

T.C.
AĞRI İBRAHİM ÇEÇEN ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI

Sinan ASLAN

EKSPONANSİYEL KONVEKS FONKSİYONLAR İÇİN
KOORDİNATLARDA İNTEGRAL EŞİTSİZLİKLER
DOKTORA TEZİ

TEZ YÖNETİCİSİ

Prof. Dr. Ahmet Ocak AKDEMİR

AĞRI – 2023

ÖZET
DOKTORA TEZİ
EKSPONANSİYEL KONVEKS FONKSİYONLAR İÇİN KOORDİNATLARDA
İNTEGRAL EŞİTSİZLİKLER

Tez Danışmanı: Prof. Dr. Ahmet Ocak AKDEMİR

Jüri: Prof. Dr. Havva KAVURMACI ÖNALAN

Doç. Dr. Mustafa GÜRBÜZ

Doç. Dr. Murat ÇAĞLAR

Doç. Dr. Mustafa Ali DOKUYUCU

Bu tezde, farklı türden bazı eksponansiyel konveks fonksiyonlar dikdörtgensel bölge üzerinde incelenerek bu fonksiyonlar için koordinatlarda çeşitli integral eşitsizlikler elde edilmiştir. Birinci bölüm konveks fonksiyonlar ve Eşitsizlik Teorisi için kısa bir tarihsel bakış içermekte olup literatürde mevcut çalışmalar ile ilgili bilgiler verilmiştir. İkinci ve üçüncü bölümlerde, konveks fonksiyonlar ve eksponansiyel konveks fonksiyonlar için detaylı bir literatür çalışması yapılmıştır. Dördüncü bölümde ilk olarak koordinatlarda eksponansiyel konveks fonksiyon ve farklı türden eksponansiyel konveks fonksiyonlar tanımlanıp özellikleri ispatlanmıştır. Daha sonra eksponansiyel konveks fonksiyonların çarpımının sonucunda elde edilen bazı integral eşitsizlikler ve literatürde var olan iki lemma kullanılarak yeni integral eşitsizlikler elde edilmiştir. Son olarak iki tane yeni lemma ispatlanıp bu lemmalara bağlı olarak yeni teoremler elde edilmiştir.

2023, 249 sayfa

Anahtar Kelimeler: Eşitsizlikler, Hermite– Hadamard Eşitsizliği, Eksponansiyel Konveks Fonksiyon, Eksponansiyel m – Konveks Fonksiyon, Eksponansiyel (α, m) – Konveks Fonksiyon, Eksponansiyel P – Fonksiyon, Eksponansiyel s – Konveks Fonksiyon, Eksponansiyel Quasi– Konveks Fonksiyon, Eksponansiyel h – Konveks Fonksiyon.

ABSTRACT

Ph. D. THESIS

INTEGRAL INEQUALITIES ON THE COORDINATES FOR EXPONENTIALLY CONVEX FUNCTIONS

Advisor of Thesis: Prof. Dr. Ahmet Ocak AKDEMİR

Jury: Prof. Dr. Havva KAVURMACI ÖNALAN

Assoc. Prof. Dr. Mustafa GÜRBÜZ

Assoc. Prof. Dr. Murat ÇAĞLAR

Assoc. Prof. Dr. Mustafa Ali DOKUYUCU

In this thesis, some exponentially convex functions of different types are examined on the rectangular region and various integral inequalities are obtained for these functions on the coordinates. The first chapter includes a brief historical overview of convex functions and inequality theory, and information about existing studies in the literature is given. In the second and third chapters, a detailed literature study has been made for convex functions and exponentially convex functions. In the fourth chapter, firstly, exponentially convex functions and different types of exponential convexity are defined and their properties are proved. Then, some integral inequalities obtained as a result of multiplication of exponentially convex functions are obtained and new inequalities are obtained by using two lemmas existing in the literature. Finally, two new lemmas are proved and new theorems are obtained.

2023, 249 pages

Key Words: Inequalities, Hermite – Hadamard inequality, Exponentially Convex Functions, Exponentially m – Convex Functions, Exponentially (α, m) – Convex Functions, Exponentially P – Function, Exponentially s – Convex Functions, Exponentially Quasi– Convex Functions, Exponentially h – Convex Functions.

TEŞEKKÜR

Doktora tezi olarak sunduğum bu çalışma Ağrı İbrahim Çeçen Üniversitesi Lisansüstü Eğitim Enstitüsü Matematik Anabilim Dalında hazırlanmıştır.

Lisans, Yüksek Lisans ve Doktora çalışmamın her aşamasında, bütün enerjisiyle ve engin birikimiyle beni motive eden, çalışmalarımnda eşsiz katkıları bulunan, bana bilimsel çalışma ve düşünme yeteneğini aşılayan saygıdeğer danışman hocam,

Sayın Prof. Dr. Ahmet Ocak AKDEMİR'e;

ve Matematik Bölümü Öğretim Üyelerine bütün samimiyetimle teşekkür ederim.

Tez çalışmama finansal katkı sunan Ağrı İbrahim Çeçen Üniversitesi Bilimsel Araştırma Projeleri Birimi (BAP)'a teşekkür ederim.

Doktora sürecinde kendilerine ayırmam gereken zamandan kısıp bu zamanı tez çalışmama ayırdığım halde beni yalnız bırakmayan sevgili eşim Fatma ASLAN'a ve çocuklarım Halil İbrahim ASLAN'a, Yusuf Eymen ASLAN'a ve Ahmet Selim ASLAN'a teşekkür ederim.

Son olarak doktora sürecünde dualarını benden esirgemeyen başta babam Erol ASLAN'a, annem Bahar ASLAN'a, "Ağrı Türk Telekom Sosyal Bilimler Lisesi" öğretmenlerine, öğrencilerine ve tüm dostlarıma teşekkür ederim.

13/04/2023

Sinan ASLAN

İçindekiler

ÖZET	ii
ABSTRACT	iii
TEŞEKKÜR	iv
SİMGELER VE KISALTMALAR DİZİNİ	vii
1. GİRİŞ	1
2. KURAMSAL TEMELLER	5
2.1. Genel Kavramlar.....	5
2.2. Bazı Önemli Eşitsizlikler.....	10
3. MATERYAL VE YÖNTEM	12
3.1. Koordinatlarda Konveks Fonksiyonlarla İlgili Genel Kavramlar.....	12
3.2. Koordinatlarda Konveks Fonksiyonlar için Bazı Lemma ve İntegral Eşitsizlikler	17
4. ARAŞTIRMA BULGULARI	29
4.1. Koordinatlarda Ekspansiyel Konveks Fonksiyon Sınıfları	29
4.1.1. Koordinatlarda ekspansiyel konveks fonksiyon	29
4.1.2. Koordinatlarda ekspansiyel m – konveks fonksiyon	34
4.1.3. Koordinatlarda ekspansiyel (α, m) – konveks fonksiyon	39
4.1.4. Koordinatlarda ekspansiyel Godunova-Levin konveks fonksiyon	45
4.1.5. Koordinatlarda ekspansiyel h – konveks fonksiyon	48
4.1.6. Koordinatlarda ekspansiyel J – konveks fonksiyon	53
4.1.7. Koordinatlarda ekspansiyel P – fonksiyon	56
4.1.8. Koordinatlarda ekspansiyel Quasi-konveks fonksiyon	61
4.1.9. Koordinatlarda birinci anlamda ekspansiyel s – konveks fonksiyon ...	65
4.1.10. Koordinatlarda ikinci anlamda ekspansiyel s – konveks fonksiyon	70
4.2. Koordinatlarda Farklı Türden Ekspansiyel Konveks Fonksiyonların Çarpımı Üzerine Eşitsizlikler	75
4.2.1. Koordinatlarda ekspansiyel konveks ve koordinatlarda ikinci anlamda ekspansiyel s – konveks fonksiyonların çarpımı üzerine eşitsizlikler	76
4.2.2. Koordinatlarda ekspansiyel konveks ve koordinatlarda ekspansiyel m – konveks fonksiyonların çarpımı üzerine eşitsizlikler.....	85

4.2.3. Koordinatlarda eksponansiyel m –konveks fonksiyonların çarpımı üzerine eşitsizlikler	95
4.2.4. Koordinatlarda ikinci anlamda eksponansiyel s – konveks fonksiyonların çarpımı üzerine eşitsizlikler.....	105
4.3. Koordinatlarda Eksponansiyel Konveks Fonksiyonlar İçin Bazı Yeni İntegral Eşitsizlikler.....	117
4.4. Koordinatlarda Eksponansiyel Konveks Fonksiyonlar için Yeni Lemma ve İlgili Eşitsizlikler.....	150
5. TARTIŞMA VE SONUÇ	241
Kaynakça	242
ÖZGEÇMİŞ	246

SİMGELER VE KISALTMALAR DİZİNİ

F	Digamma Fonksiyonu
L	Lineer Uzay
R	Reel Sayılar Kümesi
R_+	Pozitif Reel Sayılar Kümesi
I	R 'de Bir Aralık
I°	I 'nın İçi
$L_1[\kappa_1, \kappa_2]$	$[\kappa_1, \kappa_2]$ Aralığında İntegrallenebilen Fonksiyonların Kümesi
$L_2(\Delta)$	Δ Bölgesi Üzerinde İkinci Mertebeden İntegrallenebilen Fonksiyonların Kümesi
F'	F Fonksiyonun Birinci Mertebeden Türevi
$\frac{\partial^2 F}{\partial \zeta \partial \xi}$	F Fonksiyonun ζ ve ξ 'ye Göre Kısmi Türevi
$K_m(\kappa_2)$	m — Konveks Fonksiyonların Sınıfı
$K_m^\alpha(\kappa_2)$	(α, m) — Konveks Fonksiyonların Sınıfı
$SX(h, I)$	h — Konveks Fonksiyonların Sınıfı
$SV(h, I)$	h —Konkav Fonksiyonların Sınıfı
K_s^1	Birinci Anlamda s — Konveks Fonksiyonların Sınıfı
K_s^2	İkinci Anlamda s — Konveks Fonksiyonların Sınıfı
$Q(I)$	Godunova-Levin Fonksiyonu
$P(I)$	P —Fonksiyon
$C(I)$	Konveks Fonksiyonlar Sınıfı
$QC(I)$	Quasi— Konveks Fonksiyonlar Sınıfı
$JQC(I)$	Jensen—Quasi— Konveks Fonksiyonlar Sınıfı
$J(I)$	Jensen— Konveks Fonksiyonlar Sınıfı
$\beta(\tau_1, \tau_2)$	Beta Fonksiyonu

1. GİRİŞ

Eşitsizlikler sadece matematikte değil başta mühendislik olmak üzere birçok bilim dalında önemli bir yere sahiptir. Bu yüzden birçok bilim insanının ilgisini çekmiştir. Polya, Hardy ve Littlewood isimli yazarların 1934'te kaleme aldıkları "Inequalities" isimli kitap eşitsizlik teorisine ilişkin ilk çalışmadır. Bu çalışmada eşitsizlikler teorisi kapsamlı olarak incelenmiş olup daha önce elde edilen eşitsizliklerden farklı olarak yeni eşitsizlikler ve bu eşitsizliklerle ilişkili uygulamalara değinilmiştir. E.F. Beckenbach ve R. Bellman (1961) tarafından 1934-1960 yılları arasında eşitsizlikler teorisi için yeni ve farklı sonuçlar barındıran "Inequalities" adlı başka bir kitap yazmıştır. Ardından Mitrinović (1970) tarafından kaleme alınan "Analytic Inequalities" adlı kitapta daha önceki kitaplarda yer almayan yeni konulara değinilmiştir. Bu çalışmalar dışında Mitrinović vd. (1993) tarafından kaleme alınan "Classical and New Inequalities in Analysis" kitabı ve Pachpatte (2005) tarafından kaleme alınan "Mathematical Inequalities" isimli kitaplar da literatüre büyük katkı sağlayan eserlerdir. Yakın tarihte kendi alanlarında öncü olan birçok bilim insanı bu konuda çok fazla bilimsel çalışma yaparak literatürü genişletmiştir.

Eşitsizlikler teorisiyle birebir ilişkili olan konveks fonksiyon kavramı üzerine birçok bilim insanı birçok çalışma yazmış ve çok fazla integral eşitsizliği literatüre eklemişlerdir. Bu eşitsizliklerden biri de Hermite-Hadamard integral eşitsizliğidir. 1822-1901 yılları arasında yaşayan Hermite, Journal Mathesis dergisinin Ekim 1881 sayısına bu eşitsizliği ispatlamadan göndermiştir. Bu çalışmadan bir süre sonra 1905-1906 yılları arasında J.L.W.V. Jensen isimli bilim insanı Jensen eşitsizliğini ileri sürdü. Bu iki çalışmayla birlikte konveks fonksiyonlar için integral eşitsizlikler popüler hale gelmiş ve birçok bilim insanı bu konu üzerine çalışmalar yapmıştır.

20.yy'ın başından günümüze kadar eşitsizlik teorisi alanında çalışan bilim insanları tarafından yeni konveks fonksiyon sınıfları tanımlanıp özellikleri incelenmiş ve birçok sayıda yeni ve farklı integral eşitsizlikler elde edilmiştir. Mitrinović (1970) ve Beckenbach ve Bellman (1961) gibi birçok bilim insanı konveks fonksiyonlar için eşitsizlikler konusunu incelemişlerdir. J.Pečarić adlı bilim insanının kaleminden günümüze ulaşan (Convex Functions: Ine-

qualities) isimli kaynak yalnız konveks fonksiyonları kullanılarak elde edilen eşitsizlikleri barındıran ilk çalışma olarak karşımıza çıkıyor. Ayrıca Roberts and Varberg (1973), Pečarić et al. (1992), Niculescu and Persson (2006) gibi pek çok kişi konveks fonksiyonlar üzerinde eşitsizliklerle ilgi çok sayıda çalışma yapmışlardır. Bunun yanı sıra birçok bilim insanı konveks fonksiyonları kullanarak eşitsizlikler üzerine birçok bilimsel çalışma yaparak literatüre katkıda bulunmuştur.

Set (2010), “Bazı Farklı Türden Konveks Fonksiyonlar İçin İntegral Eşitsizlikleri” adlı doktora çalışmasında E -konveks ve $E-m$ -konveks fonksiyonlar, farklı türden E -konveks ve $E-m$ -konveks fonksiyonlar için Hermite-Hadamard eşitsizliklerini literatüre kazandırmıştır. Ardından s -konveks, m -konveks, log-konveks, (α, m) -konveks, quasi-konveks, h -konveks ve r -konveks fonksiyonlarını kullanarak yeni ve farklı integral eşitsizlikleri ve bunların genelleştirmelerini de elde ederek literatüre katkıda bulunmuştur.

Yıldız (2011), “Quasi Konveks Fonksiyonlar İçin Eşitsizlikler Ve Uygulamaları” adlı yüksek lisans çalışmasında quasi-konveks fonksiyonlar hakkında derinlemesine bir literatür çalışması yaparak, quasi-konveks fonksiyonlar için Ostrowski, Simpson ve Hermite-Hadamard tipli eşitsizlikleri literatüre kazandırmıştır. Ardından da bu eşitsizlikleri kullanarak yeni sonuçlar bulmuştur ve sonuçları birebir ilgilendiren bazı özel uygulamaları literatüre kazandırmıştır.

Alomari (2011), “Several Inequalities Of Hermite-Hadamard, Ostrowski And Simpson Type For Convex, Quasi Convex And Convex Mappings And Applications” adlı doktora çalışmasında quasi-konveks, r -konveks ve s -konveks fonksiyonlarını kullanarak Ostrowski, Hermite-Hadamard ve Simpson tipli integral eşitsizlikleri ve bu integral eşitsizlikleri için yeni uygulamalar elde etmiştir.

Akdemir (2012), "Farklı Türden Konveks Fonksiyonlar İçin Koordinatlarda İntegral Eşitsizlikler" adlı doktora çalışmasında ilk olarak koordinatlarda yeni konveks fonksiyon sınıflarını tanımlayıp ve bu tanımladıkları fonksiyon sınıflarının belli özelliklerini ispatlayarak yeni integral eşitsizlikleri literatüre kazandırmıştır. Ardından bazı yeni lemmalar ve bu lemmaların ispatlarını yaparak literatüre katkıda bulunmuştur. Yeni bulduğu bu lemmaları kullanarak (α, m) -konveks fonksiyon, m -konveks fonksiyon, P -fonksiyonu, h -konveks fonksiyon ve quasi-konveks fonksiyonları için koordinatlarda integral eşitsizlikleri literatüre

kazandırmıştır. Bu çalışmanın sonunda iki fonksiyonun çarpımını kullanarak yeni integral eşitsizlikleri literatüre kazandırmıştır. Sonuçları ispatlarken konveks fonksiyon sınıflarının tanımları, literatürde iyi bilinen bazı integral eşitlikleri ve iki katlı integraller için Hölder ve Power mean eşitsizlikleri kullanılmıştır.

Kavurmacı (2012), "Bazı Farklı Türden Konveks Fonksiyonlar İçin Ostrowski ve Hermite-Hadamard Tipli İntegral Eşitsizlikler" adlı doktora çalışmasında farklı türden konveks fonksiyonları kullanarak yeni baskın konveks fonksiyon kavramları tanımlamış, bu yeni konveks fonksiyon sınıfları için Hermite-Hadamard tipli integral eşitsizlikler elde etmiştir. Daha sonra sırasıyla konveks fonksiyonlar için yeni Hermite-Hadamard tipli; s -konveks ve m -konveks fonksiyonlar için yeni Ostrowski tipli integral eşitsizlikleri elde etmiştir.

Gürbüz (2013), "Farklı Türden Konveks Fonksiyonların Çarpımı Üzerine İntegral Eşitsizlikleri ve Uygulamaları" adlı doktora çalışmasında farklı türden konveks fonksiyonlar ve bunların bileşkelerine dair bazı özellikler verilerek fonksiyon sınıflarına örnekler vermiştir. Daha sonra farklı türden konveks fonksiyonların çarpımına dair bazı integral eşitsizlikler elde edilmiş elde edilen eşitsizliklerin bazılarına uygulamalar vermiştir.

Ekinci (2014), "Klasik Eşitsizlikler Yoluyla Konveks Fonksiyonlar İçin İntegral Eşitsizlikler" adlı doktora çalışmasında diferansiyellenebilir konveks fonksiyonlar için yeni eşitsizlikler ve genelleştirmeler elde etmiştir. Ayrıca konveks fonksiyonlar sınıfının daha genel halini tanımlayarak bu sınıfın temel özellikleri ve bazı fonksiyonlarla olan ilişkisini açıklayıp quasi-konveks fonksiyonlar ile ilgili eşitsizliklere yer vermiştir.

Neden Matematiksel Eşitsizlikler?

1978 yılında Richard Bellman, Almanya'da 2. Uluslararası Matematik Eşitsizlikler Konferansı'nda bu soruya şu şekilde cevap vermiştir: Eşitsizlik çalışmak için üç neden vardır. Bunlar:

1. Pratik Nedenler
2. Teorik Nedenler
3. Estetik Nedenler

dir. Pratik açıdan bakıldığında, birçok araştırmada bir niceliği diğer bir nicelikle sınırlandırmak karşımıza çıkmaktadır. Klasik eşitsizlikler de bu şekilde ortaya çıkmıştır. Teorik açıdan bakıldığında çok basit sorular sorularak tüm temel teoremler oluşturulabilir. Örneğin, negatif olmayan bir niceliğin ne zaman bir diğerini kapsadığı sorulabilir ve bu basit soru ile Pozitif Operatörler Teorisi ve Diferansiyel Eşitsizlikler Teorisi kurulur. Son olarak estetik açıdan bakıldığında genel olarak resim, müzik ve matematiğin bazı parçalarının uyumlu olduğu görülür. Elde edilen eşitsizliklerin göze hitap etmesi de eşitsizlikleri çekici hale getirir.

Bu çalışmada, farklı türden bazı eksponansiyel konveks fonksiyonlar dikdörtgensel bölge üzerinde incelenerek bu fonksiyonlar için koordinatlarda çeşitli integral eşitsizlikler elde edilmiştir. Birinci bölüm konveks fonksiyonlar ve Eşitsizlik Teorisi için kısa bir tarihsel bakış içermekte olup literatürde mevcut çalışmalar ile ilgili bilgiler verilmiştir. İkinci ve üçüncü bölümlerde, konveks fonksiyonlar ve eksponansiyel konveks fonksiyonlar için detaylı bir literatür çalışması yapılmıştır.

Dördüncü bölümde ilk olarak koordinatlarda eksponansiyel konveks fonksiyon ve farklı türden eksponansiyel konveks fonksiyonlar tanımlanıp özellikleri ispatlanmıştır. Daha sonra eksponansiyel konveks fonksiyonların çarpımının sonucunda elde edilen bazı integral eşitsizlikler ve literatürde var olan iki lemma kullanılarak yeni integral eşitsizlikler elde edilmiştir. Son olarak iki tane yeni lemma ispatlanıp bu lemmalara bağlı olarak yeni teoremler elde edilmiştir.

2. KURAMSAL TEMELLER

2.1. Genel Kavramlar

Tanım 2.1 (Konveks Küme) : L bir lineer uzay $D \subset L$ ve $\tau_1, \tau_2 \in D$ keyfi olmak üzere

$$D = \{\tau_3 \in L : \tau_3 = \zeta \tau_1 + (1 - \zeta) \tau_2, \ 0 \leq \zeta \leq 1\} \subseteq D$$

ise D kümesine konveks küme denir (Bayraktar, 2000).

Tanım 2.2 (J- Konveks Fonksiyon) : I, R 'de bir aralık olmak üzere $\forall \tau_1, \tau_2 \in I$ için

$$F\left(\frac{\tau_1 + \tau_2}{2}\right) \leq \frac{F(\tau_1) + F(\tau_2)}{2}$$

şartını sağlayan F fonksiyonuna I üzerinde Jensen anlamında konveks veya J - konveks fonksiyon denir (Mitrinovic and Vasic, 1970).

Tanım 2.3 (Kesin J- Konveks Fonksiyon) : I, R 'de bir aralık olmak üzere $\forall \tau_1, \tau_2 \in I$ ve $\tau_1 \neq \tau_2$ için

$$F\left(\frac{\tau_1 + \tau_2}{2}\right) < \frac{F(\tau_1) + F(\tau_2)}{2}$$

oluyorsa F fonksiyonuna I üzerinde kesin J - konveks fonksiyon denir (Mitrinovic and Vasic, 1970).

Tanım 2.4 (Konveks Fonksiyon) : I, R 'de bir aralık ve $F : I \rightarrow R$ bir fonksiyon olmak üzere $\forall \tau_1, \tau_2 \in I$ ve $\zeta \in [0, 1]$ için

$$F(\zeta \tau_1 + (1 - \zeta) \tau_2) \leq \zeta F(\tau_1) + (1 - \zeta) F(\tau_2)$$

şartını sağlayan F fonksiyonuna konveks fonksiyon denir (Pečarić and Tong, 1992).

Tanım 2.5 (m - Konveks Fonksiyon) : $F : [0, \kappa_2] \rightarrow R$ ve $\kappa_2 > 0$ olsun. $\forall \tau_1, \tau_2 \in [0, \kappa_2]$, $\zeta, m \in [0, 1]$ için

$$F(\zeta \tau_1 + m(1 - \zeta) \tau_2) \leq \zeta F(\tau_1) + m(1 - \zeta) F(\tau_2)$$

şartını sağlayan F fonksiyonuna m - konveks fonksiyon denir (Toader, 1984).

Tanım 2.6 $((\alpha, m) - \text{Konveks Fonksiyon}) : F : [0, \kappa_2] \rightarrow R$ ve $\kappa_2 > 0$ olsun. $\forall \tau_1, \tau_2 \in [0, \kappa_2], \zeta \in [0, 1]$ ve $(\alpha, m) \in [0, 1]^2$ için

$$F(\zeta \tau_1 + m(1 - \zeta) \tau_2) \leq \zeta^\alpha F(\tau_1) + m(1 - \zeta^\alpha) F(\tau_2)$$

şartını sağlayan F fonksiyonuna $(\alpha, m) - \text{konveks fonksiyon}$ denir (Miheşan, 1993).

Burada $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ alınırsa sırasıyla artan, $\alpha - \text{starshaped}$, starshaped , $m - \text{konveks}$, konveks ve $\alpha - \text{konveks}$ fonksiyon sınıfları elde edilir.

Tanım 2.7 (Godunova–Levin Fonksiyonu) : Negatif olmayan $F : I \rightarrow R$ fonksiyonu $\tau_1, \tau_2 \in I, \zeta \in (0, 1)$ olmak üzere

$$F(\zeta \tau_1 + (1 - \zeta) \tau_2) \leq \frac{F(\tau_1)}{\zeta} + \frac{F(\tau_2)}{1 - \zeta}$$

eşitsizliğini sağlıyorsa F 'ya Godunova–Levin fonksiyonu veya $Q(I)$ sınıfına aittir denir. Bu tanıma denk olarak; $F \in Q(I)$ ve $\tau_1, \tau_2, \tau_3 \in I$ ise bu takdirde

$$\begin{aligned} 0 &\leq F(\tau_1)(\tau_1 - \tau_2)(\tau_1 - \tau_3) \\ &\quad + F(\tau_2)(\tau_2 - \tau_1)(\tau_2 - \tau_3) \\ &\quad + F(\tau_3)(\tau_3 - \tau_1)(\tau_3 - \tau_2) \end{aligned}$$

eşitsizliği sağlanır (Godunova and Levin, 1985).

Tanım 2.8 (P –Fonksiyonu) : Negatif olmayan $F : I \rightarrow R$ fonksiyonu $\tau_1, \tau_2 \in I, \zeta \in (0, 1)$ olmak üzere

$$F(\zeta \tau_1 + (1 - \zeta) \tau_2) \leq F(\tau_1) + F(\tau_2)$$

eşitsizliğini sağlıyorsa F 'ya P –fonksiyonu veya $P(I)$ sınıfına aittir denir (Dragomir et al., 1995).

Tanım 2.9 (Birinci Anlamda s – Konveks Fonksiyon) : $\beta_1, \beta_2 \geq 0, \beta_1^s + \beta_2^s = 1$ ve $s \in (0, 1]$ olmak üzere $\forall u_1, u_2 \in R_+$ için $F : R_+ \rightarrow R$ fonksiyonu eğer

$$F(\beta_1 u_1 + \beta_2 u_2) \leq \beta_1^s F(u_1) + \beta_2^s F(u_2)$$

eşitsizliğini sağlıyorsa F 'ya birinci anlamda s – konveks fonksiyon denir. Bu fonksiyonların sınıfı K_s^1 ile gösterilir (Orlicz, 1961).

Tanım 2.10 (İkinci Anlamda s – Konveks Fonksiyon) : $\beta_1, \beta_2 \geq 0, \beta_1 + \beta_2 = 1$ ve $s \in (0, 1]$ olmak üzere $\forall u_1, u_2 \in R_+$ için $F : R_+ \rightarrow R$ fonksiyonu eğer

$$F(\beta_1 u_1 + \beta_2 u_2) \leq \beta_1^s F(u_1) + \beta_2^s F(u_2)$$

eşitsizliğini sağlıyorsa F ’ya ikinci anlamda s – konveks fonksiyon denir. Bu fonksiyonların sınıfı K_s^2 ile gösterilir (Breckner, 1978).

Tanım 2.11 (h – Konveks Fonksiyon) : $h : J \subseteq R \rightarrow R$ negatif olmayan bir fonksiyon olsun. Negatif olmayan $F : I \rightarrow R$ fonksiyonu $\forall \tau_1, \tau_2 \in I$ ve $\zeta \in (0, 1)$ olmak üzere

$$F(\zeta \tau_1 + (1 - \zeta) \tau_2) \leq h(\zeta) F(\tau_1) + h(1 - \zeta) F(\tau_2)$$

şartını sağlayan F fonksiyonuna h – konveks fonksiyon veya $SX(h, I)$ sınıfına aittir denir (Varošanec, 2007).

Tanım 2.12 (Quasi– Konveks Fonksiyon) : $F : T \rightarrow R$ bir fonksiyon ve $T \subset R^n$ boştan farklı konveks küme olsun. $\forall \tau_1, \tau_2 \in T, \zeta \in [0, 1]$ için

$$F(\zeta \tau_1 + (1 - \zeta) \tau_2) \leq \max\{F(\tau_1), F(\tau_2)\}$$

ise F ’ya quasi– konveks fonksiyon denir. Eğer

$$F(\zeta \tau_1 + (1 - \zeta) \tau_2) < \max\{F(\tau_1), F(\tau_2)\}$$

ise F ’ya kesin quasi– konveks fonksiyon denir. Aynı şartlar altında

$$F(\zeta \tau_1 + (1 - \zeta) \tau_2) \geq \max\{F(\tau_1), F(\tau_2)\}$$

ise F ’ya quasi– konkav fonksiyon ve

$$F(\zeta \tau_1 + (1 - \zeta) \tau_2) > \max\{F(\tau_1), F(\tau_2)\}$$

ise F ’ya kesin quasi – konkav fonksiyon denir (Dragomir and Pearce, 1998).

Tanım 2.13 F hem quasi – konveks hem de quasi –konkav ise F ’ya quasi monotonik denir (Greenberg and Pierskalla, 1971).

Tanım 2.14 (Jensen – Quasi – Konveks Fonksiyon) : $F : I \subseteq R \rightarrow R$ fonksiyonu eğer $\forall \tau_1, \tau_2 \in I$ için

$$F\left(\frac{\tau_1 + \tau_2}{2}\right) \leq \max\{F(\tau_1), F(\tau_2)\}$$

şartını sağlıyor ise F ’ya Jensen – quasi – konveks fonksiyon denir (Dragomir and Pearce, 1998).

Tanım 2.15 (α – Star s – Konveks Fonksiyon) : $\alpha, \zeta \in [0, 1]$ olmak üzere $\forall \tau_1, \tau_2 \in I$ için $F : I \subseteq R_+ \rightarrow R$ fonksiyonu eğer

$$F(\zeta \tau_1 + (1 - \zeta) \tau_2) \leq \zeta^s F(\tau_1) + \{(1 - \zeta) \alpha\}^s F(\tau_2)$$

eşitsizliğini sağlıyorsa F 'ya α –star s – konveks fonksiyon denir (Park, 2010).

Tanım 2.16 ((h, m) – Konveks Fonksiyon) : Negatif olmayan $h : J \subseteq R \rightarrow R$ fonksiyon olsun. Negatif olmayan $F : [0, \kappa_2] \rightarrow R$ fonksiyonu $\forall \tau_1, \tau_2 \in [0, \kappa_2], m \in [0, 1], \zeta \in (0, 1)$ olmak üzere

$$F(\zeta \tau_1 + m(1 - \zeta) \tau_2) \leq h(\zeta) F(\tau_1) + mh(1 - \zeta) F(\tau_2)$$

eşitsizliğini sağlıyorsa F 'ya (h, m) – konveks fonksiyon denir (Özdemir et al., 2016).

Tanım 2.17 (Beta Fonksiyonu) : Aşağıdaki eşitsizliği sağlayan ifadeye Beta fonksiyonu denir.

$$\beta(\tau_1, \tau_2) = \int_0^1 \zeta^{\tau_1-1} (1 - \zeta)^{\tau_2-1} d\zeta, \quad \tau_1, \tau_2 > 0.$$

Bu eşitlik Euler tipi Beta fonksiyonu ya da birinci çeşit Euler integrali olarak isimlendirilir (Dragomir et al., 1999).

Tanım 2.18 (Eksponansiyel Konveks Fonksiyon) : $F : K \subseteq R \rightarrow (0, \infty)$ pozitif değerli bir fonksiyon olsun. $\forall \tau_1, \tau_2 \in K$ ve $\zeta \in [0, 1]$ için

$$e^{F(\zeta \tau_1 + (1-\zeta)\tau_2)} \leq \zeta e^{F(\tau_1)} + (1 - \zeta) e^{F(\tau_2)}$$

eşitsizliği sağlanıyorsa F fonksiyonuna K üzerinde eksponansiyel konveks fonksiyon denir (Antczak, 2001).

Tanım 2.19 (Eksponansiyel s – Konveks Fonksiyon) : $F : K \subseteq R \rightarrow R$ bir fonksiyon olsun. $\forall \tau_1, \tau_2 \in K, \zeta \in [0, 1]$ ve $s \in (0, 1]$ için

$$e^{F(\zeta \tau_1 + (1-\zeta)\tau_2)} \leq \zeta^s e^{F(\tau_1)} + (1 - \zeta)^s e^{F(\tau_2)}$$

eşitsizliği sağlanıyorsa F fonksiyonuna K üzerinde eksponansiyel s – konveks fonksiyon denir.

Bu tanımda $\zeta = \frac{1}{2}$ alınırsa

$$e^{F(\frac{\tau_1+\tau_2}{2})} \leq \frac{e^{F(\tau_1)} + e^{F(\tau_2)}}{2^s}$$

elde edilir (Antczak, 2001).

Tanım 2.20 (Ekspansiyel m – Konveks Fonksiyon) : $F : K \subseteq R \rightarrow R$ bir fonksiyon olsun. $\forall \tau_1, \tau_2 \in K$, ve $\zeta, m \in [0, 1]$ için

$$e^{F(\zeta\tau_1+m(1-\zeta)\tau_2)} \leq \zeta e^{F(\tau_1)} + m(1-\zeta) e^{F(\tau_2)}$$

eşitsizliği sağlanıyorsa F fonksiyonuna K üzerinde ekspansiyel m – konveks fonksiyon denir.

Bu tanımda $\zeta = \frac{1}{2}$ alınırsa $\forall \tau_1, \tau_2 \in K$ için

$$e^{F\left(\frac{\tau_1+m\tau_2}{2}\right)} \leq \frac{e^{F(\tau_1)} + me^{F(\tau_2)}}{2^s}$$

elde edilir. Bu fonksiyona Jensen m – konveks denir (Rashid et al., 2019).

Tanım 2.21 (Ekspansiyel Quasi – Konveks Fonksiyon) : $F : K \subseteq R \rightarrow R$ bir fonksiyon olsun. $\forall \tau_1, \tau_2 \in K$ ve $\zeta \in [0, 1]$ için

$$e^{F(\zeta\tau_1+(1-\zeta)\tau_2)} \leq \max \{e^{F(\tau_1)}, e^{F(\tau_2)}\}$$

eşitsizliği sağlanıyorsa F fonksiyonuna K üzerinde ekspansiyel Quasi – konveks fonksiyon denir.

Yukarıdaki tanımdan;

$$\begin{aligned} e^{F(\zeta\tau_1+(1-\zeta)\tau_2)} &\leq \zeta e^{F(\tau_1)} + (1-\zeta) e^{F(\tau_2)} \\ &\leq \max \{e^{F(\tau_1)}, e^{F(\tau_2)}\} \end{aligned}$$

elde edilir ve bu eşitsizlik her ekspansiyel konveks fonksiyonun ekspansiyel quasi–konveks olduğunu ancak tersinin doğru olmadığını gösterir (Noor and Noor, 2019).

Ekspansiyel konveksliğin farklı bir versiyonu aşağıdaki gibi tanımlanmıştır.

Tanım 2.22 (Ekspansiyel Konveks Fonksiyon) : $F : K \subseteq R \rightarrow R$ bir fonksiyon olsun. $\forall \tau_1, \tau_2 \in K, \alpha \in R$ ve $\zeta \in [0, 1]$ için

$$F(\zeta\tau_1 + (1-\zeta)\tau_2) \leq \zeta \frac{F(\tau_1)}{e^{\alpha\tau_1}} + (1-\zeta) \frac{F(\tau_2)}{e^{\alpha\tau_2}}$$

eşitsizliği sağlanıyorsa F fonksiyonuna K üzerinde ekspansiyel konveks fonksiyon denir. $\alpha = 0$ alınırsa ekspansiyel konveks fonksiyon tanımı konveks fonksiyona dönüşür (Awan et al., 2018).

2.2. Bazı Önemli Eşitsizlikler

Teorem 2.1 (Hermite-Hadamard Eşitsizliği) : I, R 'de bir aralık $\kappa_1, \kappa_2 \in I$ ve $\kappa_1 < \kappa_2$ olmak üzere $F : I \rightarrow R$ konveks bir fonksiyon olsun. Bu durumda

$$F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau_1) d\tau_1 \leq \frac{F(\kappa_1) + F(\kappa_2)}{2}$$

eşitsizliği sağlanır (Pečarić and Tong, 1992).

Teorem 2.2 (İntegraller İçin Hölder Eşitsizliği) : $p > 1$ ve $p^{-1} + q^{-1} = 1$ olsun. F ve $\Phi, [\kappa_1, \kappa_2]$ aralığında tanımlı reel fonksiyonlar, $|F|^p$ ve $|\Phi|^q, [\kappa_1, \kappa_2]$ aralığında integrallenebilir fonksiyonlar ise

$$\int_{\kappa_1}^{\kappa_2} |F(\tau_1) \Phi(\tau_1)| d\tau_1 \leq \left(\int_{\kappa_1}^{\kappa_2} |F(\tau_1)|^p d\tau_1 \right)^{\frac{1}{p}} \left(\int_{\kappa_1}^{\kappa_2} |\Phi(\tau_1)|^q d\tau_1 \right)^{\frac{1}{q}}$$

eşitsizliği geçerlidir (Mitrinovic and Vasic, 1970). Benzer şekilde iki katlı integraller için Hölder eşitsizliği aşağıdaki gibi ifade edilebilir:

$$\begin{aligned} & \int_{\kappa_1}^{\kappa_2} \int_{\kappa_1}^{\kappa_2} |F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2)| d\tau_1 d\tau_2 \\ & \leq \left(\int_{\kappa_1}^{\kappa_2} \int_{\kappa_1}^{\kappa_2} |F(\tau_1, \tau_2)|^p d\tau_1 d\tau_2 \right)^{\frac{1}{p}} \left(\int_{\kappa_1}^{\kappa_2} \int_{\kappa_1}^{\kappa_2} |\Phi(\tau_1, \tau_2)|^q d\tau_1 d\tau_2 \right)^{\frac{1}{q}}. \end{aligned}$$

Teorem 2.3 (Young Eşitsizliği) : $\kappa_1, \kappa_2 > 0$ ve $p, q > 1$ için $p^{-1} + q^{-1} = 1$ olmak üzere

$$\kappa_1 \kappa_2 \leq \frac{\kappa_1^p}{p} + \frac{\kappa_2^q}{q}$$

eşitsizliği geçerlidir. Eşitlik durumu sadece $\kappa_1^p = \kappa_2^q$ durumunda sağlanır (Mitrinovic and Vasic, 1970).

Teorem 2.4 (Üçgen Eşitsizliği) : Herhangi τ_1, τ_2 reel sayıları için

$$|\tau_1 + \tau_2| \leq |\tau_1| + |\tau_2|,$$

$$||\tau_1| - |\tau_2|| \leq |\tau_1 - \tau_2|,$$

$$||\tau_1| - |\tau_2|| \leq |\tau_1 + \tau_2|,$$

ve tümevarım metoduyla

$$|\tau_1 + \tau_2 + \dots + \tau_n| \leq |\tau_1| + |\tau_2| + \dots + |\tau_n|$$

eşitsizlikleri geçerlidir (Mitrinovic et al., 1993).

Teorem 2.5 (Üçgen Eşitsizliğinin İntegral Versiyonu) : $F, [\kappa_1, \kappa_2]$ aralığında sürekli reel değerli bir fonksiyon olsun. Bu takdirde

$$\left| \int_{\kappa_1}^{\kappa_2} F(\tau_1) d\tau_1 \right| \leq \int_{\kappa_1}^{\kappa_2} |F(\tau_1)| d\tau_1 \quad (\kappa_1 < \kappa_2)$$

eşitsizliği geçerlidir (Mitrinovic et al., 1993).



3. MATERYAL VE YÖNTEM

Bu çalışmada $\Delta = [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4] \subset R^2$ ve $\kappa_1 < \kappa_2, \kappa_3 < \kappa_4$ ve $\Delta_1 = [0, \kappa_2] \times [0, \kappa_4] \subset R^2$ ve $\kappa_2, \kappa_4 > 0$ alınacaktır.

3.1. Koordinatlarda Konveks Fonksiyonlarla İlgili Genel Kavramlar

Tanım 3.1 (Koordinatlarda Konveks Fonksiyon) : $F : \Delta \rightarrow R$ bir fonksiyon olsun.

$\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta$ ve $\zeta \in [0, 1]$ olmak üzere

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq \zeta F(\tau_1, \tau_2) + (1 - \zeta)F(\tau_3, \tau_4)$$

eşitsizliği sağlanıyorsa F 'ya Δ üzerinde konvektir denir (Dragomir, 2001).

$\forall \tau_1 \in [\kappa_1, \kappa_2]$ ve $\forall \tau_2 \in [\kappa_3, \kappa_4]$ için eğer

$$F_{\tau_2} : [\kappa_1, \kappa_2] \rightarrow R, F_{\tau_2}(u) = F(u, \tau_2)$$

ve

$$F_{\tau_1} : [\kappa_3, \kappa_4] \rightarrow R, F_{\tau_1}(v) = F(\tau_1, v)$$

kısmi dönüşümleri konveks ise $F : \Delta \rightarrow R$ fonksiyonu koordinatlarda konvektir (Dragomir, 2001).

Lemma 3.1 $F : \Delta \rightarrow R$ şeklinde tanımlı bütün konveks fonksiyonlar koordinatlarda konvektir fakat tersi her zaman doğru değildir (Dragomir, 2001).

Örnek 3.1 $F : [0, 1]^2 \rightarrow R$ fonksiyonu $F(\tau_1, \tau_2) = \tau_1\tau_2$ şeklinde tanımlayalım. Bu fonksiyon koordinatlarda konveks olmasına rağmen $[0, 1]^2$ de konveks değildir (Dragomir, 2001).

