

**ACCURATE AND EFFICIENT SOLUTIONS
OF ELECTROMAGNETICS PROBLEMS
WITH THE MULTILEVEL FAST
MULTIPOLE ALGORITHM**

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FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By

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July 2009

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ABSTRACT

ACCURATE AND EFFICIENT SOLUTIONS OF ELECTROMAGNETICS PROBLEMS WITH THE MULTILEVEL FAST MULTIPOLE ALGORITHM

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The multilevel fast multipole algorithm (MLFMA) is a powerful method for the fast and efficient solution of electromagnetics problems discretized with large numbers of unknowns. This method reduces the complexity of matrix-vector multiplications required by iterative solvers and enables the solution of large-scale problems that cannot be investigated by using traditional methods. On the other hand, efficiency and accuracy of solutions via MLFMA depend on many parameters, such as the integral-equation formulation, discretization, iterative solver, preconditioning, computing platform, parallelization, and many other details of the numerical implementation. This dissertation is based on our efforts to develop sophisticated implementations of MLFMA for the solution of real-life scattering and radiation problems involving three-dimensional complicated objects with arbitrary geometries.

Keywords: Multilevel fast multipole algorithm (MLFMA), electromagnetic scattering and radiation, surface integral equations, parallelization, iterative algorithms.

ÖZET

ELEKTROMANYETİK PROBLEMLERİN ÇOK SEVİYELİ HIZLI ÇOKKUTUP YÖNTEMİYLE DOĞRU VE VERİMLİ ÇÖZÜMLERİ

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Çok seviyeli hızlı çokkutup yöntemi (ÇSHÇY), çok sayıda bilinmeyen içeren elektromanyetik problemlerinin hızlı ve etkin çözümleri için kullanılan güçlü bir metottur. Bu yöntem sayesinde, iteratif çözücülerin ihtiyaç duyduğu matrisvektör çarpımlarının karmaşıklığı düşürülebilmekte ve geleneksel yöntemlerle incelenemeyen büyük ölçekli problemlerin çözümleri mümkün hale gelmektedir. Öte yandan, ÇSHÇY ile gerçekleştirilen çözümlerin verimi ve doğruluğu, integral denklemi formülasyonu, ayrıklaştırma, iteratif çözücü, öniyileştirici, hesaplama platformu, paralelleştirme, ve sayısal uygulamanın detayları gibi pek çok etkene bağlıdır. Bu tez, gerçek yaşamda karşımıza çıkan ve üç boyutlu karmaşık cisimler içeren saçılım ve ışıınım problemlerinin çözümleri için yüksek kabiliyetli ÇSHÇY uygulamaları geliştirme konusundaki çalışmalarımız üzerinedir.

Anahtar kelimeler: Çok seviyeli hızlı çokkutup yöntemi (ÇSHÇY), elektromanyetik saçılım ve ışıınım, yüzey integral denklemleri, paralelleştirme, iteratif algoritmalar.

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To My Ayça and My Family...
Ayça'ma ve Aileme...

Chapter 1

Introduction

Solutions of electromagnetics problems are extremely important to analyze electromagnetic interactions of devices with each other and with their environments including living and nonliving objects [1]. A plethora of applications in the areas of antennas [2]–[4], radars [5], optics [6], medical imaging [7], wireless communications [8], nanotechnology, metamaterials [9]–[13], remote sensing, and electronic packaging involve scattering or radiation of electromagnetic waves. Solutions of problems are particularly useful to increase the productivity in those areas by providing remedial procedures to improve existing designs before their actual realizations and by preventing the waste of resources during design processes.

Electromagnetics problems can be formulated rigorously with Maxwell’s equations. Unfortunately, Maxwell’s equations can only be solved analytically for a few canonical objects, such as a sphere [14]. Recently, computational electromagnetics has become a hot research area, where electromagnetics problems are investigated with numerical techniques. Mathematical formulations of physical events lead to set of equations that can be solved numerically by using computers. Thanks to advances in both computer technology and solution algorithms, it has become possible to solve real-life problems involving complicated structures, such

as scattering from ordinary and radar-eluding stealth airborne targets, radiation from antennas and electronic devices into living organisms, and transmission through frequency-selective metamaterials, photonic crystals, and optical imaging systems.

Developing a fast and accurate electromagnetics solver requires a well-designed combination of diverse components in many areas, including integral equations, wave theory, fast algorithms, numerical techniques, iterative solvers, preconditioners, parallel algorithms, and computer technology. Accurate simulations of real-life problems usually require the solution of numerical problems involving large numbers of unknowns, which cannot be achieved easily, even when using the most powerful computers. Therefore, one needs to develop special acceleration algorithms, such as the multilevel fast multipole algorithm (MLFMA) [15], in order to solve those large-scale problems efficiently on relatively inexpensive computing platforms.

This dissertation presents our efforts to solve real-life problems in electromagnetics. Our ultimate aim was to develop a simulation environment based on MLFMA for efficient and accurate solutions of scattering and radiation problems involving complicated structures. By developing sophisticated implementations of MLFMA, we have made significant contributions in the area of computational electromagnetics.

1.2 Background

In this section, we provide a brief overview of the literature on surface formulations, discretization, MLFMA, iterative solvers, preconditioning, and parallelization.

1.2.1 Surface Integral Equations and Surface Formulations

Surface integral equations are commonly used for the solution of scattering and radiation problems in electromagnetics [16]–[41]. Complicated problems involving three-dimensional metallic and/or homogeneous dielectric structures are formulated rigorously by defining equivalent electric and/or magnetic currents on surfaces and applying boundary conditions. Depending on the testing scheme and boundary conditions used, there are four basic surface integral equations, namely, the tangential electric-field integral equation (T-EFIE), the normal electric-field integral equation (N-EFIE), the tangential magnetic-field integral equation (T-MFIE), and the normal magnetic-field integral equation (N-MFIE) [32]. In tangential equations, boundary conditions are tested directly by sampling tangential components of fields on surfaces. In normal equations, however, fields are tested after they are projected onto surfaces via cross-product operations with the outward normal vector.

For the solution of electromagnetics problems, various surface formulations can be derived by using diverse combinations of surface integral equations. Some of these formulations are known to be stable and provide accurate results [16]–[40], although their performances may vary significantly in terms of efficiency and accuracy. In general, surface formulations can be categorized into three groups, i.e., tangential, normal, and mixed formulations, depending on their contents. Tangential formulations involve T-EFIE and/or T-MFIE, while normal formulations involve N-EFIE and/or N-MFIE. Mixed formulations are obtained by combining tangential and normal formulations, and they contain at least one tangential equation (T-EFIE and T-MFIE) and one normal equation (N-EFIE and N-MFIE).

Homogeneous Dielectric Objects:

For dielectric objects with homogeneous material properties, surface formulations are obtained by testing boundary conditions on the two sides (i.e., inner and outer sides) of surfaces and combining equations properly. To avoid numerical internal resonances, it is preferable to derive a combined-field integral equation (CFIE) by linearly combining EFIE and MFIE [27]. For example, a combination of T-EFIE and T-MFIE leads to a T-T-CFIE formulation. Similarly, one can obtain T-N-CFIE (T-EFIE+N-MFIE), N-T-CFIE (N-EFIE+T-MFIE), and N-N-CFIE (N-EFIE+N-MFIE) by combining EFIE and MFIE appropriately. In these formulations, equations obtained for inner and outer regions are solved simultaneously to obtain the two sets of unknowns, i.e., electric and magnetic currents. We can also derive various other formulations involving triple combinations, such as TN-N-CFIE (T-EFIE+N-EFIE+N-MFIE), for more stable solutions [32].

CFIE formulations described above are based on a linear combination of EFIE and MFIE in the same way for each medium, while it is also possible to use different combinations for inner and outer regions [33]. Alternatively, we can linearly combine inner and outer equations while solving EFIE and MFIE simultaneously. In this class of formulations, Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) [17]–[19] and Müller [16] formulations are well known and commonly used in the literature. Both tangential and normal versions of these formulations are possible, but only the tangential PMCHWT (T-PMCHWT) formulation and the normal Müller formulation (NMF) are stable, whereas the normal PMCHWT and the tangential Müller formulations are usually unstable [16],[34]. Recently, remarkable efforts have been made to further improve dielectric formulations by devising novel combinations of integral equations. For example, the combined tangential formulation (CTF) [37] is similar to the T-PMCHWT formulation, but it involves a careful (and improved) scaling of T-EFIE and T-MFIE. A similar combination of N-EFIE and N-MFIE leads to the

combined normal formulation (CNF) [37]. The modified NMF (MNMF) is obtained by normalizing equations in NMF to produce better-conditioned matrix equations [36]. Finally, a mixed formulation called the electric and magnetic current combined-field integral equation (JMCFIE) is derived by combining all four types of integral equations [35],[40].

Perfectly-Conducting Objects:

When the object is a perfect electric conductor (PEC), T-EFIE, T-MFIE, N-EFIE, and N-MFIE can be solved independently to obtain the induced electric current on the surface. Especially, T-EFIE and N-MFIE are commonly used in the literature [17],[22],[26]. However, for closed conductors, those formulations suffer from internal resonances. Therefore, similar to dielectric formulations, it is essential to combine EFIE and MFIE appropriately to derive various CFIE formulations [20].

Conventional CFIE formulations are usually obtained by convex combinations of EFIE and MFIE. However, there are many studies presenting hybrid formulations, where EFIE and MFIE are coupled in diverse ways. For example, EFIE and MFIE can be applied on different parts of the object to efficiently and accurately solve electromagnetics problems involving complicated structures [24],[28],[29]. For example, in [24], a hybrid formulation is used to properly formulate electromagnetics problems with composite structures of thin and thick bodies. The same idea is employed to avoid calculating both MFIE and EFIE for all cases, thus reducing the computational cost [28]. In [29], EFIE and MFIE are applied carefully on different parts of the geometry and the two formulations are coupled via iterative updates of equivalent sources.

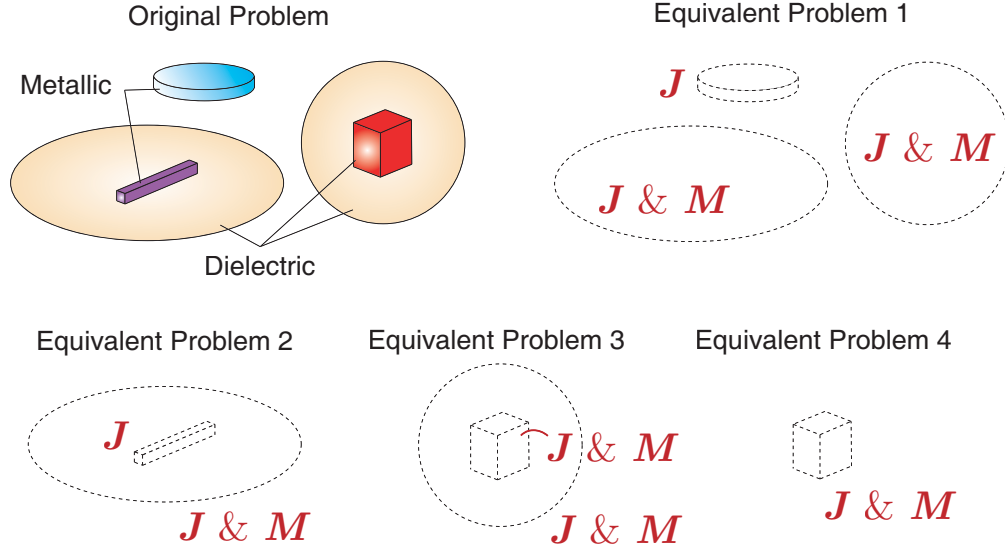


Figure 1.1: Applying the equivalence principle to a complicated problem involving multiple dielectric and metallic regions.

Composite Objects with Coexisting Dielectric and Metallic Parts:

Surface formulations can easily be applied to composite structures with coexisting dielectric and metallic parts [35],[38]. As depicted in Figure 1.1, the equivalence principle is used to decompose the original problem into equivalent problems by employing equivalent currents on surfaces. Equivalent problems are defined for all nonmetallic regions, including the host medium extending to infinity. Then, integral equations can be derived as usual, by applying operators on currents and enforcing boundary conditions for tangential fields.

Scattering from Low-Contrast Dielectric Objects:

Surface formulations tend to be less accurate as the contrast of the object decreases and electromagnetic properties change slightly across dielectric interfaces. There are various applications that involve scattering from low-contrast objects, such as red blood cells in blood plasma [42]–[44], plastic mines buried in soil [45],

polymeric materials [46], and dielectric photonic crystals [47]. When the contrast is low, however, conventional surface formulations encounter stability problems, and scattered fields cannot be calculated accurately with them. Those scattering problems can be solved accurately with volume formulations [48], which are stable when the contrast is low. On the other hand, it is also desirable to extend the applicability of surface integral equations to low-contrast problems in order to use advantages of surface formulations, which are usually discretized with fewer unknowns compared to volume formulations.

The inaccuracy of surface formulations for the solution of low-contrast problems is due to insufficient modelling of radiating parts of equivalent currents defined on objects [49]. By extracting nonradiating currents and solving modified equations only for radiating currents (similar to volume formulations), scattered fields from low-contrast objects can be calculated accurately.

1.2.2 Discretization

Electromagnetics problems involving complicated objects can be discretized and solved numerically. By means of a simultaneous discretization of the geometry and surface integral equations, equivalent currents are expanded in a series of basis functions [50]. Then, coefficients of basis functions are calculated by solving dense matrix equations, which are derived by using the method of moments (MOM) [51] and testing boundary conditions for electric and magnetic fields on the surface.

Elements of matrices obtained by the discretization integral-equation formulations can be interpreted as electromagnetic interactions between pairs of discretization elements, i.e., basis and testing functions. Applying a Galerkin scheme for the discretization of surface formulations, i.e., using the same set

of functions to expand current densities (basis functions) and to test boundary conditions (testing functions), normal and mixed formulations contain well-tested identity operators. It is well-known that matrix equations involving well-tested identity operators are diagonally dominant and they are well-conditioned [37],[52],[53]. However, tangential formulations do not contain well-tested identity operators, and their discretizations may lead to ill-conditioned matrix equations. Therefore, for the efficiency of solutions, normal and mixed formulations are preferable, especially when problems involve large objects discretized with large numbers of unknowns [54]. On the other hand, errors in the discretization of well-tested identity operators may deteriorate the accuracy of solutions obtained with normal and mixed formulations [53]. The excessive error in those formulations compared to tangential formulations can be significant, especially in conventional implementations employing the Rao-Wilton-Glisson (RWG) [22] functions. Investigations on PEC objects show that scattered fields obtained with N-MFIE and T-N-CFIE are significantly inaccurate in comparison to those obtained with T-EFIE [55],[56]. Similar observations have been made for the solution of dielectric objects [37]. In such cases, it is helpful to improve discretizations by employing more appropriate, such as higher-order [57]–[60], basis functions to obtain accurate results with normal and mixed formulations [37],[61]–[64]. Recent studies also show that the regularization of the identity operator improves the accuracy of N-MFIE [65],[66].

For PEC objects, T-EFIE suffers from the low-frequency breakdown problem when it is discretized with ordinary basis functions, such as the RWG functions [67],[68]. Specifically, matrix equations obtained from T-EFIE become increasingly ill-conditioned as the discretization is refined. T-N-CFIE is also affected by the low-frequency breakdown since it contains T-EFIE [69]–[71], whereas N-MFIE with the RWG functions is usually stable unless the frequency is extremely low [72]. To eliminate the low-frequency breakdown problem, one

can apply loop-star or loop-tree decomposition methods [73]–[75]. In addition, current and charge decomposition is introduced in [76] to derive stable formulations at low frequencies. Regularization of integral equations using Calderon identities is also commonly practiced [71],[77],[78].

1.2.3 Multilevel Fast Multipole Algorithm

It has been more than 15 years since the fast multipole method (FMM) was developed for the efficient solution of radiation and scattering problems in electromagnetics [79],[80]. Discretizations of integral-equation formulations lead to $N \times N$ dense matrix equations, which can be solved iteratively via a Krylov-subspace algorithm. FMM provides matrix-vector multiplications (MVMs) required by iterative algorithms in $\mathcal{O}(N^{3/2})$ time using $\mathcal{O}(N^{3/2})$ memory. By reducing the computational complexity from $\mathcal{O}(N^2)$ to $\mathcal{O}(N^{3/2})$, FMM enabled the solution of large-scale problems on relatively inexpensive computing platforms [81]. A few years later, the idea behind FMM was extended and applied in a recursive manner, leading to MLFMA [15], which provides the solution of larger problems by reducing the complexity of MVMs to $\mathcal{O}(N \log^2 N)$ [82] or $\mathcal{O}(N \log N)$ [1],[32],[83]. A tree structure of $L = \mathcal{O}(\log N)$ levels is constructed by placing the object in a cubic box and recursively dividing the computational domain into sub-boxes (recursive clustering). Then, MLFMA calculates interactions between radiating (basis) and receiving (testing) elements, which are far from each other, in a group-by-group manner consisting of three stages called aggregation, translation, and disaggregation. In each MVM, these stages are performed on the tree structure in a multilevel scheme.

- *Aggregation:* Radiated fields of boxes are calculated from the bottom of the tree structure to the highest level. At the lowest level, the radiated field of a box is obtained by combining radiation patterns of basis functions inside

the box. At higher levels, the radiated field of a box is the combination of radiated fields of its sub-boxes.

- *Translation:* Radiated fields computed during the aggregation stage are translated into incoming fields. For each box at any level, there are $\mathcal{O}(1)$ boxes to translate the radiated field to. Using cubic (identical) boxes, there is a maximum of $7^3 - 3^3 = 316$ different translations at each level, independent of the number of boxes [84].
- *Disaggregation:* Total incoming fields at box centers are calculated from the top of the tree structure to the lowest level. The total incoming field for a box is obtained by combining incoming fields due to translations and the incoming field to the center of its parent box, if it exists. In the lowest level, incoming fields are received by testing functions.

In MLFMA, radiated and incoming fields are sampled on the unit sphere as a function of spherical coordinates θ and ϕ . The number of samples required for each box is proportional to the size of the box as measured by the wavelength [85]. Therefore, to match different sampling rates of consecutive levels, interpolation and transpose interpolation (anterpolation) [86] are required during aggregation and disaggregation stages, respectively. There are two major ways of implementing interpolations (and anterpolations), namely, through global and local interpolation methods. Global interpolations are usually based on the fast Fourier transform (FFT) along the ϕ direction and the Legendre transform along the θ direction, performed via one-dimensional FMM [82],[87]. Using uniform sampling, FFT can also be used along the θ direction [88]. A resulting MLFMA implementation has $\mathcal{O}(N \log^2 N)$ time complexity, while interpolations are performed without error, provided that the Nyquist criterion is applied for the sampling rate. On the other hand, local interpolation methods introduce controllable errors [89], but they lead to more efficient MLFMA implementations with $\mathcal{O}(N \log N)$ complexity.

To be employed repeatedly, translation operators are calculated and stored in memory before iterations. On the other hand, direct calculation of these operators requires $\mathcal{O}(N^{3/2})$ operations, and the processing time for their setup becomes substantial as problem size grows. As a remedy, a two-step computation is suggested based on the interpolation of the translation operator [90]. First, the translation operator is expressed as a band-limited function and it is sampled at maximum $\mathcal{O}(N)$ points. Second, the operator is evaluated at required points by interpolation from the previous samples. With an efficient interpolation algorithm, processing time for the calculation of translation operators is reduced to $\mathcal{O}(N)$.

In MLFMA, there are also $\mathcal{O}(N)$ near-field interactions, which are calculated directly in the setup stage of the program and stored in memory to be used multiple times during iterations. These interactions are between basis and testing functions that are located close to each other, and they can be calculated accurately by using singularity extraction techniques [22],[91]–[94], adaptive integration methods [95], and Gaussian quadratures [96].

MLFMA introduces several error sources at the cost of the provided efficiency. These error sources, which are controllable to some extent [1],[85],[90],[97],[98], arise in addition to others due to MOM, such as the simultaneous discretization of the geometry and Maxwellian integral equations. Two errors are inherited from FMM, namely, the truncation of an infinite series and the angular integration over the unit sphere. Interpolation and antinterpolation operations constitute the third error source introduced by MLFMA.

1.2.4 Low-Frequency Multilevel Fast Multipole Algorithm

MLFMA is an efficient algorithm to solve large-scale electromagnetics problems discretized with large numbers of unknowns. However, it suffers from a low-frequency breakdown, i.e., the box size in MLFMA cannot be very small compared to wavelength. This is because MLFMA is based on the plane wave expansion, which becomes invalid for short distances. Therefore, MLFMA becomes inefficient for problems involving small objects discretized with large numbers of unknowns. Specifically, when an ordinary MLFMA is applied to those low-frequency problems, the lowest-level boxes may involve many discretization elements. This significantly increases the processing time and memory required for near-field interactions, which must be calculated directly. Even the complexity of the MLFMA implementation can be more than $\mathcal{O}(N \log N)$ due to excessively large numbers of the near-field interactions.

In the literature, there are two common approaches to solve low-frequency problems efficiently via MLFMA. The first approach is based on a spectral representation of the Green's function, where fields are divided into propagating and evanescent parts [99]–[102]. Evanescent waves are employed to compute interactions between groups separated with short distances. The second approach is based on multipole representation of fields [103]–[108]. The Green's function is factorized in a series of multipoles, but the multipoles are not converted into plane waves. In both two approaches, the box size is not restricted, and one can divide the object into sub-boxes, which can be much smaller than the wavelength.

1.2.5 Iterative Solvers and Preconditioning

In the literature, various Krylov-subspace methods are available to solve electromagnetics problems formulated with integral equations [109]. Some of them can be listed as follows:

- BiCG: Biconjugate Gradient
- BiCGStab: Biconjugate Gradient Stabilized [110]
- CG: Conjugate Gradient
- CGS: Conjugate Gradient Squared [111]
- GMRES: Generalized Minimal Residual [112]
- LSQR: Least-Squares QR [113]–[115]
- QMR: Quasi-Minimal Residual [116]
- TFQMR: Transpose-Free Quasi-Minimal Residual [117]

The performance of each algorithm may vary significantly depending on the problem, and there is not an optimal algorithm that performs the best for all cases. Therefore, for a given problem, one should investigate and compare iterative solutions provided by various algorithms. Sequential and parallel implementations of iterative algorithms can be found in [109] and [118].

Preconditioning refers to transforming a matrix equation into another, but equivalent one with more favorable conditioning characteristics and that can be solved easily with iterative methods [119]. This is usually achieved by multiplying the matrix equation with an approximate inverse of the matrix. In MLFMA implementations, near-field interactions are available to construct robust preconditioners. For example, with the direct factorization of the sparse near-field

matrix of all near-field interactions, one can obtain the near-field preconditioner (NFP) [54]. However, in most cases, near-field interactions are distributed irregularly in the full matrix and they cannot be collected in a bandlimited diagonal region, even when using reordering algorithms [120]. Hence, both the factorization performed once before iterations and the solution of the preconditioner system applied in each iteration are expensive in terms of processing time; thus the preconditioner becomes a bottleneck as the problem size grows. As a remedy, a block diagonal preconditioner (BDP) that is much more efficient can be obtained by extracting only self interactions of the lowest-level boxes [15]. Both the factorization and the application of BDP are $\mathcal{O}(N)$; this is appropriate for MLFMA since the complexity of MLFMA is $\mathcal{O}(N \log N)$ and the complexity of the preconditioner should not be higher than that. Due to its simplicity and favorable computing cost, BDP is commonly used for MLFMA solutions of normal and mixed formulations, such as T-N-CFIE. Unfortunately, it usually decelerates iterative solutions of tangential formulations, such as T-EFIE [54], except for a few special cases [121]. In addition, for large problems involving complicated objects, acceleration provided by BDP may not be sufficient, even when normal and mixed formulations are used.

Recently, there have been many efforts to improve iterative solutions of surface formulations using preconditioners. In the context of MLFMA, one of the most common preconditioning technique is the incomplete LU (ILU) method [119],[122]–[125]. During the factorization of the preconditioner matrix, the ILU method sacrifices some of elements and provides an approximation to the near-field matrix. Despite the wide success of ILU preconditioners, they are limited to sequential implementations due to their factorization phase and forward-backward solutions. Hence, the sparse-approximate-inverse (SAI) preconditioner that is well-suited for parallel implementations has been more preferable for the solution of large-scale electromagnetics problems [126],[127].

1.2.6 Parallelization of the Multilevel Fast Multipole Algorithm

For an $N \times N$ dense matrix equation, MLFMA reduces the complexity of MVMs from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$, allowing for the solution of large problems with limited computational resources. On the other hand, accurate solutions of many real-life problems require discretizations with millions of elements, leading to matrix equations with millions of unknowns, which cannot easily be solved with sequential implementations of MLFMA running on a single processor. To solve such large problems, it is helpful to increase computational resources by assembling parallel computing platforms and, at the same time, by parallelizing MLFMA.

Of the various parallelization schemes for MLFMA, the most popular use distributed-memory architectures by constructing clusters of computers with local memories connected via fast networks [84],[128]–[135]. Parallelization tools are available, such as the message passing interface (MPI), which provide many communication protocols to organize parallel solutions. However, the parallelization of MLFMA is not trivial because of the complicated structure of this algorithm [134]. Simple parallelization techniques usually fail to provide efficient solutions, due to communications among processors, poor load-balancing of the workload, and unavoidable duplications of computations over multiple processors. Consequently, there have been many efforts to improve the parallelization of MLFMA by using novel partitioning strategies, load-balancing algorithms, and optimizations for communications. This way, it has become possible to solve problems with tens of millions of unknowns on relatively inexpensive computing platforms with distributed-memory architectures [84],[132],[133].

1.3 Contributions

In this section, we summarize our major achievements during this Ph.D. period. As listed in the bibliography, results of this study have been published in 25 journal and 66 conference papers so far.

1.3.1 Solutions of Electromagnetics Problems Involving Dielectric and Dielectric/Metallic Objects

- We investigated various surface formulations for the fast and accurate solution of electromagnetics problems involving homogeneous dielectric objects via MLFMA [136]–[141].
- We applied the stabilization procedure presented in [49] to various surface formulations for the accurate solution of scattering problems involving three-dimensional low-contrast dielectric objects [142]–[144]. For both tangential and normal formulations, the stabilization technique involves the expansion of incident fields in a series of basis functions and the rearrangement of right-hand sides (RHSs) of equations. By eliminating terms related to identity operators, accuracy of formulations are improved further at the cost of increased processing time. The stabilization procedure is simple and easily applied to existing implementations of conventional formulations. On the other hand, the resulting formulations are stable only for moderately low-contrast problems and they break down as the contrast is further decreased to very low values. In addition, these quasi-stable formulations are sensitive to how accurately matrix elements are computed and the stable region is more limited when finite-precision methods, such as MLFMA, are used in order to accelerate solutions.
- We introduced a novel stabilization technique for accurate surface formulations of dielectric bodies with arbitrarily low contrasts [145],[146]. Similar

to the previous stabilization procedures, the proposed technique is based on extracting nonradiating currents and solving modified equations to obtain radiating currents. In addition, this technique involves the use of fictitious incident fields to rearrange RHSs of equations appropriately. Resulting stabilized formulations provide accurate results even for extremely low-contrast objects.

1.3.2 Solutions of Electromagnetics Problems Involving Perfectly-Conducting Objects

- We showed that transforming original matrix equations into normal equations improves the convergence of T-EFIE significantly [147]–[150]. Specifically, the LSQR algorithm, which corresponds to a stable application of the CG algorithm on normal equations, provides improved convergence rates compared to other algorithms for the solution T-EFIE. Improvements obtained with the LSQR algorithm become significant for the solution of large-scale problems involving open surfaces that must be formulated with T-EFIE.
- We presented hybrid integral equations (HIEs) by extending the applicability of T-N-CFIE to composite geometries with coexisting open and closed parts [151]–[155]. For the solution of problems containing open surfaces, the traditional T-N-CFIE is not applicable since it contains N-MFIE, which can be applied only to closed geometries. If the overall geometry is composed of open surfaces either completely or even to a minor extent (i.e., even if the majority of the geometry is closed), then T-EFIE becomes the inevitable choice in spite of its ill-conditioned nature. As the problem size gets larger, solutions of T-EFIE become extremely difficult, even when iterative techniques are used with accelerated methods, such as MLFMA for MVMs. Using HIEs, T-EFIE is still applied on open parts of the geometry,

whereas T-N-CFIE is applied on closed parts to improve the convergence of the iterative solution. Different from the previous studies on hybrid formulations, we used HIEs for iterative solutions of large problems with composite structures involving both open and closed surfaces. We also investigated the convergence characteristics of iterative solutions of large composite problems with MLFMA [156]. Following a thorough study of how the convergence characteristics depends on the problem geometry, formulation, and iterative solvers, we provided concrete guidelines for efficient solutions.

- We conducted a series of studies to investigate in detail the inaccuracy of N-MFIE, and systematically compared T-EFIE and N-MFIE implementations in order to identify the source of the error [157]–[159]. In those studies, singularities arising in outer integrals of N-MFIE were extracted for improved accuracy in numerical integrations [157], the solid-angle factor in N-MFIE (which is not present in T-EFIE) was clarified to avoid incorrect usage [158], and the testing scheme of N-MFIE was extended by allowing testing points to reside on edges of the triangulation (similar to the practice in T-EFIE) [159]. Although these studies provided a better understanding of N-MFIE and led to more robust implementations, the improvement in the accuracy was limited, indicating that the fundamental reason for the inaccuracy lied elsewhere. Hence, we focused on the use of novel basis and testing functions for more accurate MFIE implementations.
- We presented efficient and reliable schemes for the numerical discretization of N-MFIE with the curl-conforming $\hat{\mathbf{n}} \times \text{RWG}$ basis and testing functions defined on planar triangulations [160],[161]. Two different formulations to calculate elements of the impedance matrix were given in detail. In the first formulation, relations between elements of the impedance matrix obtained by using the $\hat{\mathbf{n}} \times \text{RWG}$ and the RWG functions with the Galerkin method were also established. The second formulation was presented as an

alternative and more efficient way to calculate interactions. Furthermore, we outlined the factorization of interactions for MLFMA implementations employing the $\hat{\mathbf{n}} \times \text{RWG}$ functions, similar to MLFMA implementation involving the RWG functions. Finally, we compared results obtained by using the $\hat{\mathbf{n}} \times \text{RWG}$ and the RWG functions, and demonstrated the improvement in the accuracy of solutions due to the $\hat{\mathbf{n}} \times \text{RWG}$ functions.

- Basis functions with linear variations were investigated in terms of accuracy of N-MFIE and T-N-CFIE, based on the previous reports indicating the inaccuracy of N-MFIE [162]. Specifically, two functions with linear variations along triangulation edges in both tangential and normal directions (LN-LT type) were defined. They were compared to the conventional RWG and the curl-conforming $\hat{\mathbf{n}} \times \text{RWG}$ functions. Examples were presented to demonstrate the significant improvement in the accuracy of N-MFIE and T-N-CFIE gained by replacing the commonly used RWG functions with LN-LT type functions.
- We presented the divergence-conforming linear-linear (LL) basis functions to improve the accuracy of T-N-CFIE for large scattering problems solved via MLFMA [163],[164]. We showed that accuracy problems of T-N-CFIE arising from its implementations with the conventional RWG basis functions can be mitigated by using the LL functions for the discretization. This was achieved without increasing computational requirements and with only minor modifications in existing codes based on the RWG functions.
- By setting up a computational experiment based on nonradiating currents, we proved that the identity operator is a major error source in surface integral equations [165]–[167]. This explains why normal and mixed formulations involving well-tested identity operators are significantly inaccurate compared to tangential formulations.

- We showed that, in an iterative solution, the minimization of the residual error of T-N-CFIE involves a breakpoint, where a further reduction of the residual error does not improve the compatibility of the CFIE solution with EFIE and MFIE systems [168]. Since EFIE produces more accurate results than CFIE, the breakpoint also corresponds to the last useful iteration, where the accuracy of the CFIE solution is saturated and cannot be improved any more.

1.3.3 Efficient Solutions of Electromagnetics Problems with the Multilevel Fast Multipole Algorithm

- We presented an efficient technique to reduce interpolation and antinterpolation errors during aggregation and disaggregation stages of MLFMA, which is based on the sampling of radiated and incoming fields over all possible solid angles, i.e., all directions on the sphere [169],[170]. Specifically, the error is reduced by employing additional samples at the poles. Since the interpolation error is propagated and amplified by every level of aggregation, this technique is particularly useful for large problems. Additional costs in terms of memory and processing time are negligible, and the technique can easily be adapted into existing implementations of MLFMA.
- We revisited the Lagrange interpolation of the translation operator in MLFMA [171],[172]. We optimized two parameters of the interpolation, namely, the number of interpolation points and the oversampling factor. In this way, the error becomes controllable and the processing time required to satisfy the desired level of accuracy is minimized.
- We presented a two-step Lagrange interpolation method to accelerate the solution of electromagnetics problems with MLFMA [173]. The method is based on performing the required two-dimensional interpolation as a

sequence of two one-dimensional interpolations. By also applying the two-step method for interpolations, efficiency of MLFMA can be improved significantly. The two-step method is easy to implement and is especially useful for problems involving large-scale objects.

- We developed a low-frequency MLFMA (LF-MLFMA) for the solution of electromagnetics problems involving relatively small objects discretized with large numbers of unknowns. By using the multipole representation for radiated and incoming fields, the smallest box size and the number of levels are not limited for accurate computations. LF-MLFMA was also combined with the ordinary MLFMA for broadband applications.

1.3.4 Efficient Parallelization of the Multilevel Fast Multipole Algorithm for the Solution of Large-Scale Electromagnetics Problems

- We extensively investigated the parallelization of MLFMA by focusing on different parts of the algorithm and identifying obstacles to parallelization efficiency [174]–[178]. Our approach involved load-balancing and partitioning techniques to distribute tasks equally among processors and to minimize inter-processor communications.
- We introduced a novel hierarchical partitioning strategy for the efficient parallelization of MLFMA on distributed-memory architectures [179]–[184]. Unlike the previous parallelization techniques, this strategy is based on the simultaneous distribution of boxes and their fields by considering the best possible partitioning of each level separately, and it is well suited for the multilevel structure of MLFMA. Due to the improved load-balancing

and reduced communications, the hierarchical strategy offers a higher parallelization efficiency than the previous approaches, especially when the number of processors is large.

- We developed a sophisticated simulation environment based on efficient parallelization of MLFMA. We demonstrated the effectiveness of our implementations by solving increasingly large integral-equation problems involving as many as 205 million unknowns [185]–[202].

1.3.5 Applications

- We solved scattering problems involving large-scale airborne targets with complicated geometries. We compared radar cross section (RCS) values obtained for various targets at realistic frequencies [203].
- Log-periodic (LP) antennas and their arrays were investigated in detail for broadband applications [204]–[210]. The results obtained in the simulation environment led to essential improvements in fundamental design procedures that could not be achieved with textbook-type theoretical approaches alone. Our implementations were also used to analyze other types of antennas, such as double-ridged horn antennas [211].
- We presented fast and accurate simulations of metamaterial structures constructed with large numbers of unit cells containing split-ring resonators (SRRs) and thin wires (TWs) [212]–[225]. Without resorting to homogenization approximations and periodicity assumptions, we were able to obtain accurate solutions of realistic metamaterial problems discretized with millions of unknowns. We also used our implementations to solve various electromagnetics problems involving photonic crystals [226].

1.4 Organization and Conventions

This dissertation involves seven main chapters and three appendices. In Chapters 2 and 3, solutions of electromagnetics problems involving metallic, dielectric, and dielectric/metallic objects are focused. Then, in Chapter 4, we present efficient implementations of MLFMA, as well as LF-MLFMA. Chapter 5 is devoted to efficient parallelization of MLFMA for the solution of very large electromagnetics problems. We present some of the applications investigated by using MLFMA in Chapter 6, followed by our concluding remarks in Chapter 7. Basic formulations and other important issues related to this dissertation are organized in Appendices A and B. Finally, in Appendix C, we list a few other works that need further research.

For all electromagnetic problems and their solutions presented in this dissertation, we assume time-harmonic electromagnetic waves with $\exp(-i\omega t)$ time dependence.

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Chapter 2

Solutions of Electromagnetics Problems Involving Dielectric and Dielectric/Metallic Objects

In this chapter, we consider the solution of electromagnetics problems involving three-dimensional dielectric and dielectric/metallic objects with arbitrary shapes. Various surface formulations are extensively discussed on homogeneous dielectric objects in the context of iterative solutions via MLFMA. Comparisons show that a mixed formulation called JMCIE is preferable for large and complicated structures. This formulation is extended to composite problems involving multiple dielectric and dielectric/metallic regions. Investigations on the accuracy also show that conventional surface formulations break down when the contrast of the object is low. We provide various stabilization procedures in order to obtain accurate results for arbitrarily low contrasts.

2.1 Surface Operators

For surface formulations of scattering and radiation problems, three different operators are defined as

$$\mathcal{T}\{\mathbf{X}\}(\mathbf{r}) = ik \int_S d\mathbf{r}' \left[\mathbf{X}(\mathbf{r}') + \frac{1}{k^2} \nabla' \cdot \mathbf{X}(\mathbf{r}') \nabla \right] g(\mathbf{r}, \mathbf{r}') \quad (2.1)$$

$$\mathcal{K}\{\mathbf{X}\}(\mathbf{r}) = \int_S d\mathbf{r}' \mathbf{X}(\mathbf{r}') \times \nabla' g(\mathbf{r}, \mathbf{r}') \quad (2.2)$$

$$\mathcal{I}\{\mathbf{X}\}(\mathbf{r}) = \mathbf{X}(\mathbf{r}), \quad (2.3)$$

where S is the closed surface of a three-dimensional object with an arbitrary shape. In (2.1)–(2.3), $\mathbf{X}$ is either the equivalent electric current ($\mathbf{J}$) or the equivalent magnetic current ($\mathbf{M}$) on the surface, $k = \omega \sqrt{\mu\epsilon} = 2\pi/\lambda$ is the wavenumber, and $g(\mathbf{r}, \mathbf{r}')$ denotes the homogeneous-space Green's function defined as

$$g(\mathbf{r}, \mathbf{r}') = \frac{\exp(ikR)}{4\pi R} \quad \left(R = |\mathbf{r} - \mathbf{r}'| \right) \quad (2.4)$$

in phasor notation with the $e^{-i\omega t}$ convention. The operator $\mathcal{K}$ is commonly separated into principal-value and limit parts [26] as

$$\mathcal{K}\{\mathbf{X}\}(\mathbf{r}) = \mathcal{K}_{PV}\{\mathbf{X}\}(\mathbf{r}) - \frac{\Omega_i}{4\pi} \mathcal{I}^{\times n}\{\mathbf{X}\}(\mathbf{r}), \quad (2.5)$$

where $0 \leq \Omega_i \leq 4\pi$ is the internal solid angle, which is nonzero when the observation point $\mathbf{r}$ is on the surface. In (2.5), $\mathcal{I}^{\times n}\{\mathbf{X}\}(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{X}(\mathbf{r})$, where $\hat{\mathbf{n}}$ is the outward normal unit vector.

Using equivalent surface currents, i.e.,

$$\mathbf{J}(\mathbf{r}) = \mathcal{I}^{\times n}\{\mathbf{H}\}(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r}) \quad (2.6)$$

$$\mathbf{M}(\mathbf{r}) = -\mathcal{I}^{\times n}\{\mathbf{E}\}(\mathbf{r}) = -\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r}), \quad (2.7)$$

secondary (scattered or radiated) electric and magnetic fields (outside the object) can be calculated [17] as

$$\mathbf{E}^{sec}(\mathbf{r}) = \eta \mathcal{T}\{\mathbf{J}\}(\mathbf{r}) - \mathcal{K}_{PV}\{\mathbf{M}\}(\mathbf{r}) + \frac{\Omega_i}{4\pi} \mathcal{I}^{\times n}\{\mathbf{M}\}(\mathbf{r}) \quad (2.8)$$

$$\mathbf{H}^{sec}(\mathbf{r}) = \frac{1}{\eta} \mathcal{T}\{\mathbf{M}\}(\mathbf{r}) + \mathcal{K}_{PV}\{\mathbf{J}\}(\mathbf{r}) - \frac{\Omega_i}{4\pi} \mathcal{I}^{\times n}\{\mathbf{J}\}(\mathbf{r}), \quad (2.9)$$

where $\eta = \sqrt{\mu/\epsilon}$ is the wave impedance.

2.2 Homogeneous Dielectric Objects

For dielectric objects with homogeneous material properties, the general procedure of surface formulations can be summarized as follows:

- Apply surface operators on equivalent surface currents to obtain expressions for secondary fields.
- Enforce boundary conditions for total tangential electric and magnetic fields on the surface of the object and derive integral equations for inner and outer media.
- Combine inner and outer integral equations appropriately to obtain a set of equations to solve for equivalent currents.
- Calculate radiated electric and magnetic fields from equivalent currents.

Based on items above, various formulations can be derived by using different combinations of boundary conditions, testing schemes, and scaling operations. Several of them are stable, free of the internal-resonance problem, and commonly used in the literature [16]–[40].

2.2.1 Surface Integral Equations

Consider a homogeneous dielectric object with a closed surface separating two media with different electrical properties. Incident fields exist outside the object. Using the equivalence principle, we consider inner and outer problems separately

and derive two sets of equations by employing boundary conditions, i.e.,

$$\begin{aligned} \eta_o \hat{\mathbf{n}} \times \mathcal{T}_o\{\mathbf{J}\}(\mathbf{r}) - \hat{\mathbf{n}} \times \mathcal{K}_{PV,o}\{\mathbf{M}\}(\mathbf{r}) \\ + \frac{\Omega_o}{4\pi} \mathcal{I}\{\mathbf{M}\}(\mathbf{r}) = -\hat{\mathbf{n}} \times \mathbf{E}^{inc}(\mathbf{r}) \quad (\text{N-EFIE-O}) \end{aligned} \quad (2.10)$$

$$\begin{aligned} \eta_o \hat{\mathbf{t}} \cdot \mathcal{T}_o\{\mathbf{J}\}(\mathbf{r}) - \hat{\mathbf{t}} \cdot \mathcal{K}_{PV,o}\{\mathbf{M}\}(\mathbf{r}) \\ - \frac{\Omega_o}{4\pi} \hat{\mathbf{t}} \cdot \mathcal{I}^{\times n}\{\mathbf{M}\}(\mathbf{r}) = -\hat{\mathbf{t}} \cdot \mathbf{E}^{inc}(\mathbf{r}) \quad (\text{T-EFIE-O}) \end{aligned} \quad (2.11)$$

$$\begin{aligned} \frac{1}{\eta_o} \hat{\mathbf{n}} \times \mathcal{T}_o\{\mathbf{M}\}(\mathbf{r}) + \hat{\mathbf{n}} \times \mathcal{K}_{PV,o}\{\mathbf{J}\}(\mathbf{r}) \\ - \frac{\Omega_o}{4\pi} \mathcal{I}\{\mathbf{J}\}(\mathbf{r}) = -\hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}) \quad (\text{N-MFIE-O}) \end{aligned} \quad (2.12)$$

$$\begin{aligned} \frac{1}{\eta_o} \hat{\mathbf{t}} \cdot \mathcal{T}_o\{\mathbf{M}\}(\mathbf{r}) + \hat{\mathbf{t}} \cdot \mathcal{K}_{PV,o}\{\mathbf{J}\}(\mathbf{r}) \\ + \frac{\Omega_o}{4\pi} \hat{\mathbf{t}} \cdot \mathcal{I}^{\times n}\{\mathbf{J}\}(\mathbf{r}) = -\hat{\mathbf{t}} \cdot \mathbf{H}^{inc}(\mathbf{r}) \quad (\text{T-MFIE-O}) \end{aligned} \quad (2.13)$$

for the outer medium D_o and

$$\begin{aligned} \eta_i \hat{\mathbf{n}} \times \mathcal{T}_i\{\mathbf{J}\}(\mathbf{r}) - \hat{\mathbf{n}} \times \mathcal{K}_{PV,i}\{\mathbf{M}\}(\mathbf{r}) \\ - \frac{\Omega_i}{4\pi} \mathcal{I}\{\mathbf{M}\}(\mathbf{r}) = 0 \quad (\text{N-EFIE-I}) \end{aligned} \quad (2.14)$$

$$\begin{aligned} \eta_i \hat{\mathbf{t}} \cdot \mathcal{T}_i\{\mathbf{J}\}(\mathbf{r}) - \hat{\mathbf{t}} \cdot \mathcal{K}_{PV,i}\{\mathbf{M}\}(\mathbf{r}) \\ + \frac{\Omega_i}{4\pi} \hat{\mathbf{t}} \cdot \mathcal{I}^{\times n}\{\mathbf{M}\}(\mathbf{r}) = 0 \quad (\text{T-EFIE-I}) \end{aligned} \quad (2.15)$$

$$\begin{aligned} \frac{1}{\eta_i} \hat{\mathbf{n}} \times \mathcal{T}_i\{\mathbf{M}\}(\mathbf{r}) + \hat{\mathbf{n}} \times \mathcal{K}_{PV,i}\{\mathbf{J}\}(\mathbf{r}) \\ + \frac{\Omega_i}{4\pi} \mathcal{I}\{\mathbf{J}\}(\mathbf{r}) = 0 \quad (\text{N-MFIE-I}) \end{aligned} \quad (2.16)$$

$$\begin{aligned} \frac{1}{\eta_i} \hat{\mathbf{t}} \cdot \mathcal{T}_i\{\mathbf{M}\}(\mathbf{r}) + \hat{\mathbf{t}} \cdot \mathcal{K}_{PV,i}\{\mathbf{J}\}(\mathbf{r}) \\ - \frac{\Omega_i}{4\pi} \hat{\mathbf{t}} \cdot \mathcal{I}^{\times n}\{\mathbf{J}\}(\mathbf{r}) = 0 \quad (\text{T-MFIE-I}) \end{aligned} \quad (2.17)$$

for the inner medium D_i . In (2.10)–(2.17), $\mathbf{E}^{inc}$ and $\mathbf{H}^{inc}$ are incident electric and magnetic fields, and $\Omega_o = 4\pi - \Omega_i$ denotes the external solid angle. In (2.10)–(2.17), $\hat{\mathbf{t}}$ is any tangential unit vector at the observation point $\mathbf{r}$ on the surface.

2.2.2 Surface Formulations

For homogeneous dielectric objects, tangential formulations can be obtained by combining tangential equations [37] as

$$\begin{bmatrix} a\mathbf{T}\text{-EFIE-O} + b\mathbf{T}\text{-EFIE-I} \\ c\mathbf{T}\text{-MFIE-O} + d\mathbf{T}\text{-MFIE-I} \end{bmatrix} \quad (2.18)$$

leading to

$$\begin{aligned} \hat{\mathbf{t}} \cdot [a\eta_o \mathbf{T}_o + b\eta_i \mathbf{T}_i] \{\mathbf{J}\}(\mathbf{r}) - \hat{\mathbf{t}} \cdot [a\mathcal{K}_{PV,o} + b\mathcal{K}_{PV,i}] \{\mathbf{M}\}(\mathbf{r}) \\ - \frac{1}{4\pi} \hat{\mathbf{t}} \cdot [a\Omega_o - b\Omega_i] \mathcal{I}^{\times n} \{\mathbf{M}\}(\mathbf{r}) = -a\hat{\mathbf{t}} \cdot \mathbf{E}^{inc}(\mathbf{r}) \end{aligned} \quad (2.19)$$

$$\begin{aligned} \hat{\mathbf{t}} \cdot [c\mathcal{K}_{PV,o} + d\mathcal{K}_{PV,i}] \{\mathbf{J}\}(\mathbf{r}) + \hat{\mathbf{t}} \cdot [c\Omega_o - d\Omega_i] \mathcal{I}^{\times n} \{\mathbf{J}\}(\mathbf{r}) \\ + \hat{\mathbf{t}} \cdot [c\eta_o^{-1} \mathbf{T}_o + d\eta_i^{-1} \mathbf{T}_i] \{\mathbf{M}\}(\mathbf{r}) = -c\hat{\mathbf{t}} \cdot \mathbf{H}^{inc}(\mathbf{r}). \end{aligned} \quad (2.20)$$

Among infinitely many possibilities for scalars $\{a, b, c, d\}$, several choices provide stable formulations, such as

$$a = b = c = d = 1 \quad (2.21)$$

and

$$a = \eta_o^{-1}, \quad b = \eta_i^{-1}, \quad c = \eta_o, \quad d = \eta_i, \quad (2.22)$$

which lead to the T-PMCHWT formulation [17]–[19] and CTF [37], respectively.

CTF can be written as

$$\hat{\mathbf{t}} \cdot \begin{bmatrix} \mathcal{Z}_{11}^{\text{CTF}} & \mathcal{Z}_{12}^{\text{CTF}} \\ \mathcal{Z}_{21}^{\text{CTF}} & \mathcal{Z}_{22}^{\text{CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}(\mathbf{r}) = -\hat{\mathbf{t}} \cdot \begin{bmatrix} \eta_o^{-1} \mathbf{E}^{inc} \\ \eta_o \mathbf{H}^{inc} \end{bmatrix}(\mathbf{r}), \quad (2.23)$$

where

$$\mathcal{Z}_{11}^{\text{CTF}} = \mathcal{Z}_{22}^{\text{CTF}} = \mathbf{T}_o + \mathbf{T}_i \quad (2.24)$$

$$\mathcal{Z}_{12}^{\text{CTF}} = -\eta_o^{-1} \mathcal{K}_{PV,o} - \eta_i^{-1} \mathcal{K}_{PV,i} - \frac{1}{4\pi} (\Omega_o \eta_o^{-1} - \Omega_i \eta_i^{-1}) \mathcal{I}^{\times n} \quad (2.25)$$

$$\mathcal{Z}_{21}^{\text{CTF}} = \eta_o \mathcal{K}_{PV,o} + \eta_i \mathcal{K}_{PV,i} + \frac{1}{4\pi} (\Omega_o \eta_o - \Omega_i \eta_i) \mathcal{I}^{\times n}. \quad (2.26)$$

Matrix equations obtained from CTF have identical diagonal partitions, and they are usually better conditioned than matrix equations obtained from the T-PMCHWT formulation [37].

Different from tangential formulations, normal formulations are derived by using normal integral equations [37], i.e.,

$$\begin{bmatrix} a\text{N-MFIE-O} - b\text{N-MFIE-I} \\ -c\text{N-EFIE-O} + d\text{N-EFIE-I} \end{bmatrix}, \quad (2.27)$$

leading to

$$\begin{aligned} -\frac{1}{4\pi} [a\Omega_o + b\Omega_i] \mathcal{I}\{\mathbf{J}\}(\mathbf{r}) + \hat{\mathbf{n}} \times [a\mathcal{K}_{PV,o} - b\mathcal{K}_{PV,i}] \{\mathbf{J}\}(\mathbf{r}) \\ + \hat{\mathbf{n}} \times [a\eta_o^{-1}\mathcal{T}_o - b\eta_i^{-1}\mathcal{T}_i] \{\mathbf{M}\}(\mathbf{r}) = -a\hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}) \end{aligned} \quad (2.28)$$

$$\begin{aligned} \hat{\mathbf{n}} \times [-c\eta_o\mathcal{T}_o + d\eta_i\mathcal{T}_i] \{\mathbf{J}\}(\mathbf{r}) - \frac{1}{4\pi} [a\Omega_o + b\Omega_i] \mathcal{I}\{\mathbf{M}\}(\mathbf{r}) \\ + \hat{\mathbf{n}} \times [c\mathcal{K}_{PV,o} - d\mathcal{K}_{PV,i}] \{\mathbf{M}\}(\mathbf{r}) = c\hat{\mathbf{n}} \times \mathbf{E}^{inc}(\mathbf{r}). \end{aligned} \quad (2.29)$$

Choices

$$a = \mu_o, \quad b = \mu_i, \quad c = \epsilon_o, \quad d = \epsilon_i \quad (2.30)$$

and

$$a = b = c = d = 1 \quad (2.31)$$

in (2.27) lead to NMF [16] and CNF [37], respectively. CNF can be written as

$$\hat{\mathbf{n}} \times \begin{bmatrix} \mathcal{Z}_{11}^{\text{CNF}} & \mathcal{Z}_{12}^{\text{CNF}} \\ \mathcal{Z}_{21}^{\text{CNF}} & \mathcal{Z}_{22}^{\text{CNF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}(\mathbf{r}) = -\hat{\mathbf{n}} \times \begin{bmatrix} \mathbf{H}^{inc} \\ -\mathbf{E}^{inc} \end{bmatrix}(\mathbf{r}), \quad (2.32)$$

where

$$\mathcal{Z}_{11}^{\text{CNF}} = \mathcal{Z}_{22}^{\text{CNF}} = \mathcal{K}_{PV,o} - \mathcal{K}_{PV,i} + \mathcal{I}^{\times n} \quad (2.33)$$

$$\mathcal{Z}_{12}^{\text{CNF}} = \eta_o^{-1}\mathcal{T}_o - \eta_i^{-1}\mathcal{T}_i \quad (2.34)$$

$$\mathcal{Z}_{21}^{\text{CNF}} = -\eta_o\mathcal{T}_o + \eta_i\mathcal{T}_i. \quad (2.35)$$

On the other hand, choosing scalars as

$$a = \mu_o/(\mu_o + \mu_i), \quad b = \mu_i/(\mu_o + \mu_i), \quad c = \epsilon_o/(\epsilon_o + \epsilon_i), \quad d = \epsilon_i/(\epsilon_o + \epsilon_i) \quad (2.36)$$

in (2.27) leads to MNMF [36], which usually produces better-conditioned matrix equations than NMF and CNF. MNMF can be written as

$$\begin{aligned} \hat{\mathbf{n}} \times \begin{bmatrix} \mathcal{Z}_{11}^{\text{MNMF}} & \mathcal{Z}_{12}^{\text{MNMF}} \\ \mathcal{Z}_{21}^{\text{MNMF}} & \mathcal{Z}_{22}^{\text{MNMF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}(\mathbf{r}) \\ = -\hat{\mathbf{n}} \times \begin{bmatrix} \mu_o \mathbf{H}^{\text{inc}} / (\mu_o + \mu_i) \\ -\epsilon_o \mathbf{E}^{\text{inc}} / (\epsilon_o + \epsilon_i) \end{bmatrix}(\mathbf{r}), \end{aligned} \quad (2.37)$$

where

$$\begin{aligned} \mathcal{Z}_{11}^{\text{MNMF}} &= \frac{1}{(\mu_o + \mu_i)} (\mu_o \mathcal{K}_{PV,o} - \mu_i \mathcal{K}_{PV,i}) \\ &\quad + \frac{1}{4\pi(\mu_o + \mu_i)} (\mu_o \Omega_o + \mu_i \Omega_i) \mathcal{I}^{\times n} \end{aligned} \quad (2.38)$$

$$\mathcal{Z}_{12}^{\text{MNMF}} = \frac{1}{(\mu_o + \mu_i)} (\mu_o \eta_o^{-1} \mathcal{T}_o - \mu_i \eta_i^{-1} \mathcal{T}_i) \quad (2.39)$$

$$\mathcal{Z}_{21}^{\text{MNMF}} = -\frac{1}{(\epsilon_o + \epsilon_i)} (\epsilon_o \eta_o \mathcal{T}_o - \epsilon_i \eta_i \mathcal{T}_i) \quad (2.40)$$

$$\begin{aligned} \mathcal{Z}_{22}^{\text{MNMF}} &= \frac{1}{(\epsilon_o + \epsilon_i)} (\epsilon_o \mathcal{K}_{PV,o} - \epsilon_i \mathcal{K}_{PV,i}) \\ &\quad + \frac{1}{4\pi(\epsilon_o + \epsilon_i)} (\epsilon_o \Omega_o + \epsilon_i \Omega_i) \mathcal{I}^{\times n}. \end{aligned} \quad (2.41)$$

Finally, mixed formulations involve both tangential and normal integral equations. For example, JMCfie is obtained by using all equations in (2.10)–(2.17) [35], i.e.,

$$\begin{bmatrix} \text{N-MFIE-O} - \text{N-MFIE-I} + \eta_o^{-1} \text{T-EFIE-O} + \eta_i^{-1} \text{T-EFIE-I} \\ -\text{N-EFIE-O} + \text{N-EFIE-I} + \eta_o \text{T-MFIE-O} + \eta_i \text{T-MFIE-I} \end{bmatrix}. \quad (2.42)$$

Combining operators, we derive JMCfie as

$$\begin{aligned} \begin{bmatrix} \mathcal{Z}_{11}^{\text{JMCfie}} & \mathcal{Z}_{12}^{\text{JMCfie}} \\ \mathcal{Z}_{21}^{\text{JMCfie}} & \mathcal{Z}_{22}^{\text{JMCfie}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}(\mathbf{r}) \\ = \begin{bmatrix} -\eta_o^{-1} \hat{\mathbf{t}} \cdot \mathbf{E}^{\text{inc}}(\mathbf{r}) - \hat{\mathbf{n}} \times \mathbf{H}^{\text{inc}}(\mathbf{r}) \\ -\eta_o \hat{\mathbf{t}} \cdot \mathbf{H}^{\text{inc}}(\mathbf{r}) + \hat{\mathbf{n}} \times \mathbf{E}^{\text{inc}}(\mathbf{r}) \end{bmatrix}(\mathbf{r}), \end{aligned} \quad (2.43)$$

where

$$\mathcal{Z}_{11}^{\text{JMC FIE}} = \mathcal{Z}_{22}^{\text{JMC FIE}} = \hat{\mathbf{t}} \cdot (\mathcal{T}_o + \mathcal{T}_i) + \hat{\mathbf{n}} \times (\mathcal{K}_{PV,o} - \mathcal{K}_{PV,i}) - \mathcal{I} \quad (2.44)$$

$$\begin{aligned} \mathcal{Z}_{12}^{\text{JMC FIE}} &= -\hat{\mathbf{t}} \cdot (\eta_o^{-1} \mathcal{K}_{PV,o} + \eta_i^{-1} \mathcal{K}_{PV,i}) + \hat{\mathbf{n}} \times (\eta_o^{-1} \mathcal{T}_o - \eta_i^{-1} \mathcal{T}_i) \\ &\quad - \frac{1}{4\pi} (\Omega_o \eta_o^{-1} - \Omega_i \eta_i^{-1}) \hat{\mathbf{t}} \cdot \mathcal{I}^{\times n} \end{aligned} \quad (2.45)$$

$$\begin{aligned} \mathcal{Z}_{21}^{\text{JMC FIE}} &= \hat{\mathbf{t}} \cdot (\eta_o \mathcal{K}_{PV,o} + \eta_i \mathcal{K}_{PV,i}) - \hat{\mathbf{n}} \times (\eta_o \mathcal{T}_o - \eta_i \mathcal{T}_i) \\ &\quad + \frac{1}{4\pi} (\Omega_o \eta_o - \Omega_i \eta_i) \hat{\mathbf{t}} \cdot \mathcal{I}^{\times n}. \end{aligned} \quad (2.46)$$

2.2.3 Discretization

For numerical solutions of surface formulations, we discretize surfaces by using small planar triangles and employ basis functions to expand unknown surface current densities, i.e.,

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r}) \approx \sum_{n=1}^N \mathbf{a}_J[n] \mathbf{b}_n(\mathbf{r}) \quad (2.47)$$

$$\mathbf{M}(\mathbf{r}) = -\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r}) \approx \sum_{n=1}^N \mathbf{a}_M[n] \mathbf{b}_n(\mathbf{r}), \quad (2.48)$$

where $\mathbf{b}_n$ represents the n th basis function associated with the n th edge. Testing integral equations on the surface, $2N \times 2N$ matrix equations are constructed as

$$\begin{bmatrix} \bar{\mathbf{Z}}_{11} & \bar{\mathbf{Z}}_{12} \\ \bar{\mathbf{Z}}_{21} & \bar{\mathbf{Z}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J \\ \mathbf{a}_M \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix}, \quad (2.49)$$

where matrix elements correspond to interactions of basis and testing functions, while the RHS vector is obtained by testing incident fields. We note that matrix elements involve combinations of discretized operators depending on the formulation. Considering the n th basis function $\mathbf{b}_n$ and the m th testing function $\mathbf{t}_m$, tangentially-tested and normally-tested $\mathcal{K}$, $\mathcal{T}$, and $\mathcal{I}$ operators are discretized

as

$$\hat{\mathbf{t}} \cdot \mathcal{K}_{PV} \rightarrow \bar{\mathbf{K}}_{PV}^T[m, n] = \left\langle \mathbf{t}_m(\mathbf{r}), \mathcal{K}_{PV}\{\mathbf{b}_n\}(\mathbf{r}) \right\rangle \quad (2.50)$$

$$\hat{\mathbf{n}} \times \mathcal{K}_{PV} \rightarrow \bar{\mathbf{K}}_{PV}^N[m, n] = \left\langle \mathbf{t}_m(\mathbf{r}), \hat{\mathbf{n}} \times \mathcal{K}_{PV}\{\mathbf{b}_n\}(\mathbf{r}) \right\rangle \quad (2.51)$$

$$\hat{\mathbf{t}} \cdot \mathcal{T} \rightarrow \bar{\mathbf{T}}^T[m, n] = \left\langle \mathbf{t}_m(\mathbf{r}), \mathcal{T}\{\mathbf{b}_n\}(\mathbf{r}) \right\rangle \quad (2.52)$$

$$\hat{\mathbf{n}} \times \mathcal{T} \rightarrow \bar{\mathbf{T}}^N[m, n] = \left\langle \mathbf{t}_m(\mathbf{r}), \hat{\mathbf{n}} \times \mathcal{T}\{\mathbf{b}_n\}(\mathbf{r}) \right\rangle \quad (2.53)$$

$$\hat{\mathbf{t}} \cdot \mathcal{I} \rightarrow \bar{\mathbf{I}}^T[m, n] = \left\langle \mathbf{t}_m(\mathbf{r}), \mathcal{I}\{\mathbf{b}_n\}(\mathbf{r}) \right\rangle = \left\langle \mathbf{t}_m(\mathbf{r}), \mathbf{b}_n(\mathbf{r}) \right\rangle \quad (2.54)$$

$$\mathcal{I}^{\times n} \rightarrow \bar{\mathbf{I}}^N[m, n] = \left\langle \mathbf{t}_m(\mathbf{r}), \mathcal{I}^{\times n}\{\mathbf{b}_n\}(\mathbf{r}) \right\rangle = \left\langle \mathbf{t}_m(\mathbf{r}), \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \right\rangle, \quad (2.55)$$

where

$$\left\langle \mathbf{t}_m(\mathbf{r}), \mathbf{f}(\mathbf{r}) \right\rangle = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{f}(\mathbf{r}) \quad (2.56)$$

is a surface integral on the support of the testing function S_m . Elements of RHS vectors in (2.49) also depend on the formulation. They involve tangential and normal testing of incident electric and magnetic fields, i.e.,

$$\mathbf{v}_E^T[m] = \left\langle \mathbf{t}_m(\mathbf{r}), \mathbf{E}^{inc}(\mathbf{r}) \right\rangle = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{E}^{inc}(\mathbf{r}) \quad (2.57)$$

$$\mathbf{v}_E^N[m] = \left\langle \mathbf{t}_m(\mathbf{r}), \hat{\mathbf{n}} \times \mathbf{E}^{inc}(\mathbf{r}) \right\rangle = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{E}^{inc}(\mathbf{r}) \quad (2.58)$$

$$\mathbf{v}_H^T[m] = \left\langle \mathbf{t}_m(\mathbf{r}), \mathbf{H}^{inc}(\mathbf{r}) \right\rangle = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{H}^{inc}(\mathbf{r}) \quad (2.59)$$

$$\mathbf{v}_H^N[m] = \left\langle \mathbf{t}_m(\mathbf{r}), \hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}) \right\rangle = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}). \quad (2.60)$$

Matrix partitions in (2.49) can be derived as

$$\bar{\mathbf{Z}}_{11}^{\text{CTF}} = \bar{\mathbf{Z}}_{22}^{\text{CTF}} = \bar{\mathbf{T}}_o^T + \bar{\mathbf{T}}_i^T \quad (2.61)$$

$$\bar{\mathbf{Z}}_{12}^{\text{CTF}} = -\eta_o^{-1} \bar{\mathbf{K}}_{PV,o}^T - \eta_i^{-1} \bar{\mathbf{K}}_{PV,i}^T - \frac{1}{2}(\eta_o^{-1} - \eta_i^{-1}) \bar{\mathbf{I}}^N \quad (2.62)$$

$$\bar{\mathbf{Z}}_{21}^{\text{CTF}} = \eta_o \bar{\mathbf{K}}_{PV,o}^T + \eta_i \bar{\mathbf{K}}_{PV,i}^T + \frac{1}{2}(\eta_o - \eta_i) \bar{\mathbf{I}}^N \quad (2.63)$$

for CTF,

$$\bar{\mathbf{Z}}_{11}^{\text{CNF}} = \bar{\mathbf{Z}}_{22}^{\text{CNF}} = \bar{\mathbf{K}}_{PV,o}^N - \mu_i \bar{\mathbf{K}}_{PV,i}^N - \bar{\mathbf{I}}^T \quad (2.64)$$

$$\bar{\mathbf{Z}}_{12}^{\text{CNF}} = \eta_o^{-1} \bar{\mathbf{T}}_o^N - \eta_i^{-1} \bar{\mathbf{T}}_i^N \quad (2.65)$$

$$\bar{\mathbf{Z}}_{21}^{\text{CNF}} = -\eta_o \bar{\mathbf{T}}_o^N + \eta_i \bar{\mathbf{T}}_i^N \quad (2.66)$$

for CNF,

$$\bar{\mathbf{Z}}_{11}^{\text{MNMF}} = \frac{1}{(\mu_o + \mu_i)} (\mu_o \bar{\mathbf{K}}_{PV,o}^N - \mu_i \bar{\mathbf{K}}_{PV,i}^N) - \frac{1}{2} \bar{\mathbf{I}}^T \quad (2.67)$$

$$\bar{\mathbf{Z}}_{12}^{\text{MNMF}} = \frac{1}{(\mu_o + \mu_i)} (\mu_o \eta_o^{-1} \bar{\mathbf{T}}_o^N - \mu_i \eta_i^{-1} \bar{\mathbf{T}}_i^N) \quad (2.68)$$

$$\bar{\mathbf{Z}}_{21}^{\text{MNMF}} = -\frac{1}{(\epsilon_o + \epsilon_i)} (\epsilon_o \eta_o \bar{\mathbf{T}}_o^N - \epsilon_i \eta_i \bar{\mathbf{T}}_i^N) \quad (2.69)$$

$$\bar{\mathbf{Z}}_{22}^{\text{MNMF}} = \frac{1}{(\epsilon_o + \epsilon_i)} (\epsilon_o \bar{\mathbf{K}}_{PV,o}^N - \epsilon_i \bar{\mathbf{K}}_{PV,i}^N) - \frac{1}{2} \bar{\mathbf{I}}^T \quad (2.70)$$

for MNMF, and

$$\bar{\mathbf{Z}}_{11}^{\text{JMC FIE}} = \bar{\mathbf{Z}}_{22}^{\text{JMC FIE}} = \bar{\mathbf{T}}_o^T + \bar{\mathbf{T}}_i^T + \bar{\mathbf{K}}_{PV,o}^N - \bar{\mathbf{K}}_{PV,i}^N - \bar{\mathbf{I}}^T \quad (2.71)$$

$$\begin{aligned} \bar{\mathbf{Z}}_{12}^{\text{JMC FIE}} &= -\eta_o^{-1} \bar{\mathbf{K}}_{PV,o}^T - \eta_i^{-1} \bar{\mathbf{K}}_{PV,i}^T \\ &\quad + \eta_o^{-1} \bar{\mathbf{T}}_o^N - \eta_i^{-1} \bar{\mathbf{T}}_i^N - \frac{1}{2} (\eta_o^{-1} - \eta_i^{-1}) \bar{\mathbf{I}}^N \end{aligned} \quad (2.72)$$

$$\bar{\mathbf{Z}}_{21}^{\text{JMC FIE}} = \eta_o \bar{\mathbf{K}}_{PV,o}^T + \eta_i \bar{\mathbf{K}}_{PV,i}^T - \eta_o \bar{\mathbf{T}}_o^N + \eta_i \bar{\mathbf{T}}_i^N + \frac{1}{2} (\eta_o - \eta_i) \bar{\mathbf{I}}^N \quad (2.73)$$

for JMC FIE. We note that $\Omega_o = \Omega_i = 2\pi$ on planar triangles. RHS vectors in (2.49) are derived as

$$\left\{ \mathbf{w}_1^{\text{CTF}}, \mathbf{w}_2^{\text{CTF}} \right\} = \left\{ -\eta_o^{-1} \mathbf{v}_E^T, -\eta_o \mathbf{v}_H^T \right\}, \quad (2.74)$$

$$\left\{ \mathbf{w}_1^{\text{CNF}}, \mathbf{w}_2^{\text{CNF}} \right\} = \left\{ -\mathbf{v}_H^N, \mathbf{v}_E^N \right\}, \quad (2.75)$$

$$\left\{ \mathbf{w}_1^{\text{MNMF}}, \mathbf{w}_2^{\text{MNMF}} \right\} = \left\{ -\frac{\mu_o}{(\mu_o + \mu_i)} \mathbf{v}_H^N, \frac{\epsilon_o}{(\epsilon_o + \epsilon_i)} \mathbf{v}_E^N \right\}, \quad (2.76)$$

and

$$\left\{ \mathbf{w}_1^{\text{JMC FIE}}, \mathbf{w}_2^{\text{JMC FIE}} \right\} = \left\{ -\eta_o^{-1} \mathbf{v}_E^T - \mathbf{v}_H^N, -\eta_o \mathbf{v}_H^T + \mathbf{v}_E^N \right\} \quad (2.77)$$

for CTF, CNF, MNMF, and JMC FIE, respectively.

