left(\int_0^1 \frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}\right) d\zeta\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

***İspat*** *Lemma 3.5 ve mutlak değer yardımıyla*

$$
\begin{aligned}
&\bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} \\
&-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]} \\
&\times\left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2) + (\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\bigg| \\
&\le \frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]} \\
&\times\left[\int_0^1 (1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right] \\
&\times|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta \\
&-\int_0^1 (\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(\zeta)^{\rho_1}]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta
\end{aligned}
$$

*elde edilir. Hölder eşitsizliği uygulanırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\le \frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\right.\\
&\times\left(\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\\
&\left.\times\left(\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta\right)^{\frac{1}{q}}\right].
\end{aligned}$$

*elde edilir.* $|\upsilon'|^q$ *foksiyonu eksponansiyel konveks oluşu dikkate alınırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\le \frac{\psi(\xi_2)-\psi(\xi_1)}{2}\\
&-\frac{1}{\xi_2-\xi_1}\left[\left(\int_0^1|(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\right.\\
&\times\left(\left(\int_0^1\frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1|\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\\
&\left.\times\left(\left(\int_0^1\frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}\right].
\end{aligned}$$

*elde edilir ve ispat tamamlanmış olur.*

**Teorem 4.7** *$\lambda_1$ , $\rho_1 > 0$, $\omega_1 \in \mathbb{R}$ ve $\sigma_1$ , negatif olmayan gerçek sayıların bir dizisidir. $\xi_1 < \xi_2$ ve $\lambda_1 > 0$ olmak üzere $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ , $(\xi_1, \xi_2)$ aralığında bir fonksiyondur. Eğer $\upsilon$' $\in \mathcal{L}[\xi_1,\xi_2]$ ve $|\upsilon'|^q$ eksponansiyel konveks fonksiyon ise Raina kesirli integral operatörü için*

*aşağıdaki eşitsizlik geçerlidir*

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right.$$
$$\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|$$
$$\leq\frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}$$
$$\times\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^q\right.\right.$$
$$\left.\times\left(\frac{\zeta|\upsilon(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}d\zeta\right)^{1-\frac{1}{q}}$$
$$+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^q\left(\frac{\zeta|\upsilon(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}$$
$$\left.\times\left(\int_0^1\frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}d\zeta\right)^{1-\frac{1}{q}}\right].$$

***İspat*** *Lemma 3.5 ve mutlak değer yardımıyla*

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}\right.$$
$$-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}$$
$$\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|$$
$$\leq\frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}$$
$$\times\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right]$$
$$\times|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta$$
$$-\int_0^1(\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(\zeta)^{\rho_1}]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta$$

*yazılır . Power-Mean eşitsizliği uygulanırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq \frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^q|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\right)^{\frac{1}{q}}\right.\\
&\times\left(\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^q|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\right)^{\frac{1}{q}}\\
&\left.\times\left(\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta\right)^{1-\frac{1}{q}}\right].
\end{aligned}$$

*elde edilir.* $|\upsilon'|^q$ *fonksiyonu eksponansiyel oluşu dikkate alınırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\\
&\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\Bigg|\\
&\leq \frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\left[\left(\int_0^1|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^q\right.\right.\\
&\left.\times\left(\frac{\zeta|\upsilon(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}\\
&\times\left(\int_0^1\frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^q\left(\frac{\zeta|\upsilon(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}\\
&\left.\times\left(\int_0^1\frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}d\zeta\right)^{1-\frac{1}{q}}\right].
\end{aligned}$$

*yazılır ve ispat tamamlanmış olur.*

**Teorem 4.8** $\lambda_1$ , $\rho_1 > 0$, $\omega_1 \in \mathbb{R}$ *ve* $\sigma_1$ *, negatif olmayan gerçek sayıların bir dizisidir.* $\xi_1 < \xi_2$ *ve* $\lambda_1 > 0$ *olmak üzere* $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ *,* $(\xi_1, \xi_2)$ *aralığında bir fonksiyondur. Eğer* $\upsilon$*'* $\in \mathcal{L}\ [\xi_1, \xi_2]$ *ve* $|\upsilon'|^q$ *eksponansiyel konveks fonksiyon ise Raina kesirli integral operatörü için*

*aşağıdaki eşitsizlik geçerlidir*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\right. \\
&\left.\times\left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\right| \\
&\leq \frac{\psi(\xi_2)-\psi(\xi_1)}{2} \\
&-\frac{1}{\xi_2-\xi_1}\left[\left(\frac{1}{p}\int_0^1 |(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta\right.\right. \\
&+\left(\frac{1}{q}\int_0^1 \frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta \\
&+\left(\frac{1}{p}\int_0^1 |\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta\right. \\
&\left.\left.+\frac{1}{q}\int_0^1\left(\frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)\right].
\end{aligned}$$

***İspat*** *Lemma 3.5 ve mutlak değer yardımıyla*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}\right. \\
&-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]} \\
&\left.\times\left((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)\right)\right| \\
&\leq \frac{(\xi_2-\xi_1)}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]} \\
&\times\left[\int_0^1 (1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right] \\
&\times|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta \\
&-\int_0^1 (\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(\zeta)^{\rho_1}]|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|d\zeta
\end{aligned}$$