Koordinatlarda konveks fonksiyon tanımına denk bir tanım şu şekilde verilmiştir.

Tanım 3.2 (Koordinatlarda Konveks Fonksiyon) : $F : \Delta \rightarrow R$ fonksiyonu verilsin. Eğer

$$\begin{aligned} & F(\zeta\tau_1 + (1 - \zeta)\tau_2, \xi\tau_3 + (1 - \xi)\tau_4) \\ & \leq \zeta\xi F(\tau_1, \tau_3) + \zeta(1 - \xi)F(\tau_1, \tau_4) + (1 - \zeta)\xi F(\tau_2, \tau_3) + (1 - \zeta)(1 - \xi)F(\tau_2, \tau_4) \end{aligned}$$

eşitsizliği $\forall \zeta, \xi \in [0, 1]$ ve $(\tau_1, \tau_3), (\tau_1, \tau_4), (\tau_2, \tau_3), (\tau_2, \tau_4) \in \Delta$ için sağlanıyorsa F 'ya Δ üzerinde koordinatlarda konveks fonksiyon denir (Latif and Alomari, 2009a).

Lemma 3.2 $F, \Phi : \Delta \rightarrow R$ şeklinde tanımlı koordinatlarda konveks iki fonksiyon ve $k \geq 0$ verilsin. Bu takdirde $F + \Phi$ ve kF koordinatlarda konvektir (Latif and Alomari, 2009a).

Lemma 3.3 $F, \Phi : \Delta \rightarrow R$ şeklinde tanımlı koordinatlarda konveks iki fonksiyon olsun. Bu takdirde $F \Phi$ koordinatlarda konvektir (Latif and Alomari, 2009a).

Tanım 3.3 (Koordinatlarda m - Konveks Fonksiyon) : $F : \Delta_1 \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta_1$ ve $m, \zeta \in [0, 1]$ olmak üzere

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + m(1 - \zeta)\tau_4) \leq \zeta F(\tau_1, \tau_2) + m(1 - \zeta)F(\tau_3, \tau_4)$$

eşitsizliği sağlanıyorsa F 'ya Δ_1 üzerinde m - konvektir denir (Özdemir et al., 2011a).

$\forall \tau_1 \in [0, \kappa_2]$ ve $\forall \tau_2 \in [0, \kappa_4]$ için eğer

$$F_{\tau_2} : [0, \kappa_2] \rightarrow R, F_{\tau_2}(u) = F(u, \tau_2)$$

ve

$$F_{\tau_1} : [0, \kappa_4] \rightarrow R, F_{\tau_1}(v) = F(\tau_1, v)$$

kısmi dönüşümleri m - konveks ise $F : \Delta_1 \rightarrow R$ fonksiyonu koordinatlarda m - konvektir (Özdemir et al., 2011a).

Lemma 3.4 $F : \Delta_1 \rightarrow R$ şeklinde tanımlı bütün m - konveks fonksiyon $m \in [0, 1]$ için koordinatlarda m - konvektir (Özdemir et al., 2011a).

Tanım 3.4 (Koordinatlarda (α, m) - Konveks Fonksiyon) : $F : \Delta_1 \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta_1$ ve $(\alpha, m) \in [0, 1]^2, \zeta \in [0, 1]$ için

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + m(1 - \zeta)\tau_4) \leq \zeta^\alpha F(\tau_1, \tau_2) + m(1 - \zeta^\alpha)F(\tau_3, \tau_4)$$

eşitsizliği sağlanıyorsa F 'ya Δ_1 üzerinde (α, m) - konvektir denir (Özdemir et al., 2011a).

Bu tanımda $(\alpha, m) = \{(1, m), (1, 1)\}$ alınırsa sırasıyla koordinatlarda m - konveks ve koordinatlarda konveks fonksiyonlar sınıfı elde edilir.

Lemma 3.5 $F : \Delta_1 \rightarrow R$ şeklinde tanımlı bütün (α, m) - konveks fonksiyonlar $(\alpha, m) \in [0, 1]^2$ için koordinatlarda (α, m) - konvektir (Özdemir et al., 2011a).

Tanım 3.5 (Koordinatlarda İkinci Anlamda s – Konveks Fonksiyon) : $F : \Delta \rightarrow R$ bir fonksiyon olsun. $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta$ ve $s \in (0, 1], \zeta \in [0, 1]$ için

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq \zeta^s F(\tau_1, \tau_2) + (1 - \zeta)^s F(\tau_3, \tau_4)$$

eşitsizliği sağlanıyorsa F ’ya Δ üzerinde ikinci anlamda s – konvektir denir (Alomari and Darus, 2008b).

Lemma 3.6 $F : \Delta \rightarrow [0, \infty)$ şeklinde tanımlı bütün s – konveks fonksiyon koordinatlarda s – konvektir (Alomari and Darus, 2008a).

Tanım 3.6 (Koordinatlarda Birinci Anlamda s – Konveks Fonksiyon) : $F : \Delta \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta$ ve $\alpha^s + \beta^s = 1, s \in (0, 1]$ ve $\alpha, \beta \geq 0$ için

$$F(\alpha\tau_1 + \beta\tau_3, \alpha\tau_2 + \beta\tau_4) \leq \alpha^s F(\tau_1, \tau_2) + \beta^s F(\tau_3, \tau_4)$$

eşitsizliği sağlanıyorsa F ’ya Δ üzerinde birinci anlamda s – konvektir denir (Alomari and Darus, 2008a).

Tanım 3.7 (Koordinatlarda h – Konveks Fonksiyon) : $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta, \zeta \in [0, 1]$ ve $h : J \subset R \rightarrow R$ pozitif bir fonksiyon olmak üzere negatif olmayan $F : \Delta \rightarrow R$ fonksiyonu eğer

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq h(\zeta) F(\tau_1, \tau_2) + h(1 - \zeta) F(\tau_3, \tau_4)$$

eşitsizliğini sağlıyorsa F ’ya Δ üzerinde h – konvektir denir. Δ üzerinde h –konveks fonksiyonlar sınıfı $SX(h, \Delta)$ ile gösterilir (Latif and Alomari, 2009b).

Koordinatlarda h – konveks fonksiyon tanımına denk bir tanım şu şekilde verilmiştir.

Tanım 3.8 (Koordinatlarda h – Konveks Fonksiyon) : $F : \Delta \rightarrow R$ fonksiyonu verilsin. $\forall \zeta, \xi \in [0, 1]$ ve $(\tau_1, \tau_3), (\tau_1, \tau_4), (\tau_2, \tau_3), (\tau_2, \tau_4) \in \Delta$ için

$$\begin{aligned} & F(\zeta\tau_1 + (1 - \zeta)\tau_2, \xi\tau_3 + (1 - \xi)\tau_4) \\ & \leq h(\zeta)h(\xi)F(\tau_1, \tau_3) + h(\zeta)h(1 - \xi)F(\tau_1, \tau_4) \\ & \quad + h(1 - \zeta)h(\xi)F(\tau_2, \tau_3) + h(1 - \zeta)h(1 - \xi)F(\tau_2, \tau_4) \end{aligned}$$

eşitsizliği sağlanıyorsa F ’ya Δ üzerinde koordinatlarda h – konvektir denir (Latif and Alomari, 2009b).

Tanım 3.9 (Koordinatlarda (h, m) – Konveks Fonksiyon) : $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta_1, \zeta \in (0, 1), m \in [0, 1]$ ve $h : J \subset R \rightarrow R$ pozitif bir fonksiyon olmak üzere negatif olmayan $F : \Delta_1 \rightarrow R$ fonksiyonu eğer

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + m(1 - \zeta)\tau_4) \leq h(\zeta)F(\tau_1, \tau_2) + mh(1 - \zeta)F(\tau_3, \tau_4)$$

eşitsizliğini sağlıyorsa F ’ya Δ_1 üzerinde (h, m) – konvektir denir (Akdemir, 2012).

Lemma 3.7 $F : \Delta_1 \rightarrow R$ şeklinde tanımlı her (h, m) – konveks fonksiyon koordinatlarda (h, m) – konveks fonksiyondur (Akdemir, 2012).

Önerme 3.1 F, Φ fonksiyonları koordinatlarda (h, m) –konveks fonksiyonlar ise $k > 0$ olmak üzere $F + \Phi$ ve kF fonksiyonları da koordinatlarda (h, m) – konveks fonksiyonlardır (Akdemir, 2012).

Koordinatlarda (h, m) – konveks fonksiyonuna denk bir tanım aşağıdaki gibi verilmiştir.

Tanım 3.10 (Koordinatlarda (h, m) – Konveks Fonksiyon) : $F : \Delta_1 \rightarrow R$ fonksiyonu verilsin $\forall \zeta, \xi \in [0, 1]$ ve $(\tau_1, \tau_3), (\tau_1, \tau_4), (\tau_2, \tau_3), (\tau_2, \tau_4) \in \Delta_1$ için

$$\begin{aligned} & F(\zeta\tau_1 + (1 - \zeta)\tau_2, \xi\tau_3 + m(1 - \xi)\tau_4) \\ & \leq h(\zeta)h(\xi)F(\tau_1, \tau_3) + mh(\zeta)h(1 - \xi)F(\tau_1, \tau_4) \\ & \quad + h(\xi)h(1 - \zeta)F(\tau_2, \tau_3) + mh(1 - \zeta)h(1 - \xi)F(\tau_2, \tau_4) \end{aligned}$$

eşitsizliğini sağlıyorsa F ’ya Δ_1 üzerinde (h, m) – konvektir denir (Akdemir, 2012).

Tanım 3.11 (Koordinatlarda Quasi– Konveks Fonksiyon) : $F : \Delta \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_3), (\tau_1, \tau_4), (\tau_2, \tau_3), (\tau_2, \tau_4) \in \Delta$ ve $\zeta \in [0, 1]$ için

$$F(\zeta\tau_1 + (1 - \zeta)\tau_2, \zeta\tau_3 + (1 - \zeta)\tau_4) \leq \max\{F(\tau_1, \tau_3), F(\tau_1, \tau_4), F(\tau_2, \tau_3), F(\tau_2, \tau_4)\}$$

eşitsizliği sağlanıyorsa F ’ya Δ üzerinde koordinatlarda quasi– konvektir denir (Akdemir, 2012).

Tanım 3.12 (Koordinatlarda α – Star s – Konveks Fonksiyon) : $F : \Delta \rightarrow [0, \infty)$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta$ ve $s \in (0, 1], \zeta, \alpha \in [0, 1]$ için

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq \zeta^s F(\tau_1, \tau_2) + \{(1 - \zeta)\alpha\}^s F(\tau_3, \tau_4)$$

eşitsizliğini sağlıyorsa F 'ya Δ üzerinde α -star s -konveks fonksiyon denir. $F : \Delta \rightarrow [0, \infty)$ α -star s -konveks fonksiyonu eğer

$$F_{\tau_2} : [\kappa_1, \kappa_2] \rightarrow R, F_{\tau_2}(u) = F(u, \tau_2)$$

ve

$$F_{\tau_1} : [\kappa_3, \kappa_4] \rightarrow R, F_{\tau_1}(v) = F(\tau_1, v)$$

kısmi dönüşümleri $\forall \tau_1 \in [\kappa_1, \kappa_2]$ ve $\forall \tau_2 \in [\kappa_3, \kappa_4]$ için α -star s -konveks fonksiyon ise F 'ya Δ üzerinde α -star s -konveks fonksiyon denir (Akdemir, 2012).

Lemma 3.8 $F : \Delta \rightarrow [0, \infty)$ şeklinde tanımlı bütün α -star s -konveks fonksiyonlar koordinatlarda α -star s -konvektir (Akdemir, 2012).

Tanım 3.13 (Koordinatlarda Jensen–Konveks Fonksiyon) : $F : \Delta \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_3), (\tau_1, \tau_4), (\tau_2, \tau_3), (\tau_2, \tau_4) \in \Delta$ için

$$F\left(\frac{\tau_1 + \tau_2}{2}, \frac{\tau_3 + \tau_4}{2}\right) \leq \frac{F(\tau_1, \tau_3) + F(\tau_1, \tau_4) + F(\tau_2, \tau_3) + F(\tau_2, \tau_4)}{2}$$

eşitsizliği sağlanıyorsa F 'ya Δ üzerinde Jensen–konvektir denir (Akdemir, 2012).

Tanım 3.14 (Koordinatlarda Jensen–Quasi–Konveks Fonksiyon) : $F : \Delta \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_3), (\tau_1, \tau_4), (\tau_2, \tau_3), (\tau_2, \tau_4) \in \Delta$ için

$$F\left(\frac{\tau_1 + \tau_2}{2}, \frac{\tau_3 + \tau_4}{2}\right) \leq \max\{F(\tau_1, \tau_3), F(\tau_1, \tau_4), F(\tau_2, \tau_3), F(\tau_2, \tau_4)\}$$

eşitsizliği sağlanıyorsa F 'ya Δ üzerinde koordinatlarda Jensen–Quasi–konvektir denir (Akdemir, 2012).

Tanım 3.15 (Koordinatlarda Godunova–Levin Fonksiyon) : Negatif olmayan $F : \Delta \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta$ ve $\zeta \in (0, 1)$ için

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq \frac{F(\tau_1, \tau_2)}{\zeta} + \frac{F(\tau_3, \tau_4)}{1 - \zeta}$$

eşitsizliğini sağlıyorsa F 'ya Δ üzerinde Godunova-Levin sınıfında fonksiyon denir (Akdemir and Özdemir, 2010).

Lemma 3.9 $F : \Delta \rightarrow R$ şeklinde tanımlı bütün Godunova-Levin sınıfına ait fonksiyon koordinatlarda Godunova-Levin sınıfına aittir (Akdemir and Özdemir, 2010).

Tanım 3.16 (Koordinatlarda P –Fonksiyon) : Negatif olmayan $F : \Delta \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta$ ve $\zeta \in (0, 1)$ için

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq F(\tau_1, \tau_2) + F(\tau_3, \tau_4)$$

eşitsizliğini sağlıyorsa F 'ya Δ üzerinde P –fonksiyon denir (Akdemir and Özdemir, 2010).

Lemma 3.10 $F : \Delta \rightarrow R$ şeklinde tanımlı P –fonksiyon koordinatlarda P –fonksiyondur (Akdemir and Özdemir, 2010).

3.2. Koordinatlarda Konveks Fonksiyonlar için Bazı Lemma ve İntegral Eşitsizlikler

Koordinatlarda konveks fonksiyonlar için Dragomir tarafından elde edilen Hadamard-tipi eşitsizlikler aşağıdaki teoremdedir.

Teorem 3.1 $F : \Delta \rightarrow R$ fonksiyonu koordinatlarda konveks fonksiyon olsun. Bu takdirde

$$\begin{aligned} & F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\ & \leq \frac{1}{2} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \right] \\ & \leq \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{1}{4} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_3) d\tau_1 + \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_4) d\tau_1 \right. \\ & \quad \left. + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F(\kappa_1, \tau_2) d\tau_2 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F(\kappa_2, \tau_2) d\tau_2 \right] \\ & \leq \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} \end{aligned}$$

eşitsizliği sağlanır (Dragomir, 2001).

Benzer yöntemler kullanarak koordinatlarda s –konveks fonksiyonlar için aşağıdaki eşitsizlikler elde edilmiştir.

Teorem 3.2 $F : \Delta \rightarrow [0, \infty)$ fonksiyonu koordinatlarda s –konveks fonksiyon olsun. Bu

takdirde

$$\begin{aligned}
& F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& \leq 2^{s-2} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \right] \\
& \leq \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
& \leq \frac{1}{2(s+1)} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_3) d\tau_1 + \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_4) d\tau_1 \right. \\
& \quad \left. + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F(\kappa_1, \tau_2) d\tau_2 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F(\kappa_2, \tau_2) d\tau_2 \right] \\
& \leq \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{(s+1)^2}
\end{aligned}$$

eşitsizliği sağlanır (Alomari and Darus, 2008b).

Teorem 3.3 $F : \Delta \rightarrow [0, \infty)$ fonksiyonu koordinatlarda birinci anlamda s -konveks fonksiyon olsun. Bu takdirde

$$\begin{aligned}
& F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& \leq \frac{1}{2} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \right] \\
& \leq \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
& \leq \frac{1}{2(s+1)} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_3) d\tau_1 + \frac{s}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_4) d\tau_1 \right. \\
& \quad \left. + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F(\kappa_1, \tau_2) d\tau_2 + \frac{s}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F(\kappa_2, \tau_2) d\tau_2 \right] \\
& \leq \frac{F(\kappa_1, \kappa_3) + sF(\kappa_1, \kappa_4) + sF(\kappa_2, \kappa_3) + s^2F(\kappa_2, \kappa_4)}{(s+1)^2}
\end{aligned}$$

eşitsizliği sağlanır (Alomari and Darus, 2008a).

Koordinatlarda h -konveks fonksiyonlar için aşağıdaki eşitsizlikler elde edilmiştir.

Teorem 3.4 $F : \Delta \rightarrow R$ fonksiyonu koordinatlarda h -konveks fonksiyon, $F \in L_2(\Delta)$ ve $h \in L_1[0, 1]$ olsun. Bu takdirde:

$$\begin{aligned}
& \frac{1}{4\left(h\left(\frac{1}{2}\right)\right)^2} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& \leq \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
& \leq (F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)) \left(\int_0^1 h(\zeta) d\zeta \right)^2
\end{aligned}$$

eşitsizliği sağlanır (Latif and Alomari, 2009b).

Özdemir ve arkadaşları 2010 yılındaki çalışmalarında benzer yöntemler kullanarak koordinatlarda m -konveks ve (α, m) -konveks fonksiyonlar için aşağıdaki eşitsizlikleri elde etmişlerdir.

Teorem 3.5 $F : \Delta_1 \rightarrow R$ koordinatlarda m -konveks fonksiyon olsun. Eğer $m \in (0, 1]$ ile birlikte $0 \leq \kappa_1 < \kappa_2 < \infty$ ve $0 \leq \kappa_3 < \kappa_4 < \infty$ ise

$$\begin{aligned} & \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{1}{4(\kappa_2 - \kappa_1)} \min\{v_1, v_2\} + \frac{1}{4(\kappa_4 - \kappa_3)} \min\{v_3, v_4\} \end{aligned}$$

eşitsizliği geçerlidir. Burada

$$\begin{aligned} v_1 &= \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_3) d\tau_1 + m \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_4}{m}\right) d\tau_1, \\ v_2 &= \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_4) d\tau_1 + m \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3}{m}\right) d\tau_1, \\ v_3 &= \int_{\kappa_3}^{\kappa_4} F(\kappa_1, \tau_2) d\tau_2 + m \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_2}{m}, \tau_2\right) d\tau_2, \\ v_4 &= \int_{\kappa_3}^{\kappa_4} F(\kappa_2, \tau_2) d\tau_2 + m \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1}{m}, \tau_2\right) d\tau_2, \end{aligned}$$

şeklinde (Özdemir et al., 2011a).

Teorem 3.6 $F : \Delta_1 \rightarrow R$ koordinatlarda m -konveks fonksiyon olsun. Eğer $m \in (0, 1]$, $0 \leq \kappa_1 < \kappa_2 < \infty$ ve $0 \leq \kappa_3 < \kappa_4 < \infty$ ile birlikte $F_{\tau_1} \in L_1[0, \kappa_4]$, $F_{\tau_2} \in L_1[0, \kappa_2]$ ise

$$\begin{aligned} & \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\ & \leq \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \left[\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \frac{F(\tau_1, \tau_2) + mF\left(\tau_1, \frac{\tau_2}{m}\right)}{2} d\tau_1 d\tau_2 \right. \\ & \quad \left. + \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \frac{F(\tau_1, \tau_2) + mF\left(\frac{\tau_1}{m}, \tau_2\right)}{2} d\tau_1 d\tau_2 \right] \end{aligned}$$

eşitsizliği geçerlidir (Özdemir et al., 2011a).

Teorem 3.7 $F : \Delta_1 \rightarrow R$ koordinatlarda (α, m) -konveks fonksiyon olsun. Eğer $(\alpha, m) \in [0, 1]^2$, $0 \leq \kappa_1 < \kappa_2 < \infty$ ve $0 \leq \kappa_3 < \kappa_4 < \infty$ ile birlikte $F_{\tau_1} \in L_1[0, \kappa_4]$, $F_{\tau_2} \in L_1[0, \kappa_2]$ ise

$$\begin{aligned} & \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\ & \leq \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \\ & \quad \times \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \frac{2F(\tau_1, \tau_2) + m(2^\alpha - 1)(F(\tau_1, \frac{\tau_2}{m}) + F(\frac{\tau_1}{m}, \tau_2))}{2^\alpha} d\tau_1 d\tau_2 \right) \end{aligned}$$

ve

$$\begin{aligned} & \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{1}{2(\alpha + 1)(\kappa_2 - \kappa_1)} \min\{w_1, w_2\} + \frac{1}{2(\alpha + 1)(\kappa_4 - \kappa_3)} \min\{w_3, w_4\} \end{aligned}$$

eşitsizlikleri geçerlidir. Burada

$$\begin{aligned} w_1 &= \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_3) d\tau_1 + \alpha m \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_4}{m}\right) d\tau_1, \\ w_2 &= \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_4) d\tau_1 + \alpha m \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3}{m}\right) d\tau_1, \\ w_3 &= \int_{\kappa_3}^{\kappa_4} F(\kappa_1, \tau_2) d\tau_2 + \alpha m \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_2}{m}, \tau_2\right) d\tau_2, \\ w_4 &= \int_{\kappa_3}^{\kappa_4} F(\kappa_2, \tau_2) d\tau_2 + \alpha m \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1}{m}, \tau_2\right) d\tau_2, \end{aligned}$$

şeklinde (Özdemir et al., 2011a).

Teorem 3.8 $F : \Delta_1 \rightarrow R$ koordinatlarda (α, m) – konveks fonksiyon olsun. Eğer $(\alpha, m) \in [0, 1]^2$, $0 \leq \kappa_1 < \kappa_2 < \infty$ ve $0 \leq \kappa_3 < \kappa_4 < \infty$ ile birlikte $F_{\tau_1} \in L_1[0, \kappa_4]$, $F_{\tau_2} \in L_1[0, \kappa_2]$ ise

$$\begin{aligned} & \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{1}{4(\alpha + 1)} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_3) d\tau_1 + \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_4) d\tau_1 \right. \\ & \quad + \frac{m\alpha}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3}{m}\right) d\tau_1 + \frac{m\alpha}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_4}{m}\right) d\tau_1 \\ & \quad + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F(\kappa_1, \tau_2) d\tau_2 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F(\kappa_2, \tau_2) d\tau_2 \\ & \quad \left. + \frac{m\alpha}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1}{m}, \tau_2\right) d\tau_2 + \frac{m\alpha}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_2}{m}, \tau_2\right) d\tau_2 \right] \end{aligned}$$

eşitsizliği geçerlidir (Özdemir et al., 2011a).

Koordinatlarda P – fonksiyonlar ve koordinatlarda Godunova – Levin sınıfından fonksiyonlar için aşağıdaki eşitsizlikler elde edilmiştir.

Teorem 3.9 $F : \Delta \rightarrow R$ koordinatlarda Godunova–Levin sınıfından bir fonksiyon olsun. Eğer $\zeta \in (0, 1)$ olmak üzere $F_{\tau_1} \in L_1[\kappa_3, \kappa_4]$ ve $F_{\tau_2} \in L_1[\kappa_1, \kappa_2]$ ise

$$\begin{aligned} & \frac{1}{16} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\ & \leq \frac{1}{8} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \right] \\ & \leq \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \end{aligned}$$

eşitsizliği geçerlidir (Akdemir and Özdemir, 2010).

Sonuç 3.1 Teorem 3.9’da $F : [\kappa_1, \kappa_2] \times [\kappa_1, \kappa_2] \rightarrow R$ olarak alınırsa;

$$\begin{aligned} & \frac{1}{16} F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_1 + \kappa_2}{2} \right) \\ & \leq \frac{1}{8} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \left(F \left(\tau_1, \frac{\kappa_1 + \kappa_2}{2} \right) + F \left(\frac{\kappa_1 + \kappa_2}{2}, \tau_1 \right) \right) d\tau_1 \right] \\ & \leq \frac{1}{(\kappa_2 - \kappa_1)^2} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \end{aligned}$$

eşitsizliği elde edilir (Akdemir and Özdemir, 2010).

Sonuç 3.2 Teorem 3.9’da $F : [\kappa_1, \kappa_2] \times [\kappa_1, \kappa_2] \rightarrow R$ ve $F(\tau_1, \tau_2) = F(\tau_2, \tau_1)$ olarak alınırsa;

$$\begin{aligned} & \frac{1}{16} F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_1 + \kappa_2}{2} \right) \\ & \leq \frac{1}{4} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F \left(\tau_1, \frac{\kappa_1 + \kappa_2}{2} \right) d\tau_1 \right] \\ & \leq \frac{1}{(\kappa_2 - \kappa_1)^2} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \end{aligned}$$

eşitsizliği elde edilir (Akdemir and Özdemir, 2010).

Teorem 3.10 $F : \Delta \rightarrow R$ koordinatlarda P -fonksiyon olsun. Eğer $\zeta \in (0, 1)$, $F_{\tau_1} \in L_1[\kappa_3, \kappa_4]$ ve $F_{\tau_2} \in L_1[\kappa_1, \kappa_2]$ ise

$$\begin{aligned} & F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \\ & \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F \left(\tau_1, \frac{\kappa_3 + \kappa_4}{2} \right) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F \left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2 \right) d\tau_2 \\ & \leq \frac{4}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{2}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{2}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2 \end{aligned}$$

eşitsizlikleri geçerlidir (Akdemir and Özdemir, 2010).

Sarıkaya ve arkadaşları 2010 yılında yaptıkları çalışmada aşağıdaki Lemmayı elde etmişlerdir.

Lemma 3.11 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden diferansiyellenebilen

bir dönüşüm olsun. Eğer $\frac{\partial^2 F}{\partial \zeta \partial \xi} \in L^2(\Delta)$ ise;

$$\begin{aligned}
& \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} \\
& + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
& - \frac{1}{2} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} [F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)] d\tau_1 \right. \\
& \left. + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} [F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)] d\tau_2 \right] \\
& = \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \\
& \times \int_0^1 \int_0^1 (1 - 2\zeta)(1 - 2\xi) \frac{\partial^2 F}{\partial \zeta \partial \xi} (\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi
\end{aligned}$$

eşitsizliği geçerlidir (Sarıkaya et al., 2012).

Sarıkaya ve arkadaşları 2010 yılında lemma 3.11 i kullanarak koordinatlarda konveks fonksiyonlar için aşağıdaki eşitsizlikleri elde etmişlerdir.

Teorem 3.11 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden diferensiyellenebilen bir dönüşüm olsun. Eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$ koordinatlarda konveks ise

$$\begin{aligned}
& \left| \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} \right. \\
& \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - B \right| \\
& \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{16} \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|(\kappa_1, \kappa_3) + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|(\kappa_1, \kappa_4) + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|(\kappa_2, \kappa_3) + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|(\kappa_2, \kappa_4)}{4} \right)
\end{aligned}$$

eşitsizliği geçerlidir. Burada

$$B = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

şeklinde (Sarıkaya et al., 2012).

Teorem 3.12 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden diferensiyellenebilen bir dönüşüm olsun. Eğer $p > 1$ ve $p^{-1} + q^{-1} = 1$ olmak üzere $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$ koordinatlarda konveks

ise

$$\begin{aligned}
& \left| \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} \right. \\
& \quad \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - B \right| \\
& \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \\
& \quad \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q(\kappa_1, \kappa_3) + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q(\kappa_1, \kappa_4) + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q(\kappa_2, \kappa_3) + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q(\kappa_2, \kappa_4)}{4} \right)^{\frac{1}{q}}
\end{aligned}$$

eşitsizliği geçerlidir. Burada

$$B = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

şeklindedir (Sarıkaya et al., 2012).

Teorem 3.13 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden diferensiyellenebilen bir dönüşüm olsun. Eğer $q > 1$ olmak üzere $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$ koordinatlarda konveks ise

$$\begin{aligned}
& \left| \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} \right. \\
& \quad \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - B \right| \\
& \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{16} \\
& \quad \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q(\kappa_1, \kappa_3) + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q(\kappa_1, \kappa_4) + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q(\kappa_2, \kappa_3) + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q(\kappa_2, \kappa_4)}{4} \right)^{\frac{1}{q}}
\end{aligned}$$

eşitsizliği geçerlidir. Burada

$$B = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

şeklindedir (Sarıkaya et al., 2012).

Özdemir ve arkadaşları 2011 yılında yaptıkları çalışmada aşağıdaki eşitsizlikleri elde etmişlerdir.

Teorem 3.14 $F : \Delta \rightarrow R$ koordinatlarda Jensen– konveks fonksiyon olsun. Eğer $F_{\tau_1} \in L_1[\kappa_3, \kappa_4]$ ve $F_{\tau_2} \in L_1[\kappa_1, \kappa_2]$ ise

$$\begin{aligned}
& \frac{1}{2} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \right] \\
& \leq \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 + H(\tau_1, \tau_2)
\end{aligned}$$

eşitsizliği geçerlidir ve burada

$$\begin{aligned}
& H(\tau_1, \tau_2) \\
&= \frac{1}{4(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_0^1 |F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \tau_2) - F((1 - \zeta) \kappa_1 + \zeta \kappa_2, \tau_2)| d\zeta d\tau_2 \\
&+ \frac{1}{4(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \int_0^1 |F(\tau_1, \zeta \kappa_3 + (1 - \zeta) \kappa_4) - F(\tau_1, (1 - \zeta) \kappa_3 + \zeta \kappa_4)| d\zeta d\tau_1
\end{aligned}$$

şeklindedir (Özdemir et al., 2012a).

Özdemir ve arkadaşları 2011 yılında yaptıkları çalışmada aşağıdaki lemmayı elde etmiş ve bu lemma yardımıyla koordinatlarda konveks fonksiyonlar için yeni integral eşitsizlikleri ispatlamışlardır.

Lemma 3.12 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. $\zeta \in [0, 1]$ olmak üzere eğer $\frac{\partial^2 F}{\partial \zeta \partial \xi} \in L_2(\Delta)$ ise

$$\begin{aligned}
J &= (1 - \zeta)^2 F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
&+ \zeta^2 \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} \\
&+ \frac{\zeta(1 - \zeta)}{2} \left[F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3\right) + F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4\right) \right. \\
&\left. + F\left(\kappa_1, \frac{\kappa_3 + \kappa_4}{2}\right) + F\left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2}\right) \right] \\
&- (1 - \zeta) \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\
&- (1 - \zeta) \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
&- \frac{\zeta}{2} \left[\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 \right. \\
&\left. + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2 \right] \\
&+ \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&= \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} K(\tau_1) M(\tau_2) \frac{\partial^2 F}{\partial \tau_1 \partial \tau_2} d\tau_1 d\tau_2
\end{aligned}$$

eşitsizliği geçerlidir ve burada

$$K(\tau_1) = \begin{cases} \tau_1 - \left(\kappa_1 + \zeta \frac{\kappa_2 - \kappa_1}{2}\right), & \tau_1 \in \left[\kappa_1, \frac{\kappa_1 + \kappa_2}{2}\right] \\ \tau_1 - \left(\kappa_2 - \zeta \frac{\kappa_2 - \kappa_1}{2}\right), & \tau_1 \in \left[\frac{\kappa_1 + \kappa_2}{2}, \kappa_2\right] \end{cases}$$

ve

$$M(\tau_2) = \begin{cases} \tau_2 - \left(\kappa_3 + \zeta \frac{\kappa_4 - \kappa_3}{2}\right), & \tau_2 \in \left[\kappa_3, \frac{\kappa_3 + \kappa_4}{2}\right] \\ \tau_2 - \left(\kappa_4 - \zeta \frac{\kappa_4 - \kappa_3}{2}\right), & \tau_2 \in \left[\frac{\kappa_3 + \kappa_4}{2}, \kappa_4\right] \end{cases}$$

dir (Özdemir et al., 2011b).

Teorem 3.15 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. $\zeta \in [0, 1]$ olmak üzere eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$ fonksiyonu Δ üzerinde koordinatlarda konveks ise

$$|J| \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{16} [2\zeta^2 - 2\zeta + 1]^2 \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{4} \right)$$

eşitsizliği geçerlidir (Özdemir et al., 2011b).

Teorem 3.16 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. $\zeta \in [0, 1]$ olmak üzere eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^{\frac{p}{p-1}}$ fonksiyonu Δ üzerinde koordinatlarda konveks ise, $q > 1$ ve $p^{-1} + q^{-1} = 1$ için;

$$|J| \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{4(p+1)^{\frac{2}{p}}} [2\zeta^2 - 2\zeta + 1]^2 \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{4} \right)^{\frac{1}{q}}$$

eşitsizliği geçerlidir (Özdemir et al., 2011b).

Teorem 3.17 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. $\zeta \in [0, 1]$ olmak üzere eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$ fonksiyonu Δ üzerinde koordinatlarda konveks ise, $q > 1$ için;

$$|J| \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{16} [2\zeta^2 - 2\zeta + 1]^2 \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{4} \right)^{\frac{1}{q}}$$

eşitsizliği geçerlidir (Özdemir et al., 2011b).

Özdemir ve arkadaşları 2011 yılında aşağıdaki lemmayı oluşturmuş ve koordinatlarda konveks fonksiyonlar için Hadamard–tipli eşitsizlikler elde etmişlerdir.

Lemma 3.13 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. Eğer $\frac{\partial^2 F}{\partial \zeta \partial \xi} \in L_2(\Delta)$ ise

$$\begin{aligned}
I &= F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\
&\quad - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&= \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{16} \\
&\quad \times \left[\int_0^1 \int_0^1 \zeta \xi \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\zeta \frac{\kappa_1 + \kappa_2}{2} + (1 - \zeta) \kappa_1, \xi \frac{\kappa_3 + \kappa_4}{2} + (1 - \xi) \kappa_3 \right) d\zeta d\xi \right. \\
&\quad + \int_0^1 \int_0^1 (\zeta - 1)(\xi - 1) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\zeta \kappa_2 + (1 - \zeta) \frac{\kappa_1 + \kappa_2}{2}, \xi \kappa_4 + (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
&\quad + \int_0^1 \int_0^1 \zeta(\xi - 1) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\zeta \frac{\kappa_1 + \kappa_2}{2} + (1 - \zeta) \kappa_1, \xi \kappa_4 + (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
&\quad \left. + \int_0^1 \int_0^1 (\zeta - 1)\xi \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\zeta \kappa_2 + (1 - \zeta) \frac{\kappa_1 + \kappa_2}{2}, \xi \frac{\kappa_3 + \kappa_4}{2} + (1 - \xi) \kappa_3 \right) d\zeta d\xi \right]
\end{aligned}$$

eşitsizliği geçerlidir. Burada $\zeta, \xi \in [0, 1]$ 'dir (Özdemir et al., 2021).

Teorem 3.18 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. Eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$ fonksiyonu Δ üzerinde koordinatlarda konveks ise

$$\begin{aligned}
|I| &\leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{144} \\
&\quad \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{4} \right. \\
&\quad + 4 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3 \right) \right| \\
&\quad \left. + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4 \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2} \right) \right| \right]
\end{aligned}$$

eşitsizliği geçerlidir (Özdemir et al., 2021).

Teorem 3.19 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. Eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^{\frac{p}{p-1}}$ fonksiyonu Δ üzerinde koordinatlarda konveks, $q > 1$ ve

$p^{-1} + q^{-1} = 1$ için;

$$\begin{aligned}
& |I| \\
& \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{16(p+1)^{\frac{2}{p}}} \\
& \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{4} \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q \\
& \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{2} \right]^{\frac{1}{q}}
\end{aligned}$$

eşitsizliği geçerlidir (Özdemir et al., 2021).

Teorem 3.20 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. Eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$ fonksiyonu Δ üzerinde koordinatlarda konveks ve $q > 1$ için;

$$\begin{aligned}
& |I| \\
& \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{16} \\
& \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{36} \right. \\
& + \frac{4 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{9} \\
& \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{9} \right]^{\frac{1}{q}}
\end{aligned}$$

eşitsizliği geçerlidir (Özdemir et al., 2021).

Lemma 3.14 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden diferansiyellenebilen bir dönüşüm olsun. Eğer $\frac{\partial^2 F}{\partial \zeta \partial \xi} \in L_2(\Delta)$ ise;

$$\begin{aligned}
D &= F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) - \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} F \left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2 \right) d\tau_2 \\
&\quad - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F \left(\tau_1, \frac{\kappa_3 + \kappa_4}{2} \right) d\tau_1 \\
&\quad + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&= \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} p(\tau_1, \zeta) q(\tau_2, \xi) \\
&\quad \times \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) d\tau_1 d\tau_2
\end{aligned}$$

eşitliği geçerlidir. Burada

$$p(\tau_1, \zeta) = \begin{cases} \zeta - \kappa_1, \tau_1 \in [\kappa_1, \frac{\kappa_1 + \kappa_2}{2}] \\ \kappa_2 - \zeta, \tau_1 \in (\frac{\kappa_1 + \kappa_2}{2}, \kappa_2] \end{cases}$$

ve

$$q(\tau_2, \xi) = \begin{cases} \xi - \kappa_3, \tau_2 \in [\kappa_3, \frac{\kappa_3 + \kappa_4}{2}] \\ \kappa_4 - \xi, \tau_2 \in (\frac{\kappa_3 + \kappa_4}{2}, \kappa_4] \end{cases}$$

dir (Özdemir et al., 2012b).

Teorem 3.21 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. Eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$ fonksiyonu Δ üzerinde koordinatlarda konveks ise

$$|D| \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{64} \times \left[\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right| \right]$$

eşitsizliği geçerlidir (Özdemir et al., 2012b).

Teorem 3.22 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. Eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^{\frac{p}{p-1}}$ fonksiyonu Δ üzerinde koordinatlarda konveks, $q > 1$ ve $p^{-1} + q^{-1} = 1$ için;

$$|D| \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{4(p+1)^{\frac{2}{p}}} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{4} \right]^{\frac{1}{q}}$$

eşitsizliği geçerlidir (Özdemir et al., 2012b).

Teorem 3.23 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden türevlenebilir bir fonksiyon olsun. Eğer $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$ fonksiyonu Δ üzerinde koordinatlarda konveks ve $q > 1$ için;

$$|D| \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{16} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{4} \right]^{\frac{1}{q}}$$

eşitsizliği geçerlidir (Özdemir et al., 2012b).

4. ARAŞTIRMA BULGULARI

4.1. Koordinatlarda Ekspansiyel Konveks Fonksiyon Sınıfları

4.1.1. Koordinatlarda ekspansiyel konveks fonksiyon

Tanım 4.1 $F : \Delta \longrightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta, \alpha \in R$, ve $\zeta \in [0, 1]$ için eğer

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq \zeta \frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + (1 - \zeta) \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}}$$

eşitsizliği sağlanıyorsa F 'ye Δ üzerinde koordinatlarda ekspansiyel konveks fonksiyon denir.

$\forall \tau_1 \in [\kappa_1, \kappa_2]$ ve $\forall \tau_2 \in [\kappa_3, \kappa_4]$ için eğer

$$F_{\tau_2} : [\kappa_1, \kappa_2] \rightarrow R, F_{\tau_2}(u) = e^{\alpha\tau_2} F(u, \tau_2)$$

ve

$$F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha\tau_1} F(\tau_1, v)$$

kısmi dönüşümleri ekspansiyel konveks ise $F : \Delta \longrightarrow R$ fonksiyonu koordinatlarda ekspansiyel konveks fonksiyondur (Aslan and Akdemir, 2022).

Koordinatlarda ekspansiyel konveks fonksiyon tanımına denk bir tanımlama aşağıdaki gibi verilmiştir.

Tanım 4.2 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta, \alpha \in R$ ve $\zeta, \xi \in [0, 1]$ için

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\ & \leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + (1 - \zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

eşitsizliği sağlanıyorsa F 'ye Δ üzerinde koordinatlarda ekspansiyel konveks fonksiyon denir (Aslan and Akdemir, 2022).

Lemma 4.1 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R, F_{\tau_2}(u) = e^{\alpha \tau_2} F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha \tau_1} F(\tau_1, v)$ ise F 'ye koordinatlarda eksponansiyel konveks fonksiyon denir (Aslan and Akdemir, 2022).

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned}
F_{\tau_1}(\zeta v_1 + (1 - \zeta) v_2) &= e^{\alpha \tau_1} F(\tau_1, \zeta v_1 + (1 - \zeta) v_2) \\
&= e^{\alpha \tau_1} F(\zeta \tau_1 + (1 - \zeta) \tau_1, \zeta v_1 + (1 - \zeta) v_2) \\
&\leq e^{\alpha \tau_1} \left(\zeta \frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1 + v_1)}} + (1 - \zeta) \frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1 + v_2)}} \right) \\
&= \zeta \frac{F(\tau_1, v_1)}{e^{\alpha v_1}} + (1 - \zeta) \frac{F(\tau_1, v_2)}{e^{\alpha v_2}} \\
&= \zeta \frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}} + (1 - \zeta) \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}}
\end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned}
F_{\tau_2}(\zeta u_1 + (1 - \zeta) u_2) &= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \tau_2) \\
&= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \zeta \tau_2 + (1 - \zeta) \tau_2) \\
&\leq e^{\alpha \tau_2} \left(\zeta \frac{F(u_1, \tau_2)}{e^{\alpha(u_1 + \tau_2)}} + (1 - \zeta) \frac{F(u_2, \tau_2)}{e^{\alpha(u_2 + \tau_2)}} \right) \\
&= \zeta \frac{F(u_1, \tau_2)}{e^{\alpha u_1}} + (1 - \zeta) \frac{F(u_2, \tau_2)}{e^{\alpha u_2}} \\
&= \zeta \frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}} + (1 - \zeta) \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}}
\end{aligned}$$

elde edilir.