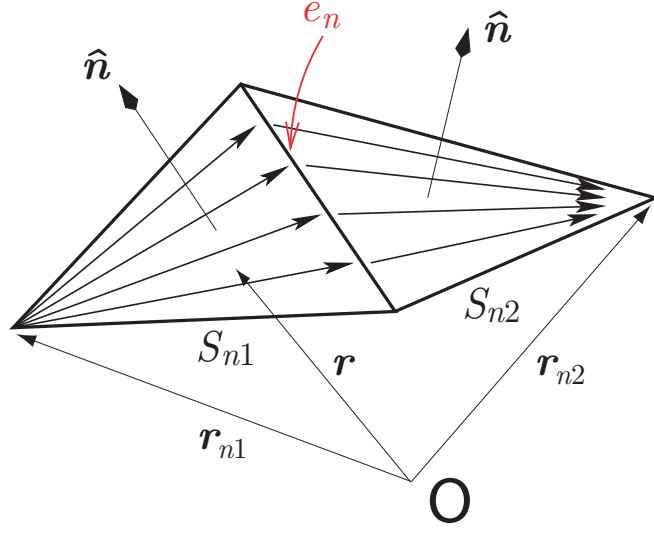


Figure 2.1: Spatial distribution of the RWG functions.

2.2.4 Direct Calculation of Interactions

Interaction between the m th testing function $\mathbf{t}_m$ and the n th basis function $\mathbf{b}_n$ are derived for different operators ($\mathcal{T}$ and $\mathcal{K}$) and testing types (T and N) as

$$\begin{aligned} \bar{\mathbf{T}}^T[m, n] &= ik \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n(\mathbf{r}') \\ &\quad + \frac{i}{k} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' \nabla g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \end{aligned} \quad (2.78)$$

$$\begin{aligned} \bar{\mathbf{T}}^N[m, n] &= ik \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n(\mathbf{r}') \\ &\quad + \frac{i}{k} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \int_{S_n} d\mathbf{r}' \nabla g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \end{aligned} \quad (2.79)$$

$$\bar{\mathbf{K}}_{PV}^T[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{PV, S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \times \nabla' g(\mathbf{r}, \mathbf{r}') \quad (2.80)$$

$$\bar{\mathbf{K}}_{PV}^N[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \int_{PV, S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \times \nabla' g(\mathbf{r}, \mathbf{r}'), \quad (2.81)$$

where S_m and S_n represent spatial supports of $\mathbf{t}_m$ and $\mathbf{b}_n$, respectively.

We discretize dielectric formulations by employing the RWG functions [22] having

a spatial distribution depicted in Figure 2.1, which can be expressed as

$$\mathbf{b}_n^{\text{RWG}}(\mathbf{r}) = \begin{cases} \frac{l_n}{2A_{n1}}(\mathbf{r} - \mathbf{r}_{n1}), & \mathbf{r} \in S_{n1} \\ \frac{l_n}{2A_{n2}}(\mathbf{r}_{n2} - \mathbf{r}), & \mathbf{r} \in S_{n2} \\ 0, & \mathbf{r} \notin S_n. \end{cases} \quad (2.82)$$

In (2.82), l_n represents the length of the edge, and A_{n1} and A_{n2} are respectively areas of the first (S_{n1}) and the second (S_{n2}) triangles associated with the edge.

For the RWG functions, direct calculations of integrals in (2.78)–(2.81) can be summarized as follows:

- Using the RWG functions, interactions in (2.80) and (2.81) can be modified as [93]

$$\bar{\mathbf{K}}_{PV}^T[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{b}_n(\mathbf{r}) \times \int_{PV, S_n} d\mathbf{r}' \nabla' g(\mathbf{r}, \mathbf{r}') \quad (2.83)$$

$$\bar{\mathbf{K}}_{PV}^N[m, n] = \int_{S_m} d\mathbf{r} (\mathbf{t}_m(\mathbf{r}) \times \hat{\mathbf{n}}) \cdot \mathbf{b}_n(\mathbf{r}) \times \int_{PV, S_n} d\mathbf{r}' \nabla' g(\mathbf{r}, \mathbf{r}'), \quad (2.84)$$

where outer integrals are evaluated numerically. Only the principal value of the inner integral is required, so that (2.83) and (2.84) are not evaluated for self interactions of triangles. However, the value of the inner integral is infinite when the testing point is on the edge of the source triangle. Since the singularity is logarithmic and it is quite mild, interaction of two near-neighborhood (touching) triangles can be calculated accurately by sampling the observation points strictly inside the testing triangle. In addition, accuracy and efficiency of calculations can be improved by extracting the singularity [157].

- Using divergence-conforming functions, such as the RWG functions, the interaction in (2.78) is modified as

$$\begin{aligned}\bar{\mathcal{T}}^T[m, n] &= ik \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n(\mathbf{r}') \\ &\quad + \frac{i}{k} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \nabla \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n(\mathbf{r}')\end{aligned}\quad (2.85)$$

$$\begin{aligned}&= ik \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n(\mathbf{r}') \\ &\quad + \frac{i}{k} \int_{S_m} d\mathbf{r} \nabla \cdot \left\{ \mathbf{t}_m(\mathbf{r}) \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \right\} \\ &\quad - \frac{i}{k} \int_{S_m} d\mathbf{r} \nabla \cdot \mathbf{t}_m(\mathbf{r}) \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n(\mathbf{r}')\end{aligned}\quad (2.86)$$

$$\begin{aligned}&= ik \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n(\mathbf{r}') \\ &\quad - \frac{i}{k} \int_{S_m} d\mathbf{r} \nabla \cdot \mathbf{t}_m(\mathbf{r}) \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n(\mathbf{r}')\end{aligned}\quad (2.87)$$

by moving the differential operator onto the testing function in the second term [22]. This way, the hyper-singularity of the $\mathcal{T}$ operator is eliminated. For the second term of (2.79), however, the differential operator is kept on the Green's function. Then, this term is calculated similarly to the interaction in (2.81). In contrast to (2.81), however, (2.79) should also be calculated for self interactions of triangles.

- During the calculation of interactions, loops are constructed over triangles instead of basis and testing functions. The integrals over basis and testing functions are divided into many basic (double) integrals that are independent from the alignment of functions.
- Basic integrals are evaluated in two steps; first, inner integrals are calculated, and then, they are used in forming integrands of outer integrals. Inner integrals are commonly shared by more than one basic integral.
- Inner integrals are performed via a decomposition into numerical and analytical parts [22],[92],[94]. Singularities arising due to the Green's function are extracted this way in order to avoid numerical difficulties and inaccuracies.

- Numerical integrations are usually performed by using adaptive methods [95] employing low-order Gaussian quadratures [96]. Such adaptive methods achieve the preset required error bounds by sampling integration points efficiently.

The processing time for the calculation of interactions in (2.78)–(2.81) depends on medium parameters. When the relative permittivity or permeability of a medium increases, it becomes difficult to evaluate interactions since integrands become more oscillatory. In such cases, accurate calculations of interactions require extracting more terms to smooth integrands or increasing the number of sampling points for numerical integrations.

Consider the interaction of two half RWG functions $\mathbf{t}_{ma}^{\text{RWG}}$ and $\mathbf{b}_{nb}^{\text{RWG}}$ associated with the a th triangle of the m th edge and the b th triangle of the n th edge, respectively. Then,

$$\bar{\mathbf{I}}^{T,\text{RWG}}[m, n, a, b] = \mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot (\mathbf{r} - \mathbf{r}_{nb}) \quad (2.88)$$

$$\bar{\mathbf{I}}^{N,\text{RWG}}[m, n, a, b] = \mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \hat{\mathbf{n}} \times (\mathbf{r} - \mathbf{r}_{nb}) \quad (2.89)$$

$$\begin{aligned} \bar{\mathbf{T}}^{T,\text{RWG}}[m, n, a, b] &= ik\mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \mathbf{I}_{in}^{(\mathbf{r}' - \mathbf{r}_{nb})} \\ &\quad - \frac{4i}{k} \mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r} I_{in}^1 \end{aligned} \quad (2.90)$$

$$\begin{aligned} \bar{\mathbf{T}}^{N,\text{RWG}}[m, n, a, b] &= ik\mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \hat{\mathbf{n}}_{ma} \times \mathbf{I}_{in}^{(\mathbf{r}' - \mathbf{r}_{nb})} \\ &\quad - \frac{2i}{k} \mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \hat{\mathbf{n}}_{ma} \times \mathbf{I}_{in}^{\nabla'} \end{aligned} \quad (2.91)$$

$$\bar{\mathbf{K}}_{PV}^{T,\text{RWG}}[m, n, a, b] = \mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot (\mathbf{r} - \mathbf{r}_{nb}) \times \mathbf{I}_{in}^{\nabla'} \quad (2.92)$$

$$\bar{\mathbf{K}}_{PV}^{N,\text{RWG}}[m, n, a, b] = \mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \times \hat{\mathbf{n}}_{ma} \cdot (\mathbf{r} - \mathbf{r}_{nb}) \times \mathbf{I}_{in}^{\nabla'}, \quad (2.93)$$

where

$$\mathcal{A}_{ma,nb} = \frac{l_m l_n}{4A_{ma}A_{nb}} \gamma_{ma} \gamma_{nb} \quad (2.94)$$

and

$$I_{in}^1 = \int_{S_{nb}} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \quad (2.95)$$

$$\mathbf{I}_{in}^{(\mathbf{r}' - \mathbf{r}_{nb})} = \int_{S_{nb}} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') (\mathbf{r}' - \mathbf{r}_{nb}) \quad (2.96)$$

$$\mathbf{I}_{in}^{\nabla'} = \int_{S_{nb}} d\mathbf{r}' \nabla' g(\mathbf{r}, \mathbf{r}'). \quad (2.97)$$

In (2.94), $\gamma_{nb} = \pm 1$ and $\gamma_{ma} = \pm 1$ depend on the direction of basis and testing functions on triangles. Interactions of full basis and testing functions can be calculated as

$$\left\{ \begin{array}{c} \bar{\mathbf{I}}^{T, \text{RWG}} \\ \bar{\mathbf{I}}^{N, \text{RWG}} \\ \bar{\mathbf{T}}^{T, \text{RWG}} \\ \bar{\mathbf{T}}^{N, \text{RWG}} \\ \bar{\mathbf{K}}_{PV}^{T, \text{RWG}} \\ \bar{\mathbf{K}}_{PV}^{N, \text{RWG}} \end{array} \right\} [m, n] = \sum_{a=1}^2 \sum_{b=1}^2 \left\{ \begin{array}{c} \bar{\mathbf{I}}^{T, \text{RWG}} \\ \bar{\mathbf{I}}^{N, \text{RWG}} \\ \bar{\mathbf{T}}^{T, \text{RWG}} \\ \bar{\mathbf{T}}^{N, \text{RWG}} \\ \bar{\mathbf{K}}_{PV}^{T, \text{RWG}} \\ \bar{\mathbf{K}}_{PV}^{N, \text{RWG}} \end{array} \right\} [m, n, a, b]. \quad (2.98)$$

We perform a coordinate transformation so that the basis triangle lies on the x - y plane with one of its edges on the x axis [95]. Such a transformation is essential in order to easily evaluate analytic integrals appearing in the singularity extraction. Then, interactions become

$$\bar{\mathbf{I}}^{T, \text{RWG}}[m, n, a, b] = \mathcal{A}_{ma, nb} \int_{S_{ma}} d\mathbf{r} (\boldsymbol{\rho} - \boldsymbol{\rho}_{ma}) \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{nb}) \quad (2.99)$$

$$\bar{\mathbf{I}}^{N, \text{RWG}}[m, n, a, b] = \mathcal{A}_{ma, nb} \int_{S_{ma}} d\mathbf{r} (\boldsymbol{\rho} - \boldsymbol{\rho}_{ma}) \cdot \hat{\mathbf{z}} \times (\boldsymbol{\rho} - \boldsymbol{\rho}_{nb}) \quad (2.100)$$

$$\begin{aligned} \bar{\mathbf{T}}^{T, \text{RWG}}[m, n, a, b] &= ik \mathcal{A}_{ma, nb} \int_{S_{ma}} d\mathbf{r} (\boldsymbol{\rho} - \boldsymbol{\rho}_{ma}) \cdot \mathbf{I}_{in}^{(\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb})} \\ &\quad - \frac{4i}{k} \mathcal{A}_{ma, nb} \int_{S_{ma}} d\mathbf{r} I_{in}^1 \end{aligned} \quad (2.101)$$

$$\begin{aligned}\bar{\mathbf{T}}^{N,\text{RWG}}[m, n, a, b] &= ik\mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r}(\mathbf{r} - \mathbf{r}_{ma}) \cdot \hat{\mathbf{n}}_{ma} \times \bar{\mathbf{I}}_{in}^{(\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb})} \\ &\quad - \frac{2i}{k}\mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r}(\mathbf{r} - \mathbf{r}_{ma}) \cdot \hat{\mathbf{n}}_{ma} \times \mathbf{I}_{in}^{\nabla'}\end{aligned}\quad (2.102)$$

$$\bar{\mathbf{K}}_{PV}^{T,\text{RWG}}[m, n, a, b] = \mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r}(\mathbf{r} - \mathbf{r}_{ma}) \cdot (\mathbf{r} - \boldsymbol{\rho}_{nb}) \times \mathbf{I}_{in}^{\nabla'} \quad (2.103)$$

$$\bar{\mathbf{K}}_{PV}^{N,\text{RWG}}[m, n, a, b] = \mathcal{A}_{ma,nb} \int_{S_{ma}} d\mathbf{r}(\mathbf{r} - \mathbf{r}_{ma}) \times \hat{\mathbf{n}}_{ma} \cdot (\mathbf{r} - \boldsymbol{\rho}_{nb}) \times \mathbf{I}_{in}^{\nabla'}, \quad (2.104)$$

where $\boldsymbol{\rho} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ and

$$\mathbf{I}_{in}^{(\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb})} = \int_{S_{nb}} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}')(\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb}). \quad (2.105)$$

For integro-differential operators, interactions are divided into basic integrals as follows:

- The tangentially-tested $\mathcal{T}$ operator involves 7 basic integrals, i.e.,

$$\begin{aligned}\bar{\mathbf{T}}^{T,\text{RWG}}[m, n, a, b] &= ik\mathcal{A}_{ma,nb} \left\{ (x_{ma}x_{nb} + y_{ma}y_{nb})I_1 \right. \\ &\quad \left. - x_{ma}I_2 - y_{ma}I_3 - x_{nb}I_4 - y_{nb}I_5 + I_6 + I_7 - \frac{4}{k^2}I_1 \right\},\end{aligned}\quad (2.106)$$

where

$$I_{\{1,4,5\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y\} I_{in}^1 \quad (2.107)$$

$$I_{\{2,6\}} = \int_{S_{ma}} d\mathbf{r} \{1, x\} I_{in}^{x'} \quad (2.108)$$

$$I_{\{3,7\}} = \int_{S_{ma}} d\mathbf{r} \{1, y\} I_{in}^{y'} \quad (2.109)$$

and

$$I_{in}^{x'} = \int_{S_{nb}} d\mathbf{r}' x' g(\mathbf{r}, \mathbf{r}') \quad (2.110)$$

$$I_{in}^{y'} = \int_{S_{nb}} d\mathbf{r}' y' g(\mathbf{r}, \mathbf{r}'). \quad (2.111)$$

- The normally-tested $\mathcal{T}$ operator involves 19 basic integrals, i.e.,

$$\begin{aligned}
\bar{\mathbf{T}}^{N,\text{RWG}}[m, n, a, b] = ik\mathcal{A}_{ma,nb} & \left\{ (n_y z_{ma} - n_z y_{ma})I_2 + (n_z x_{ma} - n_x z_{ma})I_3 \right. \\
& + (n_x z_{ma} y_{nb} - n_y z_{ma} x_{nb} + n_z y_{ma} x_{nb} - n_z x_{ma} y_{nb})I_1 \\
& + n_z y_{nb} I_4 - n_z x_{nb} I_5 + (n_y x_{nb} - n_x y_{nb})I_6 \\
& + n_z I_7 - n_z I_8 - n_y I_9 + n_x I_{10} \\
& - \frac{2}{k^2} \left[(z_{ma} n_y - y_{ma} n_z)I_{11} + (x_{ma} n_z - z_{ma} n_x)I_{14} \right. \\
& + n_z I_{12} - n_y I_{13} - n_z I_{15} + n_x I_{16} + (y_{ma} n_x - n_y x_{ma})I_{17} \\
& \left. \left. + n_y I_{18} - n_x I_{19} \right] \right\}, \tag{2.112}
\end{aligned}$$

where the normal of the testing triangle is decomposed as $\hat{\mathbf{n}}_{ma} = \hat{\mathbf{x}}n_x + \hat{\mathbf{y}}n_y + \hat{\mathbf{z}}n_z$, and

$$I_{\{1,4,5,6\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z\} I_{in}^1 \tag{2.113}$$

$$I_{\{2,7,9\}} = \int_{S_{ma}} d\mathbf{r} \{1, y, z\} I_{in}^{x'} \tag{2.114}$$

$$I_{\{3,8,10\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, z\} I_{in}^{y'} \tag{2.115}$$

$$I_{\{11,12,13\}} = \int_{S_{ma}} d\mathbf{r} \{1, y, z\} \hat{\mathbf{x}} \cdot \mathbf{I}_{in}^{\nabla'} \tag{2.116}$$

$$I_{\{14,15,16\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, z\} \hat{\mathbf{y}} \cdot \mathbf{I}_{in}^{\nabla'} \tag{2.117}$$

$$I_{\{17,18,19\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y\} \hat{\mathbf{z}} \cdot \mathbf{I}_{in}^{\nabla'}. \tag{2.118}$$

- The tangentially-tested $\mathcal{K}$ operator involves 9 basic integrals, i.e.,

$$\begin{aligned}
\bar{\mathbf{K}}_{PV}^{T,\text{RWG}}[m, n, a, b] = \mathcal{A}_{ma,nb} & \left\{ -z_{ma} y_{nb} I_1 + z_{ma} I_2 + (y_{nb} - y_{ma})I_3 \right. \\
& + z_{ma} x_{nb} I_4 - z_{ma} I_5 + (x_{ma} - x_{nb})I_6 \\
& + (x_{ma} y_{nb} - y_{ma} x_{nb})I_7 \\
& \left. + (y_{ma} - y_{nb})I_8 + (x_{nb} - x_{ma})I_9 \right\}, \tag{2.119}
\end{aligned}$$

where

$$I_{\{1,2,3\}} = \int_{S_{ma}} d\mathbf{r} \{1, y, z\} \hat{\mathbf{x}} \cdot \mathbf{I}_{in}^{\nabla'} \quad (2.120)$$

$$I_{\{4,5,6\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, z\} \hat{\mathbf{y}} \cdot \mathbf{I}_{in}^{\nabla'} \quad (2.121)$$

$$I_{\{7,8,9\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y\} \hat{\mathbf{z}} \cdot \mathbf{I}_{in}^{\nabla'}. \quad (2.122)$$

- The normally-tested $\mathcal{K}$ operator involves 24 basic integrals, i.e.,

$$\begin{aligned} \bar{\mathbf{K}}_{PV}^{N,\text{RWG}}[m, n, a, b] = \mathcal{A}_{ma,nb} & \left\{ \left[y_{nb}(n_x y_{ma} - n_y x_{ma}) \right] I_1 \right. \\ & + \left[n_y x_{ma} - n_x(y_{ma} + y_{nb}) \right] I_2 \\ & + \left[n_z x_{ma} - n_x z_{ma} \right] I_3 + \left[x_{nb}(n_y x_{ma} - n_x y_{ma}) \right] I_4 \\ & + \left[n_x y_{ma} - n_y(x_{ma} + x_{nb}) \right] I_5 + \left[n_z y_{ma} - n_y z_{ma} \right] I_6 \\ & + \left[x_{nb}(n_z x_{ma} - n_x z_{ma}) + y_{nb}(n_z y_{ma} - n_y z_{ma}) \right] I_7 \\ & + \left[z_{ma} n_x - n_z(x_{ma} + x_{nb}) \right] I_8 \\ & + \left[z_{ma} n_y - n_z(y_{ma} + y_{nb}) \right] I_9 \\ & + y_{nb} n_y I_{10} - n_y I_{11} - n_z I_{12} + n_x I_{13} + n_x I_{14} \\ & + x_{nb} n_x I_{15} - n_x I_{16} - n_z I_{17} + n_y I_{18} + n_y I_{19} \\ & - n_x I_{20} - n_y I_{21} + \left[n_x x_{nb} + n_y y_{nb} \right] I_{22} \\ & \left. + n_z I_{23} + n_z I_{24} \right\}, \quad (2.123) \end{aligned}$$

where

$$I_{\{1,10,2,3,11,12,13,14\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z, xy, xz, y^2, z^2\} \hat{\mathbf{x}} \cdot \mathbf{I}_{in}^{\nabla'} \quad (2.124)$$

$$I_{\{4,5,15,6,16,17,18,19\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z, xy, yz, x^2, z^2\} \hat{\mathbf{y}} \cdot \mathbf{I}_{in}^{\nabla'} \quad (2.125)$$

$$I_{\{7,8,9,22,20,21,23,24\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z, xz, yz, x^2, y^2\} \hat{\mathbf{z}} \cdot \mathbf{I}_{in}^{\nabla'}. \quad (2.126)$$

Finally, inner integrals are evaluated as

$$I_{in}^{\{1,x',y'\}} = \frac{1}{4\pi} \int_{S_{nb}} d\mathbf{r}' \{1, x', y'\} \frac{\exp(ikR) - 1}{R} \\ + \frac{1}{4\pi} \int_{S_{nb}} d\mathbf{r}' \{1, (x' - x), (y' - y)\} \frac{1}{R} + \frac{1}{4\pi} \{0, x, y\} \int_{S_{nb}} d\mathbf{r}' \frac{1}{R} \quad (2.127)$$

$$= \frac{1}{4\pi} I_{in,c}^{\{1,x',y'\}} + \frac{1}{4\pi} I_{in,a}^{\{1,(x'-x),(y'-y)\}} + \frac{1}{4\pi} \{0, x, y\} I_{in,a}^1 \quad (2.128)$$

and

$$\mathbf{I}_{in}^{\nabla'} = \frac{1}{4\pi} \int_{S_{nb}} d\mathbf{r}' \nabla' \left(\frac{\exp(ikR) - 1 + 0.5k^2 R^2}{R} \right) \\ + \frac{1}{4\pi} \int_{S_{nb}} d\mathbf{r}' \nabla' \left(\frac{1}{R} \right) - \frac{k^2}{8\pi} \int_{S_{nb}} d\mathbf{r}' \nabla' R \quad (2.129)$$

$$= \frac{1}{4\pi} \mathbf{I}_{in,c}^{\nabla'} + \frac{1}{4\pi} \mathbf{I}_{in,a1}^{\nabla'} - \frac{k^2}{8\pi} \mathbf{I}_{in,a2}^{\nabla'}. \quad (2.130)$$

In (2.127) and (2.129), we note that

$$\lim_{R \rightarrow 0} \frac{\exp(ikR) - 1}{R} = ik \quad (2.131)$$

$$\lim_{R \rightarrow 0} \nabla' \left(\frac{\exp(ikR) - 1 + 0.5k^2 R^2}{R} \right) = 0. \quad (2.132)$$

Therefore, $I_{in,c}^{\{1,x',y'\}}$ and $\mathbf{I}_{in,c}^{\nabla'}$ can be calculated numerically using adaptive integration methods [95] or Gaussian quadrature rules [96]. Remaining terms, i.e., $I_{in,a}^{\{1,(x'-x),(y'-y)\}}$, $\mathbf{I}_{in,a1}^{\nabla'}$, and $\mathbf{I}_{in,a2}^{\nabla'}$, are evaluated analytically [92].

2.2.5 Calculation of Interactions Using the Multilevel Fast Multipole Algorithm

By factorizing the Green's function and performing a diagonalization [79],[80], interactions in (2.78)–(2.81) can be rewritten as

$$\begin{Bmatrix} \bar{\mathbf{K}}^T \\ \bar{\mathbf{K}}^N \\ \bar{\mathbf{T}}^T \\ \bar{\mathbf{T}}^N \end{Bmatrix} [m, n] = \left(\frac{ik}{4\pi} \right)^2 \int d^2 \hat{\mathbf{k}} \begin{Bmatrix} \mathbf{F}_m^{\kappa,T} \\ \mathbf{F}_m^{\kappa,N} \\ \mathbf{F}_m^{\tau,T} \\ \mathbf{F}_m^{\tau,N} \end{Bmatrix} (\mathbf{r}_C, \mathbf{k}) \cdot \alpha_\tau(\mathbf{k}, \mathbf{r}_{CC'}) \mathbf{S}_n(\mathbf{r}_{C'}, \mathbf{k}) \quad (2.133)$$

when testing and basis functions are far from each other. In (2.133), the integral is evaluated on the unit sphere, $\mathbf{k} = \hat{\mathbf{k}}k$, and

$$\alpha_\tau(\mathbf{k}, \mathbf{r}_{CC'}) = \sum_{t=0}^{\tau} i^t (2t+1) h_t^{(1)}(kR_{CC'}) P_t(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}_{CC'}) \quad (2.134)$$

is the translation operator expressed in terms of the spherical Hankel function of the first kind $h_t^{(1)}$ and the Legendre polynomial P_t . The translation operator in (2.134) is employed to translate the radiation pattern of the n th basis function in box C' , i.e., $\mathbf{S}_n$, into incoming fields for testing functions in box C . Then, incoming fields are received by using the receiving pattern of the m th testing function, i.e., $\mathbf{F}_m$. The distance between two boxes is represented by the vector

$$\mathbf{r}_{CC'} = r_{CC'} \hat{\mathbf{r}}_{CC'} = \mathbf{r}_C - \mathbf{r}_{C'}, \quad (2.135)$$

where $\mathbf{r}_C$ and $\mathbf{r}_{C'}$ are reference points of C and C' , respectively.

The radiation pattern of a basis function $\mathbf{b}_n$ with respect to a reference point $\mathbf{r}_{C'}$ can be written as

$$\mathbf{S}_n(\mathbf{r}_{C'}, \mathbf{k}) = (\bar{\mathbf{I}}^{3 \times 3} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \int_{S_n} d\mathbf{r}' \exp[-i\mathbf{k} \cdot (\mathbf{r}' - \mathbf{r}_{C'})] \mathbf{b}_n(\mathbf{r}'), \quad (2.136)$$

where $\bar{\mathbf{I}}^{3 \times 3}$ denotes the 3×3 unit dyad. In contrast to radiation patterns, receiving patterns depend on the operator and the testing type. Using a Galerkin scheme, the receiving pattern of a testing function $\mathbf{t}_m$ with respect to a reference point $\mathbf{r}_C$ can be derived for different operators ($\mathcal{K}$ and $\mathcal{T}$) and testing types (T and N) as

$$\mathbf{F}_m^{\mathcal{K}, T}(\mathbf{r}_C, \mathbf{k}) = -\hat{\mathbf{k}} \times \int_{S_m} d\mathbf{r} \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_C)] \mathbf{t}_m(\mathbf{r}) \quad (2.137)$$

$$= -\hat{\mathbf{k}} \times \{\mathbf{S}_m(\mathbf{r}_C, \mathbf{k})\}^* \quad (2.138)$$

$$\mathbf{F}_m^{\mathcal{T},T}(\mathbf{r}_C, \mathbf{k}) = (\bar{\mathbf{I}}^{3 \times 3} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \int_{S_m} d\mathbf{r} \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_C)] \mathbf{t}_m(\mathbf{r}) \quad (2.139)$$

$$= \{\mathbf{S}_m(\mathbf{r}_C, \mathbf{k})\}^* \quad (2.140)$$

$$\mathbf{F}_m^{\mathcal{K},N}(\mathbf{r}_C, \mathbf{k}) = -\hat{\mathbf{k}} \times \int_{S_m} d\mathbf{r} \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_C)] \mathbf{t}_m(\mathbf{r}) \times \hat{\mathbf{n}} \quad (2.141)$$

$$\mathbf{F}_m^{\mathcal{T},N}(\mathbf{r}_C, \mathbf{k}) = (\bar{\mathbf{I}}^{3 \times 3} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \int_{S_m} d\mathbf{r} \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_C)] \mathbf{t}_m(\mathbf{r}) \times \hat{\mathbf{n}} \quad (2.142)$$

$$= \hat{\mathbf{k}} \times \mathbf{F}_m^{\mathcal{K},N}(\mathbf{r}_C, \mathbf{k}), \quad (2.143)$$

where “*” represents the complex-conjugate operation.

As an example, we consider a half RWG function associated with the first triangle of the m th edge. Then,

$$\mathbf{S}_{m1}^{\text{RWG}}(\mathbf{r}_C, \mathbf{k}) = \frac{l_m}{2A_{m1}} (\bar{\mathbf{I}}^{3 \times 3} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \int_{S_{m1}} d\mathbf{r} \exp[-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_C)] (\mathbf{r} - \mathbf{r}_{m1}) \quad (2.144)$$

and

$$\mathbf{F}_{m1}^{\mathcal{K},N,\text{RWG}}(\mathbf{r}_C, \mathbf{k}) = \frac{l_m}{2A_{m1}} \hat{\mathbf{k}} \times \hat{\mathbf{n}} \times \int_{S_{m1}} d\mathbf{r} \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_C)] (\mathbf{r} - \mathbf{r}_{m1}), \quad (2.145)$$

where integrals can be calculated analytically.

Using MLFMA, interactions in (2.133) are calculated in a multilevel scheme [1],[15]. During the aggregation stage, radiation patterns of boxes are calculated from the bottom to the top of the tree structure. We sample fields uniformly in the ϕ direction and use Gauss-Legendre points in the θ direction [80]. Then, a total of $[\tau(l) + 1] \times [2\tau(l) + 2]$ samples are required for each box, where $\tau(l)$ is the truncation number for the series in (2.134). To determine the value of $\tau(l)$ for each level $l = 1, 2, \dots, L$, we use the excess bandwidth formula for a one-box-buffer scheme [85]. Due to the oscillatory nature of the Helmholtz equation, the truncation number and the sampling rate for radiation and receiving patterns depend on the size of boxes with respect to the wavelength associated

with the medium. We employ local Lagrange interpolation methods to match different sampling rates of consecutive levels [1]. After the aggregation stage, translations are performed to obtain incoming fields for all boxes. Then, the disaggregation stage is performed from the top of the tree structure to the lowest level using antepolations [86]. Finally, angular integrations in (2.133) are computed to complete MVMs. An efficient implementation of MLFMA is detailed in Chapter 4.

In MLFMA, there are $\mathcal{O}(N)$ near-field interactions, which are calculated directly during the initial setup stage of the program and stored in memory, to be used multiple times during iterations. When the problem size is large, the setup of MLFMA usually requires negligible time, compared to the iterative solution part. Therefore, the long setup time of JMCIE, compared with CTF, CNF, and MNMF, is negligible when the overall time is considered for large problems. However, near-field interactions require a significant amount of memory. With nonidentical diagonal partitions, i.e., $\bar{\mathbf{Z}}_{11} \neq \bar{\mathbf{Z}}_{22}$, MNMF requires larger memory (4/3 that of others) to store near-field interactions.

Similar to near-field interactions, radiation and receiving patterns of basis and testing functions are calculated and stored in memory before iterations. Since patterns have only θ and ϕ components, they are stored in spherical coordinates. Using CTF, only one set of patterns is required for each medium, because both of the receiving patterns, $\mathbf{F}_m^{\mathcal{K},T}$ and $\mathbf{F}_m^{\mathcal{T},T}$, can be obtained from the related radiation pattern $\mathbf{S}_m$, as indicated in (2.138) and (2.140). In other words, receiving operations during MVMs can be performed by using radiation patterns (instead of receiving patterns) with small modifications involving a complex conjugation and a cross product with the angular direction $\hat{\mathbf{k}}$. However, CNF and MNMF require two sets of patterns, as receiving patterns $\mathbf{F}_m^{\mathcal{K},N}$ and $\mathbf{F}_m^{\mathcal{T},N}$ can be derived from each other, but they cannot be obtained directly from the related radiation

pattern $\mathbf{S}_m$ in spherical coordinates. Finally, JMCIE also requires two sets of patterns, considering all relations in (2.137)–(2.143).

For the solution of dielectric formulations with MLFMA, we employ the same tree structure for both inner and outer media. However, the sampling rate for radiated and incoming fields as well as the number of harmonics for translation operators depend on the wavenumber. Therefore, accurate calculation of interactions requires two separate versions of MLFMA to perform MVMs related to inner and outer media.

2.2.6 General Properties of Surface Formulations

Using a Galerkin method for the discretization of surface formulations, i.e., using the same set of functions to expand current densities and to test boundary conditions, $\mathcal{T}$ and $\mathcal{I}$ operators are well-tested with $\mathbf{t}_m$. On the other hand, $\mathcal{K}$ operator is well-tested with $\hat{\mathbf{n}} \times \mathbf{t}_m$ [37]. (Testing with $\hat{\mathbf{n}} \times \mathbf{t}_m$ is equivalent to testing with $\mathbf{t}_m$ after a projection onto the surface by using $\hat{\mathbf{n}}$.) Consequently, tangential formulations, such as CTF, contain well-tested $\mathcal{T}$ operators and/or weakly-tested $\mathcal{K}$ and $\mathcal{I}$ operators, i.e.,

$$\bar{\mathbf{T}}^T[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{T}\{\mathbf{b}_n\}(\mathbf{r}) \quad (2.146)$$

$$\bar{\mathbf{K}}_{PV}^T[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{K}_{PV}\{\mathbf{b}_n\}(\mathbf{r}) \quad (2.147)$$

$$\bar{\mathbf{I}}^N[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}). \quad (2.148)$$

On the other hand, normal formulations, such as MNMF and CNF, contain well-tested $\mathcal{K}$ and $\mathcal{I}$ operators and/or weakly-tested $\mathcal{T}$ operators, i.e.,

$$\bar{\mathbf{K}}_{PV}^N[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{K}_{PV}\{\mathbf{b}_n\}(\mathbf{r}) \quad (2.149)$$

$$\bar{\mathbf{I}}^T[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{b}_n(\mathbf{r}) \quad (2.150)$$

$$\bar{\mathbf{T}}^N[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{T}\{\mathbf{b}_n\}(\mathbf{r}). \quad (2.151)$$

Finally, in mixed formulations, such as JMCIE, all three operators are well-tested. The value of $\bar{\mathbf{I}}^T[m, n]$ in (2.150) is nonzero for overlapping testing and basis functions, and it typically has a large value for $m = n$. Therefore, matrix equations obtained with normal and mixed formulations are diagonally dominant. We also note that well-tested operators appear on diagonal partitions of matrix equations for all formulations. This is a required property in terms of stability.

In general, conditioning and accuracy properties of tangential formulations are determined by well-tested $\mathcal{T}$ operators. Using a Galerkin scheme and divergence-conforming functions, tangentially-tested $\mathcal{T}$ operator has a weakly-singular kernel. This leads to ill-conditioned matrix equations, but very accurate results, provided that the condition number of the matrix is not too large to deteriorate the accuracy of the solutions. On the other hand, normal and mixed formulations are dominated by well-tested identity operators. This is preferable in terms of efficiency since diagonally-dominant matrix equations are well-conditioned and easy to solve iteratively. However, discretization of the identity operator involves a large numerical error, which significantly contaminates the accuracy of results.

2.2.7 Decoupling for Perfectly-Conducting Surfaces

When the object is PEC, operators associated with the inner medium and the magnetic current disappear in formulations. Then, T-EFIE and T-MFIE in

tangential formulations are decoupled, and they can be solved independently to obtain the induced electric current on the surface of the object. For example, CTF in (2.23)–(2.26) reduces to two independent equations, i.e.,

$$\hat{\mathbf{t}} \cdot \mathcal{T}_o\{\mathbf{J}\}(\mathbf{r}) = -\eta_o^{-1} \hat{\mathbf{t}} \cdot \mathbf{E}^{inc}(\mathbf{r}) \quad (2.152)$$

$$\hat{\mathbf{t}} \cdot \mathcal{K}_{PV,o}\{\mathbf{J}\}(\mathbf{r}) + \frac{\Omega_o}{4\pi} \hat{\mathbf{t}} \cdot \mathcal{I}^{\times n}\{\mathbf{J}\}(\mathbf{r}) = -\hat{\mathbf{t}} \cdot \mathbf{H}^{inc}(\mathbf{r}). \quad (2.153)$$

T-MFIE in (2.153) is extremely unstable since both $\mathcal{K}$ and identity operators are weakly-tested. On the other hand, T-EFIE in (2.152) is stable and is commonly used in the literature [22]. Similarly, in normal formulations, N-MFIE and N-EFIE are decoupled for PEC objects. For example, MNMF in (2.37)–(2.41) reduces to two independent equations, i.e.,

$$\hat{\mathbf{n}} \times \mathcal{K}_{PV,o}\{\mathbf{J}\}(\mathbf{r}) - \frac{\Omega_o}{4\pi} \mathcal{I}\{\mathbf{J}\}(\mathbf{r}) = -\hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}) \quad (2.154)$$

$$\hat{\mathbf{n}} \times \mathcal{T}_o\{\mathbf{J}\}(\mathbf{r}) = -\eta_o^{-1} \hat{\mathbf{n}} \times \mathbf{E}^{inc}(\mathbf{r}). \quad (2.155)$$

N-EFIE in (2.155) is unstable since the $\mathcal{T}$ operator is weakly tested, whereas N-MFIE in (2.154) is stable and commonly used in the literature [93].

Finally, mixed formulations are decomposed into two independent formulations for PEC objects. Specifically, JMCFIE in (2.43)–(2.46) reduces to two mixed formulations, i.e., stable T-N-CFIE

$$\begin{aligned} \hat{\mathbf{t}} \cdot \mathcal{T}_o\{\mathbf{J}\}(\mathbf{r}) + \hat{\mathbf{n}} \times \mathcal{K}_{PV,o}\{\mathbf{J}\}(\mathbf{r}) - \frac{\Omega_o}{4\pi} \mathcal{I}\{\mathbf{J}\}(\mathbf{r}) \\ = -\eta_o^{-1} \hat{\mathbf{t}} \cdot \mathbf{E}^{inc}(\mathbf{r}) - \hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}) \end{aligned} \quad (2.156)$$

and unstable N-T-CFIE

$$\begin{aligned} -\hat{\mathbf{n}} \times \mathcal{T}_o\{\mathbf{J}\}(\mathbf{r}) + \hat{\mathbf{t}} \cdot \mathcal{K}_{PV,o}\{\mathbf{J}\}(\mathbf{r}) + \frac{\Omega_o}{4\pi} \hat{\mathbf{t}} \cdot \mathcal{I}^{\times n}\{\mathbf{J}\}(\mathbf{r}) \\ = \eta_o^{-1} \hat{\mathbf{n}} \times \mathbf{E}^{inc}(\mathbf{r}) - \hat{\mathbf{t}} \cdot \mathbf{H}^{inc}(\mathbf{r}). \end{aligned} \quad (2.157)$$

T-N-CFIE in (2.156) is free of the internal-resonance problem and is a preferable formulation for closed conductors [20].

2.2.8 Iterative Solutions and Preconditioning

MLFMA provides the solution of large problems by reducing the complexity of MVMs required by iterative solvers from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$ [1]. For efficient solutions, however, the number of iterations should also be small, in addition to fast MVMs. To reduce the number of iterations required for solutions, we apply two types of efficient preconditioners, namely, a two-partition block-diagonal preconditioner (2PBDP) and a four-partition block-diagonal preconditioner (4PBDP). Matrix equations in (2.49) can be preconditioned as

$$\bar{\mathbf{P}}^{-1} \cdot \begin{bmatrix} \bar{\mathbf{Z}}_{11} & \bar{\mathbf{Z}}_{12} \\ \bar{\mathbf{Z}}_{21} & \bar{\mathbf{Z}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J \\ \mathbf{a}_M \end{bmatrix} = \bar{\mathbf{P}}^{-1} \cdot \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix}, \quad (2.158)$$

where $\bar{\mathbf{P}}$ is a preconditioner matrix. In MLFMA, there are $\mathcal{O}(N)$ near-field interactions, which are calculated directly and are available for constructing preconditioners. These interactions are between basis and testing functions that are located in the same box or in two touching boxes at the lowest level of the tree structure. During solutions with MLFMA, we reorder basis/testing functions according to their positions in the multilevel tree. Let N_1 be the number of boxes at the lowest level and $N(C)$ represent the number of basis/testing functions in box $C = 1, 2, \dots, N_1$. Then, basis/testing functions in box C are indexed from $N^+(C) + 1$ to $N^+(C) + N(C)$, where

$$N^+(C) = \sum_{C'=1}^{C-1} N(C'). \quad (2.159)$$

This way, the system matrix in (2.49) has a block structure, where each block represents the interaction of a pair of boxes at the lowest level. In the sparse near-field matrix, only blocks corresponding to self interactions of boxes or interactions of two touching boxes involve nonzero elements.

BDP, which is based on using self interactions of boxes at the lowest level, is commonly used to accelerate MLFMA solutions of electromagnetics problems

involving perfectly-conducting objects [15]. The preconditioner matrix, which has a block-diagonal structure, can be inverted and used efficiently with $\mathcal{O}(N)$ complexity. Although BDP is successful in reducing iteration counts for normal and mixed formulations, such as N-MFIE and T-N-CFIE, it may not accelerate iterative solutions of tangential equations, such as T-EFIE. In fact, T-EFIE solutions are usually decelerated with BDP, and BDP is rarely useful for EFIE [121].

A direct extension of BDP for dielectric problems, which we call 2PBDP, involves self interactions of boxes at the lowest level in diagonal partitions, i.e.,

$$\bar{\mathbf{P}}_{2P} = \begin{bmatrix} \bar{\mathbf{P}}_{11} & 0 \\ 0 & \bar{\mathbf{P}}_{22} \end{bmatrix}, \quad (2.160)$$

where $\bar{\mathbf{P}}_{11} \approx \bar{\mathbf{Z}}_{11}$ and $\bar{\mathbf{P}}_{22} \approx \bar{\mathbf{Z}}_{22}$ are block-diagonal matrices. Then, a preconditioned matrix equation can be written as

$$\begin{bmatrix} \bar{\mathbf{B}}_{11} \cdot \bar{\mathbf{Z}}_{11} & \bar{\mathbf{B}}_{11} \cdot \bar{\mathbf{Z}}_{12} \\ \bar{\mathbf{B}}_{22} \cdot \bar{\mathbf{Z}}_{21} & \bar{\mathbf{B}}_{22} \cdot \bar{\mathbf{Z}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J \\ \mathbf{a}_M \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{B}}_{11} \cdot \mathbf{w}_1 \\ \bar{\mathbf{B}}_{22} \cdot \mathbf{w}_2 \end{bmatrix}, \quad (2.161)$$

where $\bar{\mathbf{B}}_{11} = \bar{\mathbf{P}}_{11}^{-1}$ and $\bar{\mathbf{B}}_{22} = \bar{\mathbf{P}}_{22}^{-1}$ are also block-diagonal matrices. Similar to BDP, 2PBDP accelerates iterative solutions of normal and mixed formulations, such as CNF, MNMF, and JMCIE.

To further improve iterative solutions of normal and mixed formulations, we use 4PBDP, which is based on using diagonal blocks, i.e., self interactions of boxes at the lowest level, in all four partitions of matrix equations. This way, some of large elements in off-diagonal matrix partitions ($\bar{\mathbf{Z}}_{12}$ and $\bar{\mathbf{Z}}_{21}$) are considered in constructing an effective preconditioner. Resulting preconditioner matrices are in the form of

$$\bar{\mathbf{P}}_{4P} \approx \begin{bmatrix} \bar{\mathbf{P}}_{11} & \bar{\mathbf{P}}_{12} \\ \bar{\mathbf{P}}_{21} & \bar{\mathbf{P}}_{22} \end{bmatrix}. \quad (2.162)$$

Since $\bar{\mathbf{P}}_{12}$ and $\bar{\mathbf{P}}_{21}$ are also block-diagonal matrices, the inverse of $\bar{\mathbf{P}}_{4P}$ can be evaluated efficiently as

$$\bar{\mathbf{P}}_{4P}^{-1} = \begin{bmatrix} \bar{\mathbf{B}}_{11} & \bar{\mathbf{B}}_{12} \\ \bar{\mathbf{B}}_{21} & \bar{\mathbf{B}}_{22} \end{bmatrix}, \quad (2.163)$$

where

$$\bar{\mathbf{B}}_{11} = \bar{\mathbf{P}}_{11}^{-1} \cdot \left[\bar{\mathbf{I}} + \bar{\mathbf{P}}_{12} \cdot \bar{\mathbf{S}}^{-1} \cdot \bar{\mathbf{P}}_{21} \cdot \bar{\mathbf{P}}_{11}^{-1} \right] \quad (2.164)$$

$$\bar{\mathbf{B}}_{12} = -\bar{\mathbf{P}}_{11}^{-1} \cdot \bar{\mathbf{P}}_{12} \cdot \bar{\mathbf{S}}^{-1} \quad (2.165)$$

$$\bar{\mathbf{B}}_{21} = -\bar{\mathbf{S}}^{-1} \cdot \bar{\mathbf{P}}_{21} \cdot \bar{\mathbf{P}}_{11}^{-1} \quad (2.166)$$

$$\bar{\mathbf{B}}_{22} = \bar{\mathbf{S}}^{-1} \quad (2.167)$$

and

$$\bar{\mathbf{S}} = \bar{\mathbf{P}}_{22} - \bar{\mathbf{P}}_{21} \cdot \bar{\mathbf{P}}_{11}^{-1} \cdot \bar{\mathbf{P}}_{12} \quad (2.168)$$

is the Schur complement of $\bar{\mathbf{P}}_{11}$. Matrix operations in (2.164)–(2.168), i.e., matrix-matrix multiplications, the inversion of $\bar{\mathbf{P}}_{11}$, and the inversion of $\bar{\mathbf{S}}$ can be performed efficiently in $\mathcal{O}(N)$ time using $\mathcal{O}(N)$ memory. Our numerical experiments show that the extra cost of 4PBDP with respect to 2PBDP is always negligible, considering the overall cost of solutions with MLFMA. Nevertheless, 4PBDP can significantly improve the efficiency of solutions by reducing iteration counts, and it is especially useful when the acceleration provided by 2PBDP is not sufficient.

In general, iterative solutions of CNF and JMCIE become difficult as the contrast increases, i.e., when permittivity and permeability change significantly across dielectric interfaces. The main reason is the existence of off-diagonal matrix partitions, which are numerically sensitive to the contrast. Off-diagonal partitions of JMCIE and CNF are significantly unbalanced due to multiplications with η^{-1} and η in $\bar{\mathbf{Z}}_{12}$ and $\bar{\mathbf{Z}}_{21}$, respectively. Although this may not be critical for low contrasts, the off-diagonal partition $\bar{\mathbf{Z}}_{21}$ dominates the overall matrix,

as the contrast of the object increases. Elements in the off-diagonal partition of CNF can be written as

$$\begin{aligned}\bar{\mathbf{Z}}_{21}^{\text{CNF}}[m, n] = i\omega \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) \times \\ \left\{ \int_{S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \left[\mu_i g_i(\mathbf{r}, \mathbf{r}') - \mu_o g_o(\mathbf{r}, \mathbf{r}') \right] \right. \\ \left. + \frac{1}{\omega^2} \int_{S_n} d\mathbf{r}' \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \nabla \left[\frac{g_i(\mathbf{r}, \mathbf{r}')}{\epsilon_i} - \frac{g_o(\mathbf{r}, \mathbf{r}')}{\epsilon_o} \right] \right\}. \quad (2.169)\end{aligned}$$

$\bar{\mathbf{Z}}_{21}^{\text{CNF}}$ is also a major contribution in $\bar{\mathbf{Z}}_{21}^{\text{JMC FIE}}$ for sufficiently high contrasts.

Using a Taylor series expansion for the exponential in the Green's function,

$$\left[\mu_i g_i(\mathbf{r}, \mathbf{r}') - \mu_o g_o(\mathbf{r}, \mathbf{r}') \right] = \frac{1}{4\pi R} \sum_{s=0}^{\infty} \frac{(i\omega R)^s}{s!} \left[\mu_i (\mu_i \epsilon_i)^{s/2} - \mu_o (\mu_o \epsilon_o)^{s/2} \right] \quad (2.170)$$

and

$$\begin{aligned}\nabla \left[\frac{g_i(\mathbf{r}, \mathbf{r}')}{\epsilon_i} - \frac{g_o(\mathbf{r}, \mathbf{r}')}{\epsilon_o} \right] = \frac{\hat{\mathbf{R}}}{4\pi R^2} \sum_{s=0}^{\infty} \frac{(i\omega R)^{s+1}}{s!} \left[\frac{(\mu_i \epsilon_i)^{s/2+1/2}}{\epsilon_i} - \frac{(\mu_o \epsilon_o)^{s/2+1/2}}{\epsilon_o} \right] \\ - \frac{\hat{\mathbf{R}}}{4\pi R^2} \sum_{s=0}^{\infty} \frac{(i\omega R)^s}{s!} \left[\frac{(\mu_i \epsilon_i)^{s/2}}{\epsilon_i} - \frac{(\mu_o \epsilon_o)^{s/2}}{\epsilon_o} \right], \quad (2.171)\end{aligned}$$

where $\mathbf{R} = (\mathbf{r} - \mathbf{r}') = \hat{\mathbf{R}}R$. We note that $1/R$ and $1/R^2$ singularities in (2.170) and (2.171) exist when $\mu_o \neq \mu_i$ and $\epsilon_o \neq \epsilon_i$, respectively. In addition, numerical values of expressions in (2.170) and (2.171), thus $\bar{\mathbf{Z}}_{21}^{\text{CNF}}$, grow rapidly with the increasing contrast. As a result, the overall matrix equation becomes significantly unbalanced and difficult to solve iteratively. For those problems formulated with CNF and JMC FIE, 4PBDP, which employs large elements in $\bar{\mathbf{Z}}_{21}$, can be very effective to accelerate iterations.

2.2.9 Comparison of Formulations: Efficiency

In this section, we investigate the efficiency of solutions when electromagnetics problems are formulated with CTF, CNF, MNMF, and JMC FIE. For all solutions, near-field interactions are calculated with at most 1% error, and far-field

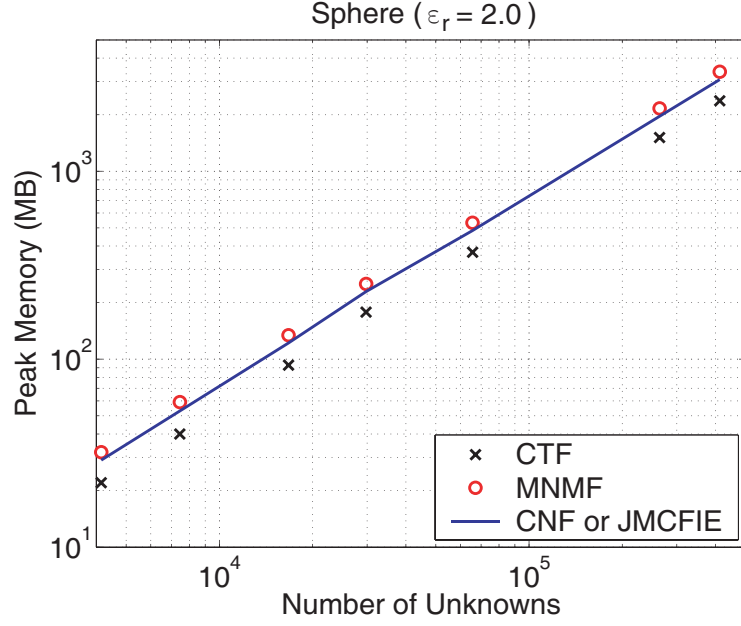


Figure 2.2: Peak memory required for MLFMA solutions of scattering problems involving a sphere with a relative permittivity of 2.0 located in free space. The radius of the sphere is in the range from $0.75\lambda_o$ to $7.5\lambda_o$.

interactions are computed by MLFMA with three digits of accuracy. Tree structures are constructed by fixing the size of boxes at the lowest level to $0.25\lambda_o$ and using a bottom-up scheme.

First, we compare the memory required for MLFMA solutions of various formulations. We consider a dielectric sphere with a radius of 0.3 m located in free space and illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. The radius of the sphere changes from $0.75\lambda_o$ to $7.5\lambda_o$, where λ_o is the wavelength in free space. Discretizations with $\lambda_o/10$ triangulations lead to 4142 and 412,998 unknowns, respectively, for radii $0.75\lambda_o$ and $7.5\lambda_o$. The relative permittivity of the sphere is $\epsilon_r = 2.0$. Figure 2.2 presents the peak memory required for MLFMA solutions with respect to the number of unknowns. The peak memory depends on the formulation type mainly because of different storage requirements for near-field interactions

Table 2.1: Number of CGS Iterations (10^{-3} Residual Error) for the Solution of Sphere Problems with $\epsilon_r = 4.0$

Sphere Problems		CTF			CNF		
Radius	Unknowns	NP	2PBDP	4PBDP	NP	2PBDP	4PBDP
$0.75\lambda_o$	4152	102	344	325	48	47	32
$1.00\lambda_o$	7446	245	623	557	87	52	42
$1.50\lambda_o$	16,728	259	1145	959	184	115	76
$2.00\lambda_o$	29,742	213	1317	1183	272	182	90
$3.00\lambda_o$	65,724	678	NC	NC	305	278	128
$6.00\lambda_o$	264,006	417	NC	NC	NC	1583	755
$7.50\lambda_o$	412,998	369	NC	NC	1153	1005	468
Sphere Problems		MNMF			JMCIE		
Radius	Unknowns	NP	2PBDP	4PBDP	NP	2PBDP	4PBDP
$0.75\lambda_o$	4152	34	26	19	46	34	28
$1.00\lambda_o$	7446	46	30	30	94	48	37
$1.50\lambda_o$	16,728	66	51	32	129	77	63
$2.00\lambda_o$	29,742	66	54	46	122	86	54
$3.00\lambda_o$	65,724	144	111	108	173	189	113
$6.00\lambda_o$	264,006	332	234	174	414	276	148
$7.50\lambda_o$	412,998	380	303	321	269	230	123

(identical or nonidentical diagonal partitions) and far-field patterns of basis and testing functions (one set or two sets of patterns). Having identical diagonal partitions and using one set of far-field patterns, CTF requires less memory than other formulations. CNF and JMCIE require two sets of far-field patterns, and their memory usage is larger than CTF. Finally, MNMF has nonidentical diagonal partitions and requires two sets of far-field patterns, leading to a larger memory usage than CTF, CNF, and JMCIE. Although the memory is not critical for small problems, it becomes more important as the problem size grows. For example, when the radius is $7.5\lambda_o$, the peak memory is 2370 MB and 3385 MB for CTF and MNMF, respectively.