*elde edilir. Young eşitsizliği uygulanırsa*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right.\\
&\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|\\
&\leq \frac{\psi(\xi_2)-\psi(\xi_1)}{2}-\frac{1}{\xi_2-\xi_1}\\
&\times\left[\left(\int_0^1\frac{|(1-\zeta)^{\lambda_1+1}\mathcal{F}^{\sigma}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta}{p}\right.\right.\\
&\left.+\int_0^1\frac{(|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta}{q}\right)\\
&+\left(\int_0^1\frac{|\zeta^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta}{p}\right.\\
&\left.+\int_0^1\frac{(|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q)d\zeta}{q}\right).
\end{aligned}$$

*yazılır.* $|\upsilon'|^q$ *fonksiyonu eksponansiyel konveks oluşu dikkate alınırsa*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}_1]}\right.\\
&\left.\times((\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1))\right|\\
&\leq \frac{\psi(\xi_2)-\psi(\xi_1)}{2}\\
&-\frac{1}{\xi_2-\xi_1}\left[\left(\frac{1}{p}\int_0^1|(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]|^p d\zeta\right.\right.\\
&\left.+\left(\frac{1}{q}\int_0^1\frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right.\\
&+\left(\frac{1}{p}\int_0^1|\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]|^p d\zeta\right.\\
&\left.\left.+\frac{1}{q}\int_0^1\left(\frac{\zeta|\upsilon(\xi_1)|^q}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)\right].
\end{aligned}$$

*elde edilir ve ispat tamamlanmış olur.*

**Teorem 4.9** *$\xi_1<\xi_2$ ise $\upsilon:[\xi_1,\xi_2]\in\mathbb{R}$, $(\xi_1,\xi_2)$ aralığında diferansiyellenbilir bir fonksiyondur. Eğer $\upsilon$' $\in\mathcal{L}[\xi_1,\xi_2]$ ve $|\upsilon'|^q$ h-konveks fonksiyon ise Raina kesirli integral operatörü için*

*aşağıdaki eşitsizlik geçerlidir*

$$
\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\right|\\
&\leq(|\upsilon'(\xi_1)|)\\
&\times\left[\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]h(\zeta)d\zeta\\
&+\int_0^1(-)(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]h(\zeta)d\zeta\\
&+\int_0^1(-)(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]h(\zeta)d\zeta\\
&+\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\left.\times\,\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]h(\zeta)d\zeta\right]\\
&+(|\upsilon'(\xi_2)|)\\
&\times\left[\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]h(1-\zeta)d\zeta\\
&+\int_0^1(-)(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]h(1-\zeta)d\zeta\\
&+\int_0^1(-)(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]h(1-\zeta)d\zeta\\
&+\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\left.\times\,\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]h(1-\zeta)d\zeta\right].
\end{aligned}
$$

***İspat*** *Lemma 3.6 kullanılarak ve mutlak değer yardımıyla*

$$
\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4k(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\right|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|d\zeta
\end{aligned}
$$

*yazılır.* $|\upsilon'|^q$ *fonksiyonunun h-konveks oluşu dikkate alınarak*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\Big[(\mathcal{J}^{\sigma_1,k,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\Big]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)(h(\zeta)|\upsilon'(\xi_1)|+h(1-\zeta)|\upsilon'(\xi)|)d\zeta.
\end{aligned}$$

*elde edilir.*

$$\begin{aligned}
&\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)=\\
&[\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]+\\
&-[\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]+\\
&-[\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]\\
&+[\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-(1-\zeta)\xi_1))^{\rho_1}].
\end{aligned}$$

*oluşu dikkate alınırsa*

$$
\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\right|\\
&\leq (|\upsilon'(\xi_1)|)\\
&\times\left[\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]h(\zeta)d\zeta\\
&+\int_0^1(-)(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]h(\zeta)d\zeta\\
&+\int_0^1(-)(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]h(\zeta)d\zeta\\
&+\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\left.\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]h(\zeta)d\zeta\right]\\
&+(|\upsilon'(\xi_2)|)\\
&\times\left[\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]h(1-\zeta)d\zeta\\
&+\int_0^1(-)(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]h(1-\zeta)d\zeta\\
&+\int_0^1(-)(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]h(1-\zeta)d\zeta\\
&+\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\left.\times\ \mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]h(1-\zeta)d\zeta\right].
\end{aligned}
$$

*sonucuna ulaşılır ve ispat tamamlanır.*

**Teorem 4.10** *$\xi_1<\xi_2$ ise $\upsilon:[\xi_1,\xi_2]\in\mathbb{R}$, $(\xi_1,\xi_2)$ aralığında diferansiyellenbilir bir fonksiyondur. Eğer $\upsilon$' $\in\mathcal{L}[\xi_1,\xi_2]$ ve $|\upsilon'|^q$ h-konveks fonksiyon ise Raina kesirli integral operatörü*