Not 4.1 Lemma 4.1'de $\alpha = 0$ seçilirse (Dragomir, 2001)'deki Lemma 1 elde edilir.

Teorem 4.1 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir (Aslan and Akdemir, 2022):

$$\begin{aligned}
&\frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&\leq \frac{1}{4} \left[\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right].
\end{aligned}$$

İspat Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
&F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
&\leq \zeta \xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + (1 - \zeta) \xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınır;

$$\begin{aligned} & \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \\ & \leq \int_0^1 \int_0^1 \zeta \xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 \zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 (1 - \zeta) \xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi \end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapıp integ-rallerin değeri hesaplanırsa istenen sonuç elde edilir.

Teorem 4.2 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonk-siyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ ve $p > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_3} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{(p+1)^{\frac{2}{p}}} \left(\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \zeta \xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + (1 - \zeta) \xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integ-rali alınır;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| \zeta \xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| (1 - \zeta) \xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| (1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned}
& \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
& \leq \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \int_0^1 \zeta^p (1 - \xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \int_0^1 (1 - \zeta)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \int_0^1 (1 - \zeta)^p (1 - \xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

yazılır. İntegrallerin değeri hesaplanırsa istenen sonuç elde edilir.

Teorem 4.3 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ ve $p, q > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \leq \frac{4}{p(p+1)^2} \\
& + \frac{1}{q} \left(\frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)
\end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\
& \leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + (1 - \zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alma işlemi uygulanırsa;

$$\begin{aligned}
& \left| \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) d\zeta d\xi \right| \\
& \leq \int_0^1 \int_0^1 \left| \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\
& \quad + \int_0^1 \int_0^1 \left| (1 - \zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| (1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanıp $\tau_1 =$

$\zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değişirmesi yapılarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1 - \xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^p (1 - \xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \end{aligned}$$

eşitsizliği elde edilir. İntegrallerin değeri hesaplanırsa istenen sonuç elde edilir.

Önerme 4.1 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyonlar olsun. Bu takdirde $F + \Phi$ fonksiyonu da koordinatlarda eksponansiyel konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) + \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \zeta \xi \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) + \zeta(1 - \xi) \left(\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right) \\ & \quad + (1 - \zeta) \xi \left(\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right) + (1 - \zeta)(1 - \xi) \left(\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \end{aligned}$$

yani;

$$\begin{aligned} & (F + \Phi)(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \zeta \xi \frac{(F + \Phi)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta(1 - \xi) \frac{(F + \Phi)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta) \xi \frac{(F + \Phi)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)(1 - \xi) \frac{(F + \Phi)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

elde edilir. Yukarıdaki eşitsizlikten dolayı $(F + \Phi)$ fonksiyonu koordinatlarda eksponansiyel konveks fonksiyondur.

Önerme 4.2 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyonlar olsun. Bu takdirde $k \geq 0$ için kF fonksiyonu da koordinatlarda eksponansiyel konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \zeta \xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + (1 - \zeta) \xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Her iki taraf k ile çarpılırsa;

$$\begin{aligned} & (kF)(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ \leq & \zeta\xi \frac{(kF)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta(1-\xi) \frac{(kF)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & + (1-\zeta)\xi \frac{(kF)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)(1-\xi) \frac{(kF)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

eşitsizliği elde edilir. Yukarıdaki eşitsizlikten dolayı (kF) fonksiyonu koordinatlarda eksponansiyel konveks fonksiyondur.

4.1.2. Koordinatlarda eksponansiyel m – konveks fonksiyon

Tanım 4.3 $F : \Delta_1 \longrightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta_1, \alpha \in R$ ve $\zeta, m \in [0, 1]$ için eğer

$$F(\zeta\tau_1 + (1-\zeta)\tau_3, \zeta\tau_2 + m(1-\zeta)\tau_4) \leq \zeta \frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1+\tau_2)}} + m(1-\zeta) \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3+\tau_4)}}$$

eşitsizliği sağlanıyorsa F 'ye Δ_1 üzerinde eksponansiyel m – konveks fonksiyon denir.

$\forall \tau_1 \in [0, \kappa_2]$ ve $\forall \tau_2 \in [0, \kappa_4]$ için eğer

$$F_{\tau_2} : [0, \kappa_2] \rightarrow R, F_{\tau_2}(u) = e^{\alpha\tau_2} F(u, \tau_2)$$

ve

$$F_{\tau_1} : [0, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha\tau_1} F(\tau_1, v)$$

kısmi dönüşümleri eksponansiyel m – konveks ise $F : \Delta_1 \longrightarrow R$ fonksiyonu koordinatlarda eksponansiyel m – konveks fonksiyondur (Aslan et al., 2022).

Koordinatlarda eksponansiyel m – konveks fonksiyon tanımına denk bir tanımlama aşağıdaki gibi yapılır.

Tanım 4.4 $F : \Delta_1 \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta_1, \alpha \in R$ ve $\zeta, \xi, m \in [0, 1]$ için

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\ \leq & \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m\zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m(1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

eşitsizliği sağlanıyorsa F 'ye Δ_1 üzerinde koordinatlarda eksponansiyel m – konveks fonksiyon denir (Aslan et al., 2022).

Lemma 4.2 $F : \Delta_1 \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R$, $F_{\tau_2}(u) = e^{\alpha \tau_2} F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R$, $F_{\tau_1}(v) = e^{\alpha \tau_1} F(\tau_1, v)$ ise F 'ye koordinatlarda eksponansiyel m -konveks fonksiyon denir (Aslan et al., 2022).

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned}
 F_{\tau_1}(\zeta v_1 + m(1 - \zeta)v_2) &= e^{\alpha \tau_1} F(\tau_1, \zeta v_1 + m(1 - \zeta)v_2) \\
 &= e^{\alpha \tau_1} F(\zeta \tau_1 + (1 - \zeta)\tau_1, \zeta v_1 + m(1 - \zeta)v_2) \\
 &\leq e^{\alpha \tau_1} \left[\zeta \frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1 + v_1)}} + m(1 - \zeta) \frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1 + v_2)}} \right] \\
 &= \zeta \frac{F(\tau_1, v_1)}{e^{\alpha v_1}} + m(1 - \zeta) \frac{F(\tau_1, v_2)}{e^{\alpha v_2}} \\
 &= \zeta \frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}} + m(1 - \zeta) \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}}
 \end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned}
 F_{\tau_2}(\zeta u_1 + m(1 - \zeta)u_2) &= e^{\alpha \tau_2} F(\zeta u_1 + m(1 - \zeta)u_2, \tau_2) \\
 &= e^{\alpha \tau_2} F(\zeta u_1 + m(1 - \zeta)u_2, \zeta \tau_2 + (1 - \zeta)\tau_2) \\
 &\leq e^{\alpha \tau_2} \left[\zeta \frac{F(u_1, \tau_2)}{e^{\alpha(u_1 + \tau_2)}} + m(1 - \zeta) \frac{F(u_2, \tau_2)}{e^{\alpha(u_2 + \tau_2)}} \right] \\
 &= \zeta \frac{F(u_1, \tau_2)}{e^{\alpha u_1}} + m(1 - \zeta) \frac{F(u_2, \tau_2)}{e^{\alpha u_2}} \\
 &= \zeta \frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}} + m(1 - \zeta) \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}}
 \end{aligned}$$

elde edilir.

Teorem 4.4 $F : \Delta_1 \rightarrow R$ fonksiyonu Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta_1)$, $\alpha \in R$ ve $m \in [0, 1]$ için aşağıdaki eşitsizlik geçerlidir (Aslan et al., 2022):

$$\begin{aligned}
 &\frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
 &\leq \frac{1}{4} \left[\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + m \left(\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \right].
 \end{aligned}$$

İspat Δ_1 üzerinde koordinatlar eksponansiyel m -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned}
 &F(\zeta \kappa_1 + (1 - \zeta)\kappa_2, \xi \kappa_3 + m(1 - \xi)\kappa_4) \\
 &\leq \zeta \xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + m \zeta (1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
 &\quad + (1 - \zeta) \xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + m(1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
 \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınır;

$$\begin{aligned} & \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m(1 - \xi) \kappa_4) d\zeta d\xi \\ & \leq \int_0^1 \int_0^1 \zeta \xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 m \zeta (1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 (1 - \zeta) \xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 m(1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi \end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + m(1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak integrallerin değeri hesaplanırsa istenen sonuç elde edilir.

Teorem 4.5 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$, $m \in [0, 1]$ ve $p > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{(p+1)^{\frac{2}{p}}} \left(\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{m|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{m|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right). \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir (Aslan et al., 2022).

İspat Δ üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m(1 - \xi) \kappa_4) \\ & \leq \zeta \xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + m \zeta (1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta) \xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + m(1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alma işlemi uygulanırsa;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m(1 - \xi) \kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| \zeta \xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| m \zeta (1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| (1 - \zeta) \xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| m(1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + m(1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned}
& \left| \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
& \leq \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \int_0^1 \zeta^p (1 - \xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \int_0^1 (1 - \zeta)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \int_0^1 (1 - \zeta)^p (1 - \xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

eşitsizliği elde edilir. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Teorem 4.6 $F : \Delta_1 \rightarrow R$ fonksiyonu Δ_1 üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta_1)$, $\alpha \in R$, $m \in [0, 1]$ ve $p, q > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \left| \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
& \leq \frac{4}{p(p+1)^2} + \frac{1}{q} \left(\frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{|mF(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{|mF(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)
\end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir (Aslan et al., 2022).

İspat Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) \\
& \leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + m\zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& \quad + (1 - \zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + m(1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alma işlemi uygulanırsa;

$$\begin{aligned}
& \left| \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) d\zeta d\xi \right| \\
& \leq \int_0^1 \int_0^1 \left| \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| m\zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\
& \quad + \int_0^1 \int_0^1 \left| (1 - \zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| m(1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + m(1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1 - \xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^p (1 - \xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \end{aligned}$$

eşitsizliği elde edilir. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Önerme 4.3 $F, \Phi : \Delta_1 \rightarrow R$ fonksiyonları Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyonlar olsun. Bu takdirde $F + \Phi$ fonksiyonunda koordinatlarda eksponansiyel m -konveks fonksiyondur (Aslan et al., 2022).

İspat Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m(1 - \xi) \kappa_4) + \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m(1 - \xi) \kappa_4) \\ & \leq \zeta \xi \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) + m\zeta(1 - \xi) \left(\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right) \\ & \quad + (1 - \zeta)\xi \left(\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right) + m(1 - \zeta)(1 - \xi) \left(\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \end{aligned}$$

yani;

$$\begin{aligned} & (F + \Phi)(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m(1 - \xi) \kappa_4) \\ & \leq \zeta \xi \frac{(F + \Phi)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + m\zeta(1 - \xi) \frac{(F + \Phi)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta)\xi \frac{(F + \Phi)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + m(1 - \zeta)(1 - \xi) \frac{(F + \Phi)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı $(F + \Phi)$ fonksiyonu koordinatlarda eksponansiyel m -konveks fonksiyondur.

Önerme 4.4 $F, \Phi : \Delta_1 \rightarrow R$ fonksiyonları Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyonlar olsun. Bu takdirde $k \geq 0$ için kF fonksiyonunda koordinatlarda eksponansiyel m -konveks fonksiyondur (Aslan et al., 2022).

İspat Δ_1 üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\ \leq & \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m\zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m(1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Her iki taraf k ile çarpılarak yeniden düzenlenir ise,

$$\begin{aligned} & (kF)(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\ \leq & \zeta\xi \frac{(kF)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m\zeta(1-\xi) \frac{(kF)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & + (1-\zeta)\xi \frac{(kF)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m(1-\zeta)(1-\xi) \frac{(kF)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

eşitsizliği elde edilir. Bu eşitsizlikten dolayı (kF) fonksiyonu koordinatlarda eksponansiyel m -konveks fonksiyondur.

4.1.3. Koordinatlarda eksponansiyel (α, m) -konveks fonksiyon

Tanım 4.5 $F : \Delta_1 \longrightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta_1, \alpha \in R, (\alpha_1, m) \in [0, 1]^2$ ve $\zeta \in [0, 1]$ için eğer

$$F(\zeta\tau_1 + (1-\zeta)\tau_3, \zeta\tau_2 + m(1-\zeta)\tau_4) \leq \zeta^{\alpha_1} \frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1+\tau_2)}} + m(1-\zeta^{\alpha_1}) \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3+\tau_4)}}$$

eşitsizliği sağlanıyorsa F 'ye Δ_1 üzerinde eksponansiyel (α_1, m) -konveks fonksiyon denir (Aslan et al., 2022).

Koordinatlarda eksponansiyel (α_1, m) -konveks fonksiyon tanımına denk bir tanımlama aşağıdaki gibi yapılır.

Tanım 4.6 $F : \Delta_1 \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta_1, \alpha \in R, (\alpha_1, m) \in [0, 1]^2$ ve $\zeta, \xi \in [0, 1]$ için

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\ \leq & \zeta^{\alpha_1} \xi^{\alpha_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m\zeta^{\alpha_1}(1-\xi^{\alpha_1}) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & + (1-\zeta^{\alpha_1})\xi^{\alpha_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m(1-\zeta^{\alpha_1})(1-\xi^{\alpha_1}) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

eşitsizliği sağlanıyorsa F 'ye Δ_1 üzerinde koordinatlarda eksponansiyel (α_1, m) -konveks fonksiyon denir (Aslan et al., 2022).

Bu tanımda $(\alpha_1, m) = (1, m)$ alınırsa koordinatlarda eksponansiyel m – konveks fonksiyonlar sınıfı ve $(\alpha_1, m) = (1, 1)$ alınırsa koordinatlarda eksponansiyel konveks fonksiyonlar sınıfı elde edilir.

Lemma 4.3 $F : \Delta_1 \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R, F_{\tau_2}(u) = e^{\alpha\tau_2}F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha\tau_1}F(\tau_1, v)$ ise F 'ye koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyon denir (Aslan et al., 2022).

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned} F_{\tau_1}(\zeta v_1 + m(1 - \zeta)v_2) &= e^{\alpha\tau_1}F(\tau_1, \zeta v_1 + m(1 - \zeta)v_2) \\ &= e^{\alpha\tau_1}F(\zeta\tau_1 + (1 - \zeta)\tau_1, \zeta v_1 + m(1 - \zeta)v_2) \\ &\leq e^{\alpha\tau_1} \left[\zeta^{\alpha_1} \frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1+v_1)}} + m(1 - \zeta^{\alpha_1}) \frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1+v_2)}} \right] \\ &= \zeta^{\alpha_1} \frac{F(\tau_1, v_1)}{e^{\alpha v_1}} + m(1 - \zeta^{\alpha_1}) \frac{F(\tau_1, v_2)}{e^{\alpha v_2}} \\ &= \zeta^{\alpha_1} \frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}} + m(1 - \zeta^{\alpha_1}) \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}} \end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned} F_{\tau_2}(\zeta u_1 + m(1 - \zeta)u_2) &= e^{\alpha\tau_2}F(\zeta u_1 + m(1 - \zeta)u_2, \tau_2) \\ &= e^{\alpha\tau_2}F(\zeta u_1 + m(1 - \zeta)u_2, \zeta\tau_2 + (1 - \zeta)\tau_2) \\ &\leq e^{\alpha\tau_2} \left[\zeta^{\alpha_1} \frac{F(u_1, \tau_2)}{e^{\alpha(u_1+\tau_2)}} + m(1 - \zeta^{\alpha_1}) \frac{F(u_2, \tau_2)}{e^{\alpha(u_2+\tau_2)}} \right] \\ &= \zeta^{\alpha_1} \frac{F(u_1, \tau_2)}{e^{\alpha u_1}} + m(1 - \zeta^{\alpha_1}) \frac{F(u_2, \tau_2)}{e^{\alpha u_2}} \\ &= \zeta^{\alpha_1} \frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}} + m(1 - \zeta^{\alpha_1}) \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}} \end{aligned}$$

elde edilir.

Teorem 4.7 $F : \Delta_1 \rightarrow R$ fonksiyonu Δ_1 üzerinde koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta_1)$, $\alpha \in R$ ve $(\alpha_1, m) \in [0, 1]^2$ için aşağıdaki eşitsizlik geçerlidir (Aslan et al., 2022):

$$\begin{aligned} &\frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ &\leq \frac{1}{(\alpha_1 + 1)^2} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{\alpha_1}{(\alpha_1 + 1)^2} \left(\frac{mF(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right) + \frac{\alpha_1^2}{(\alpha_1 + 1)^2} \frac{mF(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}. \end{aligned}$$

İspat Δ_1 üzerinde koordinatlar eksponansiyel (α_1, m) – konveks fonksiyon tanımını kullanarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) \\ & \leq \zeta^{\alpha_1} \xi^{\alpha_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + m \zeta^{\alpha_1} (1 - \xi^{\alpha_1}) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta^{\alpha_1}) \xi^{\alpha_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + m(1 - \zeta^{\alpha_1})(1 - \xi^{\alpha_1}) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınırsa;

$$\begin{aligned} & \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) d\zeta d\xi \\ & \leq \int_0^1 \int_0^1 \zeta^{\alpha_1} \xi^{\alpha_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 m \zeta^{\alpha_1} (1 - \xi^{\alpha_1}) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 (1 - \zeta^{\alpha_1}) \xi^{\alpha_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 m(1 - \zeta^{\alpha_1})(1 - \xi^{\alpha_1}) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi \end{aligned}$$

yazılır. $\tau_1 = \zeta\kappa_1 + (1 - \zeta)\kappa_2$ ve $\tau_2 = \xi\kappa_3 + (1 - \xi)\kappa_4$ değişken değiştirmesi yapılarak integrallerin değeri hesaplanırsa istenen sonuç elde edilir.

Teorem 4.8 $F : \Delta_1 \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta_1)$, $\alpha \in R$, $(\alpha_1, m) \in [0, 1]^2$ ve $p > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{(p\alpha_1 + 1)^{\frac{2}{p}}} \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{p\alpha_1}{(p\alpha_1 + 1)^2} \right)^{\frac{1}{p}} \left(\frac{m|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2 + \kappa_3)}} \right) \\ & \quad + \left(\frac{p^2\alpha_1^2}{(p\alpha_1 + 1)^2} \right)^{\frac{1}{p}} \frac{m|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir (Aslan et al., 2022).

İspat Δ_1 üzerinde koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) \\ & \leq \zeta^{\alpha_1} \xi^{\alpha_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + m \zeta^{\alpha_1} (1 - \xi^{\alpha_1}) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta^{\alpha_1}) \xi^{\alpha_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + m(1 - \zeta^{\alpha_1})(1 - \xi^{\alpha_1}) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınırsa;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m(1 - \xi) \kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| \zeta^{\alpha_1} \xi^{\alpha_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| m \zeta^{\alpha_1} (1 - \xi^{\alpha_1}) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| (1 - \zeta^{\alpha_1}) \xi^{\alpha_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| m(1 - \zeta^{\alpha_1})(1 - \xi^{\alpha_1}) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \left(\int_0^1 \int_0^1 |\zeta^{p\alpha_1} \xi^{p\alpha_1}| d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 |\zeta^{p\alpha_1} (1 - \xi^{\alpha_1})^p| d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 |(1 - \zeta^{\alpha_1})^p \xi^{p\alpha_1}| d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 |(1 - \zeta^{\alpha_1})^p (1 - \xi^{\alpha_1})^p| d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \end{aligned}$$

elde edilir. $\theta > 0, \beta > 0$ için $|1 - (1 - t)^\theta|^\beta \leq 1 - (1 - t)^{\theta\beta}$ eşitsizliği kullanılarak (Akdemir et al., 2019),

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \left(\int_0^1 \int_0^1 \zeta^{p\alpha_1} \xi^{p\alpha_1} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 \zeta^{p\alpha_1} (1 - \xi^{p\alpha_1}) d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 (1 - \zeta^{p\alpha_1}) \xi^{p\alpha_1} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 (1 - \zeta^{p\alpha_1})(1 - \xi^{p\alpha_1}) d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \end{aligned}$$

elde edilir. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Teorem 4.9 $F : \Delta_1 \rightarrow R$ fonksiyonu Δ_1 üzerinde koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta_1)$, $\alpha \in R$, $(\alpha_1, m) \in [0, 1]^2$ ve $p, q > 1$ için

aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \left(\frac{1}{p(p\alpha_1 + 1)^2} \right) + \frac{|F(\kappa_1, \kappa_3)|^q}{qe^{q\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{\alpha_1}{(p\alpha_1 + 1)^2} \right) + \left(\frac{m|F(\kappa_1, \kappa_4)|^q}{qe^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|^q}{qe^{q\alpha(\kappa_2 + \kappa_3)}} \right) \\ & \quad + \left(\frac{p\alpha_1^2}{(p\alpha_1 + 1)^2} \right) + \frac{m|F(\kappa_2, \kappa_4)|^q}{qe^{q\alpha(\kappa_2 + \kappa_4)}}. \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir (Aslan et al., 2022).

İspat Δ_1 üzerinde koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) \\ & \leq \zeta^{\alpha_1} \xi^{\alpha_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + m\zeta^{\alpha_1} (1 - \xi^{\alpha_1}) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta^{\alpha_1}) \xi^{\alpha_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + m(1 - \zeta^{\alpha_1}) (1 - \xi^{\alpha_1}) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınır;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| \zeta^{\alpha_1} \xi^{\alpha_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| m\zeta^{\alpha_1} (1 - \xi^{\alpha_1}) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| (1 - \zeta^{\alpha_1}) \xi^{\alpha_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| m(1 - \zeta^{\alpha_1}) (1 - \xi^{\alpha_1}) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanıp $\tau_1 = \zeta\kappa_1 + (1 - \zeta)\kappa_2$ ve $\tau_2 = \xi\kappa_3 + (1 - \xi)\kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{p\alpha_1} \xi^{p\alpha_1} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{p\alpha_1} (1 - \xi^{\alpha_1})^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta^{\alpha_1})^p \xi^{p\alpha_1} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta^{\alpha_1})^p (1 - \xi^{\alpha_1})^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \end{aligned}$$

eşitsizliği elde edilir ve $\theta > 0, \beta > 0$ için $|1 - (1 - t)^\theta|^\beta \leq 1 - (1 - t)^{\theta\beta}$ kullanılarak (Akdemir et al., 2019),

$$\begin{aligned}
& \left| \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
& \leq \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{p\alpha_1} \xi^{p\alpha_1} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{p\alpha_1} (1 - \xi^{p\alpha_1}) d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\
& \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta^{p\alpha_1}) \xi^{p\alpha_1} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta^{p\alpha_1})(1 - \xi^{p\alpha_1}) d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{mF(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)
\end{aligned}$$

yazılır. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Önerme 4.5 $F, \Phi : \Delta_1 \rightarrow R$ fonksiyonları Δ_1 üzerinde koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyonlar olsun. Bu takdirde $F + \Phi$ fonksiyonunda koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyondur (Aslan et al., 2022).

İspat Δ_1 üzerinde koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) + \Phi(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) \\
& \leq \zeta^{\alpha_1} \xi^{\alpha_1} \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) + m\zeta^{\alpha_1} (1 - \xi^{\alpha_1}) \left(\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right) \\
& \quad + (1 - \zeta^{\alpha_1}) \xi^{\alpha_1} \left(\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right) \\
& \quad + m(1 - \zeta^{\alpha_1})(1 - \xi^{\alpha_1}) \left(\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right)
\end{aligned}$$

yani;

$$\begin{aligned}
& (F + \Phi)(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) \\
& \leq \zeta^{\alpha_1} \xi^{\alpha_1} \frac{(F + \Phi)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + m\zeta^{\alpha_1} (1 - \xi^{\alpha_1}) \frac{(F + \Phi)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& \quad + (1 - \zeta^{\alpha_1}) \xi^{\alpha_1} \frac{(F + \Phi)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + m(1 - \zeta^{\alpha_1})(1 - \xi^{\alpha_1}) \frac{(F + \Phi)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı $(F + \Phi)$ fonksiyonu koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyondur.

Önerme 4.6 $F, \Phi : \Delta_1 \rightarrow R$ fonksiyonları Δ_1 üzerinde koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyonlar olsun. Bu takdirde $k \geq 0$ için kF fonksiyonunda koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyondur (Aslan et al., 2022).

İspat Δ_1 üzerinde koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) \\ \leq & \zeta^{\alpha_1}\xi^{\alpha_1}\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m\zeta^{\alpha_1}(1 - \xi^{\alpha_1})\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & + (1 - \zeta^{\alpha_1})\xi^{\alpha_1}\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m(1 - \zeta^{\alpha_1})(1 - \xi^{\alpha_1})\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Her iki taraf k ile çarpılarak yeniden düzenlenir ise,

$$\begin{aligned} & (kF)(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + m(1 - \xi)\kappa_4) \\ \leq & \zeta^{\alpha_1}\xi^{\alpha_1}\frac{(kF)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m\zeta^{\alpha_1}(1 - \xi^{\alpha_1})\frac{(kF)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & + (1 - \zeta^{\alpha_1})\xi^{\alpha_1}\frac{(kF)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m(1 - \zeta^{\alpha_1})(1 - \xi^{\alpha_1})\frac{(kF)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

eşitsizliği elde edilir. Bu nedenle (kF) fonksiyonu koordinatlarda eksponansiyel (α_1, m) – konveks fonksiyondur.

4.1.4. Koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyon

Tanım 4.7 $F : \Delta \longrightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta, \alpha \in R$, ve $\zeta \in (0, 1)$ için eğer

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq \frac{\frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1+\tau_2)}}}{\zeta} + \frac{\frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3+\tau_4)}}}{(1 - \zeta)}$$

eşitsizliği sağlanıyorsa F ’ye Δ üzerinde eksponansiyel Godunova-Levin konveks fonksiyon denir.

Koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyon tanımına denk bir tanımlama aşağıdaki gibi yapılabilir.

Tanım 4.8 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta, \alpha \in R$ ve $\zeta, \xi \in (0, 1)$ için

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\ \leq & \frac{\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}}{\zeta\xi} + \frac{\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}}{\zeta(1 - \xi)} + \frac{\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}}{(1 - \zeta)\xi} + \frac{\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}}{(1 - \zeta)(1 - \xi)} \end{aligned}$$

eşitsizliği sağlanıyorsa F ’ye Δ üzerinde koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyon denir.

Lemma 4.4 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R$, $F_{\tau_2}(u) = e^{\alpha \tau_2} F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R$, $F_{\tau_1}(v) = e^{\alpha \tau_1} F(\tau_1, v)$ ise F 'ye koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyon denir.

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned}
F_{\tau_1}(\zeta v_1 + (1 - \zeta) v_2) &= e^{\alpha \tau_1} F(\tau_1, \zeta v_1 + (1 - \zeta) v_2) \\
&= e^{\alpha \tau_1} F(\zeta \tau_1 + (1 - \zeta) \tau_1, \zeta v_1 + (1 - \zeta) v_2) \\
&\leq e^{\alpha \tau_1} \left(\frac{\frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1 + v_1)}}}{\zeta} + \frac{\frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1 + v_2)}}}{(1 - \zeta)} \right) \\
&= \frac{\frac{F(\tau_1, v_1)}{e^{\alpha v_1}}}{\zeta} + \frac{\frac{F(\tau_1, v_2)}{e^{\alpha v_2}}}{(1 - \zeta)} \\
&= \frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}} + \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}} \\
&= \frac{F_{\tau_1}(v_1)}{\zeta} + \frac{F_{\tau_1}(v_2)}{(1 - \zeta)}
\end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned}
F_{\tau_2}(\zeta u_1 + (1 - \zeta) u_2) &= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \tau_2) \\
&= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \zeta \tau_2 + (1 - \zeta) \tau_2) \\
&\leq e^{\alpha \tau_2} \left(\frac{\frac{F(u_1, \tau_2)}{e^{\alpha(u_1 + \tau_2)}}}{\zeta} + \frac{\frac{F(u_2, \tau_2)}{e^{\alpha(u_2 + \tau_2)}}}{(1 - \zeta)} \right) \\
&= \frac{\frac{F(u_1, \tau_2)}{e^{\alpha u_1}}}{\zeta} + \frac{\frac{F(u_2, \tau_2)}{e^{\alpha u_2}}}{(1 - \zeta)} \\
&= \frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}} + \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}} \\
&= \frac{F_{\tau_2}(u_1)}{\zeta} + \frac{F_{\tau_2}(u_2)}{(1 - \zeta)}
\end{aligned}$$

elde edilir.

Teorem 4.10 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&\leq \int_0^1 \int_0^1 \frac{\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}}{\zeta \xi} d\zeta d\xi + \int_0^1 \int_0^1 \frac{\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}}{\zeta(1 - \xi)} d\zeta d\xi \\
&\quad + \int_0^1 \int_0^1 \frac{\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}}{(1 - \zeta)\xi} d\zeta d\xi + \int_0^1 \int_0^1 \frac{\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}}{(1 - \zeta)(1 - \xi)} d\zeta d\xi.
\end{aligned}$$

İspat Δ üzerinde koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyon tanımını kullanarak;

$$\begin{aligned}
&F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
&\leq \frac{\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}}{\zeta \xi} + \frac{\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}}{\zeta(1 - \xi)} + \frac{\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}}{(1 - \zeta)\xi} + \frac{\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}}{(1 - \zeta)(1 - \xi)}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınırsa;

$$\begin{aligned} & \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \\ & \leq \int_0^1 \int_0^1 \frac{\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}}{\zeta \xi} d\zeta d\xi + \int_0^1 \int_0^1 \frac{\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}}{\zeta(1 - \xi)} d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \frac{\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}}{(1 - \zeta)\xi} d\zeta d\xi + \int_0^1 \int_0^1 \frac{\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}}{(1 - \zeta)(1 - \xi)} d\zeta d\xi \end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak integrallerin değeri hesaplanırsa istenen sonuç elde edilir.

Önerme 4.7 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyonlar olsun. Bu takdirde $F + \Phi$ fonksiyonunda koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) + \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \frac{\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}}{\zeta \xi} + \frac{\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}}{\zeta(1 - \xi)} + \frac{\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}}{(1 - \zeta)\xi} + \frac{\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}}{(1 - \zeta)(1 - \xi)} \end{aligned}$$

yani;

$$\begin{aligned} & (F + \Phi)(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \frac{\frac{(F + \Phi)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}}{\zeta \xi} + \frac{\frac{(F + \Phi)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}}{\zeta(1 - \xi)} + \frac{\frac{(F + \Phi)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}}{(1 - \zeta)\xi} + \frac{\frac{(F + \Phi)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}}{(1 - \zeta)(1 - \xi)} \end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı $(F + \Phi)$ fonksiyonu koordinatlarda eksponansiyel Godunova-Levin konveks fonksiyondur.

Önerme 4.8 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel Godunova – Levin konveks fonksiyonlar olsun. Bu takdirde $k \geq 0$ için kF fonksiyonunda koordinatlarda eksponansiyel Godunova – Levin konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda eksponansiyel Godunova – Levin konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \frac{\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}}{\zeta \xi} + \frac{\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}}{\zeta(1 - \xi)} + \frac{\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}}{(1 - \zeta)\xi} + \frac{\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}}{(1 - \zeta)(1 - \xi)} \end{aligned}$$

yazılır. Her iki taraf k ile çarpılarak yeniden düzenlenir ise,

$$\begin{aligned} & (kF)(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq \frac{(kF)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{(kF)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{(kF)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{(kF)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\ & \quad \zeta\xi + \zeta(1-\xi) + (1-\zeta)\xi + (1-\zeta)(1-\xi) \end{aligned}$$

elde edilir. Yukarıdaki eşitsizlikten dolayı (kF) fonksiyonu koordinatlarda eksponansiyel Godunova – Levin konveks fonksiyondur.

4.1.5. Koordinatlarda eksponansiyel h – konveks fonksiyon

Tanım 4.9 $F : \Delta \longrightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta, \alpha \in R, h : J \subseteq R \longrightarrow R_+$ ve $\zeta \in [0, 1]$ için eğer

$$F(\zeta\tau_1 + (1-\zeta)\tau_3, \zeta\tau_2 + (1-\zeta)\tau_4) \leq h(\zeta) \frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1+\tau_2)}} + h(1-\zeta) \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3+\tau_4)}}$$

eşitsizliği sağlanıyorsa F ’ye Δ üzerinde eksponansiyel h – konveks fonksiyon denir.

Koordinatlarda eksponansiyel h – konveks fonksiyon tanımına denk bir tanımlama aşağıdaki gibi yapılabilir.

Tanım 4.10 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta, \alpha \in R$ ve $\zeta, \xi \in [0, 1]$ için

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq h(\zeta)h(\xi) \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + h(\zeta)h(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + h(1-\zeta)h(\xi) \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + h(1-\zeta)h(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

eşitsizliği sağlanıyorsa F ’ye Δ üzerinde koordinatlarda eksponansiyel h – konveks fonksiyon denir.

Lemma 4.5 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R, F_{\tau_2}(u) = e^{\alpha\tau_2}F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha\tau_1}F(\tau_1, v)$ ise F ’ye koordinatlarda eksponansiyel h – konveks fonksiyon denir.

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned}
F_{\tau_1}(\zeta v_1 + (1 - \zeta) v_2) &= e^{\alpha \tau_1} F(\tau_1, \zeta v_1 + (1 - \zeta) v_2) \\
&= e^{\alpha \tau_1} F(\zeta \tau_1 + (1 - \zeta) \tau_1, \zeta v_1 + (1 - \zeta) v_2) \\
&\leq e^{\alpha \tau_1} \left[h(\zeta) \frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1 + v_1)}} + h(1 - \zeta) \frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1 + v_2)}} \right] \\
&= h(\zeta) \frac{F(\tau_1, v_1)}{e^{\alpha v_1}} + h(1 - \zeta) \frac{F(\tau_1, v_2)}{e^{\alpha v_2}} \\
&= h(\zeta) \frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}} + h(1 - \zeta) \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}}
\end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned}
F_{\tau_2}(\zeta u_1 + (1 - \zeta) u_2) &= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \tau_2) \\
&= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \zeta \tau_2 + (1 - \zeta) \tau_2) \\
&\leq e^{\alpha \tau_2} \left[h(\zeta) \frac{F(u_1, \tau_2)}{e^{\alpha(u_1 + \tau_2)}} + h(1 - \zeta) \frac{F(u_2, \tau_2)}{e^{\alpha(u_2 + \tau_2)}} \right] \\
&= h(\zeta) \frac{F(u_1, \tau_2)}{e^{\alpha u_1}} + h(1 - \zeta) \frac{F(u_2, \tau_2)}{e^{\alpha u_2}} \\
&= h(\zeta) \frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}} + h(1 - \zeta) \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}}
\end{aligned}$$

elde edilir.

Teorem 4.11 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel h -konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ ve $h : J \subseteq R \rightarrow R_+$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&\leq \int_0^1 \int_0^1 h(\zeta) h(\xi) \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 h(\zeta) h(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \\
&\quad + \int_0^1 \int_0^1 h(1 - \zeta) h(\xi) \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 h(1 - \zeta) h(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi.
\end{aligned}$$

İspat Δ üzerinde koordinatlarda eksponansiyel h -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned}
&F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
&\leq h(\zeta) h(\xi) \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + h(\zeta) h(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
&\quad + h(1 - \zeta) h(\xi) \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + h(1 - \zeta) h(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınırsa;

$$\begin{aligned} & \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \\ & \leq \int_0^1 \int_0^1 h(\zeta) h(\xi) \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 h(\zeta) h(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 h(1 - \zeta) h(\xi) \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 h(1 - \zeta) h(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi \end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapıp gerekli düzenlemeler yapıldığında istenen sonuç elde edilir.

Teorem 4.12 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$, $J \subseteq R \rightarrow R_+$ ve $p > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \left(\int_0^1 \int_0^1 |h(\zeta)|^p |h(\xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1 + \kappa_3)}} \\ & \quad + \left(\int_0^1 \int_0^1 |h(\zeta)|^p |h(1 - \xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + \left(\int_0^1 \int_0^1 |h(1 - \zeta)|^p |h(\xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2 + \kappa_3)}} \\ & \quad + \left(\int_0^1 \int_0^1 |h(1 - \zeta)|^p |h(1 - \xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat Δ üzerinde koordinatlarda eksponansiyel h -konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq h(\zeta) h(\xi) \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + h(\zeta) h(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + h(1 - \zeta) h(\xi) \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + h(1 - \zeta) h(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye

göre integral alınır;

$$\begin{aligned}
& \left| \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) d\zeta d\xi \right| \\
& \leq \int_0^1 \int_0^1 \left| h(\zeta) h(\xi) \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| h(\zeta) h(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\
& \quad + \int_0^1 \int_0^1 \left| h(1-\zeta) h(\xi) \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi \\
& \quad + \int_0^1 \int_0^1 \left| h(1-\zeta) h(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1-\zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1-\xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned}
& \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
& \leq \left(\int_0^1 \int_0^1 |h(\zeta)|^p |h(\xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \int_0^1 |h(\zeta)|^p |h(1-\xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \int_0^1 |h(1-\zeta)|^p |h(\xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \int_0^1 |h(1-\zeta)|^p |h(1-\xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

eşitsizliği elde edilir ve integrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Teorem 4.13 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel h -konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$, $h : J \subseteq R \rightarrow R_+$ ve $p, q > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
& \leq \frac{1}{p} \left(\int_0^1 \int_0^1 |h(\zeta)|^p |h(\xi)|^p d\zeta d\xi \right) + \frac{|F(\kappa_1, \kappa_3)|^q}{q e^{q\alpha(\kappa_1 + \kappa_3)}} \\
& \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 |h(\zeta)|^p |h(1-\xi)|^p d\zeta d\xi \right) + \frac{|F(\kappa_1, \kappa_4)|^q}{q e^{q\alpha(\kappa_1 + \kappa_4)}} \\
& \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 |h(1-\zeta)|^p |h(\xi)|^p d\zeta d\xi \right) + \frac{|F(\kappa_2, \kappa_3)|^q}{q e^{q\alpha(\kappa_2 + \kappa_3)}} \\
& \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 |h(1-\zeta)|^p |h(1-\xi)|^p d\zeta d\xi \right) + \frac{|F(\kappa_2, \kappa_4)|^q}{q e^{q\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat Δ üzerinde koordinatlarda eksponansiyel h –konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq h(\zeta)h(\xi) \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + h(\zeta)h(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + h(1-\zeta)h(\xi) \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + h(1-\zeta)h(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınırsa;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| h(\zeta)h(\xi) \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| h(\zeta)h(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| h(1-\zeta)h(\xi) \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| h(1-\zeta)h(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanıp $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2$ ve $\tau_2 = \xi\kappa_3 + (1-\xi)\kappa_4$ değişken değiştirmesi uygulanarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{p} \left(\int_0^1 \int_0^1 |h(\zeta)|^p |h(\xi)|^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 |h(\zeta)|^p |h(1-\xi)|^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 |h(1-\zeta)|^p |h(\xi)|^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 |h(1-\zeta)|^p |h(1-\xi)|^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \end{aligned}$$

eşitsizliği elde edilir ve integrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Önerme 4.9 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel h –konveks fonksiyonlar olsun. Bu takdirde $F + \Phi$ fonksiyonunda koordinatlarda eksponansiyel h –konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda eksponansiyel h – konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) + \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
\leq & h(\zeta)h(\xi) \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right) + h(\zeta)h(1-\xi) \left(\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right) \\
& + h(1-\zeta)h(\xi) \left(\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right) \\
& + h(1-\zeta)h(1-\xi) \left(\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right)
\end{aligned}$$

yani;

$$\begin{aligned}
& (F + \Phi)(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
\leq & h(\zeta)h(\xi) \frac{(F + \Phi)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + h(\zeta)h(1-\xi) \frac{(F + \Phi)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + h(1-\zeta)h(\xi) \frac{(F + \Phi)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + h(1-\zeta)h(1-\xi) \frac{(F + \Phi)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı $(F + \Phi)$ fonksiyonu koordinatlarda eksponansiyel h – konveks fonksiyondur.

Önerme 4.10 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel h – konveks fonksiyonlar olsun. Bu takdirde $k \geq 0$ için kF fonksiyonunda koordinatlarda eksponansiyel h – konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda eksponansiyel h – konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
\leq & h(\zeta)h(\xi) \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + h(\zeta)h(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + h(1-\zeta)h(\xi) \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + h(1-\zeta)h(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. Her iki taraf k ile çarpılarak yeniden düzenlenir ise,

$$\begin{aligned}
& F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
\leq & h(\zeta)h(\xi) \frac{(kF)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + h(\zeta)h(1-\xi) \frac{(kF)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + h(1-\zeta)h(\xi) \frac{(kF)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + h(1-\zeta)h(1-\xi) \frac{(kF)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

elde edilir. Yukarıdaki eşitsizlikten dolayı (kF) fonksiyonu koordinatlarda eksponansiyel h – konveks fonksiyondur.