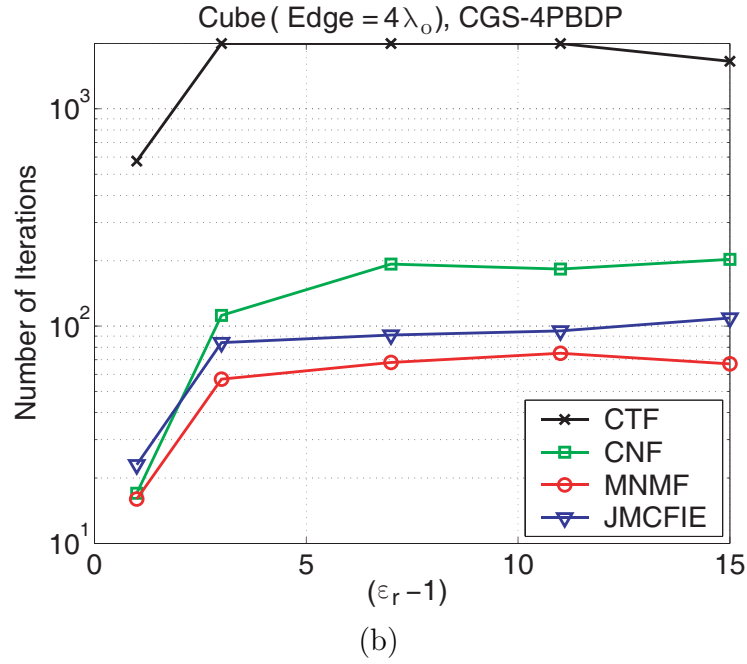
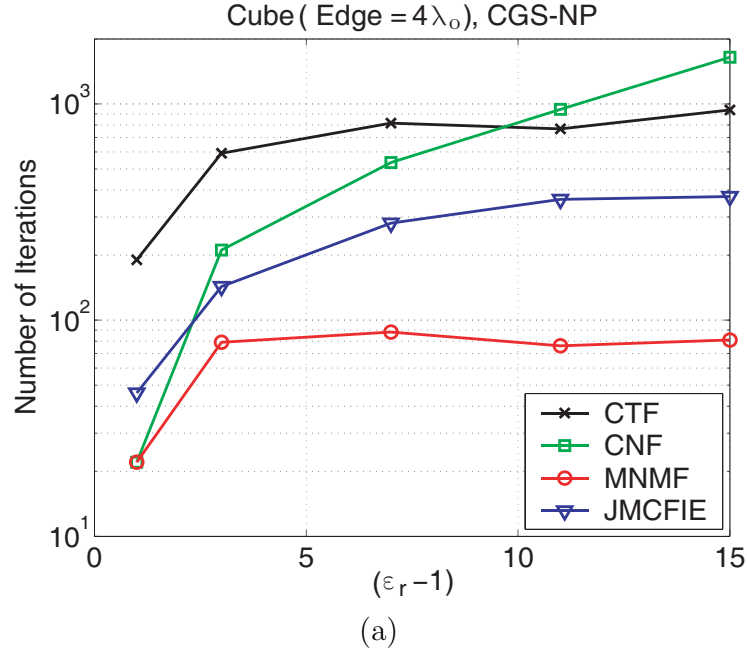
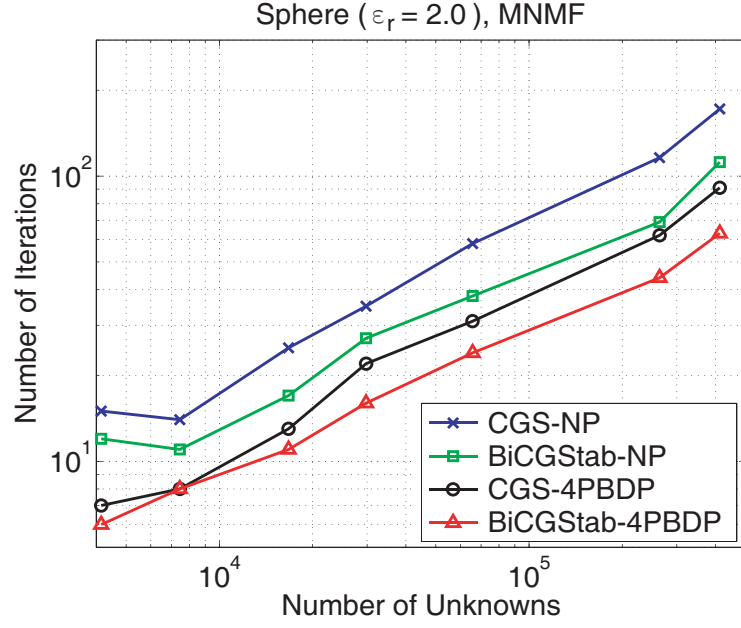
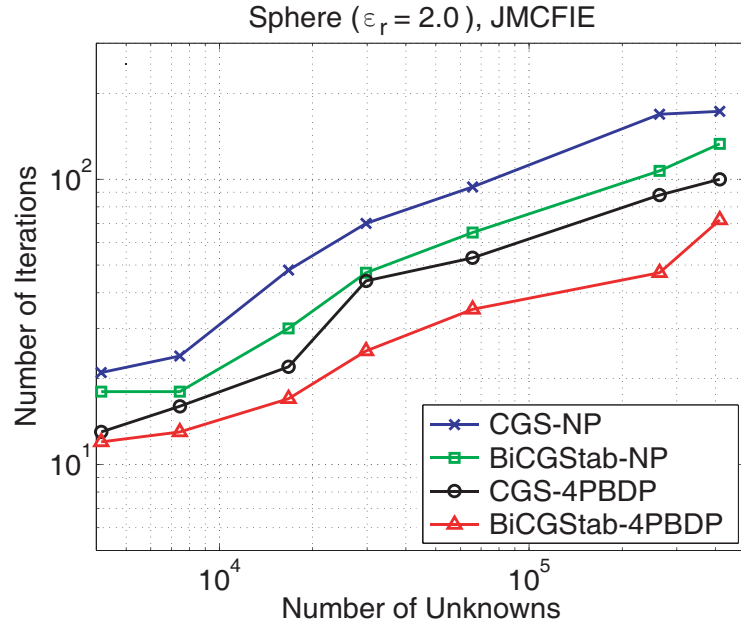


Figure 2.3: Iteration counts for the solution of scattering problems involving a cube with edges of $4\lambda_o$ located in free space. The relative permittivity of the cube changes from 2.0 to 16.0. Iterative solutions are performed by CGS (a) without preconditioning and (b) accelerated with 4PBDP.

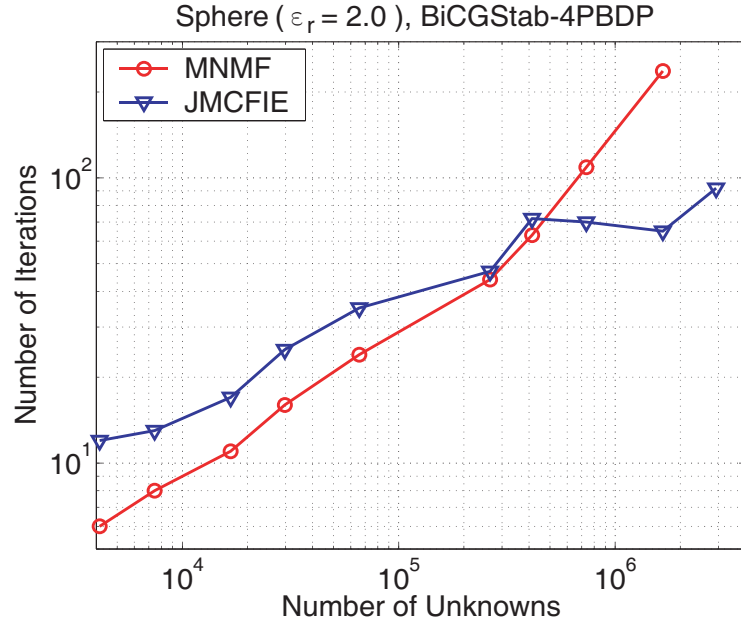


(a)

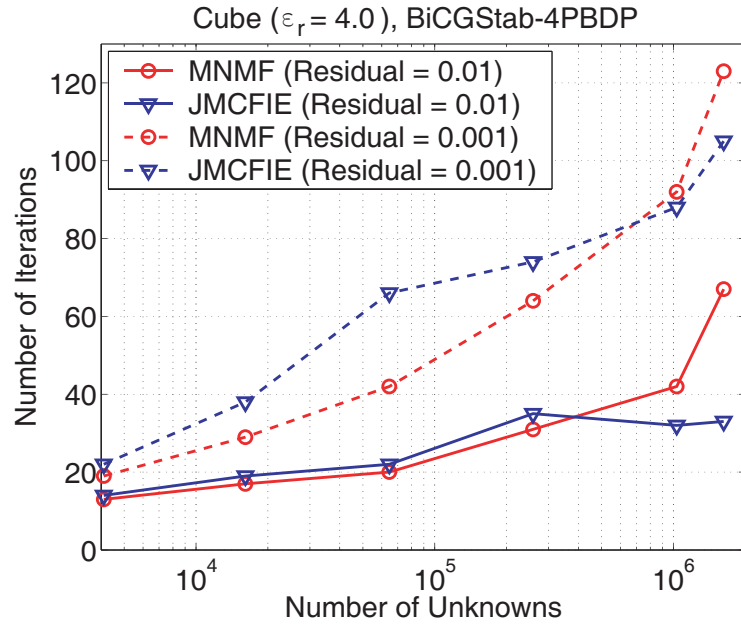


(b)

Figure 2.4: Iteration counts for the solution of scattering problems involving a sphere with a relative permittivity of 2.0 located in free space, when problems are formulated with (a) MNMF and (b) JMCfIE. The radius of the sphere is in the range from $0.75\lambda_o$ to $7.5\lambda_o$.

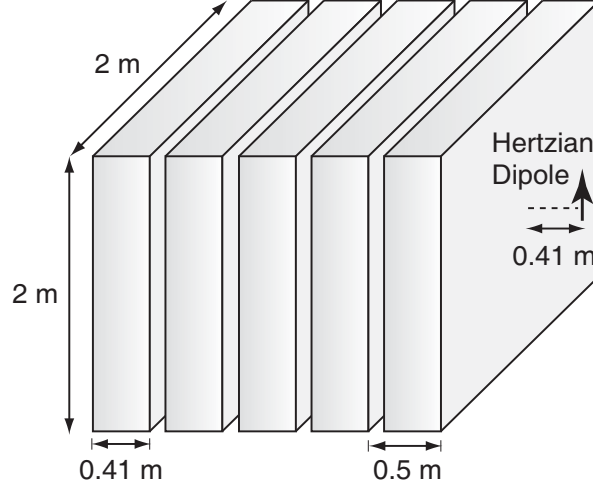


(a)

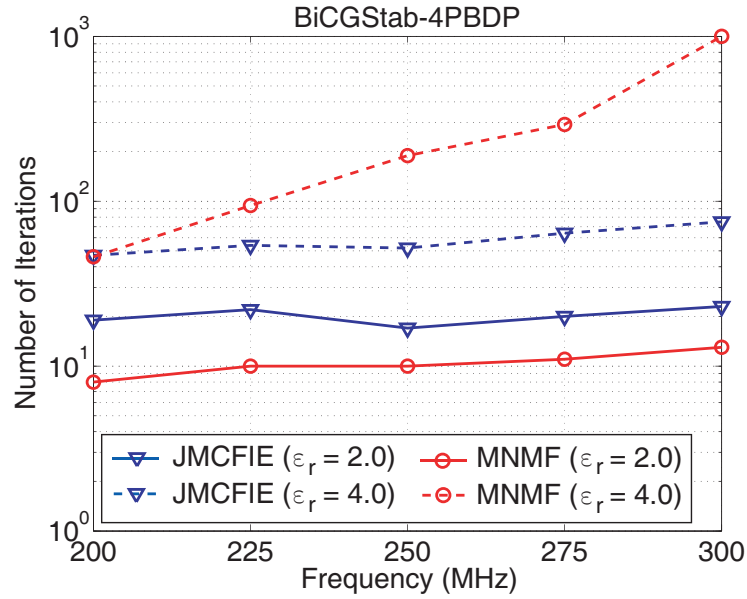


(b)

Figure 2.5: Iteration counts for the solution of scattering problems involving (a) a sphere with a relative permittivity of 2.0 and (b) a cube with a relative permittivity of 4.0 located in free space. The radius of the sphere is in the range from $0.75\lambda_o$ to $20\lambda_o$ and the edge length of the cube is in the range from λ_o to $20\lambda_o$. Iterative solutions are performed by employing BiCGStab accelerated with 4PBDP.



(a)



(b)

Figure 2.6: (a) A 5-layer periodic dielectric structure illuminated by a Hertizian dipole in free space. (b) Iteration counts (using BiCGStab accelerated with 4PBDP) for the solution of the problem in Figure 2.6(a) when the frequency changes from 200 MHz to 300 MHz, and the relative permittivity of the structure is 2.0 and 4.0.

Next, we focus on the processing time required for MLFMA solutions of dielectric problems. Once again, we consider the solution of scattering problems involving a dielectric sphere with a radius of 0.3 m located in free space and illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. As the problem size gets larger, the setup time becomes negligible, compared with the time required for iterations. Then, the processing time of MLFMA solutions is directly proportional to iteration counts. Table 2.1 lists the number of CGS iterations to reduce the residual error below 10^{-3} when the radius of the sphere is in the range from $0.75\lambda_o$ to $7.5\lambda_o$ and $\epsilon_r = 4.0$. For all formulations, we apply two efficient preconditioners, i.e., 2PBDP and 4PBDP, in addition to the no-preconditioner (NP) case. Our comments for iteration counts in Table 2.1 are as follows:

- Preconditioning matrix equations with 2PBDP and 4PBDP does not accelerate the iterative convergence for CTF. In fact, these low-cost preconditioners decelerate the convergence and increase the number of iterations for this formulation. Especially for large problems, convergence cannot be achieved within 2000 iterations, when 2PBDP and 4PBDP are used for CTF. These are denoted as “no convergence (NC)” in Table 2.1. A negative effect of efficient preconditioners was also observed for other tangential formulations, such as T-EFIE for the solution of PEC objects [54]. However, 2PBDP and 4PBDP significantly accelerate the iterative convergence for normal formulations and JMCIE.
- Although they are both normal formulations, iteration counts for CNF and MNMF differ significantly; convergence is consistently faster for MNMF. For large problems, CNF fails to provide quick convergence even when compared to CTF without preconditioning.

- Using 4PBDP provides faster convergence than 2PBDP, especially for JMC FIE. This is due to strong off-diagonal partitions of JMC FIE, in addition to its strong diagonal partitions involving well-tested identity operators.
- For small problems, iteration counts for MNMF are lower than JMC FIE. This can be observed in Table 2.1, when the radius of the sphere is in the range from $0.75\lambda_o$ to $3\lambda_o$. For larger problems, however, the convergence for JMC FIE becomes faster than the convergence for MNMF. When the radius of the sphere is $7.5\lambda_o$, iteration counts are consistently lower for JMC FIE with and without preconditioning.

To further compare dielectric formulations in terms of efficiency, we consider scattering problems involving a dielectric cube with edges of $4\lambda_o$ located in free space. Similar to sphere problems, the cube is illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. Discretization of the problem with $\lambda_o/10$ triangulation leads to 64,548 unknowns. The relative permittivity of the cube changes from 2.0 to 16.0. Figure 2.3(a) depicts the number of CGS iterations with respect to the contrast of the cube, i.e., $(\epsilon_r - 1)$, to reduce the residual error below 10^{-3} . For small contrasts, CTF has the slowest convergence, while normal formulations, i.e., CNF and MNMF, offer the fastest convergence. As the contrast increases, however, convergence of CNF decelerates significantly and this formulation has the poorest convergence when $\epsilon_r = 16.0$. Figure 2.3(a) also shows that MNMF provides the most efficient solutions and the number of iterations for this formulation is almost constant when the contrast increases from 3.0 to 15.0. Figure 2.3(b) presents iteration counts when 4PBDP is used to accelerate the convergence of solutions. Similar to sphere problems, CTF solutions are decelerated with 4PBDP and the convergence cannot be achieved within 2000 iterations for contrasts from 3.0 to 11.0. Comparing Figure 2.3(b) with Figure 2.3(a), we also observe that 4PBDP significantly reduces the number of iterations for CNF and JMC FIE, but it provides

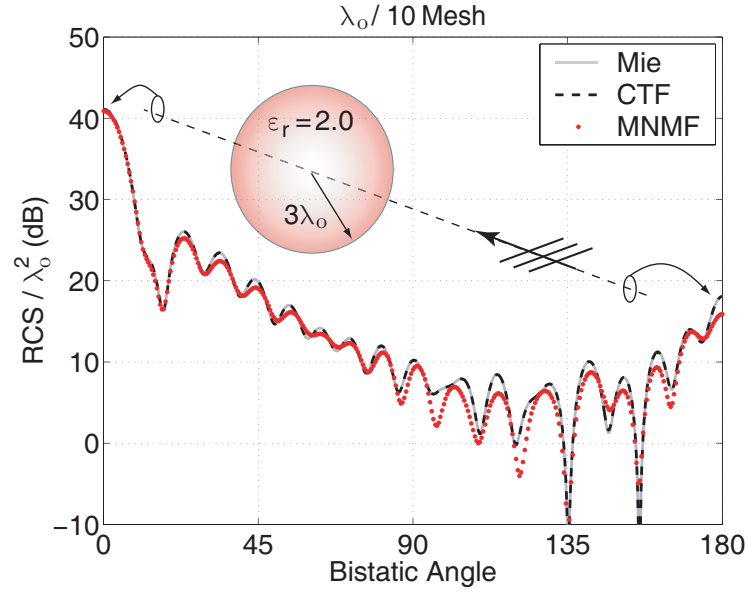
less improvement for MNMF. Using 4PBDP, MNMF still has the lowest iteration counts, but the efficiency of JMCIE becomes close to the efficiency of MNMF.

Table 2.1 and Figure 2.3 show that MNMF and JMCIE are the most appropriate formulations for efficient solutions of large dielectric problems. These formulations are further investigated by considering both various preconditioning schemes and different iterative algorithms to improve the efficiency of solutions. As an example, Figure 2.4 presents iteration counts for sphere problems when $\epsilon_r = 2.0$. In addition to CGS with no preconditioning and with 4PBDP, we also present solutions obtained by employing a BiCGStab algorithm. BiCGStab is known to provide rapid convergence for normal and combined formulations. Figure 2.4 shows that the number of iterations is reduced for both MNMF and JMCIE, if BiCGStab is employed, instead of CGS.

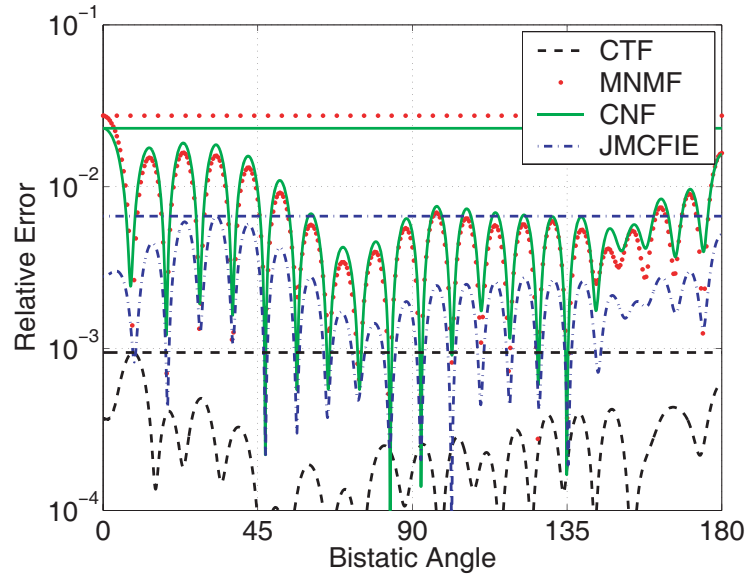
Finally, MNMF and JMCIE are compared for the solution of very large dielectric problems. Figure 2.5(a) presents iteration counts for sphere problems ($\epsilon_r = 2.0$), when solutions are performed by BiCGStab accelerated with 4PBDP. This time, the frequency is extended to 20 GHz and the radius of the sphere grows up to $20\lambda_o$. At 20 GHz, discretization of the sphere with $\lambda_o/10$ triangulation leads to 2,925,708 unknowns. Figure 2.5(a) shows that solutions with JMCIE become significantly faster than MNMF for large problems. Using JMCIE, we are able to solve a 3-million-unknown problem, which is one of the largest integral-equation problems involving dielectric objects ever solved. Figure 2.5(b) presents results of a similar experiment, where scattering problems involving a dielectric cube ($\epsilon_r = 4.0$) is solved by using BiCGStab accelerated with 4PBDP. The size of the cube changes from λ_o to $20\lambda_o$, where λ_o is the wavelength in free space. The number of unknowns due to $\lambda_o/10$ triangulations is in the range from 4104 to 1,624,320. Iteration counts required for both 10^{-2} and 10^{-3} residual errors are plotted with respect to the number of unknowns.

Similar to the previous case, solutions with JMCIE become faster than MNMF for large problems.

In general, JMCIE leads to more efficient solutions than MNMF, when the problem size is sufficiently large. Our investigations further show that the better performance of JMCIE becomes more evident when a problem involves complicated targets. As an example, Figure 2.6 presents results of a problem involving a 5-layer periodic structure in free space excited by a Hertzian dipole. Dimensions of the structure and the position of the source are detailed in Figure 2.6(a). Discretization of the structure with 10 cm triangulation size leads to 38,700 unknowns. We consider two different values for the relative permittivity of the structure, i.e., $\epsilon_r = 2.0$ and $\epsilon_r = 4.0$. Figure 2.6(b) depicts iteration counts for 10^{-3} residual error as a function of frequency from 200 MHz to 300 MHz, when problems are solved by using BiCGStab accelerated with 4PBDP. We observe that MNMF offers faster solutions when $\epsilon_r = 2.0$, and the number of iterations is halved compared to JMCIE. When $\epsilon_r = 4.0$, however, the number of iterations for MNMF increases rapidly as the frequency changes from 200 MHz to 300 MHz. At 300 MHz, convergence cannot be achieved within 1000 iterations by using MNMF. On the other hand, JMCIE is more stable in the same frequency range, and it provides significantly faster solutions when $\epsilon_r = 4.0$ and the frequency is larger than 200 MHz.

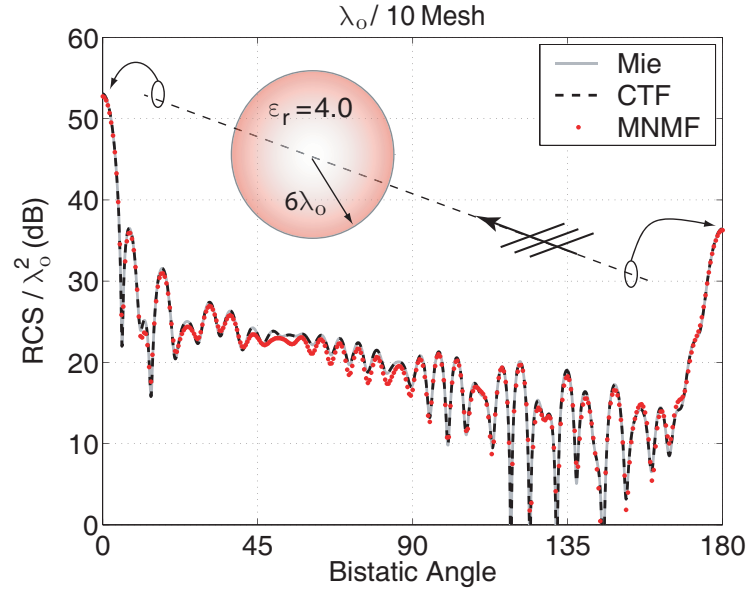


(a)

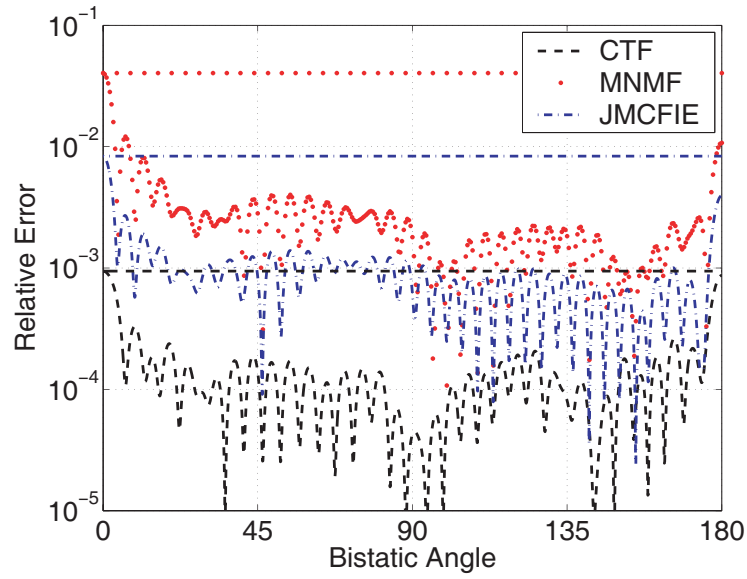


(b)

Figure 2.7: (a) Normalized bistatic RCS (RCS/λ_o^2) of a sphere with a radius of $3\lambda_o$ and with a relative permittivity of 2.0 located in free space. (b) Relative error defined in (2.172) for different formulations as a function of the bistatic angle.

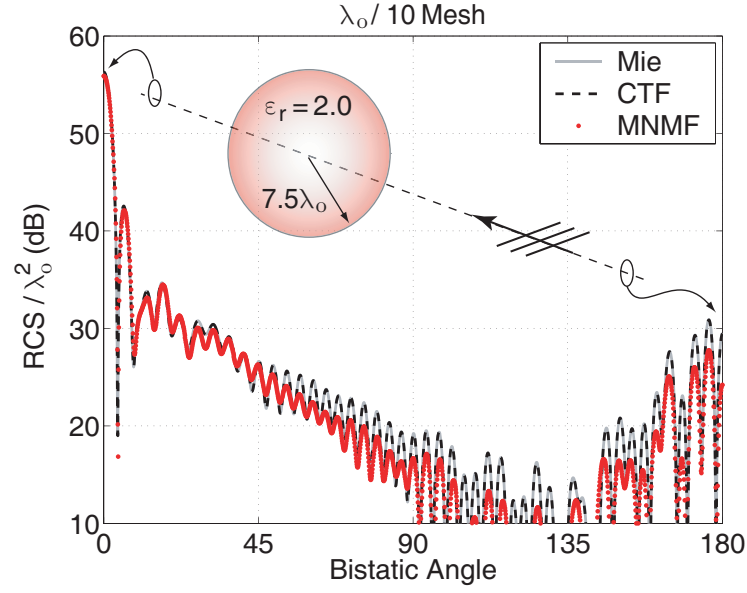


(a)

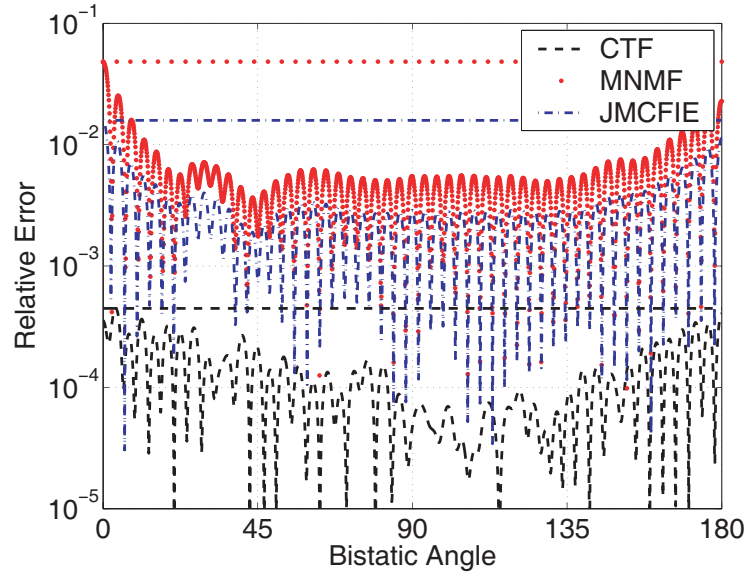


(b)

Figure 2.8: (a) Normalized bistatic RCS (RCS/λ_o^2) of a sphere with a radius of $6\lambda_o$ and with a relative permittivity of 4.0 located in free space. (b) Relative error defined in (2.172) for different formulations as a function of the bistatic angle. CNF is omitted in this figure since its accuracy is very close to that of MNMF, as depicted in Figure 2.7(b).



(a)



(b)

Figure 2.9: (a) Normalized bistatic RCS (RCS/λ_o^2) of a sphere with a radius of $7.5\lambda_o$ and with a relative permittivity of 2.0 located in free space. (b) Relative error defined in (2.172) for different formulations as a function of the bistatic angle. CNF is omitted in this figure since its accuracy is very close to that of MNMF, as depicted in Figure 2.7(b).

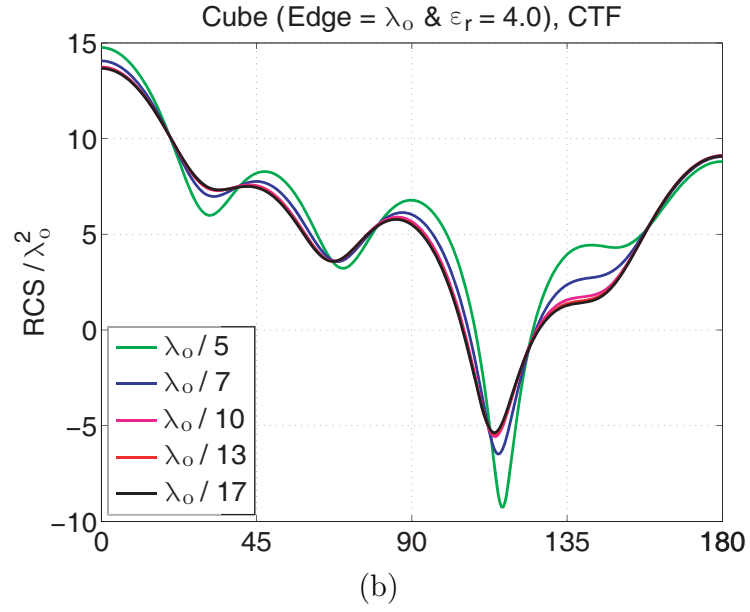
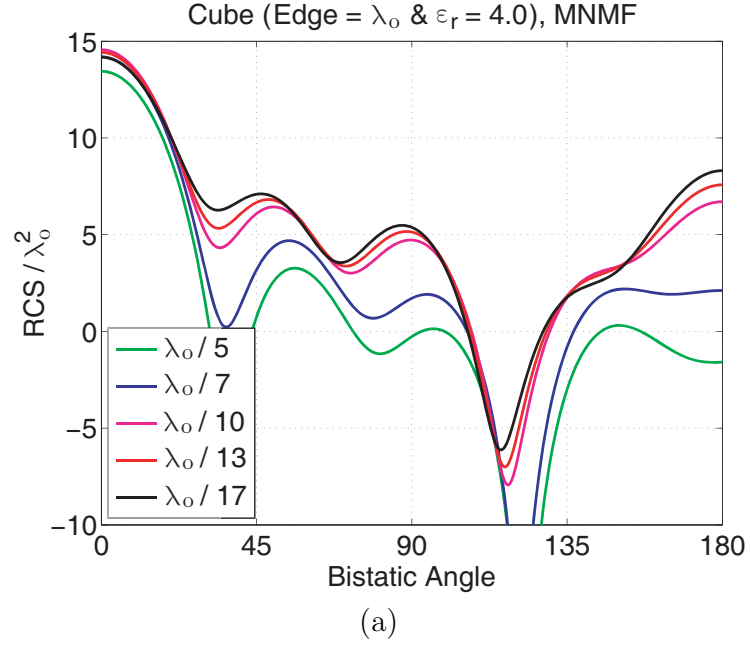


Figure 2.10: Normalized bistatic RCS (RCS/λ_o^2) of a cube with edges of λ_o and with a relative permittivity of 4.0 located in free space. RCS values are obtained for different discretizations and by using (a) MNMF and (b) CTF.

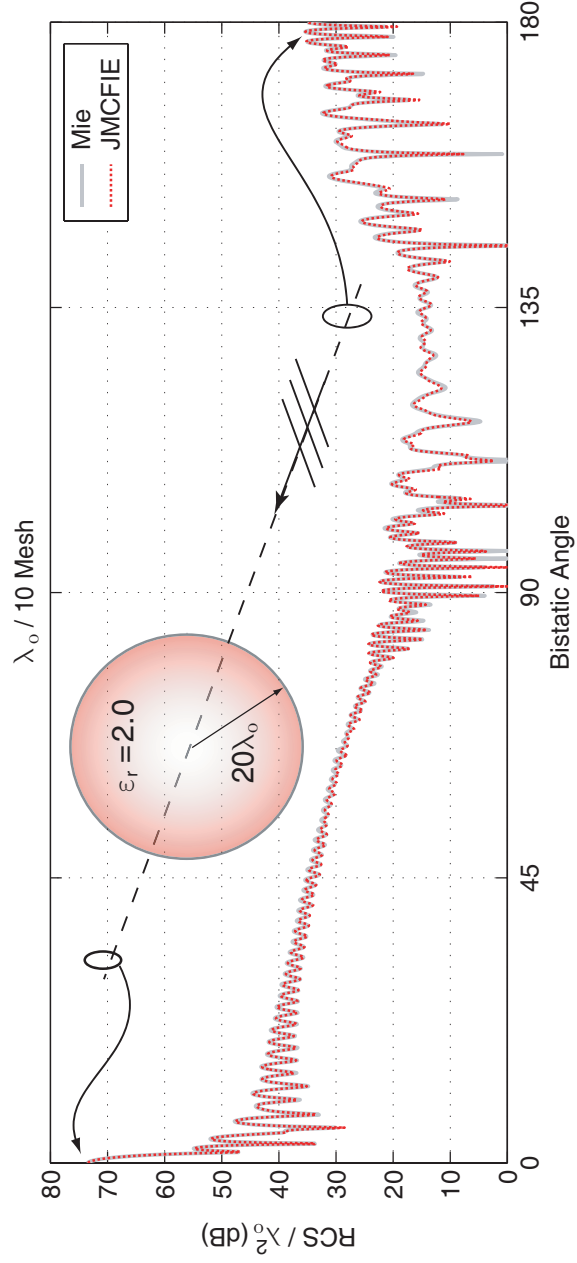


Figure 2.11: Normalized bistatic RCS (RCS/λ_0^2) of a sphere with a radius of $20\lambda_0$ and with a relative permittivity of 2.0 located in free space.

2.2.10 Comparison of Formulations: Accuracy

Figure 2.7(a) depicts bistatic RCS values for a sphere with a radius of $3\lambda_o$ and $\epsilon_r = 2.0$. Normalized RCS (RCS/λ_o^2 in dB) is plotted as a function of the observation angle from 0° to 180° on the z - x plane, where 0° corresponds to the forward-scattering direction. Figure 2.7(a) shows that computational values calculated by using CTF are in agreement with the analytical curve obtained by a Mie-series solution. Although results obtained by MNMF are also close to the analytical curve, they are significantly inaccurate compared to CTF. For more quantitative information, Figure 2.7(b) presents the relative error in computational results with respect to the reference analytical solution. In addition to CTF and MNMF, we also consider the error for CNF and JMCFIE. The relative error as a function of bistatic angle θ is defined as

$$\Delta_R(\theta) = \frac{|E_{C,\theta}^\infty(\theta) - E_{A,\theta}^\infty(\theta)|}{\max_\theta |E_{A,\theta}^\infty(\theta)|}, \quad (2.172)$$

where $E_{C,\theta}^\infty$ and $E_{A,\theta}^\infty$ are computational and analytical values of the far-zone co-polar electric field, i.e.,

$$E_\theta^\infty(\theta) = \lim_{r \rightarrow \infty} \left\{ r E_\theta(r, \theta, \phi = 0) \right\}. \quad (2.173)$$

The maximum value of the relative error is also indicated by a horizontal line in the figure for each formulation. Figure 2.7(b) shows that CTF provides the most accurate results, while normal formulations (MNMF and CNF) are significantly inaccurate compared to CTF. JMCFIE is also worse than CTF, but it is more accurate than normal formulations. In Figure 2.8, we present bistatic RCS values and the relative error for a sphere with a radius of $6\lambda_o$ and $\epsilon_r = 4.0$. Results are very similar to the previous case. Finally, Figure 2.9 presents similar results for a sphere with a radius of $7.5\lambda_o$ and $\epsilon_r = 2.0$.

Figure 2.10 depicts the normalized bistatic RCS (RCS/λ_o^2 in dB) of a cube with edges of λ_o and with a relative permittivity of 4.0. The cube is located in free

space and illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. The bistatic RCS is plotted on the z - x plane, where 0° corresponds to the forward-scattering direction. The cube is discretized by using various mesh sizes from $\lambda_o/5$ to $\lambda_o/17$. We investigate convergence to correct RCS values as the discretization is refined. Figure 2.10 shows that RCS values obtained by using MNMF converge slower than RCS values obtained by using CTF. Specifically, MNMF discretized with $\lambda_o/5$ and $\lambda_o/7$ triangles is unacceptably inaccurate. Using MNMF, there are also significant discrepancies between results obtained with $\lambda_o/10$ and finer discretizations. Using CTF, however, RCS values converge rapidly as the discretization is refined.

In general, normal formulations MNMF and CNF are consistently inaccurate, compared to CTF and JMCfie. As discussed in Chapter 3, this excessive error in normal formulations is caused by well-tested identity operators discretized with low-order functions, such as the RWG functions. As demonstrated in Figures 2.7–2.9, the discrepancy between the accuracy of results obtained with normal, mixed, and tangential formulations can be significant even for large dielectric objects with smooth surfaces.

Being a combination of CTF and CNF, solutions with JMCfie are contaminated with the inaccuracy of CNF. Therefore, CTF is preferable to JMCfie in terms of accuracy. However, JMCfie is more suitable for the solution of large problems, which cannot easily be obtained with CTF. For example, Figure 2.11 presents bistatic RCS values for a sphere with a radius of $20\lambda_o$ and $\epsilon_r = 2.0$ discretized with 2,925,708 unknowns. Computational values obtained with JMCfie are close to the analytical Mie-series solution, and the maximum relative error is 2.4%. This problem cannot be solved in a reasonable number of iterations when it is formulated with CTF, CNF, or MNMF.

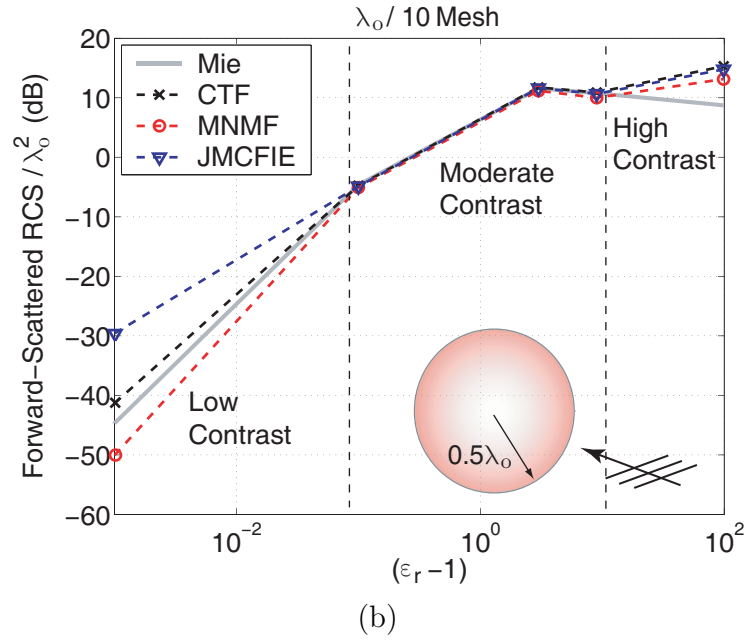
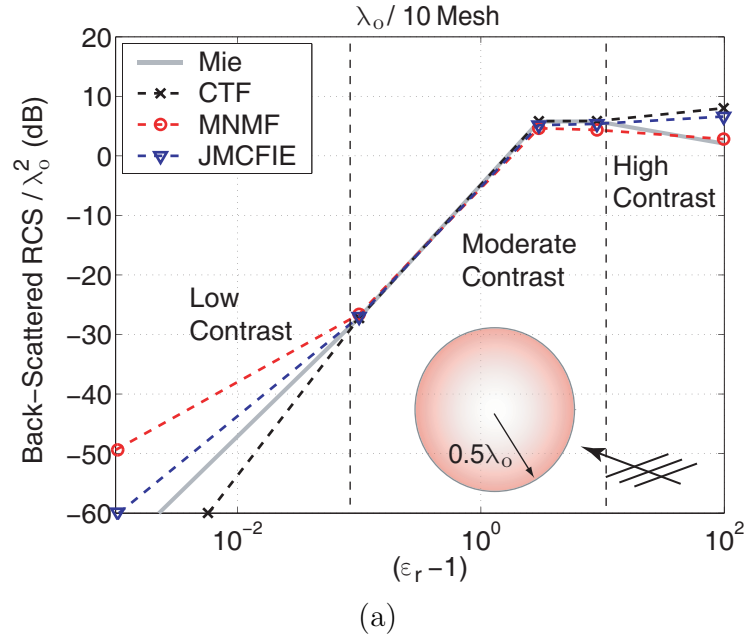


Figure 2.12: Normalized (a) back-scattered and (b) forward-scattered RCS (RCS/λ_0^2) of a dielectric sphere with a radius of $0.5\lambda_0$ located in free space. The relative permittivity of the sphere changes from $1.0 + 10^{-3}$ to 100.0.

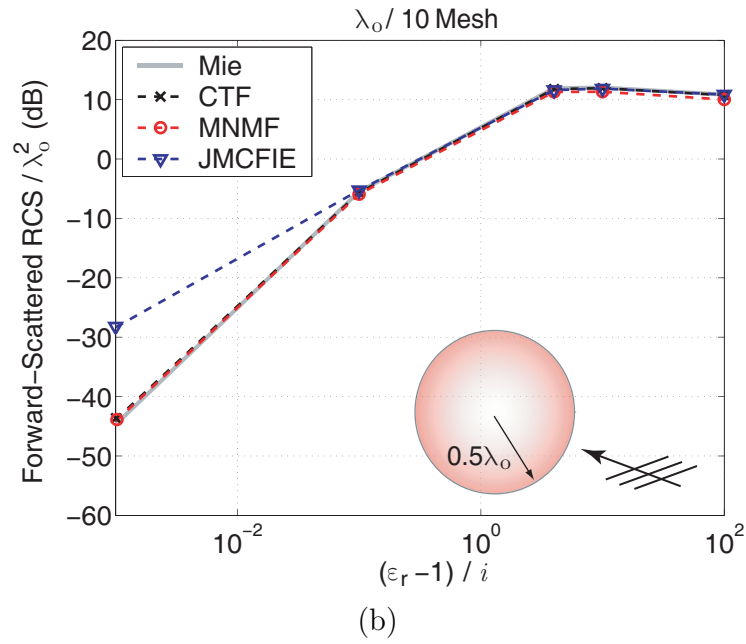
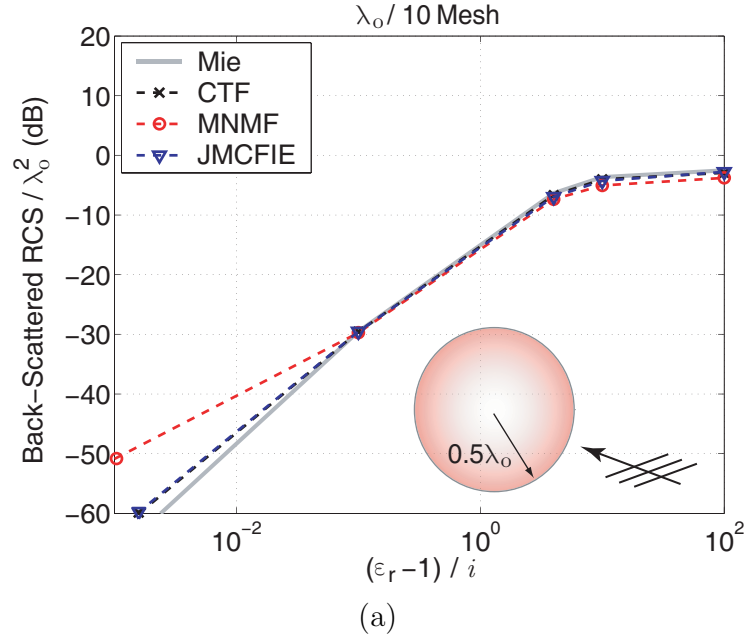


Figure 2.13: Normalized (a) back-scattered and (b) forward-scattered RCS (RCS/λ_o^2) of a lossy sphere with a radius of $0.5\lambda_o$ located in free space. The relative complex permittivity of the sphere changes from $1.0 + i10^{-3}$ to $1.0 + i100.0$.

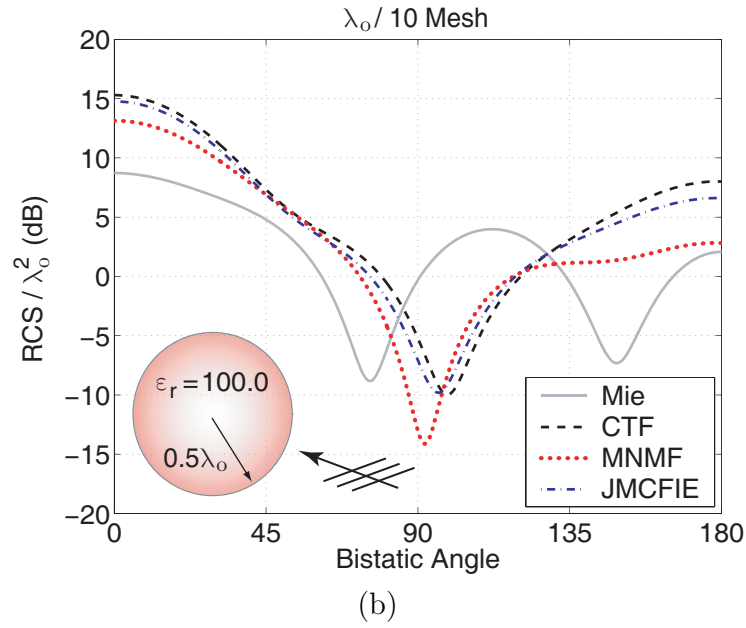
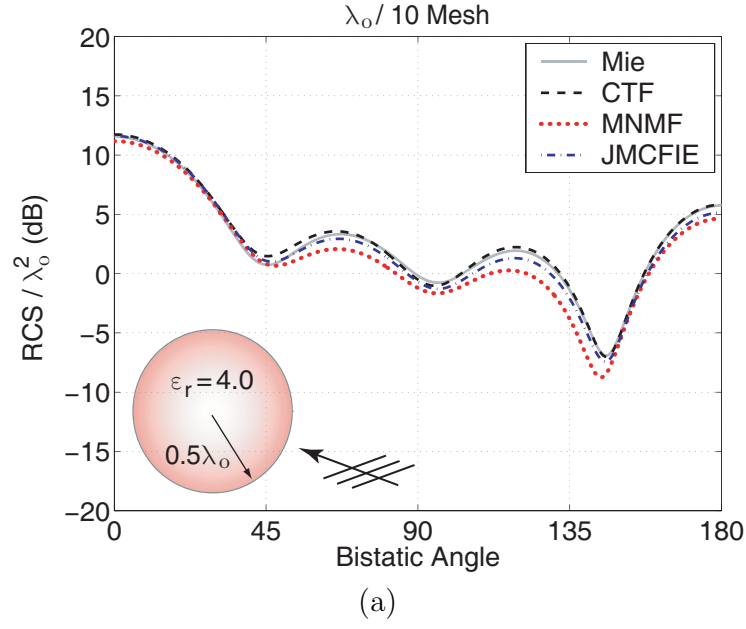


Figure 2.14: Normalized bistatic RCS (RCS/λ_o^2) of a sphere with a radius of $0.5\lambda_o$ located in free space, when the relative permittivity of the sphere is (a) 4.0 and (b) 100.0. The sphere is discretized with $\lambda_o/10$ triangles.

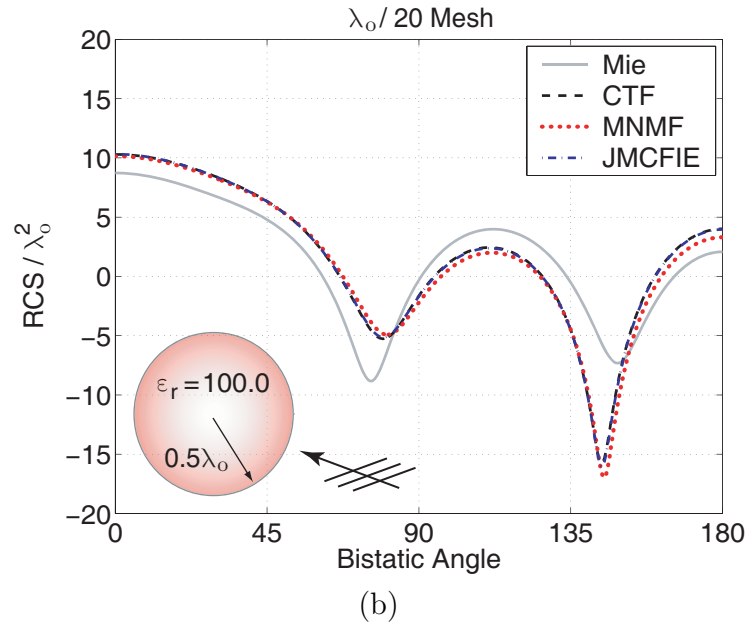
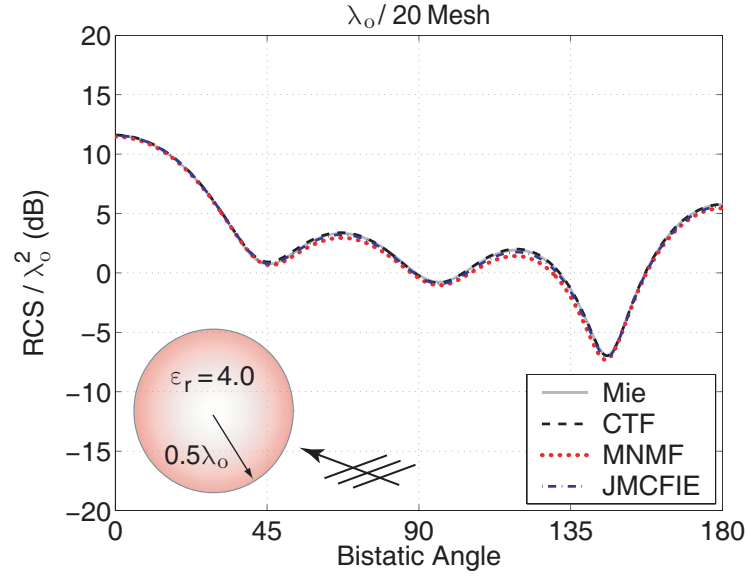


Figure 2.15: Normalized bistatic RCS (RCS/λ_o^2) of a sphere with a radius of $0.5\lambda_o$ located in free space, when the relative permittivity of the sphere is (a) 4.0 and (b) 100.0. The sphere is discretized with $\lambda_o/20$ triangles.

2.2.11 Accuracy with Respect to Contrast

Figure 2.12 presents back-scattered and forward-scattered RCS of a dielectric sphere with a radius of $0.5\lambda_o$ located in free space. Normalized RCS (RCS/λ_o^2) is plotted as a function contrast ($\epsilon_r - 1$) when the relative permittivity of the sphere changes from $1.0 + 10^{-3}$ to 100.0. For numerical solutions, discretization of the sphere with $\lambda_o/10$ triangulation lead to 1860 unknowns. Computational results obtained by using CTF, MNMF, and JMCIE are compared with analytical values obtained by a Mie-series solution. Figure 2.12 shows that computational values are consistent with analytical solutions for moderate contrasts from 0.1 to 10.0. However, they become inaccurate for very low and high contrasts.

In general, conventional surface formulations are unstable when the contrast of the object is very low. To obtain accurate results, those formulations should be stabilized as discussed in Section 2.3. On the other hand, inaccuracy of surface formulations for relatively high contrasts is due to the approximation of the actual geometry during numerical solutions. Since we use small planar triangles for the discretization, the computational model and the actual geometry of a smooth object are different. When the permittivity or permeability of an object is high, small discrepancies between the model and the actual geometry may create large errors. We emphasize that the error is not due to the insufficient discretization of equivalent currents since the discretized model is solved with a desired level of accuracy using $\lambda_o/10$ triangles. The error is due to the geometric difference between the actual object and the computational model.

Figure 2.13 presents back-scattered and forward-scattered RCS of a lossy sphere with a radius of $0.5\lambda_o$ located in free space. The relative complex permittivity of the sphere changes from $1.0 + i10^{-3}$ to $1.0 + i100.0$. We again compare computational values obtained by using CTF, MNMF, and JMCIE with analytical results. As opposed to the previous case, surface formulations are accurate when

the conductivity is high. In fact, we know that $\lambda_o/10$ discretizations provide accurate results even for smooth PEC objects with infinite conductivity. However, similar to the low-permittivity case, surface formulations become unstable when the conductivity is low and they provide inaccurate results as depicted in Figure 2.13.

In order to show that the geometric discretization is critical for high contrasts, we present results of scattering problems involving a sphere with a radius of $0.5\lambda_o$ located in free space. We consider two different values for the relative permittivity of the sphere, i.e., $\epsilon_r = 4.0$ (moderate contrast) and $\epsilon_r = 100.0$ (high contrast). The sphere is illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. Figure 2.14 presents the normalized bistatic RCS (RCS/λ_o^2) on the z - x plane. When the contrast is moderate, computational values obtained by using $\lambda_o/10$ discretization are in agreement with the Mie-series result. However, as depicted Figure 2.14(b), computational and analytical results become inconsistent for the high contrast case. Figure 2.15 presents the normalized bistatic RCS, when the discretization of the sphere is refined by using $\lambda_o/20$ triangles. We observe that the accuracy of computational results is improved for both moderate and high contrasts. However, the improvement is more significant for the high-contrast case. In addition to the better discretization of the equivalent currents, the geometric discretization of the sphere is enhanced by using smaller triangles; this improvement is more visible when the contrast is high.

2.3 Low-Contrast Breakdown and Its Solution

Conventional surface formulations are stable and provide accurate solutions (with various levels of accuracy, depending on the existence of well-tested identity operators, types of basis and testing functions, discretization method, geometry of the object, etc.), for problems involving objects with moderate contrasts. However, those formulations become inaccurate to calculate scattered fields as the contrast of the object decreases, i.e., when the electromagnetic material properties of the object and the host medium become close to each other. This breakdown, which limits the applicability of surface formulations, does not arise in volume integral equations. Therefore, volume formulations can be used to solve such problems accurately. However, it is also desirable to extend the applicability of surface integral equations to low-contrast problems in order to use advantages of surface formulations, which may require lower numbers of unknowns compared to volume formulations for some problems.

2.3.1 A Combined Tangential Formulation without Identity Operators

Stabilization procedures to eliminate the low-contrast breakdown are applied to a tangential formulation obtained by using

$$a = b = 1, \quad c = d = \eta_o \eta_i \quad (2.174)$$

in (2.19)–(2.20), which is slightly different from the original CTF in (2.23)–(2.26).

With these coefficients, the modified CTF (M-CTF) is derived as

$$\hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{Z}_{11}^{\text{M-CTF}} & \mathbf{Z}_{12}^{\text{M-CTF}} \\ \mathbf{Z}_{21}^{\text{M-CTF}} & \mathbf{Z}_{22}^{\text{M-CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}(\mathbf{r}) = -\hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{E}^{\text{inc}} \\ \eta_o \eta_i \mathbf{H}^{\text{inc}} \end{bmatrix}(\mathbf{r}), \quad (2.175)$$

where

$$\mathcal{Z}_{11}^{\text{M-CTF}} = \eta_o \mathcal{T}_o + \eta_i \mathcal{T}_i \quad (2.176)$$

$$\mathcal{Z}_{12}^{\text{M-CTF}} = -\mathcal{K}_{PV,o} - \mathcal{K}_{PV,i} - \frac{1}{4\pi}(\Omega_o - \Omega_i) \mathcal{I}^{\times n} \quad (2.177)$$

$$= -\mathcal{K}_{PV,o} - \mathcal{K}_{PV,i} - \left(1 - \frac{\Omega_i}{2\pi}\right) \mathcal{I}^{\times n} \quad (2.178)$$

$$\mathcal{Z}_{21}^{\text{M-CTF}} = \eta_o \eta_i (\mathcal{K}_{PV,o} + \mathcal{K}_{PV,i}) + \frac{1}{4\pi}(\Omega_o - \Omega_i) \eta_o \eta_i \mathcal{I}^{\times n} \quad (2.179)$$

$$= \eta_o \eta_i (\mathcal{K}_{PV,o} + \mathcal{K}_{PV,i}) + \eta_o \eta_i \left(1 - \frac{\Omega_i}{2\pi}\right) \mathcal{I}^{\times n} \quad (2.180)$$

$$\mathcal{Z}_{22}^{\text{M-CTF}} = \eta_i \mathcal{T}_o + \eta_o \mathcal{T}_i. \quad (2.181)$$

Then, matrix partitions for M-CTF are derived as

$$\bar{\mathcal{Z}}_{11}^{\text{M-CTF}} = \eta_o \bar{\mathcal{T}}_o^T + \eta_i \bar{\mathcal{T}}_i^T \quad (2.182)$$

$$\bar{\mathcal{Z}}_{12}^{\text{M-CTF}} = -\bar{\mathcal{K}}_{PV,o}^T - \bar{\mathcal{K}}_{PV,i}^T \quad (2.183)$$

$$\bar{\mathcal{Z}}_{21}^{\text{M-CTF}} = \eta_o \eta_i (\bar{\mathcal{K}}_{PV,o}^T + \bar{\mathcal{K}}_{PV,i}^T) \quad (2.184)$$

$$\bar{\mathcal{Z}}_{22}^{\text{M-CTF}} = \eta_i \bar{\mathcal{T}}_o^T + \eta_o \bar{\mathcal{T}}_i^T. \quad (2.185)$$

When discretized, M-CTF is free of identity operators (like the PMCHWT formulation) and involves well-balanced diagonal partitions (like the original CTF) for low contrasts. In addition to M-CTF, we also consider CNF in (2.32)–(2.35) in the context of stabilization.

2.3.2 Nonradiating Currents

In surface formulations of closed objects, equivalent surface currents can be decomposed as

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{H}^{\text{inc}}(\mathbf{r}) + \hat{\mathbf{n}} \times \mathbf{H}^{\text{rad}}(\mathbf{r}) \quad (2.186)$$

$$\mathbf{M}(\mathbf{r}) = -\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r}) = -\hat{\mathbf{n}} \times \mathbf{E}^{\text{inc}}(\mathbf{r}) - \hat{\mathbf{n}} \times \mathbf{E}^{\text{rad}}(\mathbf{r}), \quad (2.187)$$

where $\{\mathbf{J}^{inc}, \mathbf{M}^{inc}\} = \{\hat{\mathbf{n}} \times \mathbf{H}^{inc}, -\hat{\mathbf{n}} \times \mathbf{E}^{inc}\}$ do not radiate outside the surface [30], i.e.,

$$\eta_o \mathcal{T}_o \{\mathbf{J}^{inc}\}(\mathbf{r}) - \mathcal{K}_{PV,o} \{\mathbf{M}^{inc}\}(\mathbf{r}) + \frac{\Omega_i}{4\pi} \mathcal{I}^{\times n} \{\mathbf{M}^{inc}\}(\mathbf{r}) = 0 \quad (2.188)$$

$$\frac{1}{\eta_o} \mathcal{T}_o \{\mathbf{M}^{inc}\}(\mathbf{r}) + \mathcal{K}_{PV,o} \{\mathbf{J}^{inc}\}(\mathbf{r}) - \frac{\Omega_i}{4\pi} \mathcal{I}^{\times n} \{\mathbf{J}^{inc}\}(\mathbf{r}) = 0, \quad (2.189)$$

for $\mathbf{r} \in D_o$. Equations (2.188) and (2.189) can be projected onto the surface as

$$\begin{aligned} \begin{Bmatrix} \hat{\mathbf{t}} \cdot \\ \hat{\mathbf{n}} \times \end{Bmatrix} \begin{bmatrix} \eta_o \mathcal{T}_o & -\mathcal{K}_{PV,o} \\ \mathcal{K}_{PV,o} & \eta_o^{-1} \mathcal{T}_o \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{n}} \times \mathbf{H}^{inc} \\ -\hat{\mathbf{n}} \times \mathbf{E}^{inc} \end{bmatrix}(\mathbf{r}) \\ - \frac{\Omega_i}{4\pi} \begin{Bmatrix} \hat{\mathbf{t}} \cdot \\ \hat{\mathbf{n}} \times \end{Bmatrix} \begin{bmatrix} \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E}^{inc} \\ \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H}^{inc} \end{bmatrix}(\mathbf{r}) = 0 \end{aligned} \quad (2.190)$$

or

$$\begin{aligned} \begin{Bmatrix} \hat{\mathbf{t}} \cdot \\ \hat{\mathbf{n}} \times \end{Bmatrix} \begin{bmatrix} \eta_o \mathcal{T}_o & -\mathcal{K}_{PV,o} \\ \mathcal{K}_{PV,o} & \eta_o^{-1} \mathcal{T}_o \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{n}} \times \mathbf{H}^{inc} \\ -\hat{\mathbf{n}} \times \mathbf{E}^{inc} \end{bmatrix}(\mathbf{r}) \\ = -\frac{\Omega_i}{4\pi} \begin{Bmatrix} \hat{\mathbf{t}} \cdot \\ \hat{\mathbf{n}} \times \end{Bmatrix} \begin{bmatrix} \mathbf{E}^{inc} \\ \mathbf{H}^{inc} \end{bmatrix}(\mathbf{r}) \end{aligned} \quad (2.191)$$

for $\mathbf{r} \in S$.