*için aşağıdaki eşitsizlik geçerlidir*

$$
\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\Bigg[\Bigg(\int_0^1|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\Bigg)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|^q+h(1-\zeta)\upsilon'(\xi_2)|^q)d\zeta\right)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\Bigg)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|^q+h(1-\zeta)\upsilon'(\xi_2)|^q)d\zeta\right)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^p d\zeta\Bigg)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_2)|^q+h(1-\zeta)\upsilon'(\xi_1)|^q)d\zeta\right)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta\Bigg)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_2)|^q+h(1-\zeta)\upsilon'(\xi_1)|^q)d\zeta\right)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

***İspat*** *Lemma 3.6 kullanılarak ve mutlar değer yardımıyla*

$$
\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|d\zeta
\end{aligned}
$$

*yazılır. Hölder eşitsizliği uygulanırsa*

$$
\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Bigg|\\
&\le\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\Bigg[\Bigg(\int_0^1|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

*elde edilir.* $|\upsilon'|^q$ *fonksiyonunun h-konveks fonksiyon oluşu dikkate alınarak*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\right|\\
&\le \frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[\left(\int_0^1 |(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\right.\right.\\
&\left.\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|^q+h(1-\zeta)\upsilon'(\xi_2)|^q)d\zeta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 |(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\\
&\left.\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|^q+h(1-\zeta)\upsilon'(\xi_2)|^q)d\zeta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 |(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\right.\\
&\left.\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_2)|^q+h(1-\zeta)\upsilon'(\xi_1)|^q)d\zeta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 |(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\\
&\left.\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta\right)^{\frac{1}{p}}\\
&\left.\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_2)|^q+h(1-\zeta)\upsilon'(\xi_1)|^q)d\zeta\right)^{\frac{1}{q}}\right]
\end{aligned}$$

*sonucu elde edilir ve ispat tamamlanmış olur.*

**Teorem 4.11** $\xi_1<\xi_2$ *ise* $\upsilon:[\xi_1,\xi_2]\in\mathbb{R}$ *,* $(\xi_1,\xi_2)$ *aralığında diferansiyellenbilir bir fonksiyondur. Eğer* $\upsilon$*'* $\in\mathcal{L}[\xi_1,\xi_2]$ *ve* $|\upsilon'|^q$ *h-konveks fonksiyon ise Raina kesirli integral operatörü*

*için aşağıdaki eşitsizlik geçerlidir*

$$
\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\right|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[\left(\int_0^1|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^q\\
&\times(|h(\zeta)\upsilon'(\xi_1)|+|h(1-\zeta)\upsilon'(\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|+h(1-\zeta)\upsilon'(\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^q\\
&\times(|h(\zeta)\upsilon'(\xi_1)|+|h(1-\zeta)\upsilon'(\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|+h(1-\zeta)\upsilon'(\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{k}}\right.\\
&\times\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^q\\
&\times(|h(\zeta)\upsilon'(\xi_1)|+|h(1-\zeta)\upsilon'(\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|+h(1-\zeta)\upsilon'(\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^q\\
&\times(|h(\zeta)\upsilon'(\xi_1)|+|h(1-\zeta)\upsilon'(\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\left.\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|+h(1-\zeta)\upsilon'(\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\right].
\end{aligned}
$$

***İspat*** *Lemma 3.6 ve mutlak değer yardımıyla*

$$\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4k(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$
$$\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Bigg|$$
$$\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|d\zeta$$

*yazılır. Power-Mean eşitsizliği uygulanırsa*

$$
\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\right|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[\left(\int_0^1|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^q\\
&\times(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^q\\
&\times(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{k}}\right.\\
&\times\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^q\\
&\times(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\\
&+\left(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right.\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^q\\
&\times(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\left.\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\right].
\end{aligned}
$$

*elde edilir.* $|\upsilon'|^q$ *fonksiyonunun h-konveks fonksiyon oluşu kullanılırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\Bigg[\Bigg(\int_0^1|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^q\\
&\times(|h(\zeta)\upsilon'(\xi_1)|+|h(1-\zeta)\upsilon'(\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|+h(1-\zeta)\upsilon'(\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^q\\
&\times(|h(\zeta)\upsilon'(\xi_1)|+|h(1-\zeta)\upsilon'(\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|+h(1-\zeta)\upsilon'(\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{k}}\\
&\times\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^q\\
&\times(|h(\zeta)\upsilon'(\xi_1)|+|h(1-\zeta)\upsilon'(\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|+h(1-\zeta)\upsilon'(\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^q\\
&\times(|h(\zeta)\upsilon'(\xi_1)|+|h(1-\zeta)\upsilon'(\xi_2)|)d\zeta)^{\frac{1}{q}}\\
&\times\left(\int_0^1(|h(\zeta)\upsilon'(\xi_1)|+h(1-\zeta)\upsilon'(\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\Bigg].
\end{aligned}$$

*elde edilir ve ispat tamamlanmış olur.*

**Teorem 4.12** *$\xi_1<\xi_2$ ise $\upsilon:[\xi_1,\xi_2]\in\mathbb{R}$, $(\xi_1,\xi_2)$ aralığında diferansiyellenbilir bir fonksiyondur. Eğer $\upsilon$' $\in\mathcal{L}\,[\xi_1,\xi_2]$ ve $|\upsilon'|^q$ h-konveks fonksiyon ise Raina kesirli integral operatörü*