4.1.6. Koordinatlarda eksponansiyel J – konveks fonksiyon

Tanım 4.11 $F : \Delta \longrightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta, \alpha \in R, \forall \tau_1 \in [\kappa_1, \kappa_2]$ ve $\forall \tau_2 \in [\kappa_3, \kappa_4], \zeta \in [0, 1]$ için eğer

$$F\left(\frac{\tau_1 + \tau_3}{2}, \frac{\tau_2 + \tau_4}{2}\right) \leq \frac{1}{2} \left(\frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}} \right)$$

eşitsizliği sağlanıyorsa F ’ye Δ üzerinde eksponansiyel J – konveks fonksiyon denir.

Koordinatlarda eksponansiyel J – konveks fonksiyon tanımına denk bir tanımlama aşağıdaki gibi yapılabilir.

Tanım 4.12 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta, \alpha \in R$ ve $\zeta, \xi \in [0, 1]$ için

$$\begin{aligned} & F\left(\frac{\zeta \kappa_1 + (1 - \zeta) \kappa_2 + \zeta \kappa_2 + (1 - \zeta) \kappa_1}{2}, \frac{\xi \kappa_3 + (1 - \xi) \kappa_4 + \xi \kappa_4 + (1 - \xi) \kappa_3}{2}\right) \\ & \leq \frac{\zeta \xi}{4} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\zeta(1 - \xi)}{4} \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{(1 - \zeta)\xi}{4} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{(1 - \zeta)(1 - \xi)}{4} \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

eşitsizliği sağlanıyorsa F ’ye Δ üzerinde koordinatlarda eksponansiyel J – konveks fonksiyon denir.

Lemma 4.6 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R, F_{\tau_2}(u) = e^{\alpha \tau_2} F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha \tau_1} F(\tau_1, v)$ ise F ’ye koordinatlarda eksponansiyel J – konveks fonksiyon denir.

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned} F_{\tau_1}\left(\frac{v_1 + v_2}{2}\right) &= e^{\alpha \tau_1} F\left(\tau_1, \frac{v_1 + v_2}{2}\right) \\ &= e^{\alpha \tau_1} F\left(\frac{\tau_1 + \tau_1}{2}, \frac{v_1 + v_2}{2}\right) \\ &\leq e^{\alpha \tau_1} \frac{1}{2} \left(\frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1 + v_1)}} + \frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1 + v_2)}} \right) \\ &= \frac{1}{2} \left(\frac{F(\tau_1, v_1)}{e^{\alpha v_1}} + \frac{F(\tau_1, v_2)}{e^{\alpha v_2}} \right) \\ &= \frac{1}{2} \left(\frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}} + \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}} \right) \end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned}
F_{\tau_2} \left(\frac{u_1 + u_2}{2} \right) &= e^{\alpha \tau_2} F_{\tau_2} \left(\frac{u_1 + u_2}{2}, \tau_2 \right) \\
&= e^{\alpha \tau_2} F \left(\frac{u_1 + u_2}{2}, \frac{\tau_2 + \tau_2}{2} \right) \\
&\leq e^{\alpha \tau_2} \frac{1}{2} \left(\frac{F(u_1, \tau_2)}{e^{\alpha(u_1 + \tau_2)}} + \frac{F(u_2, \tau_2)}{e^{\alpha(u_2 + \tau_2)}} \right) \\
&= \frac{1}{2} \left(\frac{F(u_1, \tau_2)}{e^{\alpha u_1}} + \frac{F(u_2, \tau_2)}{e^{\alpha u_2}} \right) \\
&= \frac{1}{2} \left(\frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}} + \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}} \right)
\end{aligned}$$

elde edilir.

Teorem 4.14 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel J -konveks fonksiyon olsun. Bu takdirde $F \in L(\Delta)$, $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \\
&\leq \frac{1}{16} \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right).
\end{aligned}$$

İspat Δ üzerinde koordinatlarda eksponansiyel J -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned}
&F \left(\frac{\zeta \kappa_1 + (1 - \zeta) \kappa_2 + \zeta \kappa_2 + (1 - \zeta) \kappa_1}{2}, \frac{\xi \kappa_3 + (1 - \xi) \kappa_4 + \xi \kappa_4 + (1 - \xi) \kappa_3}{2} \right) \\
&\leq \frac{\zeta \xi}{4} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\zeta(1 - \xi)}{4} \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{(1 - \zeta)\xi}{4} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{(1 - \zeta)(1 - \xi)}{4} \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınır;

$$\begin{aligned}
&\int_0^1 \int_0^1 F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
&\leq \int_0^1 \int_0^1 \frac{\zeta \xi}{4} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 \frac{\zeta(1 - \xi)}{4} \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \\
&\quad + \int_0^1 \int_0^1 \frac{(1 - \zeta)\xi}{4} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 \frac{(1 - \zeta)(1 - \xi)}{4} \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi
\end{aligned}$$

yazılır. İntegrallerin değeri hesaplanırsa istenen sonuç elde edilir.

Önerme 4.11 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel J -konveks fonksiyonlar olsun. Bu takdirde $F + \Phi$ fonksiyonunda koordinatlarda eksponansiyel konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda eksponansiyel J – konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F\left(\frac{\tau_1 + \tau_3}{2}, \frac{\tau_2 + \tau_4}{2}\right) + \Phi\left(\frac{\tau_1 + \tau_3}{2}, \frac{\tau_2 + \tau_4}{2}\right) \\ & \leq \frac{1}{2} \left(\frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}} \right) + \frac{1}{2} \left(\frac{\Phi(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + \frac{\Phi(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}} \right) \end{aligned}$$

yani;

$$\begin{aligned} & (F + \Phi)\left(\frac{\tau_1 + \tau_3}{2}, \frac{\tau_2 + \tau_4}{2}\right) \\ & \leq \frac{1}{2} \left(\frac{(F + \Phi)(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + \frac{(F + \Phi)(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}} \right) \end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı $(F + \Phi)$ fonksiyonu koordinatlarda eksponansiyel J – konveks fonksiyondur.

Önerme 4.12 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel J – konveks fonksiyonlar olsun. Bu takdirde $k \geq 0$ için kF fonksiyonunda koordinatlarda eksponansiyel J – konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda eksponansiyel J – konveks fonksiyon tanımı kullanılarak;

$$F\left(\frac{\tau_1 + \tau_3}{2}, \frac{\tau_2 + \tau_4}{2}\right) \leq \frac{1}{2} \left(\frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}} \right)$$

yazılır. Her iki taraf k ile çarpılarak yeniden düzenlenir ise,

$$(kF)\left(\frac{\tau_1 + \tau_3}{2}, \frac{\tau_2 + \tau_4}{2}\right) \leq \frac{1}{2} \left(\frac{(kF)(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + \frac{(kF)(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}} \right)$$

elde edilir. Yukarıdaki eşitsizlikten dolayı (kF) fonksiyonu koordinatlarda eksponansiyel konveks fonksiyondur.

4.1.7. Koordinatlarda eksponansiyel P – fonksiyon

Tanım 4.13 $F : \Delta \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta, \alpha \in R$, ve $\zeta \in [0, 1]$ için eğer

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq \frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}}$$

eşitsizliği sağlanıyorsa F ’ye Δ üzerinde koordinatlarda eksponansiyel P – fonksiyon denir (Akdemir et al., 2022a).

Koordinatlarda eksponansiyel P – fonksiyon tanımına denk bir tanımlama aşağıdaki gibi yapılabilir.

Tanım 4.14 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta, \alpha \in R$ ve $\zeta, \xi \in [0, 1]$ için

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

eşitsizliği sağlanıyorsa F 'ye Δ üzerinde koordinatlarda eksponansiyel P -fonksiyon denir (Akdemir et al., 2022a).

Lemma 4.7 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R, F_{\tau_2}(u) = e^{\alpha \tau_2} F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha \tau_1} F(\tau_1, v)$ ise F 'ye koordinatlarda eksponansiyel P -fonksiyon denir (Akdemir et al., 2022a).

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned} F_{\tau_1}(\zeta v_1 + (1 - \zeta) v_2) &= e^{\alpha \tau_1} F(\tau_1, \zeta v_1 + (1 - \zeta) v_2) \\ &= e^{\alpha \tau_1} F(\zeta \tau_1 + (1 - \zeta) \tau_1, \zeta v_1 + (1 - \zeta) v_2) \\ &\leq e^{\alpha \tau_1} \left(\frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1 + v_1)}} + \frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1 + v_2)}} \right) \\ &= \frac{F(\tau_1, v_1)}{e^{\alpha v_1}} + \frac{F(\tau_1, v_2)}{e^{\alpha v_2}} \\ &= \frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}} + \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}} \end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned} F_{\tau_2}(\zeta u_1 + (1 - \zeta) u_2) &= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \tau_2) \\ &= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \zeta \tau_2 + (1 - \zeta) \tau_2) \\ &\leq e^{\alpha \tau_2} \left(\frac{F(u_1, \tau_2)}{e^{\alpha(u_1 + \tau_2)}} + \frac{F(u_2, \tau_2)}{e^{\alpha(u_2 + \tau_2)}} \right) \\ &= \frac{F(u_1, \tau_2)}{e^{\alpha u_1}} + \frac{F(u_2, \tau_2)}{e^{\alpha u_2}} \\ &= \frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}} + \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}} \end{aligned}$$

elde edilir.

Teorem 4.15 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel P -fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir (Akdemir et al., 2022a):

$$\begin{aligned} & \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right). \end{aligned}$$

İspat Δ üzerinde koordinatlarda eksponansiyel P – fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ \leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınır;

$$\begin{aligned} & \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \\ \leq & \int_0^1 \int_0^1 \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \\ & + \int_0^1 \int_0^1 \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi \end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılır ve integrallerin değeri hesaplanırsa istenen sonuç elde edilir.

Teorem 4.16 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel P – fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ ve $p > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ \leq & \left(\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir (Akdemir et al., 2022a).

İspat Δ üzerinde koordinatlarda eksponansiyel P – fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ \leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınır;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \right| \\ \leq & \int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\ & + \int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi uygulanarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \left(\int_0^1 \int_0^1 d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \end{aligned}$$

eşitsizliği elde edilir. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Teorem 4.17 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel P -fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ ve $p, q > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{4}{p} + \frac{1}{q} \left(\frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right) \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir (Akdemir et al., 2022a).

İspat Δ üzerinde koordinatlarda eksponansiyel P -fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınır;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanıp $\tau_1 =$

$\zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{p} \left(\int_0^1 \int_0^1 d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \end{aligned}$$

eşitsizliği elde edilir. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Önerme 4.13 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel P -fonksiyonlar olsun. Bu takdirde $F + \Phi$ fonksiyonunda koordinatlarda eksponansiyel P -fonksiyondur (Akdemir et al., 2022a).

İspat Δ üzerinde koordinatlarda eksponansiyel P -fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) + \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) + \left(\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right) \\ & \quad + \left(\frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right) + \left(\frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \end{aligned}$$

yani;

$$\begin{aligned} & (F + \Phi)(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \frac{(F + \Phi)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{(F + \Phi)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{(F + \Phi)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{(F + \Phi)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı $(F + \Phi)$ fonksiyonu koordinatlarda eksponansiyel P -fonksiyondur.

Önerme 4.14 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel P -fonksiyonlar olsun. Bu takdirde $k \geq 0$ için kF fonksiyonunda koordinatlarda eksponansiyel P -fonksiyondur (Akdemir et al., 2022a).

İspat Δ üzerinde koordinatlarda eksponansiyel P -fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Her iki taraf k ile çarpılarak yeniden düzenlenir ise,

$$\begin{aligned} & (kF)(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ \leq & \frac{(kF)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{(kF)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{(kF)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{(kF)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı (kF) fonksiyonu koordinatlarda eksponansiyel P -fonksiyondur.

4.1.8. Koordinatlarda eksponansiyel Quasi-konveks fonksiyon

Tanım 4.15 $F : \Delta \longrightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta, \alpha \in R$, ve $\zeta \in [0, 1]$ için eğer

$$F(\zeta\tau_1 + (1-\zeta)\tau_3, \zeta\tau_2 + (1-\zeta)\tau_4) \leq \max \left\{ \frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1+\tau_2)}} + \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3+\tau_4)}} \right\}$$

eşitsizliği sağlanıyorsa F 'ya Δ üzerinde koordinatlarda eksponansiyel Quasi-konveks fonksiyon denir (Akdemir et al., 2022b).

Koordinatlarda eksponansiyel quasi-konveks fonksiyon tanımına denk bir tanımlama aşağıdaki gibi yapılır.

Tanım 4.16 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta, \alpha \in R$ ve $\zeta, \xi \in [0, 1]$ için

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ \leq & \max \left\{ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}, \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}, \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}, \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right\} \end{aligned}$$

eşitsizliği sağlanıyorsa F 'ye Δ üzerinde koordinatlarda eksponansiyel quasi-konveks fonksiyon denir (Akdemir et al., 2022b).

Lemma 4.8 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R, F_{\tau_2}(u) = e^{\alpha\tau_2}F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha\tau_1}F(\tau_1, v)$ ise F 'ye koordinatlarda eksponansiyel Quasi-konveks fonksiyon denir (Akdemir et al., 2022b).

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned}
F_{\tau_1}(\zeta v_1 + (1 - \zeta) v_2) &= e^{\alpha \tau_1} F(\tau_1, \zeta v_1 + (1 - \zeta) v_2) \\
&= e^{\alpha \tau_1} F(\zeta \tau_1 + (1 - \zeta) \tau_1, \zeta v_1 + (1 - \zeta) v_2) \\
&\leq e^{\alpha \tau_1} \left(\max \left\{ \frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1 + v_1)}}, \frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1 + v_2)}} \right\} \right) \\
&= \max \left\{ \frac{F(\tau_1, v_1)}{e^{\alpha v_1}}, \frac{F(\tau_1, v_2)}{e^{\alpha v_2}} \right\} \\
&= \max \left\{ \frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}}, \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}} \right\}
\end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned}
F_{\tau_2}(\zeta u_1 + (1 - \zeta) u_2) &= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \tau_2) \\
&= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \zeta \tau_2 + (1 - \zeta) \tau_2) \\
&\leq e^{\alpha \tau_2} \max \left\{ \frac{F(u_1, \tau_2)}{e^{\alpha(u_1 + \tau_2)}}, \frac{F(u_2, \tau_2)}{e^{\alpha(u_2 + \tau_2)}} \right\} \\
&= \max \left\{ \frac{F(u_1, \tau_2)}{e^{\alpha u_1}}, \frac{F(u_2, \tau_2)}{e^{\alpha u_2}} \right\} \\
&= \max \left\{ \frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}}, \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}} \right\}
\end{aligned}$$

elde edilir.

Teorem 4.18 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel quasi – konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir (Akdemir et al., 2022b):

$$\begin{aligned}
&\frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&\leq \max \left\{ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}, \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}, \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}, \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right\}.
\end{aligned}$$

İspat Δ üzerinde koordinatlarda eksponansiyel quasi-konveks fonksiyon tanımını kullanarak;

$$\begin{aligned}
&F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
&\leq \max \left\{ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}, \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}, \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}, \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right\}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınır;

$$\begin{aligned}
&\int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \\
&\leq \int_0^1 \int_0^1 \max \left\{ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}, \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}, \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}, \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right\} d\zeta d\xi
\end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapıp integ-rallerin değeri hesaplanırsa istenen sonuç elde edilir.

Teorem 4.19 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel quasi-konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ ve $p > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \leq \left(\max \left\{ \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}}, \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}}, \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}}, \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right\} \right)^{\frac{1}{q}}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat Δ üzerinde koordinatlarda eksponansiyel quasi-konveks fonksiyon tanımı kullanıla-rak;

$$F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \leq \max \left\{ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}}, \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}}, \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}}, \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right\}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınır;

$$\left| \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \right| \leq \int_0^1 \int_0^1 \max \left\{ \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1 + \kappa_3)}}, \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1 + \kappa_4)}}, \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2 + \kappa_3)}}, \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right\} d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi uygulanarak;

$$\left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \leq \left(\int_0^1 \int_0^1 d\zeta d\xi \right)^{\frac{1}{p}} \times \left(\int_0^1 \int_0^1 \max \left\{ \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}}, \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}}, \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}}, \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right\} d\zeta d\xi \right)^{\frac{1}{q}}$$

eşitsizliği elde edilir. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Teorem 4.20 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda eksponansiyel quasi-konveks fonksiyon olsun. Bu takdirde $F \in L_2(\Delta)$, $\alpha \in R$ ve $p, q > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \leq \frac{1}{p} + \frac{1}{q} \left(\frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir (Akdemir et al., 2022b).

İspat Δ üzerinde koordinatlarda eksponansiyel quasi-konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq \max \left\{ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}, \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}, \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}, \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right\} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınırsa;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \max \left\{ \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}}, \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}}, \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}}, \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \right\} d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanıp $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2$ ve $\tau_2 = \xi\kappa_3 + (1-\xi)\kappa_4$ değişken değiştirmesi uygulanarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{p} \left(\int_0^1 \int_0^1 d\zeta d\xi \right) \\ & \quad + \frac{1}{q} \left(\int_0^1 \int_0^1 \max \left\{ \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}, \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}}, \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}, \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \right\} d\zeta d\xi \right) \end{aligned}$$

eşitsizliği elde edilir. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Önerme 4.15 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda eksponansiyel Quasi-konveks fonksiyonlar olsun. Bu takdirde $k \geq 0$ için (kF) fonksiyonunda koordinatlarda eksponansiyel Quasi-konveks fonksiyondur (Akdemir et al., 2022b).

İspat Δ üzerinde koordinatlarda eksponansiyel quasi-konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq \max \left\{ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}, \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}, \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}, \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right\} \end{aligned}$$

yazılır. Her iki taraf k ile çarpılarak yeniden düzenlenir ise,

$$\begin{aligned} & (kF)(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq \max \left\{ \frac{(kF)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}, \frac{(kF)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}, \frac{(kF)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}, \frac{(kF)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right\} \end{aligned}$$

elde edilir. Yukarıdaki eşitsizlikten dolayı (kF) fonksiyonu koordinatlarda eksponansiyel Quasi-konveks fonksiyondur.

4.1.9. Koordinatlarda birinci anlamda eksponansiyel s – konveks fonksiyon

Tanım 4.17 $F : \Delta \longrightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta, \alpha \in R, s \in (0, 1]$ ve $\zeta \in [0, 1]$ için eğer

$$F(\zeta\tau_1 + (1 - \zeta)\tau_3, \zeta\tau_2 + (1 - \zeta)\tau_4) \leq \zeta^s \frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + (1 - \zeta^s) \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}}$$

eşitsizliği sağlanıyorsa F ’ye Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s – konveks fonksiyon denir.

Koordinatlarda birinci anlamda eksponansiyel s – konveks tanımına denk bir tanımlama aşağıdaki gibi yapılabilir.

Tanım 4.18 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta, \alpha \in R, s \in (0, 1]$ ve $\zeta, \xi \in [0, 1]$ için

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\ & \leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi^s) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta^s) \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta^s)(1 - \xi^s) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

eşitsizliği sağlanıyorsa F ’ya Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s – konveks fonksiyon denir.

Lemma 4.9 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R, F_{\tau_2}(u) = e^{\alpha\tau_2} F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha\tau_1} F(\tau_1, v)$ ise F ’ya koordinatlarda birinci anlamda eksponansiyel s – konveks fonksiyon denir.

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned} F_{\tau_1}(\zeta v_1 + (1 - \zeta)v_2) &= e^{\alpha\tau_1} F(\tau_1, \zeta v_1 + (1 - \zeta)v_2) \\ &= e^{\alpha\tau_1} F(\zeta\tau_1 + (1 - \zeta)\tau_1, \zeta v_1 + (1 - \zeta)v_2) \\ &\leq e^{\alpha\tau_1} \left(\zeta^s \frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1 + v_1)}} + (1 - \zeta^s) \frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1 + v_2)}} \right) \\ &= \zeta^s \frac{F(\tau_1, v_1)}{e^{\alpha v_1}} + (1 - \zeta^s) \frac{F(\tau_1, v_2)}{e^{\alpha v_2}} \\ &= \zeta^s \frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}} + (1 - \zeta^s) \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}} \end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned}
F_{\tau_2}(\zeta u_1 + (1 - \zeta) u_2) &= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \tau_2) \\
&= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \zeta \tau_2 + (1 - \zeta) \tau_2) \\
&\leq e^{\alpha \tau_2} \left(\zeta^s \frac{F(u_1, \tau_2)}{e^{\alpha(u_1 + \tau_2)}} + (1 - \zeta^s) \frac{F(u_2, \tau_2)}{e^{\alpha(u_2 + \tau_2)}} \right) \\
&= \zeta^s \frac{F(u_1, \tau_2)}{e^{\alpha u_1}} + (1 - \zeta^s) \frac{F(u_2, \tau_2)}{e^{\alpha u_2}} \\
&= \zeta^s \frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}} + (1 - \zeta^s) \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}}
\end{aligned}$$

elde edilir.

Teorem 4.21 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $F \in L(\Delta)$, $\alpha \in R$ ve $s \in (0, 1]$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&\leq \frac{1}{(s+1)^2} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{s}{(s+1)^2} \left(\frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right) + \frac{s^2}{(s+1)^2} \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}.
\end{aligned}$$

İspat Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned}
&F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
&\leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi^s) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
&\quad + (1 - \zeta^s) \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta^s) (1 - \xi^s) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınır;

$$\begin{aligned}
&\int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \\
&\leq \int_0^1 \int_0^1 \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 \zeta^s (1 - \xi^s) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \\
&\quad + \int_0^1 \int_0^1 (1 - \zeta^s) \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 (1 - \zeta^s) (1 - \xi^s) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi
\end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapıp integralerin değeri hesaplanırsa istenen sonuç elde edilir.

Teorem 4.22 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $F \in L(\Delta)$, $\alpha \in R$, $s \in (0, 1]$ ve $p > 1$

için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \left(\frac{1}{(ps+1)^2} \right)^{\frac{1}{p}} \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \left(\frac{ps}{(ps+1)^2} \right)^{\frac{1}{p}} \left(\frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right) \\ & \quad + \left(\frac{p^2 s^2}{(ps+1)^2} \right)^{\frac{1}{p}} \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^s (1-\xi^s) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + (1-\zeta^s) \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta^s)(1-\xi^s) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınırsa;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \zeta^s (1-\xi^s) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| (1-\zeta^s) \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| (1-\zeta^s)(1-\xi^s) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanıp $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2$ ve $\tau_2 = \xi\kappa_3 + (1-\xi)\kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \left(\int_0^1 \int_0^1 \zeta^{sp} \xi^{sp} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 \zeta^{sp} (1-\xi^s)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 (1-\zeta^s)^p \xi^{sp} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 (1-\zeta^s)^p (1-\xi^s)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \end{aligned}$$

eşitsizliği elde edilir. $\theta > 0, \beta > 0$ için $\left|1 - (1 - t)^\theta\right|^\beta \leq 1 - (1 - t)^{\theta\beta}$ kullanılarak (Akdemir et al., 2019),

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \left(\int_0^1 \int_0^1 \zeta^{sp} \xi^{sp} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 \zeta^{sp} (1 - \xi^{sp}) d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 (1 - \zeta^{sp}) \xi^{sp} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 (1 - \zeta^{sp})(1 - \xi^{sp}) d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \end{aligned}$$

bulunur ve integrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Teorem 4.23 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $F \in L(\Delta)$, $\alpha \in R$, $s \in (0, 1]$ ve $p, q > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1 + ps + p^2 s^2}{p(ps + 1)^2} + \frac{1}{q} \left(\frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right) \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\ & \leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi^s) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta^s) \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta^s)(1 - \xi^s) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınır;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \zeta^s (1 - \xi^s) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| (1 - \zeta^s) \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| (1 - \zeta^s)(1 - \xi^s) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{sp} \xi^{sp} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{sp} (1 - \xi^s)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta^s)^p \xi^{sp} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta^s)^p (1 - \xi^s)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \end{aligned}$$

eşitsizliği elde edilir. $\theta > 0, \beta > 0$ için $|1 - (1 - t)^\theta|^\beta \leq 1 - (1 - t)^{\theta\beta}$ kullanılarak (Akdemir et al., 2019),

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{sp} \xi^{sp} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{sp} (1 - \xi^{sp}) d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta^{sp}) \xi^{sp} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta^{sp}) (1 - \xi^{sp}) d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \end{aligned}$$

bulunur ve integrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Önerme 4.16 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyonlar olsun. Bu takdirde $F + \Phi$ fonksiyonunda koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) + \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi^s) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + (1 - \zeta^s) \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \\ & \quad + (1 - \zeta^s) (1 - \xi^s) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} + \zeta^s \xi^s \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi^s) \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta^s) \xi^s \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta^s) (1 - \xi^s) \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yani;

$$\begin{aligned}
& (F + \Phi)(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
\leq & \zeta^s \xi^s \frac{(F + \Phi)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi^s) \frac{(F + \Phi)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + (1 - \zeta^s) \xi^s \frac{(F + \Phi)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta^s)(1 - \xi^s) \frac{(F + \Phi)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı $(F + \Phi)$ fonksiyonu koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyondur.

Önerme 4.17 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyonlar olsun. Bu takdirde $k \geq 0$ için (kF) fonksiyonunda koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned}
& F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
\leq & \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi^s) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + (1 - \zeta^s) \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \\
& + (1 - \zeta^s)(1 - \xi^s) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Her iki taraf k ile çarpılarak yeniden düzenlenir ise,

$$\begin{aligned}
& (kF)(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
\leq & \zeta^s \xi^s \frac{(kF)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi^s) \frac{(kF)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + (1 - \zeta^s) \xi^s \frac{(kF)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta^s)(1 - \xi^s) \frac{(kF)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

elde edilir. Yukarıdaki eşitsizlikten dolayı (kF) fonksiyonu koordinatlarda birinci anlamda eksponansiyel s -konveks fonksiyondur.

4.1.10. Koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon

Tanım 4.19 $F : \Delta \rightarrow R$ fonksiyonu $\forall (\tau_1, \tau_2), (\tau_3, \tau_4) \in \Delta, \alpha \in R, s \in (0, 1]$ ve $\zeta \in [0, 1]$ için eğer

$$F(\zeta \tau_1 + (1 - \zeta) \tau_3, \zeta \tau_2 + (1 - \zeta) \tau_4) \leq \zeta^s \frac{F(\tau_1, \tau_2)}{e^{\alpha(\tau_1 + \tau_2)}} + (1 - \zeta)^s \frac{F(\tau_3, \tau_4)}{e^{\alpha(\tau_3 + \tau_4)}}$$

eşitsizliği sağlanıyorsa F 'ye Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon denir.

Koordinatlarda ikinci anlamda eksponansiyel s -konveks tanımına denk bir tanımlama aşağıdaki gibi yapılabilir.

Tanım 4.20 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall (\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta, \alpha \in R, s \in (0, 1]$ ve $\zeta, \xi \in [0, 1]$ için

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta)^s \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

eşitsizliği sağlanıyorsa F 'ye Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon denir.

Lemma 4.10 $F : \Delta \longrightarrow R$ fonksiyonu verilsin. Eğer $\forall \tau_2 \in [\kappa_3, \kappa_4]$ ve $\forall \tau_1 \in [\kappa_1, \kappa_2]$ için $F_{\tau_2} : [\kappa_1, \kappa_2] \longrightarrow R, F_{\tau_2}(u) = e^{\alpha \tau_2} F(u, \tau_2)$ ve $F_{\tau_1} : [\kappa_3, \kappa_4] \longrightarrow R, F_{\tau_1}(v) = e^{\alpha \tau_1} F(\tau_1, v)$ ise F 'ye koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon denir.

İspat F kısmi dönüşüm tanımı kullanılarak;

$$\begin{aligned} F_{\tau_1}(\zeta v_1 + (1 - \zeta) v_2) &= e^{\alpha \tau_1} F(\tau_1, \zeta v_1 + (1 - \zeta) v_2) \\ &= e^{\alpha \tau_1} F(\zeta \tau_1 + (1 - \zeta) \tau_1, \zeta v_1 + (1 - \zeta) v_2) \\ &\leq e^{\alpha \tau_1} \left(\zeta^s \frac{F(\tau_1, v_1)}{e^{\alpha(\tau_1 + v_1)}} + (1 - \zeta)^s \frac{F(\tau_1, v_2)}{e^{\alpha(\tau_1 + v_2)}} \right) \\ &= \zeta^s \frac{F(\tau_1, v_1)}{e^{\alpha v_1}} + (1 - \zeta)^s \frac{F(\tau_1, v_2)}{e^{\alpha v_2}} \\ &= \zeta^s \frac{F_{\tau_1}(v_1)}{e^{\alpha v_1}} + (1 - \zeta)^s \frac{F_{\tau_1}(v_2)}{e^{\alpha v_2}} \end{aligned}$$

yazılır. Benzer şekilde,

$$\begin{aligned} F_{\tau_2}(\zeta u_1 + (1 - \zeta) u_2) &= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \tau_2) \\ &= e^{\alpha \tau_2} F(\zeta u_1 + (1 - \zeta) u_2, \zeta \tau_2 + (1 - \zeta) \tau_2) \\ &\leq e^{\alpha \tau_2} \left(\zeta^s \frac{F(u_1, \tau_2)}{e^{\alpha(u_1 + \tau_2)}} + (1 - \zeta)^s \frac{F(u_2, \tau_2)}{e^{\alpha(u_2 + \tau_2)}} \right) \\ &= \zeta^s \frac{F(u_1, \tau_2)}{e^{\alpha u_1}} + (1 - \zeta)^s \frac{F(u_2, \tau_2)}{e^{\alpha u_2}} \\ &= \zeta^s \frac{F_{\tau_2}(u_1)}{e^{\alpha u_1}} + (1 - \zeta)^s \frac{F_{\tau_2}(u_2)}{e^{\alpha u_2}} \end{aligned}$$

elde edilir.

Teorem 4.24 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyon olsun. Bu takdirde $F \in L(\Delta)$, $\alpha \in R$ ve $s \in (0, 1]$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{1}{(s+1)^2} \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right). \end{aligned}$$

İspat Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^s (1-\xi)^s \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + (1-\zeta)^s \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)^s (1-\xi)^s \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integrali alınır;

$$\begin{aligned} & \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) d\zeta d\xi \\ & \leq \int_0^1 \int_0^1 \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 \zeta^s (1-\xi)^s \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 (1-\zeta)^s \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi + \int_0^1 \int_0^1 (1-\zeta)^s (1-\xi)^s \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \end{aligned}$$

yazılır. $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2$ ve $\tau_2 = \xi\kappa_3 + (1-\xi)\kappa_4$ değişken değiştirmesi yapıp integralerin değeri hesaplanırsa istenen sonuç elde edilir.

Teorem 4.25 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyon olsun. Bu takdirde $F \in L(\Delta)$, $\alpha \in R$, $s \in (0, 1]$ ve $p > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{(ps+1)^{\frac{2}{p}}} \left(\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \right) \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^s (1-\xi)^s \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + (1-\zeta)^s \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)^s (1-\xi)^s \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınırsa;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \zeta^s (1 - \xi)^s \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| (1 - \zeta)^s \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| (1 - \zeta)^s (1 - \xi)^s \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \left(\int_0^1 \int_0^1 \zeta^{sp} \xi^{sp} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 \zeta^{sp} (1 - \xi)^{sp} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 (1 - \zeta)^{sp} \xi^{sp} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 \int_0^1 (1 - \zeta)^{sp} (1 - \xi)^{sp} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \end{aligned}$$

eşitsizliği elde edilir. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Teorem 4.26 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $F \in L(\Delta)$, $\alpha \in R$, $s \in (0, 1]$ ve $p, q > 1$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{4}{p(ps + 1)^2} + \frac{1}{q} \left(\frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right) \end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta)^s \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Yukarıdaki eşitsizliğin her iki tarafının mutlak değeri alınıp $[0, 1]^2$ üzerinde ζ ve ξ 'ye göre integral alınırsa;

$$\begin{aligned} & \left| \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \right| \\ & \leq \int_0^1 \int_0^1 \left| \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \zeta^s (1 - \xi)^s \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right| d\zeta d\xi \\ & \quad + \int_0^1 \int_0^1 \left| (1 - \zeta)^s \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| (1 - \zeta)^s (1 - \xi)^s \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned} & \left| \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{sp} \xi^{sp} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{sp} (1 - \xi)^{sp} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^{sp} \xi^{sp} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\ & \quad + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^{sp} (1 - \xi)^{sp} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \end{aligned}$$

eşitsizliği elde edilir. İntegrallerin değeri hesaplanırsa ispat tamamlanmış olur.

Önerme 4.18 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyonlar olsun. Bu takdirde $F + \Phi$ fonksiyonunda koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) + \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \leq \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta)^s \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \\ & \quad + \zeta^s \xi^s \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta)^s \xi^s \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yani;

$$\begin{aligned}
& (F + \Phi)(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\
\leq & \zeta^s \xi^s \frac{(F + \Phi)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{(F + \Phi)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + (1 - \zeta)^s \xi^s \frac{(F + \Phi)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{(F + \Phi)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı $(F + \Phi)$ fonksiyonu koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyondur.

Önerme 4.19 $F, \Phi : \Delta \rightarrow R$ fonksiyonları Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyonlar olsun. Bu takdirde $k \geq 0$ için (kF) fonksiyonunda koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyondur.

İspat Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\
\leq & \zeta^s \xi^s \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + (1 - \zeta)^s \xi^s \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Her iki taraf k ile çarpılarak yeniden düzenlenir ise,

$$\begin{aligned}
& (kF)(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\
\leq & \zeta^s \xi^s \frac{(kF)(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{(kF)(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + (1 - \zeta)^s \xi^s \frac{(kF)(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{(kF)(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}}
\end{aligned}$$

yazılır. Yukarıdaki eşitsizlikten dolayı (kF) fonksiyonu koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyondur.

4.2. Koordinatlarda Farklı Türden Ekspansiyel Konveks Fonksiyonların Çarpımı Üzerine Eşitsizlikler

4.2.1. Koordinatlarda ekspansiyel konveks ve koordinatlarda ikinci anlamda ekspansiyel s — konveks fonksiyonların çarpımı üzerine eşitsizlikler

Teorem 4.27 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda ekspansiyel konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda ikinci anlamda ekspansiyel s — konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta), s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{1}{(2+s)^2} L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + \frac{1}{(1+s)(2+s)^2} M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \\ & \quad + \frac{1}{(1+s)^2(2+s)^2} N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \end{aligned}$$

burada

$$\begin{aligned} L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\ M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \\ & \quad + \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_3)}} \\ & \quad + \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_4)}} \\ N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

İspat F, Δ üzerinde koordinatlarda ekspansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Φ, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s - konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ \leq & \zeta^s \xi^s \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & + (1 - \zeta)^s \xi^s \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak, $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınır;

$$\begin{aligned} & \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\ & \times \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \\ \leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^{1+s} \xi^{1+s} d\zeta d\xi \\ & + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^{1+s} (1 - \xi)^s \xi d\zeta d\xi \\ & + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta)^s \xi^{1+s} d\zeta d\xi \\ & + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta)^s \xi (1 - \xi)^s d\zeta d\xi \\ & + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^{1+s} \xi^s (1 - \xi) d\zeta d\xi \\ & + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^{1+s} (1 - \xi)^{1+s} d\zeta d\xi \\ & + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta)^s \xi^s (1 - \xi) d\zeta d\xi \\ & + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta)^s (1 - \xi)^{1+s} d\zeta d\xi \\ & + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^s (1 - \zeta) \xi^{1+s} d\zeta d\xi \\ & + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^s (1 - \zeta) (1 - \xi)^s \xi d\zeta d\xi \\ & + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta)^{1+s} \xi^{1+s} d\zeta d\xi \\ & + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)^{1+s} (1 - \xi)^s \xi d\zeta d\xi \\ & + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^s (1 - \zeta) (1 - \xi) \xi^s d\zeta d\xi \\ & + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^s (1 - \zeta) (1 - \xi)^{1+s} d\zeta d\xi \end{aligned}$$

$$\begin{aligned}
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^{1+s} \xi^s (1-\xi) d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^{1+s} (1-\xi)^{1+s} d\zeta d\xi
\end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + (1-\zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1-\xi) \kappa_4$ değişken değiştirmesi yapıp integ-rallerin değeri hesaplanırsa;

$$\begin{aligned}
& \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{1}{(2+s)^2} \left[\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right] \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \Big] \\
& + \frac{1}{(1+s)(2+s)^2} \left[\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& \left. + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right] \\
& + \frac{1}{(1+s)^2(2+s)^2} \left[\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& \left. + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right]
\end{aligned}$$

elde edilir.

Teorem 4.28 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta), s \in (0, 1], q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{1}{(ps+p+1)^{\frac{2}{p}}} L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + \left(\frac{\beta(ps+1, p+1)}{ps_1+p+1} \right)^{\frac{1}{p}} M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \\
& + \beta^{\frac{2}{p}} (ps+1, p+1) N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)
\end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ ve

$$\begin{aligned}
L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = & \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

$$\begin{aligned}
M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = & \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}}
\end{aligned}$$

$$\begin{aligned}
N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = & \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}}
\end{aligned}$$

İspat F, Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
\leq & \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. Φ, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
\leq & \zeta^s \xi^s \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^s (1-\xi)^s \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + (1-\zeta)^s \xi^s \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)^s (1-\xi)^s \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak, $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınır;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
& \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) d\zeta d\xi \\
\leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{1+s} \xi^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{1+s} (1-\xi)^s \xi d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta)^s \xi^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta)^s \xi (1-\xi)^s d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{1+s} \xi^s (1-\xi) d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{1+s} (1-\xi)^{1+s} d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta(1-\zeta)^s \xi^s (1-\xi) d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta(1-\zeta)^s (1-\xi)^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^s (1-\zeta) \xi^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^s (1-\zeta) (1-\xi)^s \xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^{1+s} \xi^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^{1+s} (1-\xi)^s \xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^s (1-\zeta) (1-\xi) \xi^s d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^s (1-\zeta) (1-\xi)^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^{1+s} \xi^s (1-\xi) d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^{1+s} (1-\xi)^{1+s} d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) d\zeta d\xi \\
& \leq \left(\int_0^1 \int_0^1 \zeta^{(1+s)p} \xi^{(1+s)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{(1+s)p} (1-\xi)^{ps} \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^{ps} \xi^{(1+s)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p (1-\zeta)^{ps} (1-\xi)^{ps} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{(1+s)p} \xi^{ps} (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^{ps} (1-\xi)^{(1+s)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^{ps} \xi^{ps} (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^{ps} (1-\xi)^{(1+s)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \zeta^{ps} (1-\zeta)^p \xi^{(1+s)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps} (1-\zeta)^p (1-\xi)^{ps} \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{(1+s)p} \xi^{(1+s)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{(1+s)p} (1-\xi)^{ps} \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps} (1-\zeta)^p (1-\xi)^p \xi^{ps} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps} (1-\zeta)^p (1-\xi)^{(1+s)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{(1+s)p} \xi^{ps} (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{(1+s)p} (1-\xi)^{(1+s)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

eşitsizliği elde edilir. $\tau_1 = \zeta \kappa_1 + (1-\zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + (1-\xi) \kappa_4$ değişken değiştirmesi yapıp integrallerin değeri hesaplanırsa aşağıdaki sonuca ulaşılır.

$$\begin{aligned}
& \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{1}{(ps + p + 1)^{\frac{2}{p}}} \left[\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \left. \right] \\
& + \left(\frac{\beta(ps + 1, p + 1)}{ps + p + 1} \right)^{\frac{1}{p}} \left[\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \left. \right] \\
& + \beta^{\frac{2}{p}} (ps + 1, p + 1) \left[\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \left. \right].
\end{aligned}$$

Teorem 4.29 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde F, Φ ,

$F \Phi \in L_2(\Delta)$, $s \in (0, 1]$, $q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{4}{p(ps + p + 1)^2} + \frac{8\beta(ps + 1, p + 1)}{p(ps + p + 1)} + \frac{4\beta^2(ps + 1, p + 1)}{p} \\ & \quad + \frac{L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)}{q} \end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ ve

$$\begin{aligned} L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \\ M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \\ & \quad + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \\ & \quad + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \\ & \quad + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \\ N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \\ & \quad + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \end{aligned}$$

İspat F, Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\ & \leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta(1 - \xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} + (1 - \zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)(1 - \xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. Φ, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & \Phi(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\ & \leq \zeta^s \xi^s \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \\ & \quad + (1 - \zeta)^s \xi^s \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \end{aligned}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak, $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki

tarafın integrali alınırsa;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi \\
\leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^{1+s} \xi^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^{1+s} (1 - \xi)^s \xi d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta)^s \xi^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta)^s \xi (1 - \xi)^s d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^{1+s} \xi^s (1 - \xi) d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^{1+s} (1 - \xi)^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta)^s \xi^s (1 - \xi) d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta)^s (1 - \xi)^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^s (1 - \zeta) \xi^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^s (1 - \zeta) (1 - \xi)^s \xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta)^{1+s} \xi^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)^{1+s} (1 - \xi)^s \xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^s (1 - \zeta) (1 - \xi) \xi^s d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^s (1 - \zeta) (1 - \xi)^{1+s} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta)^{1+s} \xi^s (1 - \xi) d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)^{1+s} (1 - \xi)^{1+s} d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{(1+s)p} \xi^{(1+s)p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{(1+s)p} (1-\xi)^{ps} \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^{ps} \xi^{(1+s)p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p (1-\zeta)^{ps} (1-\xi)^{ps} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{(1+s)p} \xi^{ps} (1-\xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^{ps} (1-\xi)^{(1+s)p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^{ps} \xi^{ps} (1-\xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^{ps} (1-\xi)^{(1+s)p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps} (1-\zeta)^p \xi^{(1+s)p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps} (1-\zeta)^p (1-\xi)^{ps} \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{(1+s)p} \xi^{(1+s)p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{(1+s)p} (1-\xi)^{ps} \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps} (1-\zeta)^p (1-\xi)^{ps} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps} (1-\zeta)^p (1-\xi)^{(1+s)p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{(1+s)p} \xi^{ps} (1-\xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
&+ \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{(1+s)p} (1-\xi)^{(1+s)p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)
\end{aligned}$$

yazılır. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$

ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned}
&\frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&\leq \frac{4}{p(ps + p + 1)^2} \\
&+ \frac{1}{q} \left[\frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \right. \\
&\left. + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{8\beta(ps+1, p+1)}{p(ps+p+1)} \\
& \frac{1}{q} \left[\frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)} e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)} e^{q\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)} e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)} e^{q\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)} e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)} e^{q\alpha(\kappa_2+\kappa_4)}} \\
& \left. + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)} e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_4)} e^{q\alpha(\kappa_2+\kappa_3)}} \right] \\
& + \frac{4\beta^2(ps+1, p+1)}{p} \\
& \frac{1}{q} \left[\frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)} e^{q\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)} e^{q\alpha(\kappa_2+\kappa_3)}} \right. \\
& \left. + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)} e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_4)} e^{q\alpha(\kappa_1+\kappa_3)}} \right]
\end{aligned}$$

elde edilir.