2.3.3 Conventional Formulations in the Limit Case

As the contrast goes to zero, i.e., $\epsilon_i \rightarrow \epsilon_o$ and $\mu_i \rightarrow \mu_o$, M-CTF in (2.175)–(2.181) becomes

$$\begin{aligned} \hat{\mathbf{t}} \cdot \begin{bmatrix} 0 & (\Omega_i/2\pi - 1) \mathcal{I}^{\times n} \\ \eta_o^2 (1 - \Omega_i/2\pi) \mathcal{I}^{\times n} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}(\mathbf{r}) \\ + \hat{\mathbf{t}} \cdot \begin{bmatrix} 2\eta_o \mathcal{T}_o & -2\mathcal{K}_{PV,o} \\ 2\eta_o^2 \mathcal{K}_{PV,o} & 2\eta_o \mathcal{T}_o \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}(\mathbf{r}) = -\hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{E}^{inc} \\ \eta_o^2 \mathbf{H}^{inc} \end{bmatrix}(\mathbf{r}) \end{aligned} \quad (2.192)$$

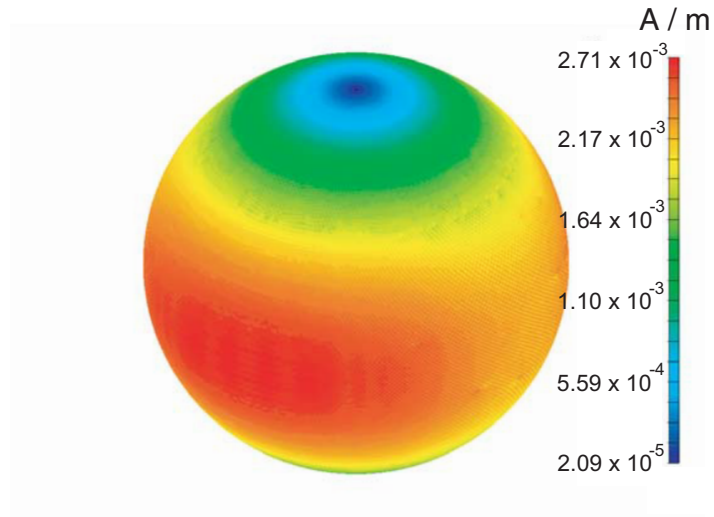
or

$$\hat{\mathbf{t}} \cdot \begin{bmatrix} \eta_o \mathcal{T}_o & -\mathcal{K}_{PV,o} \\ \eta_o^2 \mathcal{K}_{PV,o} & \eta_o \mathcal{T}_o \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}(\mathbf{r}) = -\frac{\Omega_i}{4\pi} \hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{E}^{inc} \\ \eta_o^2 \mathbf{H}^{inc} \end{bmatrix}(\mathbf{r}). \quad (2.193)$$

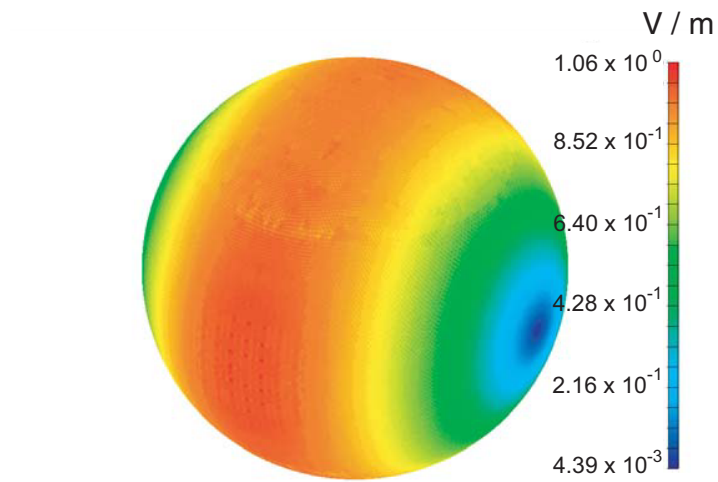
Considering the tangential form of (2.191), the solution of (2.193) can be obtained as $\mathbf{J} = \hat{\mathbf{n}} \times \mathbf{H}^{inc}$ and $\mathbf{M} = -\hat{\mathbf{n}} \times \mathbf{E}^{inc}$. When the contrast is zero, CNF in (2.32)–(2.35) reduces to a simpler form, i.e.,

$$-\begin{bmatrix} \boldsymbol{\mathcal{I}} & 0 \\ 0 & \boldsymbol{\mathcal{I}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix}(\mathbf{r}) = -\hat{\mathbf{n}} \times \begin{bmatrix} \mathbf{H}^{inc} \\ -\mathbf{E}^{inc} \end{bmatrix}(\mathbf{r}), \quad (2.194)$$

where the same solution can be obtained trivially. We note that “incident currents” $\{\hat{\mathbf{n}} \times \mathbf{H}^{inc}, -\hat{\mathbf{n}} \times \mathbf{E}^{inc}\}$ do not radiate and conventional forms of surface formulations satisfy the limit case mathematically. On the other hand, when they are discretized, these formulations fail to provide accurate results for scattered fields from low-contrast objects.

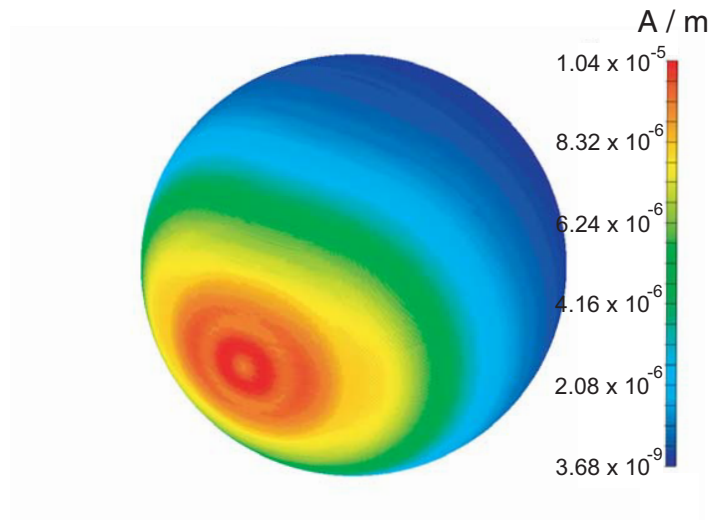


(a)

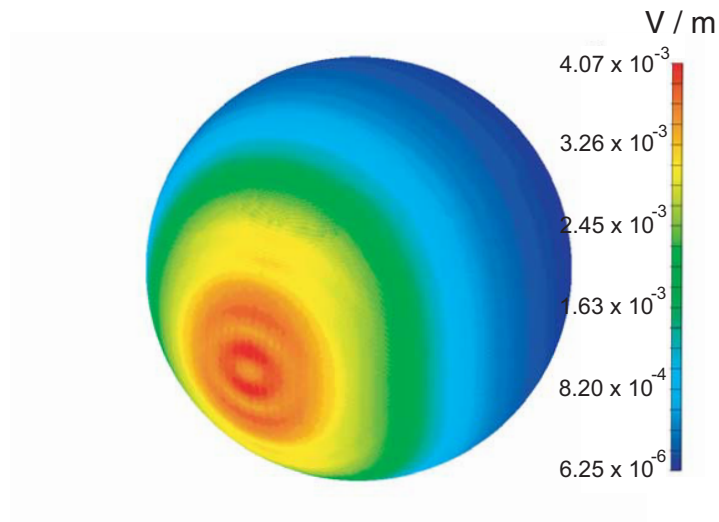


(b)

Figure 2.16: Equivalent (a) electric and (b) magnetic currents on a sphere with a radius of 0.3 m illuminated by a plane wave at 6 GHz. The sphere is located in free space and has a relative permittivity of $\epsilon_r = 1.0 + 10^{-4}$.



(a)



(b)

Figure 2.17: Radiating parts of the (a) electric and (b) magnetic currents depicted in Figure 2.16.

2.3.4 Low-Contrast Breakdown

As the contrast goes to zero, nonradiating currents dominate total currents, while radiating currents, i.e., $\{\hat{\mathbf{n}} \times \mathbf{H}^{rad}, -\hat{\mathbf{n}} \times \mathbf{E}^{rad}\}$, tend to vanish. As an example, Figure 2.16 presents equivalent electric and magnetic currents on the surface of a dielectric sphere of radius 0.3 m illuminated by a plane wave at 6 GHz. The sphere is in free space and has a relative dielectric constant of $\epsilon_r = 1.0 + 10^{-4}$. Radiating parts of currents are also depicted in Figure 2.17. Comparing Figure 2.17 to Figure 2.16, we observe that radiating currents form very small portions of total currents. Therefore, when total currents are solved by employing conventional surface formulations, it becomes difficult to perform calculations accurately enough to capture small radiating currents properly. In other words, even though surface currents $\mathbf{J}$ and $\mathbf{M}$ are computed with relatively small error, radiated fields may not be obtained accurately from them, since radiating currents are numerically insignificant compared to nonradiating currents.

2.3.5 Stabilization of Surface Formulations by Extracting Nonradiating Currents

In order to calculate scattered fields from low-contrast objects accurately, we extract nonradiating parts of currents and solve only radiating currents as unknowns of the problem [49]. Applying the stabilization procedure to the tangential formulation in (2.175)–(2.181), we obtain a stable CTF (S-CTF) as

$$\begin{aligned} \hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{Z}_{11}^{\text{M-CTF}} & \mathbf{Z}_{12}^{\text{M-CTF}} \\ \mathbf{Z}_{21}^{\text{M-CTF}} & \mathbf{Z}_{22}^{\text{M-CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{rad} \\ \mathbf{M}^{rad} \end{bmatrix}(\mathbf{r}) = -\hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{E}^{inc} \\ \eta_o \eta_i \mathbf{H}^{inc} \end{bmatrix}(\mathbf{r}) \\ - \hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{Z}_{11}^{\text{M-CTF}} & \mathbf{Z}_{12}^{\text{M-CTF}} \\ \mathbf{Z}_{21}^{\text{M-CTF}} & \mathbf{Z}_{22}^{\text{M-CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{inc} \\ \mathbf{M}^{inc} \end{bmatrix}(\mathbf{r}). \quad (2.195) \end{aligned}$$

Using (2.176)–(2.181) and (2.191),

$$\begin{aligned} \hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{Z}_{11}^{\text{M-CTF}} & \mathbf{Z}_{12}^{\text{M-CTF}} \\ \mathbf{Z}_{21}^{\text{M-CTF}} & \mathbf{Z}_{22}^{\text{M-CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{\text{rad}} \\ \mathbf{M}^{\text{rad}} \end{bmatrix}(\mathbf{r}) \\ = \hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{\text{inc}} \\ \mathbf{M}^{\text{inc}} \end{bmatrix}(\mathbf{r}), \end{aligned} \quad (2.196)$$

where

$$\mathbf{y}_{11} = \frac{\Omega_o}{\Omega_i} \eta_o \mathbf{T}_o - \eta_i \mathbf{T}_i \quad (2.197)$$

$$\mathbf{y}_{12} = -\frac{\Omega_o}{\Omega_i} \mathbf{K}_{PV,o} + \mathbf{K}_{PV,i} + \left(1 - \frac{\Omega_i}{2\pi}\right) \mathbf{I}^{\times n} \quad (2.198)$$

$$\mathbf{y}_{21} = \eta_o \eta_i \left(\frac{\Omega_o}{\Omega_i} \mathbf{K}_{PV,o} - \mathbf{K}_{PV,i} \right) - \eta_o \eta_i \left(1 - \frac{\Omega_i}{2\pi} \right) \quad (2.199)$$

$$\mathbf{y}_{22} = \frac{\Omega_o}{\Omega_i} \eta_i \mathbf{T}_o - \eta_o \mathbf{T}_i. \quad (2.200)$$

When the contrast goes to zero, S-CTF reduces to

$$\begin{aligned} \hat{\mathbf{t}} \cdot \begin{bmatrix} \eta_o \mathbf{T}_o & -\mathbf{K}_{PV,o} \\ \mathbf{K}_{PV,o} & \eta_o^{-1} \mathbf{T}_o \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{\text{rad}} \\ \mathbf{M}^{\text{rad}} \end{bmatrix}(\mathbf{r}) = \frac{(\Omega_o - \Omega_i)}{\Omega_i} \times \\ \hat{\mathbf{t}} \cdot \begin{bmatrix} \eta_o \mathbf{T}_o & -\mathbf{K}_{PV,o} + (\Omega_i/4\pi) \mathbf{I}^{\times n} \\ \mathbf{K}_{PV,o} - (\Omega_i/4\pi) \mathbf{I}^{\times n} & \eta_o^{-1} \mathbf{T}_o \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{\text{inc}} \\ \mathbf{M}^{\text{inc}} \end{bmatrix}(\mathbf{r}), \end{aligned} \quad (2.201)$$

or

$$\hat{\mathbf{t}} \cdot \begin{bmatrix} \eta_o \mathbf{T}_o & -\mathbf{K}_{PV,o} \\ \mathbf{K}_{PV,o} & \eta_o^{-1} \mathbf{T}_o \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{\text{rad}} \\ \mathbf{M}^{\text{rad}} \end{bmatrix}(\mathbf{r}) = 0 \quad (2.202)$$

using (2.191), and the solution is zero.

Discretization of S-CTF in (2.196) leads to

$$\begin{bmatrix} \bar{\mathbf{Z}}_{11}^{\text{M-CTF}} & \bar{\mathbf{Z}}_{12}^{\text{M-CTF}} \\ \bar{\mathbf{Z}}_{21}^{\text{M-CTF}} & \bar{\mathbf{Z}}_{22}^{\text{M-CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{\text{rad}} \\ \mathbf{a}_M^{\text{rad}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{Y}}_{11} & \bar{\mathbf{Y}}_{12} \\ \bar{\mathbf{Y}}_{21} & \bar{\mathbf{Y}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{\text{inc}} \\ \mathbf{a}_M^{\text{inc}} \end{bmatrix}, \quad (2.203)$$

where

$$\bar{\mathbf{Y}}_{11} = \eta_o \bar{\mathbf{T}}_o^T - \eta_i \bar{\mathbf{T}}_i^T \quad (2.204)$$

$$\bar{\mathbf{Y}}_{12} = -\bar{\mathbf{K}}_{PV,o}^T + \bar{\mathbf{K}}_{PV,i}^T \quad (2.205)$$

$$\bar{\mathbf{Y}}_{21} = \eta_o \eta_i \bar{\mathbf{K}}_{PV,o}^T - \eta_o \eta_i \bar{\mathbf{K}}_{PV,i}^T \quad (2.206)$$

$$\bar{\mathbf{Y}}_{22} = \eta_i \bar{\mathbf{T}}_o^T - \eta_o \bar{\mathbf{T}}_i^T. \quad (2.207)$$

In (2.203), $\{\mathbf{a}_J^{rad}, \mathbf{a}_M^{rad}\}$ and $\{\mathbf{a}_J^{inc}, \mathbf{a}_M^{inc}\}$ are two sets of vectors involving coefficients expanding radiating and nonradiating currents, respectively, i.e.,

$$\left\{ \hat{\mathbf{n}} \times \mathbf{H}^{rad}(\mathbf{r}), -\hat{\mathbf{n}} \times \mathbf{E}^{rad}(\mathbf{r}) \right\} = \sum_{n=1}^N \left\{ \mathbf{a}_J^{rad}[n], \mathbf{a}_M^{rad}[n] \right\} \mathbf{b}_n(\mathbf{r}) \quad (2.208)$$

$$\left\{ \hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}), -\hat{\mathbf{n}} \times \mathbf{E}^{inc}(\mathbf{r}) \right\} = \sum_{n=1}^N \left\{ \mathbf{a}_J^{inc}[n], \mathbf{a}_M^{inc}[n] \right\} \mathbf{b}_n(\mathbf{r}), \quad (2.209)$$

and

$$\mathbf{a}_J = \mathbf{a}_J^{rad} + \mathbf{a}_J^{inc} \quad (2.210)$$

$$\mathbf{a}_M = \mathbf{a}_M^{rad} + \mathbf{a}_M^{inc}. \quad (2.211)$$

We note that only $\{\mathbf{a}_J^{rad}, \mathbf{a}_M^{rad}\}$ contribute to radiated fields.

Extracting nonradiating parts of currents in CNF, we obtain a stable CNF (S-CNF) as

$$\begin{aligned} \hat{\mathbf{n}} \times \begin{bmatrix} \mathbf{Z}_{11}^{CNF} & \mathbf{Z}_{12}^{CNF} \\ \mathbf{Z}_{21}^{CNF} & \mathbf{Z}_{22}^{CNF} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{rad} \\ \mathbf{M}^{rad} \end{bmatrix}(\mathbf{r}) &= -\hat{\mathbf{n}} \times \begin{bmatrix} \mathbf{H}^{inc} \\ -\mathbf{E}^{inc} \end{bmatrix}(\mathbf{r}) \\ &- \hat{\mathbf{n}} \times \begin{bmatrix} \mathbf{Z}_{11}^{CNF} & \mathbf{Z}_{12}^{CNF} \\ \mathbf{Z}_{21}^{CNF} & \mathbf{Z}_{22}^{CNF} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{inc} \\ \mathbf{M}^{inc} \end{bmatrix}(\mathbf{r}). \end{aligned} \quad (2.212)$$

Using (2.33)–(2.35),

$$\begin{aligned} \hat{\mathbf{n}} \times \begin{bmatrix} \mathbf{Z}_{11}^{CNF} & \mathbf{Z}_{12}^{CNF} \\ \mathbf{Z}_{21}^{CNF} & \mathbf{Z}_{22}^{CNF} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{rad} \\ \mathbf{M}^{rad} \end{bmatrix}(\mathbf{r}) \\ = -\hat{\mathbf{n}} \times \begin{bmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} \\ \mathbf{v}_{21} & \mathbf{v}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{inc} \\ \mathbf{M}^{inc} \end{bmatrix}(\mathbf{r}), \end{aligned} \quad (2.213)$$

where

$$\mathcal{V}_{11} = \mathcal{V}_{22} = \mathcal{K}_{PV,o} - \mathcal{K}_{PV,i} \quad (2.214)$$

$$\mathcal{V}_{12} = \mathcal{Z}_{12}^{\text{CNF}} = \eta_o^{-1} \mathcal{T}_o - \eta_i^{-1} \mathcal{T}_i \quad (2.215)$$

$$\mathcal{V}_{21} = \mathcal{Z}_{21}^{\text{CNF}} = -\eta_o \mathcal{T}_o + \eta_i \mathcal{T}_i. \quad (2.216)$$

When the contrast goes to zero, S-CNF reduces to

$$\begin{bmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{\text{rad}} \\ \mathbf{M}^{\text{rad}} \end{bmatrix}(\mathbf{r}) = 0 \quad (2.217)$$

and the solution is again zero. S-CNF is discretized as

$$\begin{bmatrix} \bar{\mathbf{Z}}_{11}^{\text{CNF}} & \bar{\mathbf{Z}}_{12}^{\text{CNF}} \\ \bar{\mathbf{Z}}_{21}^{\text{CNF}} & \bar{\mathbf{Z}}_{22}^{\text{CNF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{\text{rad}} \\ \mathbf{a}_M^{\text{rad}} \end{bmatrix} = - \begin{bmatrix} \bar{\mathbf{V}}_{11} & \bar{\mathbf{V}}_{12} \\ \bar{\mathbf{V}}_{21} & \bar{\mathbf{V}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{\text{inc}} \\ \mathbf{a}_M^{\text{inc}} \end{bmatrix}, \quad (2.218)$$

where

$$\bar{\mathbf{V}}_{11} = \bar{\mathbf{V}}_{22} = \bar{\mathbf{K}}_{PV,o}^N - \bar{\mathbf{K}}_{PV,i}^N \quad (2.219)$$

$$\bar{\mathbf{V}}_{12} = \bar{\mathbf{Z}}_{12}^{\text{CNF}} = \eta_o^{-1} \bar{\mathbf{T}}_o^N - \eta_i^{-1} \bar{\mathbf{T}}_i^N \quad (2.220)$$

$$\bar{\mathbf{V}}_{21} = \bar{\mathbf{Z}}_{21}^{\text{CNF}} = -\eta_o \bar{\mathbf{T}}_o^N + \eta_i \bar{\mathbf{T}}_i^N. \quad (2.221)$$

Left-hand sides (LHSs) of stable formulations are the same as those of their conventional forms. In other words, the stabilization procedure only requires a modification on RHSs of formulations. On RHSs, both S-CTF and S-CNF involve operators applied on incident fields. Since these operators are already available, the stabilization does not require a significant cost in terms of the processing time and memory usage. The extra cost is only due to the calculation of modified RHSs before iterative solutions, which can be performed in negligible time compared to other parts of implementations.

In their discrete forms, a direct approach to apply operators on incident fields is

to expand fields in a series of basis functions and perform MVMs. The expansion can be achieved by using identity operators and solving the equation

$$\begin{bmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{n}} \times \mathbf{H}^{inc} \\ -\hat{\mathbf{n}} \times \mathbf{E}^{inc} \end{bmatrix}(\mathbf{r}) = \begin{bmatrix} \hat{\mathbf{n}} \times \mathbf{H}^{inc} \\ -\hat{\mathbf{n}} \times \mathbf{E}^{inc} \end{bmatrix}(\mathbf{r}) \quad (2.222)$$

with MOM. Using local basis functions, such as the RWG functions, identity operators lead to sparse matrices and the discrete form of (2.222) can be written as

$$\begin{bmatrix} \bar{\mathbf{I}}^T & 0 \\ 0 & \bar{\mathbf{I}}^T \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{inc} \\ \mathbf{a}_M^{inc} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_H^N \\ -\mathbf{v}_E^N \end{bmatrix}. \quad (2.223)$$

The solution of (2.223) usually requires negligible time; however, the use of discretized identity operators may deteriorate the accuracy of results.

Using (2.223) in (2.203) and (2.218), discretized S-CTF and S-CNF can be written as

$$\begin{aligned} & \begin{bmatrix} \bar{\mathbf{Z}}_{11}^{\text{M-CTF}} & \bar{\mathbf{Z}}_{12}^{\text{M-CTF}} \\ \bar{\mathbf{Z}}_{21}^{\text{M-CTF}} & \bar{\mathbf{Z}}_{22}^{\text{M-CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{rad} \\ \mathbf{a}_M^{rad} \end{bmatrix} \\ &= \begin{bmatrix} \bar{\mathbf{Y}}_{11} & \bar{\mathbf{Y}}_{12} \\ \bar{\mathbf{Y}}_{21} & \bar{\mathbf{Y}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathbf{I}}^T & 0 \\ 0 & \bar{\mathbf{I}}^T \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{v}_H^N \\ -\mathbf{v}_E^N \end{bmatrix} \end{aligned} \quad (2.224)$$

and

$$\begin{aligned} & \begin{bmatrix} \bar{\mathbf{Z}}_{11}^{\text{CNF}} & \bar{\mathbf{Z}}_{12}^{\text{CNF}} \\ \bar{\mathbf{Z}}_{21}^{\text{CNF}} & \bar{\mathbf{Z}}_{22}^{\text{CNF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{rad} \\ \mathbf{a}_M^{rad} \end{bmatrix} \\ &= - \begin{bmatrix} \bar{\mathbf{V}}_{11} & \bar{\mathbf{V}}_{12} \\ \bar{\mathbf{V}}_{21} & \bar{\mathbf{V}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathbf{I}}^T & 0 \\ 0 & \bar{\mathbf{I}}^T \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{v}_H^N \\ -\mathbf{v}_E^N \end{bmatrix}, \end{aligned} \quad (2.225)$$

respectively.

2.3.6 Double-Stabilized Combined Tangential Formulation

The presence of discretized identity operators in (2.223) deteriorates the accuracy of results obtained with S-CTF and S-CNF. Although this is not critical for S-CNF, which already involves identity operators on the LHS, the accuracy of S-CTF can be affected significantly. Therefore, to obtain more accurate results, we developed a double-stabilized CTF (DS-CTF) based on calculating coefficients related to incident fields by discretizing and solving the tangential form of (2.191), i.e.,

$$\begin{bmatrix} \eta_o \bar{\mathbf{T}}_o^T & -\bar{\mathbf{K}}_{PV,o}^T \\ \bar{\mathbf{K}}_{PV,o}^T & \eta_o^{-1} \bar{\mathbf{T}}_o^T \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{inc} \\ \mathbf{a}_M^{inc} \end{bmatrix} = -0.5 \begin{bmatrix} \mathbf{v}_E^T \\ \mathbf{v}_H^T \end{bmatrix}. \quad (2.226)$$

This formulation is completely free of identity operators. On the other hand, the improved accuracy comes at the cost of reduced efficiency since it is necessary to solve an additional dense matrix equation rather than the extremely sparse equation in (2.223). Using (2.226) in (2.203), discretized DS-CTF can be written as

$$\begin{bmatrix} \bar{\mathbf{Z}}_{11}^{M-CTF} & \bar{\mathbf{Z}}_{12}^{M-CTF} \\ \bar{\mathbf{Z}}_{21}^{M-CTF} & \bar{\mathbf{Z}}_{22}^{M-CTF} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{rad} \\ \mathbf{a}_M^{rad} \end{bmatrix} = -0.5 \begin{bmatrix} \bar{\mathbf{Y}}_{11} & \bar{\mathbf{Y}}_{12} \\ \bar{\mathbf{Y}}_{21} & \bar{\mathbf{Y}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \eta_o \bar{\mathbf{T}}_o^T & -\bar{\mathbf{K}}_{PV,o}^T \\ \bar{\mathbf{K}}_{PV,o}^T & \eta_o^{-1} \bar{\mathbf{T}}_o^T \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{v}_E^T \\ \mathbf{v}_H^T \end{bmatrix}. \quad (2.227)$$

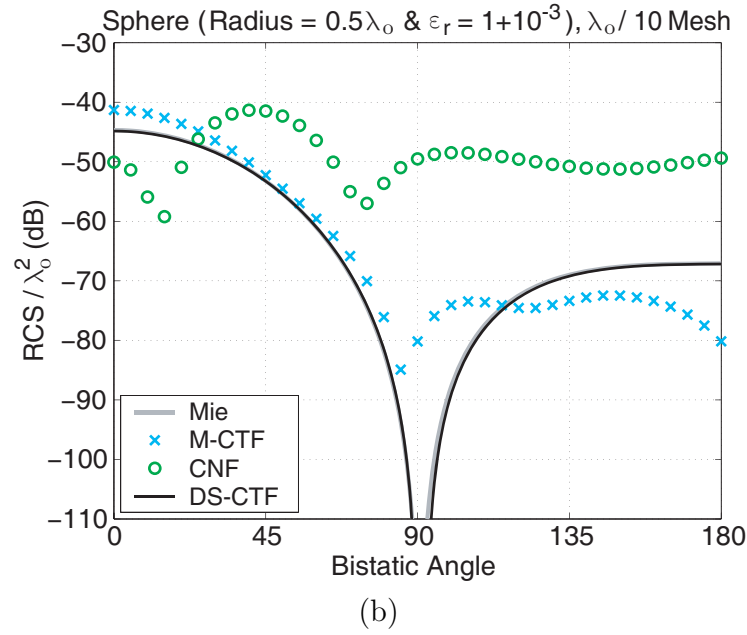
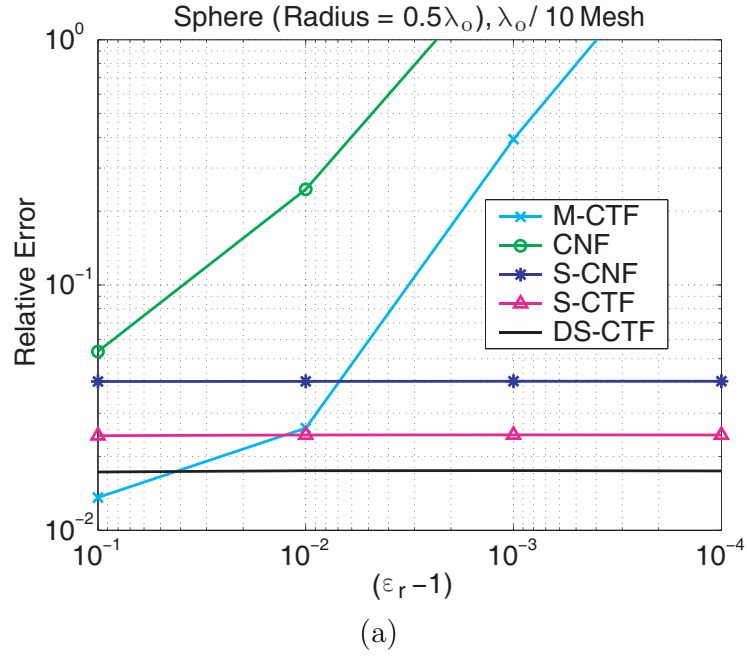


Figure 2.18: (a) Relative error in the solution of scattering problems involving a sphere with a radius of $0.5\lambda_o$ located in free space. (b) Normalized bistatic RCS (RCS/λ_o^2) of a sphere with a radius of $0.5\lambda_o$ and with a relative permittivity of $1.0 + 10^{-3}$ located in free space.

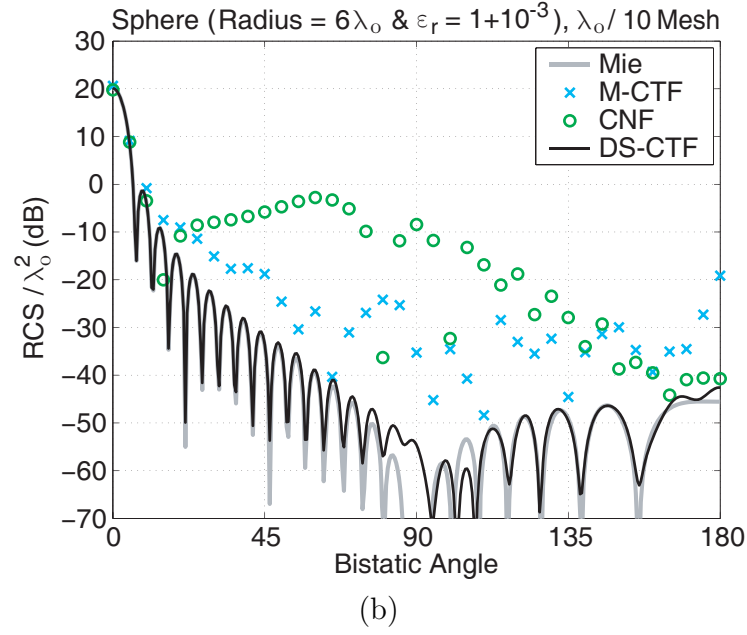
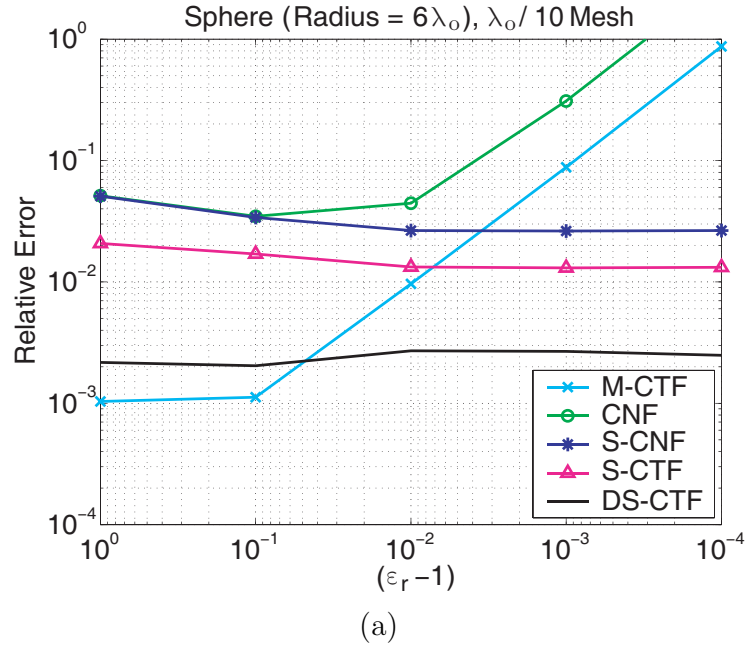


Figure 2.19: (a) Relative error in the solution of scattering problems involving a sphere with a radius of $6\lambda_o$ located in free space. (b) Normalized bistatic RCS (RCS/λ_o^2) of a sphere with a radius of $6\lambda_o$ and with a relative permittivity of $1.0 + 10^{-3}$ located in free space.

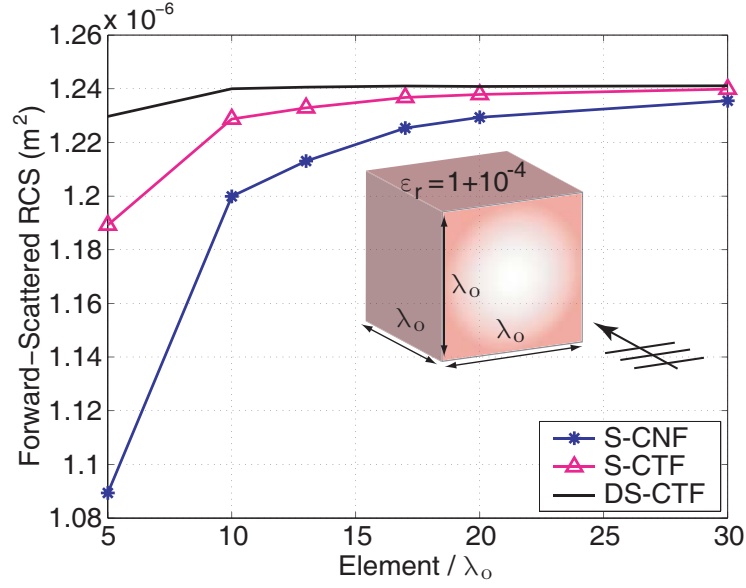


Figure 2.20: Forward-scattered RCS (m²) of a cube with edges of λ_o and with a relative permittivity of $1.0 + 10^{-4}$ located in free space with respect to various discretizations of the scattering problem.

2.3.7 Numerical Results

First, we demonstrate the improved accuracy of stable formulations in the solution of scattering problems involving a small sphere with a radius of $0.5\lambda_o$. The sphere is located in free space and illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. Discretization of the surface with $\lambda_o/10$ mesh size produces 620 triangles and 2×930 RWG functions are employed to expand the unknown surface current densities. Matrix equations involving 1860 unknowns and obtained by using conventional and stable formulations are solved iteratively. Matrix elements are computed directly with 10^{-2} relative error. Figure 2.18(a) shows the relative error in computed values of the far-zone electric field on the z - x plane with respect to different contrasts $(\epsilon_r - 1)$ from 10^{-4} to 10^{-1} . The far-zone co-polar electric field is sampled at 360 points as

$$\mathbf{E}_\theta^\infty[n] = \lim_{r \rightarrow \infty} \left\{ r E_\theta(r, \theta_n, \phi = 0) \right\}, \quad (2.228)$$

where $\theta_n = (n-1)\pi/360$ for $n = 1, 2, \dots, 360$. We compute (2.228) by using different integral-equation formulations and by a Mie-series method as a reference. Then, the relative error is defined as

$$\Delta_R = \frac{\|\mathbf{E}_{C,\theta}^\infty - \mathbf{E}_{A,\theta}^\infty\|_2}{\|\mathbf{E}_{A,\theta}^\infty\|_2}, \quad (2.229)$$

where $\mathbf{E}_{C,\theta}^\infty$ and $\mathbf{E}_{A,\theta}^\infty$ are computational and analytical values, respectively, and $\|\cdot\|_2$ represents the 2-norm of complex arrays involving 360 elements. Figure 2.18(a) shows that M-CTF and CNF formulations encounter stability problems as contrast decreases and they fail to provide accurate results for low-contrast levels. With stable formulations, however, accuracy does not depend on the contrast and we obtain accurate results even for $\epsilon_r = 1.0 + 10^{-4}$. We also observe that S-CTF is more accurate than S-CNF. In addition, DS-CTF provides the most accurate results for low-contrast problems, since its implementation is completely free of identity terms. Figure 2.18(b) presents the RCS of the sphere when the contrast is 10^{-3} . Normalized RCS (RCS/λ_o^2) is plotted on the z - x plane as a function of the observation angle, where 0° corresponds to the forward-scattering direction. We observe that traditional formulations are inaccurate, while values obtained by DS-CTF are in agreement with the reference Mie-series solution.

Figure 2.19 shows the solution of a scattering problem involving a sphere of radius $6\lambda_o$ discretized with 264,006 unknowns. In Figure 2.19(a), we again consider the relative error for different formulations with respect to the contrast of the sphere. In MLFMA solutions, near-field interactions are calculated with at most 1% error, and far-field interactions are computed with three digits of accuracy. Similar to the previous example, stable formulations do not break down as contrast decreases. In Figure 2.19(b), bistatic RCS values are plotted as a function of the observation angle when the contrast is 10^{-3} . Conventional formulations are accurate only in the forward-scattering direction (0°), while the accuracy is improved significantly by using DS-CTF.

For the solution of the scattering problem involving a sphere of radius $6\lambda_o$ with 0.1 contrast, Table 2.2 lists iteration counts of the CGS method to reduce the residual error to less than 10^{-6} . We note that extra costs of S-CTF and S-CNF are negligible compared to M-CTF and CNF, respectively. These stable formulations require only 8–9 iterations to solve the extremely sparse matrix equation in (2.223). On the other hand, using DS-CTF doubles the processing time compared to S-CTF since the solution of the dense matrix equation in (2.226) requires 670 iterations, in addition to 616 iterations for the solution of the main equation. Table 2.2 also shows that both conventional and stable tangential formulations (M-CTF, S-CTF, and DS-CTF) require more iterations than normal formulations (CNF and S-CNF). On the other hand, tangential formulations are generally more accurate to calculate scattered fields compared to normal formulations, as usual.

Finally, Figure 2.20 presents results of a scattering problem involving a cube with edges of 1 m illuminated by a plane wave at 300 MHz. The cube has a relative permittivity of $1.0 + 10^{-4}$ and it is located in free space. Figure 2.20 shows forward-scattered RCS values (m^2) for different discretizations of the geometry. Triangulation of the geometry with $\lambda_o/5$ and $\lambda_o/30$ mesh sizes leads to 1008 and 36,756 unknowns, respectively. In the MLFMA implementation, near-field interactions are again calculated with 1% error and far-field interactions are computed with three digits of accuracy. Figure 2.20 indicates that all stable formulations tend to converge to a limit (correct) value as the discretization becomes denser. However, convergence characteristics depend on the formulation type. We observe that S-CNF and DS-CTF have the slowest and the fastest convergence rates, respectively. Another interpretation of Figure 2.20 is that the most accurate results are provided by DS-CTF for a given mesh size. Specifically, for the conventional $\lambda_o/10$ triangulation, the error of S-CNF is more than 3%;

Table 2.2: Number of CGS Iterations (10^{-6} Residual Error) for the Solution of a Sphere Problem Discretized with 264,006 Unknowns

	CNF	M-CTF	S-CNF	S-CTF	DS-CTF
Full System	35	638	33	655	616+670
Sparse System	–	–	8	9	–

this may not be acceptable since matrix elements are calculated with at most 1% error.

2.3.8 Breakdown of Stable Formulations for Very Low Contrasts

In the three stable formulations described above, namely, S-CNF, S-CTF, and DS-CTF, RHSs are obtained by applying inner and outer operators on incident fields. In addition, operators are subtracted from each other so that RHSs go to zero as the contrast decreases. Hence, we call these formulations operator-based stabilization formulations (OBSFs). Using OBSFs, radiating parts of currents can be computed accurately for low-contrast objects, i.e., when radiating currents are numerically insignificant compared to nonradiating currents. This is because relatively large nonradiating currents are extracted, and only radiating currents are solved for. Consequently, OBSFs provide accurate results for low-contrast problems that cannot be solved accurately with conventional formulations. However, even OBSFs break down and fail to provide accurate results for very low contrasts. The reason is numerical errors arising during the computation of RHSs of OBSFs, which become significant when the contrast decreases to very low values and renders the RHS vanishingly small.

To explain numerical problems in OBSFs, we consider S-CTF in (2.196)–(2.200)

and its discretization in (2.203)–(2.207). The RHS of S-CTF is obtained by applying integro-differential operators on nonradiating currents, i.e., we compute

$$\begin{aligned} \begin{bmatrix} \mathbf{w}_1^{\text{S-CTF}} \\ \mathbf{w}_2^{\text{S-CTF}} \end{bmatrix} &= \begin{bmatrix} \eta_o \bar{\mathbf{T}}_o^T & -\bar{\mathbf{K}}_{PV,o}^T \\ \eta_o \eta_i \bar{\mathbf{K}}_{PV,o}^T & \eta_i \bar{\mathbf{T}}_o^T \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{\text{inc}} \\ \mathbf{a}_M^{\text{inc}} \end{bmatrix} \\ &\quad - \begin{bmatrix} \eta_i \bar{\mathbf{T}}_i^T & -\bar{\mathbf{K}}_{PV,i}^T \\ \eta_o \eta_i \bar{\mathbf{K}}_{PV,i}^T & \eta_o \bar{\mathbf{T}}_i^T \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{\text{inc}} \\ \mathbf{a}_M^{\text{inc}} \end{bmatrix}. \end{aligned} \quad (2.230)$$

Operators related to outer and inner media are applied on tangential incident fields (nonradiating currents) in the first and the second terms of (2.230), respectively. Then, the second term is subtracted from the first term to compute the RHS. When the contrast is low, the RHS is small, but it is obtained by the subtraction of two terms that are relatively large. Then, depending on the discretization and the accuracy of MVMs, the RHS of S-CTF may not be calculated accurately when the contrast is very low, i.e., when the result of the subtraction is very small.

In general, OBSFs fail to provide accurate results when the contrast is decreased to very low values. Therefore, it is desirable to obtain a robust formulation that is valid for arbitrarily low contrasts. We achieve this by introducing fictitious fields and forming RHSs based on the difference of fields. This method leads to accurate calculation of RHSs, even when the contrast is extremely low.

2.3.9 Field-Based Stabilized Combined-Tangential Formulation

When incident fields are extracted from the LHS, S-CTF in (2.195) can be rewritten as

$$\begin{aligned} \hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{Z}_{11}^{\text{M-CTF}} & \mathbf{Z}_{12}^{\text{M-CTF}} \\ \mathbf{Z}_{21}^{\text{M-CTF}} & \mathbf{Z}_{22}^{\text{M-CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{\text{rad}} \\ \mathbf{M}^{\text{rad}} \end{bmatrix}(\mathbf{r}) = -0.5\hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{E}^{\text{inc}} \\ \eta_o\eta_i\mathbf{H}^{\text{inc}} \end{bmatrix}(\mathbf{r}) \\ -\hat{\mathbf{t}} \cdot \begin{bmatrix} \eta_i\mathcal{T}_i & -\mathcal{K}_{PV,i} \\ \eta_o\eta_i\mathcal{K}_{PV,i} & \eta_o\mathcal{T}_i \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{\text{inc}} \\ \mathbf{M}^{\text{inc}} \end{bmatrix}(\mathbf{r}). \end{aligned} \quad (2.231)$$

At this stage, we consider incident fields as functions of medium parameters, i.e.,

$$\mathbf{E}^{\text{inc}}(\mathbf{r}) = \mathbf{e}(\mathbf{r}, \epsilon_o, \mu_o) \quad (2.232)$$

$$\mathbf{H}^{\text{inc}}(\mathbf{r}) = \mathbf{h}(\mathbf{r}, \epsilon_o, \mu_o). \quad (2.233)$$

Then, we define fictitious incident fields as

$$\mathbf{E}_f^{\text{inc}}(\mathbf{r}) = \mathbf{e}(\mathbf{r}, \epsilon_i, \mu_i) \quad (2.234)$$

$$\mathbf{H}_f^{\text{inc}}(\mathbf{r}) = \mathbf{h}(\mathbf{r}, \epsilon_i, \mu_i) \quad (2.235)$$

by using parameters of the inner medium for the outside. This way, similar to identities in (2.191), we have

$$\begin{aligned} \begin{Bmatrix} \hat{\mathbf{t}} \cdot \\ \hat{\mathbf{n}} \times \end{Bmatrix} \begin{bmatrix} \eta_i\mathcal{T}_i & -\mathcal{K}_{PV,i} \\ \mathcal{K}_{PV,i} & \eta_i^{-1}\mathcal{T}_i \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{n}} \times \mathbf{H}_f^{\text{inc}} \\ -\hat{\mathbf{n}} \times \mathbf{E}_f^{\text{inc}} \end{bmatrix}(\mathbf{r}) \\ = -\frac{\Omega_i}{4\pi} \begin{Bmatrix} \hat{\mathbf{t}} \cdot \\ \hat{\mathbf{n}} \times \end{Bmatrix} \begin{bmatrix} \mathbf{E}_f^{\text{inc}} \\ \mathbf{H}_f^{\text{inc}} \end{bmatrix}(\mathbf{r}). \end{aligned} \quad (2.236)$$

Finally, by adding and subtracting terms of (2.236) in (2.231), we obtain

$$\begin{aligned}
& \hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{Z}_{11}^{\text{M-CTF}} & \mathbf{Z}_{12}^{\text{M-CTF}} \\ \mathbf{Z}_{21}^{\text{M-CTF}} & \mathbf{Z}_{22}^{\text{M-CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{J}^{\text{rad}} \\ \mathbf{M}^{\text{rad}} \end{bmatrix}(\mathbf{r}) \\
&= -0.5\hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{E}^{\text{inc}} - \mathbf{E}_f^{\text{inc}} \\ \eta_o\eta_i(\mathbf{H}^{\text{inc}} - \mathbf{H}_f^{\text{inc}}) \end{bmatrix}(\mathbf{r}) \\
&- \hat{\mathbf{t}} \cdot \begin{bmatrix} \eta_i\mathcal{T}_i & -\mathcal{K}_{PV,i} \\ \eta_o\eta_i\mathcal{K}_{PV,i} & \eta_o\mathcal{T}_i \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{n}} \times (\mathbf{H}^{\text{inc}} - \mathbf{H}_f^{\text{inc}}) \\ -\hat{\mathbf{n}} \times (\mathbf{E}^{\text{inc}} - \mathbf{E}_f^{\text{inc}}) \end{bmatrix}(\mathbf{r}), \quad (2.237)
\end{aligned}$$

which we call field-based-stabilized CTF (FBS-CTF). In contrast to OBSFs, FBS-CTF has a RHS obtained by subtracting real and fictitious incident fields from each other. These subtractions can be performed analytically in the continuous space before the discretization. Then, operators related to the inner medium are applied to compute the second term of the RHS in (2.237). We note that the RHS of FBS-CTF is obtained as the sum of two terms, i.e.,

$$\begin{bmatrix} \mathbf{w}_{1,1}^{\text{FBS-CTF}} \\ \mathbf{w}_{2,1}^{\text{FBS-CTF}} \end{bmatrix} = -0.5\hat{\mathbf{t}} \cdot \begin{bmatrix} \mathbf{E}^{\text{inc}} - \mathbf{E}_f^{\text{inc}} \\ \eta_o\eta_i(\mathbf{H}^{\text{inc}} - \mathbf{H}_f^{\text{inc}}) \end{bmatrix}(\mathbf{r}) \quad (2.238)$$

and

$$\begin{bmatrix} \mathbf{w}_{1,2}^{\text{FBS-CTF}} \\ \mathbf{w}_{2,2}^{\text{FBS-CTF}} \end{bmatrix} = -\hat{\mathbf{t}} \cdot \begin{bmatrix} \eta_i\mathcal{T}_i & -\mathcal{K}_{PV,i} \\ \eta_o\eta_i\mathcal{K}_{PV,i} & \eta_o\mathcal{T}_i \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{n}} \times (\mathbf{H}^{\text{inc}} - \mathbf{H}_f^{\text{inc}}) \\ -\hat{\mathbf{n}} \times (\mathbf{E}^{\text{inc}} - \mathbf{E}_f^{\text{inc}}) \end{bmatrix}(\mathbf{r}), \quad (2.239)$$

which are both small when the contrast is low. Consequently, the RHS can be calculated accurately for arbitrarily low contrasts, and it is sensitive to neither the discretization errors nor numerical errors arising during MVMs. FBS-CTF can easily be obtained from the conventional CTF implementation and, similar to S-CTF and S-CNF, its extra cost is also negligible.

Using the expansion in (2.222), discretization of FBS-CTF leads to

$$\begin{aligned} \begin{bmatrix} \bar{\mathbf{Z}}_{11}^{\text{M-CTF}} & \bar{\mathbf{Z}}_{12}^{\text{M-CTF}} \\ \bar{\mathbf{Z}}_{21}^{\text{M-CTF}} & \bar{\mathbf{Z}}_{22}^{\text{M-CTF}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J^{\text{rad}} \\ \mathbf{a}_M^{\text{rad}} \end{bmatrix} = -0.5 \begin{bmatrix} \bar{\mathbf{I}}^T & 0 \\ 0 & \bar{\mathbf{I}}^T \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{v}_{E,f}^T \\ \eta_o \eta_i \mathbf{v}_{M,f}^T \end{bmatrix} \\ - \begin{bmatrix} \eta_i \bar{\mathbf{T}}_i^T & -\bar{\mathbf{K}}_{PV,i}^T \\ \eta_o \eta_i \bar{\mathbf{K}}_{PV,i}^T & \eta_o \bar{\mathbf{T}}_i^T \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathbf{I}}^T & 0 \\ 0 & \bar{\mathbf{I}}^T \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{v}_{M,f}^N \\ -\mathbf{v}_{E,f}^N \end{bmatrix}, \quad (2.240) \end{aligned}$$

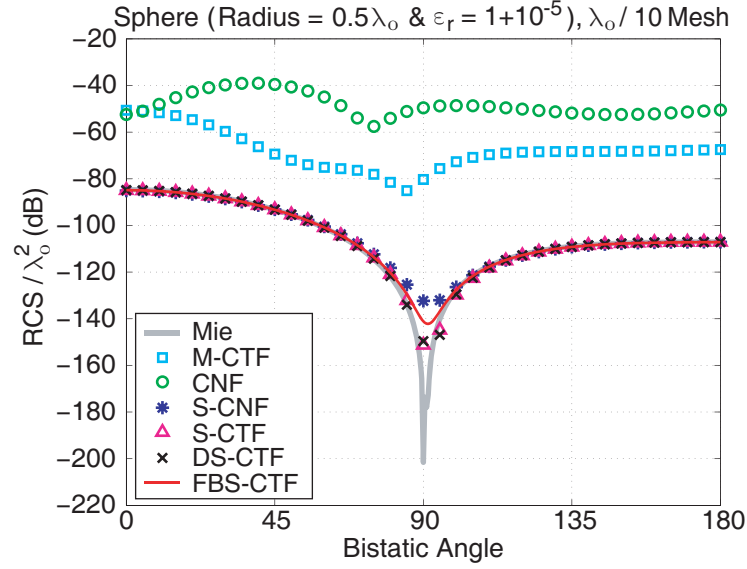
where

$$\mathbf{v}_{E,f}^T[m] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot [\mathbf{E}^{\text{inc}}(\mathbf{r}) - \mathbf{E}_f^{\text{inc}}(\mathbf{r})] \quad (2.241)$$

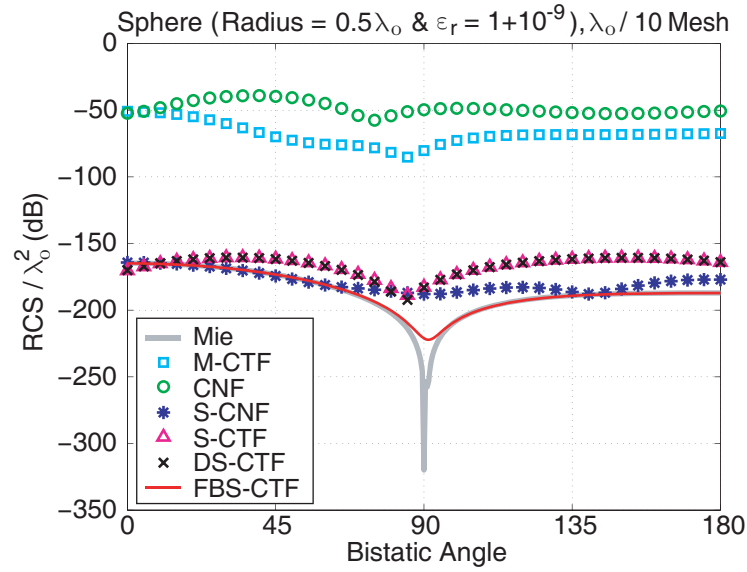
$$\mathbf{v}_{M,f}^T[m] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot [\mathbf{H}^{\text{inc}}(\mathbf{r}) - \mathbf{H}_f^{\text{inc}}(\mathbf{r})] \quad (2.242)$$

$$\mathbf{v}_{E,f}^N[m] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times [\mathbf{E}^{\text{inc}}(\mathbf{r}) - \mathbf{E}_f^{\text{inc}}(\mathbf{r})] \quad (2.243)$$

$$\mathbf{v}_{M,f}^N[m] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times [\mathbf{H}^{\text{inc}}(\mathbf{r}) - \mathbf{H}_f^{\text{inc}}(\mathbf{r})]. \quad (2.244)$$



(a)



(b)

Figure 2.21: Normalized bistatic RCS (RCS/λ_o^2) of a sphere with a radius of $0.5\lambda_o$ located in free space, when the relative permittivity of the sphere is (a) $1.0+10^{-5}$ and (b) $1.0+10^{-9}$.

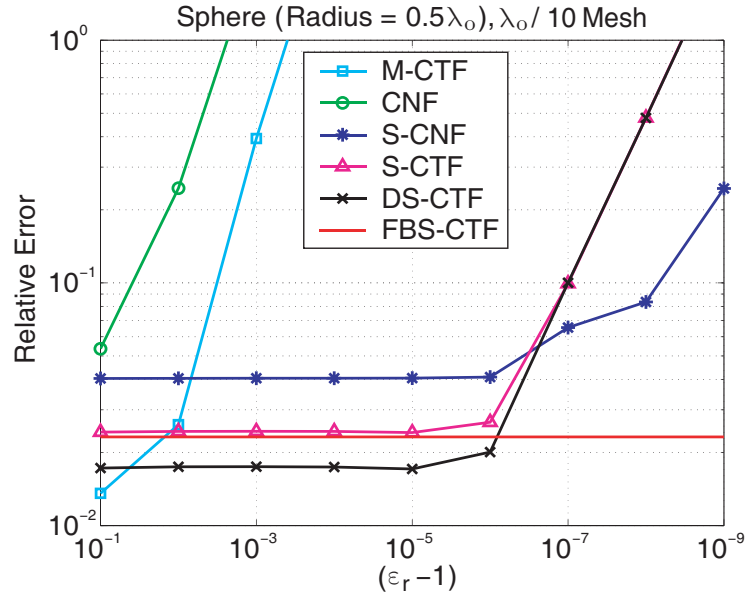


Figure 2.22: Relative error in the solution of scattering problems involving a sphere with a radius of $0.5\lambda_o$ located in free space.

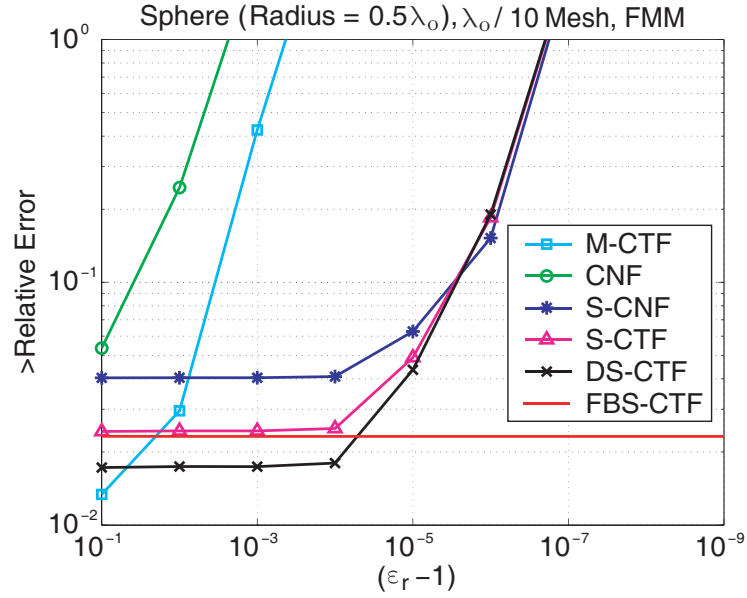
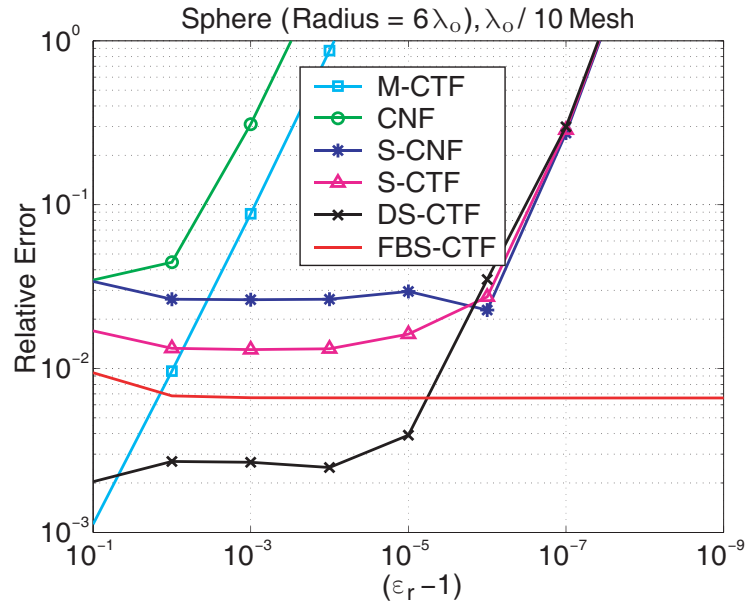
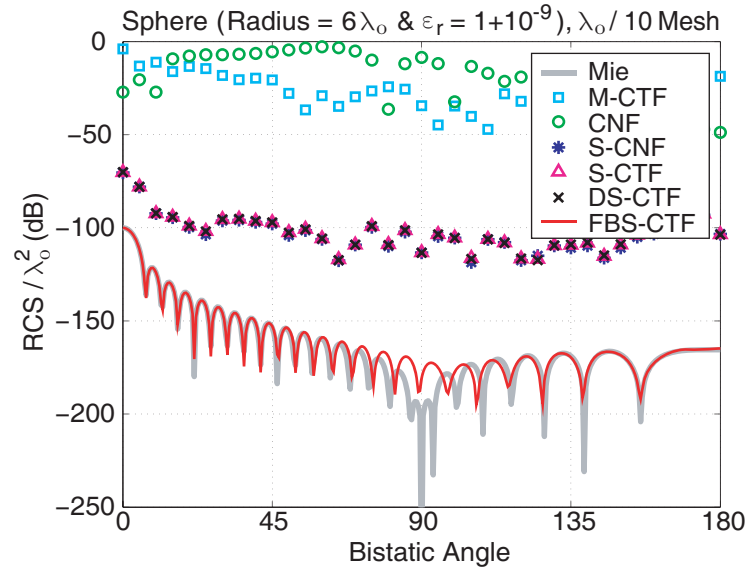


Figure 2.23: Relative error in the solution of scattering problems involving a sphere with a radius of $0.5\lambda_o$ located in free space. Scattering problems are solved by using FMM.



(a)



(b)

Figure 2.24: (a) Relative error in the solution of scattering problems involving a sphere with a radius of $6\lambda_o$ located in free space. (b) Normalized bistatic RCS (RCS/λ_o^2) of a sphere with a radius of $6\lambda_o$ and with a relative permittivity of $1.0 + 10^{-9}$ located in free space.

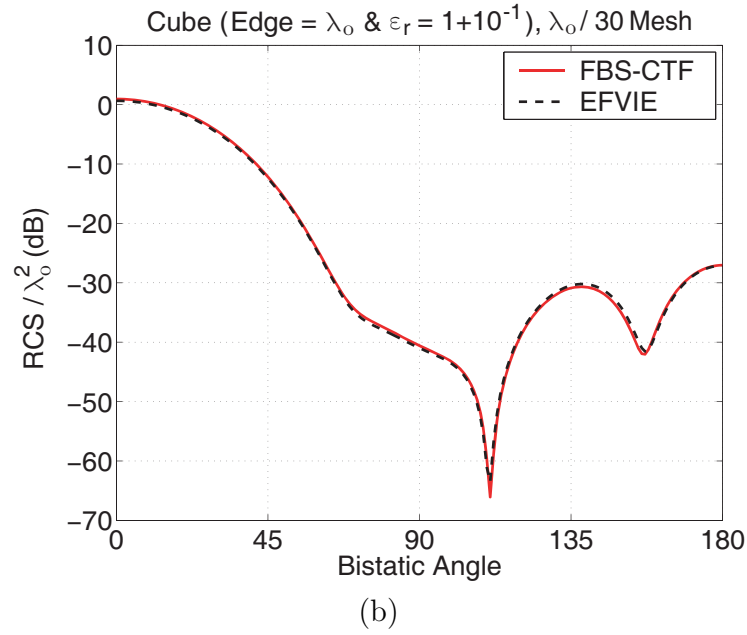
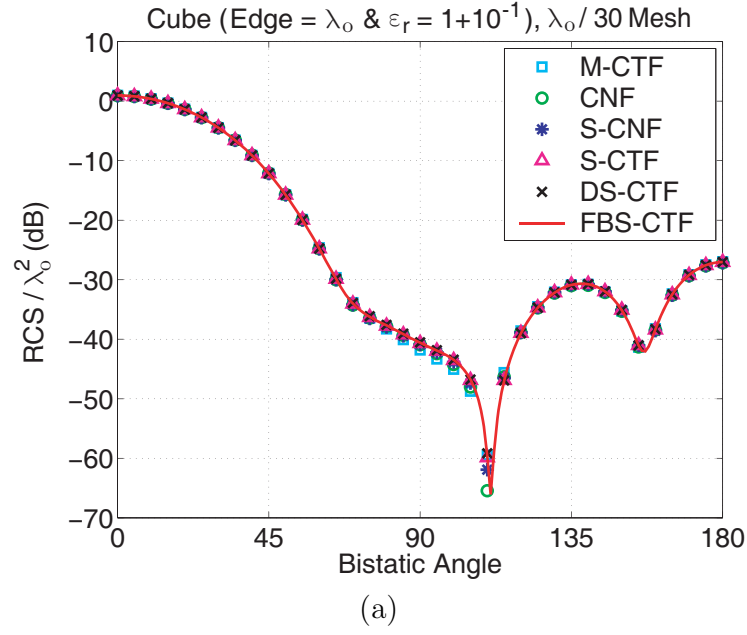


Figure 2.25: (a) Normalized bistatic RCS (RCS/λ_o^2) of a cube with edges of λ_o located in free space. The relative permittivity of the cube is $1.0 + 10^{-1}$. RCS values are obtained by using surface formulations when the mesh size is $\lambda_o/30$. (b) RCS values obtained with FBS-CTF and EFVIE agree with each other.