*için aşağıdaki eşitsizlik geçerlidir*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}\right]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\Bigg[\Bigg(\int_0^1\frac{|\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho}]|^p d\zeta}{p}\\
&+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q}{q}\Bigg)\\
&+\Bigg(\int_0^1\frac{|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho}])|^p d\zeta}{p}\\
&+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q}{q}\Bigg)\\
&+\Bigg(\int_0^1\frac{|\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho}]|^p d\zeta}{p}\\
&+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q}{q}\Bigg)\\
&+\Bigg(\int_0^1\frac{|\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{k}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta}{p}\\
&+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_2)|^q+|h(1-\zeta)\upsilon'(\xi_1)|^q}{q}\Bigg)\Bigg].
\end{aligned}$$

***İspat*** *Lemma 3.6 kullanılarak ve mutlak değer kullaılarak*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|d\zeta
\end{aligned}$$

*yazılır. Young eşitsizliği uygulanırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\right|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\Bigg[\Bigg(\int_0^1\frac{|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^p d\zeta}{p}\\
&+\int_0^1\frac{|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|^q d\zeta}{q}\Bigg)\\
&+\Bigg(\int_0^1\frac{|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho}]|^p d\zeta}{p}\\
&+\int_0^1\frac{|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|^q d\zeta}{q}\Bigg)\\
&+\Bigg(\int_0^1\frac{|(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho}]|^p d\zeta}{p}\\
&+\int_0^1\frac{|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|^q d\zeta}{q}\Bigg)\\
&+\Bigg(\int_0^1\frac{|(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta}{p}\\
&+\int_0^1\frac{|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|^q d\zeta}{q}\Bigg)\Bigg].
\end{aligned}$$

*bulunur.* $|\upsilon'|^q$ *h-konveks fonksiyon oluşu kullanılırsa*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times\left[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}\right]\right|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[\left(\int_0^1\frac{|\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho}]|^p d\zeta}{p}\right.\right.\\
&\left.+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q}{q}\right)\\
&+\left(\int_0^1\frac{|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho}])|^p d\zeta}{p}\right.\\
&\left.+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q}{q}\right)\\
&+\left(\int_0^1\frac{|\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho}]|^p d\zeta}{p}\right.\\
&\left.+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q}{q}\right)\\
&+\left(\int_0^1\frac{|\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{k}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta}{p}\right.\\
&\left.\left.+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_2)|^q+|h(1-\zeta)\upsilon'(\xi_1)|^q}{q}\right)\right].
\end{aligned}$$

*elde edilir ve ispat tamamlanmış olur.*

**Sonuç 4.3** *Teorem 4.12 de h(ζ)=ζ alınırsa aşağıdaki eşitsizlik geçerlidir :*

$$
\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times\,[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\right|\\
&\leq \frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\times\\
&(-)|\upsilon'(\xi_1)|\\
&\times\Big[\int_0^1\kappa_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]d\zeta\\
&+\Big(\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&-\kappa_1\int_0^1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\,\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]d\zeta\Big)\\
&+\Big(\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&-\kappa_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\,\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]d\zeta\Big)\\
&+\int_0^1\kappa_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\,\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]d\zeta\Big]\\
&+|\upsilon'(\xi_2)|\\
&\times\Big[-\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&+\kappa_1\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-(\psi(\xi_1))^{\rho_1}]d\zeta\\
&(-)\kappa_1\int_0^1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]d\zeta\\
&+\kappa_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]d\zeta\\
&+\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma,2\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&+\kappa_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\,\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]d\zeta\Big].
\end{aligned}
$$

**Sonuç 4.4** *Teorem 4.12 de $h(\zeta)$=$\zeta^s$ alınırsa aşağıdaki eşitsizlik geçerlidir :*

$$\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$
$$\times[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\Bigg|$$
$$\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$
$$\times|\upsilon'(\xi_1)|$$
$$\times\Bigg[\Bigg(-\kappa_1 t\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\Bigg)$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]\zeta^{\iota_1}d\zeta$$
$$+\Big(-\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$+\kappa_1\iota_1\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]\zeta^{\iota_1}d\zeta\Big)$$
$$+\Big(-\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$+\kappa_1\iota_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]\zeta^{\iota_1}d\zeta\Big)$$
$$+\Bigg(-\kappa_1\iota_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]\zeta^{\iota_1}d\zeta\Big)$$
$$+|\upsilon'(\xi_2)|$$
$$\times\Big[-\Big(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}$$
$$+\kappa_1\iota_1\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}](1-\zeta)^{\iota_1}d\zeta\Big)$$
$$+\Bigg(-\kappa_1\iota_1\int_0^1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}](1-\zeta)^{\iota_1}d\zeta\Big)$$
$$+\Bigg(-\kappa_1\iota_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}](1-\zeta)^{\iota_1}d\zeta\Big)$$
$$+\Big[-\Big(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}$$
$$+\kappa_1\iota_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}](1-\zeta)^{\iota_1}d\zeta\Big].$$