4.2.2. Koordinatlarda eksponansiyel konveks ve koordinatlarda eksponansiyel m -konveks fonksiyonların çarpımı üzerine eşitsizlikler

Teorem 4.30 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta), m \in [0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{m\kappa_4} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
& \leq \frac{4L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + 2M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)}{36}
\end{aligned}$$

burada

$$\begin{aligned}
L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)} e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)} e^{\alpha(\kappa_1+\kappa_4)}} \\
&+ \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)} e^{\alpha(\kappa_2+\kappa_3)}} + m \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)} e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

$$\begin{aligned}
M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \\
&+ \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&+ \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&+ m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}
\end{aligned}$$

$$\begin{aligned}
N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \\
&+ m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}
\end{aligned}$$

İspat F, Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
&F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
&\leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. Φ, Δ üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
&\Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\
&\leq \zeta\xi \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m\zeta(1-\xi) \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
&+ (1-\zeta)\xi \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m(1-\zeta)(1-\xi) \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak;

$$\begin{aligned}
&F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\
&\leq \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^2 \xi^2 + m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^2 (1-\xi) \xi \\
&+ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta (1-\zeta) \xi^2 + m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta \xi (1-\zeta) (1-\xi) \\
&+ \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^2 \xi (1-\xi) + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta \zeta (1-\xi)^2 \\
&+ \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta (1-\zeta) \xi (1-\xi) + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta (1-\zeta) (1-\xi)^2 \\
&+ \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta (1-\zeta) \xi^2 + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta (1-\zeta) (1-\xi) \xi \\
&+ \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^2 \xi^2 + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^2 (1-\xi) \xi \\
&+ \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta (1-\zeta) (1-\xi) \xi + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta (1-\zeta) (1-\xi)^2 \\
&+ \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^2 \xi (1-\xi) + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^2 (1-\xi)^2
\end{aligned}$$

elde edilir. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınır;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m(1 - \xi) \kappa_4) d\zeta d\xi \\
\leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi^2 d\zeta d\xi \\
& + m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1 - \xi) \xi d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) \xi^2 d\zeta d\xi \\
& + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) (1 - \xi) \xi d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi (1 - \xi) d\zeta d\xi \\
& + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1 - \xi)^2 d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) \xi (1 - \xi) d\zeta d\xi \\
& + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) (1 - \xi)^2 d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) \xi^2 d\zeta d\xi \\
& + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) (1 - \xi) \xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta)^2 \xi^2 d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)^2 (1 - \xi) \xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) (1 - \xi) \xi d\zeta d\xi \\
& + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) (1 - \xi)^2 d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta)^2 \xi (1 - \xi) d\zeta d\xi \\
& + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)^2 (1 - \xi)^2 d\zeta d\xi
\end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + m(1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak integrallerin değeri hesaplanırsa;

$$\frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2$$

$$\begin{aligned}
&\leq \frac{1}{9} \left[\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right. \\
&\quad + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \left. \right] \\
&\quad + \frac{1}{18} \left[m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
&\quad + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&\quad + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&\quad + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \left. \right] \\
&\quad + \frac{1}{36} \left[m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
&\quad + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \left. \right]
\end{aligned}$$

elde edilir.

Teorem 4.31 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta)$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
&\leq \frac{L(\kappa_1, \kappa_2, \kappa_3, \kappa_4)}{(2p+1)^{\frac{2}{p}}} + \left(\frac{\beta(p+1, p+1)}{(2p+1)} \right)^{\frac{1}{p}} M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \\
&\quad + \beta^{\frac{2}{p}}(p+1, p+1) N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)
\end{aligned}$$

burada $m \in (0, 1], q > 1, q^{-1} + p^{-1} = 1$ ve

$$\begin{aligned}
L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
&\quad + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} + m \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= m \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
&\quad + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&\quad + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&\quad + m \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}}
\end{aligned}$$

$$N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = m \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \\ + m \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}}$$

İspat F, Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ \leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}$$

yazılır. Φ, Δ üzerinde koordinatlarda eksponansiyel m – konveks fonksiyon tanımı kullanılarak;

$$\Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\ \leq \zeta\xi \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m\zeta(1-\xi) \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ + (1-\zeta)\xi \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m(1-\zeta)(1-\xi) \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak;

$$F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\ \leq \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^2 \xi^2 + m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^2 (1-\xi) \xi \\ + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta (1-\zeta) \xi^2 + m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta \xi (1-\zeta) (1-\xi) \\ + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^2 \xi (1-\xi) + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta \xi (1-\xi)^2 \\ + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta (1-\zeta) \xi (1-\xi) + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta (1-\zeta) (1-\xi)^2 \\ + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta (1-\zeta) \xi^2 + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta (1-\zeta) (1-\xi) \xi \\ + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^2 \xi^2 + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^2 (1-\xi) \xi \\ + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta (1-\zeta) (1-\xi) \xi + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta (1-\zeta) (1-\xi)^2 \\ + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^2 \xi (1-\xi) + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^2 (1-\xi)^2$$

yazılır. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınır;

$$\int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) d\zeta d\xi$$

$$\begin{aligned}
&\leq \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi^2 d\zeta d\xi \\
&+ m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1-\xi) \xi d\zeta d\xi \\
&+ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi^2 d\zeta d\xi \\
&+ m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi) \xi d\zeta d\xi \\
&+ \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi (1-\xi) d\zeta d\xi \\
&+ m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1-\xi)^2 d\zeta d\xi \\
&+ \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi (1-\xi) d\zeta d\xi \\
&+ m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi)^2 d\zeta d\xi \\
&+ \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi^2 d\zeta d\xi \\
&+ m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi) \xi d\zeta d\xi \\
&+ \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^2 \xi^2 d\zeta d\xi \\
&+ \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^2 (1-\xi) \xi d\zeta d\xi \\
&+ \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi) \xi d\zeta d\xi \\
&+ m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi)^2 d\zeta d\xi \\
&+ \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^2 \xi (1-\xi) d\zeta d\xi \\
&+ m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^2 (1-\xi)^2 d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
&\int_0^1 \int_0^1 F(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) \\
&\times \Phi(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + m(1-\xi) \kappa_4) d\zeta d\xi \\
&\leq \left(\int_0^1 \int_0^1 \zeta^{2p} \xi^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
&+ \left(\int_0^1 \int_0^1 \zeta^{2p} (1-\xi)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p \xi^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{2p} \xi^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p (1-\xi)^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p \xi^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p \xi^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} \xi^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} (1-\xi)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} \xi^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} (1-\xi)^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

yazılır. $[0, 1]$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınıp $\tau_1 = \zeta \kappa_1 + (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + m(1 - \xi) \kappa_4$ değişken değiştirmesi yapılırsa;

$$\begin{aligned}
& \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
& \leq \frac{1}{(2p+1)^{\frac{2}{p}}} \left[\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \right. \\
& \quad \left. + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} + m \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\beta(p+1, p+1)}{2p+1} \right)^{\frac{1}{p}} \left[m \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& \left. + m \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right] \\
& + \beta^{\frac{2}{p}}(p+1, p+1) \left[m \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& \left. + m \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \right]
\end{aligned}$$

elde edilir.

Teorem 4.32 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta), m \in [0, 1], q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
& \leq \frac{4}{p(2p+1)^2} + \frac{8\beta(p+1, p+1)}{p(2p+1)^2} + \frac{4\beta^2(p+1, p+1)}{p} \\
& + \frac{L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)}{q}
\end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ ve

$$\begin{aligned}
L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) & = \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + m \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

$$\begin{aligned}
M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) & = m \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \\
& + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \\
& + m \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}
\end{aligned}$$

$$\begin{aligned}
N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) & = m \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \\
& + m \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}
\end{aligned}$$

İspat F, Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ \leq & \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Φ, Δ üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\ \leq & \zeta\xi \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m\zeta(1-\xi) \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & + (1-\zeta)\xi \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m(1-\zeta)(1-\xi) \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) \\ \leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^2 \xi^2 + m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^2 (1-\xi) \xi \\ & + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta (1-\zeta) \xi^2 + m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta \xi (1-\zeta) (1-\xi) \\ & + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^2 \xi (1-\xi) + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta \xi (1-\xi)^2 \\ & + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta (1-\zeta) \xi (1-\xi) + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta (1-\zeta) (1-\xi)^2 \\ & + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta (1-\zeta) \xi^2 + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta (1-\zeta) (1-\xi) \xi \\ & + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^2 \xi^2 + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^2 (1-\xi) \xi \\ & + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta (1-\zeta) (1-\xi) \xi + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta (1-\zeta) (1-\xi)^2 \\ & + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^2 \xi (1-\xi) + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^2 (1-\xi)^2 \end{aligned}$$

elde edilir. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınırsa;

$$\begin{aligned} & \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ & \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) d\zeta d\xi \\ \leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi^2 d\zeta d\xi \\ & + m \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1-\xi) \xi d\zeta d\xi \end{aligned}$$

$$\begin{aligned}
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta(1-\zeta)\xi^2 d\zeta d\xi \\
& + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta(1-\zeta)(1-\xi)\xi d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi(1-\xi) d\zeta d\xi \\
& + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^2(1-\xi)^2 d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta(1-\zeta)\xi(1-\xi) d\zeta d\xi \\
& + m \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta(1-\zeta)(1-\xi)^2 d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta(1-\zeta)\xi^2 d\zeta d\xi \\
& + m \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta(1-\zeta)(1-\xi)\xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^2 \xi^2 d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^2(1-\xi)\xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta(1-\zeta)(1-\xi)\xi d\zeta d\xi \\
& + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta(1-\zeta)(1-\xi)^2 d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^2 \xi(1-\xi) d\zeta d\xi \\
& + m \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^2(1-\xi)^2 d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
& \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m(1-\xi)\kappa_4) d\zeta d\xi \\
\leq & \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{2p} \xi^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{2p} (1-\xi)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p \xi^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{2p} \xi^p (1-\xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{2p} (1-\xi)^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p \xi^p (1-\xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p \xi^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} \xi^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} (1-\xi)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} \xi^p (1-\xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} (1-\xi)^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)
\end{aligned}$$

yazılır. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınıp $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2$

ve $\tau_2 = \xi\kappa_3 + m(1-\xi)\kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned}
& \frac{1}{(\kappa_2 - \kappa_1)(m\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{4}{p(2p+1)^2} + \frac{1}{q} \left[\frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + m \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + m \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \left. \right] \\
& + \frac{8\beta(p+1, p+1)}{p(2p+1)^2} + \frac{1}{q} \left[m \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \right. \\
& + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + m \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \\
& + m \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + m \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} + \left. \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \right] \\
& + \frac{4\beta^2(p+1, p+1)}{p} + \frac{1}{q} \left[m \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \right. \\
& + m \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \left. \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \right]
\end{aligned}$$

sonucuna ulaşılır.

4.2.3. Koordinatlarda eksponansiyel m —konveks fonksiyonların çarpımı üzerine eşitsizlikler

Teorem 4.33 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel m_1 —konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel m_2 —konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta), m_1, m_2 \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \frac{1}{(m_1\kappa_2 - \kappa_1)(m_2\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m_2\kappa_4} \int_{\kappa_1}^{m_1\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{4L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + 2M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)}{36} \end{aligned}$$

burada

$$\begin{aligned} L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ &+ \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m_1 m_2 \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\ M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= m_2 \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \\ &+ m_1 \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\ &+ \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m_2 \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\ &+ m_1 m_2 \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \\ N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= m_2 \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} + m_1 \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \\ &+ m_2 \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \end{aligned}$$

İspat F, Δ üzerinde koordinatlarda eksponansiyel m_1 —konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_1(1-\xi)\kappa_4) \\ & \leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m_1\zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m_2(1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Φ, Δ üzerinde koordinatlarda eksponansiyel m_2 —konveks fonksiyon tanımı kullanılarak;

larak;

$$\begin{aligned}
& \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_2(1-\xi)\kappa_4) \\
\leq & \zeta\xi \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m_2\zeta(1-\xi) \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + (1-\zeta)\xi \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m_2(1-\zeta)(1-\xi) \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınır;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_1(1-\xi)\kappa_4) \\
& \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_2(1-\xi)\kappa_4) d\zeta d\xi \\
\leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi^2 d\zeta d\xi \\
& + m_2 \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1-\xi) \xi d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi^2 d\zeta d\xi \\
& + m_2 \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi (1-\xi) d\zeta d\xi \\
& + m_1 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi (1-\xi) d\zeta d\xi \\
& + m_1 m_2 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1-\xi)^2 d\zeta d\xi \\
& + m_1 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi (1-\xi) d\zeta d\xi \\
& + m_1 m_2 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi)^2 d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi^2 d\zeta d\xi \\
& + m_2 \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi) \xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^2 \xi^2 d\zeta d\xi \\
& + m_2 \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^2 (1-\xi) \xi d\zeta d\xi \\
& + m_1 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi) \xi d\zeta d\xi \\
& + m_1 m_2 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi)^2 d\zeta d\xi \\
& + m_1 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^2 \xi (1-\xi) d\zeta d\xi \\
& + m_1 m_2 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^2 (1-\xi)^2 d\zeta d\xi
\end{aligned}$$

yazılır. $\tau_1 = \zeta \kappa_1 + m_1 (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + m_2 (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak integrallerin değeri hesaplanırsa aşağıdaki sonuca ulaşılır.

$$\begin{aligned}
& \frac{1}{(m_1 \kappa_2 - \kappa_1)(m_2 \kappa_4 - \kappa_3)} \int_{\kappa_3}^{m_2 \kappa_4} \int_{\kappa_1}^{m_1 \kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{1}{9} \left[\frac{F(\kappa_1, \kappa_3) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)} e^{\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)} e^{\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)} e^{\alpha(\kappa_2+\kappa_3)}} + m_1 m_2 \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)} e^{\alpha(\kappa_2+\kappa_4)}} \Big] \\
& + \frac{1}{18} \left[m_2 \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_3)} e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)} e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + m_1 \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_4)} e^{\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)} e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)} e^{\alpha(\kappa_1+\kappa_3)}} + m_2 \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_3)} e^{\alpha(\kappa_2+\kappa_4)}} \\
& + m_1 m_2 \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)} e^{\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_4)} e^{\alpha(\kappa_2+\kappa_3)}} \Big] \\
& + \frac{1}{36} \left[m_2 \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_3)} e^{\alpha(\kappa_2+\kappa_4)}} + m_1 \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_4)} e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + m_2 \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_3)} e^{\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_4)} e^{\alpha(\kappa_1+\kappa_3)}} \Big].
\end{aligned}$$

Teorem 4.34 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel m_1 -konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel m_2 -konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta)$, $m_1, m_2 \in [0, 1]$, $q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \frac{1}{(m_1 \kappa_2 - \kappa_1)(m_2 \kappa_4 - \kappa_3)} \int_{\kappa_3}^{m_2 \kappa_4} \int_{\kappa_1}^{m_1 \kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{L(\kappa_1, \kappa_2, \kappa_3, \kappa_4)}{(2p+1)^{\frac{2}{p}}} + \left(\frac{\beta(p+1, p+1)}{(2p+1)} \right)^{\frac{1}{p}} M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \\
& + \beta^{\frac{2}{p}}(p+1, p+1) N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)
\end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ ve

$$\begin{aligned}
L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
&+ \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= m_2 \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
&+ m_1 \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&+ \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m_2 \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&+ m_1 m_2 \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}}
\end{aligned}$$

$$N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = m_2 \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} + m_1 \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \\ + m_2 \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}}.$$

İspat F, Δ üzerinde koordinatlarda eksponansiyel m_1 – konveks fonksiyon tanımı kullanılarak;

$$F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_1(1-\xi)\kappa_4) \\ \leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m_1\zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m_1(1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}$$

yazılır. Φ, Δ üzerinde koordinatlarda eksponansiyel m_2 – konveks fonksiyon tanımı kullanılarak;

$$\Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_2(1-\xi)\kappa_4) \\ \leq \zeta\xi \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m_2\zeta(1-\xi) \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ + (1-\zeta)\xi \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m_2(1-\zeta)(1-\xi) \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınır;

$$\int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_1(1-\xi)\kappa_4) \\ \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_2(1-\xi)\kappa_4) d\zeta d\xi \\ \leq \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi^2 d\zeta d\xi \\ + m_2 \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1-\xi) \xi d\zeta d\xi \\ + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi^2 d\zeta d\xi \\ + m_2 \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi (1-\xi) d\zeta d\xi \\ + m_1 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi (1-\xi) d\zeta d\xi \\ + m_1 m_2 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1-\xi)^2 d\zeta d\xi \\ + m_1 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi (1-\xi) d\zeta d\xi \\ + m_1 m_2 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta (1-\zeta) (1-\xi)^2 d\zeta d\xi \\ + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta (1-\zeta) \xi^2 d\zeta d\xi$$

$$\begin{aligned}
& +m_2 \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta(1-\zeta)(1-\xi)\xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^2 \xi^2 d\zeta d\xi \\
& +m_2 \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^2(1-\xi)\xi d\zeta d\xi \\
& +m_1 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta(1-\zeta)(1-\xi)\xi d\zeta d\xi \\
& +m_1 m_2 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta(1-\zeta)(1-\xi)^2 d\zeta d\xi \\
& +m_1 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^2 \xi(1-\xi) d\zeta d\xi \\
& +m_1 m_2 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^2(1-\xi)^2 d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_1(1-\xi)\kappa_4) \\
& \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_2(1-\xi)\kappa_4) d\zeta d\xi \\
& \leq \left(\int_0^1 \int_0^1 \zeta^{2p} \xi^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{2p} (1-\xi)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_2 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p \xi^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_2 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{2p} \xi^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p (1-\xi)^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_1 m_2 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p \xi^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_1 m_2 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p \xi^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} \xi^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} (1-\xi)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_2 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p (1-\zeta)^p (1-\xi)^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_1 m_2 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} \xi^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{2p} (1-\xi)^{2p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 m_1 m_2 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

yazılır. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınıp $\tau_1 = \zeta \kappa_1 + m_1 (1 - \zeta) \kappa_2$ ve $\tau_2 = \xi \kappa_3 + m_2 (1 - \xi) \kappa_4$ değişken değiştirmesi yapılırsa;

$$\begin{aligned}
& \frac{1}{(m_1 \kappa_2 - \kappa_1)(m_2 \kappa_4 - \kappa_3)} \int_{\kappa_3}^{m_2 \kappa_4} \int_{\kappa_1}^{m_1 \kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{1}{(2p+1)^{\frac{2}{p}}} \left[\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \left. \right] \\
& + \frac{\beta^{\frac{1}{p}} (p+1, p+1)}{(2p+1)^{\frac{1}{p}}} \left[m_2 \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + m_1 \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + m_2 \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + m_1 m_2 \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \left. \right] \\
& + \beta^{\frac{2}{p}} (p+1, p+1) \left[m_2 \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} + m_1 \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + m_2 \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \left. \right]
\end{aligned}$$

elde edilir.

Teorem 4.35 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel m_1 -konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda eksponansiyel m_2 -konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta), m_1, m_2 \in [0, 1], q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \frac{1}{(m_1\kappa_2 - \kappa_1)(m_2\kappa_4 - \kappa_3)} \int_{\kappa_3}^{m_2\kappa_4} \int_{\kappa_1}^{m_1\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
& \leq \frac{4}{p(2p+1)^2} + \frac{8\beta(p+1, p+1)}{p(2p+1)^2} + \frac{4\beta^2(p+1, p+1)}{p} \\
& \quad + \frac{L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)}{q}
\end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ ve

$$\begin{aligned}
L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \\
&\quad + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \\
M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= m_2 \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \\
&\quad + m_1 \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \\
&\quad + \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + m_2 \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \\
&\quad + m_1 m_2 \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \\
N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= m_2 \frac{|F(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} + m_1 \frac{|F(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \\
&\quad + m_2 \frac{|F(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + m_1 \frac{|F(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}.
\end{aligned}$$

İspat F, Δ üzerinde koordinatlarda eksponansiyel m_1 – konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_1(1-\xi)\kappa_4) \\
& \leq \zeta\xi \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m_1\zeta(1-\xi) \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& \quad + (1-\zeta)\xi \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m_1(1-\zeta)(1-\xi) \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. Φ, Δ üzerinde koordinatlarda eksponansiyel m_2 – konveks fonksiyon tanımı kullanılarak;

$$\begin{aligned}
& \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + m_2(1-\xi)\kappa_4) \\
& \leq \zeta\xi \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + m_2\zeta(1-\xi) \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& \quad + (1-\zeta)\xi \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + m_2(1-\zeta)(1-\xi) \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınırsa;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m_1 (1 - \xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m_2 (1 - \xi) \kappa_4) d\zeta d\xi \\
\leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi^2 d\zeta d\xi \\
& + m_2 \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1 - \xi) \xi d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) \xi^2 d\zeta d\xi \\
& + m_2 \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) \xi (1 - \xi) d\zeta d\xi \\
& + m_1 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^2 \xi (1 - \xi) d\zeta d\xi \\
& + m_1 m_2 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^2 (1 - \xi)^2 d\zeta d\xi \\
& + m_1 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) \xi (1 - \xi) d\zeta d\xi \\
& + m_1 m_2 \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) (1 - \xi)^2 d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) \xi^2 d\zeta d\xi \\
& + m_2 \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) (1 - \xi) \xi d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta)^2 \xi^2 d\zeta d\xi \\
& + m_2 \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)^2 (1 - \xi) \xi d\zeta d\xi \\
& + m_1 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) (1 - \xi) \xi d\zeta d\xi \\
& + m_1 m_2 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta (1 - \zeta) (1 - \xi)^2 d\zeta d\xi \\
& + m_1 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta)^2 \xi (1 - \xi) d\zeta d\xi \\
& + m_1 m_2 \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)^2 (1 - \xi)^2 d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m_1 (1 - \xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + m_2 (1 - \xi) \kappa_4) d\zeta d\xi \\
\leq & \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{2p} \xi^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{2p} (1 - \xi)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_2 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1 - \zeta)^p \xi^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p (1 - \zeta)^p (1 - \xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_2 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{2p} \xi^p (1 - \xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{2p} (1 - \xi)^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_1 m_2 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1 - \zeta)^p \xi^p (1 - \xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1 - \zeta)^p (1 - \xi)^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_1 m_2 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1 - \zeta)^p \xi^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1 - \zeta)^p (1 - \xi)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_2 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^{2p} \xi^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^{2p} (1 - \xi)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_2 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1 - \zeta)^p (1 - \xi)^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p (1 - \zeta)^p (1 - \xi)^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_1 m_2 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1 + \kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^{2p} \xi^p (1 - \xi)^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2 + \kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^{2p} (1 - \xi)^{2p} d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 m_1 m_2 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2 + \kappa_4)}} \right|^q d\zeta d\xi \right)
\end{aligned}$$

yazılır. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınıp $\tau_1 = \zeta \kappa_1 + m_1 (1 - \zeta) \kappa_2$

ve $\tau_2 = \xi \kappa_3 + m_2 (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak;

$$\begin{aligned}
& \frac{1}{(m_1 \kappa_2 - \kappa_1)(m_2 \kappa_4 - \kappa_3)} \int_{\kappa_3}^{m_2 \kappa_4} \int_{\kappa_1}^{m_1 \kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{4}{p(2p+1)^2} + \frac{1}{q} \left[\frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + m_1 m_2 \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \left. \right] \\
& + \frac{8\beta(p+1, p+1)}{p(2p+1)^2} + \frac{1}{q} \left[m_2 \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \right. \\
& + m_1 \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + m_1 m_2 \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + m_2 \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \\
& + m_1 m_2 \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} + m_1 \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \left. \right] \\
& + \frac{4\beta^2(p+1, p+1)}{p} + \frac{1}{q} \left[m_2 \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + m_1 \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \right. \\
& + m_2 \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + m_1 \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \left. \right]
\end{aligned}$$

elde edilir.

4.2.4. Koordinatlarda ikinci anlamda eksponansiyel s — konveks fonksiyonların çarpımı üzerine eşitsizlikler

Teorem 4.36 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda ikinci anlamda eksponansiyel s_1 — konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda ikinci anlamda eksponansiyel s_2 — konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta)$, $s_1, s_2 \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{1}{(s_2 + s_1 + 1)^2} L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + \frac{\beta(s_1 + 1, s_2 + 1)}{(s_2 + s_1 + 1)} M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \\
& + \beta^2(s_1 + 1, s_2 + 1) N(\kappa_1, \kappa_2, \kappa_3, \kappa_4).
\end{aligned}$$

burada

$$\begin{aligned}
L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = & \frac{F(\kappa_1, \kappa_3) \Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_4) \Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \frac{F(\kappa_2, \kappa_3) \Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{F(\kappa_2, \kappa_4) \Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

$$\begin{aligned}
M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \\
&+ \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&+ \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&+ \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \\
N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \\
&+ \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}
\end{aligned}$$

İspat F, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s_1 – konveks fonksiyon tanımını kullanarak;

$$\begin{aligned}
&F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
&\leq \zeta^{s_1} \xi^{s_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^{s_1} (1-\xi)^{s_1} \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
&+ (1-\zeta)^{s_1} \xi^{s_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)^{s_1} (1-\xi)^{s_1} \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

yazılır. Φ, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s_2 – konveks fonksiyon tanımını kullanarak;

$$\begin{aligned}
&\Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
&\leq \zeta^{s_2} \xi^{s_2} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^{s_2} (1-\xi)^{s_2} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\
&+ (1-\zeta)^{s_2} \xi^{s_2} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)^{s_2} (1-\xi)^{s_2} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

F ve Φ fonksiyonları taraf tarafa çarpılarak;

$$\begin{aligned}
&F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
&\leq \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_1+s_2} \xi^{s_1+s_2} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} \\
&+ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_1+s_2} \\
&+ \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta^{s_1} \xi^{s_1} (1-\zeta)^{s_2} (1-\xi)^{s_2} \\
&+ \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} \\
&+ \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_1} \xi^{s_2} (1-\xi)^{s_1+s_2} \\
&+ \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_2} (1-\xi)^{s_1}
\end{aligned}$$

$$\begin{aligned}
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta^{s_1} (1-\zeta)^{s_2} (1-\xi)^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_2} (1-\zeta)^{s_1} \xi^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_2} \xi^{s_1} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^{s_1+s_2} \xi^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1} \xi^{s_2} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^{s_1+s_2} (1-\xi)^{s_1+s_2}
\end{aligned}$$

yazılır. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınırsa;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) d\zeta d\xi \\
\leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_1} (1-\xi)^{s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} (1-\xi)^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_2} (1-\xi)^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} (1-\xi)^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_2} \xi^{s_1} d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1} \xi^{s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} (1-\xi)^{s_1+s_2} d\zeta d\xi
\end{aligned}$$

yazılır. $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2$ ve $\tau_2 = \xi\kappa_3 + (1-\xi)\kappa_4$ değişken değiştirmesi yapılarak integrallerin değeri hesaplanırsa aşağıdaki sonuca ulaşılır.

$$\begin{aligned}
& \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{1}{(s_2 + s_1 + 1)^2} \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \Big) \\
& + \frac{\beta(s_1 + 1, s_2 + 1)}{(s_2 + s_1 + 1)} \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \Big) \\
& + \beta^2(s_1 + 1, s_2 + 1) \left(\frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \Big).
\end{aligned}$$

Teorem 4.37 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda ikinci anlamda eksponansiyel s_1 -konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda ikinci anlamda eksponansiyel s_2 -konveks fonksiyon olsun. Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta)$, $s_1, s_2 \in (0, 1]$, $q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{1}{(ps_2 + ps_1 + 1)^{\frac{2}{p}}} L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + \frac{\beta^{\frac{1}{p}}(ps_1 + 1, ps_2 + 1)}{(ps_2 + ps_1 + 1)^{\frac{1}{p}}} M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \\
& + \beta^{\frac{2}{p}}(ps_1 + 1, ps_2 + 1) N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)
\end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ ve

$$L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \\ + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}}$$

$$M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \\ + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\ + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\ + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}}$$

$$N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \\ + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}}$$

İspat F, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s_1 – konveks fonksiyon tanımını kullanarak;

$$F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ \leq \zeta^{s_1} \xi^{s_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^{s_1} (1-\xi)^{s_1} \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ + (1-\zeta)^{s_1} \xi^{s_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)^{s_1} (1-\xi)^{s_1} \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}$$

yazılır. Φ, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s_2 – konveks fonksiyon tanımını kullanarak;

$$\Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ \leq \zeta^{s_2} \xi^{s_2} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^{s_2} (1-\xi)^{s_2} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ + (1-\zeta)^{s_2} \xi^{s_2} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)^{s_2} (1-\xi)^{s_2} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak;

$$F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ \leq \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_1+s_2} \xi^{s_1+s_2} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} \\ + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_1+s_2} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta^{s_1} \xi^{s_1} (1-\zeta)^{s_2} (1-\xi)^{s_2}$$

$$\begin{aligned}
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_1} \zeta^{s_2} (1-\xi)^{s_1+s_2} \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_2} (1-\xi)^{s_1} \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta^{s_1} (1-\zeta)^{s_2} (1-\xi)^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_2} (1-\zeta)^{s_1} \xi^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_2} \xi^{s_1} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^{s_1+s_2} \xi^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1} \xi^{s_2} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^{s_1+s_2} (1-\xi)^{s_1+s_2}
\end{aligned}$$

elde edilir. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınır;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) d\zeta d\xi \\
\leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_1} (1-\xi)^{s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} (1-\xi)^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_2} (1-\xi)^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} (1-\xi)^{s_1+s_2} d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_2} \xi^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1} \xi^{s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} (1-\xi)^{s_1+s_2} d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) d\zeta d\xi \\
\leq & \left(\int_0^1 \int_0^1 \zeta^{(s_1+s_2)p} \xi^{(s_1+s_2)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{(s_1+s_2)p} (1-\xi)^{ps_2} \xi^{ps_1} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps_1} (1-\zeta)^{ps_2} \xi^{(s_1+s_2)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps_1} \xi^{ps_1} (1-\zeta)^{ps_2} (1-\xi)^{ps_2} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{(s_1+s_2)p} \xi^{ps_2} (1-\xi)^{ps_1} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps_1} \xi^{ps_2} (1-\xi)^{(s_1+s_2)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps_1} (1-\zeta)^{ps_2} \xi^{ps_2} (1-\xi)^{ps_1} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps_1} (1-\zeta)^{ps_2} (1-\xi)^{(s_1+s_2)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps_2} (1-\zeta)^{ps_1} \xi^{(s_1+s_2)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps_2} (1-\zeta)^{ps_1} (1-\xi)^{ps_2} \xi^{ps_1} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{(s_1+s_2)p} \xi^{(s_1+s_2)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{(s_1+s_2)p} (1-\xi)^{ps_2} \xi^{ps_1} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps_2} (1-\zeta)^{ps_1} (1-\xi)^{ps_2} \xi^{ps_1} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^{ps_2} (1-\zeta)^{ps_1} (1-\xi)^{(s_1+s_2)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{(s_1+s_2)p} \xi^{ps_2} (1-\xi)^{ps_1} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^{(s_1+s_2)p} (1-\xi)^{(s_1+s_2)p} d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

yazılır. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınıp $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2$

ve $\tau_2 = \xi\kappa_3 + (1-\xi)\kappa_4$ değişken değiştirmesi yapılarak aşağıdaki sonuca ulaşılır.

$$\begin{aligned}
& \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{1}{(ps_2 + ps_1 + 1)^{\frac{2}{p}}} \left[\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \left. \right] \\
& + \left(\frac{\beta(ps_1 + 1, ps_2 + 1)}{ps_2 + ps_1 + 1} \right)^{\frac{1}{p}} \left[\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \left. \right] \\
& + \beta^{\frac{2}{p}}(ps_1 + 1, ps_2 + 1) \left[\frac{|F(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{|\Phi(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{|\Phi(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \frac{|F(\kappa_2, \kappa_3)|}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{|\Phi(\kappa_1, \kappa_4)|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{|\Phi(\kappa_1, \kappa_3)|}{e^{\alpha(\kappa_1+\kappa_3)}} \left. \right]
\end{aligned}$$

burada

$$\beta(\tau_1, \tau_2) = \int_0^1 \zeta^{\tau_1-1} (1-\zeta)^{\tau_2-1} d\zeta, \quad \tau_1 > 0, \tau_2 > 0$$

dir.

Teorem 4.38 $F, \Phi : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonlar olmak üzere F, Δ üzerinde negatif olmayan koordinatlarda ikinci anlamda eksponansiyel s_1 -konveks ve Φ, Δ üzerinde negatif olmayan koordinatlarda ikinci anlamda eksponansiyel s_2 -konveks fonksiyon olsun.

Bu takdirde $F, \Phi, F\Phi \in L_2(\Delta)$, $s_1, s_2 \in (0, 1]$, $q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ & \leq \frac{4}{p(ps_2 + ps_1 + 1)^2} + \frac{8\beta(ps_2 + 1, ps_1 + 1)}{p(ps_2 + ps_1 + 1)} + \frac{4\beta^2(ps + 1, p + 1)}{p} \\ & \quad + \frac{L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) + N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)}{q} \end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ ve

$$\begin{aligned} L(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \\ & \quad + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \\ M(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \\ & \quad + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \\ & \quad + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \\ & \quad + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \\ N(\kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \\ & \quad + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

İspat F, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s_1 – konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & F(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\ & \leq \zeta^{s_1} \xi^{s_1} \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^{s_1} (1 - \xi)^{s_1} \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + (1 - \zeta)^{s_1} \xi^{s_1} \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1 - \zeta)^{s_1} (1 - \xi)^{s_1} \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. Φ, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s_2 – konveks fonksiyon tanımını kullanılarak;

$$\begin{aligned} & \Phi(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \\ & \leq \zeta^{s_2} \xi^{s_2} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta^{s_2} (1 - \xi)^{s_2} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \\ & \quad + (1 - \zeta)^{s_2} \xi^{s_2} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1 - \zeta)^{s_2} (1 - \xi)^{s_2} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

yazılır. F ve Φ fonksiyonları taraf tarafa çarpılarak;

$$\begin{aligned}
& F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
\leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_1+s_2} \xi^{s_1+s_2} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_1+s_2} + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta^{s_1} \xi^{s_1} (1-\zeta)^{s_2} (1-\xi)^{s_2} \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_1} \xi^{s_2} (1-\xi)^{s_1+s_2} \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_2} (1-\xi)^{s_1} \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \zeta^{s_1} (1-\zeta)^{s_2} (1-\xi)^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_2} (1-\zeta)^{s_1} \xi^{s_1+s_2} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_2} \xi^{s_1} \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^{s_1+s_2} \xi^{s_1+s_2} + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1} \xi^{s_2} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1+s_2} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} (1-\zeta)^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} (1-\zeta)^{s_1+s_2} (1-\xi)^{s_1+s_2}
\end{aligned}$$

elde edilir. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınır;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\
& \times \Phi(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) d\zeta d\xi \\
\leq & \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_1} (1-\xi)^{s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1+s_2} (1-\xi)^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} \xi^{s_2} (1-\xi)^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_1} (1-\zeta)^{s_2} (1-\xi)^{s_1+s_2} d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_2} \xi^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} \xi^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} (1-\xi)^{s_2} \xi^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1} \xi^{s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta^{s_2} (1-\zeta)^{s_1} (1-\xi)^{s_1+s_2} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} \xi^{s_2} (1-\xi)^{s_1} d\zeta d\xi \\
& + \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)^{s_1+s_2} (1-\xi)^{s_1+s_2} d\zeta d\xi
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned}
& \int_0^1 \int_0^1 F(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) \\
& \times \Phi(\zeta \kappa_1 + (1-\zeta) \kappa_2, \xi \kappa_3 + (1-\xi) \kappa_4) d\zeta d\xi \\
& \leq \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{(s_1+s_2)p} \xi^{(s_1+s_2)p} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{(s_1+s_2)p} (1-\xi)^{ps_2} \xi^{ps_1} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps_1} (1-\zeta)^{ps_2} \xi^{(s_1+s_2)p} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps_1} \xi^{ps_1} (1-\zeta)^{ps_2} (1-\xi)^{ps_2} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{(s_1+s_2)p} \xi^{ps_2} (1-\xi)^{ps_1} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps_1} \xi^{ps_2} (1-\xi)^{(s_1+s_2)p} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps_1} (1-\zeta)^{ps_2} \xi^{ps_2} (1-\xi)^{ps_1} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps_1} (1-\zeta)^{ps_2} (1-\xi)^{(s_1+s_2)p} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps_2} (1-\zeta)^{ps_1} \xi^{(s_1+s_2)p} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps_2} (1-\zeta)^{ps_1} (1-\xi)^{ps_2} \xi^{ps_1} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{(s_1+s_2)p} \xi^{(s_1+s_2)p} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{(s_1+s_2)p} (1-\xi)^{ps_2} \xi^{ps_1} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps_2} (1-\zeta)^{ps_1} (1-\xi)^{ps_1} \xi^{ps_2} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^{ps_2} (1-\zeta)^{ps_1} (1-\xi)^{(s_1+s_2)p} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{(s_1+s_2)p} \xi^{ps_2} (1-\xi)^{ps_1} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \right|^q d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^{(s_1+s_2)p} (1-\xi)^{(s_1+s_2)p} d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{F(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q \left| \frac{\Phi(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \right|^q d\zeta d\xi \right)
\end{aligned}$$

elde edilir. $[0, 1]^2$ aralığında ζ ve ξ 'ye göre her iki tarafın integrali alınıp $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2$

ve $\tau_2 = \xi \kappa_3 + (1 - \xi) \kappa_4$ değişken değiştirmesi yapılarak aşağıdaki sonuca ulaşılır.

$$\begin{aligned}
& \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) \Phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \\
\leq & \frac{4}{p(ps_2 + ps_1 + 1)^2} + \frac{1}{q} \left[\frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \left. \right] \\
& + \frac{8\beta(ps_2 + 1, ps_1 + 1)}{p(ps_2 + ps_1 + 1)} + \frac{1}{q} \left[\frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \right. \\
& + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \\
& + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \\
& + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \left. \right] \\
& + \frac{4\beta^2(ps_2 + 1, ps_1 + 1)}{p} + \frac{1}{q} \left[\frac{|F(\kappa_1, \kappa_3)|^q |\Phi(\kappa_2, \kappa_4)|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{|F(\kappa_1, \kappa_4)|^q |\Phi(\kappa_2, \kappa_3)|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \frac{|F(\kappa_2, \kappa_3)|^q |\Phi(\kappa_1, \kappa_4)|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{|F(\kappa_2, \kappa_4)|^q |\Phi(\kappa_1, \kappa_3)|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \left. \right]
\end{aligned}$$

burada

$$\beta(\tau_1, \tau_2) = \int_0^1 \zeta^{\tau_1-1} (1 - \zeta)^{\tau_2-1} d\zeta, \quad \tau_1 > 0, \tau_2 > 0$$

dir.