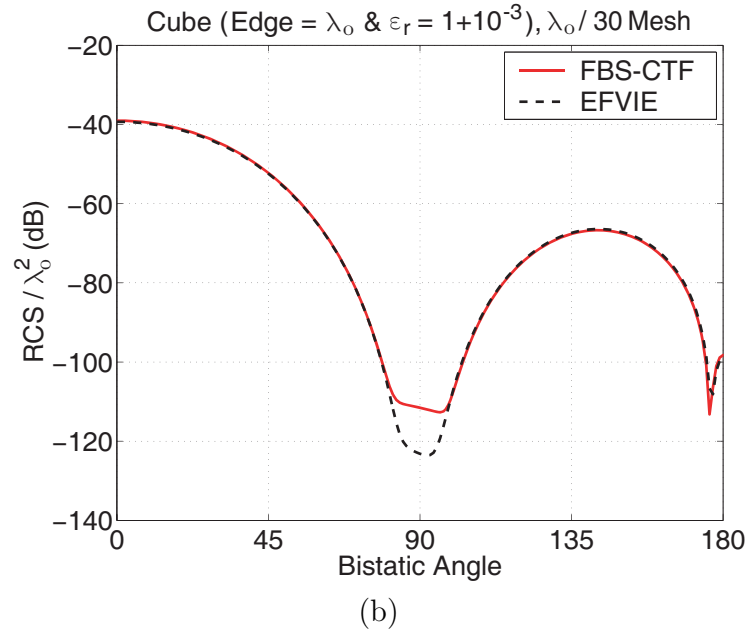
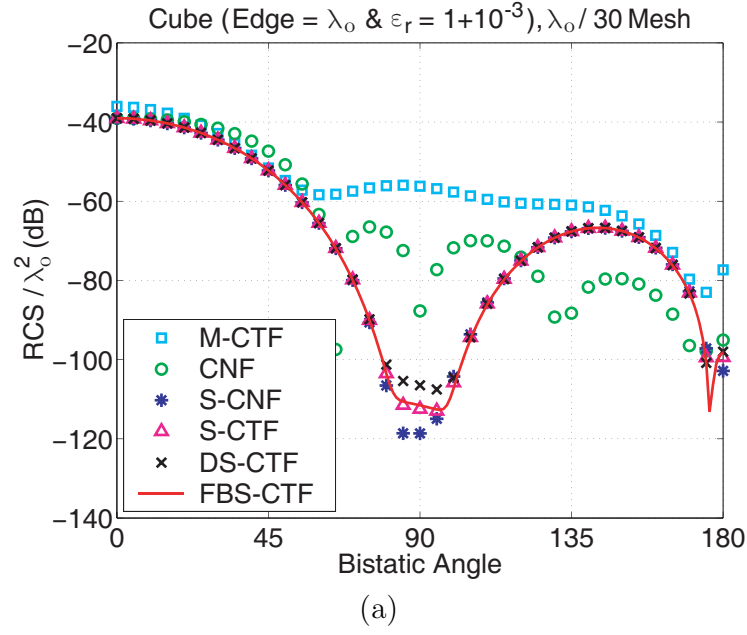


Figure 2.26: (a) Normalized bistatic RCS (RCS/λ_o^2) of a cube with edges of λ_o located in free space. The relative permittivity of the cube is $1.0 + 10^{-3}$. RCS values are obtained by using surface formulations when the mesh size is $\lambda_o/30$. (b) RCS values obtained with FBS-CTF and EFVIE agree with each other.

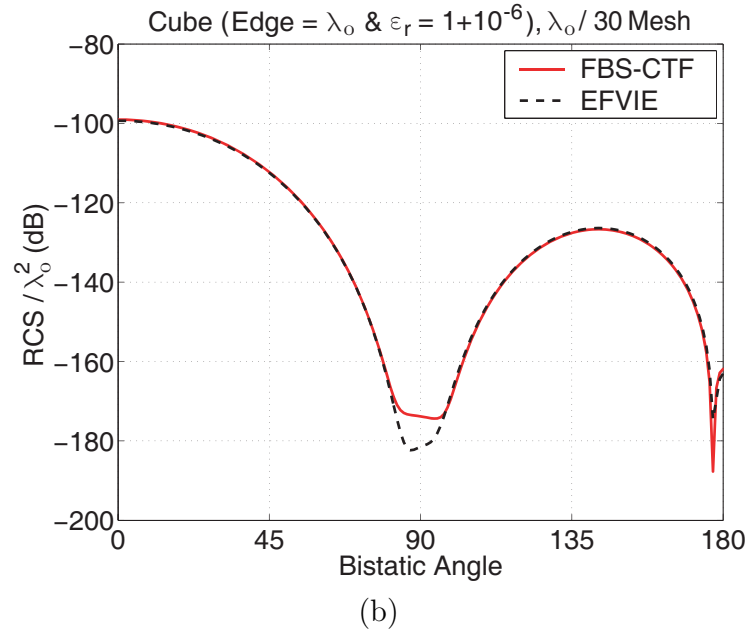
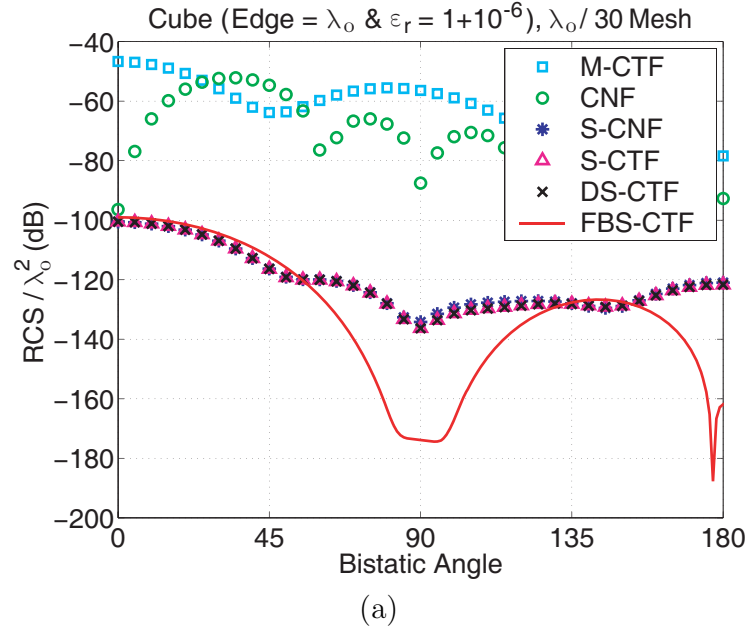


Figure 2.27: (a) Normalized bistatic RCS (RCS/λ_o^2) of a cube with edges of λ_o located in free space. The relative permittivity of the cube is $1.0 + 10^{-6}$. RCS values are obtained by using surface formulations when the mesh size is $\lambda_o/30$. (b) RCS values obtained with FBS-CTF and EFVIE agree with each other.

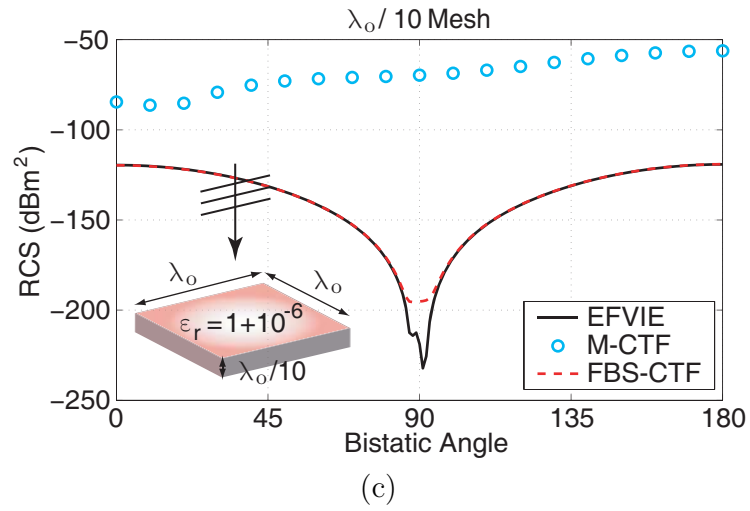
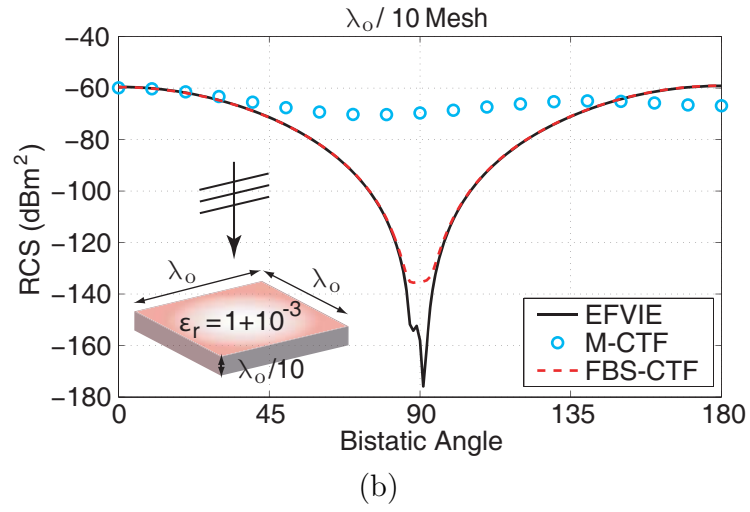
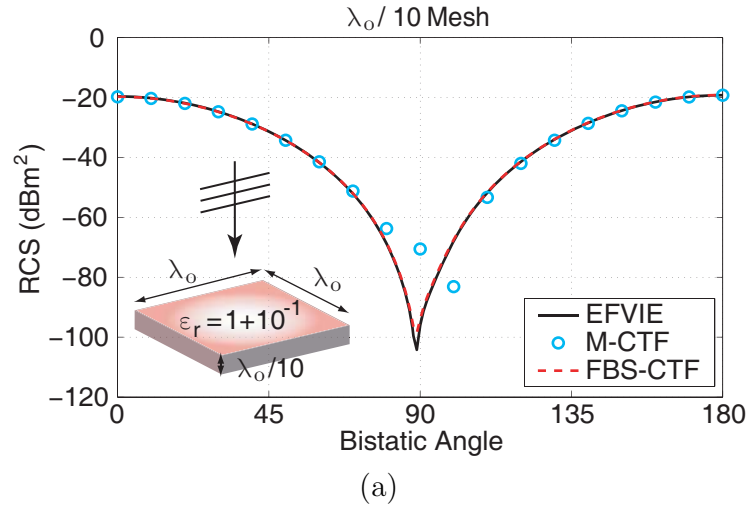


Figure 2.28: Bistatic RCS (dBm²) of a $\lambda_o \times \lambda_o \times \lambda_o/10$ slab located in free space, when the relative permittivity of the slab is (a) 1.1, (b) $1.0 + 10^{-3}$, and (c) $1.0 + 10^{-6}$.

2.3.10 Numerical Results for Very Low Contrasts

Figure 2.21 presents bistatic RCS results for a sphere with a radius of $0.5\lambda_o$. The sphere is located in free space and illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. Discretization of the surface with $\lambda_o/10$ mesh size leads to matrix equations with 1860 unknowns. Matrix elements are computed directly with 10^{-2} relative error. In Figure 2.21, the normalized RCS (RCS/λ_o^2) is plotted as a function of the observation angle on the z - x plane, where 0° corresponds to the forward-scattering direction. The contrast of the sphere is 10^{-5} and 10^{-9} in Figure 2.21(a) and Figure 2.21(b), respectively. For reference, RCS values are also computed analytically by a Mie-series solution. When the contrast is 10^{-5} , conventional formulations, i.e., M-CTF and CNF, are inaccurate, while stable formulations provide accurate results that are close to the analytical curve. When the contrast is further reduced to 10^{-9} , OBSFs also fail to agree with the analytical curve. For this extremely low contrast, the only formulation that provides accurate results is FBS-CTF. To further compare formulations, Figure 2.22 presents the relative error defined in (2.229) as a function of the contrast. We observe that errors of M-CTF and CNF increase sharply when the contrast decreases below 10^{-1} – 10^{-2} , while OBSFs break down when the contrast is about 10^{-5} – 10^{-6} . On the other hand, the error of FBS-CTF is almost constant with respect to the contrast. Finally, Figure 2.23 presents the relative error, when computational values are obtained by FMM, instead of MOM. Comparing Figure 2.23 with Figure 2.22, we observe that the extra error introduced by FMM affects the stability of OBSFs significantly. Specifically, using FMM, those formulations break down earlier, i.e., when the contrast is about 10^{-4} . This is because OBSFs are very sensitive to numerical errors arising during MVMs. On the other hand, FBS-CTF provides stable results even with FMM, as depicted in Figure 2.23.

Figure 2.24 presents results of a scattering problem involving a sphere of $6\lambda_o$ discretized with $\lambda_o/10$ triangles leading to 264,006 unknowns. Problems are solved by MLFMA, where near-field interactions are calculated with 1% error and far-field interactions are computed with three digits of accuracy. Figure 2.24(a) depicts bistatic RCS values on the z - x plane when the sphere is illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction and the contrast of the sphere is 10^{-9} . We observe that all formulations except for the FBS-CTF fail to provide accurate results compared to Mie-series solution. RCS values provided by FBS-CTF deviate from analytical results only around 90° , where scattering is very low. As presented in Figure 2.24(b), the accuracy of FBS-CTF is stable for all values of the contrast from 10^{-1} to 10^{-9} . OBSFs are also stable in the 10^{-1} – 10^{-5} range, while they offer different levels of accuracy depending on the use of the identity operator. However, they break down when the contrast decreases below 10^{-5} . Finally, as in the previous examples, conventional CTF and CNF break down immediately below 10^{-1} contrast, testifying to the need for stabilized formulations.

Next, we consider the solution of scattering problems involving a cube with edges of λ_o discretized with $\lambda_o/30$ triangles. The cube is in free space and it is illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. We consider three different relative permittivities for the cube, i.e., 1.1, $1.0 + 10^{-3}$, and $1.0 + 10^{-6}$, corresponding to 10^{-1} , 10^{-3} , and 10^{-6} contrasts, respectively. Scattering problems are solved by FMM, where near-field interactions are calculated with 1% error and far-field interactions are computed with three digits of accuracy. Figure 2.25 presents the normalized RCS (RCS/λ_o^2) as a function of the observation angle on the z - x plane, where 0° corresponds to the forward-scattering direction. The contrast of the cube is 10^{-1} . As depicted in Figure 2.25(a), there are relatively small discrepancies among results obtained by using different formulations. In Figure 2.25(b), we compare RCS

values obtained by using FBS-CTF with those obtained by using the electric-field volume integral equation (EFVIE) [48], which is immune to low-contrast problems. Figure 2.25(b) confirms that FBS-CTF and other surface formulations are accurate when the contrast of the cube is 10^{-1} . Figure 2.26 compares bistatic RCS values obtained by various formulations as the contrast of the cube is decreased to 10^{-3} . In this case, RCS values obtained by using conventional formulations, namely, M-CTF and CNF, are inconsistent with values obtained by using stable formulations, i.e., S-CNF, S-CTF, DS-CTF, and FBS-CTF. As demonstrated in Figure 2.26(d), FBS-CTF (and other stable formulations) are consistent with VIE. Hence, we conclude that stable formulations are accurate, while M-CTF and CNF break down when the contrast of the cube is 10^{-3} . Finally, Figure 2.27 presents bistatic RCS results, when the contrast of the cube is very low, i.e., 10^{-6} . As opposed to the previous examples, RCS values obtained with OBSFs (S-CNF, S-CTF, and DS-CTF) and FBS-CTF are not consistent. In Figure 2.27(d), we again compare FBS-CTF with VIE, where the two formulations agree well with each other. This proves that only FBS-CTF provides accurate results, while other surface formulations break down when the contrast is 10^{-6} .

Finally, Figure 2.28 presents results of scattering problems involving a $\lambda_o \times \lambda_o \times \lambda_o/10$ dielectric slab, where $\lambda_o = 1$ m is the wavelength outside the object (free space). The slab is located at the origin as depicted in the insets of Fig 2.28, and it is illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. We consider three different relative permittivities for the slab, i.e., 1.1, $1.0 + 10^{-3}$, and $1.0 + 10^{-6}$, corresponding to 0.1, 10^{-3} , and 10^{-6} contrasts, respectively. The slab is discretized with $\lambda_o/20$ mesh size leading to matrix equations with 11,424 unknowns. Scattering problems are solved by using FMM, where near-field interactions are calculated with 1% error and far-field interactions are computed with three digits of accuracy. We plot

the bistatic RCS in dBm^2 as a function of the observation angle on the z - x plane. RCS values obtained by using M-CTF and FBS-CTF are compared with those obtained by using EFVIE. As depicted in Figure 2.28(a), M-CTF and FBS-CTF are consistent with EFVIE when the contrast of the slab is relatively large (0.1). As the contrast decreases to 10^{-3} and 10^{-6} , however, RCS values obtained with M-CTF become inconsistent with values obtained with FBS-CTF and EFVIE. Similar to the previous examples, FBS-CTF is accurate and agrees well with the reference EFVIE for all contrasts.

2.4 Composite Objects with Multiple Dielectric and Metallic Regions

Surface formulations can easily be extended to those problems including composite objects with multiple dielectric and PEC regions. The generalized procedure consists of the following main stages:

- Formulate an equivalent problem for each nonmetallic domain by defining equivalent currents, applying operators on currents, and enforcing boundary conditions for tangential fields.
- Perform a discretization for each domain by employing oriented basis and testing functions.
- Combine related unknowns on boundaries and corresponding equations to form a single matrix equation to solve.
- Calculate radiated electric and magnetic fields from equivalent currents.

This generalized procedure is extensively discussed in [38] in the context of a EFIE-CFIE-PMCHWT formulation, together with various techniques to handle

junctions, i.e., those locations where three or more domains intersect.

As demonstrated in Section 2.2, JMCFIE is a preferable formulation in terms of accuracy and efficiency for the solution of problems involving homogeneous dielectric objects. In numerical solutions employing the RWG functions, JMCFIE is more accurate than normal formulations, such as CNF and MNMF. In addition, iterative solutions of problems involving large and complicated objects are more efficient with JMCFIE, which requires fewer iterations than CNF and MNMF. For a given discretization with the RWG functions, tangential formulations, such as CTF, may provide more accurate results than JMCFIE. On the other hand, matrix equations obtained with tangential formulations are difficult to solve iteratively. In fact, improving the accuracy of JMCFIE solutions to levels of tangential formulations by refining the discretization can be more efficient than using tangential formulations with coarse discretizations. Moreover, JMCFIE becomes essential for large problems, which might not easily be solved with tangential formulations. In this section, we extend the MLFMA solution of JMCFIE to those problems involving composite structures with multiple dielectric and PEC regions. We mainly focus on the efficiency of solutions and investigate the number of iterations for increasingly large objects.

Consider the general case involving U regions, namely, $D_0, D_1, \dots, D_{U-1}$, and D_0 is a region extending to infinity. Each region D_u for $u = 0, 1, \dots, (U - 1)$ is either metallic with perfect conductivity or lossless dielectric with constant electromagnetic parameters, i.e., permittivity ϵ_u and permeability μ_u . We assume that each region D_u has a nonzero volume and is bounded by a closed surface S_u . Then,

$$S_u = \sum_{\substack{v=1 \\ v \neq u}}^U S_{uv}, \quad (2.245)$$

where $S_{uv} = S_{vu}$ is the interface between regions D_u and D_v . We also assume that there is no junction where three or more regions intersect. For an interface S_{uv} for $u < v$, the unit normal vector $\hat{\mathbf{n}}$ is directed into the region D_u .

2.4.1 Discretization

For numerical solutions, surface currents are expanded in a series of basis functions. Since we assume that there is no junction, each basis function is located on the interface of two regions, such as D_u and D_v . For $u < v$, D_u and D_v are called the first and the second regions, respectively, of the basis function. In addition, basis functions are indexed by first considering nonmetallic surfaces, which involve $N_D \leq N$ basis functions. On these surfaces, which separate two dielectric regions, both electric and magnetic currents are expanded in a series of the same set of basis functions $\mathbf{b}_n$ for $n = 1, 2, \dots, N_D$. Remaining $(N - N_D)$ basis functions, if any, are defined on metallic surfaces to expand the electric current. Using a Galerkin scheme for the discretization, we employ the same set of functions as testing functions, i.e., there are N testing functions to test boundary conditions.

In general, discretizations of JMCIE lead to $(N + N_D) \times (N + N_D)$ dense matrix equations in the form of

$$\begin{bmatrix} \begin{bmatrix} \bar{\mathbf{Z}}_{11}^{\text{JMCIE}} \end{bmatrix}_{N \times N} & \begin{bmatrix} \bar{\mathbf{Z}}_{12}^{\text{JMCIE}} \end{bmatrix}_{N \times N_D} \\ \begin{bmatrix} \bar{\mathbf{Z}}_{21}^{\text{JMCIE}} \end{bmatrix}_{N_D \times N} & \begin{bmatrix} \bar{\mathbf{Z}}_{22}^{\text{JMCIE}} \end{bmatrix}_{N_D \times N_D} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J \\ \mathbf{a}_M \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^{\text{JMCIE}} \\ \mathbf{w}_2^{\text{JMCIE}} \end{bmatrix}. \quad (2.246)$$

Consider the interaction between a basis function $\mathbf{b}_n$ and a testing function $\mathbf{t}_m$, and let a dielectric region D_u be common for the two functions. Then, the corresponding matrix element in the diagonal partition $\bar{\mathbf{Z}}_{11}^{\text{JMCIE}}$ in (2.246) can

be written as

$$\begin{aligned}
\bar{\mathbf{Z}}_{11}^{\text{JMCFIE}}[m, n] \leftarrow^+ & - \frac{\xi_n \xi_m}{2} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{b}_n(\mathbf{r}) \\
& + \xi_n \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{K}_{PV,u}\{\mathbf{b}_n\}(\mathbf{r}) \\
& + \xi_n \xi_m \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{T}_u\{\mathbf{b}_n\}(\mathbf{r}), \tag{2.247}
\end{aligned}$$

where we use the $\leftarrow^+$ notation to indicate the cumulative addition operation, since the value in (2.247) may not be the only contribution to $\bar{\mathbf{Z}}_{11}^{\text{JMCFIE}}[m, n]$. Specifically, if $\mathbf{t}_m$ and $\mathbf{b}_n$ are on the same nonmetallic surface, both regions of these functions are common, and the corresponding matrix element $\bar{\mathbf{Z}}_{11}^{\text{JMCFIE}}[m, n]$ involves two sets of contributions, i.e., interactions of basis and testing functions through the two regions. In (2.247), signs $\xi_m = \pm 1$ and $\xi_n = \pm 1$ are determined by the orientation of basis and testing functions. If the common region D_u is the “first” region for the basis (testing) function, then $\xi_n = +1$ ($\xi_m = +1$); otherwise, $\xi_n = -1$ ($\xi_m = -1$).

When the basis function $\mathbf{b}_n$ is not on a metallic surface, i.e., $n \leq N_D$, there exists a matrix element $\bar{\mathbf{Z}}_{12}^{\text{JMCFIE}}[m, n]$ in (2.246). A contribution to this element due to the interaction of basis and testing functions through the common region D_u can be written as

$$\begin{aligned}
\bar{\mathbf{Z}}_{12}^{\text{JMCFIE}}[m, n] \leftarrow^+ & - \frac{\xi_n}{2} \eta_u^{-1} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\
& + \xi_n \eta_u^{-1} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{T}_u\{\mathbf{b}_n\}(\mathbf{r}) \\
& - \xi_n \xi_m \eta_u^{-1} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{K}_{PV,u}\{\mathbf{b}_n\}(\mathbf{r}). \tag{2.248}
\end{aligned}$$

When the testing function $\mathbf{t}_m$ is not on a metallic surface, i.e., $m \leq N_D$, there exists a matrix element $\bar{\mathbf{Z}}_{21}^{\text{JMC FIE}}[m, n]$ with a contribution as

$$\begin{aligned} \bar{\mathbf{Z}}_{21}^{\text{JMC FIE}}[m, n] \stackrel{+}{\leftarrow} & \frac{\xi_n}{2} \eta_u \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\ & - \xi_n \eta_u \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{T}_u\{\mathbf{b}_n\}(\mathbf{r}) \\ & + \xi_n \xi_m \eta_u \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{K}_{PV,u}\{\mathbf{b}_n\}(\mathbf{r}). \end{aligned} \quad (2.249)$$

Finally, when both basis and testing functions are not on metallic surfaces, there exists a matrix element $\bar{\mathbf{Z}}_{22}^{\text{JMC FIE}}[m, n]$, which is equal to the corresponding element of $\bar{\mathbf{Z}}_{11}^{\text{JMC FIE}}[m, n]$, i.e.,

$$\bar{\mathbf{Z}}_{22}^{\text{JMC FIE}}[m, n] = \bar{\mathbf{Z}}_{11}^{\text{JMC FIE}}[m, n], \quad (2.250)$$

for $m \leq N_D$ and $n \leq N_D$.

In general, each nonmetallic region may host some external sources that produce incident electric and magnetic fields, i.e., $\mathbf{E}_u^{\text{inc}}$ and $\mathbf{H}_u^{\text{inc}}$. Consider a testing function $\mathbf{t}_m$ on the surface of a nonmetallic region D_u . Incident fields in D_u are tested by $\mathbf{t}_m$ and added to the related element in the upper partition of the RHS vector, i.e.,

$$\begin{aligned} \mathbf{w}_1^{\text{JMC FIE}}[m] \stackrel{+}{\leftarrow} & - \xi_m \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{H}_u^{\text{inc}}(\mathbf{r}) \\ & - \xi_m \eta_u^{-1} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{E}_u^{\text{inc}}(\mathbf{r}), \end{aligned} \quad (2.251)$$

where $\xi_m = \pm 1$ is determined by the orientation of $\mathbf{t}_m$. In addition, when the testing function $\mathbf{t}_m$ is not on a metallic surface, i.e., $m < N_D$, there exists a corresponding element in the second partition of the RHS vector in (2.246). Contribution to this element due to incident fields in region D_u can be written as

$$\begin{aligned} \mathbf{w}_2^{\text{JMC FIE}}[m] \stackrel{+}{\leftarrow} & \xi_m \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{E}_u^{\text{inc}}(\mathbf{r}) \\ & - \xi_m \eta_u \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{H}_u^{\text{inc}}(\mathbf{r}). \end{aligned} \quad (2.252)$$

2.4.2 Special Case: Coated Dielectric or Metallic Object

In the case of a coated dielectric object, there are three nonmetallic regions, namely, D_0 , D_1 , and D_2 , while D_1 is between D_0 and D_2 . Since there is no metallic surface, $N_D = N$, and the size of the resulting matrix equation is $2N \times 2N$. Let the first N_{01} basis/testing functions and the remaining $N_{02} = (N - N_{01})$ basis/testing functions be defined on surfaces S_{01} and S_{02} , respectively. Each partition in (2.246) can be divided into four sub-partitions, i.e.,

$$\bar{\mathbf{Z}}_{ab}^{\text{JMC FIE}} = \begin{bmatrix} \bar{\mathbf{Z}}_{ab,11}^{\text{JMC FIE}} & \bar{\mathbf{Z}}_{ab,12}^{\text{JMC FIE}} \\ \bar{\mathbf{Z}}_{ab,21}^{\text{JMC FIE}} & \bar{\mathbf{Z}}_{ab,22}^{\text{JMC FIE}} \end{bmatrix}_{N \times N} \quad (2.253)$$

for $a = 1, 2$ and $b = 1, 2$. In (2.253), $\bar{\mathbf{Z}}_{ab,11}^{\text{JMC FIE}}$ and $\bar{\mathbf{Z}}_{ab,22}^{\text{JMC FIE}}$ represent $N_{01} \times N_{01}$ and $N_{02} \times N_{02}$ matrices containing interactions of basis and testing functions located on S_{01} and S_{02} , respectively. For $m, n = 1, 2, \dots, N_{01}$,

$$\begin{aligned} \bar{\mathbf{Z}}_{11,11}^{\text{JMC FIE}}[m, n] &= \bar{\mathbf{Z}}_{22,11}^{\text{JMC FIE}}[m, n] = - \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{b}_n(\mathbf{r}) \\ &+ \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times [\mathcal{K}_{PV,0} - \mathcal{K}_{PV,1}] \{\mathbf{b}_n\}(\mathbf{r}) \\ &+ \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot [\mathcal{T}_0 + \mathcal{T}_1] \{\mathbf{b}_n\}(\mathbf{r}) \end{aligned} \quad (2.254)$$

$$\begin{aligned} \bar{\mathbf{Z}}_{12,11}^{\text{JMC FIE}}[m, n] &= \frac{1}{2}(\eta_1^{-1} - \eta_0^{-1}) \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\ &+ \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times [\eta_0^{-1} \mathcal{T}_0 - \eta_1^{-1} \mathcal{T}_1] \{\mathbf{b}_n\}(\mathbf{r}) \\ &- \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot [\eta_0^{-1} \mathcal{K}_{PV,0} + \eta_1^{-1} \mathcal{K}_{PV,1}] \{\mathbf{b}_n\}(\mathbf{r}) \end{aligned} \quad (2.255)$$

$$\begin{aligned} \bar{\mathbf{Z}}_{21,11}^{\text{JMC FIE}}[m, n] &= \frac{1}{2}(\eta_0 - \eta_1) \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\ &- \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times [\eta_0 \mathcal{T}_0 - \eta_1 \mathcal{T}_1] \{\mathbf{b}_n\}(\mathbf{r}) \\ &+ \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot [\eta_0 \mathcal{K}_{PV,0} + \eta_1 \mathcal{K}_{PV,1}] \{\mathbf{b}_n\}(\mathbf{r}). \end{aligned} \quad (2.256)$$

For $m, n = (N_{01} + 1), (N_{01} + 2), \dots, (N_{01} + N_{02})$, and $(m', n') = (m - N_{01}, n - N_{01})$,

$$\begin{aligned}\bar{\mathbf{Z}}_{11,22}^{\text{JMCFIE}}[m', n'] &= \bar{\mathbf{Z}}_{22,22}^{\text{JMCFIE}}[m', n'] = - \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{b}_n(\mathbf{r}) \\ &+ \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times [\mathcal{K}_{PV,1} - \mathcal{K}_{PV,2}] \{\mathbf{b}_n\}(\mathbf{r}) \\ &+ \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot [\mathcal{T}_1 + \mathcal{T}_2] \{\mathbf{b}_n\}(\mathbf{r})\end{aligned}\quad (2.257)$$

$$\begin{aligned}\bar{\mathbf{Z}}_{12,22}^{\text{JMCFIE}}[m', n'] &= \frac{1}{2}(\eta_2^{-1} - \eta_1^{-1}) \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\ &+ \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times [\eta_1^{-1} \mathcal{T}_1 - \eta_2^{-1} \mathcal{T}_2] \{\mathbf{b}_n\}(\mathbf{r}) \\ &- \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot [\eta_1^{-1} \mathcal{K}_{PV,1} + \eta_2^{-1} \mathcal{K}_{PV,2}] \{\mathbf{b}_n\}(\mathbf{r})\end{aligned}\quad (2.258)$$

$$\begin{aligned}\bar{\mathbf{Z}}_{21,22}^{\text{JMCFIE}}[m', n'] &= \frac{1}{2}(\eta_1 - \eta_2) \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\ &- \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times [\eta_1 \mathcal{T}_1 - \eta_2 \mathcal{T}_2] \{\mathbf{b}_n\}(\mathbf{r}) \\ &+ \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot [\eta_1 \mathcal{K}_{PV,1} + \eta_2 \mathcal{K}_{PV,2}] \{\mathbf{b}_n\}(\mathbf{r}).\end{aligned}\quad (2.259)$$

On the other hand, the off-diagonal sub-partitions, i.e., $\bar{\mathbf{Z}}_{ab,12}^{\text{JMCFIE}}$ and $\bar{\mathbf{Z}}_{ab,21}^{\text{JMCFIE}}$, involve interactions of basis and testing functions that are located on different surfaces. These basis and testing functions interact only through the region D_1 . For $m = 1, 2, \dots, N_{01}$, $n = (N_{01} + 1), (N_{01} + 2), \dots, (N_{01} + N_{02})$, and $n' = n - N_{01}$,

$$\begin{aligned}\bar{\mathbf{Z}}_{11,12}^{\text{JMCFIE}}[m, n'] &= \bar{\mathbf{Z}}_{22,12}[m, n'] = \frac{1}{2} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{b}_n(\mathbf{r}) \\ &+ \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{K}_{PV,1} \{\mathbf{b}_n\}(\mathbf{r}) \\ &- \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{T}_1 \{\mathbf{b}_n\}(\mathbf{r})\end{aligned}\quad (2.260)$$

$$\begin{aligned}\bar{\mathbf{Z}}_{12,12}^{\text{JMCFIE}}[m, n'] &= -\frac{\eta_1^{-1}}{2} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\ &+ \eta_1^{-1} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{T}_1 \{\mathbf{b}_n\}(\mathbf{r}) \\ &+ \eta_1^{-1} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{K}_{PV,1} \{\mathbf{b}_n\}(\mathbf{r})\end{aligned}\quad (2.261)$$

$$\begin{aligned}
\bar{\mathbf{Z}}_{21,12}^{\text{JMC FIE}}[m, n'] &= \frac{\eta_1}{2} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\
&\quad - \eta_1 \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{T}_1\{\mathbf{b}_n\}(\mathbf{r}) \\
&\quad - \eta_1 \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{K}_{PV,1}\{\mathbf{b}_n\}(\mathbf{r}). \tag{2.262}
\end{aligned}$$

For $m = (N_{01}+1), (N_{01}+2), \dots, (N_{01}+N_{02})$, $n = 1, 2, \dots, N_{01}$, and $m' = m - N_{01}$,

$$\begin{aligned}
\bar{\mathbf{Z}}_{11,21}^{\text{JMC FIE}}[m', n] &= \bar{\mathbf{Z}}_{22,21}[m', n] = \frac{1}{2} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{b}_n(\mathbf{r}) \\
&\quad - \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{K}_{PV,1}\{\mathbf{b}_n\}(\mathbf{r}) \\
&\quad - \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{T}_1\{\mathbf{b}_n\}(\mathbf{r}) \tag{2.263}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathbf{Z}}_{12,21}^{\text{JMC FIE}}[m', n] &= \frac{\eta_1^{-1}}{2} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\
&\quad - \eta_1^{-1} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{T}_1\{\mathbf{b}_n\}(\mathbf{r}) \\
&\quad + \eta_1^{-1} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{K}_{PV,1}\{\mathbf{b}_n\}(\mathbf{r}) \tag{2.264}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathbf{Z}}_{21,21}^{\text{JMC FIE}}[m', n] &= -\frac{\eta_1}{2} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{b}_n(\mathbf{r}) \\
&\quad + \eta_1 \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathcal{T}_1\{\mathbf{b}_n\}(\mathbf{r}) \\
&\quad - \eta_1 \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathcal{K}_{PV,1}\{\mathbf{b}_n\}(\mathbf{r}). \tag{2.265}
\end{aligned}$$

Next, we consider a coated metallic object in homogeneous space. In this case, there are two nonmetallic regions, i.e., D_0 and D_1 , and the size of the matrix equation is $(N + N_D) \times (N + N_D)$, where $N_D < N$. Let the first N_D basis/testing functions be located on the surface S_{01} and the remaining $N_{02} = (N - N_D)$ basis/testing functions be located on the metallic surface S_{02} . Then, the expressions for matrix elements in the diagonal partition $\bar{\mathbf{Z}}_{11}^{\text{JMC FIE}}$ are the same as the expressions in (2.254), (2.257), (2.260), and (2.263), which are derived for the coated dielectric object. On the other hand, the off-diagonal partition $\bar{\mathbf{Z}}_{12}^{\text{JMC FIE}}$

becomes a rectangular matrix with two sub-partitions, i.e.,

$$\bar{\mathbf{Z}}_{12}^{\text{JMC FIE}} = \begin{bmatrix} \bar{\mathbf{Z}}_{12,11}^{\text{JMC FIE}} \\ \bar{\mathbf{Z}}_{12,21}^{\text{JMC FIE}} \end{bmatrix}_{N \times N_D}, \quad (2.266)$$

where matrix elements $\bar{\mathbf{Z}}_{12,11}^{\text{JMC FIE}}[m, n]$ and $\bar{\mathbf{Z}}_{12,21}^{\text{JMC FIE}}[m', n]$ for $m, n = 1, 2, \dots, N_D$ and $m' = 1, 2, \dots, N_{02}$ are calculated as in (2.255) and (2.264). The off-diagonal partition $\bar{\mathbf{Z}}_{21}^{\text{JMC FIE}}$ also becomes a rectangular matrix, i.e.,

$$\bar{\mathbf{Z}}_{21}^{\text{JMC FIE}} = \begin{bmatrix} \bar{\mathbf{Z}}_{21,11}^{\text{JMC FIE}} & \bar{\mathbf{Z}}_{21,12}^{\text{JMC FIE}} \end{bmatrix}_{N_D \times N}, \quad (2.267)$$

where sub-partitions $\bar{\mathbf{Z}}_{21,11}^{\text{JMC FIE}}[m, n]$ and $\bar{\mathbf{Z}}_{21,12}^{\text{JMC FIE}}[m, n']$ for $m, n = 1, 2, \dots, N_D$, and $n' = 1, 2, \dots, N_{02}$ are calculated as in (2.256) and (2.262). Finally, the diagonal partition $\bar{\mathbf{Z}}_{22}^{\text{JMC FIE}}$ is an $N_D \times N_D$ matrix with elements $\bar{\mathbf{Z}}_{22}^{\text{JMC FIE}}[m, n] = \bar{\mathbf{Z}}_{11}^{\text{JMC FIE}}[m, n]$ for $m, n = 1, 2, \dots, N_D$.

2.4.3 Solutions of Composite Problems with the Multi-level Fast Multipole Algorithm

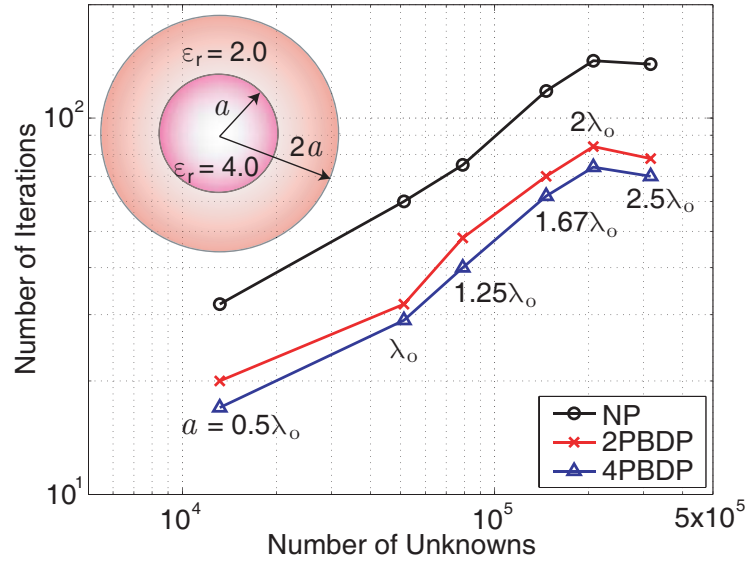
For a general problem involving U regions, MLFMA must be applied for each nonmetallic region separately [83]. This is because radiated and incoming fields of boxes are defined and sampled on the unit sphere, and the number of samples depends on the box size as measured by the wavelength [1]. For each region D_u , we perform four MVMs with four partitions of the system matrix, i.e.,

$$\mathbf{y}_1 = \bar{\mathbf{Z}}_{11,u}^{\text{JMC FIE}} \cdot \mathbf{x}_J + \bar{\mathbf{Z}}_{12,u}^{\text{JMC FIE}} \cdot \mathbf{x}_M \quad (2.268)$$

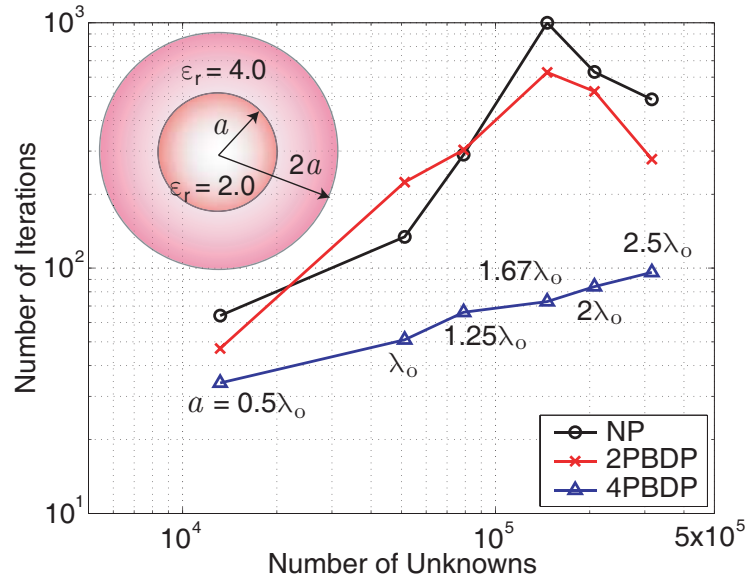
$$\mathbf{y}_2 = \bar{\mathbf{Z}}_{21,u}^{\text{JMC FIE}} \cdot \mathbf{x}_J + \bar{\mathbf{Z}}_{22,u}^{\text{JMC FIE}} \cdot \mathbf{x}_M, \quad (2.269)$$

where coefficients $\mathbf{x}_J$ and $\mathbf{x}_M$ are provided by the iterative algorithm. We note that a set of aggregation, translation, and disaggregation stages is performed once for a multiplication with a partition, although each partition involves some combination of integro-differential operators. This is possible, since radiated and

incoming fields do not depend on the type of the integro-differential operator. Only receiving patterns of testing functions depend on the operator and the testing type as given in (2.136)–(2.143). At the beginning of an aggregation stage, radiation patterns of basis functions are multiplied with coefficients provided by the iterative algorithm and combined to obtain radiated fields of boxes at the lowest level. In the aggregation stage performed for a region D_u , only basis functions located on the surface of the region (S_u) are considered. Besides, for partitions $\bar{\mathbf{Z}}_{12,u}^{\text{JMC FIE}}$ and $\bar{\mathbf{Z}}_{22,u}^{\text{JMC FIE}}$, basis functions located on metallic surfaces are omitted. At the end of a disaggregation stage, incoming fields are received by testing functions. Similar to the aggregation stage, a disaggregation stage performed for a region D_u involves only testing functions located on the surface of the region. In addition, testing functions located on metallic surfaces do not receive incoming fields for partitions $\bar{\mathbf{Z}}_{21,u}^{\text{JMC FIE}}$ and $\bar{\mathbf{Z}}_{22,u}^{\text{JMC FIE}}$. Finally, signs ξ_m and ξ_n in (2.247), (2.248), (2.249), and (2.250), are introduced when radiation patterns of basis functions are combined or when incoming fields are multiplied with receiving patterns of testing functions at the lowest level.

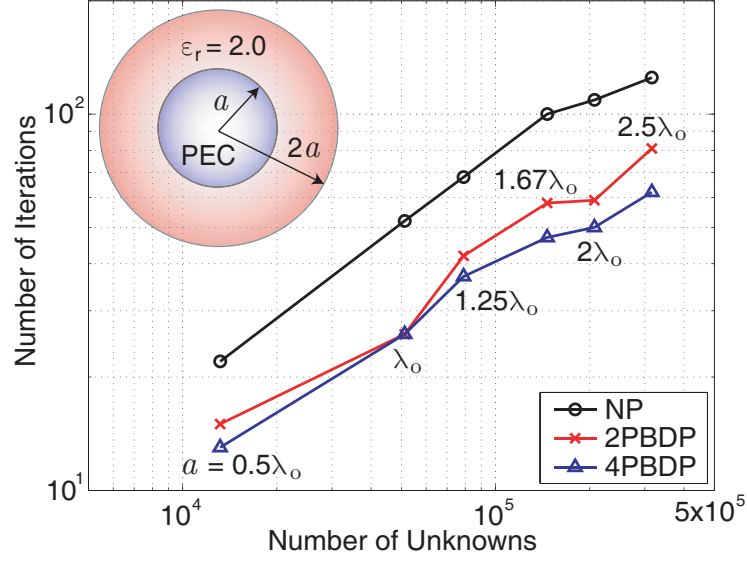


(a)

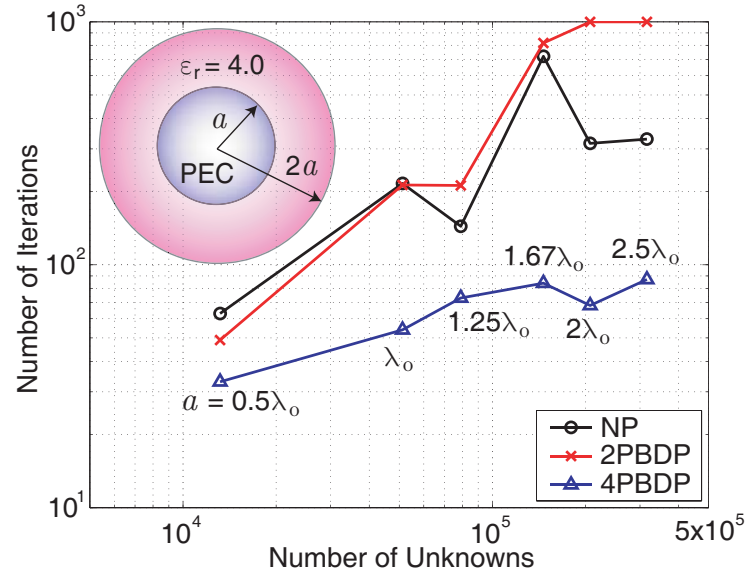


(b)

Figure 2.29: Iteration counts for the solution of scattering problems involving a dielectric sphere of radius a coated with a dielectric shell of radius $2a$, where a changes from $0.5\lambda_o$ to $2.5\lambda_o$. (a) Low-contrast case when relative permittivities of the core and the shell are 4.0 and 2.0, respectively. (b) High-contrast case when relative permittivities of the core and the shell are 2.0 and 4.0, respectively.

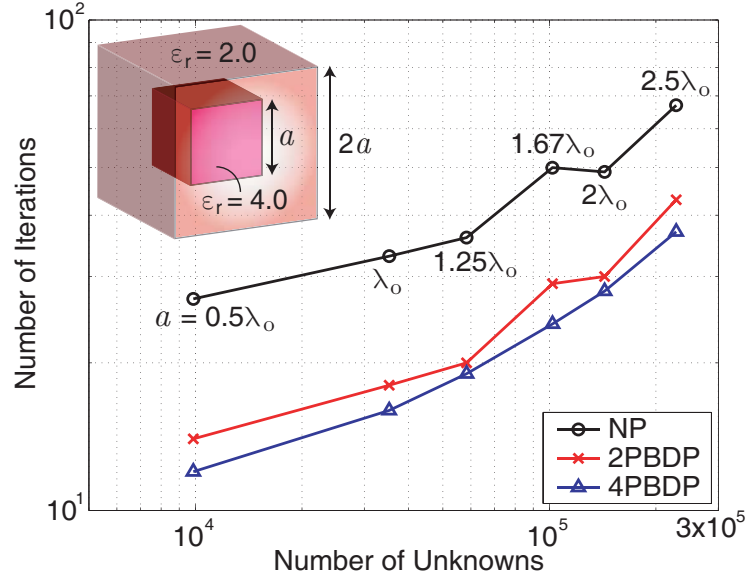


(a)

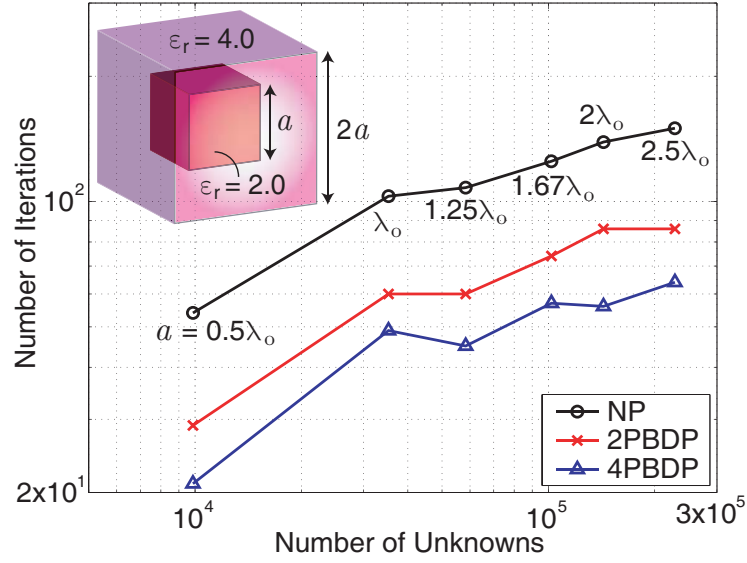


(b)

Figure 2.30: Iteration counts for the solution of scattering problems involving a PEC sphere of radius a coated with a dielectric shell of radius $2a$, where a changes from $0.5\lambda_0$ to $2.5\lambda_0$. The relative permittivity of the shell is (a) 2.0 and (b) 4.0.

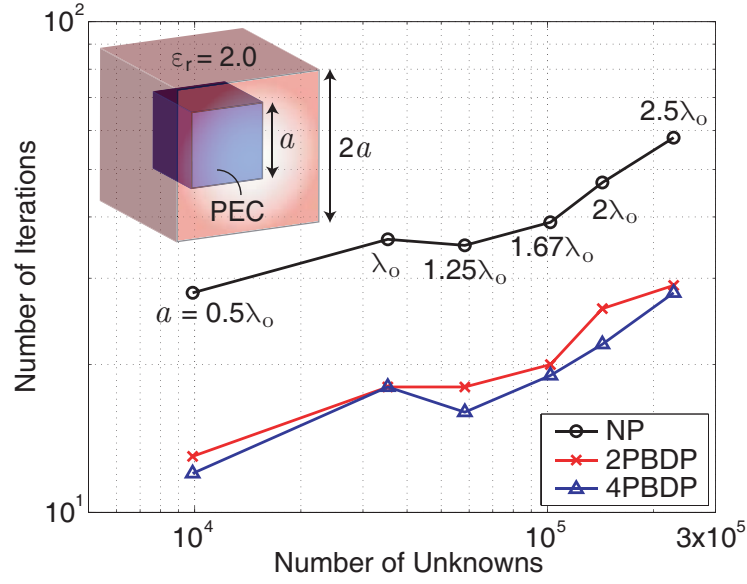


(a)

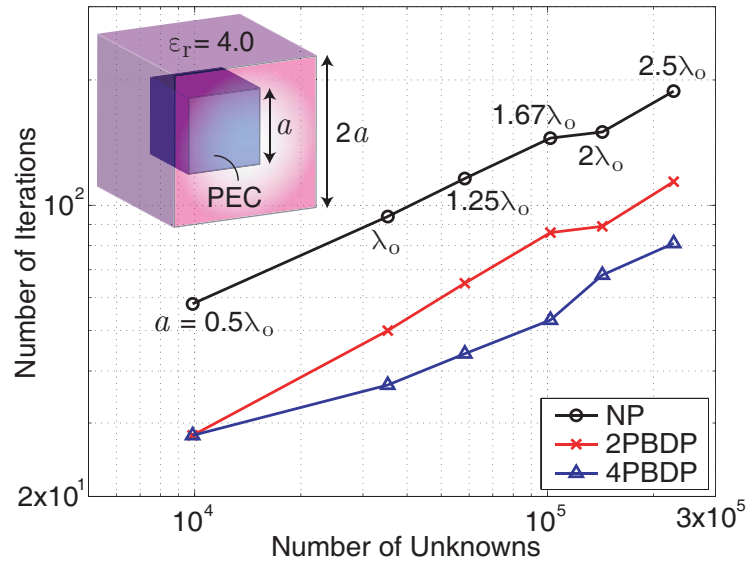


(b)

Figure 2.31: Iteration counts for the solution of scattering problems involving a dielectric cube coated with a dielectric shell. The core and the shell have edges of a and $2a$, respectively, where a changes from $0.5\lambda_o$ to $2.5\lambda_o$. (a) Low-contrast case when relative permittivities of the core and the shell are 4.0 and 2.0, respectively. (b) High-contrast case when relative permittivities of the core and the shell are 2.0 and 4.0, respectively.



(a)



(b)

Figure 2.32: Iteration counts for the solution of scattering problems involving a PEC cube coated with a dielectric shell. The core and the shell have edges of a and $2a$, respectively, where a changes from $0.5\lambda_0$ to $2.5\lambda_0$. The relative permittivity of the shell is (a) 2.0 and (b) 4.0.

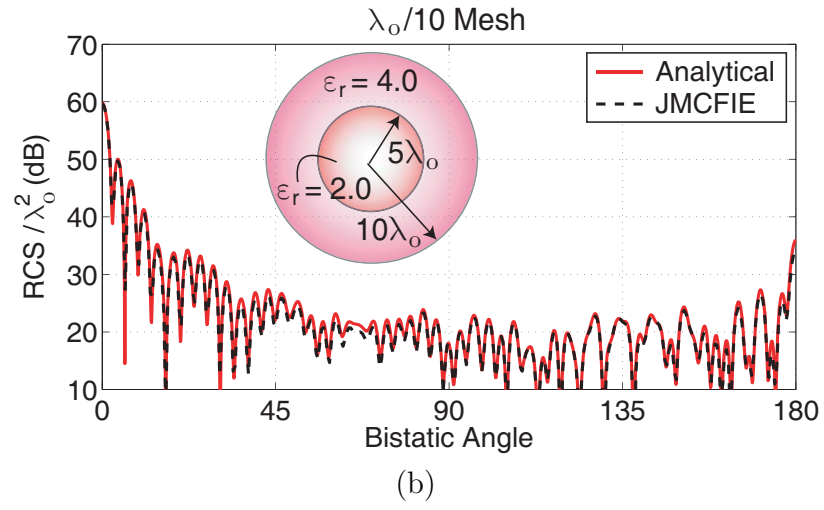
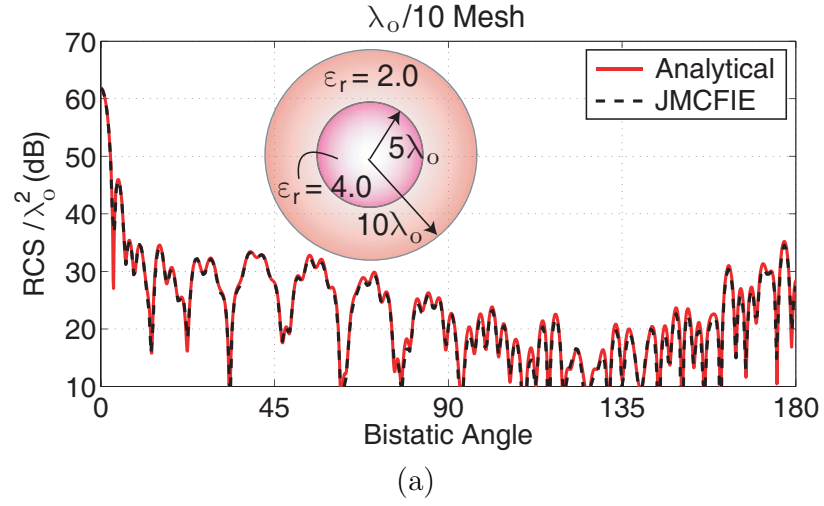


Figure 2.33: Normalized bistatic RCS (RCS/λ_o^2) of a structure involving spheres of radii $5\lambda_o$ and $10\lambda_o$, when (a) relative permittivities of the core and the shell are 4.0 and 2.0, respectively, and (b) relative permittivities of the core and the shell are 2.0 and 4.0, respectively.

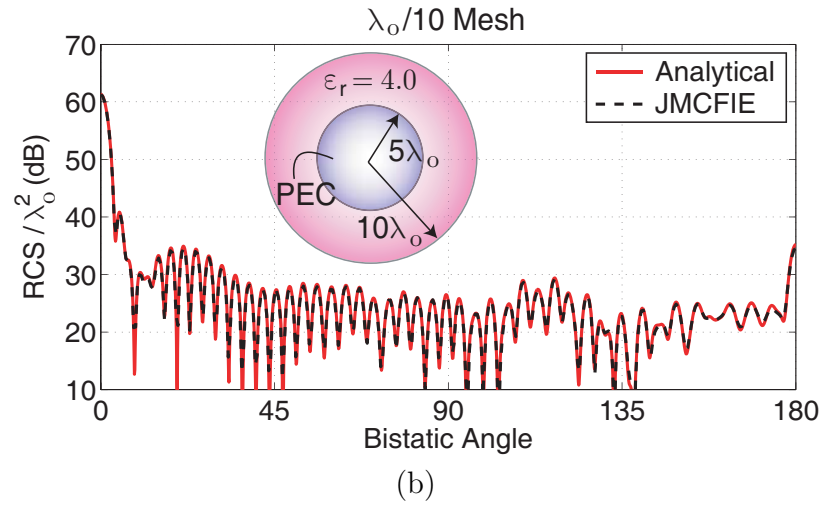
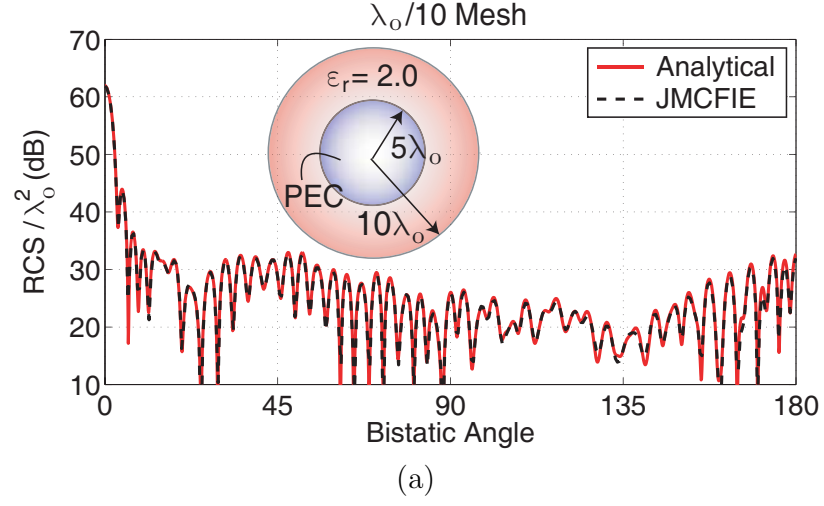


Figure 2.34: Normalized bistatic RCS (RCS/λ_o^2) of a structure involving spheres of radii $5\lambda_o$ and $10\lambda_o$, when (a) the core is metallic and the relative permittivity of the shell is 2.0 and (b) the core is metallic and the relative permittivity of the shell is 4.0.

2.4.4 Numerical Results

In this section, we present iterative solutions for various scattering problems and investigate their iteration counts when solutions are accelerated with 2PBDP and 4PBDP, in addition to the NP case. For those problems involving metallic surfaces, off-diagonal partitions of 4PBDP are rectangular matrices. In addition, blocks in off-diagonal partitions are not necessarily square, and some of them can be rectangular, depending on the object and the recursive clustering scheme in MLFMA. Scatterers are located in free space and illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction. Surfaces are discretized with the RWG functions on $\lambda_o/10$ triangulations, where λ_o is the wavelength outside objects (free space), and iterative solutions are performed using the BiCGStab algorithm. In all solutions, MVMs are accelerated via MLFMA, and the relative residual error for the iterative convergence is set to 10^{-3} .

Figure 2.29 depicts iteration counts for the solution of scattering problems involving a spherical composite object with multiple dielectric regions. A dielectric sphere of radius a is coated with a dielectric shell of radius $2a$, where a changes from $0.5\lambda_o$ to $2.5\lambda_o$. Discretizations of problems lead to 13,176 and 316,032 unknowns, respectively, when $a = 0.5\lambda_o$ and $a = 2.5\lambda_o$. Figure 2.29(a) presents iteration counts with respect to the number of unknowns when relative permittivities of the core and the shell are 4.0 and 2.0, respectively. In this case, 2PBDP reduces iteration counts substantially in comparison to the NP case, while 4PBDP does not provide a significant improvement over 2PBDP. On the other hand, when permittivities of the shell and the core are exchanged, we obtain iteration counts depicted in Figure 2.29(b), where 4PBDP presents a superior performance in comparison to 2PBDP. Due to the relatively high contrast between the shell and free space, solutions of JMCIE become difficult without preconditioning. For example, when $a = 1.67\lambda_o$, convergence cannot be achieved

in 1000 iterations. 2PBDP accelerates the convergence for large problems, but the improvement is not sufficient. Using 4PBDP, the number of iterations is less than 100 for all solutions in Figure 2.29.

Next, we consider iterative solutions of scattering problems involving a spherical composite object with dielectric and metallic parts. In this case, a metallic sphere of radius a is coated with a dielectric shell of radius $2a$, where a changes from $0.5\lambda_o$ to $2.5\lambda_o$. Figure 2.30 presents iteration counts with respect to the number of iterations. Similar to the previous example, 2PBDP reduces iteration counts significantly for the low-contrast case, i.e., when the relative permittivity of the shell is 2.0, as depicted in Figure 2.30(a). In this case, 4PBDP provides some improvement over 2PBDP, as the problem size grows. When the relative permittivity of the shell is 4.0, however, 4PBDP accelerates iterative solutions significantly, compared to 2PBDP. In fact, 2PBDP decelerates solutions for large problems, and there is a large discrepancy between performances of 2PBDP and 4PBDP. Using 4PBDP, the number of iterations is again less than 100 for all solutions in Figure 2.30.

We also consider electromagnetics problems involving composite objects with sharp edges and corners. Figure 2.31 presents iteration counts for the solution of scattering problems involving a coated dielectric cube. The core and the shell have edges of a and $2a$, respectively, where a changes from $0.5\lambda_o$ to $2.5\lambda_o$. Faces of the object are parallel to the coordinate axes. Discretizations of problems lead to matrix equations with 9864 to 228,132 unknowns. Figure 2.31(a) depicts iteration counts as a function of the number of unknowns, when relative permittivities of the core and the shell are 4.0 and 2.0, respectively. Results are similar to those for the spherical object depicted in Figure 2.29(a), i.e., 2PBDP accelerates iterative solutions significantly, and 4PBDP further reduces iteration counts slightly compared to 2PBDP. When relative permittivities of the core and

the shell are exchanged, however, 4PBDP performs much better than 2PBDP, as depicted in Figure 2.31(b). On the other hand, unlike solutions of the spherical object in Figure 2.29(b), 2PBDP is also effective for the high-contrast case in Figure 2.31(b). This is probably due to the larger numbers of elements being used in constructing 2PBDP for the cubic object, i.e., boxes at the lowest level of the multilevel tree are more populated for the cubic object than for the spherical object. Nevertheless, 4PBDP is again preferable for all solutions in Figure 2.31.

Figure 2.32 presents the solution of scattering problems involving a coated metallic cube. Sizes of the core and the shell are the same as those in the coated dielectric cube. Similar to the previous examples, 4PBDP provides the most efficient results, and it presents improved convergence in comparison to 2PBDP, when the contrast is high.

Finally, to demonstrate the effectiveness of an implementation involving MLFMA, JMCIE, and 4PBDP, we present the solution of large scattering problems discretized with 1,264,128 unknowns. A sphere of radius $5\lambda_o$ is placed inside another sphere of radius $10\lambda_o$. We consider four different cases:

- Relative permittivities of the core and the shell are 4.0 and 2.0, respectively.
- Relative permittivities of the core and the shell are 2.0 and 4.0, respectively.
- The core is metallic and the relative permittivity of the shell is 2.0.
- The core is metallic and the relative permittivity of the shell is 4.0.