**Teorem 4.13** $\xi_1 < \xi_2$ *ise* $\upsilon :$ $[\xi_1, \xi_2] \in \mathbb{R}$, $(\xi_1, \xi_2)$ *aralığında diferansiyellenbilir bir fonksiyondur. Eğer* $\upsilon$*'* $\in \mathcal{L}$ $[\xi_1, \xi_2]$ *ve* $|\upsilon'|^q$ *eksponansiyel konveks fonksiyon ise Raina kesirli* *integral operatörü için aşağıdaki eşitsizlik geçerlidir*

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.$$
$$\left.\times\left[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}\right]\right|$$
$$\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\times$$
$$(-)\frac{|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}$$
$$\times\left[\int_0^1 k(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\right.$$
$$\times\mathcal{F}^{\sigma_1;\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]d\zeta$$
$$+\Big(\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$-\kappa_1\int_0^1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]d\zeta\Big)$$
$$+\Big(\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$-\kappa_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]d\zeta\Big)$$
$$+\int_0^1\kappa_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\left.\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]d\zeta\right]$$
$$+\frac{|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}$$
$$\times\left[-\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\right.$$
$$+\kappa_1\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1;\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-(\psi(\xi_1))^{\rho_1}]d\zeta$$
$$(-)\kappa_1\int_0^1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1;\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]d\zeta$$
$$+\kappa_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]d\zeta$$
$$+\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma,2\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$+\kappa_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}$$
$$\left.\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]d\zeta\right].$$

*eşitsizliği geçerlidir.*

***İspat*** *Lemma 3.6 kullanılarak ve mutlak değer yardımıyla*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|d\zeta
\end{aligned}$$

*yazılır. Eksponansiyel konveks fonksiyon tanımı dikkate alınarak*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[(\mathcal{J}^{\sigma_1,k,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi)|}{e^{\omega_1\xi_2}}\right)d\zeta
\end{aligned}$$

*elde edilir ve*

$$\begin{aligned}
&\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)=\\
&[\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&-[\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]\\
&-[\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]\\
&+[\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1)]^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-(1-\zeta)\xi_1))^{\rho_1}].
\end{aligned}$$

*oluşu dikkate alınırsa*

$$
\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times\left[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}\right]\right|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\times\\
&(-)\frac{|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}\\
&\times\left[\int_0^1 k(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\right.\\
&\times\mathcal{F}^{\sigma_1;\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]d\zeta\\
&+\Big(\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&-\kappa_1\int_0^1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]d\zeta\Big)\\
&+\Big(\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&-\kappa_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]d\zeta\Big)\\
&+\int_0^1\kappa_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\left.\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]d\zeta\right]\\
&+\frac{|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\\
&\times\left[-\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\right.\\
&+\kappa_1\int_0^1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\mathcal{F}^{\sigma_1;\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-(\psi(\xi_1))^{\rho_1}]d\zeta\\
&(-)\kappa_1\int_0^1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\mathcal{F}^{\sigma_1;\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]d\zeta\\
&+\kappa_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]d\zeta\\
&+\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\mathcal{F}^{\sigma,2\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&+\kappa_1\int_0^1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}+1}\\
&\left.\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+2\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]d\zeta\right].
\end{aligned}
$$

*sonucuna ulaşılır ve ispat tamamlanır.*

**Teorem 4.14** *$\xi_1 < \xi_2$ ise $\upsilon : [\xi_1, \xi_2] \in \mathbb{R}$ , $(\xi_1, \xi_2)$ aralığında diferansiyellenbilir bir fonksiyondur. Eğer $\upsilon$' $\in \mathcal{L}$ $[\xi_1, \xi_2]$ ve $|\upsilon'|^q$ eksponansiyel konveks fonksiyon ise Raina kesirli integral operatörü için aşağıdaki eşitsizlik geçerlidir*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1} + \mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}\right]\Bigg|\\
&\leq \frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\Bigg[\bigg(\int_0^1 |(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times \mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\bigg)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon'(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}\\
&+\bigg(\int_0^1 |(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times \mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\bigg)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon'(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}\\
&+\bigg(\int_0^1 |(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times \mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^p d\zeta\bigg)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon'(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}\\
&+\bigg(\int_0^1 |(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{k}}\\
&\times \mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta\bigg)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon'(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

***İspat*** *Lemma 3.6 kullanılarak ve mutlak değer yardımıyla*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4k(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\\
&\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|d\zeta
\end{aligned}$$

*yazılır. Hölder eşitsizliği uygulanırsa*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\Bigg|\\
&\times\Bigg[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\Bigg[\Bigg(\int_0^1|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|\zeta\upsilon'(\xi_1)|^q+|(1-\zeta)\upsilon'(\xi_2)|^q)d\zeta\right)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|\zeta\upsilon'(\xi_1)|^q+|(1-\zeta)\upsilon'(\xi_2)|^q)d\zeta\right)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|\zeta\upsilon'(\xi_1)|^q+|(1-\zeta)\upsilon'(\xi_2)|^q)d\zeta\right)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}}\\
&\times\left(\int_0^1(|\zeta\upsilon'(\xi_1)|^q+|(1-\zeta)\upsilon'(\xi_2)|^q)d\zeta\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