4.3. Koordinatlarda Eksponansiyel Konveks Fonksiyonlar İçin Bazı Yeni İntegral Eşitsizlikler

Teorem 4.39 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \right. \\
& \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
\leq & \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{64} \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right).
\end{aligned}$$

İspat

$$T = \left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right|$$

Lemma 3.14 ve integraller için mutlak değer özelliğini kullanarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right| d\zeta d\xi$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel konveks özelliği kullanılarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \left[\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left[\left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right) \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right) \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right) \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right) \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right] d\zeta d\xi \right]$$

yazılır. Gerekli düzenlemeler yapılarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} |p(\tau_1, \zeta) q(\tau_2, \xi)| \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right) \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} |p(\tau_1, \zeta) q(\tau_2, \xi)| \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right) \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} |p(\tau_1, \zeta) q(\tau_2, \xi)| \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right) \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} |p(\tau_1, \zeta) q(\tau_2, \xi)| \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right) \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right) d\zeta d\xi \right]$$

yazılır. ξ ' göre integral alınırsa,

$$\begin{aligned}
T &\leq \frac{1}{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2} \\
&\times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\kappa_2 - \zeta)| \right. \\
&\times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)(\kappa_4 - \xi) d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^2 d\xi \right\} d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\kappa_2 - \zeta)| \\
&\times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^2 d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\xi - \kappa_3)(\kappa_4 - \xi) d\xi \right\} d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\zeta - \kappa_1)| \\
&\times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)(\kappa_4 - \xi) d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^2 d\xi \right\} d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\zeta - \kappa_1)| \\
&\times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^2 d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\xi - \kappa_3)(\kappa_4 - \xi) d\xi \right\} d\zeta \Big] \\
&= \frac{1}{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2} \\
&\times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} (\kappa_4 - \kappa_3) \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\kappa_2 - \zeta)| d\zeta \right. \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} (\kappa_4 - \kappa_3) \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\kappa_2 - \zeta)| \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} (\kappa_4 - \kappa_3) \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\zeta - \kappa_1)| d\zeta \\
&+ \left. \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} (\kappa_4 - \kappa_3) \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\zeta - \kappa_1)| d\zeta \right]
\end{aligned}$$

elde edilir. ζ 'ye göre integral alınırsa,

$$\begin{aligned}
T &\leq \frac{1}{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2} \\
&\times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} (\kappa_4 - \kappa_3) \left(\int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1) (\kappa_2 - \zeta) d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^2 d\zeta \right) \right. \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} (\kappa_4 - \kappa_3) \left(\int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1) (\kappa_2 - \zeta) d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^2 d\zeta \right) \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} (\kappa_4 - \kappa_3) \left(\int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1)^2 d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\zeta - \kappa_1) (\kappa_2 - \zeta) d\zeta \right) \\
&+ \left. \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} (\kappa_4 - \kappa_3) \left(\int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1)^2 d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\zeta - \kappa_1) (\kappa_2 - \zeta) d\zeta \right) \right] \\
&= \frac{(\kappa_4 - \kappa_3) (\kappa_2 - \kappa_1)}{64} \\
&\times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right)
\end{aligned}$$

sonucu bulunur.

Not 4.2 Yukarıdaki eşitsizlikte $\alpha = 0$ alınırsa Teorem 3.21 elde edilir.

Teorem 4.40 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $p > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \right. \\
&\quad \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
&\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)^{\frac{1}{q}}
\end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat

$$T = \left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right|$$

Lemma 3.14 ve modülün özelliğini kullanılarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right| d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \times \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right|^q d\zeta d\xi \right)^{\frac{1}{q}}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ üzerinde koordinatlarda eksponansiyel konveks özelliği kullanılır ve gerekli integraller hesaplanırsa aşağıdaki eşitsizlik yazılır.

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)^{1+\frac{1}{q}} (\kappa_2 - \kappa_1)^{1+\frac{1}{q}}} \times \left(\int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)|^p \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^p d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^p d\xi \right\} \right)^{\frac{1}{p}} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)(\kappa_4 - \xi) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)(\xi - \kappa_3) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)(\kappa_4 - \xi) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)(\xi - \kappa_3) d\zeta d\xi \right]^{\frac{1}{q}}.$$

$$\begin{aligned}
&= \frac{1}{(\kappa_4 - \kappa_3)^{1+\frac{1}{q}} (\kappa_2 - \kappa_1)^{1+\frac{1}{q}}} \\
&\times \left(\frac{(\kappa_4 - \kappa_3)^{p+1}}{2^p (p+1)} \left\{ \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} (\zeta - \kappa_1)^p d\zeta + \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^p d\zeta \right\} \right)^{\frac{1}{p}} \\
&\times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \right. \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \\
&\left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \right]^{\frac{1}{q}} \\
&= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \times \left(\frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}}{4} \right)^{\frac{1}{q}}.
\end{aligned}$$

Not 4.3 Yukarıdaki eşitsizlikte $\alpha = 0$ alınırsa Teorem 3.22 elde edilir.

Teorem 4.41 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $p > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \right. \\
&- \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
&\left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
&\leq \frac{(\kappa_4 - \kappa_3)^p (\kappa_2 - \kappa_1)^p}{4^p p (p+1)^2} + \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4q} \\
&\times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \right)
\end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat

$$T = \left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right|$$

Lemma 3.14 ve modülün özelliğini kullanılarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right| d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left[\frac{1}{p} \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)|^p d\zeta d\xi \right) + \frac{1}{q} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right|^q d\zeta d\xi \right]$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ üzerinde koordinatlarda eksponansiyel konveks özelliği kullanılır ve gerekli integraller hesaplanırsa;

$$T \leq \frac{1}{p(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left[\left(\int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)|^p \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^p d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^p d\xi \right\} \right) + \frac{1}{q(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)(\kappa_4 - \xi) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)(\xi - \kappa_3) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)(\kappa_4 - \xi) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)(\xi - \kappa_3) d\zeta d\xi \right) \right].$$

$$\begin{aligned}
&= \frac{1}{p(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \left[\left(\frac{(\kappa_4 - \kappa_3)^{p+1}}{2^p(p+1)} \left\{ \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1)^p d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^p d\zeta \right\} \right) \right. \\
&\quad + \frac{1}{q(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \right. \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \\
&\quad \left. \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \right) \right] \\
&= \frac{(\kappa_4 - \kappa_3)^p (\kappa_2 - \kappa_1)^p}{4^p p (p+1)^2} + \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4q} \\
&\quad \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)
\end{aligned}$$

ispat tamamlanır.

Teorem 4.42 $F : \Delta_1 \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta_1)$, $m \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \right. \\
&\quad - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
&\quad \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
&\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{64} \\
&\quad \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right).
\end{aligned}$$

İspat

$$\begin{aligned}
T &= \left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \right. \\
&\quad - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
&\quad \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right|
\end{aligned}$$

Lemma 3.14 ve modülün özelliğini kullanarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right| d\zeta d\xi$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks özelliği kullanılarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \left[\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left(\left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right) \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right. \right. \\ + m \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right) \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \\ + \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right) \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \\ \left. \left. + m \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right) \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right) d\zeta d\xi \right]$$

yazılır. Gerekli düzenlemeler yapılarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right) \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right) d\zeta d\xi \right. \\ + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right) \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right) d\zeta d\xi \\ + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right) \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right) d\zeta d\xi \\ \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right) \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right) d\zeta d\xi \right]$$

yazılır. ξ ' göre integral alınırsa;

$$\begin{aligned}
T &\leq \frac{1}{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2} \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\kappa_2 - \zeta)| \right. \\
&\quad \times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)(\kappa_4 - \xi) d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^2 d\xi \right\} d\zeta \\
&\quad + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\kappa_2 - \zeta)| \\
&\quad \times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^2 d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\xi - \kappa_3)(\kappa_4 - \xi) d\xi \right\} d\zeta \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\zeta - \kappa_1)| \\
&\quad \times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)(\kappa_4 - \xi) d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^2 d\xi \right\} d\zeta \\
&\quad + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\zeta - \kappa_1)| \\
&\quad \times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^2 d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\xi - \kappa_3)(\kappa_4 - \xi) d\xi \right\} d\zeta \Big] \\
&= \frac{1}{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2} \\
&\quad \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} (\kappa_4 - \kappa_3) \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\kappa_2 - \zeta)| d\zeta \right. \\
&\quad + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} (\kappa_4 - \kappa_3) \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\kappa_2 - \zeta)| \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} (\kappa_4 - \kappa_3) \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\zeta - \kappa_1)| d\zeta \\
&\quad + \left. \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} (\kappa_4 - \kappa_3) \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)(\zeta - \kappa_1)| d\zeta \right]
\end{aligned}$$

sonucu elde edilir. ζ 'ye göre integral alınırsa,

$$\begin{aligned}
T &\leq \frac{1}{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2} \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} (\kappa_4 - \kappa_3) \right. \\
&\quad \times \left(\int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1) (\kappa_2 - \zeta) d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^2 d\zeta \right) \\
&\quad + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} (\kappa_4 - \kappa_3) \left(\int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1) (\kappa_2 - \zeta) d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^2 d\zeta \right) \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} (\kappa_4 - \kappa_3) \left(\int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1)^2 d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\zeta - \kappa_1) (\kappa_2 - \zeta) d\zeta \right) \\
&\quad + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} (\kappa_4 - \kappa_3) \times \left(\int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1)^2 d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\zeta - \kappa_1) (\kappa_2 - \zeta) d\zeta \right) \Big] \\
&= \frac{(\kappa_4 - \kappa_3) (\kappa_2 - \kappa_1)}{64} \\
&\quad \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right)
\end{aligned}$$

sonucu bulunur.

Not 4.4 Yukarıdaki eşitsizlikte $\alpha = 0$ ve $m = 1$ alınrsa Teorem 3.21 elde edilir.

Not 4.5 Yukarıdaki eşitsizlikte $m = 1$ alınrsa Teorem 4.39 elde edilir.

Teorem 4.43 $F : \Delta_1 \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta_1)$, $p > 1$, $m \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \right. \\
&\quad - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
&\quad \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
&\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \\
&\quad \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right)^{\frac{1}{q}}
\end{aligned}$$

ve burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat

$$T = \left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right|$$

Lemma 3.14 ve modülün özelliğini kullanılarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right| d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \times \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right|^q d\zeta d\xi \right)^{\frac{1}{q}}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks özelliği kullanılır ve gerekli integraller hesaplanırsa aşağıdaki eşitsizlik yazılır.

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)^{1+\frac{1}{q}} (\kappa_2 - \kappa_1)^{1+\frac{1}{q}}} \times \left(\int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)|^p \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^p d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^p d\xi \right\} \right)^{\frac{1}{p}} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)(\kappa_4 - \xi) d\zeta d\xi + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)(\xi - \kappa_3) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)(\kappa_4 - \xi) d\zeta d\xi + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)(\xi - \kappa_3) d\zeta d\xi \right]^{\frac{1}{q}}$$

$$\begin{aligned}
&= \frac{1}{(\kappa_4 - \kappa_3)^{1+\frac{1}{q}} (\kappa_2 - \kappa_1)^{1+\frac{1}{q}}} \\
&\quad \times \left(\frac{(\kappa_4 - \kappa_3)^{p+1}}{2^p (p+1)} \left\{ \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} (\zeta - \kappa_1)^p d\zeta + \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^p d\zeta \right\} \right)^{\frac{1}{p}} \\
&\quad \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \right. \\
&\quad + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1+\frac{\kappa_4}{m})}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \\
&\quad \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2+\frac{\kappa_4}{m})}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \right]^{\frac{1}{q}} \\
&= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \\
&\quad \times \left(\frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1+\frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2+\frac{\kappa_4}{m})}}}{4} \right)^{\frac{1}{q}}
\end{aligned}$$

ispat tamamlanır.

Not 4.6 Yukarıdaki eşitsizlikte $\alpha = 0$ ve $m = 1$ alınrsa Teorem 3.22 elde edilir.

Not 4.7 Yukarıdaki eşitsizlikte $m = 1$ alınrsa Teorem 4.40 elde edilir.

Teorem 4.44 $F : \Delta_1 \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta_1)$, $p > 1$, $m \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \right. \\
&\quad - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
&\quad \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
&\leq \frac{(\kappa_4 - \kappa_3)^p (\kappa_2 - \kappa_1)^p}{4^p p (p+1)^2} + \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4q} \\
&\quad \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1+\frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2+\frac{\kappa_4}{m})}} \right)^{\frac{1}{q}}
\end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat

$$T = \left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right|$$

Lemma 3.14 ve modülün özelliğini kullanılarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right| d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left[\frac{1}{p} \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)|^p d\zeta d\xi \right) + \frac{1}{q} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right|^q d\zeta d\xi \right]$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks özelliği kullanılır ve gerekli integraller hesaplanırsa aşağıdaki eşitsizlik yazılır.

$$T \leq \frac{1}{p(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left[\left(\int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)|^p \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^p d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^p d\xi \right\} \right) + \frac{1}{q(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)(\kappa_4 - \xi) d\zeta d\xi + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)(\xi - \kappa_3) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)(\kappa_4 - \xi) d\zeta d\xi + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)(\xi - \kappa_3) d\zeta d\xi \right) \right]$$

$$\begin{aligned}
&= \frac{1}{p(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \\
&\times \left[\left(\frac{(\kappa_4 - \kappa_3)^{p+1}}{2^p(p+1)} \left\{ \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1)^p d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^p d\zeta \right\} \right) \right. \\
&+ \frac{1}{q(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \right. \\
&+ \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \\
&\left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \frac{(\kappa_4 - \kappa_3)^2}{2} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \right) \Bigg] \\
&= \frac{(\kappa_4 - \kappa_3)^p (\kappa_2 - \kappa_1)^p}{4^p p (p+1)^2} + \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4q} \\
&\times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right)^{\frac{1}{q}}
\end{aligned}$$

ispat tamamlanır.

Not 4.8 Yukarıdaki eşitsizlikte $m = 1$ alınırsa Teorem 4.41 elde edilir.

Teorem 4.45 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
&\left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \right. \\
&\left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\
&\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4^{1+s}} \left(\frac{(1+s)(1-2^{1+s}) + (2^{1+s}-1)(2+s)}{(1+s)(2+s)} \right)^2 \\
&\times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right).
\end{aligned}$$

İspat

$$T = \left| F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right|$$

Lemma 3.14 ve modülün özelliğini kullanarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right| d\zeta d\xi$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks özelliği kullanılarak;

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \left[\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \times \left(\left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right)^s \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \right)^s \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right)^s \left(\frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left(\frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \right)^s \left(\frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) d\zeta d\xi \right]$$

böylece

$$T \leq \frac{1}{(\kappa_4 - \kappa_3)^{1+s} (\kappa_2 - \kappa_1)^{1+s}} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| (\kappa_2 - \zeta)^s (\kappa_4 - \xi)^s d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| (\kappa_2 - \zeta)^s (\xi - \kappa_3)^s d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| (\zeta - \kappa_1)^s (\kappa_4 - \xi)^s d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| (\zeta - \kappa_1)^s (\xi - \kappa_3)^s d\zeta d\xi \right]$$

yazılır. ξ ve ζ 'ye göre integral alınırsa,

$$\begin{aligned}
T &\leq \frac{1}{(\kappa_4 - \kappa_3)^{1+s} (\kappa_2 - \kappa_1)^{1+s}} \\
&\times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) (\kappa_2 - \zeta)^s| \right. \\
&\times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3) (\kappa_4 - \xi)^s d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^{1+s} d\xi \right\} d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) (\kappa_2 - \zeta)^s| \\
&\times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^{1+s} d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\xi - \kappa_3)^s (\kappa_4 - \xi) d\xi \right\} d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) (\zeta - \kappa_1)^s| \\
&\times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3) (\kappa_4 - \xi)^s d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^{1+s} d\xi \right\} d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) (\zeta - \kappa_1)^s| \\
&\times \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^{1+s} d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\xi - \kappa_3)^s (\kappa_4 - \xi) d\xi \right\} d\zeta \Big] \\
&= \frac{1}{(\kappa_4 - \kappa_3)^{1+s} (\kappa_2 - \kappa_1)^{1+s}} \frac{(\kappa_4 - \kappa_3)^{2+s}}{2^{1+s}} \left(\frac{(1+s)(1-2^{1+s}) + (2^{1+s}-1)(2+s)}{(1+s)(2+s)} \right) \\
&\times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1) (\kappa_2 - \zeta)^s d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^{1+s} d\zeta \right. \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1) (\kappa_2 - \zeta)^s d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^{1+s} d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1)^{1+s} d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta) (\zeta - \kappa_1)^s d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1)^{1+s} d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta) (\zeta - \kappa_1)^s d\zeta \Big] \\
&= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4^{(1+s)}} \left(\frac{(1+s)(1-2^{1+s}) + (2^{1+s}-1)(2+s)}{(1+s)(2+s)} \right)^2 \\
&\times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right)
\end{aligned}$$

elde edilir.

Not 4.9 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.39 elde edilir.

Not 4.10 Yukarıdaki eşitsizlikte $s = 1$ ve $\alpha = 0$ alınırsa sonuç Teorem 3.21 elde edilir.

Teorem 4.46 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta), p > 1, s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F \left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2 \right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F \left(\tau_1, \frac{\kappa_3 + \kappa_4}{2} \right) d\tau_1 \right. \\ & \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)^{\frac{1}{q}} \end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat

$$\begin{aligned} T &= \left| F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F \left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2 \right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F \left(\tau_1, \frac{\kappa_3 + \kappa_4}{2} \right) d\tau_1 \right. \\ & \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \end{aligned}$$

Lemma 3.14 ve modülün özelliğini kullanılarak;

$$\begin{aligned} T &\leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \\ & \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned} T &\leq \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \\ & \times \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right|^q d\zeta d\xi \right)^{\frac{1}{q}} \end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q, \Delta$ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks özelliği kullanılır ve gerekli integraller hesaplanırsa aşağıdaki eşitsizlik yazılır.

$$\begin{aligned}
T &\leq \frac{1}{(\kappa_4 - \kappa_3)^{1+\frac{s}{q}} (\kappa_2 - \kappa_1)^{1+\frac{s}{q}}} \\
&\times \left(\int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)|^p \left\{ \int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}} (\xi - \kappa_3)^p d\xi + \int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^p d\xi \right\} \right)^{\frac{1}{p}} \\
&\times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)^s (\kappa_4 - \xi)^s d\zeta d\xi \right. \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)^s (\xi - \kappa_3)^s d\zeta d\xi \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)^s (\kappa_4 - \xi)^s d\zeta d\xi \\
&\left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)^s (\xi - \kappa_3)^s d\zeta d\xi \right]^{\frac{1}{q}} \\
&= \frac{1}{(\kappa_4 - \kappa_3)^{1+\frac{s}{q}} (\kappa_2 - \kappa_1)^{1+\frac{s}{q}}} \\
&\times \left(\frac{(\kappa_4 - \kappa_3)^{p+1}}{2^p (p+1)} \left\{ \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} (\zeta - \kappa_1)^p d\zeta + \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^p d\zeta \right\} \right)^{\frac{1}{p}} \\
&\times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \frac{(\kappa_4 - \kappa_3)^{1+s}}{1+s} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \right. \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \frac{(\kappa_4 - \kappa_3)^{1+s}}{1+s} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta) d\zeta \\
&+ \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \frac{(\kappa_4 - \kappa_3)^{1+s}}{1+s} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \\
&\left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \frac{(\kappa_4 - \kappa_3)^{1+s}}{1+s} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1) d\zeta \right]^{\frac{1}{q}} \\
&= \frac{1}{(\kappa_4 - \kappa_3)^{1+\frac{s}{q}} (\kappa_2 - \kappa_1)^{1+\frac{s}{q}}} \left(\frac{(\kappa_4 - \kappa_3)^{p+1}}{2^p (p+1)} \frac{(\kappa_2 - \kappa_1)^{p+1}}{2^p (p+1)} \right)^{\frac{1}{p}} \times \\
&\left(\frac{(\kappa_4 - \kappa_3)^{1+s}}{1+s} \frac{(\kappa_2 - \kappa_1)^{1+s}}{1+s} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \right)^{\frac{1}{q}} \\
&= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \left(\frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}}{(1+s)^2} \right)^{\frac{1}{q}}
\end{aligned}$$

ispat tamamlanır.

Not 4.11 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.40 elde edilir.

Not 4.12 Yukarıdaki eşitsizlikte $s = 1$ ve $\alpha = 0$ alınırsa Teorem 3.22 elde edilir.

Teorem 4.47 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta), p > 1, s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} & \left| F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F \left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2 \right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F \left(\tau_1, \frac{\kappa_3 + \kappa_4}{2} \right) d\tau_1 \right. \\ & \quad \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \\ & \leq \frac{(\kappa_4 - \kappa_3)^p (\kappa_2 - \kappa_1)^p}{4^p p (p+1)^2} + \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{q(1+s)^2} \\ & \quad \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right) \end{aligned}$$

burada $q^{-1} + p^{-1} = 1$ şeklindedir.

İspat

$$\begin{aligned} T = & \left| F \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) - \frac{1}{(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} F \left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2 \right) d\tau_2 - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F \left(\tau_1, \frac{\kappa_3 + \kappa_4}{2} \right) d\tau_1 \right. \\ & \left. + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| \end{aligned}$$

Lemma 3.14 ve modülün özelliğini kullanılarak;

$$\begin{aligned} T \leq & \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)| \\ & \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right| d\zeta d\xi \end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned} T \leq & \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \times \left[\frac{1}{p} \left(\int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta) q(\tau_2, \xi)|^p d\zeta d\xi \right) \right. \\ & \left. + \frac{1}{q} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_2 - \zeta}{\kappa_2 - \kappa_1} \kappa_1 + \frac{\zeta - \kappa_1}{\kappa_2 - \kappa_1} \kappa_2, \frac{\kappa_4 - \xi}{\kappa_4 - \kappa_3} \kappa_3 + \frac{\xi - \kappa_3}{\kappa_4 - \kappa_3} \kappa_4 \right) \right|^q d\zeta d\xi \right] \end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks özelliği kullanılır ve gerekli integraller hesaplanırsa aşağıdaki eşitsizlik yazılır.

$$\begin{aligned}
& T \\
& \leq \frac{1}{p(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \left(\int_{\kappa_1}^{\kappa_2} |p(\tau_1, \zeta)|^p \left\{ \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} (\xi - \kappa_3)^p d\xi + \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} (\kappa_4 - \xi)^p d\xi \right\} \right. \\
& \quad + \frac{1}{q(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)^s (\kappa_4 - \xi)^s d\zeta d\xi \right. \\
& \quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)^s (\xi - \kappa_3)^s d\zeta d\xi \\
& \quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)^s (\kappa_4 - \xi)^s d\zeta d\xi \\
& \quad \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)^s (\xi - \kappa_3)^s d\zeta d\xi \right] \Bigg) \\
& = \frac{1}{p(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \left(\frac{(\kappa_4 - \kappa_3)^{p+1}}{2^p(p+1)} \left\{ \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} (\zeta - \kappa_1)^p d\zeta + \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} (\kappa_2 - \zeta)^p d\zeta \right\} \right. \\
& \quad + \frac{1}{q(\kappa_4 - \kappa_3)^{1+s}(\kappa_2 - \kappa_1)^{1+s}} \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \frac{(\kappa_4 - \kappa_3)^{1+s}}{1+s} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)^s d\zeta \right. \\
& \quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \frac{(\kappa_4 - \kappa_3)^{1+s}}{1+s} \int_{\kappa_1}^{\kappa_2} (\kappa_2 - \zeta)^s d\zeta \\
& \quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \frac{(\kappa_4 - \kappa_3)^{1+s}}{1+s} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)^s d\zeta \\
& \quad \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \frac{(\kappa_4 - \kappa_3)^{1+s}}{1+s} \int_{\kappa_1}^{\kappa_2} (\zeta - \kappa_1)^s d\zeta \right] \Bigg) \\
& = \frac{(\kappa_4 - \kappa_3)^p (\kappa_2 - \kappa_1)^p}{4^p p (p+1)^2} + \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{q(1+s)^2} \\
& \quad \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)
\end{aligned}$$

sonucu bulunur.

Not 4.13 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.41 elde edilir.

Teorem 4.48 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$ ve $\alpha \in R$ için aşağıdaki

eşitsizlik geçerlidir:

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{64} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right)$$

burada

$$E = \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - A$$

ve

$$2A = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

dir.

İspat Lemma 3.11 ve mutlak değer özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \right| d\zeta d\xi.$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel konveks özelliği kullanılarak;

$$\begin{aligned} |E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \\ &\times \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \left[\zeta\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \right. \\ &\quad \left. + (1 - \zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right] d\zeta d\xi \\ &= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \zeta\xi d\zeta d\xi \right. \\ &\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \zeta(1 - \xi) d\zeta d\xi \\ &\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| (1 - \zeta)\xi d\zeta d\xi \\ &\quad \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| (1 - \zeta)(1 - \xi) d\zeta d\xi \right] \end{aligned}$$

yazılır. ζ ve ξ 'ye göre integral alınarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{64} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right)$$

sonucuna ulaşılır.

Not 4.14 Yukarıdaki eşitsizlikte $\alpha = 0$ alınırsa Teorem 3.11 elde edilir.

Teorem 4.49 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $p, q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)^{\frac{1}{q}}$$

burada

$$E = \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - A$$

ve

$$2A = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

ve $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 3.11 ve mutlak değer özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \right| d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \left(\int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \right|^q d\zeta d\xi \right)^{\frac{1}{q}}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ üzerinde koordinatlarda eksponansiyel konveks özelliği kullanılarak;

$$\begin{aligned}
|E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \left[\zeta \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \right. \\
&\quad \left. + (1-\zeta) \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \right]^{\frac{1}{q}} d\zeta d\xi \\
&= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} \int_0^1 \int_0^1 \zeta \xi d\zeta d\xi \right. \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} \int_0^1 \int_0^1 \zeta(1-\xi) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} \int_0^1 \int_0^1 (1-\zeta) \xi d\zeta d\xi \\
&\quad \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \int_0^1 \int_0^1 (1-\zeta)(1-\xi) d\zeta d\xi \right]^{\frac{1}{q}}
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \\
&\quad \times \left(\frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}}{4} \right)^{\frac{1}{q}}
\end{aligned}$$

elde edilir. İspat tamamlanmış olur.

Not 4.15 Yukarıdaki eşitsizlikte $\alpha = 0$ seçilirse Teorem 3.12 elde edilir.

Teorem 4.50 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $p, q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \\
&\quad \times \left(\frac{1}{p(p+1)^2} + \frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}}{4q} \right)
\end{aligned}$$

burada

$$\begin{aligned}
E &= \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} \\
&\quad + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - A
\end{aligned}$$

ve

$$2A = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

ve $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 3.11 ve mutlak değer özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \right| d\zeta d\xi.$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \left[\frac{1}{p} \left(\int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)|^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \right|^q d\zeta d\xi \right) \right]$$

eşitsizliği yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ üzerinde koordinatlarda eksponansiyel konveks özelliği kullanılarak;

$$\begin{aligned} |E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \\ &\times \left[\frac{1}{p(p+1)^2} + \int_0^1 \int_0^1 \frac{1}{q} \left(\zeta \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \right. \right. \\ &\quad \left. \left. + (1 - \zeta) \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right) d\zeta d\xi \right] \\ &= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \\ &\times \left[\frac{1}{p(p+1)^2} + \frac{1}{q} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta \xi d\zeta d\xi \right. \right. \\ &\quad \left. \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta(1 - \xi) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta) \xi d\zeta d\xi \right. \right. \\ &\quad \left. \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) d\zeta d\xi \right) \right] \end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \left(\frac{1}{p(p+1)^2} + \frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}}}{4q} \right)$$

sonucuna ulaşılır.

Teorem 4.51 $F : \Delta_1 \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta_1)$, $m \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{16} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right)$$

burada

$$E = \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - A$$

ve

$$2A = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

dır.

İspat Lemma 3.11 ve mutlak değer özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \right| d\zeta d\xi$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|, \Delta_1$ üzerinde koordinatlarda eksponansiyel m -konveks özelliği kullanılarak;

$$\begin{aligned} |E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \\ &\times \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \left[\zeta\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + m\zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \right. \\ &\left. + (1 - \zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + m(1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right] d\zeta d\xi \\ &= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \zeta\xi d\zeta d\xi \right. \\ &\left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \zeta(1 - \xi) d\zeta d\xi \right. \\ &\left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| (1 - \zeta)\xi d\zeta d\xi \right. \\ &\left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| (1 - \zeta)(1 - \xi) d\zeta d\xi \right] \end{aligned}$$

$$\left. \begin{aligned} & + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 |(1-2\zeta)(1-2\xi)|(1-\zeta)\xi d\zeta d\xi \\ & + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \int_0^1 \int_0^1 |(1-2\zeta)(1-2\xi)|(1-\zeta)(1-\xi) d\zeta d\xi \end{aligned} \right] \quad \left. \right]$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{16} \times \left(\frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}}}{4} \right)$$

sonucuna ulaşılır.

Not 4.16 Yukarıdaki eşitsizlikte $\alpha = 0$ ve $m = 1$ alınırsa Teorem 3.11 elde edilir.

Not 4.17 Yukarıdaki eşitsizlikte $m = 1$ alınırsa Teorem 4.48 elde edilir.

Teorem 4.52 $F : \Delta_1 \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta_1)$, $p, q > 1$, $m \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \times \left(\frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}}}{4} \right)^{\frac{1}{q}}$$

burada

$$E = \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - A$$

ve

$$2A = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

ve $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 3.11 ve mutlak değer özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \int_0^1 \int_0^1 |(1-2\zeta)(1-2\xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \right| d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \left(\int_0^1 \int_0^1 |(1-2\zeta)(1-2\xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \\ \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta \kappa_1 + (1-\zeta)\kappa_2, \xi \kappa_3 + (1-\xi)\kappa_4) \right|^q d\zeta d\xi \right)^{\frac{1}{q}}$$

eşitsizliği yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q, \Delta_1$ üzerinde koordinatlarda eksponansiyel m -konveks özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \times \left[\zeta \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \zeta(1-\xi) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \right. \\ \left. + (1-\zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + (1-\zeta)(1-\xi) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right]^{\frac{1}{q}} d\zeta d\xi \\ = \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta \xi d\zeta d\xi \right. \\ \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \int_0^1 \int_0^1 \zeta(1-\xi) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1-\zeta)\xi d\zeta d\xi \right. \\ \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \int_0^1 \int_0^1 (1-\zeta)(1-\xi) d\zeta d\xi \right]^{\frac{1}{q}}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \\ \times \left(\frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}}}{4} \right)^{\frac{1}{q}}$$

sonucuna ulaşılır.

Not 4.18 Yukarıdaki eşitsizlikte $\alpha = 0$ ve $m = 1$ alınrsa Teorem 3.12 elde edilir.

Not 4.19 Yukarıdaki eşitsizlikte $m = 1$ alınrsa Teorem 4.49 elde edilir.

Teorem 4.53 $F : \Delta_1 \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L(\Delta_1), p, q > 1, m \in$

$(0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \left(\frac{1}{p(p+1)^2} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right) / 4q$$

burada

$$E = \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - A$$

ve

$$2A = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

ve $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 3.11 ve mutlak değer özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \right| d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \left[\frac{1}{p} \left(\int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)|^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta\kappa_1 + (1 - \zeta)\kappa_2, \xi\kappa_3 + (1 - \xi)\kappa_4) \right|^q d\zeta d\xi \right) \right]$$

eşitsizliği elde edilir. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ_1 üzerinde koordinatlarda eksponansiyel m -konveks özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \left[\frac{1}{p(p+1)^2} + \int_0^1 \int_0^1 \frac{1}{q} \left(\zeta \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + m \zeta (1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + (1 - \zeta) \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + m (1 - \zeta) (1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right) d\zeta d\xi \right]$$

$$\begin{aligned}
&= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \left[\frac{1}{p(p+1)^2} + \frac{1}{q} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta \xi d\zeta d\xi \right. \right. \\
&\quad + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \int_0^1 \int_0^1 \zeta (1 - \xi) d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta) \xi d\zeta d\xi \\
&\quad \left. \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) d\zeta d\xi \right) \right]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \\
&\quad \times \left(\frac{1}{p(p+1)^2} + \frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}}}{4q} \right)
\end{aligned}$$

sonucuna ulaşılır.

Not 4.20 Yukarıdaki eşitsizlikte $m = 1$ alınırsa Teorem 4.50 elde edilir.

Teorem 4.54 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(1+s)^2} \times \left(\frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}}}{4} \right)$$

burada

$$\begin{aligned}
E &= \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} \\
&\quad + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - A
\end{aligned}$$

ve

$$2A = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

dır.

İspat Lemma 3.11 ve mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \\
&\quad \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \right| d\zeta d\xi
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks özelliği kullanılarak;

$$\begin{aligned}
|E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \\
&\times \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \left[\zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \right. \\
&\quad \left. + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right] d\zeta d\xi \\
&= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \zeta^s \xi^s d\zeta d\xi \right. \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \zeta^s (1 - \xi)^s d\zeta d\xi \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| (1 - \zeta)^s \xi^s d\zeta d\xi \\
&\quad \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| (1 - \zeta)^s (1 - \xi)^s d\zeta d\xi \right]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(1+s)^2} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right)$$

sonucuna ulaşılır.

Not 4.21 Yukarıdaki eşitsizlikte $\alpha = 0$ ve $s = 1$ alınırsa Teorem 3.11 elde edilir.

Not 4.22 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.48 elde edilir.

Teorem 4.55 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $p, q > 1$, $s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)^{\frac{1}{q}}$$

burada

$$\begin{aligned}
E &= \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} \\
&\quad + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - A
\end{aligned}$$

ve

$$2A = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

ve $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 3.11 ve mutlak değer özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \right| d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \left(\int_0^1 \int_0^1 |(1 - 2\zeta)(1 - 2\xi)|^p d\zeta d\xi \right)^{\frac{1}{p}} \times \left(\int_0^1 \int_0^1 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta \kappa_1 + (1 - \zeta) \kappa_2, \xi \kappa_3 + (1 - \xi) \kappa_4) \right|^q d\zeta d\xi \right)^{\frac{1}{q}}$$

eşitsizliği yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks özelliği kullanılarak;

$$\begin{aligned} |E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \times \int_0^1 \int_0^1 \left[\zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \right. \\ &\quad \left. + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + (1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right]^{\frac{1}{q}} d\zeta d\xi \\ &= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \times \left[\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^s \xi^s d\zeta d\xi \right. \\ &\quad \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^s (1 - \xi)^s d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta)^s \xi^s d\zeta d\xi \right. \\ &\quad \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)^s (1 - \xi)^s d\zeta d\xi \right]^{\frac{1}{q}} \end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4(p+1)^{\frac{2}{p}}} \left(\frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}}}{(1+s)^2} \right)^{\frac{1}{q}}$$

sonucuna ulaşılır.

Not 4.23 Yukarıdaki eşitsizlikte $\alpha = 0$ ve $s = 1$ alınırsa Teorem 3.12 elde edilir.

Not 4.24 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.49 elde edilir.

Teorem 4.56 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L(\Delta)$, $p, q > 1$, $s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \left(\frac{1}{p(p+1)^2} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right) \frac{1}{q(1+s)^2}$$

burada

$$E = \frac{F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)}{4} + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 - A$$

ve

$$2A = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} (F(\tau_1, \kappa_3) + F(\tau_1, \kappa_4)) d\tau_1 + \frac{1}{\kappa_4 - \kappa_3} \int_{\kappa_3}^{\kappa_4} (F(\kappa_1, \tau_2) + F(\kappa_2, \tau_2)) d\tau_2$$

ve $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 3.11 ve mutlak değer özelliği kullanılarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \int_0^1 \int_0^1 |(1-2\zeta)(1-2\xi)| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \right| d\zeta d\xi$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \left[\frac{1}{p} \left(\int_0^1 \int_0^1 |(1-2\zeta)(1-2\xi)|^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \right|^q d\zeta d\xi \right) \right]$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks kullanılırsa;

$$|E| \leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \left[\frac{1}{4p(p+1)^2} + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \zeta^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + (1-\zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + (1-\zeta)^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right) d\zeta d\xi \right]$$

$$\begin{aligned}
&= \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \times \left[\frac{1}{4p(p+1)^2} + \frac{1}{q} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \int_0^1 \int_0^1 \zeta^s \xi^s d\zeta d\xi \right. \right. \\
&\quad + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} \int_0^1 \int_0^1 \zeta^s (1 - \xi)^s d\zeta d\xi + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} \int_0^1 \int_0^1 (1 - \zeta)^s \xi^s d\zeta d\xi \\
&\quad \left. \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \int_0^1 \int_0^1 (1 - \zeta)^s (1 - \xi)^s d\zeta d\xi \right) \right]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre integral alınırsa;

$$\begin{aligned}
|E| &\leq \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{4} \\
&\quad \times \left(\frac{1}{p(p+1)^2} + \frac{\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}}}{q(1+s)^2} \right)
\end{aligned}$$

bulunur. İspat tamamlanmış olur.

Not 4.25 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.50 elde edilir.

4.4. Koordinatlarda Eksponansiyel Konveks Fonksiyonlar için Yeni Lemma ve İlgili Eşitsizlikler

Bu çalışma boyunca

$$\begin{aligned}
J_1 &= \frac{41}{576} \left[F\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}\right) + F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \right] \\
&\quad + \frac{1}{32} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
&\quad + \frac{5}{192} \left[F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}\right) + F\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}\right) \right] \\
&\quad + F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}\right) + F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \\
&\quad - \frac{1}{6(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\frac{5\kappa_3 + \kappa_4}{6}} F\left(\frac{5\kappa_1 + \kappa_2}{6}, \tau_2\right) d\tau_2 \\
&\quad - \frac{1}{6(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\frac{5\kappa_1 + \kappa_2}{6}} F\left(\tau_1, \frac{5\kappa_3 + \kappa_4}{6}\right) d\tau_1 \\
&\quad - \frac{1}{6(\kappa_4 - \kappa_3)} \int_{\frac{\kappa_3 + 5\kappa_4}{6}}^{\kappa_4} F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \tau_2\right) d\tau_2 \\
&\quad - \frac{1}{6(\kappa_2 - \kappa_1)} \int_{\frac{\kappa_1 + 5\kappa_2}{6}}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + 5\kappa_4}{6}\right) d\tau_1
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8(\kappa_4 - \kappa_3)} \int_{\frac{5\kappa_3 + \kappa_4}{6}}^{\frac{\kappa_3 + 5\kappa_4}{6}} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{8(\kappa_2 - \kappa_1)} \int_{\frac{5\kappa_1 + \kappa_2}{6}}^{\frac{\kappa_1 + 5\kappa_2}{6}} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
& -\frac{5}{24(\kappa_4 - \kappa_3)} \int_{\frac{5\kappa_3 + \kappa_4}{6}}^{\frac{\kappa_3 + \kappa_4}{2}} F\left(\frac{5\kappa_1 + \kappa_2}{6}, \tau_2\right) d\tau_2 - \frac{5}{24(\kappa_2 - \kappa_1)} \int_{\frac{5\kappa_1 + \kappa_2}{6}}^{\frac{\kappa_1 + \kappa_2}{2}} F\left(\tau_1, \frac{5\kappa_3 + \kappa_4}{6}\right) d\tau_1 \\
& -\frac{5}{24(\kappa_4 - \kappa_3)} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\frac{\kappa_3 + 5\kappa_4}{6}} F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \tau_2\right) d\tau_2 - \frac{5}{24(\kappa_2 - \kappa_1)} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\frac{\kappa_1 + 5\kappa_2}{6}} F\left(\tau_1, \frac{\kappa_3 + 5\kappa_4}{6}\right) d\tau_1 \\
& + \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} F(\tau_1, \tau_2) d\tau_1 d\tau_2
\end{aligned}$$

olarak alınacaktır.