Scattering problems are solved via 6-level MLFMA, and iteration counts are 101, 283, 75, and 187, respectively, for 10^{-3} residual error. Figures 2.33 and 2.34 presents normalized bistatic RCS (RCS/λ_o^2) values on the z - x plane, where 0° and 180° correspond to forward-scattering and back-scattering directions, respectively. Computational values are compared with analytical values obtained

by Mie-series solutions. We observe that computational and analytical results agree perfectly.

○

Chapter 3

Solutions of Electromagnetics Problems Involving Perfectly-Conducting Objects

In this chapter, we consider the solution of electromagnetics problems involving three-dimensional PEC objects. Using a Galerkin scheme for the discretization, T-EFIE, N-MFIE, and T-N-CFIE in (2.152), (2.154), and (2.156), respectively, are stable formulations for PEC surfaces. For the sake of brevity, those formulations are called EFIE, MFIE, and CFIE. We also introduce hybrid formulations for the solution of electromagnetics problems involving composite structures with coexisting open and closed metallic surfaces. Our investigations show that normal and mixed formulations, such as MFIE and CFIE, are noticeably and consistently more inaccurate than tangential formulations, such as EFIE. Computational experiments also show that the source of the excessive error in those formulations is well-tested identity operators. We improve the accuracy of MFIE and CFIE to levels of EFIE by employing more appropriate basis functions instead of the conventional RWG functions.

3.1 Integral-Equation Formulations of Perfectly-Conducting Objects

For PEC objects, the boundary condition for the tangential electric field can be used to derive EFIE as

$$\hat{\mathbf{t}} \cdot ik \int_S d\mathbf{r}' \left[\mathbf{J}(\mathbf{r}') + \frac{1}{k^2} \nabla' \cdot \mathbf{J}(\mathbf{r}') \nabla \right] g(\mathbf{r}, \mathbf{r}') = -\hat{\mathbf{t}} \cdot \eta^{-1} \mathbf{E}^{inc}(\mathbf{r}), \quad (3.1)$$

where the scattered or radiated electric field is expressed in terms of the induced (unknown) surface current $\mathbf{J}$. Similar to EFIE, MFIE is derived by using the boundary condition for the tangential magnetic field on the surface as

$$-\frac{\Omega_o}{4\pi} \mathbf{J}(\mathbf{r}) + \hat{\mathbf{n}} \times \int_{PV,S} d\mathbf{r}' \mathbf{J}(\mathbf{r}') \times \nabla' g(\mathbf{r}, \mathbf{r}') = -\hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}), \quad (3.2)$$

where the observation point approaches the closed surface S from the outside. Finally, the conventional CFIE is a convex combination of EFIE and MFIE [20], i.e.,

$$\text{CFIE} = \alpha \text{EFIE} + (1 - \alpha) \text{MFIE}, \quad (3.3)$$

where α is a parameter between 0 and 1.

Discretizations of EFIE and MFIE lead to $N \times N$ matrix equations as

$$\bar{\mathbf{Z}}_E \cdot \mathbf{a} = \mathbf{w}_E \quad (3.4)$$

$$\bar{\mathbf{Z}}_M \cdot \mathbf{a} = \mathbf{w}_M. \quad (3.5)$$

Elements of $\bar{\mathbf{Z}}$ and $\mathbf{w}$ are derived in Chapter 2. Specifically,

$$\bar{\mathbf{Z}}_E[m, n] = \bar{\mathbf{T}}^T[m, n] \quad (3.6)$$

$$\bar{\mathbf{Z}}_M[m, n] = -\frac{1}{2} \bar{\mathbf{I}}^T[m, n] + \bar{\mathbf{K}}_{PV}^N[m, n] \quad (3.7)$$

$$\mathbf{w}_E[m] = -\eta^{-1} \mathbf{v}_E^T[m] \quad (3.8)$$

$$\mathbf{w}_M[m] = -\mathbf{v}_H^N[m] \quad (3.9)$$

for $m, n = 1, 2, \dots, N$, where $\bar{\mathbf{K}}_{PV}^N$, $\bar{\mathbf{T}}^T$, $\bar{\mathbf{I}}^T$, $\mathbf{v}_E^T$, and $\mathbf{v}_H^N$ are defined in (2.51), (2.52), (2.54), (2.57), and (2.60), respectively. Using the definition in (3.3), matrix equations obtained from CFIE can be derived as

$$\bar{\mathbf{Z}}_C \cdot \mathbf{a} = \mathbf{w}_C, \quad (3.10)$$

where

$$\bar{\mathbf{Z}}_C[m, n] = \alpha \bar{\mathbf{Z}}_E[m, n] + (1 - \alpha) \bar{\mathbf{Z}}_M[m, n] \quad (3.11)$$

$$\mathbf{w}_C[m] = \alpha \mathbf{w}_E[m] + (1 - \alpha) \mathbf{w}_M[m]. \quad (3.12)$$

3.1.1 Comments on Integral-Equation Formulations

In surface formulations of electromagnetic scattering and radiation problems involving metallic objects, EFIE, MFIE, and CFIE can be employed either at our discretion or due to limitations imposed by the problem. We list advantages and disadvantages of those formulations as follows:

- EFIE can be used to formulate both open and closed surfaces. An open surface can be thought of as the limit case of squeezing a closed surface into zero thickness so that the open surface is formed by merging the two opposite surfaces of an infinitely thin geometry. Then, the current distribution on the two sides of the thin structure becomes indistinguishable, and they are simply combined into a single equivalent current to be solved as the unknown of the problem [31],[77]. For open surfaces, EFIE in (3.1) is still applicable since the implied boundary condition remains valid in this limit case. Although the current term $\mathbf{J}$ in (3.1) and (3.2) are the same, MFIE is not applicable to open surfaces. Since it contains MFIE, CFIE in (3.3) is also not applicable to open surfaces.
- For closed objects, EFIE and MFIE suffer from internal resonances. In vicinities of discrete frequencies, matrix equations obtained from EFIE and

MFIE have null-space solutions, i.e.,

$$\bar{\mathbf{Z}}_{E,M} \cdot \mathbf{a} = 0, \quad \mathbf{a} \neq 0. \quad (3.13)$$

As a major advantage, CFIE is free of internal resonances experienced by EFIE and MFIE [20].

- EFIE is a tangential formulation and its discretization usually produces ill-conditioned matrix equations that are difficult to solve iteratively. Being a normal formulation, MFIE involves well-tested identity operators, which leads to well-conditioned matrix equations that are crucial for iterative solvers, such as MLFMA. Finally, choosing a proper coupling parameter α , CFIE leads to better-conditioned matrix equations, compared to both EFIE and MFIE [52],[54].
- EFIE suffers from the low-frequency breakdown problem [67],[68]. Matrix equations obtained from EFIE discretized with ordinary functions, such as the RWG functions, become increasingly ill-conditioned as the discretization is refined. Since it contains EFIE, CFIE is also affected by the low-frequency breakdown [69],[70]. However, MFIE is stable unless the frequency is extremely low [72].
- MFIE is suitable for hybridization with the physical optics (PO) method since both are based on the same magnetic field boundary condition [29].
- Using a Galerkin scheme, EFIE produces symmetric matrix equations, whereas MFIE and CFIE do not.
- MFIE and CFIE involving well-tested identity operators are significantly inaccurate compared to EFIE, especially when they are discretized with low-order functions, such as the RWG functions.

For closed objects, CFIE is preferred since it is free of internal resonances and it leads to better-conditioned matrix equations, compared to EFIE and MFIE. For open surfaces, however, EFIE becomes an inevitable choice.

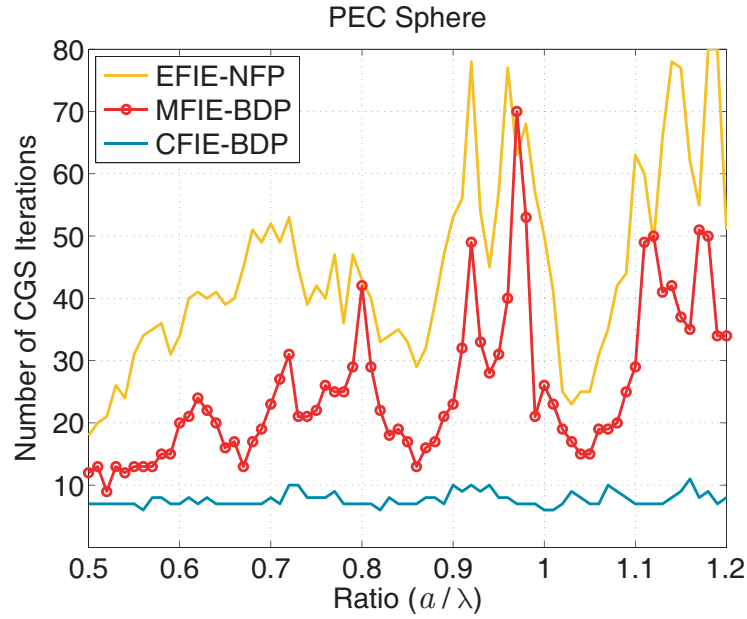
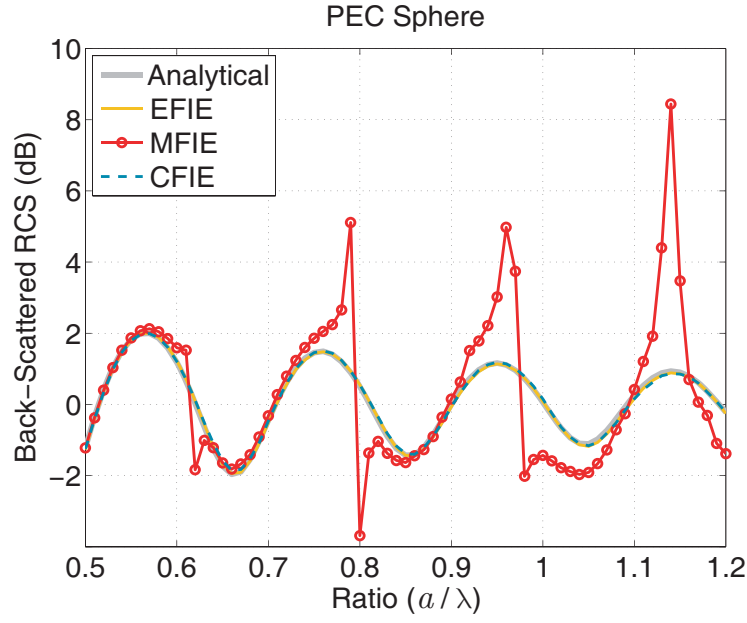


Figure 3.1: Solutions of scattering problems involving a conducting sphere with a radius of 0.3 m. (a) Normalized back-scattered RCS ($RCS/\pi a^2$) and (b) the number of CGS iterations (10^{-6} residual error) when the radius of the sphere (a) changes from 0.5λ to 1.2λ .

3.1.2 Internal-Resonance Problem

In order to demonstrate internal resonances encountered in surface formulations of PEC objects, we consider the solution of scattering problems involving a sphere with a radius of 0.3 m. The sphere is illuminated by a plane wave and discretized with 3 cm triangles, on which the RWG functions are defined. Figure 3.1(a) presents normalized back-scattered RCS values ($\text{RCS}/\pi a^2$, where a is the radius of the sphere) obtained with EFIE, MFIE, and CFIE, in addition to the analytical Mie-series solutions, when a changes from 0.5λ to 1.2λ . We observe that RCS values obtained with MFIE are significantly inaccurate in vicinities of discrete frequencies due to internal resonances. Although EFIE also suffers from internal-resonance problems, it produces accurate RCS results since null-space solutions of EFIE do not radiate.

Figure 3.1(b) presents the number of CGS iterations to reduce the residual error below 10^{-6} for EFIE, MFIE, and CFIE. Iterative solutions of EFIE are accelerated with NFP, whereas MFIE and CFIE solutions are accelerated with BDP. In Figure 3.1(b), we observe peaks at resonance frequencies for both EFIE and MFIE. However, CFIE is free of internal resonances and its iterative solution is stable in the entire frequency range.

3.1.3 Formulations of Open Surfaces

Consider a PEC open surface, which is obtained by merging the two opposite surfaces of an infinitely thin geometry. Using the same discretization for opposite surfaces, the electric current is expanded in a series of N basis functions, i.e.,

$$\mathbf{J}(\mathbf{r}) \approx \sum_{n=1}^N \mathbf{a}_+[n] \mathbf{b}_n(\mathbf{r}) + \sum_{n=1}^N \mathbf{a}_-[n] \mathbf{b}_n(\mathbf{r}), \quad (3.14)$$

where $\mathbf{a}_+[n] = \mathbf{a}[n]$ and $\mathbf{a}_-[n] = \mathbf{a}[n + N]$ represent coefficients (one for each side) of the n th basis function $\mathbf{b}_n$. Applying a Galerkin scheme, we employ the

same set of functions as testing functions. Then, the discretization of EFIE leads to a $2N \times 2N$ matrix equation in the form of

$$\begin{bmatrix} \bar{\mathbf{T}}^T & \bar{\mathbf{T}}^T \\ \bar{\mathbf{T}}^T & \bar{\mathbf{T}}^T \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_+ \\ \mathbf{a}_- \end{bmatrix} = -\eta^{-1} \begin{bmatrix} \mathbf{v}_E^T \\ \mathbf{v}_E^T \end{bmatrix}, \quad (3.15)$$

which is reduced to

$$\begin{bmatrix} \bar{\mathbf{T}}^T & \bar{\mathbf{T}}^T \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_+ \\ \mathbf{a}_- \end{bmatrix} = -\eta^{-1} \mathbf{v}_E^T. \quad (3.16)$$

Instead of solving the under-determined system in (3.16), we combine coefficients and obtain an $N \times N$ matrix equation, i.e.,

$$\bar{\mathbf{T}}^T \cdot (\mathbf{a}_+ + \mathbf{a}_-) = -\eta^{-1} \mathbf{v}_E^T. \quad (3.17)$$

Then $(\mathbf{a}_+ + \mathbf{a}_-)$ can be used to compute electromagnetic fields everywhere. On the other hand, discretization of MFIE for the same problem leads to

$$\begin{bmatrix} -0.5\bar{\mathbf{I}}^T + \bar{\mathbf{K}}_{PV}^N & 0.5\bar{\mathbf{I}}^T + \bar{\mathbf{K}}_{PV}^N \\ -0.5\bar{\mathbf{I}}^T + \bar{\mathbf{K}}_{PV}^N & 0.5\bar{\mathbf{I}}^T + \bar{\mathbf{K}}_{PV}^N \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_+ \\ \mathbf{a}_- \end{bmatrix} = - \begin{bmatrix} \mathbf{v}_H^N \\ \mathbf{v}_H^N \end{bmatrix}, \quad (3.18)$$

which is reduced to

$$\begin{bmatrix} -0.5\bar{\mathbf{I}}^T + \bar{\mathbf{K}}_{PV}^N & 0.5\bar{\mathbf{I}}^T + \bar{\mathbf{K}}_{PV}^N \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_+ \\ \mathbf{a}_- \end{bmatrix} = -\mathbf{v}_H^N \quad (3.19)$$

or

$$\bar{\mathbf{K}}_{PV}^N \cdot (\mathbf{a}_+ + \mathbf{a}_-) - 0.5\bar{\mathbf{I}}^T \cdot (\mathbf{a}_+ - \mathbf{a}_-) = -\mathbf{v}_H^N. \quad (3.20)$$

Obviously the under-determined system obtained with MFIE in (3.20) cannot be solved to obtain $(\mathbf{a}_+ + \mathbf{a}_-)$.

Similar to MFIE, the conventional CFIE, which is obtained by the convex combination of EFIE and MFIE, is not applicable to open surfaces. Nevertheless, one can derive various combined formulations based on simultaneous solutions of EFIE and MFIE for open surfaces, i.e.,

$$\begin{bmatrix} \alpha_1 \bar{\mathbf{T}}^T & 0 \\ \alpha_2 \bar{\mathbf{K}}_{PV}^N & -0.5\alpha_2 \bar{\mathbf{I}}^T \end{bmatrix} \cdot \begin{bmatrix} (\mathbf{a}_+ + \mathbf{a}_-) \\ (\mathbf{a}_+ - \mathbf{a}_-) \end{bmatrix} = - \begin{bmatrix} \alpha_1 \eta^{-1} \mathbf{v}_E^T \\ \alpha_2 \mathbf{v}_H^N \end{bmatrix} \quad (3.21)$$

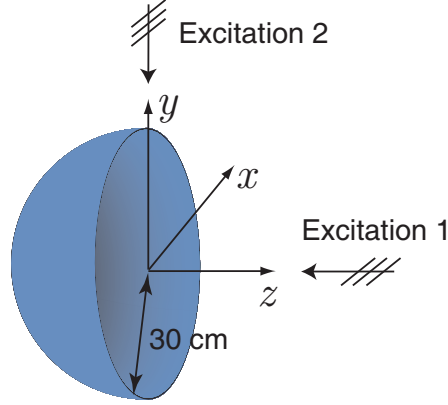


Figure 3.2: Scattering problems involving a half sphere.

or

$$\begin{bmatrix} \alpha_1 \bar{\mathbf{T}}^T & \alpha_1 \bar{\mathbf{T}}^T \\ \alpha_2 \left(\bar{\mathbf{K}}_{PV}^N - 0.5 \bar{\mathbf{I}}^T \right) & \alpha_2 \left(\bar{\mathbf{K}}_{PV}^N + 0.5 \bar{\mathbf{I}}^T \right) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_+ \\ \mathbf{a}_- \end{bmatrix} = - \begin{bmatrix} \alpha_1 \eta^{-1} \mathbf{v}_E^T \\ \alpha_2 \mathbf{v}_H^N \end{bmatrix}, \quad (3.22)$$

where α_1 and α_2 are combination parameters. On the other hand, our investigations show that using combined formulations, such as (3.21) and (3.22), instead of EFIE does not provide any advantage in terms of iterative solutions.

As an example, we present the solution of a scattering problem involving a PEC half sphere, as depicted in Figure 3.2. The object is illuminated by a plane wave propagating in the $-z$ direction with the electric field polarized in the x direction (Excitation 1 in Figure 3.2) at 2 GHz. Discretization of the object with $\lambda/10$ triangles involves 9,991 RWG functions. The problem is formulated with EFIE and with the combined formulation in (3.21) using various values for $\alpha = \alpha_1 = (1 - \alpha_2)$. All solutions are performed iteratively using MLFMA. Table 3.1 lists the number of CGS iterations to reach 10^{-6} residual error. We observe that using the combined formulation instead of EFIE does not accelerate iterative solutions. In fact, EFIE is almost the best formulation in Table 3.1.

Table 3.1: Number of CGS Iterations (10^{-6} Residual Error) for the Solution of a Scattering Problem Involving a Half Sphere in Figure 3.2 at 2 GHz

	Combined Formulation (α)								
EFIE	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
399	549	409	438	413	396	478	461	479	604

3.1.4 Low-Frequency Breakdown of the Electric-Field Integral Equation

In general, the electric field due to an electric current $\mathbf{J}$ involves two parts, namely, the vector-potential part

$$\mathbf{E}^V(\mathbf{r}) = i\omega\mu \int_S d\mathbf{r}' \mathbf{J}(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') \quad (3.23)$$

and the scalar-potential part

$$\mathbf{E}^S(\mathbf{r}) = \frac{i}{\omega\epsilon} \int_S d\mathbf{r}' \nabla' \cdot \mathbf{J}(\mathbf{r}') \nabla g(\mathbf{r}, \mathbf{r}'), \quad (3.24)$$

where $\mathbf{E}(\mathbf{r}) = \mathbf{E}^V(\mathbf{r}) + \mathbf{E}^S(\mathbf{r})$. As the frequency decreases, the scalar-potential part dominates the vector-potential part. In fact, using a finite-precision arithmetic, the vector-potential part can be lost during numerical solutions. On the other hand, (3.24) is not a one-to-one mapping between the electric current and the electric field. Specifically, for a given $\mathbf{E}^S$, there are infinitely many $\mathbf{J}$. This is because, for a nonzero frequency, $\mathbf{J}$ has a solenoidal part $\mathbf{J}^{sol}$ such that

$$\nabla' \cdot \mathbf{J}^{sol}(\mathbf{r}') = 0. \quad (3.25)$$

We note that the solenoidal part contributes to the electric field only through the vector potential part in (3.23).

A major problem is that the low-frequency breakdown of EFIE can be encountered at an arbitrary frequency as the discretization is refined. A matrix equation obtained with EFIE can be decomposed as

$$\bar{\mathbf{Z}}_E^V \cdot \mathbf{a} + \bar{\mathbf{Z}}_E^S \cdot \mathbf{a} = \mathbf{w}_E, \quad (3.26)$$

where

$$\bar{\mathbf{Z}}_E^V[m, n] = ik \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n(\mathbf{r}') \quad (3.27)$$

$$\bar{\mathbf{Z}}_E^S[m, n] = \frac{i}{k} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' \nabla g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \quad (3.28)$$

for $m, n = 1, 2, \dots, N$. It can be shown that $\bar{\mathbf{Z}}_E^S$ dominates $\bar{\mathbf{Z}}_E^V$ when the mesh size decreases. On the other hand, using ordinary basis and testing functions, such as the RWG functions, $\bar{\mathbf{Z}}_E^S$ is a rank-deficient matrix. Hence, the overall matrix equation becomes increasingly ill-conditioned as the mesh size is decreased.

A rigorous solution of the low-frequency breakdown of EFIE is using loop-star or loop-tree decomposition methods [73]–[75]. However, one can also develop other types of decompositions to construct better-conditioned matrix equations than those obtained from EFIE. In this section, we present a simple approach based on separate discretizations of current and charge distributions for stable solutions of EFIE. Similar formulations are extensively discussed in [76].

Consider a discretization of EFIE, where the electric current is expanded in a series of RWG functions. We discretize the electric charge density separately by using pulse functions on triangles, i.e.,

$$\frac{i\rho_e(\mathbf{r})}{\sqrt{\mu\epsilon}} = \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{k} \approx \sum_{n=1}^{N_\Delta} \mathbf{a}_\rho[n] b_n^{\text{PLS}}(\mathbf{r}), \quad (3.29)$$

where N_Δ is the number of triangles,

$$b_n^{\text{PLS}}(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in S_n^\Delta \\ 0, & \mathbf{r} \notin S_n^\Delta \end{cases}, \quad (3.30)$$

and S_n^Δ is the surface of the n th triangle. Testing EFIE with the RWG functions, we obtain an $N \times (N + N_\Delta)$ matrix equation

$$\begin{bmatrix} \bar{\mathbf{Z}}_E^{V, \text{RWG}} & \bar{\mathbf{Z}}_E^{S, \text{R/P}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{a}_\rho \end{bmatrix} = \frac{i}{k\eta} \mathbf{v}_E^T, \quad (3.31)$$

where

$$\bar{\mathbf{Z}}_E^{V,\text{RWG}}[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \quad (3.32)$$

$$\bar{\mathbf{Z}}_E^{S,\text{PLS}}[m, n'] = \int_{S_m} d\mathbf{r} \frac{\nabla \cdot \mathbf{t}_m^{\text{RWG}}(\mathbf{r})}{k} \int_{S_{n'}^\Delta} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') b_{n'}^{\text{PLS}}(\mathbf{r}') \quad (3.33)$$

for $m, n = 1, 2, \dots, N$ and $n' = 1, 2, \dots, N_\Delta$. To solve the under-determined system in (3.31), we consider the Lorentz gauge

$$\nabla \cdot \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') = i\omega \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \rho_e(\mathbf{r}'), \quad (3.34)$$

which describes the relation between vector and scalar potentials. Discretization of (3.34) leads to a $N_\Delta \times (N + N_\Delta)$ matrix equation

$$\begin{bmatrix} \bar{\mathbf{Z}}_E^{V,\text{P/R}} & \bar{\mathbf{Z}}_E^{S,\text{PLS}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{a}_\rho \end{bmatrix} = 0, \quad (3.35)$$

where

$$\bar{\mathbf{Z}}_E^{V,\text{P/R}} = \int_{S_{m'}^\Delta} d\mathbf{r} t_{m'}^{\text{PLS}}(\mathbf{r}) \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \frac{\nabla' \cdot \mathbf{b}_n^{\text{RWG}}(\mathbf{r}')}{k} \quad (3.36)$$

$$\bar{\mathbf{Z}}_E^{S,\text{PLS}} = \int_{S_{m'}^\Delta} d\mathbf{r} t_{m'}^{\text{PLS}}(\mathbf{r}) \int_{S_{n'}^\Delta} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') b_{n'}^{\text{PLS}}(\mathbf{r}') \quad (3.37)$$

for $m', n' = 1, 2, \dots, N_\Delta$ and $n = 1, 2, \dots, N$. Finally, we combine (3.31) and (3.35) to obtain an $(N + N_\Delta) \times (N + N_\Delta)$ matrix equation as

$$\begin{bmatrix} \bar{\mathbf{Z}}_E^{V,\text{RWG}} & \bar{\mathbf{Z}}_E^{S,\text{R/P}} \\ \bar{\mathbf{Z}}_E^{V,\text{P/R}} & \bar{\mathbf{Z}}_E^{S,\text{PLS}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{a}_\rho \end{bmatrix} = \frac{i}{k\eta} \begin{bmatrix} \mathbf{v}_E^T \\ 0 \end{bmatrix}. \quad (3.38)$$

We emphasize that the regularized EFIE (REFIE) in (3.38) does not eliminate the low-frequency breakdown. However, it is more stable than the conventional EFIE with respect to the mesh size.

As an example, we consider the solution of scattering problems involving a square patch with edges of 30 cm and discretized with 1.5 cm triangles. The patch is located on the x - y plane and illuminated by a plane wave propagating in the $-z$ direction. We choose the frequency in the range from 7.8125 MHz to 2000 MHz.

Figure 3.3(a) presents the 1-norm condition number for matrices obtained from EFIE and REFIE with respect to the number of triangles per wavelength. We observe that the condition number grows as the mesh size decreases for both EFIE and REFIE. However, REFIE provides better-conditioned matrix equations, especially for dense discretizations. Figure 3.3(b) presents iteration counts (10^{-6} residual error) for the solution of patch problems formulated with EFIE and REFIE. Using BiCGStab for EFIE, convergence cannot be achieved within 2000 iterations when the mesh size is smaller than $\lambda/100$. However, the number of iterations is less than 400 for all problems formulated with REFIE. Figure 3.3(b) also shows that the CGS algorithm is quite successful to solve ill-conditioned matrix equations obtained from EFIE, and the large gap between iteration counts for EFIE and REFIE disappears when using CGS.

In general, the number of iterations in an iterative solution does not only depend on the condition number of the matrix. This is also confirmed in the example depicted in Figure 3.3, where the number of CGS iterations for EFIE and REFIE are close to each other, even though REFIE produces better-conditioned matrix equations than EFIE. In the next section, we show that a transformation into normal equations improves the iterative solutions of EFIE although the condition number is squared.

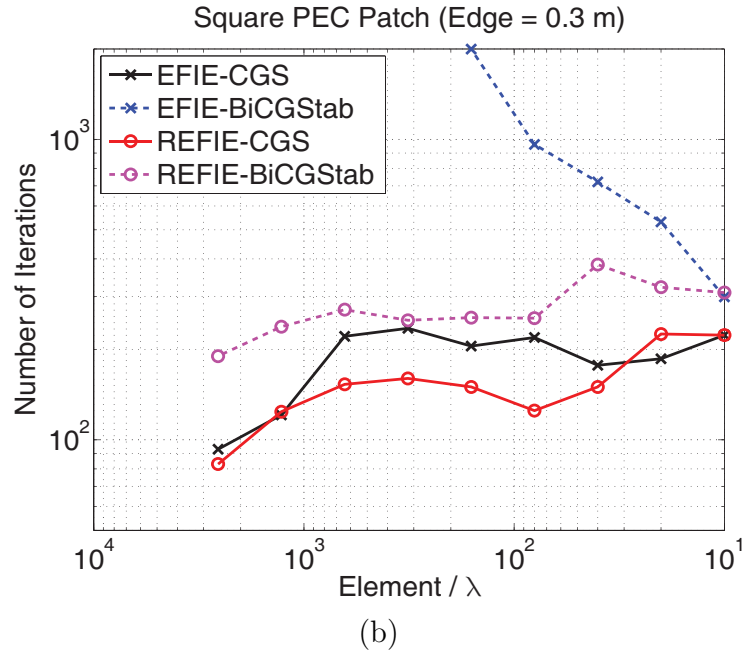
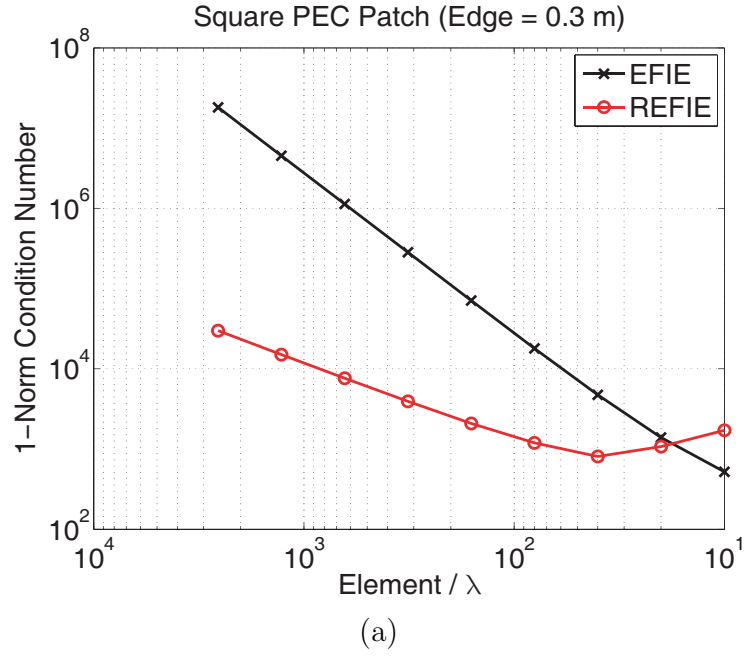


Figure 3.3: Solutions of scattering problems involving a square patch discretized with 1.5 cm triangles. (a) Condition number (1-norm) and (b) iteration counts (10^{-6} residual error) with respect to the number of triangles per wavelength.

3.2 Iterative Solutions of Normal Equations

In iterative solutions of matrix equations obtained from the discretization of surface integral equations, the number of iterations naturally depends on the type of the iterative solver. Among different kinds of algorithms, Krylov subspace methods, such as BiCG, BiCGStab, CG, CGS, GMRES, QMR, and TFQMR, can be employed for the solution of electromagnetics problems formulated with surface integral equations. For a given problem, the performance of each algorithm may vary significantly, depending on the shape of the geometry, discretization, and the type of the formulation. Since reducing the number of iterations is extremely important to obtain efficient solutions, we investigate and compare iterative solutions provided by various algorithms.

In this section, we show that transforming original equations into normal equations improves the convergence of EFIE significantly. We present the solution of EFIE by employing the LSQR algorithm [113],[114], which corresponds to a stable application of the CG algorithm on normal equations. Despite the squaring of the condition number due to such a transformation into normal equations, LSQR improves the convergence rate of iterative solutions of EFIE and performs better than many other iterative algorithms. We also show that the improved convergence is peculiar to EFIE, and LSQR does not have a good performance for CFIE solutions. Finally, we reduce the memory required for the GMRES algorithm by using the acceleration due to the transformation into normal equations.

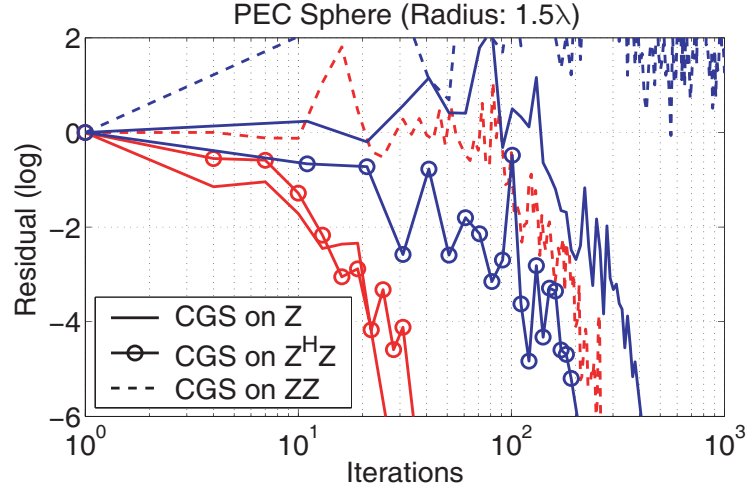


Figure 3.4: Iterative solutions of a scattering problem involving a PEC sphere with a radius of 1.5λ formulated by CFIE (red) and EFIE (blue). Matrix equations have 8364 unknowns and solutions are performed by the CGS algorithm applied on the ordinary equation and transformed equations in (3.39) and (3.42), labeled by ‘ $Z^H Z$ ’ and ‘ ZZ ’, respectively.

3.2.1 Improving the Iterative Convergence via Transformation into Normal Equations

Matrix equations in (3.4), (3.5), and (3.10) can be transformed into normal equations as

$$\bar{\mathbf{Z}} \cdot \mathbf{a} = \mathbf{w} \quad \rightarrow \quad \bar{\mathbf{Z}}^H \cdot \bar{\mathbf{Z}} \cdot \mathbf{a} = \bar{\mathbf{Z}}^H \cdot \mathbf{w} \quad (3.39)$$

with

$$\bar{\mathbf{Z}}^H = (\bar{\mathbf{Z}}^\dagger)^*, \quad (3.40)$$

where ‘ $\dagger$ ’ and ‘ $*$ ’ represent transpose and complex conjugate operations, respectively. We note that $\bar{\mathbf{Y}} = \bar{\mathbf{Z}}^H \cdot \bar{\mathbf{Z}}$ is a Hermitian (self-adjoint) matrix, i.e., $\bar{\mathbf{Y}}^H = \bar{\mathbf{Y}}$. Using a Galerkin scheme, the EFIE matrix becomes symmetric so that only the complex conjugate operation is required in (3.39). Then, the ordinary MLFMA can be employed to perform MVMS related to the complex-conjugate matrix as

$$\mathbf{y} = \bar{\mathbf{Z}}_E^* \cdot \mathbf{x} = (\bar{\mathbf{Z}}_E \cdot \mathbf{x}^*)^*, \quad (3.41)$$

where only input ($\mathbf{x}$) and output ($\mathbf{y}$) vectors are required to be modified. In the case of non-Galerkin EFIE, however, transpose operation is required in (3.39), which can be implemented carefully by reversing steps of the ordinary MLFMA. The resulting transpose MLFMA is also required for MFIE and CFIE, which lead to nonsymmetric matrices, even with the Galerkin method.

Figure 3.4 presents iterative solutions of a scattering problem involving a PEC sphere with a radius of 1.5λ , where the residual error is plotted with respect to the iteration count. The sphere is illuminated by a plane wave and matrix equations with 8364 unknowns are solved by the CGS algorithm using MLFMA. For both CFIE and EFIE formulations, we compare the rate of convergence of ordinary matrix equations in (3.4) and (3.10) and transformed equations in the form of (3.39). In addition, we consider the transformation

$$\bar{\mathbf{Z}} \cdot \mathbf{a} = \mathbf{w} \quad \rightarrow \quad \bar{\mathbf{Z}} \cdot \bar{\mathbf{Z}} \cdot \mathbf{a} = \bar{\mathbf{Z}} \cdot \mathbf{w}, \quad (3.42)$$

where both sides of the ordinary matrix equation are multiplied with the impedance matrix.

We note that new matrices in (3.39) and (3.42) obtained by transformations, i.e., $\bar{\mathbf{Z}}^H \cdot \bar{\mathbf{Z}}$ and $\bar{\mathbf{Z}} \cdot \bar{\mathbf{Z}}$, have condition numbers that are squares of condition numbers of the impedance matrix $\bar{\mathbf{Z}}$. With higher condition numbers, both transformations might decrease convergence rates. However, Figure 3.4 shows that this is not true and we obtain different results from the two transformations, especially for EFIE:

- The transformation in (3.42) decreases convergence rates significantly for both CFIE and EFIE compared to ordinary solutions. Transformed matrix equations require more iterations, especially in the case of EFIE, where convergence to an error of less than 0.1 cannot be achieved within 1000 iterations.

- With the transformation in (3.39), CFIE has a faster convergence compared to the solution obtained by the transformation in (3.42), but it is still slower than the ordinary solution. However, in the case of EFIE, the transformation in (3.39) leads to an even faster convergence than the ordinary solution.

We also confirmed results presented in Figure 3.4 by considering various scattering problems involving different geometries with closed and open surfaces. We observed that the transformation in (3.39) consistently improved the convergence rate of iterative solutions of EFIE.

It is shown in [77],[78] that a transformation similar to (3.42) improves the conditioning of EFIE and provides faster convergence in iterative solutions, but not in the discrete form as in (3.42), and subject to a stabilization procedure that is valid only for closed surfaces. Here, we show that iterative solutions of EFIE can be improved by the transformation in (3.39) using the transpose complex conjugate of the impedance matrix. This improvement is obtained in the discrete form of the transformation as in (3.39), and it also exists for the solution of problems involving open surfaces.

In Figure 3.4, the transformed equation in (3.39) offers a faster convergence rate compared to the ordinary equation for EFIE. However, it does not increase the efficiency of the iterative solution since the transformed equation requires four MVMS per CGS iteration while the ordinary solution requires only two. As a consequence, we suggest solving the transformed equation in (3.39) using the CG algorithm in order to reduce the number of MVMS to two per iteration, while maintaining benefits of the transformation. Among different types of CG algorithms for least-squares equations, we choose the LSQR algorithm, which is a stable method based on Lanczos bidiagonalization as detailed in [113],[114].

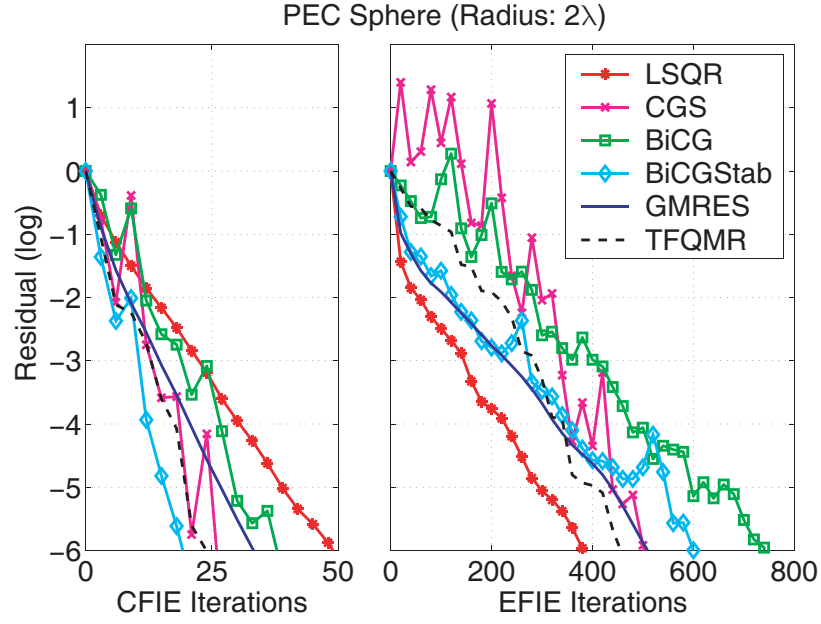


Figure 3.5: Iterative solutions of a scattering problem involving a sphere with a radius of 2λ formulated by (a) CFIE and (b) EFIE. Matrix equations with 14,871 unknowns are solved by various iterative algorithms.

Then, without having to consider the transformation in (3.39) explicitly, we simply feed the LSQR algorithm with MVMs related to the impedance matrix $\bar{\mathbf{Z}}$ and its transpose complex conjugate $\bar{\mathbf{Z}}^H$. Then, LSQR performs CG solutions of the transformed equation in an implicit and stable way.

3.2.2 Least-Squares QR Algorithm

Figure 3.5 presents iterative solutions of a scattering problem with 14,871 unknowns. The problem involves a sphere with a radius of 2λ illuminated by a plane wave. Ordinary equations obtained with CFIE and EFIE are solved iteratively by employing various algorithms, including LSQR. We note that the solution of ordinary equations in (3.4) and (3.10) using LSQR corresponds to a stable solution of the transformed equation in (3.39) using the CG algorithm [115]. All iterative algorithms in Figure 3.5 require two MVMs for each iteration except for

GMRES, which requires only one MVM per iterations at the cost of increased memory.

Figure 3.5(a) shows that the most efficient solutions of the CFIE formulation are performed by BiCGStab and the no-restart GMRES. In terms of the processing time, GMRES is better than other iterative algorithms since it requires only 34 MVMs to reduce the residual error to less than 10^{-6} . However, BiCGStab is preferable since it requires only 20 iterations, or 40 MVMs, without using the extra memory needed by GMRES. On the other hand, LSQR is inefficient compared to all other iterative algorithms by requiring 50 iterations, or 100 MVMs, to reduce the error to same levels. Therefore, LSQR appears to be a bad choice for the CFIE solution of the problem.

In contrast to CFIE, Figure 3.5(b) shows that LSQR has significantly different convergence properties for the EFIE solution of the problem; it is not the slowest converging iterative algorithm as in Figure 3.5(a). Indeed, it requires 383 iterations, or 766 MVMs, to reduce the error to less than 10^{-6} without using the extra memory needed by GMRES, which requires 510 MVMs. In other words, LSQR provides faster convergence compared to BiCG, BiCGStab, CGS, and TFQMR, and it is a good alternative to GMRES, which offers a faster convergence at the cost of extra memory. The improved convergence provided by LSQR is due to favorable properties of the transformation in (3.39) for EFIE. The same improvement cannot be observed in CFIE solutions since the transformation in (3.39) does not improve the convergence of CFIE, as also depicted in Figure 3.4.

In Figure 3.5, we observe that LSQR provides an improved convergence for EFIE, but it is still inefficient compared to CFIE solutions. In general, for the solution

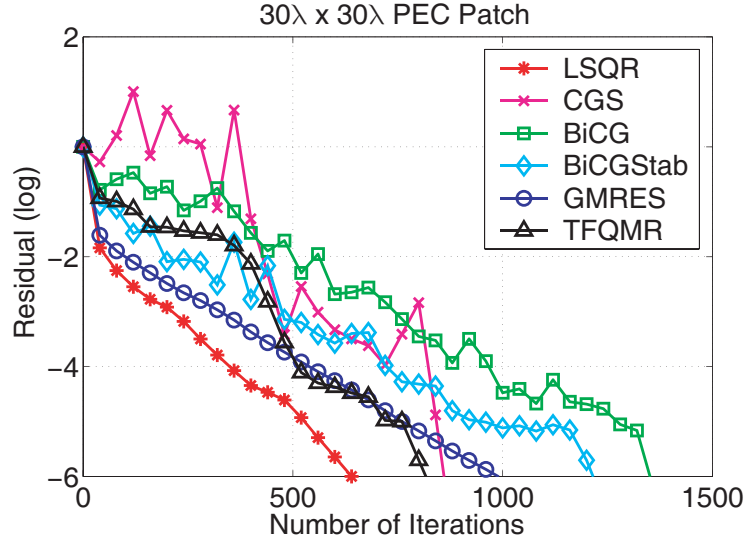


Figure 3.6: Iterative solutions of a scattering problem involving a $30\lambda \times 30\lambda$ patch discretized with 310,383 unknowns.

of problems involving closed surfaces, CFIE is very efficient and negates the necessity of employing LSQR with EFIE. On the other hand, for problems with open surfaces, EFIE is the inevitable choice and the improved convergence provided by LSQR becomes important. As an example, Figures 3.6 and 3.7 present the solution of scattering problems involving a $d \times d$ conducting patch, where $d = 30$ cm and corresponds to various values from 12λ to 30λ . The patch is located on the x - y plane and illuminated by a plane wave propagating in the $-z$ direction. Discretizations of the geometry for different frequencies lead to relatively large matrix equations with the number of unknowns ranging from 49,200 to 310,383. Scattering problems are solved by MLFMA parallelized into 16 processes on a cluster of AMD Opteron 870 processors. Figure 3.6 presents iteration counts when $d = 30\lambda$ and the number of unknowns is 310,383. In terms of iteration counts, LSQR provides the fastest convergence compared to other iterative algorithms and it requires 713 iterations, or 1426 MVMs, to reduce the error to less than 10^{-6} . On the other hand, since GMRES requires only one MVM per

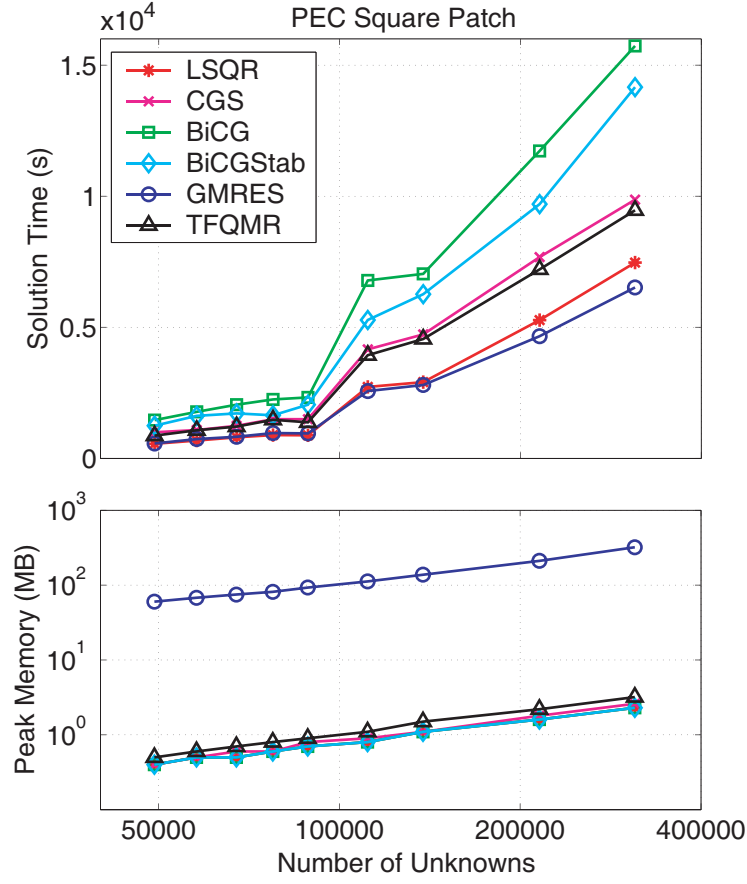


Figure 3.7: Processing time and peak memory per processor as a function of the number of unknowns for the solution of scattering problems involving a patch geometry with various dimensions from $12\lambda \times 12\lambda$ to $30\lambda \times 30\lambda$. Only the time and memory required for iterative solutions are considered.

iteration, a fair comparison should be based on processing times of iterative solutions. In addition, such a comparison is necessary in order to consider different setup times required by iterative algorithms, which become considerable as the number of unknowns increases. In this manner, Figure 3.7 presents processing times for solutions with respect to the number of unknowns. We ignore the fixed setup time for MLFMA and report only the time required to perform iterative solutions to reduce the residual error to less than 10^{-6} . It can be observed that no-restart GMRES provides the fastest solution, especially as the problem size grows. However, as also seen in Figure 3.7, it requires a peak memory that

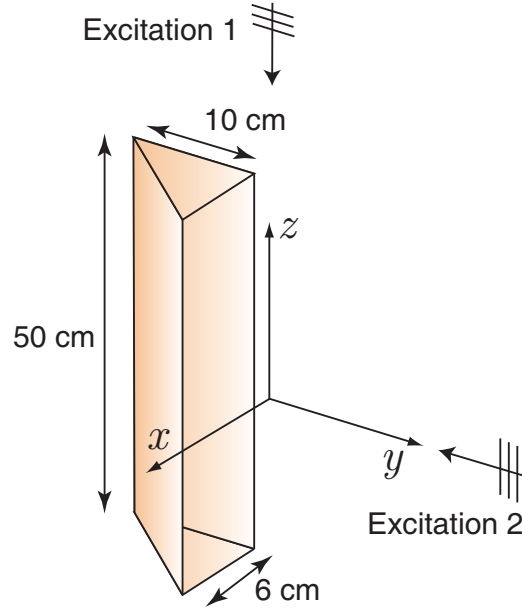
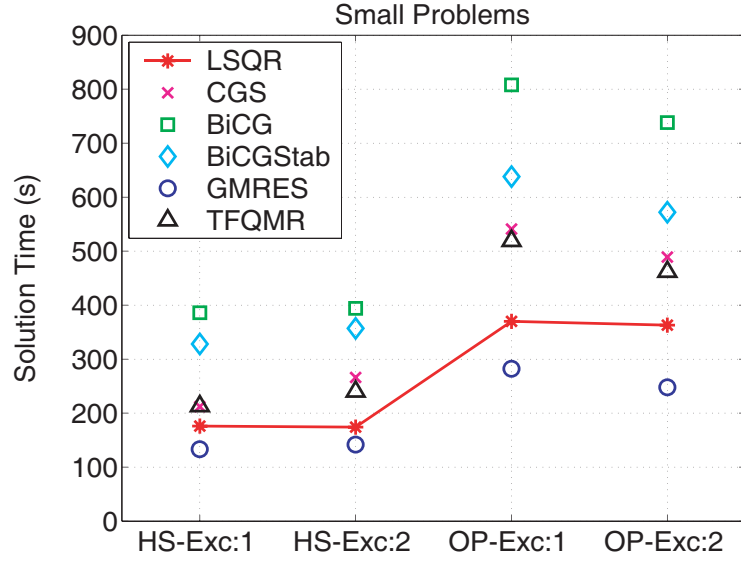
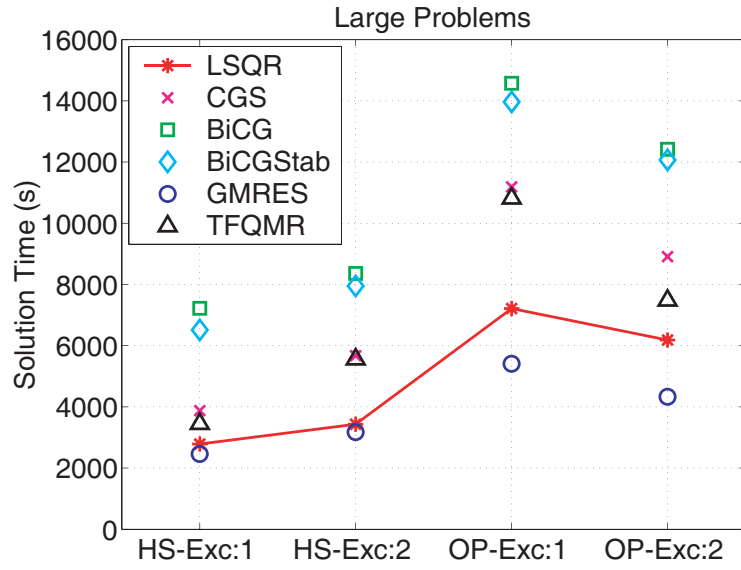


Figure 3.8: Scattering problems involving an open prism.

is significantly larger than other iterative algorithms. As an example, for the solution of the largest problem in Figure 3.7 with 310,383 unknowns, GMRES requires 321 MB memory per processor, while the total memory usage (including the memory required by MLFMA) is only 456 MB per processor. Thus, leaving the memory-hungry GMRES aside, among memory-efficient solutions provided by other algorithms, LSQR requires the minimum processing time, as depicted in Figure 3.7. In other words, LSQR performs better than CGS, BiCG, BiCGStab, and TFQMR, and it is a strong alternative to no-restart GMRES, which has considerably larger memory requirements.



(a)



(b)

Figure 3.9: Processing times required by various iterative algorithms for MLFMA solutions of scattering problems described in Figures 3.2 and 3.8.

We further demonstrate the improved convergence provided by LSQR on two scattering problems involving a half sphere (HS) and an open prism (OP), as depicted in Figures 3.2 and 3.8, respectively. The half-sphere problem is solved at 2 GHz and 7.5 GHz, while the open-prism problem is solved at 5 GHz and 16 GHz. For lower frequencies, discretizations of problems with about $\lambda/10$ triangulation lead to 9,911 and 11,351 unknowns for the half-sphere and the open-prism problems, respectively (small problems). For higher frequencies, similar discretizations with $\lambda/10$ mesh size lead to 116,596 and 127,925 unknowns (large problems). We employ a parallel MLFMA implementation to perform MVMs. For each geometry, we solve two scattering problems involving plane-wave excitations described in Figures 3.2 and 3.8. As an iterative solver, we employ the LSQR algorithm in addition to GMRES, CGS, BiCG, BiCGStab, and TFQMR. Figure 3.9 presents the processing time required by various iterative algorithms for the solution of scattering problems. Processing times are measured again on a cluster of 16 AMD Opteron 870 processors for the convergence of the residual error to less than 10^{-6} . We observe that LSQR outperforms other iterative algorithms, except for GMRES with a restart parameter of 1000. However, GMRES requires considerably larger memory than all other algorithms. For example, for the solution of the large half-sphere problem, the memory requirement of GMRES is about 140 MB per processor, while it is only 1–1.5 MB for other algorithms.

Table 3.2: Solutions of Scattering Problems Involving a Patch Geometry of Various Sizes

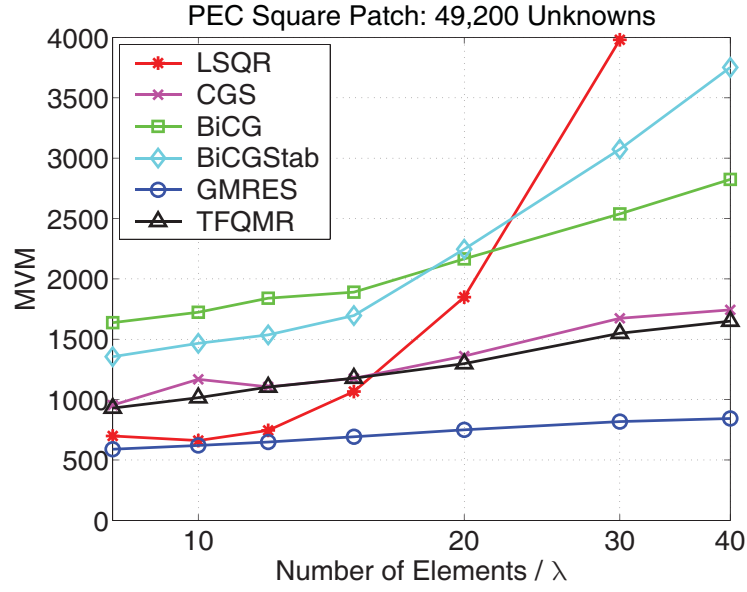
Size	N	GMRES		
		Iterations	Time (s)	Memory (MB)
$12\lambda \times 12\lambda$	49,200	620	568	30
$18\lambda \times 18\lambda$	111,405	776	2372	81
$20\lambda \times 20\lambda$	137,792	793	2635	107
$25\lambda \times 25\lambda$	215,338	877	4232	180
$30\lambda \times 30\lambda$	310,383	991	6070	290

Size	N	GMRES-NE		
		Iterations	Time (s)	Memory (MB)
$12\lambda \times 12\lambda$	49,200	254	445	13
$18\lambda \times 18\lambda$	111,405	335	1959	36
$20\lambda \times 20\lambda$	137,792	360	2269	49
$25\lambda \times 25\lambda$	215,338	413	3620	87
$30\lambda \times 30\lambda$	310,383	467	5016	139

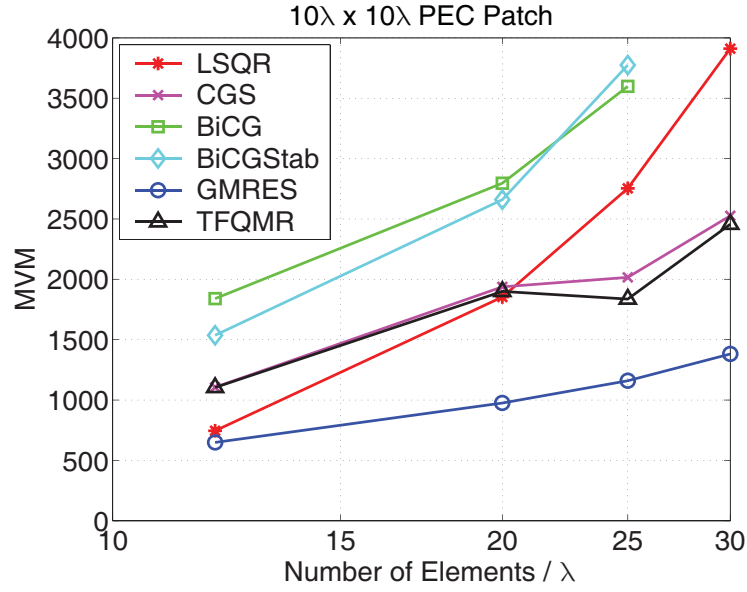
3.2.3 Solutions of Normal Equations Using the Generalized Minimal Residual Algorithm

The improved convergence provided by LSQR is due to favorable properties of normal equations in (3.39) for EFIE. We also use benefits of normal equations by employing GMRES. As an example, we revisit the solution of scattering problems involving a square patch with various sizes from $12\lambda \times 12\lambda$ to $30\lambda \times 30\lambda$. Problems are solved on a cluster of 16 AMD Opteron 870 processors. Table 3.2 lists the number of iterations for 10^{-6} residual error, solution time, and memory per processor, when GMRES is used to solve original and transformed (normal) matrix equations. Only the memory required for GMRES is considered. Compared to conventional solutions, the number of iterations is reduced by 53–59% when GMRES is applied on normal equations (denoted as GMRES-NE). The processing time, however, does not decrease by the same amount, since the number of MVMs per iteration is two for GMRES-NE. Nevertheless, for the largest problem discretized with 310,383 unknowns, the processing time is reduced by

17%, while the number of MVMs decreases only by 6%. This is because the cost of GMRES itself increases quadratically with the number of iterations [112]. Hence, reducing the number of iterations is preferable even when the number of MVMs does not decrease significantly. Finally, Table 3.2 shows that the memory required for GMRES is reduced by 52–57%. For the largest problem, MLFMA itself requires only 135 MB memory per processor. Then, the total memory usage (including GMRES and MLFMA) per processor is reduced from 425 MB to 274 MB, just by solving the normal equation instead of the original equation with GMRES.



(a)



(b)

Figure 3.10: Number of MVMs as a function of number of elements per wavelength for the solution of scattering problems involving a square patch with edges of 30 cm. (a) Patch is discretized with 49,200 unknowns and the frequency changes from 3 GHz to 15 GHz. (b) The frequency is 10 GHz and the patch is discretized with 49,200 to 310,383 unknowns.

Table 3.3: Number of MVMs (10^{-6} Residual Error) for the Solution of Scattering Problems Involving a Half Sphere and Open Prism in Figures 3.2 and 3.8

Problem	CGS		BiCGStab		LSQR	
	NP	NFP	NP	NFP	NP	NFP
Half Sphere, Excitation 1 at 2 GHz	798	38	1240	36	657	174
Half Sphere, Excitation 2 at 2 GHz	988	38	1346	38	655	172
Open Prism, Excitation 1 at 5 GHz	1342	72	1606	60	935	448
Open Prism, Excitation 2 at 5 GHz	1220	56	1426	54	899	422

3.2.4 Limitations

Transformation into normal equations improves iterative solutions of EFIE. However, the improvement is usually restricted to ordinary discretizations with $\lambda/10$ triangles. In Figure 3.10, we revisit the iterative solution of scattering problems involving a square patch with edges of 30 cm. In Figure 3.10(a), the patch is discretized with 49,200 unknowns and the frequency changes from 3 GHz to 15 GHz. The mesh size corresponds to $\lambda/8$ and $\lambda/40$ at 15 GHz and 3 GHz, respectively. Due to the low-frequency breakdown of EFIE, the iterative convergence becomes difficult as the mesh size decreases with respect to wavelength. However, as depicted in Figure 3.10(a), the number of iterations increases faster for LSQR, compared to other iterative algorithms. In other words, the improved convergence provided by LSQR vanishes for dense discretizations. Figure 3.10(b) presents another experiment, where a $10\lambda \times 10\lambda$ patch is discretized with various mesh sizes from $\lambda/12$ to $\lambda/30$. In this case, the frequency is fixed to 10 GHz, but the number of unknowns changes from 49,200 to 310,383. Similar to the previous example, rapid convergence of the LSQR algorithm compared to other algorithms deteriorates as the mesh size decreases.

In addition to low performance for dense discretizations, LSQR algorithm is difficult to accelerate with preconditioners. For example, Table 3.3 lists the number of MVMs required for the half-sphere and the open-prism problems discretized

with 9,911 and 11,351 unknowns, respectively. We compare CGS, BiCGStab, and LSQR when solutions are accelerated with NFP, in addition to the NP case. Without preconditioning, LSQR performs better than other algorithms. However, using NFP, both CGS and BiCGStab are accelerated much more than LSQR, which becomes the worst choice among the three algorithms.

Despite many limitations that prevent the use of LSQR, advantages of this algorithm and the transformation into normal equations for EFIE cannot be underestimated. It needs further research to reveal how such a transformation that squares the condition number can improve the solution of EFIE and how it can be generalized to other types of problems.

3.3 Hybrid Formulations for Coexisting Open and Closed Surfaces

Real-life electromagnetics problems often involve both thin and thick conductors that need to be modeled with open and closed surfaces, respectively. Since MFIE is not applicable to open surfaces, conventional solutions employ EFIE even when only a small part of the geometry is an open surface. Unfortunately, EFIE produces ill-conditioned matrix equations that are difficult to solve with an iterative solver, especially when the problem includes some closed parts [54]. Consequently, as problem size increases, the solution of composite problems with coexisting open and closed surfaces becomes extremely difficult. Although iterations are accelerated via efficient algorithms, such as MLFMA, convergence is usually not reached within a reasonable number of iterations. To remedy this, we propose HIE formulations by using an extended definition of CFIE. Resulting hybrid formulations allow for more efficient solutions of composite problems without sacrificing accuracy.