*elde edilir.* $|\upsilon'|^q$ *eksponansiyel konveks fonksiyon oluşu dikkate alınarak*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right| \\
&\times\left[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1} + \mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\right| \\
&\leq \frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]} \\
&\times\left[\left(\int_0^1 |(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right. \\
&\times \mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}} \\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon'(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 |(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right. \\
&\times \mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}} \\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon'(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 |(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\right. \\
&\times \mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}} \\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon'(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 |(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\right. \\
&\times \mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta\Big)^{\frac{1}{p}} \\
&\left.\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|^q}{e^{\omega_1\xi_1}} + \frac{(1-\zeta)|\upsilon'(\xi_2)|^q}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{\frac{1}{q}}\right].
\end{aligned}$$

*elde edilir ve ispat tamamlanmış olur.*

**Teorem 4.15** $\xi_1 < \xi_2$ *ise* $\upsilon : [\xi_1, \xi_2] \in \mathbb{R}$, $(\xi_1, \xi_2)$ *aralığında diferansiyellenbilir bir fonksiyondur. Eğer* $\upsilon$*'* $\in \mathcal{L}[\xi_1, \xi_2]$ *ve* $|\upsilon'|^q$ *eksponansiyel konveks fonksiyon ise Raina kesirli integral* *operatörü için aşağıdaki eşitsizlik geçerlidir*

$$
\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho}]}\\
&\times\Bigg[\Bigg(\int_0^1|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]^q\\
&\times\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\Bigg)^{\frac{1}{q}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{1-\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]^q\\
&\times\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\Bigg)^{\frac{1}{q}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{1-\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]^q\\
&\times\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\Bigg)^{\frac{1}{q}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{1-\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]^q\\
&\times\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\Bigg)^{\frac{1}{q}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{1-\frac{1}{q}}\Bigg].
\end{aligned}
$$

***İspat*** *Lemma 3.6 kullanılarak ve mutlak değer yardımıyla*

$$\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4k(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$
$$\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Bigg|$$
$$\le\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]$$
$$\times\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|d\zeta$$

*elde edilir. Power-Mean eşitsizliği uygulanırsa*

$$\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$
$$\times[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\Bigg|$$
$$\le\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$
$$\times\Bigg[\Bigg(\int_0^1|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^q(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\Bigg)^{\frac{1}{q}}$$
$$\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}$$
$$+\Bigg(\int_0^1|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^q(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\Bigg)^{\frac{1}{q}}$$
$$\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}$$
$$+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^q(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\Bigg)^{\frac{1}{q}}$$
$$\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}$$
$$+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}$$
$$\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^q(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\Bigg)^{\frac{1}{q}}$$
$$\times\left(\int_0^1(|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|)d\zeta\right)^{1-\frac{1}{q}}\Bigg].$$

*yazılır.* $|\upsilon'|^q$ *fonksiyonunun eksponansiyel konveks oluşu dikkate alınarak*

$$\begin{aligned}
&\Bigg|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\Bigg|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho}]}\\
&\times\Bigg[\Bigg(\int_0^1|(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]^q\\
&\times\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\Bigg)^{\frac{1}{q}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{1-\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]^q\\
&\times\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\Bigg)^{\frac{1}{q}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{1-\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]^q\\
&\times\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\Bigg)^{\frac{1}{q}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{1-\frac{1}{q}}\\
&+\Bigg(\int_0^1|(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\frac{\lambda_1}{\kappa_1}}\\
&\times\mathcal{F}^{\sigma,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]^q\\
&\times\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\Bigg)^{\frac{1}{q}}\\
&\times\left(\int_0^1\left(\frac{\zeta|\upsilon'(\xi_1)|}{e^{\omega_1\xi_1}}+\frac{(1-\zeta)|\upsilon'(\xi_2)|}{e^{\omega_1\xi_2}}\right)d\zeta\right)^{1-\frac{1}{q}}\Bigg].
\end{aligned}$$

*elde edilir ve ispat tamamlanmış olur.*

**Teorem 4.16** $\xi_1<\xi_2$ *ise* $\upsilon:[\xi_1,\xi_2]\in\mathbb{R}$, $(\xi_1,\xi_2)$ *aralığında diferansiyellenbilir bir fonksiyondur.Eğer* $\upsilon$*'* $\in\mathcal{L}[\xi_1,\xi_2]$ *ve* $|\upsilon'|^q$ *eksponansiyel konveks fonksiyon ise Raina kesirli integral* *operatörü için aşağıdaki eşitsizlik geçerlidir*

$$\Big|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$

$$\times\,[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\Big|$$

$$\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$

$$\times\left[\left(\int_0^1\frac{|\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}]|^p d\zeta}{p}\right.\right.$$

$$\left.+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q d\zeta}{q}\right)$$

$$+\left(\int_0^1\frac{|\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}]|^p d\zeta}{p}\right)$$

$$\left.+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q d\zeta}{q}\right)$$

$$+\left(\int_0^1\frac{|\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho_1}]|^p d\zeta}{p}\right.$$

$$\left.+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q d\zeta}{q}\right)$$

$$+\left(\int_0^1\frac{|\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho_1}]|^p d\zeta}{p}\right.$$

$$\left.\left.+\int_0^1\frac{|h(\zeta)\upsilon'(\xi_2)|^q+|h(1-\zeta)\upsilon'(\xi_1)|^q d\zeta}{q}\right)\right]$$