Lemma 4.11 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden kısmi türevlenebilen bir dönüşüm olsun. Eğer $\frac{\partial^2 F}{\partial \zeta \partial \xi} \in L_2(\Delta)$ ise aşağıdaki eşitsizlik geçerlidir.

$$\begin{aligned}
J_1 = & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \frac{\partial^2 F}{\partial \zeta \partial \xi} \left((1 - \zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1 - \xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left(4\zeta - \frac{5}{2} \right) \left(4\xi - \frac{5}{2} \right) \\
& \times \frac{\partial^2 F}{\partial \zeta \partial \xi} \left((1 - \zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1 - \xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left(4\zeta - \frac{3}{2} \right) \left(4\xi - \frac{3}{2} \right) \\
& \times \frac{\partial^2 F}{\partial \zeta \partial \xi} \left((1 - \zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left. \frac{\partial^2 F}{\partial \zeta \partial \xi} \left((1 - \zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1 - \xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \right].
\end{aligned}$$

İspat

$$\begin{aligned}
G_1 = & \int_0^1 \int_0^1 \zeta \xi \frac{\partial^2 F}{\partial \zeta \partial \xi} \left((1 - \zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1 - \xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \\
G_2 = & \int_0^1 \int_0^1 \left(4\zeta - \frac{5}{2} \right) \left(4\xi - \frac{5}{2} \right) \\
& \times \frac{\partial^2 F}{\partial \zeta \partial \xi} \left((1 - \zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1 - \xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
G_3 = & \int_0^1 \int_0^1 \left(4\zeta - \frac{3}{2} \right) \left(4\xi - \frac{3}{2} \right) \\
& \times \frac{\partial^2 F}{\partial \zeta \partial \xi} \left((1 - \zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi
\end{aligned}$$

$$G_4 = \int_0^1 \int_0^1 (1-\zeta)(1-\xi) \times \frac{\partial^2 F}{\partial \zeta \partial \xi} \left((1-\zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1-\xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi$$

olsun. G_1, G_2, G_3 ve G_4 'e iki katlı integrallerde kısmi integrasyon uygulayarak;

$$G_1 = \frac{36}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}\right) - \frac{216}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)^2} \int_{\kappa_3}^{\frac{5\kappa_3 + \kappa_4}{6}} F\left(\frac{5\kappa_1 + \kappa_2}{6}, \tau_2\right) d\tau_2 - \frac{216}{(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)} \int_{\kappa_1}^{\frac{5\kappa_1 + \kappa_2}{6}} F\left(\tau_1, \frac{5\kappa_3 + \kappa_4}{6}\right) d\tau_1 + \frac{1296}{(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)^2} \int_{\kappa_1}^{\frac{5\kappa_1 + \kappa_2}{6}} \int_{\kappa_3}^{\frac{5\kappa_3 + \kappa_4}{6}} F(\tau_1, \tau_2) d\tau_1 d\tau_2,$$

$$G_2 = \frac{81}{4(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + \frac{135}{4(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}\right) - \frac{324}{2(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)^2} \int_{\frac{5\kappa_3 + \kappa_4}{6}}^{\frac{\kappa_3 + \kappa_4}{2}} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 + \frac{135}{4(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}\right) + \frac{225}{4(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}\right) - \frac{540}{2(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)^2} \int_{\frac{5\kappa_3 + \kappa_4}{6}}^{\frac{\kappa_3 + \kappa_4}{2}} F\left(\frac{5\kappa_1 + \kappa_2}{6}, \tau_2\right) d\tau_2 - \frac{324}{2(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)} \int_{\frac{5\kappa_1 + \kappa_2}{6}}^{\frac{\kappa_1 + \kappa_2}{2}} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 - \frac{540}{2(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)} \int_{\frac{5\kappa_1 + \kappa_2}{6}}^{\frac{\kappa_1 + \kappa_2}{2}} F\left(\tau_1, \frac{5\kappa_3 + \kappa_4}{6}\right) d\tau_1 + \frac{1296}{(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)^2} \int_{\frac{5\kappa_3 + \kappa_4}{6}}^{\frac{\kappa_3 + \kappa_4}{2}} \int_{\frac{5\kappa_1 + \kappa_2}{6}}^{\frac{\kappa_1 + \kappa_2}{2}} F(\tau_1, \tau_2) d\tau_1 d\tau_2,$$

$$\begin{aligned}
G_3 = & \frac{225}{4(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \\
& + \frac{135}{4(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& - \frac{540}{2(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)^2} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\frac{\kappa_3 + 5\kappa_4}{6}} F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \tau_2\right) d\tau_2 \\
& + \frac{135}{4(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \\
& + \frac{81}{4(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& - \frac{324}{2(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)^2} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\frac{\kappa_3 + 5\kappa_4}{6}} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\
& - \frac{540}{2(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\frac{\kappa_1 + 5\kappa_2}{6}} F\left(\tau_1, \frac{\kappa_3 + 5\kappa_4}{6}\right) d\tau_1 \\
& - \frac{324}{2(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\frac{\kappa_1 + 5\kappa_2}{6}} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
& + \frac{1296}{(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)^2} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\frac{\kappa_3 + 5\kappa_4}{6}} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\frac{\kappa_1 + 5\kappa_2}{6}} F(\tau_1, \tau_2) d\tau_1 d\tau_2,
\end{aligned}$$

$$\begin{aligned}
G_4 = & \frac{36}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \\
& - \frac{216}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)^2} \int_{\frac{\kappa_3 + \kappa_4}{6}}^{\kappa_4} F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \tau_2\right) \\
& - \frac{216}{(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)} \int_{\frac{\kappa_1 + 5\kappa_2}{6}}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + 5\kappa_4}{6}\right) d\tau_1 \\
& + \frac{1296}{(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)^2} \int_{\frac{\kappa_3 + 5\kappa_4}{6}}^{\kappa_4} \int_{\frac{\kappa_1 + 5\kappa_2}{6}}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2
\end{aligned}$$

elde edilir. G_1, G_2, G_3 ve G_4 toplanıp, her iki taraf $\frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296}$ ile çarpılarak;

$$\begin{aligned}
& \frac{41}{576} \left[F\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}\right) + F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \right] \\
& + \frac{1}{32} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + \frac{5}{192} \left[F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}\right) \right. \\
& \left. + F\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}\right) + F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}\right) + F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \right] \\
& - \frac{1}{6(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\frac{5\kappa_3 + \kappa_4}{6}} F\left(\frac{5\kappa_1 + \kappa_2}{6}, \tau_2\right) d\tau_2 - \frac{1}{6(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\frac{5\kappa_1 + \kappa_2}{6}} F\left(\tau_1, \frac{5\kappa_3 + \kappa_4}{6}\right) d\tau_1 \\
& - \frac{1}{6(\kappa_4 - \kappa_3)} \int_{\frac{\kappa_3 + 5\kappa_4}{6}}^{\kappa_4} F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \tau_2\right) d\tau_2 - \frac{1}{6(\kappa_2 - \kappa_1)} \int_{\frac{\kappa_1 + 5\kappa_2}{6}}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + 5\kappa_4}{6}\right) d\tau_1
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8(\kappa_4 - \kappa_3)} \int_{\frac{5\kappa_3 + \kappa_4}{6}}^{\frac{\kappa_3 + 5\kappa_4}{6}} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 - \frac{1}{8(\kappa_2 - \kappa_1)} \int_{\frac{5\kappa_1 + \kappa_2}{6}}^{\frac{\kappa_1 + 5\kappa_2}{6}} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
& -\frac{5}{24(\kappa_4 - \kappa_3)} \int_{\frac{5\kappa_3 + \kappa_4}{6}}^{\frac{\kappa_3 + \kappa_4}{2}} F\left(\frac{5\kappa_1 + \kappa_2}{6}, \tau_2\right) d\tau_2 - \frac{5}{24(\kappa_2 - \kappa_1)} \int_{\frac{5\kappa_1 + \kappa_2}{6}}^{\frac{\kappa_1 + \kappa_2}{2}} F\left(\tau_1, \frac{5\kappa_3 + \kappa_4}{6}\right) d\tau_1 \\
& -\frac{5}{24(\kappa_4 - \kappa_3)} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\frac{\kappa_3 + 5\kappa_4}{6}} F\left(\frac{\kappa_1 + 5\kappa_2}{6}, \tau_2\right) d\tau_2 - \frac{5}{24(\kappa_2 - \kappa_1)} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\frac{\kappa_1 + 5\kappa_2}{6}} F\left(\tau_1, \frac{\kappa_3 + 5\kappa_4}{6}\right) d\tau_1 \\
& + \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2
\end{aligned}$$

elde edilir.

Teorem 4.57 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\frac{1}{36} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \right. \\
& + \frac{1}{9} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}\right) \right|}{e^{\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \right|}{e^{\alpha\left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6}\right)}} \right) \\
& + \frac{1}{18} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6}\right) \right|}{e^{\alpha\left(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3\right) \right|}{e^{\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3\right)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4\right) \right|}{e^{\alpha\left(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}\right) \right|}{e^{\alpha\left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6}\right)}} \Bigg) \\
& + \frac{63001}{147456} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}\right) \right|}{e^{\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \right|}{e^{\alpha\left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6}\right)}} \right) \\
& + \frac{49298}{147456} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \right|}{e^{\alpha\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2}\right)}} \right) + \frac{39407}{147456} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}\right) \right|}{e^{\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2}\right)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}\right) \right|}{e^{\alpha\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6}\right) \right|}{e^{\alpha\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}\left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}\right) \right|}{e^{\alpha\left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2}\right)}} \Bigg) \Bigg].
\end{aligned}$$

İspat Lemma 4.11 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1 - \xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1-\xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1-\xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1-\zeta)(1-\xi) \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1-\xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel konveks kullanılırsa;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + (1-\zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6})}} \right. \\
& + \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \left. \zeta\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + (1-\zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} \right. \\
& + \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \left. \zeta\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} + (1-\zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6})}} \right. \\
& + \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} + \left. \zeta\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1-\zeta)(1-\xi) \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} + (1-\zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4)}} \right)
\end{aligned}$$

$$+ \left| \frac{\frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right)}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} \right| + \left| \frac{\frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right)}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \right| + \left| \frac{\frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right)}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} \right| \right]$$

elde edilir.

Sonuç 4.1 *Teorem 4.57 nin şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.*

$$\begin{aligned} |J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\frac{1}{36} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right| \right) \right. \\ & + \frac{1}{9} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \right) \\ & + \frac{1}{18} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right| \right. \\ & + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \left. \right) \\ & + \frac{63001}{147456} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \right) \\ & + \frac{49298}{147456} \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| \\ & + \frac{39407}{147456} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \right. \\ & + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| \right) \left. \right]. \end{aligned}$$

Teorem 4.58 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $p > 0$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned} |J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296 (q+1)^{\frac{2}{q}}} \times \left[\frac{1}{(p+1)^{\frac{2}{p}}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6} \right)}} \right. \right. \\ & + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3 \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \left. \right. \\ & + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4 \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \left. \right) \\ & + \left(\frac{\left(\frac{5}{2} \right)^{p+1} + \left(\frac{3}{2} \right)^{p+1}}{4(p+1)} \right)^{\frac{2}{p}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} \right. \\ & + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + 2 \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} \left. \right. \\ & + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \left. \right) \left. \right] \end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.11 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1 - \xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1 - \xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1 - \xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel konveks olduğu kullanılırsa;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + (1 - \zeta) \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \zeta \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} \Big) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + (1 - \zeta) \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \zeta \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} \Big) d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + (1-\zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \right. \\
& + \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \left. \zeta\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1-\zeta)(1-\xi) \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + (1-\zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4 \right)}} \right. \\
& + \left. \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + \left. \zeta\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \kappa_4 \right) \right|}{e^{\alpha \left(\kappa_2 + \kappa_4 \right)}} \right) d\zeta d\xi \Bigg]
\end{aligned}$$

elde edilir. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
|J_1| & \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \right. \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1-\xi)^q \zeta^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3 \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \kappa_4 \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6}+\kappa_4\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\xi)^q \zeta^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\kappa_2+\frac{\kappa_3+5\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \Bigg]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre integral alınarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296(q+1)^{\frac{2}{q}}} \\
& \times \left[\frac{1}{(p+1)^{\frac{2}{p}}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha\left(\kappa_1+\frac{5\kappa_3+\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \kappa_3 \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\kappa_3\right)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\frac{5\kappa_3+\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6}+\frac{\kappa_3+5\kappa_4}{6}\right)}} \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \kappa_4 \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6}+\kappa_4\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\kappa_2+\frac{\kappa_3+5\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \Bigg) \\
& + \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4(p+1)} \right)^{\frac{2}{p}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\frac{5\kappa_3+\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\frac{\kappa_3+\kappa_4}{2}\right)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{5\kappa_3+\kappa_4}{6}\right)}} + 2 \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}\right)}} \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+5\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6}+\frac{\kappa_3+\kappa_4}{2}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6}+\frac{\kappa_3+5\kappa_4}{6}\right)}} \Bigg) \Bigg]
\end{aligned}$$

elde edilir.

Sonuç 4.2 *Teorem 4.58 nin şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda konvekslik*

sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296(q+1)^{\frac{2}{q}}} \times \left[\frac{1}{(p+1)^{\frac{2}{p}}} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| \right. \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right| \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right| \right) \\
& + \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4(p+1)} \right)^{\frac{2}{p}} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \\
& + 2 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \\
& + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \right) \Big]
\end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

Teorem 4.59 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $p > 0$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{8}{p(p+1)^2} + \frac{1}{q(q+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6}\right)}} \right. \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6}\right)}} \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6}\right)}} + \left. \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right) \\
& + 8 \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4p(p+1)} \right)^2 + \frac{1}{q(q+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6}\right)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6}\right)}} + 2 \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2}\right)}} \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2}\right)}} + \left. \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6}\right)}} \right) \Big]
\end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.11 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1 - \xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1 - \xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1 - \xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel konveks olduğu kullanılırsa;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + (1 - \zeta) \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \left. \zeta \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + (1 - \zeta) \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} \right. \\
& + \left. \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \zeta \xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} \right) d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_3+\kappa_4}{2}\right)}} + (1-\zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_3+5\kappa_4}{6}\right)}} \right. \\
& + \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2}\right)}} + \left. \zeta\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6}\right)}} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1-\zeta)(1-\xi) \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6}\right)}} + (1-\zeta)\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \kappa_4 \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \kappa_4\right)}} \right. \\
& + \left. \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\kappa_2 + \frac{\kappa_3+5\kappa_4}{6}\right)}} + \zeta\xi \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \kappa_4 \right) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right) d\zeta d\xi \Bigg]
\end{aligned}$$

elde edilir. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned}
& = \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right) \right. \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\kappa_1 + \frac{5\kappa_3+\kappa_4}{6}\right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\xi)^q \zeta^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \kappa_3 \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \kappa_3\right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \frac{5\kappa_3+\kappa_4}{6}\right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \frac{5\kappa_3+\kappa_4}{6}\right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2}\right)}} d\zeta d\xi \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right) \Bigg]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre integral alınarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{8}{p(p+1)^2} + \frac{1}{q(q+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{5\kappa_3+\kappa_4}{6}) \right|^q}{e^{q\alpha(\kappa_1+\frac{5\kappa_3+\kappa_4}{6})}} \right. \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{5\kappa_1+\kappa_2}{6}, \kappa_3) \right|^q}{e^{q\alpha(\frac{5\kappa_1+\kappa_2}{6}+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6}) \right|^q}{e^{q\alpha(\frac{5\kappa_1+\kappa_2}{6}+\frac{5\kappa_3+\kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6}) \right|^q}{e^{q\alpha(\frac{\kappa_1+5\kappa_2}{6}+\frac{\kappa_3+5\kappa_4}{6})}} \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{\kappa_1+5\kappa_2}{6}, \kappa_4) \right|^q}{e^{q\alpha(\frac{\kappa_1+5\kappa_2}{6}+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3+5\kappa_4}{6}) \right|^q}{e^{q\alpha(\kappa_2+\frac{\kappa_3+5\kappa_4}{6})}} + \left. \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} \right) \\
& + 8 \left(\frac{\left(\frac{5}{2} \right)^{p+1} + \left(\frac{3}{2} \right)^{p+1}}{4p(p+1)} \right)^2 + \frac{1}{q(q+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6}) \right|^q}{e^{q\alpha(\frac{5\kappa_1+\kappa_2}{6}+\frac{5\kappa_3+\kappa_4}{6})}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{5\kappa_1+\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2}) \right|^q}{e^{q\alpha(\frac{5\kappa_1+\kappa_2}{6}+\frac{\kappa_3+\kappa_4}{2})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{\kappa_1+\kappa_2}{2}, \frac{5\kappa_3+\kappa_4}{6}) \right|^q}{e^{q\alpha(\frac{\kappa_1+\kappa_2}{2}+\frac{5\kappa_3+\kappa_4}{6})}} + 2 \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}) \right|^q}{e^{q\alpha(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2})}} \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+5\kappa_4}{6}) \right|^q}{e^{q\alpha(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+5\kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2}) \right|^q}{e^{q\alpha(\frac{\kappa_1+5\kappa_2}{6}+\frac{\kappa_3+\kappa_4}{2})}} + \left. \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6}) \right|^q}{e^{q\alpha(\frac{\kappa_1+5\kappa_2}{6}+\frac{\kappa_3+5\kappa_4}{6})}} \right) \Bigg]
\end{aligned}$$

elde edilir. İspat tamamlanmış olur.

Sonuç 4.3 *Teorem 4.59 nin şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda konvekslik sonucunu aşağıdaki gibi karşımıza çıkar.*

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{8}{p(p+1)^2} + \frac{1}{q(q+1)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q \right. \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \kappa_3 \right) \right|^q \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \kappa_4 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q \right)
\end{aligned}$$

$$\begin{aligned}
& +8 \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4p(p+1)} \right)^2 + \frac{1}{q(q+1)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \\
& + 2 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q \\
& \left. + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q \right)
\end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

Teorem 4.60 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $m \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\frac{1}{36} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha \left(\kappa_2 + \frac{\kappa_4}{m} \right)}} \right) \right. \\
& + \frac{1}{9} \left(\frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \right) \\
& + \frac{1}{18} \left(\frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right|}{e^{\alpha \left(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3 \right)}} \right. \\
& + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \left. \right) \\
& + \frac{63001}{147456} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} \right) \\
& + \frac{24649}{147456} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} \right) \\
& + \frac{39407}{147456} \left(\frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} \right. \\
& \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} \right) \left. \right].
\end{aligned}$$

İspat Lemma 4.11 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1 - \xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1 - \xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1 - \xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel m -konveks olduğu kullanılırsa;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m})}} \Big) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m})}} \Big) d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + (1-\zeta)\xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} \right. \\
& \left. \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \zeta\xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1-\zeta)(1-\xi) \\
& \times \left((1-\zeta)(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + (1-\zeta)\xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m} \right)}} \right. \\
& \left. \zeta(1-\xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + \zeta\xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha \left(\kappa_2 + \frac{\kappa_4}{m} \right)}} \right) d\zeta d\xi \Bigg]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre integral alınarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\frac{1}{36} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{1}{18} \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right|}{e^{\alpha \left(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m} \right)}} \right. \\
& + \frac{1}{18} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3 \right)}} + \frac{1}{9} \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m} \right)}} \\
& + \frac{63001}{147456} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{39407}{147456} \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} \\
& + \frac{39407}{147456} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{24649}{147456} \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} \\
& + \frac{24649}{147456} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \frac{39407}{147456} \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} \\
& + \frac{39407}{147456} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \frac{63001}{147456} \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} \\
& + \frac{1}{9} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + \frac{1}{18} \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m} \right)}} \\
& \left. + \frac{1}{18} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + \frac{1}{36} \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha \left(\kappa_2 + \frac{\kappa_4}{m} \right)}} \right]
\end{aligned}$$

elde edilir. Gerekli düzenlemeler yapılarak;

$$\begin{aligned}
&= \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\frac{1}{36} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right) \right. \\
&\quad + \frac{1}{9} \left(\frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} \right) \\
&\quad + \frac{1}{18} \left(\frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} \right. \\
&\quad \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} \right) \\
&\quad + \frac{63001}{147456} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m})}} \right) \\
&\quad + \frac{24649}{147456} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m})}} \right) \\
&\quad + \frac{39407}{147456} \left(\frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} \right. \\
&\quad \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} \right) \Bigg]
\end{aligned}$$

elde edilir. İspat tamamlanmış olur.

Not 4.26 Yukarıdaki eşitsizlikte $m = 1$ alınırsa Teorem 4.57 elde edilir.

Sonuç 4.4 Teorem 4.60'ın şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda m -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
|J_1| &\leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\frac{1}{36} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right| \right) \right. \\
&\quad + \frac{1}{9} \left(m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m}) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right| \right) \\
&\quad + \frac{1}{18} \left(m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m}) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right| \right. \\
&\quad \left. + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m}) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right| \right) \\
&\quad + \frac{63001}{147456} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right| + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right| \right) \\
&\quad + \frac{24649}{147456} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right| + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m}) \right| \right)
\end{aligned}$$

$$+ \frac{39407}{147456} \left(m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \right. \\ \left. + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| \right).$$

Teorem 4.61 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $p > 0$, $m \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|J_1| \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296(q+1)^{\frac{2}{q}}} \left[\frac{1}{(p+1)^{\frac{2}{p}}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m})}} \right. \right. \\ \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} \right. \\ \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right) \\ + \left(\frac{(\frac{5}{2})^{p+1} + (\frac{3}{2})^{p+1}}{4(p+1)} \right)^{\frac{2}{p}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m})}} \right. \\ \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} \right. \\ \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m})}} \right) \Bigg]$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.11 ve integraller için mutlak değer özelliği kullanılarak;

$$|J_1| \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\ \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1 - \xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \\ + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\ \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1 - \xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\ + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\ \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\ + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\ \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1 - \xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \Bigg]$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel m -konveks olduğu kullanılırsa;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \left. \right) d\zeta d\xi \Big]
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
&= \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
&\quad \times \left[\left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1 - \zeta)^q (1 - \xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1 - \zeta)^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m})}} d\zeta d\xi \right)^{\frac{1}{q}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1-\xi)^q \zeta^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \kappa_3 \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\kappa_3\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6m} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\frac{5\kappa_3+\kappa_4}{6m}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\frac{5\kappa_3+\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2m} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\frac{\kappa_3+\kappa_4}{2m}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{5\kappa_3+\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2m} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2m}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+5\kappa_4}{6m} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+5\kappa_4}{6m}\right)}} d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6m} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6m}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_4}{m}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\xi)^q \zeta^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\kappa_2 + \frac{\kappa_3+5\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha\left(\kappa_2 + \frac{\kappa_4}{m}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \Bigg]
\end{aligned}$$

elde edilir. ζ ve ξ 'ye göre integral alınarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296(q+1)^{\frac{2}{q}}} \left[\frac{1}{(p+1)^{\frac{2}{p}}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3+\kappa_4}{6m} \right) \right|}{e^{\alpha\left(\kappa_1 + \frac{5\kappa_3+\kappa_4}{6m}\right)}} \right. \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \kappa_3 \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \kappa_3\right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6m} \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \frac{5\kappa_3+\kappa_4}{6m}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6}\right)}} \\
& \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_4}{m}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\kappa_2 + \frac{\kappa_3+5\kappa_4}{6}\right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha\left(\kappa_2 + \frac{\kappa_4}{m}\right)}} \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4(p+1)} \right)^{\frac{2}{p}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} \\
& \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} \right) \Bigg]
\end{aligned}$$

elde edilir.

Not 4.27 Yukarıdaki eşitsizlikte $m = 1$ alınırsa Teorem 4.58 elde edilir.

Sonuç 4.5 Teorem 4.61 in şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda m -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
|J_1| & \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296(q+1)^{\frac{2}{q}}} \\
& \times \left[\frac{1}{(p+1)^{\frac{2}{p}}} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right| \right. \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right| + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right| \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m} \right) \right| \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right| \Bigg) \\
& + \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4(p+1)} \right)^{\frac{2}{p}} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \right. \\
& + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \\
& + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| \\
& + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right| \\
& \left. + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right| \right) \Bigg].
\end{aligned}$$

Teorem 4.62 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $p > 0$, $m \in (0, 1]$ ve

$\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{8}{p(p+1)^2} + \frac{1}{q(q+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m})}} \right. \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|^q}{e^{q\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m}) \right|^q}{e^{q\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|^q}{e^{q\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} \\
& + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \left. \right) \\
& + 8 \left(\frac{\left(\frac{5}{2} \right)^{p+1} + \left(\frac{3}{2} \right)^{p+1}}{4p(p+1)} \right)^2 + \frac{1}{q(q+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|^q}{e^{q\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} \right. \\
& + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m}) \right|^q}{e^{q\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|^q}{e^{q\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} \\
& + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m}) \right|^q}{e^{q\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|^q}{e^{q\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|^q}{e^{q\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m})}} \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|^q}{e^{q\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|^q}{e^{q\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m})}} \left. \right) \Bigg]
\end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.11 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1 - \xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1 - \xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1 - \xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \Bigg]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel m - konveks olduğu kullanılırsa;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m})}} \right. \\
& + \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left((1 - \zeta)(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} + (1 - \zeta) \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m})}} \right. \\
& \left. \left. \zeta(1 - \xi) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} + \zeta \xi \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right) d\zeta d\xi \right]
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned}
& = \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 (1 - \zeta)^q (1 - \xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi \right) \right. \\
& \left. + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 (1 - \zeta)^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m})}} d\zeta d\xi \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\xi)^q \zeta^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3 \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^q (1-\xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} d\zeta d\xi \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^p (1 - \xi)^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1 - \zeta)^q (1 - \xi)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^p (1 - \xi)^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1 - \zeta)^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^p (1 - \xi)^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1 - \xi)^q \zeta^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 (1 - \zeta)^p (1 - \xi)^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^q \xi^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha \left(\kappa_2 + \frac{\kappa_4}{m} \right)}} d\zeta d\xi \right) \Bigg]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre integral alınarak;

$$\begin{aligned}
|J_1| & \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{8}{p(p+1)^2} + \frac{1}{q(q+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right|^q}{e^{q\alpha \left(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6m} \right)}} \right. \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3 \right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \\
& \left. + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_4}{m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha \left(\kappa_2 + \frac{\kappa_4}{m} \right)}} \right)
\end{aligned}$$

$$\begin{aligned}
& +8 \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4p(p+1)} \right)^2 + \frac{1}{q(q+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} \right. \\
& + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} \\
& + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2m} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \frac{\left| m \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} \\
& \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6m} \right)}} \right) \Bigg]
\end{aligned}$$

elde edilir. İspat tamamlanmış olur.

Not 4.28 Yukarıdaki eşitsizlikte $m = 1$ alınırsa Teorem 4.59 elde edilir.

Sonuç 4.6 Teorem 4.62 in şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda m -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
|J_1| & \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{8}{p(p+1)^2} + \frac{1}{q(q+1)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right|^q \right. \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|^q + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6m} \right) \right|^q \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_4}{m} \right) \right|^q \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|^q \Bigg) \\
& + 8 \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4p(p+1)} \right)^2 + \frac{1}{q(q+1)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \right. \\
& + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \\
& + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2m} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q \\
& + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|^q \\
& \left. + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6m} \right) \right|^q \right) \Bigg].
\end{aligned}$$

Teorem 4.63 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\frac{1}{(1+s)^2(2+s)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \right. \\
& + \frac{1}{(1+s)(2+s)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} \left. \right) \\
& + \left(\frac{1 + \frac{5s}{2} + 3\left(\frac{3}{8}\right)^{1+s}}{(1+s)(2+s)} \right)^2 \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} \right) \\
& + \left(\frac{1 + \frac{5s}{2} + 3\left(\frac{3}{8}\right)^{1+s}}{(1+s)(2+s)} \right) \left(\frac{3s + 5\left(\frac{5}{8}\right)^{1+s} - 1}{(1+s)(2+s)} \right) \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} \left. \right) \\
& + \left(\frac{1 - \frac{3s}{2} - 5\left(\frac{5}{8}\right)^{1+s}}{(1+s)(2+s)} \right)^2 \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} \right) \left. \right].
\end{aligned}$$

İspat Lemma 4.11 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\int_0^1 \int_0^1 \zeta \xi \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta)\kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1-\xi)\kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \right. \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1-\xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1-\xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1-\zeta)(1-\xi) \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1-\xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \left. \right]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s —konveks olduğu

kullanılırsa;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left((1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6})}} \right. \\
& + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left((1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} \right. \\
& + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left((1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6})}} \right. \\
& + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left((1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4)}} \right. \\
& + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \left. \right) d\zeta d\xi \Big]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre integral alınarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\frac{1}{(1+s)^2(2+s)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \right. \\
& + \frac{1}{(1+s)(2+s)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} \left. \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1 + \frac{5s}{2} + 3 \left(\frac{3}{8} \right)^{1+s}}{(1+s)(2+s)} \right)^2 \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \right) \\
& + \left(\frac{1 + \frac{5s}{2} + 3 \left(\frac{3}{8} \right)^{1+s}}{(1+s)(2+s)} \right) \left(\frac{\frac{3s}{2} + 5 \left(\frac{5}{8} \right)^{1+s} - 1}{(1+s)(2+s)} \right) \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2} \right)}} \left. \right) \\
& + \left(\frac{1 - \frac{3s}{2} - 5 \left(\frac{5}{8} \right)^{1+s}}{(1+s)(2+s)} \right)^2 \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2} \right)}} \right) \Bigg]
\end{aligned}$$

sonucu elde edilir.

Not 4.29 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.57 elde edilir.

Sonuç 4.7 Teorem 4.63 ün şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda ikinci anlamda s -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \left[\frac{1}{(1+s)^2(2+s)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right| \right) \right. \\
& + \frac{1}{(1+s)(2+s)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right| \right. \\
& + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \right) \\
& + \left(\frac{1 + \frac{5s}{2} + 3 \left(\frac{3}{8} \right)^{1+s}}{(1+s)(2+s)} \right)^2 \\
& \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \right) \\
& + \left(\frac{1 + \frac{5s}{2} + 3 \left(\frac{3}{8} \right)^{1+s}}{(1+s)(2+s)} \right) \left(\frac{\frac{3s}{2} + 5 \left(\frac{5}{8} \right)^{1+s} - 1}{(1+s)(2+s)} \right) \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| \right) \\
& + \left(\frac{1 - \frac{3s}{2} - 5 \left(\frac{5}{8} \right)^{1+s}}{(1+s)(2+s)} \right)^2 \\
& \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| \right) \Bigg].
\end{aligned}$$

Teorem 4.64 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s –konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta), p > 0, s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296(qs + 1)^{\frac{2}{q}}} \\
& \times \left[\frac{1}{(p+1)^{\frac{2}{p}}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6})}} \right. \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} + \left. \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \\
& + \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4(p+1)} \right)^{\frac{2}{p}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + 2 \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} \\
& + \left. \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} \right) \Bigg]
\end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.11 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\int_0^1 \int_0^1 \zeta \xi \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1 - \xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \right. \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1 - \xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1 - \xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1 - \zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1 - \xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \Bigg]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s – konveks olduğu kullanılırsa;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left((1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6})}} \right. \\
& + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3)}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left((1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6})}} + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{5\kappa_1 + \kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} \right. \\
& + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{5\kappa_3 + \kappa_4}{6})}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left((1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + \kappa_4}{2})}} + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + \kappa_2}{2} + \frac{\kappa_3 + 5\kappa_4}{6})}} \right. \\
& + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + \kappa_4}{2})}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} \left. \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1 - \zeta)(1 - \xi) \\
& \times \left((1 - \zeta)^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6})}} + (1 - \zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4) \right|}{e^{\alpha(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4)}} \right. \\
& + \zeta^s (1 - \xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6})}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \left. \right) d\zeta d\xi \Bigg]
\end{aligned}$$

elde edilir. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1 - \zeta)^{qs} (1 - \xi)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\kappa_1+\frac{5\kappa_3+\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1-\xi)^{qs} \zeta^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \kappa_3 \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\kappa_3\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \zeta^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\frac{5\kappa_3+\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} (1-\xi)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\frac{5\kappa_3+\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6}+\frac{\kappa_3+\kappa_4}{2}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^{qs} (1-\xi)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{5\kappa_3+\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} (1-\xi)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+5\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^{qs} (1-\xi)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right|^p \left| 4\xi - \frac{3}{2} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \zeta^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} (1-\xi)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \kappa_4 \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \kappa_4\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 (1-\xi)^{qs} \zeta^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\kappa_2 + \frac{\kappa_3+5\kappa_4}{6}\right)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 (1-\zeta)^p (1-\xi)^p d\zeta d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \zeta^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \Bigg]
\end{aligned}$$

elde edilir. ζ ve ξ 'ye göre integral alınarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296 (qs + 1)^{\frac{2}{q}}} \times \left[\frac{1}{(p+1)^{\frac{2}{p}}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha\left(\kappa_1 + \frac{5\kappa_3+\kappa_4}{6}\right)}} \right. \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \kappa_3 \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \kappa_3\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \frac{5\kappa_3+\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6}\right)}} \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \kappa_4 \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \kappa_4\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\kappa_2 + \frac{\kappa_3+5\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \Bigg) \\
& + \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4(p+1)} \right)^{\frac{2}{p}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \frac{5\kappa_3+\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2}\right)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+\kappa_2}{2} + \frac{5\kappa_3+\kappa_4}{6}\right)}} + 2 \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_3+\kappa_4}{2}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_3+5\kappa_4}{6}\right)}} \\
& \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6}\right)}} \right) \Bigg]
\end{aligned}$$

sonucuna ulaşılır.

Not 4.30 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.58 elde edilir.

Sonuç 4.8 Teorem 4.64'ün şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda ikinci anlamda s -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
 |J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296 (qs + 1)^{\frac{2}{q}}} \\
 & \times \left[\frac{1}{(p+1)^{\frac{2}{p}}} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \right. \right. \\
 & + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \\
 & + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right| \\
 & + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right| \right) \\
 & + \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4(p+1)} \right)^{\frac{2}{p}} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \right. \\
 & + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right| \\
 & + 2 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \\
 & + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right| \right) \Bigg].
 \end{aligned}$$

Teorem 4.65 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $p > 0$, $s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
 |J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
 & \times \left[\frac{8}{p(p+1)^2} + \frac{1}{q(qs+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6} \right)}} \right. \right. \\
 & + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3 \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6} \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} \\
 & + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right|^q}{e^{q\alpha \left(\frac{\kappa_1 + 5\kappa_2}{6} + \kappa_4 \right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\kappa_2 + \frac{\kappa_3 + 5\kappa_4}{6} \right)}} + \left. \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} \right)
 \end{aligned}$$

$$\begin{aligned}
& +8 \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4p(p+1)} \right)^2 + \frac{1}{q(qs+1)^2} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \frac{5\kappa_3+\kappa_4}{6}\right)}} \right. \\
& + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{5\kappa_1+\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2} + \frac{5\kappa_3+\kappa_4}{6}\right)}} + 2 \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_3+\kappa_4}{2}\right)}} \\
& \left. + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_3+5\kappa_4}{6}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2}\right)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|^q}{e^{q\alpha\left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6}\right)}} \right) \Bigg]
\end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.11 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_1| & \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\int_0^1 \int_0^1 \zeta \xi \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \kappa_1 + \zeta \frac{5\kappa_1 + \kappa_2}{6}, (1-\xi) \kappa_3 + \xi \frac{5\kappa_3 + \kappa_4}{6} \right) d\zeta d\xi \right. \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \frac{5\kappa_1 + \kappa_2}{6} + \zeta \frac{\kappa_1 + \kappa_2}{2}, (1-\xi) \frac{5\kappa_3 + \kappa_4}{6} + \xi \frac{\kappa_3 + \kappa_4}{2} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \frac{\kappa_1 + \kappa_2}{2} + \zeta \frac{\kappa_1 + 5\kappa_2}{6}, (1-\xi) \frac{\kappa_3 + \kappa_4}{2} + \xi \frac{\kappa_3 + 5\kappa_4}{6} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1-\zeta)(1-\xi) \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \left((1-\zeta) \frac{\kappa_1 + 5\kappa_2}{6} + \zeta \kappa_2, (1-\xi) \frac{\kappa_3 + 5\kappa_4}{6} + \xi \kappa_4 \right) d\zeta d\xi \Bigg]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda ikinci anlamada eksponansiyel s -konveks olduğu kullanılırsa;

$$\begin{aligned}
|J_1| & \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \times \left[\int_0^1 \int_0^1 \zeta \xi \right. \\
& \times \left((1-\zeta)^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + (1-\zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha\left(\kappa_1 + \frac{5\kappa_3 + \kappa_4}{6}\right)}} \right. \\
& \left. + \zeta^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|}{e^{\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \kappa_3\right)}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|}{e^{\alpha\left(\frac{5\kappa_1 + \kappa_2}{6} + \frac{5\kappa_3 + \kappa_4}{6}\right)}} \right) d\zeta d\xi
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right| \left| 4\xi - \frac{5}{2} \right| \\
& \times \left((1-\zeta)^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1+\kappa_2}{6} + \frac{5\kappa_3+\kappa_4}{6} \right)}} + (1-\zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{5\kappa_1+\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2} \right)}} \right. \\
& + \zeta^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1+\kappa_2}{2} + \frac{5\kappa_3+\kappa_4}{6} \right)}} + \left. \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_3+\kappa_4}{2} \right)}} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| 4\zeta - \frac{3}{2} \right| \left| 4\xi - \frac{3}{2} \right| \\
& \times \left((1-\zeta)^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_3+\kappa_4}{2} \right)}} + (1-\zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_3+5\kappa_4}{6} \right)}} \right. \\
& + \zeta^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+\kappa_4}{2} \right) \right|}{e^{\alpha \left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+\kappa_4}{2} \right)}} + \left. \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6} \right)}} \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 (1-\zeta)(1-\xi) \\
& \times \left((1-\zeta)^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\frac{\kappa_1+5\kappa_2}{6} + \frac{\kappa_3+5\kappa_4}{6} \right)}} + (1-\zeta)^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1+5\kappa_2}{6}, \kappa_4 \right) \right|}{e^{\alpha \left(\frac{\kappa_1+5\kappa_2}{6} + \kappa_4 \right)}} \right. \\
& \left. \zeta^s (1-\xi)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3+5\kappa_4}{6} \right) \right|}{e^{\alpha \left(\kappa_2 + \frac{\kappa_3+5\kappa_4}{6} \right)}} + \zeta^s \xi^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \kappa_4 \right) \right|}{e^{\alpha \left(\kappa_2 + \kappa_4 \right)}} \right) d\zeta d\xi \Bigg]
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned}
|J_1| & \leq \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} (1-\xi)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right) \right. \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\kappa_1 + \frac{5\kappa_3+\kappa_4}{6} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\xi)^{qs} \zeta^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \kappa_3 \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1+\kappa_2}{6} + \kappa_3 \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \zeta^p \xi^p d\zeta d\xi \right) + \frac{1}{q} \left(\int_0^1 \int_0^1 \zeta^{qs} \xi^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1+\kappa_2}{6} + \frac{5\kappa_3+\kappa_4}{6} \right)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| 4\zeta - \frac{5}{2} \right|^p \left| 4\xi - \frac{5}{2} \right|^p \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 (1-\zeta)^{qs} (1-\xi)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1+\kappa_2}{6}, \frac{5\kappa_3+\kappa_4}{6} \right) \right|^q}{e^{q\alpha \left(\frac{5\kappa_1+\kappa_2}{6} + \frac{5\kappa_3+\kappa_4}{6} \right)}} d\zeta d\xi \right)
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre integral alınarak;

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{8}{p(p+1)^2} + \frac{1}{q(qs+1)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \right. \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q \right) \\
& + 8 \left(\frac{\left(\frac{5}{2} \right)^{p+1} + \left(\frac{3}{2} \right)^{p+1}}{4p(p+1)} \right)^2 + \frac{1}{q(qs+1)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q + 2 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q \right) \right]
\end{aligned}$$

sonucuna ulaşılır.

Not 4.31 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.59 elde edilir.

Sonuç 4.9 Teorem 4.65'in şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda ikinci anlamda s -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
|J_1| \leq & \frac{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}{1296} \\
& \times \left[\frac{8}{p(p+1)^2} + \frac{1}{q(qs+1)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \right. \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \kappa_3 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \kappa_4 \right) \right|^q \\
& + \left. \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|^q \right)
\end{aligned}$$

$$\begin{aligned}
& + 8 \left(\frac{\left(\frac{5}{2}\right)^{p+1} + \left(\frac{3}{2}\right)^{p+1}}{4p(p+1)} \right)^2 + \frac{1}{q(qs+1)^2} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \right. \\
& + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{5\kappa_1 + \kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{5\kappa_3 + \kappa_4}{6} \right) \right|^q \\
& + 2 \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q \\
& \left. + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + \kappa_4}{2} \right) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{\kappa_1 + 5\kappa_2}{6}, \frac{\kappa_3 + 5\kappa_4}{6} \right) \right|^q \right).
\end{aligned}$$

Bu çalışma boyunca

$$\begin{aligned}
J_2 = & \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{72} [F(\kappa_1, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_2, \kappa_4)] \\
& + \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{18} \left[F\left(\kappa_1, \frac{\kappa_3 + \kappa_4}{2}\right) + F\left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2}\right) \right] \\
& - \frac{(\kappa_2 - \kappa_1)}{12} \left(\int_{\kappa_3}^{\kappa_4} F(\kappa_1, \tau_2) d\tau_2 + \int_{\kappa_3}^{\kappa_4} F(\kappa_2, \tau_2) d\tau_2 \right) \\
& + \frac{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{18} \left[F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3\right) + F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4\right) \right] \\
& + \frac{2(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)}{9} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& - \frac{(\kappa_2 - \kappa_1)}{3} \int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\
& - \frac{(\kappa_4 - \kappa_3)}{12} \left(\int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_3) d\tau_1 + \int_{\kappa_1}^{\kappa_2} F(\tau_1, \kappa_4) d\tau_1 \right) \\
& - \frac{(\kappa_4 - \kappa_3)}{3} \int_{\kappa_1}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 + \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} F(\tau_1, \tau_2) d\tau_1 d\tau_2
\end{aligned}$$

olarak alınacaktır.