Extending the definition of CFIE in (3.3), HIEs can be written as

$$\text{HIE} = \alpha(\mathbf{r})\text{EFIE} + [1 - \alpha(\mathbf{r})]\text{MFIE}, \quad (3.43)$$

where the parameter α depends on the observation point. HIEs employ MFIE or CFIE on closed parts of the geometry, while EFIE is applied on open parts, as usual. Discretization of (3.43) leads to

$$\bar{\mathbf{Z}}_H \cdot \mathbf{a} = \mathbf{w}_H, \quad (3.44)$$

where

$$\bar{\mathbf{Z}}_H[m, n] = \alpha_m \bar{\mathbf{Z}}_E[m, n] + (1 - \alpha_m) \bar{\mathbf{Z}}_M[m, n] \quad (3.45)$$

$$\mathbf{w}_H[m] = \alpha_m \mathbf{w}_E[m] + (1 - \alpha_m) \mathbf{w}_M[m]. \quad (3.46)$$

The matrix equation in (3.44)–(3.46) has a variable α_m depending on the index of the row (m) in the impedance matrix, i.e., the index of the testing function. This provides the freedom to choose different linear combinations for different testing functions, even using pure EFIE ($\alpha_m = 1$) or pure MFIE ($\alpha_m = 0$) for some m . In general, we obtain better-conditioned matrix equations by including CFIE interactions in the solution of composite problems. We also note that the traditional CFIE can be interpreted as a special case of HIEs for closed surfaces, where EFIE and MFIE are combined with a fixed parameter for each testing case.

MLFMA solutions of HIEs, where α_m depends on the location of the testing function, is similar to MLFMA solutions of EFIE, MFIE, and CFIE, and do not require any special procedures. This is because, in MLFMA, radiated and incoming fields do not depend on the type of the formulation and α_m appears only in receiving patterns of testing functions.

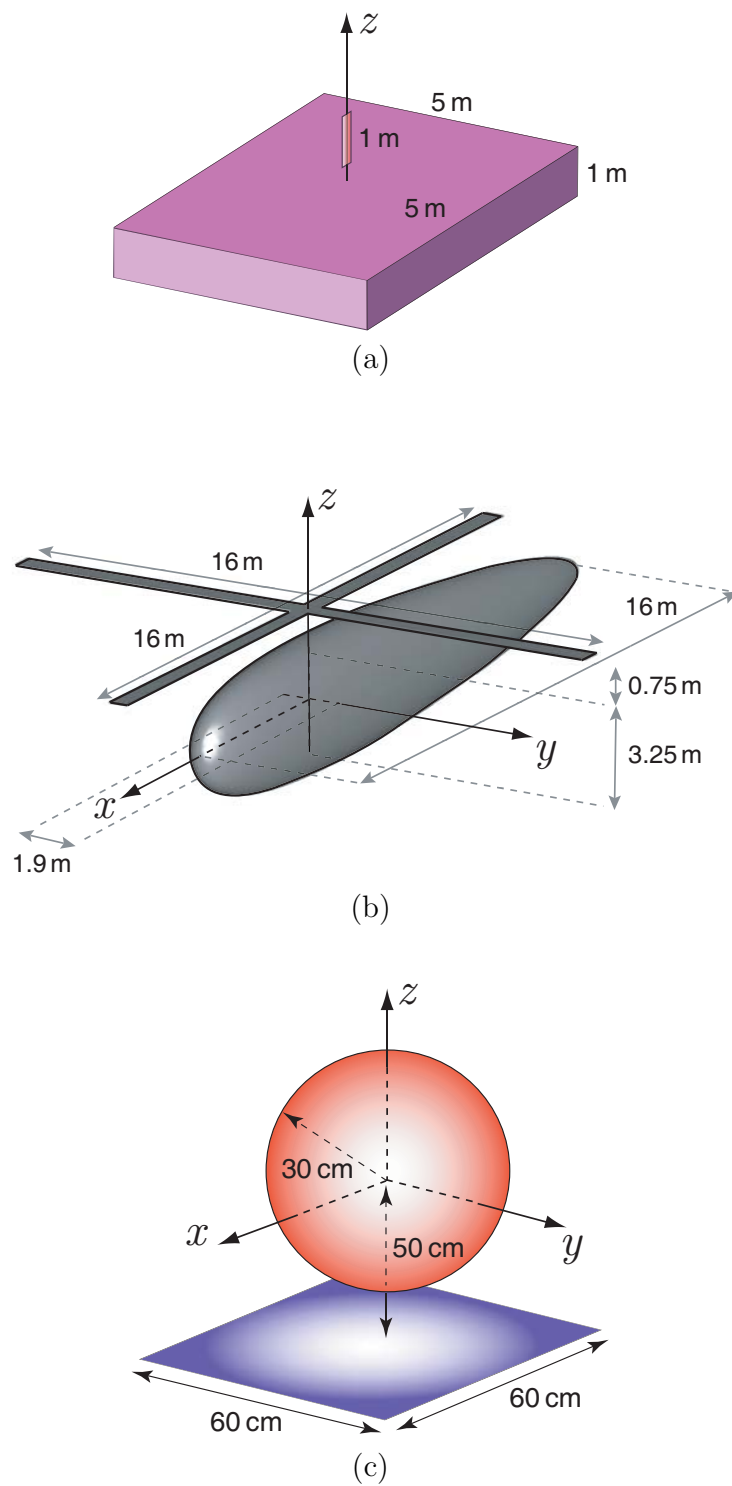


Figure 3.11: Various composite problems (a) with almost completely closed geometries, except for very minor open parts, (b) involving more closed parts than open parts, and (c) with comparable amounts of open and closed parts.

3.3.1 Examples to Composite Problems

Figure 3.11 shows three different kinds of composite problems, where relative amounts of open and closed surfaces are different. All objects are located in free space and discretized with the RWG functions. Problems are also summarized in Table 3.4. The first problem in Figure 3.11(a) is a radiation problem involving a dipole antenna (open surface) of length 1 m located over a PEC rectangular box (closed surface) with dimensions of 1 m $\times$ 5 m $\times$ 5 m. This structure is an example of problems with almost completely closed geometries, except for very minor open parts. The problem is solved at 300 MHz, and the discretization of the object leads to 24,186 unknowns on the box and only 19 unknowns on the antenna. We simulated the feed of the antenna by a delta-gap source located at the center of the strip.

Figure 3.11(b) shows an example of problems involving mostly closed parts but with some open parts. A cross-shaped open surface is placed over a closed ellipsoid surface with dimensions shown in the figure. The problem is solved at 500 MHz and illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. Discretization of the object leads to 115,023 unknowns on the ellipsoid (closed surface) and 16,608 unknowns on the cross-shaped part (open surface).

Finally, as an example of problems with comparable amounts of open and closed parts, Figure 3.11(c) shows a sphere with radius 30 cm placed over a 60 cm $\times$ 60 cm square patch. The structure is illuminated by the same plane wave as in the second problem. The problem is solved at two frequencies; at 1 GHz, the number of unknowns is 3768 on the sphere and 1301 on the patch, whereas at 6 GHz, there are 132,336 unknowns on the sphere and 49,200 on the patch.

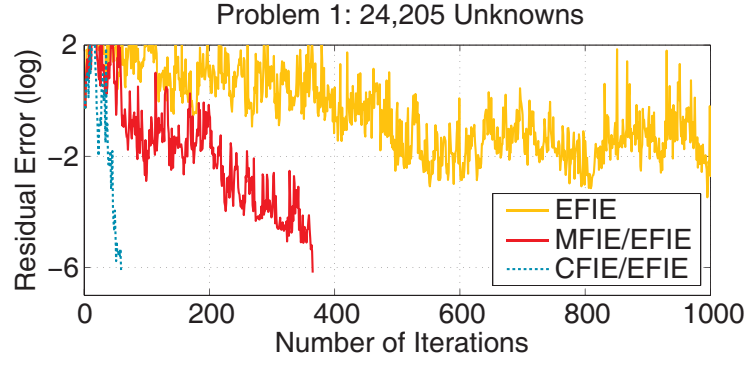
Table 3.4: Composite Problems

Problem	Frequency	Number of Unknowns	
		Closed Parts	Open Parts
Problem 1: Figure 3.11(a)	300 MHz	24,186	19
Problem 2: Figure 3.11(b)	500 MHz	115,023	16,608
Problem 3: Figure 3.11(c)	1 GHz	3768	1301
Problem 3: Figure 3.11(c)	6 GHz	132,336	49,200

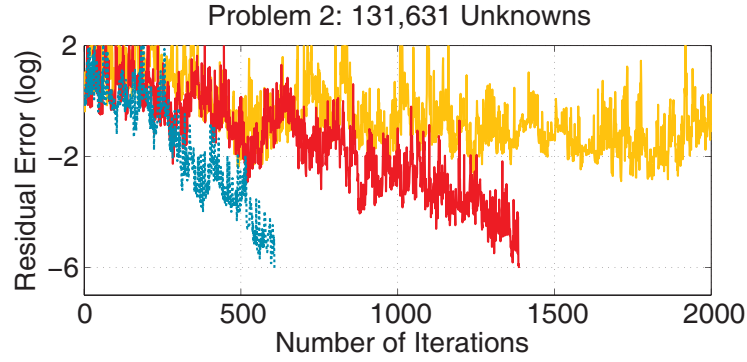
3.3.2 Iterative Convergence for Various Formulations

In Figures 3.12 and 3.13, convergence characteristics are depicted for CGS algorithm and for different formulations applied to the composite problems. In all examples, iterative convergence is the slowest for the conventional EFIE. Especially for the first two problems, EFIE does not reach the 10^{-6} residual error within a reasonable number of iterations. As shown in Figures 3.12(a) and 3.12(b), the hybrid CFIE/EFIE formulation dramatically improves the convergence of these two problems, when CFIE with $\alpha_m = 0.2$ is applied to closed parts. On the other hand, when MFIE is applied to closed parts, leading to a hybrid MFIE/EFIE formulation, improvement is not sufficient.

For the third problem containing comparable amounts of open and closed surfaces, there is improvement with the hybrid CFIE/EFIE formulation, but it is less than those in the previous two problems. For this problem, EFIE converges better than the first and the second problems due to the larger portion of open surface. The CFIE/EFIE formulation is still much better than EFIE for the larger problem in Figure 3.13(b). However, the acceleration provided by CFIE/EFIE is not very significant for the smaller problem in Figure 3.13(a). In fact, iterations in Figure 3.13(a) are at the frequency where EFIE shows quite good performance. In Figure 3.14, a frequency sweep from 500 MHz to 1200 MHz is demonstrated for the same problem. EFIE suffers from internal resonances, and the number of iterations peaks at various frequencies. On the other hand,



(a)



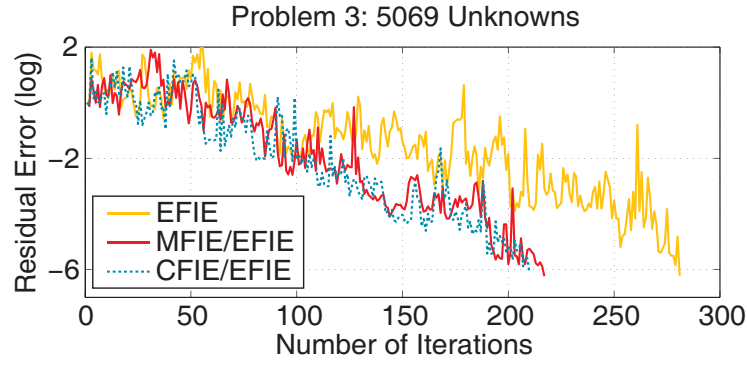
(b)

Figure 3.12: Convergence characteristics of problems in Figure 3.11(a) and Figure 3.11(b) for different formulations, i.e., EFIE, MFIE/EFIE, and CFIE/EFIE, solved with CGS.

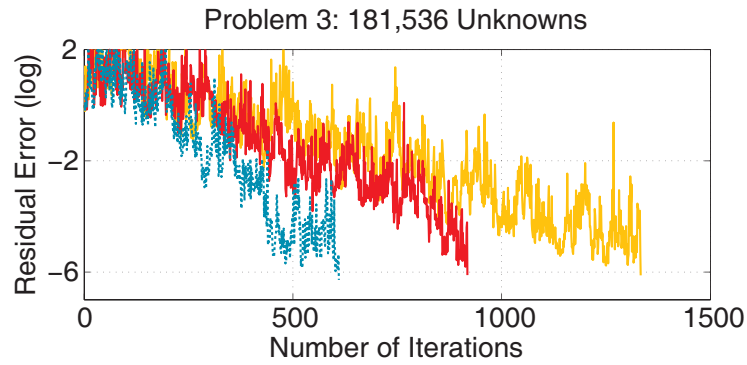
the hybrid CFIE/EFIE formulation is free of the internal resonances so that the number of iterations is steady and consistently in the same low range.

3.3.3 Iterative Convergence with Various Iterative Methods

Figures 3.15 and 3.16 demonstrate convergence characteristics for the solution of Problem 1 using various iterative methods. In general, we observe that the hybrid CFIE/EFIE formulation consistently leads to the fastest convergence. Table 3.5 lists the number of iterations for the same problem to reach 10^{-6} residual



(a)



(b)

Figure 3.13: Convergence characteristics of problem in Figure 3.11(c) for different formulations, i.e., EFIE, MFIE/EFIE, and CFIE/EFIE, solved with CGS.

error. When the EFIE formulation is applied to the entire problem, CGS, LSQR, and TFQMR converge within 2000 iterations, while BiCG, BiCGStab, and GMRES (with a restart parameter of 200 iterations) do not. With CFIE/EFIE, the number of iterations drops for all cases, but GMRES performs the best in terms of the number of MVMs. This is because GMRES requires only one MVM per iteration while others require two. On the other hand, GMRES requires more memory than other iterative methods in Table 3.5; hence BiCGStab might be preferred as problem size grows and it becomes critical to reduce memory.

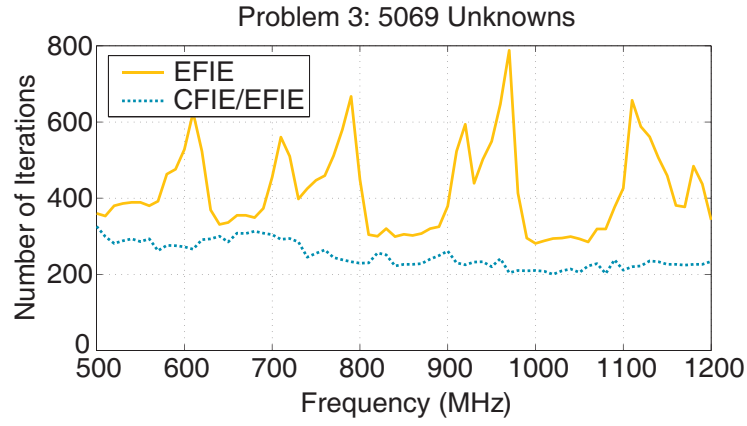


Figure 3.14: Iteration counts to reach 10^{-6} residual error for the problem in Figure 3.11(c) in the frequency range from 500 MHz to 1200 MHz.

Table 3.5: Number of Iterations (10^{-6} Residual Error) for the Solution of Problem 1 in Table 3.4

Method	EFIE	MFIE/EFIE	CFIE
BiCG	NC	495	72
BiCGStab	NC	377	47
CGS	1488	372	63
GMRES	NC	388	60
LSQR	1307	568	101
TFQMR	1407	358	52

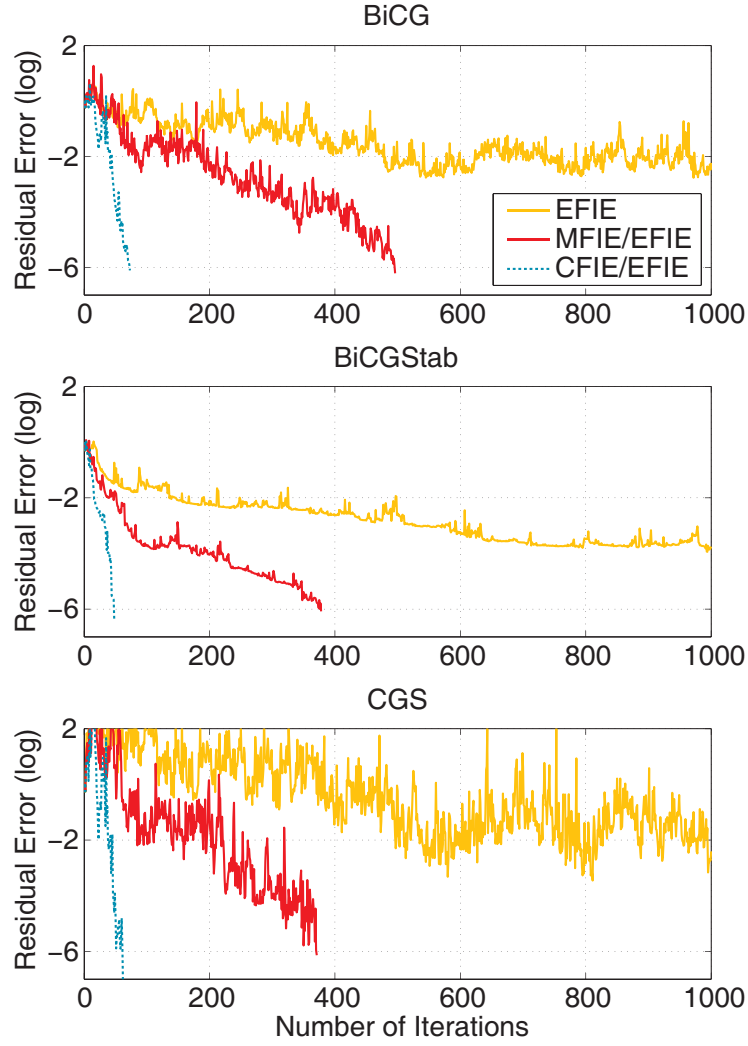


Figure 3.15: Convergence characteristics of the problem in Figure 3.11(a) for various iterative methods and different formulations, i.e., EFIE, MFIE/EFIE, and CFIE/EFIE.

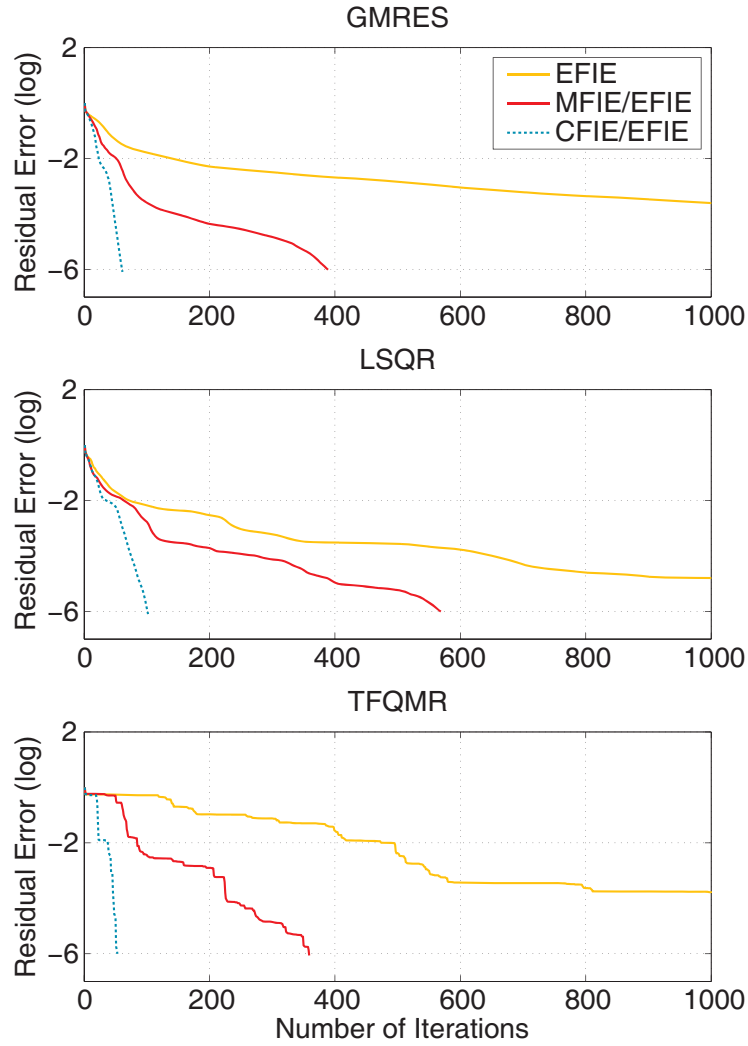
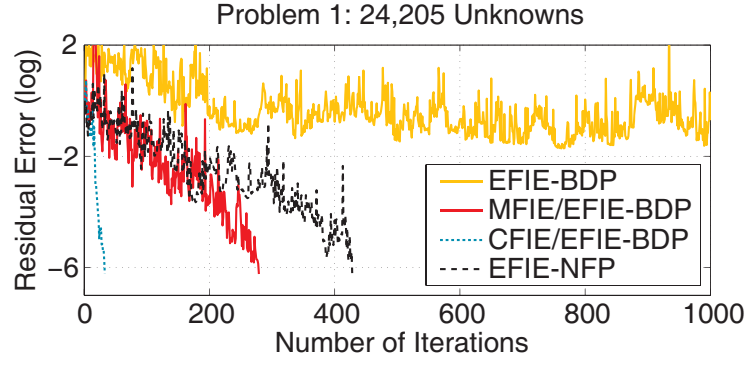
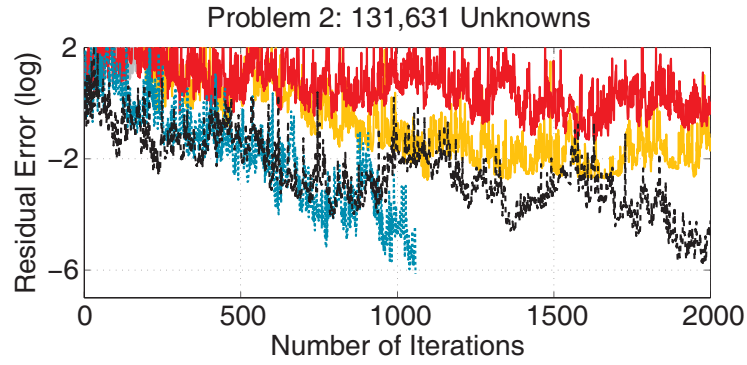


Figure 3.16: Convergence characteristics of the problem in Figure 3.11(a) for various iterative methods and different formulations, i.e., EFIE, MFIE/EFIE, and CFIE/EFIE.



(a)

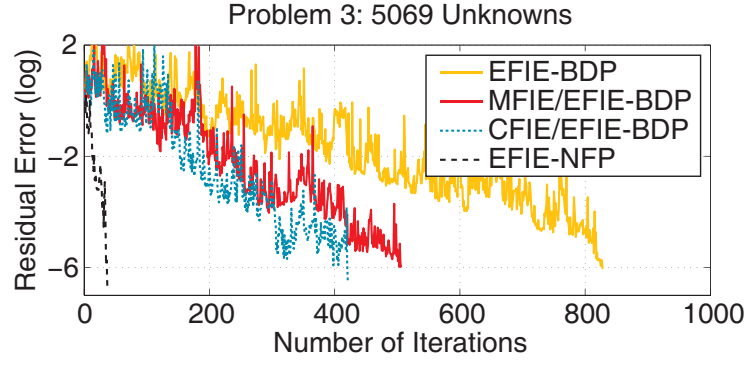


(b)

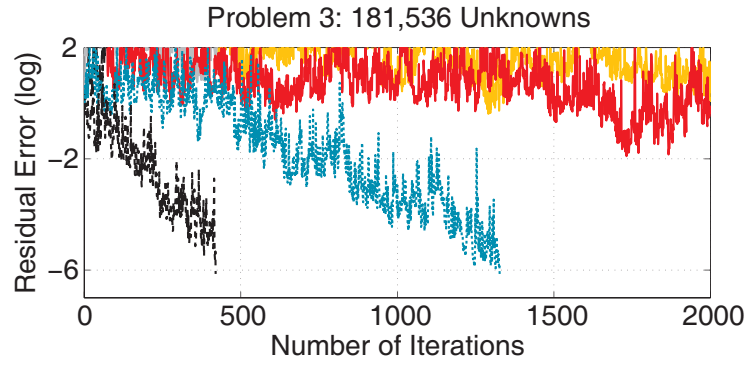
Figure 3.17: Convergence characteristics of problems in Figures 3.11(a) and 3.11(b) for the preconditioned CGS and different formulations, i.e., EFIE, MFIE/EFIE, and CFIE/EFIE.

3.3.4 Effect of Preconditioning

Convergence characteristics for preconditioned systems are depicted in Figures 3.17 and 3.18. For the three problems, EFIE and hybrid formulations are solved with CGS preconditioned by BDPs. EFIE solutions with NFP are also presented in the same figures for comparison. To easily check the effect of preconditioning on different problems and formulations, Table 3.6 lists the number of iterations to reach 10^{-6} residual error. Looking at Figure 3.17, Figure 3.18, and Table 3.6, we note the following observations:



(a)



(b)

Figure 3.18: Convergence characteristics of problem in Figure 3.11(c) for the preconditioned CGS and different formulations, i.e., EFIE, MFIE/EFIE, and CFIE/EFIE.

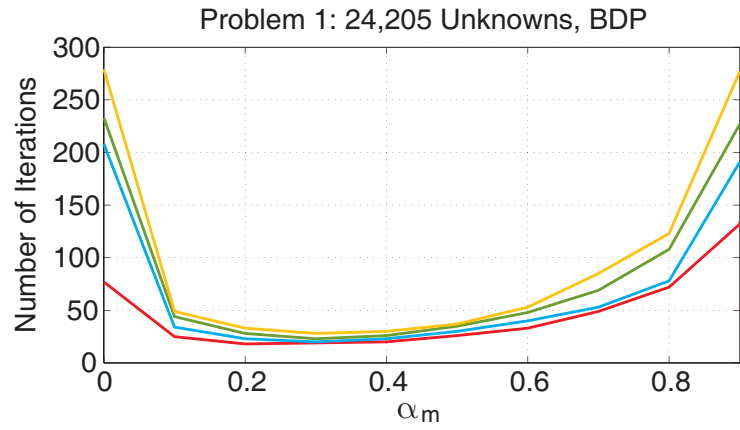
- For the first problem with a completely closed geometry except for a very minor open part, the convergence of hybrid formulations is significantly improved by the efficient BDP. However, for EFIE, BDP does not work and the convergence is worse than the NP case. The strong NFP improves the converge of EFIE, although the number of iterations is still high, compared to hybrid formulations.
- For the second problem, where there are more closed than open parts, BDP does not work even with hybrid formulations. This can be explained with the worsening of the condition of the system since more open parts need to be formulated with EFIE, compared to the first problem. Nevertheless,

Table 3.6: Number of CGS Iterations to Reach 10^{-6} Residual Error

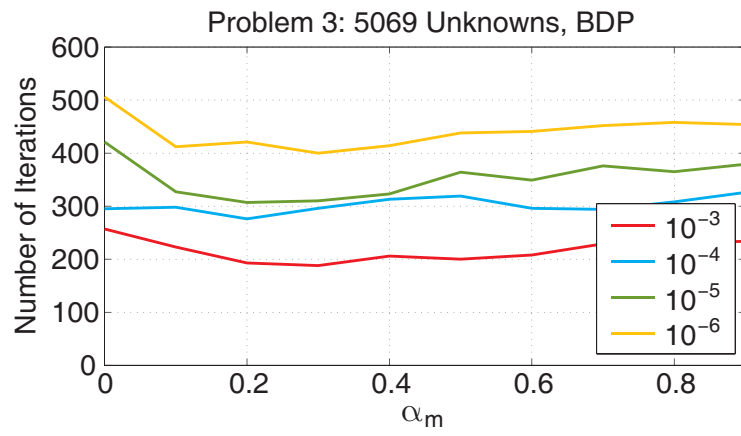
Problem/Unknowns	Preconditioner	EFIE	MFIE/EFIE	CFIE/EFIE
Problem 1 / 24,205	NP	1488	372	63
Problem 1 / 24,205	BDP	NC	280	33
Problem 1 / 24,205	NFP	429	-	-
Problem 2 / 131,631	NP	NC	1387	607
Problem 2 / 131,631	BDP	NC	NC	1058
Problem 2 / 131,631	NFP	NC	-	-
Problem 3 / 5069	NP	280	218	219
Problem 3 / 5069	BDP	828	507	421
Problem 3 / 5069	NFP	37	-	-
Problem 3 / 181,536	NP	1334	919	611
Problem 3 / 181,536	BDP	NC	NC	1327
Problem 3 / 181,536	NFP	420	-	-

the hybrid CFIE/EFIE formulation is preferable since its NP case is still better than EFIE with a strong NFP.

- For the third problem, where open and closed parts are comparable, the effect of EFIE becomes significant. As in the second problem, BDP does not work even for hybrid formulations. In addition to this, EFIE with NFP performs better than hybrid formulations with NP. Consequently, NFP provides a trade-off between memory and number of iterations. There is an alternative way using the hybrid CFIE/EFIE formulation with NFP and leading to fewer iterations. However, due to the non-symmetric impedance matrix, this would be more expensive in terms of memory than the EFIE case.

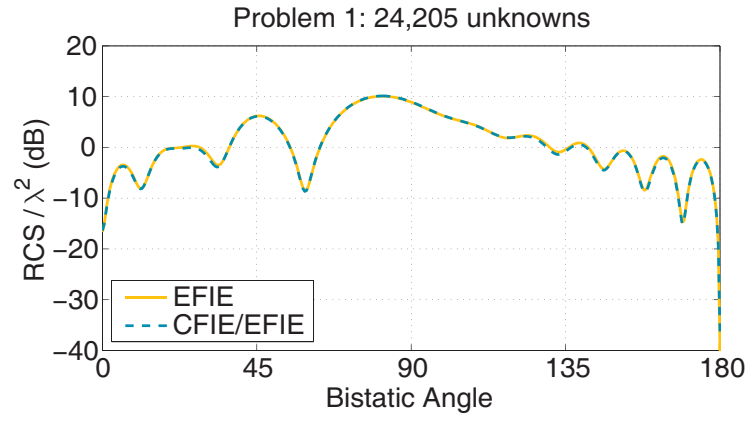


(a)

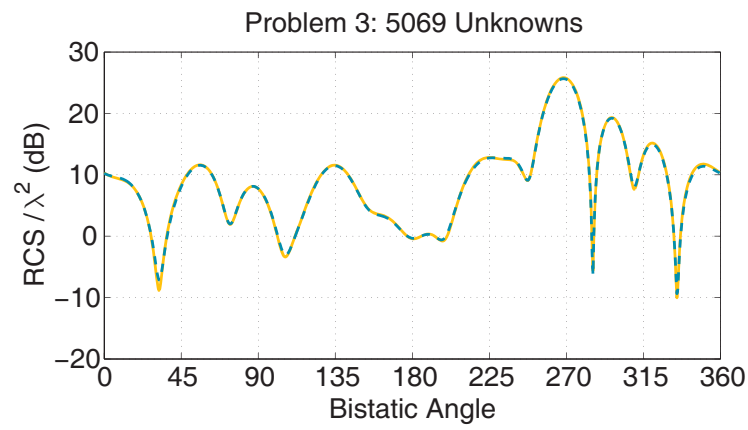


(b)

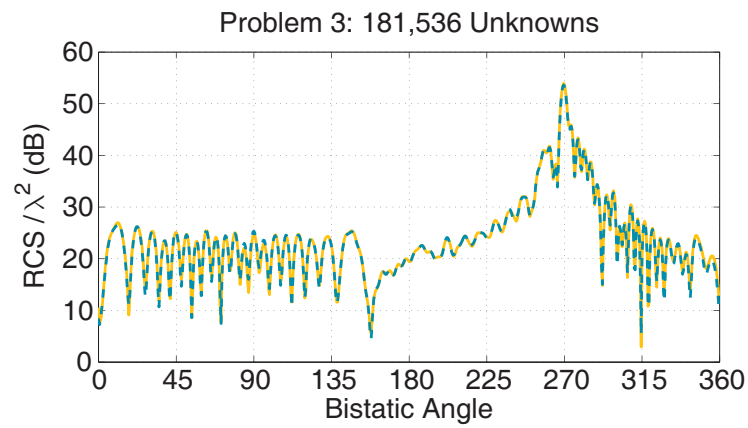
Figure 3.19: Iteration counts with respect to the variable α_m , which is applied to closed parts of problems in Figures 3.11(a) and 3.11(c).



(a)



(b)



(c)

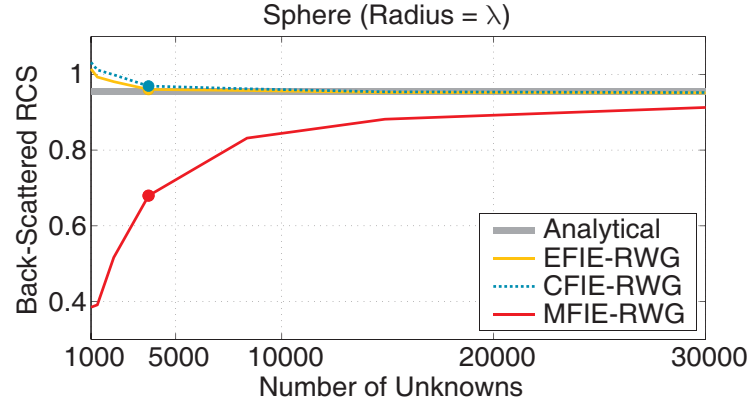
Figure 3.20: Normalized bistatic RCS (RCS/λ^2) on the z - x plane for structures in Figures 3.11(a) and 3.11(c).

3.3.5 Effect of the Combination Parameter

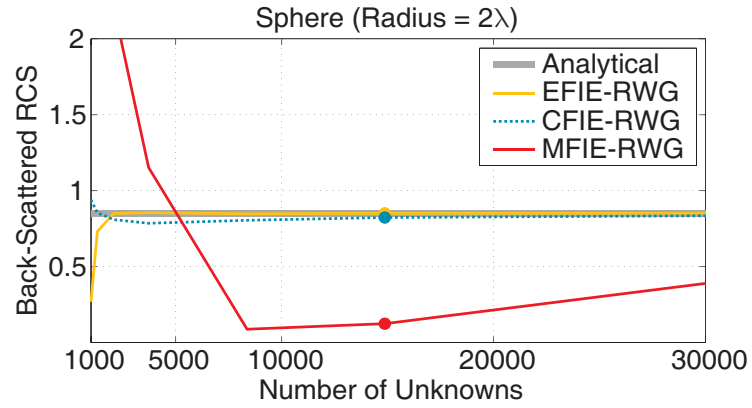
Next, we focus on the variable α_m employed in the convex combination of EFIE and MFIE in the CFIE part of hybrid formulations. We demonstrate two different cases in Figure 3.19, where the number of iterations are depicted with respect to the variable α_m applied on closed parts of problems. The four curves in each figure represent residual errors from 10^{-3} to 10^{-6} . Comparing Figures 3.19(a) and (b), we observe that the dependence on the α_m is directly based on the type of geometry. In Problem 1, where only a small portion of the geometry is open, there is an optimal α_m of about 0.2–0.4 for the best convergence. In both directions to MFIE/EFIE ($\alpha_m = 0$) and to EFIE ($\alpha_m = 1$), the number of iterations increases. On the other hand, in the case of Problem 3, the open part is relatively larger, and it is difficult to set an optimal α_m for $\alpha_m \neq 1$. We conclude that choosing α_m around 0.2 does not guarantee the exact optimal convergence for all cases, but it provides a number of iterations close to the optimal case.

3.3.6 Radar-Cross-Section Results

Finally, to check the accuracy of hybrid formulations, RCS results are presented for problems in Figures 3.11(a) and 3.11(c). In Figure 3.20, normalized RCS values (RCS/λ^2) are plotted on the z - x plane with respect to the θ angle defined as increasing from the positive z axis to the positive x axis. The highest RCS values in Figures 3.20(b) and 3.20(c) appear at 270° , which corresponds to the forward-scattering direction. For both radiation and scattering problems, we observe that the hybrid CFIE/EFIE formulation provides accurate results that agree well with reference EFIE solutions.

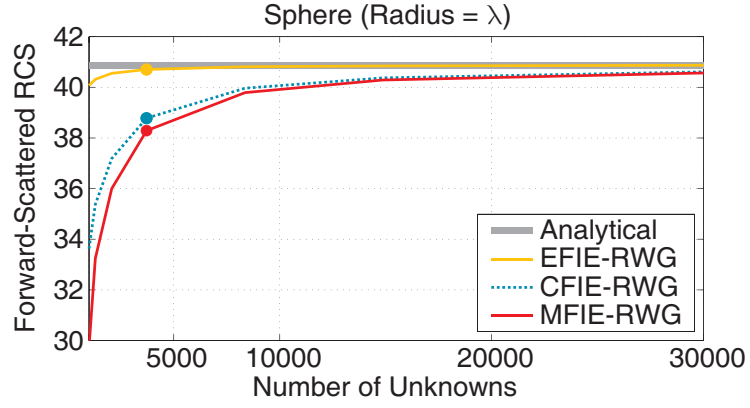


(a)

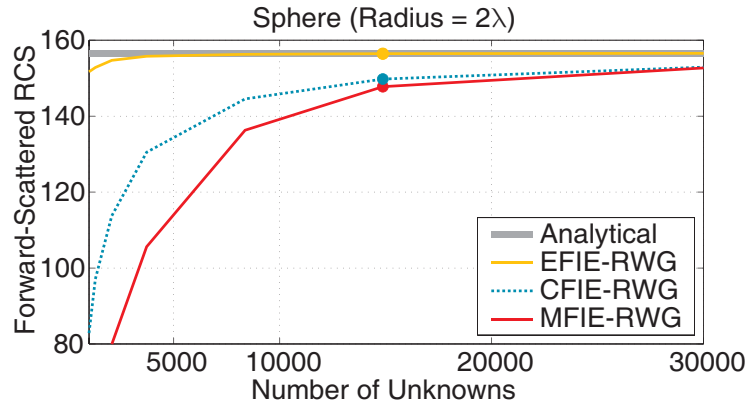


(b)

Figure 3.21: Normalized back-scattered RCS ($\text{RCS}/\pi a^2$) of a PEC sphere with a radius of (a) λ and (b) 2λ as a function of the number of unknowns. The dots on the curves correspond to the $\lambda/10$ discretization.



(a)



(b)

Figure 3.22: Normalized forward-scattered RCS ($\text{RCS}/\pi a^2$) of a PEC sphere with a radius of (a) λ and (b) 2λ as a function of the number of unknowns. The dots on the curves correspond to the $\lambda/10$ discretization.

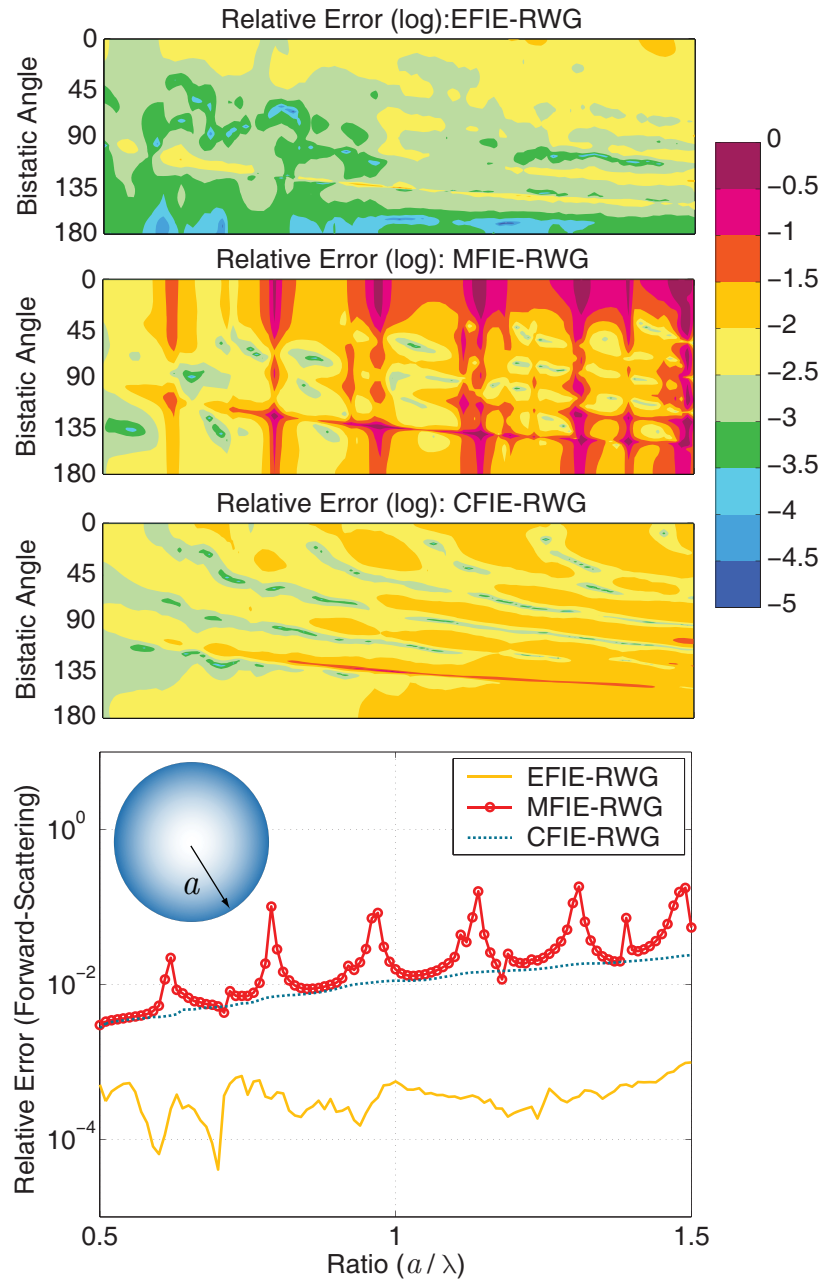


Figure 3.23: Relative error in the bistatic RCS of a PEC sphere with a radius of a , when a changes from 0.5λ to 1.5λ .

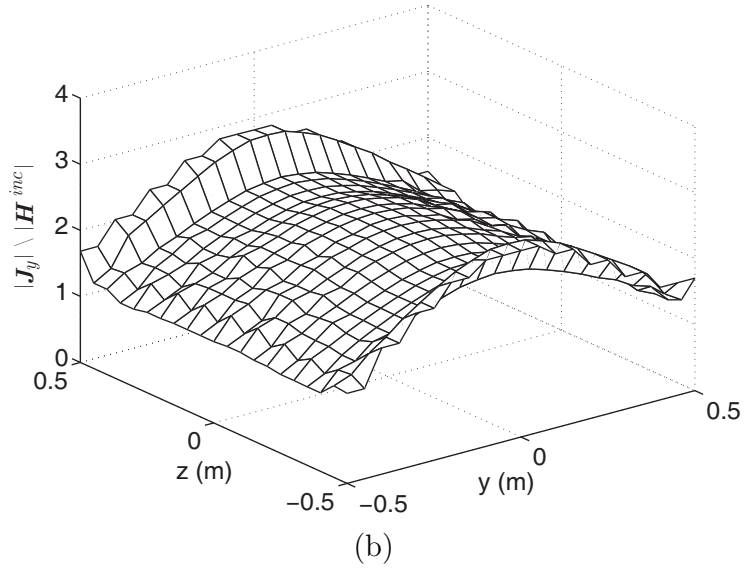
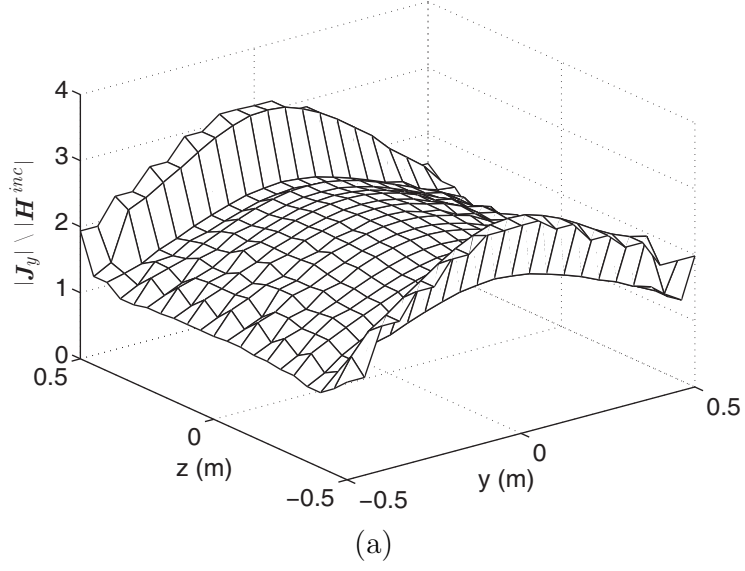


Figure 3.24: Magnitude of the y component of the normalized surface current ($|\mathbf{J}_y|/|\mathbf{H}^{inc}|$) induced on the front surface (at $x = 0.5$ m) of a PEC cube with edges of λ , obtained by using (a) EFIE with RWG and (b) MFIE with RWG.

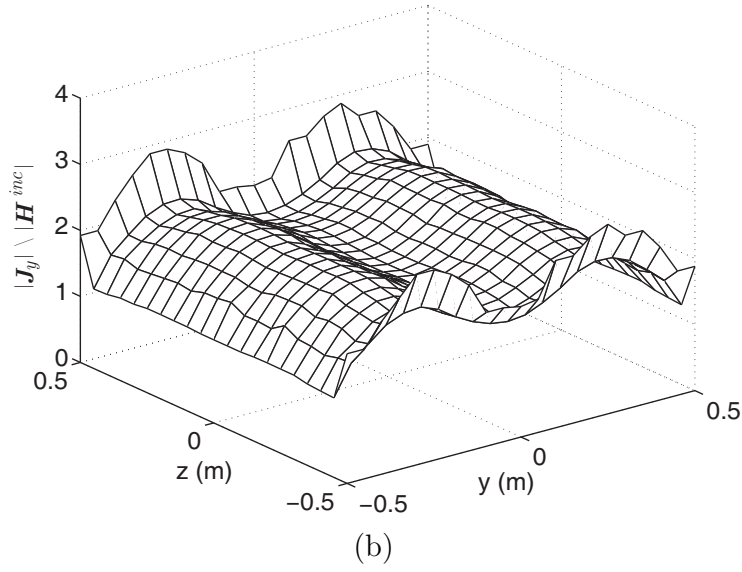
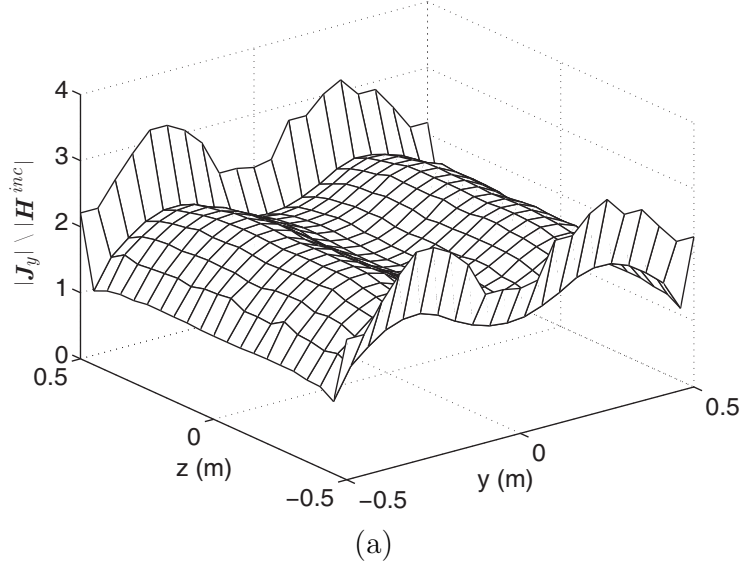


Figure 3.25: Magnitude of the y component of the normalized surface current ($|\mathbf{J}_y|/|\mathbf{H}^{inc}|$) induced on the front surface (at $x = 0.5$ m) of a PEC cube with edges of 2λ , obtained by using (a) EFIE with RWG and (b) MFIE with RWG.

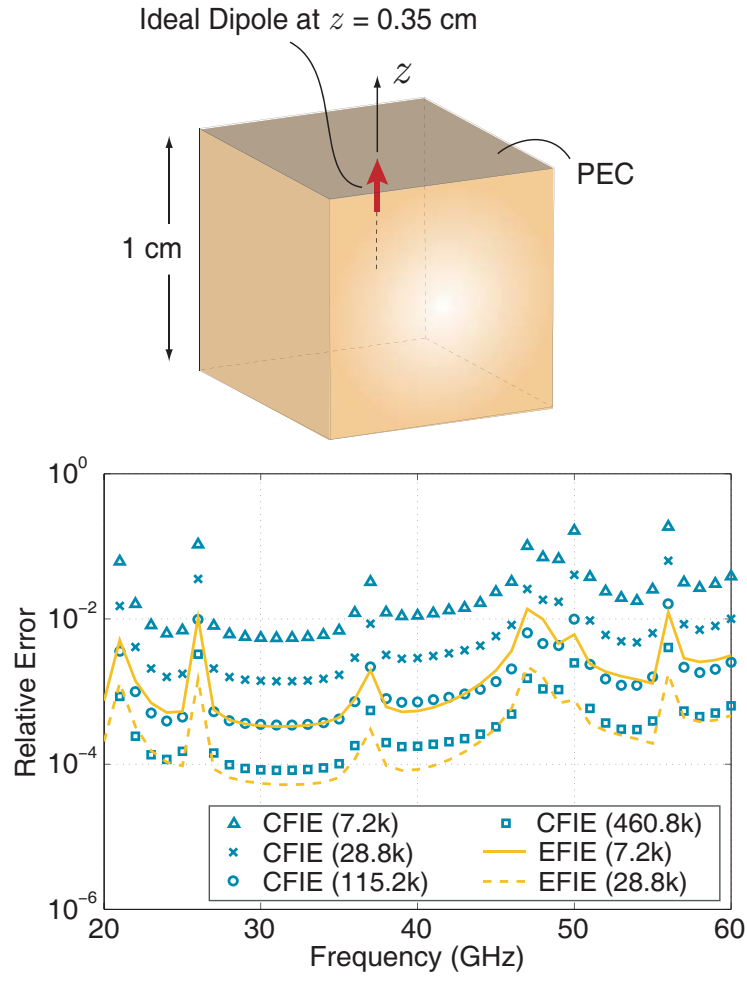


Figure 3.26: Solutions of a radiation problem involving a $1\text{ cm} \times 1\text{ cm} \times 1\text{ cm}$ PEC box excited by a Hertzian dipole located inside the box at $z = 0.35\text{ cm}$. The relative error is plotted as a function of frequency from 20 GHz to 60 GHz.

3.4 On the Accuracy of Normal and Mixed Formulations Discretized with the Rao-Wilton-Glisson Functions

Normal and mixed formulations, such as MFIE and CFIE, provide inaccurate results, compared to tangential formulations, such as EFIE. The discrepancy between results obtained with different formulations is evident especially when they are discretized with low-order basis functions, such as the RWG functions. Although the same physical current distribution is expanded by an identical set of RWG functions, the same level of accuracy cannot be obtained with EFIE, MFIE, and CFIE. Independent of the shape and the size of the geometry, computational results obtained with MFIE and CFIE employing RWG functions are plagued with a persistent inaccuracy problem.

In order to demonstrate the inaccuracy of MFIE and CFIE compared to EFIE discretized with the RWG functions, we consider scattering problems involving a PEC sphere with a radius of a . The sphere is illuminated by a plane wave at two different frequencies corresponding to $a = \lambda$ and $a = 2\lambda$. In addition to analytical Mie-series solutions, scattering problems are also solved with EFIE, MFIE, and CFIE. For this purpose, the surface of the sphere is discretized with various triangulations of coarse to fine densities, corresponding to 1000 to 30,000 unknowns, respectively. Figures 3.21 and 3.22 present back-scattered and forward-scattered RCS, respectively, obtained with various mesh sizes. The dots in the plots correspond to the $\lambda/10$ triangulation density at each frequency. Clearly, EFIE results readily converge to reference analytical results for mesh densities even smaller than $\lambda/10$, whereas MFIE results manifest significant discrepancies with analytical results not only for $\lambda/10$ triangulation, but also for much higher

mesh densities. CFIE solutions in Figures 3.21 and 3.22 are obtained by combining EFIE and MFIE with $\alpha = 0.2$. Even though back-scattered RCS values presented in Figure 3.21 do not display a significant inaccuracy for CFIE solutions, it should be emphasized that back-scattered and forward-scattered RCS in Figures 3.21 and 3.22 are obtained from the same CFIE solutions. Therefore, based on the comparison of Mie series, EFIE, and CFIE solutions presented in Figure 3.22, we conclude that CFIE solutions bear inaccuracies caused by MFIE.

In Figure 3.23, we consider scattering problems involving a PEC sphere with a radius of a , when a changes from 0.5λ to 1.5λ . The sphere is discretized with the RWG functions on 2 cm triangles. In two-dimensional plots, the relative error in bistatic RCS values obtained with EFIE, MFIE, and CFIE ($\alpha = 0.2$) formulations with respect to analytical Mie-series solutions is investigated as a function of bistatic angle and frequency. In plots, 0° and 180° correspond to back-scattering and forward-scattering directions, respectively. We also focus the relative error in the forward-scattering direction as a function of frequency in the last plot of Figure 3.23. In the two-dimensional plot, the relative error of EFIE is rarely above 10^{-2} , which occurs due to zero-crossings of the bistatic RCS. In the forward-scattering direction, the relative error of the EFIE solution is below 10^{-3} . On the other hand, MFIE and CFIE results obtained by the same discretization of the geometry have considerably larger errors. The excessive error exists both for MFIE and CFIE, but MFIE also suffers from internal-resonance problems, since null-space solutions of MFIE can radiate to far zone. Those resonances of MFIE appear in the last plot as peaks above a base formed by the CFIE curve.

Next, we consider scattering problems involving a PEC cube with edges of 1 m, when the frequency is 300 MHz and 600 MHz. The cube is located at the origin and it is illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. Various discretizations with

20 cm to 3 cm triangles lead to matrix equations with 500 to 18,000 unknowns. Figures 3.24 and 3.25 show the magnitude of the y component of the normalized surface current ($|\mathbf{J}_y|/|\mathbf{H}^{inc}|$) induced on the front surface of the cube (at $x = 0.5$ m). In Figure 3.24, where $d = \lambda$ and the triangulation size is about $\lambda/10$, and in Figure 3.25, where $d = 2\lambda$ and the triangulation size is about $\lambda/12.5$, we observe that the modelling of the surface current is different for EFIE and MFIE implementations using the RWG functions. Even though plots look similar, significant differences can be noticed by focusing on calculated values at $z = \pm 0.5$ m, where the induced current is expected to be singular. The combination of the geometry discretization and the RWG functions does not allow the computed current values to become singular at those two edges of the cube. Since the physically correct solution is singular, the more accurate computed solutions get, the higher values they would assume at those edges in order to match the singularity as closely as possible. Comparing Figure 3.24(a) to Figure 3.24(b) and Figure 3.25(a) to Figure 3.25(b), EFIE solutions are seen to have consistently higher values than MFIE solutions in spite of employing exactly the same discretization and the RWG functions. Since the only difference is in the integral-equation formulation, this comparison demonstrates the inaccuracy of MFIE relative to EFIE.

To further demonstrate the inaccuracy of CFIE compared to EFIE, Figure 3.26 presents results of a radiation problem involving a $1\text{ cm} \times 1\text{ cm} \times 1\text{ cm}$ PEC box located at the origin. As depicted in Figure 3.26, the box is excited with a Hertzian dipole oriented in the z direction and located inside the box at $z = 0.35$ cm. Ideally, the radiated field outside the box should be zero due to the shielding provided by the closed PEC surface. We calculate the radiated field in the far zone on the x - y plane at $(r = 3\text{ m}, \theta = \pi/2, \phi_n)$, where $\phi_n = (n - 1)\pi/180$ for $n = 1, 2, \dots, 360$. The relative error is defined as the 2-norm of the total electric field divided by the 2-norm of the incident electric field. The total electric field

is obtained by adding the incident field due to the Hertzian dipole and the secondary field due to the induced electric current on the cube. Figure 3.26 presents the relative error as a function of frequency from 20 GHz to 60 GHz. In this range of frequency, the size of the box varies from 0.67λ to 2λ . The radiation problem is discretized with 7200, 28,800, 115,200, and 460,800 unknowns, and solved by LF-MLFMA. We observe that CFIE with $\alpha = 0.2$ is significantly less accurate than EFIE. In order to obtain the same accuracy, the number of unknowns of CFIE should be 16 times larger than that of EFIE.

3.5 Alternative Implementations of the Magnetic-Field Integral Equation

Since EFIE in (3.1) contains electric charge density, i.e.,

$$\nabla \cdot \mathbf{J}(\mathbf{r}) = i\omega\rho_e(\mathbf{r}), \quad (3.47)$$

divergence-conforming basis functions must be used to expand the current density in EFIE. In other words, basis functions employed for current modelling should also be capable of properly modelling the inherent charge distribution. If an attempt was made to use basis functions that are not divergence conforming, the divergence in (3.47) would create singular blade functions on edges of the entire discretization that would mathematically undermine computations and that could physically be interpreted as artificial line-charge accumulation and gaps in the conducting surface. In addition, the hyper-singularity in EFIE is usually eliminated by moving the differential operator onto the testing function, as shown in (2.85)–(2.87). Then, testing functions are also restricted to be divergence conforming in EFIE.

In the conventional discretization of the normally-tested $\mathcal{K}$ operator in (2.81), the differential operator is kept on the Green's function. Therefore, there is no

any restriction on basis and testing functions in MFIE. On the other hand, one can move the differential operator from the Green's function onto either the basis function or the testing function. By using standard vector identities and the

$$\nabla' = \nabla'_S + \hat{\mathbf{n}}' \frac{\partial}{\partial n'} \quad (3.48)$$

decomposition, we can manipulate (2.81) to obtain

$$\begin{aligned} \bar{\mathbf{K}}_{PV}^N[m, n] = & \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \int_{PV, S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla'_S \times \mathbf{b}_n(\mathbf{r}') \\ & + \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \int_{PV, S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \times \hat{\mathbf{n}}' \frac{\partial g(\mathbf{r}, \mathbf{r}')}{\partial n'} \\ & - \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \int_{PV, S_n} d\mathbf{r}' \nabla'_S \times [g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n(\mathbf{r}')], \end{aligned} \quad (3.49)$$

where the differential operator is partially transferred onto the basis function. In this discretization of MFIE, the basis function is restricted to be curl conforming, however, no restriction exists on the testing function. Alternatively, (2.81) can be manipulated to obtain a third form as

$$\begin{aligned} \bar{\mathbf{K}}_{PV}^N[m, n] = & \int_{S_m} d\mathbf{r} \left[\nabla_S \times (\mathbf{t}_m(\mathbf{r}) \times \hat{\mathbf{n}}) \right] \cdot \int_{PV, S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') \\ & + \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \int_{PV, S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \frac{\partial g(\mathbf{r}, \mathbf{r}')}{\partial n} \\ & - \int_{S_m} d\mathbf{r} \nabla_S \cdot \left[(\mathbf{t}_m(\mathbf{r}) \times \hat{\mathbf{n}}) \times \int_{PV, S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') \right], \end{aligned} \quad (3.50)$$

where the differential operator is partially applied to the testing function. In this discretization, there is no restriction on the basis function; however, the testing function is restricted to be a divergence-conforming function since

$$\nabla_S \times (\mathbf{t}_m(\mathbf{r}) \times \hat{\mathbf{n}}) = -\hat{\mathbf{n}} \nabla \cdot \mathbf{t}_m(\mathbf{r}). \quad (3.51)$$

For divergence-conforming functions, such as the RWG functions, we use the expression in (2.81) in order to evaluate matrix elements. Although the same expression is also valid for curl-conforming functions, we prefer the modified expression in (3.49) for these functions. In (3.49), the inner integral of the third term can be manipulated into a line integral over edges around the basis function. In addition, the first term in (3.49) is usually simplified by using the expression for the curl of basis functions.

3.6 Improving the Accuracy with Curl-Conforming Basis Functions

Our careful investigations on the inaccuracy of MFIE showed that neither the numerical evaluation of the singular MFIE integrals, nor the incorrect use of the solid-angle factor, is the fundamental cause of the inaccuracy [56],[61],[157]–[159]. This finding stimulates further studies, especially for a better understanding of how the discretization of MFIE and the modelling of the surface current affects the accuracy of results. In this section, we present implementations of MFIE with the curl-conforming $\hat{\mathbf{n}} \times \text{RWG}$ basis functions. We show that the accuracy of MFIE can be improved significantly by employing $\hat{\mathbf{n}} \times \text{RWG}$ functions, instead of the conventional RWG functions.

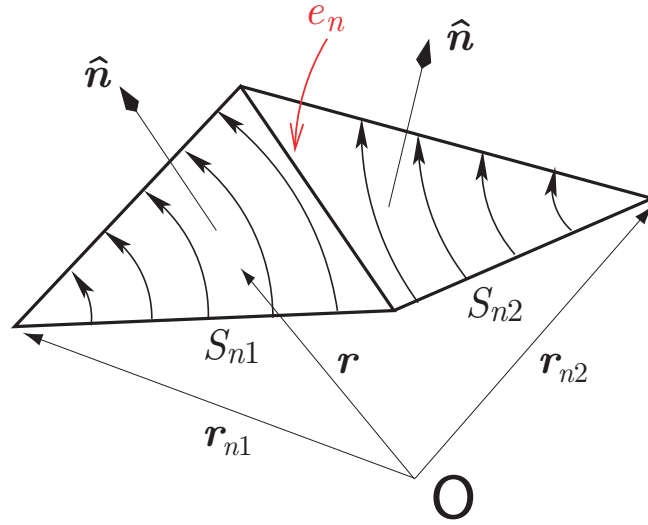


Figure 3.27: Spatial distribution of the $\hat{\mathbf{n}} \times \text{RWG}$ functions.

3.6.1 Matrix Elements

The $\hat{\mathbf{n}} \times \text{RWG}$ functions have a spatial distribution shown in Figure 3.27, which can be expressed as

$$\mathbf{b}_n^{\text{nRWG}}(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{b}_n^{\text{RWG}}(\mathbf{r}), \quad (3.52)$$

where $\mathbf{b}_n^{\text{RWG}}(\mathbf{r})$ is the associated RWG function shown in Figure 2.1 and defined in (2.82). Using (3.52), matrix elements are derived as

$$\bar{\mathbf{Z}}_M^{\text{nRWG}}[m, n] = -\frac{1}{2} \bar{\mathbf{I}}^{T, \text{nRWG}}[m, n] + \bar{\mathbf{K}}_{PV}^{N, \text{nRWG}}[m, n] \quad (3.53)$$

$$= -\frac{1}{2} \bar{\mathbf{I}}^{T, \text{RWG}}[m, n] - \bar{\mathbf{K}}_{PV}^{N, \text{RWG}}[n, m] \quad (3.54)$$

for $m, n = 1, 2, \dots, N$, where $\bar{\mathbf{K}}_{PV}^{N, \text{RWG}}$ and $\bar{\mathbf{I}}^{T, \text{RWG}}$ are normally-tested $\mathcal{K}$ and tangentially-tested $\mathcal{I}$ operators discretized with the RWG functions. We note that interactions calculated for the $\hat{\mathbf{n}} \times \text{RWG}$ and the RWG functions are closely related. When basis and testing functions related to edges m and n do not overlap in space, only the principal-value term remains in (3.53) and (3.54), and we obtain

$$\bar{\mathbf{Z}}_M^{\text{nRWG}}[m, n] = \bar{\mathbf{K}}_{PV}^{N, \text{nRWG}}[m, n] = -\bar{\mathbf{K}}_{PV}^{N, \text{RWG}}[n, m] = -\bar{\mathbf{Z}}_M^{\text{RWG}}[n, m], \quad (3.55)$$

where $\bar{\mathbf{Z}}_M^{\text{RWG}}$ represents matrix elements for the RWG functions.