***İspat*** *Lemma 3.6 kullanılarak ve mutlak değer yardımıyla*

$$\Big|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4k(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+k}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}$$

$$\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\Upsilon)(\xi_1)+(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\Upsilon)(\xi_2)\right]\Big|$$

$$\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]\int_0^1\Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)|\upsilon'(\zeta\xi_1+(1-\zeta\xi_2)|d\zeta$$

*elde edilir. Young eşitsizliği kullanılarak*

$$
\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.\\
&\left.\times[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}]\right|\\
&\leq\frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\\
&\times\left[\left(\int_0^1\frac{(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho_1}])^p d\zeta}{p}\right.\right.\\
&\left.+\int_0^1\frac{|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta}{q}\right)\\
&+\left(\int_0^1\frac{(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho_1}])^p d\zeta}{p}\right)\\
&\left.+\int_0^1\frac{|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta}{q}\right)\\
&+\left(\int_0^1\frac{(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho}])^p d\zeta}{p}\right.\\
&\left.+\int_0^1\frac{|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta}{q}\right)\\
&+\left(\int_0^1\frac{(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho}])^p d\zeta}{p}\right.\\
&\left.\left.+\int_0^1\frac{|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta}{q}\right)\right]
\end{aligned}
$$

*elde edilir.* $|\psi'|^q$ *fonksiyonu eksponansiyel konveks oluşu dikkate alınarak*

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}[\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1}]}\right.$$
$$\left.\times\left[\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,b-;\omega_1}+\mathcal{S}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,a+;\omega_1}\right]\right|$$
$$\leq \frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\mathcal{F}^{\sigma,k}_{\rho,\lambda_1+k}[\omega(\psi(\xi_2)-\psi(\xi_1))^{\rho}]}$$
$$\times\left[\left(\int_0^1 \frac{|\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega(\psi(\zeta\xi_1+(1-\zeta)\xi_2)-\psi(\xi_1))^{\rho}]|^p d\zeta}{p}\right.\right.$$
$$\left.+\int_0^1 \frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q d\zeta}{q}\right)$$
$$+\left(\int_0^1 \frac{|\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\frac{\lambda_1}{k}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega(\psi(\zeta\xi_2+(1-\zeta)\xi_1)-\psi(\xi_1))^{\rho}]|^p d\zeta}{p}\right)$$
$$\left.+\int_0^1 \frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q d\zeta}{q}\right)$$
$$+\left(\int_0^1 \frac{|\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{k}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega(\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\rho}]|^p d\zeta}{p}\right.$$
$$\left.+\int_0^1 \frac{|h(\zeta)\upsilon'(\xi_1)|^q+|h(1-\zeta)\upsilon'(\xi_2)|^q d\zeta}{q}\right)$$
$$+\left(\int_0^1 \frac{|\psi(\xi_1)-\psi(\zeta\xi_1+(1-\zeta)\xi_2))^{\frac{\lambda_1}{k}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho,\lambda_1+\kappa_1}[\omega(\psi(\xi_2)-\psi(\zeta\xi_2+(1-\zeta)\xi_1))^{\rho}]|^p d\zeta}{p}\right.$$
$$\left.\left.+\int_0^1 \frac{|h(\zeta)\upsilon'(\xi_2)|^q+|h(1-\zeta)\upsilon'(\xi_1)|^q d\zeta}{q}\right)\right]$$

*elde edilir ve ispat tamamlanmış olur.*

# 5. TARTIŞMA VE SONUÇ

Bu tezde, Raina kesirli integral operatörlerini kullanarak farklı konveks fonksiyon sınıflarını içeren birkaç yeni integral eşitsizlik elde edilmiştir. Elde edilen sonuçlar, klasik analizdeki bilinen birçok eşitsizliği genişletip genelleştirmekle kalmayıp, aynı zamanda genelleştirilmiş konvekslik merceğinden kesirli eşitsizliklere birleşik bir yaklaşım da sunmaktadır. Tez çalışmasının yaklaşımları, h-konveks ve eksponansiyel konveks fonksiyonlar gibi iyi bilinen konvekslik sınıflarına ait fonksiyonlara odaklanmıştır. Genelleştirilmiş Raina kesirli integral operatörünü uygulayarak, Hermite-Hadamard, Ostrowski ve Simpson tipi eşitsizliklerin yeni biçimlerini türeterek, kesirli hesaplamanın sınırları iyileştirmek ve klasik sonuçların daha genelleştirilmiş versiyonlarını oluşturmak için nasıl kullanılabilecegi gösterilmiştir. Genelleştirilmiş çekirdeği ve esnekligi nedeniyle Raina operatörü, kesirli integral eşitsizliklerinin incelenmesinde güçlü bir araç olduğunu kanıtlamılştır. Operatöre ek parametrelerin dahil edilmesi, özellikle konveksligin doğası ve genellemeleriyle ilgili olarak daha geniş bir fonksiyonel davranış yelpazesini yaklamayı sağlamıştır. Bu bulgular, kesirli hesaplama ve konveks analizin kesişimindeki giderek artan araştırma literatürüne katkıda bulunmakta ve matematiksel eşitsizlikler, yaklaşım teorisi ve uygulamalı matematiksel modellemede potansiyel uygulamalar sunmaktadır. Dahası, burada geliştirilen eşitsizlikler, sayısal analiz, optimizasyon ve konvekslik ile kesirli operatörlerin önemli bir rol oynadığı çeşitli problemlerde faydalı araçlar olarak kullanılabilir. Gelecekteki çalışmalar için, diger genelleştirilmiş kesirli operatörleri kullanarak benzer eşitsizlikleri keşfetmek ve bu sonuçların kesirli mertebeden diferansiyel denklemleri ve integral denklemleri çözmedeki potansiyel uygulamalarını araştırmak ilgi çekici olacaktır. Daha ileri araştırmalar ayrıca çok boyutlu analoglara odaklanabilir veya bu sonuçları ayrık kesirli ortamlara genişletebilir.