Lemma 4.12 $F : \Delta \rightarrow R$ fonksiyonu Δ üzerinde ikinci mertebeden kısmi diferensiyellenebilen bir dönüşüm olsun. Eğer $\frac{\partial^2 F}{\partial \zeta \partial \xi} \in L_2(\Delta)$ ise aşağıdaki eşitsizlik geçerlidir.

$$\begin{aligned}
& J_2 \\
= & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left(\zeta - \frac{2}{3} \right) \left(\xi - \frac{2}{3} \right) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) d\zeta d\xi \right. \\
& + \int_0^1 \int_0^1 \left(\zeta - \frac{2}{3} \right) \left(\xi - \frac{2}{3} \right) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left(\zeta - \frac{2}{3} \right) \left(\frac{2}{3} - \xi \right) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) d\zeta d\xi \\
& \left. + \int_0^1 \int_0^1 \left(\frac{2}{3} - \zeta \right) \left(\xi - \frac{2}{3} \right) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) d\zeta d\xi \right].
\end{aligned}$$

İspat

$$G_1 = \int_0^1 \int_0^1 \left(\zeta - \frac{2}{3} \right) \left(\xi - \frac{2}{3} \right) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) d\zeta d\xi$$

$$\begin{aligned}
G_2 &= \int_0^1 \int_0^1 \left(\zeta - \frac{2}{3} \right) \left(\xi - \frac{2}{3} \right) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) d\zeta d\xi \\
G_3 &= \int_0^1 \int_0^1 \left(\zeta - \frac{2}{3} \right) \left(\frac{2}{3} - \xi \right) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) d\zeta d\xi \\
G_4 &= \int_0^1 \int_0^1 \left(\frac{2}{3} - \zeta \right) \left(\xi - \frac{2}{3} \right) \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) d\zeta d\xi
\end{aligned}$$

olsun. İki katlı integrallerde kısmi integrasyon kullanılarak aşağıdaki eşitlikler elde edilir.

$$\begin{aligned}
G_1 &= \frac{4}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F(\kappa_2, \kappa_4) + \frac{8}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2}\right) \\
&\quad - \frac{8}{3(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} F(\kappa_2, \tau_2) d\tau_2 \\
&\quad + \frac{8}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4\right) \\
&\quad + \frac{16}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
&\quad - \frac{16}{3(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\
&\quad - \frac{8}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)^2} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} F(\tau_1, \kappa_4) d\tau_1 \\
&\quad - \frac{16}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)^2} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
&\quad + \frac{16}{(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)^2} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} F(\tau_1, \tau_2) d\tau_1 d\tau_2,
\end{aligned}$$

$$\begin{aligned}
G_2 &= \frac{4}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F(\kappa_1, \kappa_3) + \frac{8}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\kappa_1, \frac{\kappa_3 + \kappa_4}{2}\right) \\
&\quad - \frac{8}{3(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} F(\kappa_1, \tau_2) d\tau_2 \\
&\quad + \frac{8}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3\right) \\
&\quad + \frac{16}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
&\quad - \frac{16}{3(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\
&\quad - \frac{8}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)^2} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} F(\tau_1, \kappa_3) d\tau_1 \\
&\quad - \frac{16}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)^2} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
&\quad + \frac{16}{(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)^2} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} F(\tau_1, \tau_2) d\tau_1 d\tau_2,
\end{aligned}$$

$$\begin{aligned}
G_3 = & \frac{4}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F(\kappa_2, \kappa_3) + \frac{8}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& - \frac{8}{3(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} F(\kappa_2, \tau_2) d\tau_2 \\
& + \frac{8}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3\right) \\
& + \frac{16}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& - \frac{16}{3(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)} \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\
& - \frac{8}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)^2} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} F(\tau_1, \kappa_3) d\tau_1 \\
& - \frac{16}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)^2} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
& + \frac{16}{(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)^2} \int_{\frac{\kappa_1 + \kappa_2}{2}}^{\kappa_2} \int_{\kappa_3}^{\frac{\kappa_3 + \kappa_4}{2}} F(\tau_1, \tau_2) d\tau_1 d\tau_2,
\end{aligned}$$

$$\begin{aligned}
G_4 = & \frac{4}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F(\kappa_1, \kappa_4) + \frac{8}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\kappa_1, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& - \frac{8}{3(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} F(\kappa_1, \tau_2) d\tau_2 \\
& + \frac{8}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4\right) \\
& + \frac{16}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) \\
& - \frac{16}{3(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} F\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right) d\tau_2 \\
& - \frac{8}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)^2} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} F(\tau_1, \kappa_4) d\tau_1 \\
& - \frac{16}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)^2} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} F\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right) d\tau_1 \\
& + \frac{16}{(\kappa_4 - \kappa_3)^2(\kappa_2 - \kappa_1)^2} \int_{\kappa_1}^{\frac{\kappa_1 + \kappa_2}{2}} \int_{\frac{\kappa_3 + \kappa_4}{2}}^{\kappa_4} F(\tau_1, \tau_2) d\tau_1 d\tau_2,
\end{aligned}$$

G_1, G_2, G_3 ve G_4 toplanıp her iki taraf $\frac{(\kappa_2 - \kappa_1)^2(\kappa_4 - \kappa_3)^2}{32}$ ile çarpılırsa istenen sonuç elde edilir.

Teorem 4.66 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \xi \partial \xi} \right| \in L_2(\Delta)$ ve $\alpha \in R$ için aşağıdaki

eşitsizlik geçerlidir:

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2 5^2}{3 \cdot 2^{12}} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right).$$

İspat Lemma 4.12 ve integrallerin mutlak değer özelliği kullanılarak;

$$\begin{aligned} |J_2| &\leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\ &\times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\ &\times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \\ &+ \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\ &\times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\ &+ \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\ &\times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\ &+ \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\ &\times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \Big] \end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel konveks özelliği kullanılırsa;

$$\begin{aligned} |J_2| &\leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\ &\times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \right. \\ &+ \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\ &+ \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\ &\times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \right. \\ &+ \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \Big] \end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \right) \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \right) \Bigg]
\end{aligned}$$

yazılır. Gerekli düzenlemeler yapılırsa;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right. \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \\
& + \left. \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \Big]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2 5^2}{3 \cdot 2^{12}} \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right)
\end{aligned}$$

elde edilir.

Sonuç 4.10 *Teorem 4.66'nın şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.*

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2 5^2}{3 \cdot 2^{12}} \\
& \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right| \right)
\end{aligned}$$

Teorem 4.67 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{128} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \left(\frac{(2^{q+1} - 1)^{\frac{1}{q}} + 1}{(q+1)^{\frac{1}{q}}} \right)^2 \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right)
\end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.12 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel konveks özelliği kullanılırsa;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \\
& \Big]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \right) \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \right) \Bigg]
\end{aligned}$$

yazılır. Gerekli düzenlemeler yapılırsa;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \left. \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \Bigg]
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \right. \\
& \times \left. \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \right. \\
& + \left. \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \right. \\
& \times \left. \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \right. \\
& + \left. \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \right. \\
& \times \left. \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \Bigg]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \\
& \times \left[\left(\frac{2^{q+1} - 1}{2^q (q+1)} \right)^{\frac{2}{q}} \right. \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \left(\frac{2^{q+1} - 1}{4^q (q+1)^2} \right)^{\frac{1}{q}} \\
& \times 2 \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right)
\end{aligned}$$

$$+ \left(\frac{1}{2^q (q+1)} \right)^{\frac{2}{q}} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \Bigg]$$

sonucuna ulaşılır. Gerekli düzenlemeler yapılarak;

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{128} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \left(\frac{(2^{q+1} - 1)^{\frac{1}{q}} + 1}{(q+1)^{\frac{1}{q}}} \right)^2 \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right)$$

sonucuna ulaşılır.

Sonuç 4.11 *Teorem 4.67 nin şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.*

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{128} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \left(\frac{(2^{q+1} - 1)^{\frac{1}{q}} + 1}{(q+1)^{\frac{1}{q}}} \right)^2 \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right| \right).$$

Teorem 4.68 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $q > 1$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 + \frac{4}{q (q+1)^2} \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \Bigg]$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.12 ve integraller için mutlak değer özelliği kullanılarak;

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \right]$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel konveks özelliği kullanılırsa;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \right)
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right. \\
& \left. + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \right) \Bigg]
\end{aligned}$$

yazılır. Gerekli düzenlemeler yapılrsa;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \Bigg]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \Big]
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \right. \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right) \Big]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 \right. \\
& + \frac{1}{q} \left(\frac{2^{q+1} - 1}{2^q (q+1)} \right)^2 \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \frac{1}{q} \left(\frac{2^{q+1} - 1}{4^q (q+1)^2} \right) \\
& \times 2 \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \frac{1}{q} \left(\frac{1}{2^q (q+1)} \right)^2 \\
& \times \left. \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \right]
\end{aligned}$$

elde edilir. Gerekli düzenlemeler yapılırsa;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 + \frac{4}{q (q+1)^2} \right. \\
& \times \left. \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \right]
\end{aligned}$$

sonucuna ulaşılır.

Sonuç 4.12 *Teorem 4.68 şartları altında $\alpha = 0$ alınrsa sonuç koordinatlarda konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.*

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 + \frac{4}{q (q+1)^2} \right. \\
& \times \left. \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right| \right) \right].
\end{aligned}$$

Teorem 4.69 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel m -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $m \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{305}{5832} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) + \frac{145}{5832} m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \right]$$

İspat Lemma 4.12 ve integraller için mutlak değer özelliği kullanılarak;

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \right]$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel m -konveks özelliği kullanılarak;

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + m \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + m \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \right]$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + m \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_4}{m})}} \right. \\
& + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + m \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_4}{m})}} \Bigg) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + m \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_4}{m})}} \right. \\
& + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + m \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_4}{m})}} \Bigg) \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + m \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_3}{m})}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + m \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_3}{m})}} \right) \Bigg]
\end{aligned}$$

yazılır. Gerekli düzenlemeler yapılırsa;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_3}{m})}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_3}{m})}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \left. \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_4}{m})}} \right]
\end{aligned}$$

elde edilir. Gerekli düzenlemeler yapılarak;

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{305}{5832} \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \right. \\ \left. + \frac{145}{5832} m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \right]$$

sonucuna ulaşılır.

Not 4.32 Yukarıdaki eşitsizlikte $m = 1$ alınırsa Teorem 4.66 elde edilir.

Sonuç 4.13 Teorem 4.69'un şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda m -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{305}{5832} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right| \right) \right. \\ \left. + \frac{145m}{5832} \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right| \right) \right]$$

Teorem 4.70 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $q > 1$, $m \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{128} \times \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \\ \times \left[\frac{(2^{q+1} - 1)^{\frac{2}{q}} + (2^{q+1} - 1)^{\frac{1}{q}}}{(q+1)^{\frac{2}{q}}} \right. \\ \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\ \left. + \frac{(2^{q+1} - 1)^{\frac{1}{q}} + 1}{(q+1)^{\frac{2}{q}}} \right. \\ \left. \times m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \right]$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.12 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel m -konveks özelliği kullanılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + m \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} \right. \\
& + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + m \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \Big) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + m \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \right. \\
& + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + m \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \Big) \\
& \Big]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + m \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_4}{m})}} \right. \\
& + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + m \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_4}{m})}} \Bigg) \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + m \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_3}{m})}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + m \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_3}{m})}} \right) \Bigg]
\end{aligned}$$

yazılır. Gerekli düzenlemeler yapılırsa;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_3}{m})}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_3}{m})}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_4}{m})}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_4}{m})}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \Bigg]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \kappa_3 \right) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \kappa_4 \right) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \kappa_4 \right) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} \Bigg]
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \right. \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \kappa_4 \right) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_3}{m})}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \kappa_4 \right) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_3}{m})}} d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha(\kappa_1+\frac{\kappa_4}{m})}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha(\kappa_2+\frac{\kappa_4}{m})}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha(\kappa_2+\frac{\kappa_4}{m})}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha(\kappa_1+\frac{\kappa_4}{m})}} d\zeta d\xi \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|^q}{e^{q\alpha(\kappa_1+\frac{\kappa_3}{m})}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|^q}{e^{q\alpha(\kappa_2+\frac{\kappa_3}{m})}} d\zeta d\xi \right)^{\frac{1}{q}} \Bigg]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \\
& \times \left[\left(\frac{2^{q+1} - 1}{2^q (q+1)} \right)^{\frac{2}{q}} \right. \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \left(\frac{2^{q+1} - 1}{4^q (q+1)^2} \right)^{\frac{1}{q}} \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \left(\frac{2^{q+1} - 1}{4^q (q+1)^2} \right)^{\frac{1}{q}} \\
& \times m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2+\frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1+\frac{\kappa_3}{m})}} \right)
\end{aligned}$$

$$+ \left(\frac{1}{2^q (q+1)} \right)^{\frac{2}{q}} \\ \times m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \Bigg]$$

elde edilir. Gerekli düzenlemeler yapılarak;

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{128} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \\ \times \left[\frac{(2^{q+1} - 1)^{\frac{2}{q}} + (2^{q+1} - 1)^{\frac{1}{q}}}{(q+1)^{\frac{2}{q}}} \right. \\ \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\ \left. + \frac{(2^{q+1} - 1)^{\frac{1}{q}} + 1}{(q+1)^{\frac{2}{q}}} \right. \\ \times m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \Bigg]$$

sonucuna elde edilir.

Not 4.33 Yukarıdaki eşitsizlikte $m = 1$ alınırsa Teorem 4.67 elde edilir.

Sonuç 4.14 Teorem 4.70'in şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda m -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$|J_2| \\ \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{128} \times \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \\ \times \left[\frac{(2^{q+1} - 1)^{\frac{2}{q}} + (2^{q+1} - 1)^{\frac{1}{q}}}{(q+1)^{\frac{2}{q}}} \right. \\ \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right| \right) \\ \left. + \frac{(2^{q+1} - 1)^{\frac{1}{q}} + 1}{(q+1)^{\frac{2}{q}}} \right. \\ \times m \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right| \right) \Bigg].$$

Teorem 4.71 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda eksponansiyel konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $q > 1$, $m \in (0, 1]$ ve

$\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 \right. \\
& + \frac{(2^{q+1} - 1) 2^{q+1}}{q 4^q (q+1)^2} \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \right) \\
& + \frac{2^{q+1}}{q 4^q (q+1)^2} \\
& \times m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \Big]
\end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.12 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda eksponansiyel m -konveks özelliği kullanılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} \right. \\
& + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \right. \\
& + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \right. \\
& + \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} \right) \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} \right) \Bigg]
\end{aligned}$$

yazılır. Gerekli düzenlemeler yapılırsa;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1+\xi}{2} \right) \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \kappa_3 \right) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}}
\end{aligned}$$

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \right. \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_3}{m})}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_3}{m})}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_1 + \frac{\kappa_4}{m})}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q \frac{m \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_4}{m}) \right|^q}{e^{q\alpha(\kappa_2 + \frac{\kappa_4}{m})}} d\zeta d\xi \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q m \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha(\kappa_2+\frac{\kappa_4}{m})}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q m \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right|^q}{e^{q\alpha(\kappa_1+\frac{\kappa_4}{m})}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q m \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right|^q}{e^{q\alpha(\kappa_1+\frac{\kappa_3}{m})}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1+\xi}{2} \right)^q \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^q \left(\frac{1-\xi}{2} \right)^q m \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right|^q}{e^{q\alpha(\kappa_2+\frac{\kappa_3}{m})}} d\zeta d\xi \right) \Bigg]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 \right. \\
& + \frac{1}{q} \left(\frac{2^{q+1} - 1}{2^q (q+1)} \right)^2 \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\
& + \frac{1}{q} \left(\frac{2^{q+1} - 1}{4^q (q+1)^2} \right) \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\
& + \frac{1}{q} \left(\frac{2^{q+1} - 1}{4^q (q+1)^2} \right) \\
& \times m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \\
& + \frac{1}{q} \left(\frac{1}{2^q (q+1)} \right)^2 \\
& \times m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \Big]
\end{aligned}$$

elde edilir. Gerekli düzenlemeler yapılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 \right. \\
& + \frac{(2^{q+1} - 1) 2^{q+1}}{q 4^q (q+1)^2} \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_3)}} \right) \\
& + \frac{2^{q+1}}{q 4^q (q+1)^2} \\
& \times m \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_3}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \frac{\kappa_4}{m}) \right|}{e^{\alpha(\kappa_2 + \frac{\kappa_4}{m})}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \frac{\kappa_3}{m}) \right|}{e^{\alpha(\kappa_1 + \frac{\kappa_3}{m})}} \right) \Big]
\end{aligned}$$

sonucu elde edilir.

Not 4.34 Yukarıdaki eşitsizlikte $m = 1$ alınırsa Teorem 4.68 elde edilir.

Sonuç 4.15 Teorem 4.71'in şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda m -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 \right. \\
& + \frac{(2^{q+1} - 1) 2^{q+1}}{q 4^q (q+1)^2} \\
& \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q \right) \\
& + \frac{2^{q+1}}{q 4^q (q+1)^2} \\
& \times m \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_3}{m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_4}{m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_2, \frac{\kappa_4}{m} \right) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\kappa_1, \frac{\kappa_3}{m} \right) \right| \right) \Big].
\end{aligned}$$

Teorem 4.72 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right| \in L_2(\Delta)$, $s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \left(\frac{2 \cdot 5^{s+2} + 6^{s+1} (s-4) - 3^{s+1} 6 + 2}{6^s 9 (s+1) (s+2)} \right)^2 \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right).
\end{aligned}$$

İspat Lemma 4.12 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks özelliği

kullanılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \right. \\
& + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \right. \\
& + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \right) \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \right) \Bigg]
\end{aligned}$$

yazılır. Gerekli düzenlemeler yapılırsa;

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\left(\frac{2 \cdot 5^{s+2} + 6^{s+1} (s-4) - 3^{s+1} (2s+7)}{6^s 9 (s+1) (s+2)} \right)^2 \right. \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\
& + \frac{2 \cdot 5^{s+2} + 6^{s+1} (s-4) - 3^{s+1} (2s+7)}{6^s 9 (s+1) (s+2)} \times \frac{3^{s+1} (2s+1) + 2}{6^s 9 (s+1) (s+2)} \\
& \times 2 \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\
& + \left(\frac{3^{s+1} (2s+1) + 2}{6^s 9 (s+1) (s+2)} \right)^2 \\
& \times \left. \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \right]
\end{aligned}$$

elde edilir. Gerekli düzenlemeler yapılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left(\frac{2 \cdot 5^{s+2} + 6^{s+1} (s-4) - 3^{s+1} 6 + 2}{6^s 9 (s+1) (s+2)} \right)^2 \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right)
\end{aligned}$$

sonucuna ulaşılır.

Not 4.35 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.66 elde edilir.

Sonuç 4.16 Teorem 4.72'nin şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda ikinci anlamda s -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \left(\frac{2 \cdot 5^{s+2} + 6^{s+1} (s-4) - 3^{s+1} 6 + 2}{6^s 9 (s+1) (s+2)} \right)^2 \\
& \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right| \right).
\end{aligned}$$

Teorem 4.73 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $q > 1$, $s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{128} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \left(\frac{(2^{qs+1} - 1)^{\frac{1}{q}} + 1}{(qs+1)^{\frac{1}{q}}} \right)^2 \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right)
\end{aligned}$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.12 ve integraller için mutlak değer özelliği kullanılarak;

$$\begin{aligned}
 |J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
 & \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
 & \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \\
 & + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
 & \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
 & + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
 & \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
 & + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
 & \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \Big]
 \end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks özelliği kullanılarak;

$$\begin{aligned}
 |J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
 & \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
 & \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \right. \\
 & + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\
 & + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
 & \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \right. \\
 & + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \\
 & \Big]
 \end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \right) \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \right) \Bigg]
\end{aligned}$$

yazılır. Gerekli düzenlemeler yapılarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \left. \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \Bigg]
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Hölder eşitsizliği uygulanarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \right. \\
& \times \left. \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^{qs} \left(\frac{1+\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \right. \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left. \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^{qs} \left(\frac{1-\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2 + \kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \right. \\
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left. \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^{qs} \left(\frac{1+\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1 + \kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^{qs} \left(\frac{1-\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^{qs} \left(\frac{1+\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^{qs} \left(\frac{1-\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^{qs} \left(\frac{1+\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^{qs} \left(\frac{1-\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right)^{\frac{1}{q}} \Bigg]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^{\frac{2}{p}} \\
& \times \left[\left(\frac{2^{qs+1} - 1}{2^{qs} (qs+1)} \right)^{\frac{2}{q}} \right. \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \left(\frac{2^{qs+1} - 1}{4^{qs} (qs+1)^2} \right)^{\frac{1}{q}} \\
& \times 2 \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right)
\end{aligned}$$

$$+ \left(\frac{1}{2^{qs}(qs+1)} \right)^{\frac{2}{q}} \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| \right) \Bigg]$$

elde edilir. Gerekli düzenlemeler yapılarak;

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{128} \left(\frac{2^{p+1} + 1}{3^{p+1}(p+1)} \right)^{\frac{2}{p}} \left(\frac{(2^{qs+1} - 1)^{\frac{1}{q}} + 1}{(qs+1)^{\frac{1}{q}}} \right)^2 \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| \right)$$

sonucuna ulaşılır.

Not 4.36 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.67 elde edilir.

Sonuç 4.17 Teorem 4.73 ün şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda ikinci anlamda s -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{128} \left(\frac{2^{p+1} + 1}{3^{p+1}(p+1)} \right)^{\frac{2}{p}} \left(\frac{(2^{qs+1} - 1)^{\frac{1}{q}} + 1}{(qs+1)^{\frac{1}{q}}} \right)^2 \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| \right)$$

Teorem 4.74 $F : \Delta \rightarrow R$ kısmi türevlenebilir fonksiyonu Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks fonksiyon olsun. Bu takdirde $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|^q \in L_2(\Delta)$, $q > 1$, $s \in (0, 1]$ ve $\alpha \in R$ için aşağıdaki eşitsizlik geçerlidir:

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1}(p+1)} \right)^2 + \frac{4}{q(qs+1)^2} \times \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right| \right) \right]$$

burada $p^{-1} + q^{-1} = 1$ dir.

İspat Lemma 4.12 ve integraller için mutlak değer özelliği kullanılarak;

$$|J_2| \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \right]$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_2 + \frac{1-\zeta}{2} \kappa_1, \frac{1+\xi}{2} \kappa_3 + \frac{1-\xi}{2} \kappa_4 \right) \right| d\zeta d\xi \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \left(\frac{1+\zeta}{2} \kappa_1 + \frac{1-\zeta}{2} \kappa_2, \frac{1+\xi}{2} \kappa_4 + \frac{1-\xi}{2} \kappa_3 \right) \right| d\zeta d\xi \Big]
\end{aligned}$$

yazılır. $\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} \right|$, Δ üzerinde koordinatlarda ikinci anlamda eksponansiyel s -konveks özelliği kullanılarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \right. \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} \right) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right) \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2 + \kappa_3)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2 + \kappa_4)}} \right. \\
& + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1 + \kappa_3)}} + \left. \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1 + \kappa_4)}} \right)
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \\
& \times \left(\left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right. \\
& \left. + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \right) \Big]
\end{aligned}$$

yazılır. Gerekli düzenlemeler yapılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \right. \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right| \left| \frac{2}{3} - \xi \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi}(\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}}
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1+\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1+\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& + \int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right| \left| \xi - \frac{2}{3} \right| \left(\frac{1-\zeta}{2} \right)^s \left(\frac{1-\xi}{2} \right)^s \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} \Bigg]
\end{aligned}$$

yazılır. Eşitsizliğin sağ tarafına iki katlı integraller için Young eşitsizliği uygulanarak;

$$\begin{aligned}
|J_2| & \leq \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \\
& \times \left[\frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \right. \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^{qs} \left(\frac{1+\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^{qs} \left(\frac{1-\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^{qs} \left(\frac{1+\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^{qs} \left(\frac{1-\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^{qs} \left(\frac{1+\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \zeta - \frac{2}{3} \right|^p \left| \xi - \frac{2}{3} \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1+\zeta}{2} \right)^{qs} \left(\frac{1-\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}} d\zeta d\xi \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^{qs} \left(\frac{1+\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}} d\zeta d\xi \right) \\
& + \frac{1}{p} \left(\int_0^1 \int_0^1 \left| \frac{2}{3} - \zeta \right|^p \left| \frac{2}{3} - \xi \right|^p d\zeta d\xi \right) \\
& + \frac{1}{q} \left(\int_0^1 \int_0^1 \left(\frac{1-\zeta}{2} \right)^{qs} \left(\frac{1-\xi}{2} \right)^{qs} \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}} d\zeta d\xi \right) \Bigg]
\end{aligned}$$

yazılır. ζ ve ξ 'ye göre ayrı ayrı integral alınarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 \right. \\
& + \frac{1}{q} \left(\frac{2^{qs+1} - 1}{2^{qs} (qs+1)} \right)^2 \\
& \times \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \frac{1}{q} \left(\frac{2^{qs+1} - 1}{4^{qs} (qs+1)^2} \right) \\
& \times 2 \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \\
& + \frac{1}{q} \left(\frac{1}{2^{qs} (qs+1)} \right)^2 \\
& \times \left. \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \right]
\end{aligned}$$

elde edilir. Gerekli düzenlemeler yapılarak;

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \times \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 + \frac{4}{q (qs+1)^2} \right. \\
& \times \left. \left(\frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right|}{e^{\alpha(\kappa_2+\kappa_3)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right|}{e^{\alpha(\kappa_1+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right|}{e^{\alpha(\kappa_2+\kappa_4)}} + \frac{\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right|}{e^{\alpha(\kappa_1+\kappa_3)}} \right) \right]
\end{aligned}$$

sonucuna ulaşılır.

Not 4.37 Yukarıdaki eşitsizlikte $s = 1$ alınırsa Teorem 4.68 elde edilir.

Sonuç 4.18 Teorem 4.74 şartları altında $\alpha = 0$ alınırsa sonuç koordinatlarda ikinci anlamda s -konvekslik sonucu olarak aşağıdaki gibi karşımıza çıkar.

$$\begin{aligned}
|J_2| \leq & \frac{(\kappa_4 - \kappa_3)^2 (\kappa_2 - \kappa_1)^2}{32} \left[\frac{16}{p} \left(\frac{2^{p+1} + 1}{3^{p+1} (p+1)} \right)^2 + \frac{4}{q (qs+1)^2} \right. \\
& \times \left. \left(\left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_3) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_2, \kappa_4) \right| + \left| \frac{\partial^2 F}{\partial \zeta \partial \xi} (\kappa_1, \kappa_3) \right| \right) \right].
\end{aligned}$$

5. TARTIŞMA VE SONUÇ

Bu çalışmanın ana bölümünü oluşturan Araştırma Bulgular bölümünde, farklı türden koordinatlarda eksponansiyel konveks fonksiyonlar ile ilgili iki katlı integraller için Hölder ve Young eşitsizlikleri kullanılarak elde edilmiş koordinatlarda çeşitli integral eşitsizlikler, yeni koordinatlarda eksponansiyel konveks fonksiyon sınıfları ve bu sınıfların bazı özellikleri verilmiştir. Bulunan sonuçların bazı özel halleri literatürde daha önce mevcut olan sonuçlarla uyushmaktadır.

Bu tez çalışmasında verilen lemmalara benzer lemmalar üretilerek koordinatlarda yeni integral eşitsizlikler elde edilebilir.

Konuyla ilgilenen araştırmacılar, dördüncü bölüm olan Araştırma Bulgular bölümünde verilen ve kullanılan lemmaları farklı konveks fonksiyon sınıflarına uygulayabilir ve yeni integral eşitsizlikler elde edebilirler.

Kaynakça

- Akdemir, A. O. (2012). *Farklı Türden Konveks Fonksiyonlar İçin Koordinatlarda İntegral Eşitsizlikler*. PhD thesis, Doktora Tezi, Fen Bilimleri Enstitüsü, Atatürk Üniversitesi, Erzurum.
- Akdemir, A. O., Aslan, S., and Ekinçi, A. (2022a). Some new inequalities for exponentially P –functions on the coordinates. In *5th International Conference on Mathematical and Related Sciences Book of Proceedings*, page 94.
- Akdemir, A. O., Aslan, S., and Set, E. (2022b). Some new inequalities for exponentially Quasi-convex functions on the coordinates and related Hadamard type integral inequalities. In *5th International Conference on Mathematical and Related Sciences Book of Proceedings*, page 120.
- Akdemir, A. O., Deniz, E., and Yüksel, E. (2019). Some new generalizations for m –convexity via new conformable fractional integral operators. *Mathematica Moravica*, 23(2):69–77.
- Akdemir, A. O. and Özdemir, M. E. (2010). Some Hadamard-type inequalities for coordinated P –convex functions and Godunova-Levin functions. In *AIP Conference Proceedings*, volume 1309, pages 7–15.
- Alomari, M. (2011). *Several Inequalities of Hermite-Hadamard, Ostrowski and Simpson type for s –convex, Quasi-convex and r –convex mappings and applications*. PhD thesis, Citeseer.
- Alomari, M. and Darus, M. (2008a). Co-ordinated s –convex function in the first sense with some Hadamard-type inequalities. *Int. J. Contemp. Math. Sci*, 3(32):1557–1567.
- Alomari, M. and Darus, M. (2008b). The Hadamard’s inequality for s –convex function of 2-variables on the co-ordinates. *Int. J. Math. Anal*, 2(13):629–638.
- Antczak, T. (2001). (p, r) –invex sets and functions. *Journal of Mathematical Analysis and Applications*, 263(2):355–379.

- Aslan, S. and Akdemir, A. O. (2022). Exponentially convex functions on the co-ordinates and related integral inequalities. In *Proceedings of the 8th International Conference on Control and Optimization with Industrial Applications*, volume 2, pages 120–122.
- Aslan, S., Akdemir, A. O., and Dokuyucu, M. A. (2022). Exponentially m – and (α, m) –convex functions on the coordinates and related inequalities. *Turkish Journal of Science*, 7(3):231–244.
- Awan, M. U., Noor, M. A., and Noor, K. I. (2018). Hermite-Hadamard inequalities for exponentially convex functions. *Appl. Math. Inf. Sci*, 12(2):405–409.
- Bayraktar, M. (2000). *Fonksiyonel Analiz*. Gazi Kitabevi.
- Beckenbach, E. and Bellman, R. (1961). *Inequalities* springer/verlag. Berlin, Heidelberg, New York.
- Breckner, W. W. (1978). Stetigkeitsaussagen für eine klasse aller gemeinerter konvexer funktionen in topologisch linearen räumen. *Publ. Inst. Math.(Beograd)(NS)*, 23(37):13–20.
- Dragomir, S. S. (2001). On the Hadamard’s inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese Journal of Mathematics*, pages 775–788.
- Dragomir, S. S., Agarwal, R. P., and Barnett, N. S. (1999). Inequalities for Beta and Gamma functions via some classical and new integral inequalities. *RGMIA research report collection*, 2(3).
- Dragomir, S. S. and Pearce, C. E. M. (1998). Quasi-convex functions and Hadamard’s inequality. *Bulletin of the Australian Mathematical Society*, 57(3):377–385.
- Dragomir, S. S., Pečarić, J., and Persson, L. E. (1995). Some inequalities of Hadamard type. *Soochow J. Math*, 21(3):335–341.
- Ekinci, A. (2014). *Klasik Eşitsizlikler Yoluyla Konveks Fonksiyonlar için İntegral Eşitsizlikler*. PhD thesis, Doktora Tezi, Atatürk Üniversitesi, Erzurum.

- Godunova, E. K. and Levin, V. I. (1985). Neravenstva dlja funkicii širokogo klassa, soderzascego vypuklye, monotonnnye i nekotorye drugie vidy funkii. vycislitel. mat. i. Fiz. Mezhvuzov. Sb. Nauc. Trudov, MGPI, Moskva, 9:138–142.
- Gürbüz, M. (2013). *Farklı Türden Konveks Fonksiyonların Çarpımı Üzerine İntegral Eşitsizlikleri ve Uygulamaları*. PhD thesis, Doktora Tezi, Fen Bilimleri Enstitüsü, Atatürk Üniversitesi, Erzurum.
- Greenberg, H. J. and Pierskalla, W. P. (1971). A review of Quasi-convex functions. *Operations research*, 19(7):1553–1570.
- Kavurmacı, H. (2012). *Bazı farklı türden konveks fonksiyonlar için Ostrowski ve Hermite-Hadamard tipli integral eşitsizlikleri*. PhD thesis, Doktora Tezi, Atatürk Üniversitesi, Erzurum.
- Latif, M. A. and Alomari, M. (2009a). Hadamard-type inequalities for product two convex functions on the co-ordinates. In *Int. Math. Forum*, volume 4, pages 2327–2338.
- Latif, M. A. and Alomari, M. (2009b). On Hadamard-type inequalities for h –convex functions on the co-ordinates. *Int. J. of Math. Analysis*, 3(33):1645–1656.
- Mihesan, V. G. (1993). A generalization of the convexity. In *Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania*.
- Mitrinovic, D. S., Pecaric, J., and Fink, A. M. (1993). *Classical and new inequalities in analysis*, volume 61. Springer Science & Business Media.
- Mitrinovic, D. S. and Vasic, P. M. (1970). *Analytic inequalities*, volume 1. Springer.
- Niculescu, C. and Persson, L. (2006). *Convex functions and their applications*, volume 23. Springer.
- Noor, M. A. and Noor, K. I. (2019). On exponentially convex functions. *Journal of Orissa Mathematical Society ISSN*, 975:2323.
- Orlicz, W. (1961). A note on modular spaces i. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astro-*

nom. Phys, 9:157–162.

- Özdemir, M. E., Akdemir, A. O., and Ekinçi, A. (2021). New integral inequalities for co-ordinated convex functions. *Fundamentals of Contemporary Mathematical Sciences*, 2(1):52–69.
- Özdemir, M. E., Akdemir, A. O., and Set, E. (2016). On $(h - m)$ -convexity and Hadamard-type inequalities. *Transylv. J. Math. Mech.*, 8:51–58.
- Özdemir, M. E., Akdemir, A. O., and Yıldız, Ç. (2012a). On co-ordinated Quasi-convex functions. *Czech Math J*, 62(4):889–900.
- Özdemir, M. E., Kavurmaci, H., Akdemir, A. O., and Avcı, M. (2012b). Inequalities for convex and s -convex functions on $\delta=[a, b] \times [c, d]$. *Journal of Inequalities and Applications*, 2012(1):1–19.
- Özdemir, M. E., Set, E., and Sarıkaya, M. Z. (2011a). Some new Hadamard type inequalities for co-ordinated m -convex and (α, m) -convex functions. *Hacettepe Journal of Mathematics and Statistics*, 40(2):219–229.
- Özdemir, M. E., Yıldız, Ç., and Akdemir, A. O. (2011b). On the coordinated convex functions. *arXiv preprint arXiv:1107.4033*.
- Pachpatte, B. (2005). On some inequalities for convex functions. *RGMIA Res. Rep. Coll*, 6(1):1–9.
- Park, J. K. (2010). Hermite-Hadamard-type inequalities for real α -star s -convex mappings. *Journal of Applied Mathematics & Informatics*, 28(5_6):1507–1518.
- Pečarić, J. E. and Tong, Y. L. (1992). *Convex functions, partial orderings, and statistical applications*. Academic press.
- Rashid, S., Noor, M. A., and Noor, K. I. (2019). Fractional exponentially m -convex functions and inequalities. *International Journal of Analysis and Applications*, 17(3):464–478.
- Roberts, A. and Varberg, D. (1973). *Convex functions* academic press. New York, 62.

- Sarikaya, M. Z., Set, E., Özdemir, M. E., and Dragomir, S. S. (2012). New some Hadamard's type inequalities for co-ordinated convex functions. *Tamsui Oxford Journal of Information and Mathematical Sciences*, 28(2):137–152.
- Set, E. (2010). *Bazı Farklı Türden Konveks Fonksiyonlar İçin İntegral Eşitsizlikleri*. PhD thesis, Doktora Tezi, Fen Bilimleri Enstitüsü, Atatürk Üniversitesi, Erzurum.
- Toader, G. H. (1984). Some generalizations of the convexity. In *Proceedings of the Colloquium on Approximation and Optimization*, volume 329, page 338. University of Cluj-Napoca Cluj-Napoca, Romania.
- Varošanec, S. (2007). On h –convexity. *Journal of Mathematical Analysis and Applications*, 326(1):303–311.
- Yıldız, Ç. (2011). *Quasi Konveks Fonksiyonlar İçin Eşitsizlikler ve Uygulamaları*. PhD thesis, Yüksek Lisans Tezi, Fen Bilimleri Enstitüsü, Atatürk Üniversitesi, Erzurum.

ÖZGEÇMİŞ

1984 tarihinde Ağrı’da doğdu. İlk, orta ve lise öğrenimini Ağrı’da tamamladı. 2002’de kayıt yaptırdığı Karadeniz Teknik Üniversitesi Fatih Eğitim Fakültesi İlköğretim Matematik Öğretmenliği bölümünden 2006’da mezun oldu. Aynı yıl İsmet Ömeroğlu İlköğretim Okulunda Matematik Öğretmeni olarak göreve başladı. 2011 yılında Ağrı İbrahim Çeçen Üniversitesi Fen Edebiyat Fakültesi Matematik Bölümüne kayıt yaptırıp 2015’te mezun oldu. 2015-2017 yılları arasında Ağrı İbrahim Çeçen Üniversitesi Fen Bilimleri Enstitüsü Matematik Anabilim Dalında yüksek lisan öğrenimini tamamladı. 2019’da Giresun Üniversitesinde başladığı doktora öğrenimini 2021’de Ağrı İbrahim Çeçen Üniversitesine yatay geçiş yaparak 2023’te tamamladı. Görevine Ağrı Türk Telekom Sosyal Bilimler Lisesinde uzman öğretmen olarak devam etmektedir. Evli ve üç çocuk babasıdır.

YAPTIĞI BİLİMSEL ÇALIŞMLAR

1. İşcan, İ., Set, E., Akdemir, A. O., Ekinici, A., and Aslan, S. (2023). Some new integral inequalities via generalized proportional fractional integral operators for the classes of m –logarithmically convex functions. In *Fractional Order Systems and Applications in Engineering* (pp. 157-173). Academic Press.
2. Akdemir, A. O., Aslan, S., Dokuyucu, M. A., and Çelik, E. Exponentially Convex Functions on the Coordinates and Novel Estimations via Riemann-Liouville Fractional Operator. *Journal of Function Spaces*, 2023.
3. Aslan, S., and Akdemir, A. O. (2022, August). Exponentially convex functions on the co-ordinates and related integral inequalities. In *Proceedings of the 8th International Conference on Control and Optimization with Industrial Applications* (Vol. 2, pp. 120-122).
4. Dokuyucu, M. A., and Aslan, S. (2022). Some New Approaches for Geometrically Convex Functions. *9Roxph*, 156.
5. Çakaloğlu, M. N., Aslan, S., and Akdemir, A. O. (2021). Hadamard Type Integral Inequalities for Differentiable $(h - m)$ –Convex Functions. *Eastern Anatolian Journal of Science*, 7(1), 12-18.

6. Butt, S. I., Akdemir, A. O., Aslan, S., İşcan, İ., and Agarwal, P. (2021). Some new integral inequalities via general forms of proportional fractional integral operators. *Contrib. Math*, 4, 6-11.
7. Aslan, S., Akdemir, A. O., and Dokuyucu, M. A. Exponentially m - and (α, m) -Convex Functions on the Coordinates and Related Inequalities. *Turkish Journal of Science*, 7(3), 231-244.
8. Akdemir, A. O., Aslan, S., and Ekinici, A. (2022). Novel Approaches for s -convex functions via Caputo-Fabrizio Fractional Integrals. *Proceedings of IAM*, 11(1), 3-16.
9. Akdemir, A. O., Aslan, S., and Ekinici, A. (2022, October). Some New Inequalities for Exponentially P - Functions on the Coordinates. *5th International Conference on Mathematics and Related Sciences Book of Proceedings* (p.94).
10. Akdemir, A. O., Aslan, S., and Set, E. (2022, October). Some New Inequalities for Exponentially Quasi-Convex Functions on the Coordinates and Related Hadamard Type Integral Inequalities. *5th International Conference on Mathematics and Related Sciences Book of Proceedings* (p. 120).
11. Aslan, S., Butt, S. I., and Akdemir, A. O. Some Novel Integral Inequalities for Product of Geometrically Convex Functions. *5th International Conference on Life and Engineering Sciences (ICOLES-2022)*, Alanya, Turkey (p.466-469)
12. Akdemir, A. O., and Aslan, S.(2021) Some New Fractional Order Integral Inequalities for Logarithmically Convex Functions. In *5th International Conference on Advances in Natural and Applied Sciences* (p.123)
13. Akdemir, A. O., Set, E., Ekinici, A., and Aslan, S.(2021) On some integral inequalities via generalized proportional fractional integral operators. *5th International Conference on Computational Mathematics and Engineering Sciences* (p.136)
14. Akdemir, A.O., Aslan, S., Çakaloğlu, M. N., and Set, E. New Hadamard Type Integral Inequalities via Caputo-Fabrizio Fractional Operators. *4th International Conference on Mathematical and Related Sciences. (ICMRS-2021)*. (p.91)
15. Çakaloğlu, M. N., Akdemir, A. O., Set, E and Aslan, S. New Integral Inequalities of Hadamard Type via Katugampola Fractional Operator for m -Convex Functions.

- 4th International Conference on Mathematical and Related Sciences. (ICMRS-2021). (p.93)
16. Akdemir, A. O., Aslan, S., Çakaloğlu, M. N., and Ekinçi A. Some New Results for Different Kinds of Convex Functions Caputo- Fabrizio Fractional Operators. 4th International Conference on Mathematical and Related Sciences. (ICMRS-2021). (p.92)
 17. İşcan, İ., Akdemir, A. O., Aslan, S., and Butt S. I. Some New Integral Inequalities via General Forms of Proportional Fractional Integral Operators. 3rd International Conference on Life and Engineering Sciences, İstanbul, Turkey (ICOLES-2020). (p.25)
 18. Aslan, S., Akdemir, A. O. On Some Hadamard Type Inequalities on A Rectangular Box. 3rd International Conference on Life and Engineering Sciences, İstanbul, Turkey Icoles 2020. (p.9)
 19. Butt, S. I., Ekinçi, A., Akdemir, A. O., and Aslan, S. (2020, November). Inequalities for Geometrically Convex Functions. 3rd International Conference on Mathematical and Related Sciences: Current Trends and Developments Proceedings Book (Vol. 1, p. 238).
 20. Rashid, S., Akdemir, A. O., Ekinçi, A., and Aslan, S. (2020, November). Some Fractional Integral Inequalities for Geometrically Convex Functions. 3rd International Conference on Mathematical and Related Sciences: Current Trends and Developments Proceedings Book (p. 246).
 21. Khan, T. U., Aslan, S., Özdemir, M. E., and Ekinçi, A. (2020, November). Some New Fractional Inequalities for Quasi-Convex Functions. 3rd International Conference on Mathematical and Related Sciences: Current Trends and Developments Proceedings Book (p. 291).
 22. Tariq, M., Set, E., Akdemir, A. O. and Aslan, S. Generalized Proportional Fractional Integral Operators and Related Inequalities. 3rd International Conference on Mathematical and Related Sciences: Current Trends and Developments ICMRS 2020. (p.42)
 23. Aslan, S., Akdemir, A. O., and Çinar, Y. (2016, April). On Hadamard-type integral inequalities for co-ordinated convex functions. In AIP Conference Proceedings (Vol. 1726, No. 1, p. 020081). AIP Publishing LLC.