# Kaynakça


Akdemir, A., Nazik, M., and Ozdemir, M. (2022). Simpson's type inequalities for s- convex functions in the fourth sense. *COIA 2022, Bakü, Azerbaycan, Abstract Book*, page 57.

Alomari, M. and Darus, M. (2010). Some ostrowski type inequalities for quasi-convex functions with applications to special means. *RGMIA Res. Rep. Coll*, 13(2):6.

Awan, F. and et al (2018). On exponential convexity and related inequalities. *Journal Name*, XX(YY):ZZZ–ZZZ.

Bayraktar, M. (2000). *Fonksiyonel Analiz*. Yayıncı Bilgisi, Yayın Yeri.

Bohner, M. and Matthews, K. (2008). *Dynamic Inequalities on Time Scales*. Birkhäuser Boston.

Breckner, W. W. (1978). Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen. *Publ. Inst. Math.(Beograd)(NS)*, 23(37):13–20.

Dragomir, S. and Agarwal, R. (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Applied mathematics letters*, 11(5):91–95.

Gao, Z., Li, M., and Wang, J. (2017). On some fractional hermite–hadamard inequalities via s-convex and s-godunova–levin functions and their applications. *Boletín de la Sociedad Matemática Mexicana*, 23(2):691–711.

Gorenflo, R., Loutchko, J., and Luchko, Y. (2020). *Mittag-Leffler Functions, Related Topics and Applications*. Springer.

Gradshteyn, I. S. and Ryzhik, I. M. (1994). *Tables of Integrals, Series and Products*. Academic Press, New York, english translation edition.

Hernández, J. E. and Vivas-Cortez, M. (2019). Hermite-hadamard inequalities type for raina'fractional integral operator using $\eta$- convex functions. *Revista de Matemática Teoría* *y Aplicaciones*, 26(1):1–20.

Hudzik, H. and Maligranda, L. (1994). Some remarks on s-convex functions. *Aequationes mathematicae*, 48(1):100–111.

Kadakal, M. and İşcan, İ. (2020). Exponential type convexity and some related inequalities. *Journal of Inequalities and Applications*, 2020(1):82.

Kadakal, M., Iscan, I., Kadakal, H., and Bekar, K. (2021). On improvements of some integral inequalities. *Honam Mathematical Journal*, 43(3):441–452.

Khalil, R., Al Horani, M., Yousef, A., and Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, 264:65–70.

Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier Science B.V., Amsterdam.

Kokologiannaki, C. G. and Krasniqi, V. (2013). Some properties of the k-gamma function. *Le Matematiche*, 68(1):13–22.

Mitrinović, D. S., Pečarić, J. E., and Fink, A. M. (2013). *Classical and New Inequalities in Analysis*. Springer Science & Business Media.

Mitrinović, D. S., Pečarić, J. E., and Fink, A. M. (1993). *Classical and New Inequalities in Analysis*. Kluwer Academic Publishers, Dordrecht.

Pečarić, J. and Tong, Y. (1992). *Convex functions, partial orderings, and statistical applications*. Academic press.

Raina, R. K. (1987). Certain operators involving the generalized hypergeometric function. *Indian Journal of Pure and Applied Mathematics*, 18(8):713–720.

Samko, S., Kilbas, A., and Marichev, O. (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science Publishers.

Srivastava, H. and Choi, J. (2012). *Zeta and q-Zeta Functions and Associated Series and*

*Integrals*. Elsevier Science Publishers, Amsterdam, London and New York.

Tariq, M., Sahoo, S. K., and Ntouyas, S. K. (2023). Some refinements of hermite–hadamard type integral inequalities involving refined convex function of the raina type. *Axioms*, 12(2):124.

Tunç, T., Budak, H., Usta, F., and Sarıkaya, M. Z. (2020). On new generalized fractional integral operators and related fractional inequalities. *Konuralp Journal of Mathematics*, 8(2):268–278.

Uçar, D. and Korkmaz, F. (2019). Farklı türden fonksiyonlar İçin uyumlu kesirli İntegral eşitsizlikleri. *Uşak Üniversitesi Fen ve Doğa Bilimleri Dergisi*, 3(2):51–64.

Varošanec, S. (2007). On h-convexity. *Journal of Mathematical Analysis and Applications*, 326(1):303–311.

Yıldız, Ç. and Yergöz, A. (2023). An investigation into inequalities obtained through the new lemma for exponential type convex functions. *Filomat*, 37(15):5149–5163.