One can also derive an alternative implementation for the $\hat{\mathbf{n}} \times \text{RWG}$ functions by considering (3.49), i.e.,

$$\begin{aligned} \bar{\mathbf{Z}}_M^{\text{nRWG}}[m, n] &= \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{nRWG}}(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \int_{PV, S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla'_S \times \mathbf{b}_n^{\text{nRWG}}(\mathbf{r}') \\ &\quad + \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{nRWG}}(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \int_{PV, S_n} d\mathbf{r}' \mathbf{b}_n^{\text{nRWG}}(\mathbf{r}') \times \hat{\mathbf{n}}' \frac{\partial g(\mathbf{r}, \mathbf{r}')}{\partial n'} \\ &\quad - \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{nRWG}}(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \int_{PV, S_n} d\mathbf{r}' \nabla'_S \times \left[g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n^{\text{nRWG}}(\mathbf{r}') \right] \\ &\quad - \frac{1}{2} \bar{\mathbf{I}}^{T, \text{nRWG}}[m, n]. \end{aligned} \quad (3.56)$$

Using (3.52) in (3.56), we obtain

$$\begin{aligned}
\bar{\mathbf{Z}}_M^{\text{nRWG}}[m, n] &= \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \cdot \int_{PV, S_n} d\mathbf{r}' \hat{\mathbf{n}}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \\
&+ \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \cdot \int_{PV, S_n} d\mathbf{r}' \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \frac{\partial g(\mathbf{r}, \mathbf{r}')}{\partial n'} \\
&- \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \cdot \int_{PV, S_n} d\mathbf{r}' \hat{\mathbf{n}}' \nabla' \cdot \left[g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \right] \\
&- \frac{1}{2} \bar{\mathbf{I}}^{T, \text{RWG}}[m, n].
\end{aligned} \tag{3.57}$$

3.6.2 Direct Calculation of Interactions

For the direct calculation of interactions between the $\hat{\mathbf{n}} \times \text{RWG}$ functions, we prefer the alternative implementation in (3.57). Consider the interaction of two half $\hat{\mathbf{n}} \times \text{RWG}$ functions $\mathbf{t}_{ma}^{\text{nRWG}}$ and $\mathbf{b}_{nb}^{\text{nRWG}}$ associated with the a th triangle of the m th edge and the b th triangle of the n th edge, respectively. Then,

$$\begin{aligned}
\bar{\mathbf{K}}_{PV}^{N, \text{nRWG}}[m, n, a, b] &= \mathcal{A}_{ma, nb} \left\{ 2 \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \int_{PV, S_{nb}} d\mathbf{r}' \hat{\mathbf{n}}' g(\mathbf{r}, \mathbf{r}') \right. \\
&+ \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \int_{PV, S_{nb}} d\mathbf{r}' (\mathbf{r}' - \mathbf{r}_{nb}) \frac{\partial g(\mathbf{r}, \mathbf{r}')}{\partial n'} \\
&\left. - \int_{S_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \int_{PV, S_{nb}} d\mathbf{r}' \hat{\mathbf{n}}' \nabla' \cdot \left[g(\mathbf{r}, \mathbf{r}') (\mathbf{r}' - \mathbf{r}_{nb}) \right] \right\},
\end{aligned} \tag{3.58}$$

where $\mathcal{A}_{ma, nb}$ is defined in (2.94). After a coordinate transformation, we obtain

$$\begin{aligned}
\bar{\mathbf{K}}_{PV}^{N, \text{nRWG}}[m, n, a, b] &= \mathcal{A}_{ma, nb} \left\{ 2 \int_{S_{ma}} d\mathbf{r} (z - z_{ma}) \int_{PV, S_{nb}} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \right. \\
&+ \int_{S_{ma}} d\mathbf{r} (\boldsymbol{\rho} - \boldsymbol{\rho}_{ma}) \cdot \int_{PV, S_{nb}} d\mathbf{r}' (\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb}) \frac{\partial g(\mathbf{r}, \mathbf{r}')}{\partial z} \\
&\left. - \int_{S_{ma}} d\mathbf{r} (z - z_{ma}) \int_{PV, S_{nb}} d\mathbf{r}' \nabla' \cdot \left[g(\mathbf{r}, \mathbf{r}') (\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb}) \right] \right\}.
\end{aligned} \tag{3.59}$$

Using the surface divergence theorem, the inner integral of the third term in (3.59) can be converted into a line integral, i.e.,

$$\begin{aligned} & \int_{S_{ma}} d\mathbf{r}(z - z_{ma}) \int_{PV, S_{nb}} d\mathbf{r}' \nabla' \cdot \left[g(\mathbf{r}, \mathbf{r}')(\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb}) \right] \\ &= \int_{S_{ma}} d\mathbf{r}(z - z_{ma}) \int_{PV, \partial S_{nb}} d\mathbf{r}' \hat{\mathbf{u}}' \cdot (\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb}) g(\mathbf{r}, \mathbf{r}'), \end{aligned} \quad (3.60)$$

where ∂S_{nb} represents the contour around the basis triangle and $\hat{\mathbf{u}}'$ is the unit vector directed perpendicularly outward the triangle. We note that

$$\hat{\mathbf{u}}' \cdot (\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb}) = \frac{2A_n}{l_n} \quad (3.61)$$

on the n th edge, but it evaluates to zero on the two other edges of the basis triangle. Then, (3.59) becomes

$$\begin{aligned} \bar{\mathbf{K}}_{PV}^{N, n\text{RWG}}[m, n, a, b] &= \mathcal{A}_{ma, nb} \left\{ 2 \int_{S_{ma}} d\mathbf{r}(z - z_{ma}) \int_{PV, S_{nb}} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \right. \\ &\quad + \int_{S_{ma}} d\mathbf{r}(\boldsymbol{\rho} - \boldsymbol{\rho}_{ma}) \cdot \int_{PV, S_{nb}} d\mathbf{r}'(\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb}) \frac{\partial g(\mathbf{r}, \mathbf{r}')}{\partial z} \\ &\quad \left. - \frac{2A_n}{l_n} \int_{S_{ma}} d\mathbf{r}(z - z_{ma}) \int_{PV, e_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \right\}. \end{aligned} \quad (3.62)$$

The line integral of the Green's function along the n th edge e_n is singular when the observation point approaches to this edge. Similar to (2.83) and (2.84), the singularity is logarithmic and do not create numerical problems when observation points are sampled inside the testing triangle for near-neighboring interactions. Nevertheless, a singularity-extraction method can be used to improve the evaluation of those interactions [157]. In addition, the order of integrals can be exchanged as [94]

$$\int_{S_{ma}} d\mathbf{r}(z - z_{ma}) \int_{PV, e_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') = \int_{PV, e_n} d\mathbf{r}' \int_{S_{ma}} d\mathbf{r}(z - z_{ma}) g(\mathbf{r}, \mathbf{r}'), \quad (3.63)$$

where the inner integral is now bounded for every case including the sampling of testing points on the edge of the basis triangle.

The interaction of two half $\hat{\mathbf{n}} \times \text{RWG}$ functions can be divided into basic integrals as

$$\begin{aligned} \bar{\mathbf{K}}_{PV}^{N,\text{nRWG}}[m, n, a, b] = \mathcal{A}_{ma,nb} \bigg\{ & -x_{ma}I_6 - y_{ma}I_7 + I_8 + I_9 \\ & + (x_{ma}x_{nb} + y_{ma}y_{nb})I_3 - x_{nb}I_4 - y_{nb}I_5 - 2(z_{ma}I_1 - I_2) \\ & - \frac{2A_{nb}}{l_n}(I_{11} - z_{ma}I_{10}) \bigg\}, \end{aligned} \quad (3.64)$$

where

$$I_{\{1,2\}} = \int_{S_{ma}} d\mathbf{r} \{1, z\} I_{in}^1 \quad (3.65)$$

$$I_{\{3,4,5\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y\} \hat{\mathbf{z}} \cdot \mathbf{I}_{in}^{\nabla'} \quad (3.66)$$

$$I_{\{6,8\}} = \int_{S_{ma}} d\mathbf{r} \{1, x\} \hat{\mathbf{z}} \cdot \mathbf{I}_{in}^{x'\nabla'} \quad (3.67)$$

$$I_{\{7,9\}} = \int_{S_{ma}} d\mathbf{r} \{1, y\} \hat{\mathbf{z}} \cdot \mathbf{I}_{in}^{y'\nabla'} \quad (3.68)$$

$$I_{\{10,11\}} = \int_{S_{ma}} d\mathbf{r} \{1, z\} I_{in}^{e,1}. \quad (3.69)$$

Inner integrals I_{in}^1 and $\mathbf{I}_{in}^{\nabla'}$ are evaluated as in (2.127)–(2.130). In addition, we compute

$$\mathbf{I}_{in}^{x'\nabla'} = \mathbf{I}_{in}^{(x'-x)\nabla'} + x\mathbf{I}_{in}^{\nabla'} \quad (3.70)$$

$$\mathbf{I}_{in}^{y'\nabla'} = \mathbf{I}_{in}^{(y'-y)\nabla'} + y\mathbf{I}_{in}^{\nabla'}, \quad (3.71)$$

where

$$\begin{aligned} \mathbf{I}_{in}^{(x'-x)\nabla'} = & \frac{1}{4\pi} \int_{S_{nb}} d\mathbf{r}' (x' - x) \nabla' \left(\frac{\exp(ikR) - 1 + 0.5k^2R^2}{R} \right) \\ & + \frac{1}{4\pi} \int_{S_{nb}} d\mathbf{r}' (x' - x) \nabla' \left(\frac{1}{R} \right) - \frac{k^2}{8\pi} \int_{S_{nb}} d\mathbf{r}' (x' - x) \nabla' R \end{aligned} \quad (3.72)$$

and

$$\begin{aligned} \mathbf{I}_{in}^{(y'-y)\nabla'} = & \frac{1}{4\pi} \int_{S_{nb}} d\mathbf{r}' (y' - y) \nabla' \left(\frac{\exp(ikR) - 1 + 0.5k^2R^2}{R} \right) \\ & + \frac{1}{4\pi} \int_{S_{nb}} d\mathbf{r}' (y' - y) \nabla' \left(\frac{1}{R} \right) - \frac{k^2}{8\pi} \int_{S_{nb}} d\mathbf{r}' (y' - y) \nabla' R. \end{aligned} \quad (3.73)$$

We note that only z components of $\mathbf{I}_{in}^{\nabla'}$, $\mathbf{I}_{in}^{(x'-x)\nabla'}$, and $\mathbf{I}_{in}^{(y'-y)\nabla'}$ are required. Finally, basic integrals I_{10} and I_{11} involve a line integral

$$I_{in}^{e,1} = \frac{1}{4\pi} \int_{e_n} d\mathbf{r}' \frac{\exp(ikR)}{R}, \quad (3.74)$$

which depends on the orientation of the basis function, as opposed to other inner integrals.

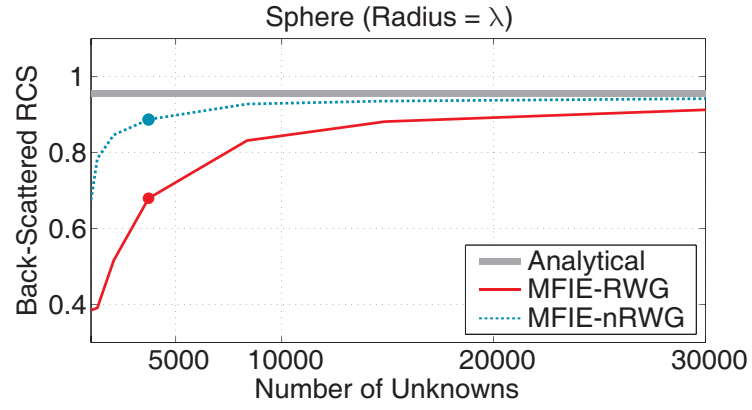
3.6.3 Calculation of Interactions Using the Multilevel Fast Multipole Algorithm

Consider a half $\hat{\mathbf{n}} \times \text{RWG}$ function associated with the first triangle of the m th edge. Radiation and receiving patterns required for MLFMA implementations employing the $\hat{\mathbf{n}} \times \text{RWG}$ functions are derived as

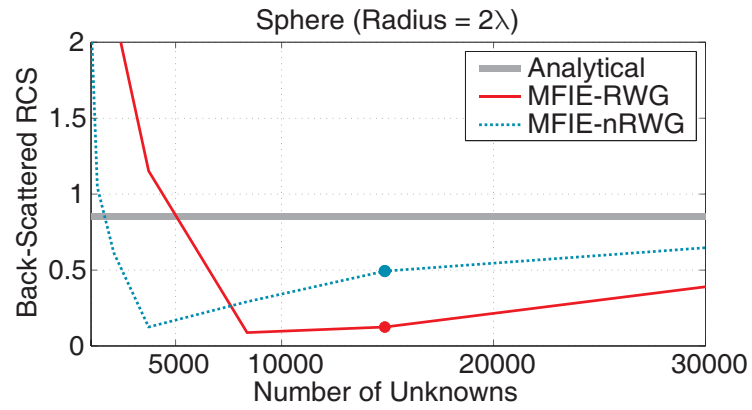
$$\mathbf{S}_{m1}^{\text{nRWG}}(\mathbf{r}_C, \mathbf{k}) = \frac{l_m}{2A_{m1}} (\bar{\mathbf{I}}^{3 \times 3} - \hat{\mathbf{k}}\hat{\mathbf{k}}). \quad (3.75)$$

$$\mathbf{F}_{m1}^{\mathbf{K},N,\text{nRWG}}(\mathbf{r}_C, \mathbf{k}) = -\frac{l_m}{2A_{m1}} \hat{\mathbf{k}} \times \int_{S_{m1}} d\mathbf{r} \exp[-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_C)] (\mathbf{r} - \mathbf{r}_{m1}), \quad (3.76)$$

where integrals are the same as those in (2.144) and (2.145).

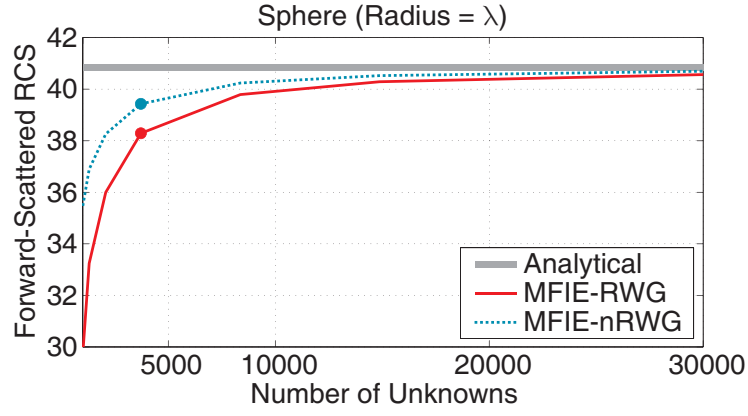


(a)

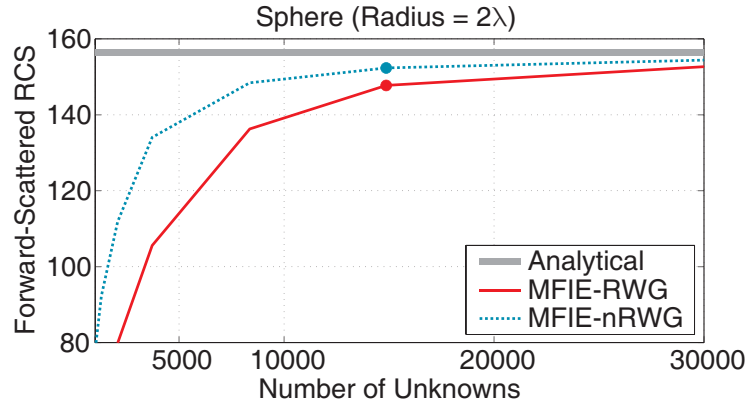


(b)

Figure 3.28: Normalized back-scattered RCS ($\text{RCS}/\pi a^2$) of a PEC sphere with a radius of (a) λ and (b) 2λ as a function of the number of unknowns. The dots on the curves correspond to the $\lambda/10$ discretization.



(a)



(b)

Figure 3.29: Normalized forward-scattered RCS ($\text{RCS}/\pi a^2$) of a PEC sphere with a radius of (a) λ and (b) 2λ as a function of the number of unknowns. The dots on the curves correspond to the $\lambda/10$ discretization.

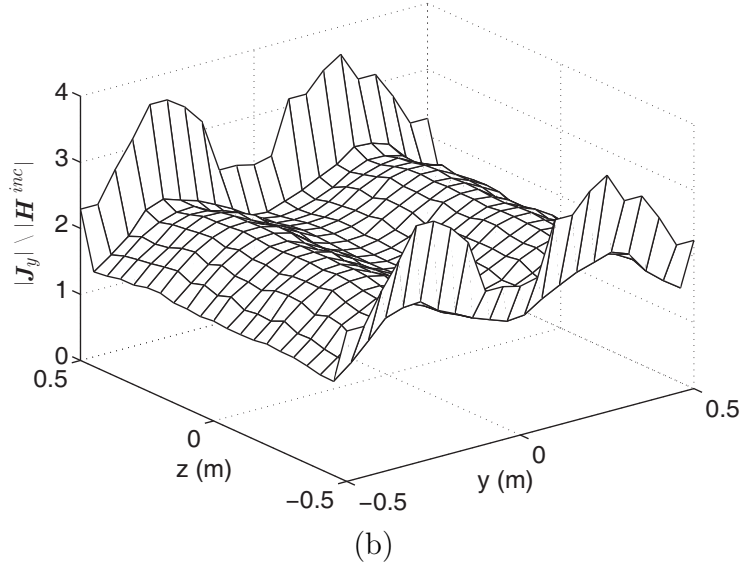
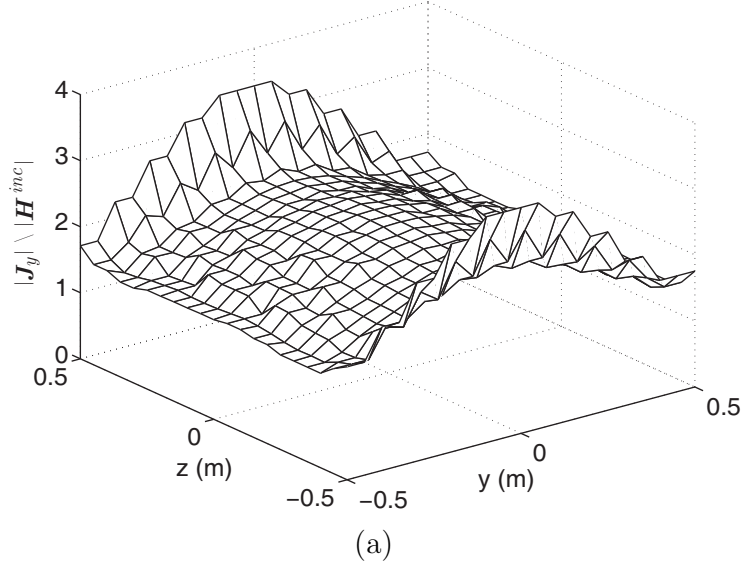
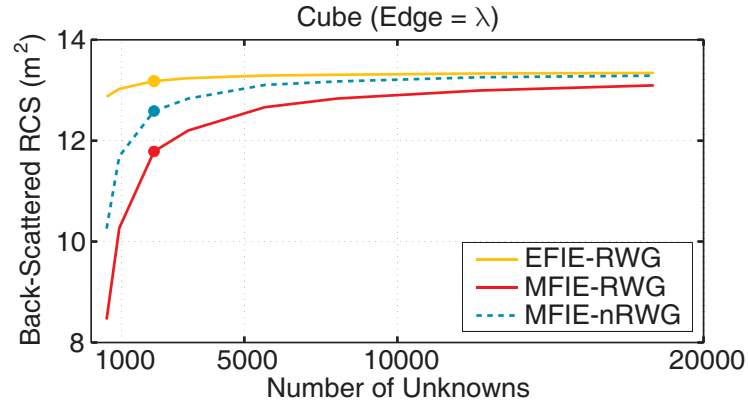
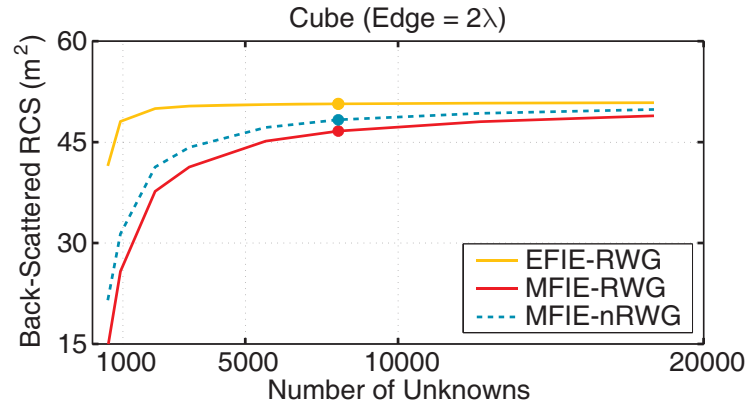


Figure 3.30: Magnitude of the y component of the normalized surface current ($|\mathbf{J}_y|/|\mathbf{H}^{inc}|$) induced on the front surface (at $x = 0.5$ m) of a PEC cube with edges of (a) λ and (b) 2λ , obtained by using MFIE with $\hat{\mathbf{n}} \times \text{RWG}$.

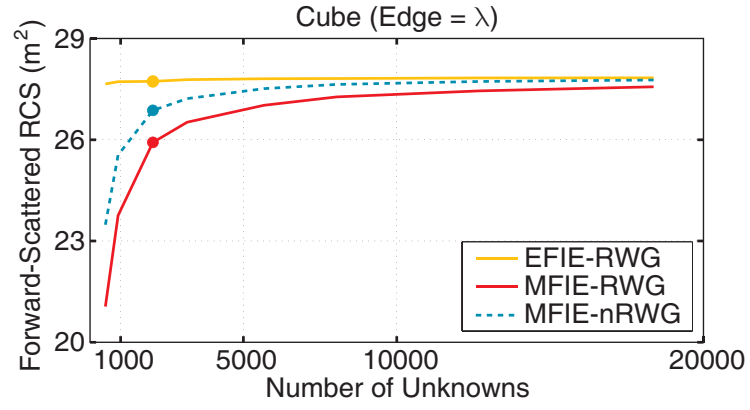


(a)

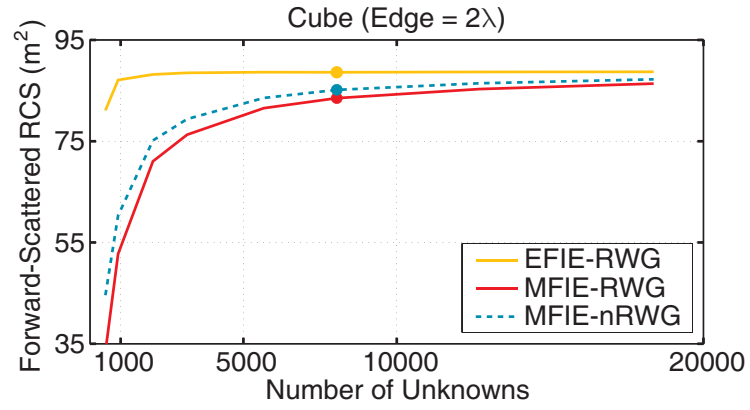


(b)

Figure 3.31: Back-scattered RCS (m^2) of a PEC cube with edges of (a) λ and (b) 2λ as a function of the number of unknowns. The dots on the curves correspond to the $\lambda/10$ discretization.



(a)



(b)

Figure 3.32: Forward-scattered RCS (m²) of a PEC cube with edges of (a) λ and (b) 2λ as a function of the number of unknowns. The dots on the curves correspond to the $\lambda/10$ discretization.

3.6.4 Numerical Results

To compare the accuracy of solutions obtained by using the conventional RWG and the $\hat{\mathbf{n}} \times \text{RWG}$ functions, we consider scattering problems involving a PEC sphere, when the radius of the sphere is λ and 2λ . Figures 3.28 and 3.29 illustrate normalized back-scattered and forward-scattered RCS ($\text{RCS}/\pi a^2$, where a is the radius of the sphere), respectively, obtained by FMM implementations employing the RWG and the $\hat{\mathbf{n}} \times \text{RWG}$ functions. Scattering problems are solved for various triangulations involving approximately 1000 to 30,000 unknowns. In both figures, we observe that, as N increases, RCS values obtained with the $\hat{\mathbf{n}} \times \text{RWG}$ functions converge to exact Mie-series results faster than those obtained with the RWG functions, i.e., RCS values obtained with the $\hat{\mathbf{n}} \times \text{RWG}$ functions are more accurate for all discretization densities. The improvement obtained with the $\hat{\mathbf{n}} \times \text{RWG}$ functions over the RWG functions is significant even for the $\lambda/10$ discretization, which is commonly used as a rule of thumb in numerical solutions.

Next, we revisit the solution of scattering problems involving a PEC cube with edges of 1 m, when the frequency is 300 MHz and 600 MHz. Figures 3.30(a) and 3.30(b) show that, using the $\hat{\mathbf{n}} \times \text{RWG}$ functions, the modelling of the surface current is visibly improved compared to the RWG functions depicted in Figures 3.24 and 3.25, especially in the representation of singular currents flowing along edges. We observe that the use of the $\hat{\mathbf{n}} \times \text{RWG}$ functions in the MFIE solution provides higher values at $z = \pm 0.5$ m than the use of the RWG functions with either EFIE or MFIE. It should be remarked that this is merely an indicator for the improved accuracy of MFIE with the use of the $\hat{\mathbf{n}} \times \text{RWG}$ functions, but not an absolute proof since the exact current distribution is not available for comparisons. Nevertheless, RCS results are also supporting inferences obtained from current solutions. Figures 3.31 and 3.32 demonstrate back-scattered and forward-scattered RCS (in m^2), respectively, when edges of the cube are λ and 2λ . Comparing MFIE results, we observe faster convergence (with respect to the

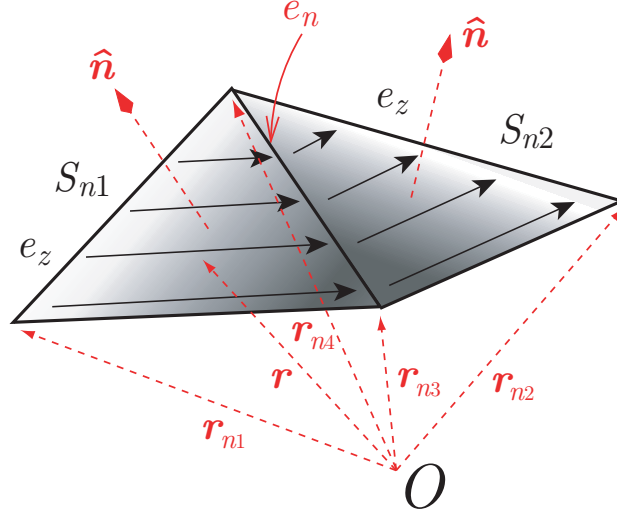
improved discretization and the increasing N) of RCS values with the $\hat{\mathbf{n}} \times \text{RWG}$ functions. The improvement is significant especially for relatively coarse discretizations, even for the commonly used $\lambda/10$ triangulation, as marked on the curves.

3.7 Improving the Accuracy with LN-LT Type Basis Functions

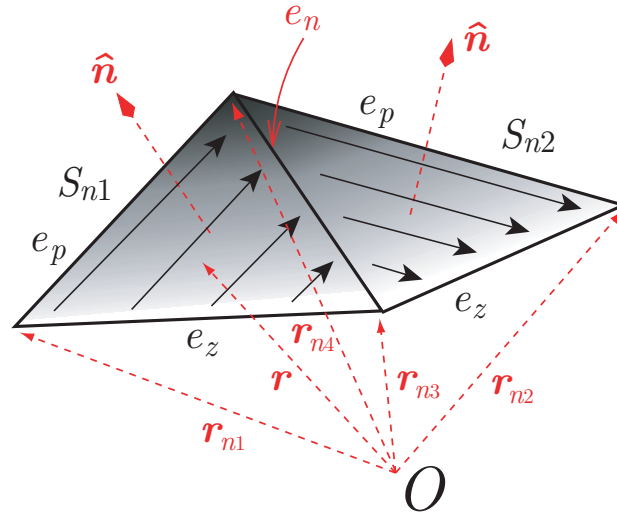
In this section, we present LN-LT type basis functions to further improve the accuracy of MFIE and CFIE implementations. Specifically, we introduce divergence-conforming LL and curl-conforming $\hat{\mathbf{n}} \times \text{LL}$ functions, which allow linear variation of the current along edges of triangulations [60]. We show that LN-LT type functions improve the accuracy of MFIE and CFIE to levels of EFIE, without increasing computational requirements. Improving the accuracy of normal and mixed formulations of dielectric objects with the LL functions is presented in [37].

Table 3.7: Linear Basis and Testing Functions

Function	Conformity	Type	Function/Edge
RWG	Divergence	CN-LT	1
$\hat{\mathbf{n}} \times \text{RWG}$	Curl	LN-CT	1
LL	Divergence	LN-LT	2
$\hat{\mathbf{n}} \times \text{LL}$	Curl	LN-LT	2

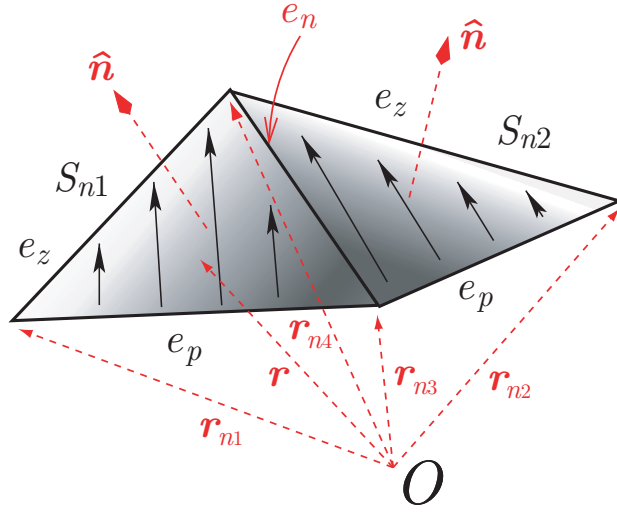


(a)

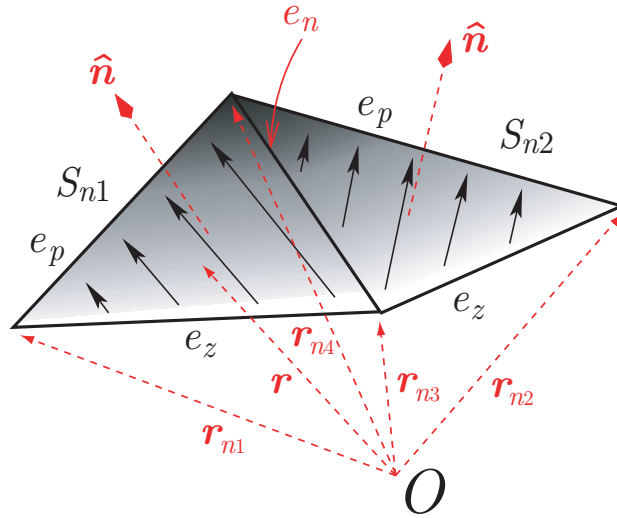


(b)

Figure 3.33: (a) First-kind and (b) second-kind LL functions defined at the n th edge (e_n). Arrows show the direction while the shading indicates the magnitude of the vector distribution. Light and dark colors represent the low and high values, respectively, and white corresponds to zero.



(a)



(b)

Figure 3.34: (a) First-kind and (b) second-kind $\hat{n} \times \text{LL}$ functions defined at the n th edge (e_n). Arrows show the direction while the shading indicates the magnitude of the vector distribution. Light and dark colors represent the low and high values, respectively, and white corresponds to zero.

3.7.1 Linear Basis Functions

Table 3.7 lists functions under investigation in this section with some of their essential properties. For comparisons, we consider the conventional divergence-conforming RWG functions, as well as the curl-conforming $\hat{\mathbf{n}} \times$ RWG functions presented in Section 3.6. The RWG functions can be classified as CN-LT type [31], since their spatial distribution depicted in Figure 2.1 has constant normal (CN) and linear tangential (LT) components at the main edge. On the other hand, the $\hat{\mathbf{n}} \times$ RWG functions having spatial distributions shown in Figure 3.27 are LN-CT (linear normal, constant tangential) type. Besides the RWG and the $\hat{\mathbf{n}} \times$ RWG functions, we investigate the divergence-conforming LL functions. Spatial distributions of the LL functions are depicted in Figure 3.33, where two kinds of LL functions are defined on the same edge e_n with expressions

$$\mathbf{b}_n^{\text{LL}(1)}(\mathbf{r}) = \begin{cases} \frac{l_n}{4(A_{n1})^2}(\mathbf{r} - \mathbf{r}_{n1}) \cdot [(\mathbf{r}_{n4} - \mathbf{r}_{n1}) \times \hat{\mathbf{n}}](\mathbf{r}_{n3} - \mathbf{r}_{n1}), & r \in S_{n1} \\ \frac{l_n}{4(A_{n2})^2}(\mathbf{r} - \mathbf{r}_{n2}) \cdot [(\mathbf{r}_{n4} - \mathbf{r}_{n2}) \times \hat{\mathbf{n}}](\mathbf{r}_{n3} - \mathbf{r}_{n2}), & r \in S_{n2} \\ 0, & r \notin S_n \end{cases} \quad (3.77)$$

for the LL function of the first kind, and

$$\mathbf{b}_n^{\text{LL}(2)}(\mathbf{r}) = \begin{cases} \frac{l_n}{4(A_{n1})^2}(\mathbf{r} - \mathbf{r}_{n1}) \cdot [(\mathbf{r}_{n3} - \mathbf{r}_{n1}) \times \hat{\mathbf{n}}](\mathbf{r}_{n4} - \mathbf{r}_{n1}), & r \in S_{n1} \\ \frac{l_n}{4(A_{n2})^2}(\mathbf{r} - \mathbf{r}_{n2}) \cdot [(\mathbf{r}_{n3} - \mathbf{r}_{n2}) \times \hat{\mathbf{n}}](\mathbf{r}_{n4} - \mathbf{r}_{n2}), & r \in S_{n2} \\ 0, & r \notin S_n \end{cases} \quad (3.78)$$

for the LL function of the second kind. In Figure 3.33, vertex locations $\mathbf{r}_{n1}$, $\mathbf{r}_{n2}$, $\mathbf{r}_{n3}$, and $\mathbf{r}_{n4}$ are labeled as well as the three edges of the triangles e_n , e_p , and e_z depending on the kind of the LL function. For both kinds of the LL functions, the spatial distribution

- is parallel to one of the side edges e_p over both of the triangles,

- changes linearly along the edge e_p and becomes maximum at the intersection of that edge with the main edge e_n ,
- is exactly zero at the other side edge e_z ,
- has a continuous normal component across the main edge e_n , and
- varies linearly at the main edge e_n in both perpendicular and tangential directions.

Because of the final property above, LL functions are classified as LN-LT type, providing linear variation along the main edge in both directions.

The LL functions can actually decompose the corresponding RWG function as

$$\mathbf{b}_n^{\text{RWG}}(\mathbf{r}) = \mathbf{b}_n^{\text{LL}(1)}(\mathbf{r}) + \mathbf{b}_n^{\text{LL}(2)}(\mathbf{r}), \quad (3.79)$$

where $\mathbf{b}_n^{\text{RWG}}$ represents the RWG function associated with the n th edge. The charge distribution implied by expressions in (3.77) and (3.78) is

$$\nabla \cdot \mathbf{b}_n^{\text{LL}(1)}(\mathbf{r}) = \nabla \cdot \mathbf{b}_n^{\text{LL}(2)}(\mathbf{r}) = \begin{cases} \frac{l_n}{2A_{n1}}, & r \in S_{n1} \\ \frac{-l_n}{2A_{n2}}, & r \in S_{n2} \\ 0, & \text{otherwise,} \end{cases} \quad (3.80)$$

which is constant and exactly half of that for the RWG functions. Equation (3.80) indicates that the LL functions are divergence-conforming since their divergence is finite. Although this property is not crucial for MFIE, which also allows the use of curl-conforming functions depending on the formulation type, divergence-conforming property of the LL functions becomes essential for CFIE implementations.

The LL functions can also be identified as a subset of LN/QT or QN/LT type vector functions discussed in [31]. Contrary to the RWG functions, the LL functions are first-order complete to represent vectors by providing six degrees of

freedom to model the linear variation on triangles. As depicted in Figure 3.33, there are two kinds of LL functions defined on the same edge simultaneously to expand the current density better than the RWG functions that are only zeroth-order complete. Improved current modelling with higher-order completeness is achieved at the cost of doubling the number of basis functions compared to the RWG functions for the same triangulation of the geometry. On the other hand, we show that the LL functions provide more accurate RCS results with MFIE and CFIE for the same number of unknowns as the RWG functions. The improvement is present even for curved geometries, where the discretization for the LL functions is coarser and leads to more deviation from the actual geometry to keep the number of unknowns approximately the same as when the RWG functions are used on the same geometry with a finer triangulation.

Although the LL functions are first-order complete to represent vectors, they are only zeroth-order complete to represent the divergence or curl of a vector. The LL functions improve the accuracy of MFIE to levels of EFIE, while they are similar to the RWG functions by providing linear variations for the induced current on triangular domains. In this manner, the LL functions are different from higher-order techniques in computational electromagnetics appeared in the literature to improve the accuracy of integral equations [57]–[59]. Advantages of the LL functions can be listed as follows:

- The LL functions can be implemented with simple modifications on existing codes employing the RWG functions. Most of numerical procedures in MLFMA, such as singularity extractions for near-field interactions and Fourier transforms for far-field interactions, can be adapted from those of the RWG functions.

- For the same number of unknowns, employing the LL functions in MLFMA implementations does not require extra computational load compared to employing the RWG functions.

Finally, the curl-conforming $\hat{\mathbf{n}} \times \text{LL}$ functions are derived from the LL functions as

$$\begin{aligned}\mathbf{b}^{\text{nLL}(1)}(\mathbf{r}) &= \hat{\mathbf{n}} \times \mathbf{b}^{\text{LL}(1)}(\mathbf{r}) \\ \mathbf{b}^{\text{nLL}(2)}(\mathbf{r}) &= \hat{\mathbf{n}} \times \mathbf{b}^{\text{LL}(2)}(\mathbf{r}),\end{aligned}\tag{3.81}$$

and we note that

$$\begin{aligned}\nabla_S \times \mathbf{b}^{\text{nLL}(1)}(\mathbf{r}) &= \hat{\mathbf{n}} \nabla \cdot \mathbf{b}^{\text{LL}(1)}(\mathbf{r}) \\ \nabla_S \times \mathbf{b}^{\text{nLL}(2)}(\mathbf{r}) &= \hat{\mathbf{n}} \nabla \cdot \mathbf{b}^{\text{LL}(2)}(\mathbf{r}).\end{aligned}\tag{3.82}$$

Similar to the LL functions, the $\hat{\mathbf{n}} \times \text{LL}$ functions are also defined in pairs located at the same edge with spatial distributions illustrated in Figure 3.34. For both kinds of the $\hat{\mathbf{n}} \times \text{LL}$ functions, the spatial distribution

- is perpendicular to one of the side edges e_p over both of the triangles,
- changes linearly along the edge e_p and becomes maximum at the intersection of that edge with the main edge e_n ,
- is exactly zero at the other side edge e_z ,
- has a continuous normal component across the main edge e_n , and
- varies linearly at the main edge e_n in both perpendicular and tangential directions.

Similar to the LL functions, the $\hat{\mathbf{n}} \times \text{LL}$ functions are LN-LT type.

3.7.2 Direct Calculation of Interactions

The divergence-conforming LL functions can be employed for EFIE, MFIE, and CFIE. As an example, we consider the interaction of two half LL functions $\mathbf{t}_{ma}^{\text{LL}(1)}$ and $\mathbf{b}_{nb}^{\text{LL}(1)}$ associated with the a th triangle of the m th edge and the b th triangle of the n th edge, respectively. We assume that both functions are the first kind. Using (3.77) and (3.80), matrix elements are derived as

$$\bar{\mathbf{Z}}_E^{\text{LL}(1)}[m, n, a, b] = \bar{\mathbf{T}}^{T, \text{LL}(1)}[m, n, a, b] \quad (3.83)$$

$$\bar{\mathbf{Z}}_M^{\text{LL}(1)}[m, n, a, b] = \bar{\mathbf{K}}_{PV}^{N, \text{LL}(1)}[m, n, a, b] - \frac{1}{2} \bar{\mathbf{I}}^{T, \text{LL}(1)}[m, n, a, b], \quad (3.84)$$

where

$$\begin{aligned} \bar{\mathbf{I}}^{T, \text{LL}(1)}[m, n, a, b] &= \mathcal{B}_{ma, nb}(\mathbf{r}_{m3} - \mathbf{r}_{ma}) \cdot (\mathbf{r}_{n3} - \mathbf{r}_{nb}) \\ &\quad \int_{S_{ma}} d\mathbf{r}(\mathbf{r} - \mathbf{r}_{ma}) \cdot [(\mathbf{r}_{m4} - \mathbf{r}_{ma}) \times \hat{\mathbf{n}}] (\mathbf{r} - \mathbf{r}_{nb}) \cdot [(\mathbf{r}_{n4} - \mathbf{r}_{nb}) \times \hat{\mathbf{n}}] \end{aligned} \quad (3.85)$$

$$\begin{aligned} \bar{\mathbf{T}}^{T, \text{LL}(1)}[m, n, a, b] &= ik\mathcal{B}_{ma, nb}(\mathbf{r}_{m3} - \mathbf{r}_{ma}) \cdot (\mathbf{r}_{n3} - \mathbf{r}_{nb}) \\ &\quad \int_{S_m} d\mathbf{r}(\mathbf{r} - \mathbf{r}_{ma}) \cdot [(\mathbf{r}_{m4} - \mathbf{r}_{ma}) \times \hat{\mathbf{n}}_{ma}] \\ &\quad \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') (\mathbf{r}' - \mathbf{r}_{nb}) \cdot [(\mathbf{r}_{n4} - \mathbf{r}_{nb}) \times \hat{\mathbf{n}}_{nb}] \\ &\quad - \frac{4i}{k} \mathcal{B}_{ma, nb} \int_{S_m} d\mathbf{r} \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \end{aligned} \quad (3.86)$$

$$\begin{aligned} \bar{\mathbf{K}}_{PV}^{N, \text{LL}(1)}[m, n, a, b] &= \mathcal{B}_{ma, nb}(\mathbf{r}_{m3} - \mathbf{r}_{ma}) \cdot \hat{\mathbf{n}}_{ma} \times (\mathbf{r}_{n3} - \mathbf{r}_{nb}) \times \\ &\quad \int_{S_{ma}} d\mathbf{r}(\mathbf{r} - \mathbf{r}_{ma}) \cdot [(\mathbf{r}_{m4} - \mathbf{r}_{ma}) \times \hat{\mathbf{n}}_{ma}] \\ &\quad \int_{PV, S_{nb}} d\mathbf{r}' (\mathbf{r}' - \mathbf{r}_{nb}) \cdot [(\mathbf{r}_{n4} - \mathbf{r}_{nb}) \times \hat{\mathbf{n}}_{nb}] \nabla' g(\mathbf{r}, \mathbf{r}') \end{aligned} \quad (3.87)$$

and

$$\mathcal{B}_{ma, nb} = \frac{l_m l_n}{16(A_{ma})^2(A_{nb})^2}. \quad (3.88)$$

After a coordinate transformation,

$$\begin{aligned} \bar{\mathbf{I}}^{T, \text{LL}(1)}[m, n, a, b] &= \mathcal{B}_{ma, nb}(\boldsymbol{\rho}_{m3} - \boldsymbol{\rho}_{ma}) \cdot (\boldsymbol{\rho}_{n3} - \boldsymbol{\rho}_{nb}) \\ &\quad \int_{S_{ma}} d\boldsymbol{\rho}(\boldsymbol{\rho} - \boldsymbol{\rho}_{ma}) \cdot [(\boldsymbol{\rho}_{m4} - \boldsymbol{\rho}_{ma}) \times \hat{\mathbf{z}}] (\boldsymbol{\rho} - \boldsymbol{\rho}_{nb}) \cdot [(\boldsymbol{\rho}_{n4} - \boldsymbol{\rho}_{nb}) \times \hat{\mathbf{z}}] \end{aligned} \quad (3.89)$$

$$\begin{aligned}
\bar{\mathbf{T}}^{T,LL(1)}[m, n, a, b] = & ik\mathcal{B}_{ma,nb}(\boldsymbol{\rho}_{m3} - \boldsymbol{\rho}_{ma}) \cdot (\boldsymbol{\rho}_{n3} - \boldsymbol{\rho}_{nb}) \\
& \int_{S_m} d\mathbf{r}(\mathbf{r} - \mathbf{r}_{ma}) \cdot [(\mathbf{r}_{m4} - \mathbf{r}_{ma}) \times \hat{\mathbf{n}}_{ma}] \\
& \int_{S_n} d\mathbf{r}'g(\mathbf{r}, \mathbf{r}')(\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb}) \cdot [(\boldsymbol{\rho}_{n4} - \boldsymbol{\rho}_{nb}) \times \hat{\mathbf{z}}] \\
& - \frac{4i}{k}\mathcal{B}_{ma,nb} \int_{S_m} d\mathbf{r} \int_{S_n} d\mathbf{r}'g(\mathbf{r}, \mathbf{r}')
\end{aligned} \tag{3.90}$$

$$\begin{aligned}
\bar{\mathbf{K}}_{PV}^{N,LL(1)}[m, n, a, b] = & \mathcal{B}_{ma,nb}(\mathbf{r}_{m3} - \mathbf{r}_{ma}) \cdot \hat{\mathbf{n}}_{ma} \times (\boldsymbol{\rho}_{n3} - \boldsymbol{\rho}_{nb}) \times \\
& \int_{S_{ma}} d\mathbf{r}(\mathbf{r} - \mathbf{r}_{ma}) \cdot [(\mathbf{r}_{m4} - \mathbf{r}_{ma}) \times \hat{\mathbf{n}}_{ma}] \\
& \int_{PV, S_{nb}} d\mathbf{r}'(\boldsymbol{\rho}' - \boldsymbol{\rho}_{nb}) \cdot [(\boldsymbol{\rho}_{n4} - \boldsymbol{\rho}_{nb}) \times \hat{\mathbf{z}}] \nabla' g(\mathbf{r}, \mathbf{r}').
\end{aligned} \tag{3.91}$$

For integro-differential operators, interactions are further divided into basic integrals as

$$\begin{aligned}
\bar{\mathbf{T}}^{T,LL(1)}[m, n, a, b] = & ik\mathcal{B}_{ma,nb}(\boldsymbol{\rho}_{m3} - \boldsymbol{\rho}_{ma}) \cdot (\boldsymbol{\rho}_{n3} - \boldsymbol{\rho}_{nb}) \\
& \left\{ (y_{n4} - y_{nb}) \left[(y_{m4} - y_{ma})n_z - (z_{m4} - z_{ma})n_y \right] \left[x_{ma}(x_{nb}I_1 - I_2) - x_{nb}I_4 + I_5 \right] \right. \\
& + (x_{n4} - x_{nb}) \left[(z_{m4} - z_{ma})n_y - (y_{m4} - y_{ma})n_z \right] \left[x_{ma}(y_{nb}I_1 - I_3) - y_{nb}I_4 + I_6 \right] \\
& + (y_{n4} - y_{nb}) \left[(z_{m4} - z_{ma})n_x - (x_{m4} - x_{ma})n_z \right] \left[y_{ma}(x_{nb}I_1 - I_2) - x_{nb}I_7 + I_8 \right] \\
& + (x_{n4} - x_{nb}) \left[(x_{m4} - x_{ma})n_z - (z_{m4} - z_{ma})n_x \right] \left[y_{ma}(y_{nb}I_1 - I_3) - y_{nb}I_7 + I_9 \right] \\
& + (y_{n4} - y_{nb}) \left[(x_{m4} - x_{ma})n_y - (y_{m4} - y_{ma})n_x \right] \left[z_{ma}(x_{nb}I_1 - I_2) - x_{nb}I_{10} + I_{11} \right] \\
& \left. + (x_{n4} - x_{nb}) \left[(y_{m4} - y_{ma})n_x - (x_{m4} - x_{ma})n_y \right] \left[z_{ma}(y_{nb}I_1 - I_3) - y_{nb}I_{10} + I_{12} \right] \right\} \\
& - \frac{4i}{k}\mathcal{B}_{ma,nb}I_1
\end{aligned} \tag{3.92}$$

and

$$\begin{aligned}
\bar{\mathbf{K}}_{PV}^{N,LL(1)}[m, n, a, b] = & \mathcal{B}_{ma,nb}(\mathbf{r}_{m3} - \mathbf{r}_{ma}) \cdot \hat{\mathbf{n}}_{ma} \times (\boldsymbol{\rho}_{n3} - \boldsymbol{\rho}_{nb}) \times \\
& \left\{ (y_{n4} - y_{nb}) \left[(y_{m4} - y_{ma})n_z - (z_{m4} - z_{ma})n_y \right] \left[x_{ma}(x_{nb}\mathbf{I}_1 - \mathbf{I}_2) - x_{nb}\mathbf{I}_4 + \mathbf{I}_5 \right] \right. \\
& + (x_{n4} - x_{nb}) \left[(z_{m4} - z_{ma})n_y - (y_{m4} - y_{ma})n_z \right] \left[x_{ma}(y_{nb}\mathbf{I}_1 - \mathbf{I}_3) - y_{nb}\mathbf{I}_4 + \mathbf{I}_6 \right] \\
& + (y_{n4} - y_{nb}) \left[(z_{m4} - z_{ma})n_x - (x_{m4} - x_{ma})n_z \right] \left[y_{ma}(x_{nb}\mathbf{I}_1 - \mathbf{I}_2) - x_{nb}\mathbf{I}_7 + \mathbf{I}_8 \right] \\
& + (x_{n4} - x_{nb}) \left[(x_{m4} - x_{ma})n_z - (z_{m4} - z_{ma})n_x \right] \left[y_{ma}(y_{nb}\mathbf{I}_1 - \mathbf{I}_3) - y_{nb}\mathbf{I}_7 + \mathbf{I}_9 \right] \\
& + (y_{n4} - y_{nb}) \left[(x_{m4} - x_{ma})n_y - (y_{m4} - y_{ma})n_x \right] \left[z_{ma}(x_{nb}\mathbf{I}_1 - \mathbf{I}_2) - x_{nb}\mathbf{I}_{10} + \mathbf{I}_{11} \right] \\
& \left. + (x_{n4} - x_{nb}) \left[(y_{m4} - y_{ma})n_x - (x_{m4} - x_{ma})n_y \right] \left[z_{ma}(y_{nb}\mathbf{I}_1 - \mathbf{I}_3) - y_{nb}\mathbf{I}_{10} + \mathbf{I}_{12} \right] \right\}.
\end{aligned} \tag{3.93}$$

We note similarities between (3.92) and (3.93). Finally, basic integrals can be listed as

$$I_{\{1,4,7,10\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z\} I_{in}^1 \tag{3.94}$$

$$I_{\{2,5,8,11\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z\} I_{in}^{x'} \tag{3.95}$$

$$I_{\{3,6,9,12\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z\} I_{in}^{y'} \tag{3.96}$$

$$\mathbf{I}_{\{1,4,7,10\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z\} \mathbf{I}_{in}^{\nabla'} \tag{3.97}$$

$$\mathbf{I}_{\{2,5,8,11\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z\} \mathbf{I}_{in}^{x'\nabla'} \tag{3.98}$$

$$\mathbf{I}_{\{3,6,9,12\}} = \int_{S_{ma}} d\mathbf{r} \{1, x, y, z\} \mathbf{I}_{in}^{y'\nabla'}, \tag{3.99}$$

where I_{in}^1 , $I_{in}^{x'}$, $I_{in}^{y'}$, $\mathbf{I}_{in}^{\nabla'}$, $\mathbf{I}_{in}^{x'\nabla'}$, and $\mathbf{I}_{in}^{y'\nabla'}$ are defined in (2.95), (2.110), (2.111), (2.97), (3.70), and (3.71), respectively.

The curl-conforming $\hat{\mathbf{n}} \times \text{LL}$ functions can be employed only for MFIE. As in the case of the RWG and the $\hat{\mathbf{n}} \times \text{RWG}$ functions, matrix elements derived for the LL and the $\hat{\mathbf{n}} \times \text{LL}$ functions are closely related. On the other hand, similar

to the $\hat{\mathbf{n}} \times \text{RWG}$ functions, we derive an alternative and more efficient implementation for the $\hat{\mathbf{n}} \times \text{LL}$ functions. Using (3.81) in (3.49), we obtain

$$\begin{aligned} \bar{\mathbf{K}}_{PV}^{N,\text{nLL}(1,2)}[m, n] = & \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{LL}(1,2)}(\mathbf{r}) \cdot \int_{PV, S_n} d\mathbf{r}' \hat{\mathbf{n}}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n^{\text{LL}(1,2)}(\mathbf{r}') \\ & + \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{LL}(1,2)}(\mathbf{r}) \cdot \int_{PV, S_n} d\mathbf{r}' \mathbf{b}_n^{\text{LL}(1,2)}(\mathbf{r}') \frac{\partial g(\mathbf{r}, \mathbf{r}')}{\partial n'} \\ & - \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{LL}(1,2)}(\mathbf{r}) \cdot \int_{PV, S_n} d\mathbf{r}' \hat{\mathbf{n}}' \nabla' \cdot \left[g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n^{\text{LL}(1,2)}(\mathbf{r}') \right]. \end{aligned} \quad (3.100)$$

As in the previous cases, expression in (3.100) is decomposed into basic integrals for the implementation.

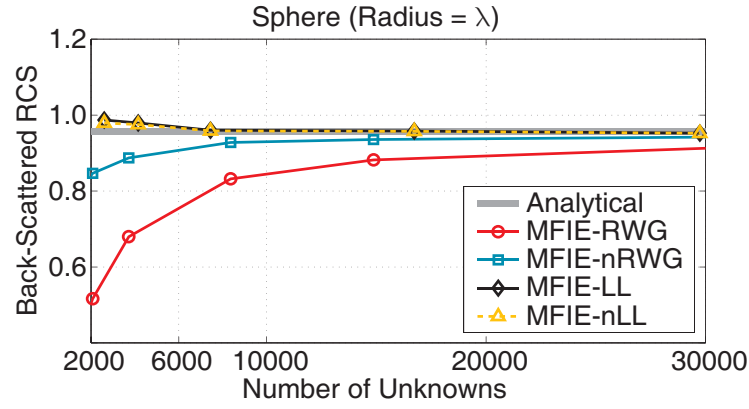
3.7.3 Calculation of Interactions Using the Multilevel Fast Multipole Algorithm

In MLFMA implementations employing LN-LT type functions, radiation and receiving patterns are calculated again by using (2.136)–(2.143). As an example, we consider the LL functions of the first kind and take only the first triangle of the m th edge. We obtain

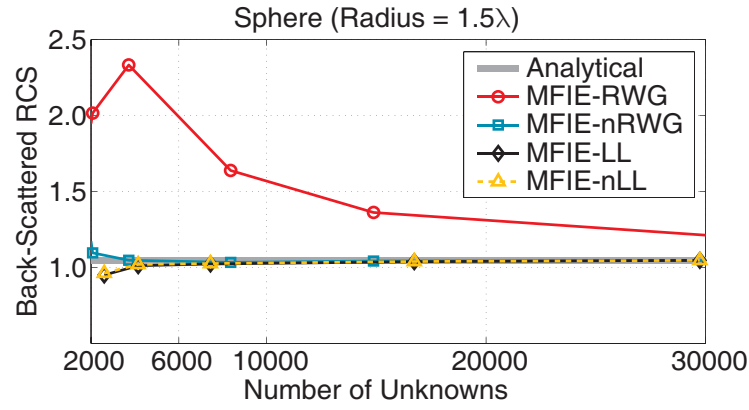
$$\begin{aligned} \mathbf{S}_{m1}^{\text{LL}(1)}(\mathbf{r}_C, \mathbf{k}) = & \frac{l_m}{4(A_{m1})^2} (\bar{\mathbf{I}}^{3 \times 3} - \hat{\mathbf{k}} \hat{\mathbf{k}}) \cdot (\mathbf{r}_{m3} - \mathbf{r}_{m1}) \\ & [(\mathbf{r}_{m4} - \mathbf{r}_{m1}) \times \hat{\mathbf{n}}] \cdot \int_{S_{m1}} d\mathbf{r} \exp[-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_C)] (\mathbf{r} - \mathbf{r}_{m1}) \end{aligned} \quad (3.101)$$

$$\begin{aligned} \mathbf{F}_{m1}^{\mathcal{K}, N, \text{LL}(1)}(\mathbf{r}_C, \mathbf{k}) = & -\frac{l_m}{4(A_{m1})^2} \hat{\mathbf{k}} \times [(\mathbf{r}_{m3} - \mathbf{r}_{m1}) \times \hat{\mathbf{n}}] \\ & [(\mathbf{r}_{m4} - \mathbf{r}_{m1}) \times \hat{\mathbf{n}}] \cdot \int_{S_{m1}} d\mathbf{r} \exp[i\mathbf{k} \cdot (\mathbf{r}' - \mathbf{r}_C)] (\mathbf{r}' - \mathbf{r}_{m1}). \end{aligned} \quad (3.102)$$

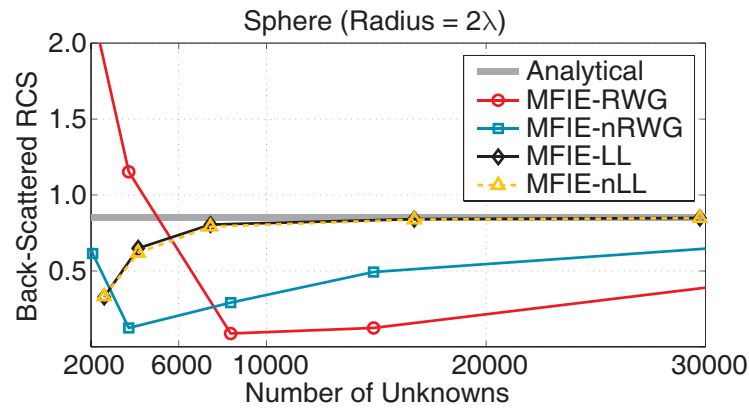
Derivations for the LL functions of the second kind and the $\hat{\mathbf{n}} \times \text{LL}$ functions are similar.



(a)

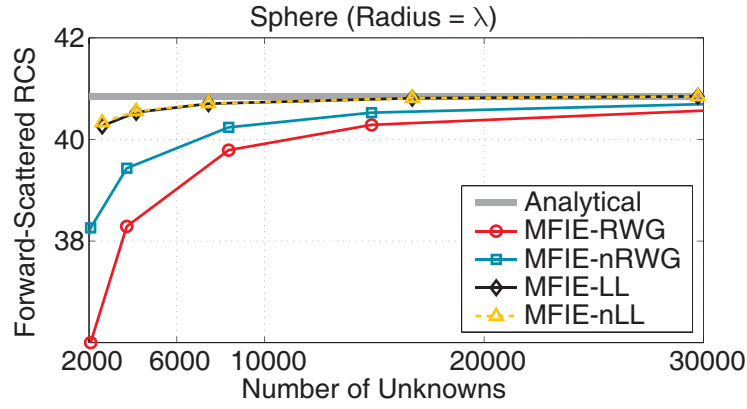


(b)

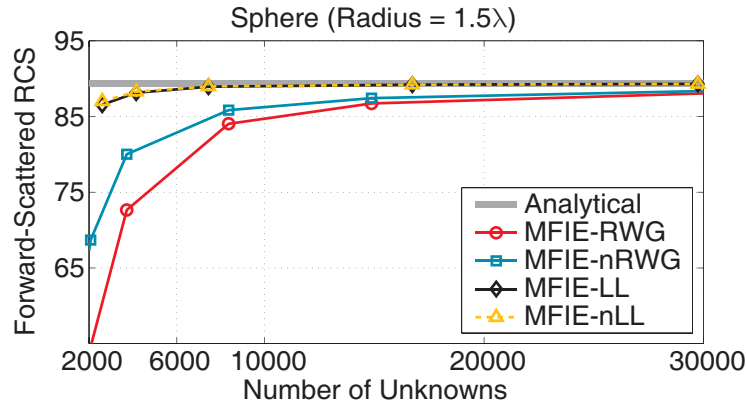


(c)

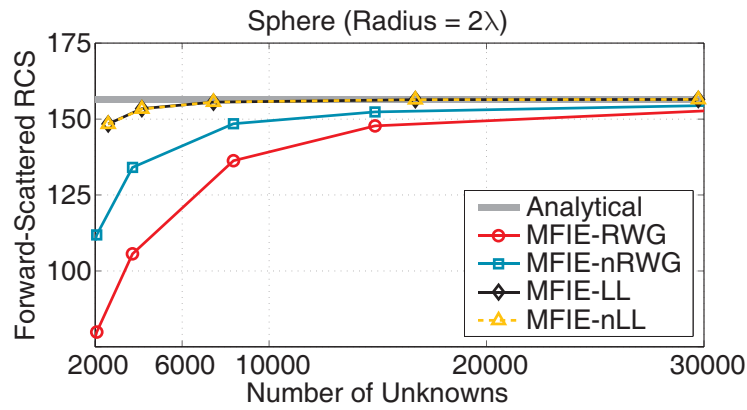
Figure 3.35: Normalized back-scattered RCS ($\text{RCS}/\pi a^2$) of a PEC sphere with a radius of (a) λ , (b) 1.5λ , and (c) 2λ as a function of the number of unknowns.



(a)



(b)



(c)

Figure 3.36: Normalized forward-scattered RCS ($\text{RCS}/\pi a^2$) of a PEC sphere with a radius of (a) λ , (b) 1.5λ , and (c) 2λ as a function of the number of unknowns.

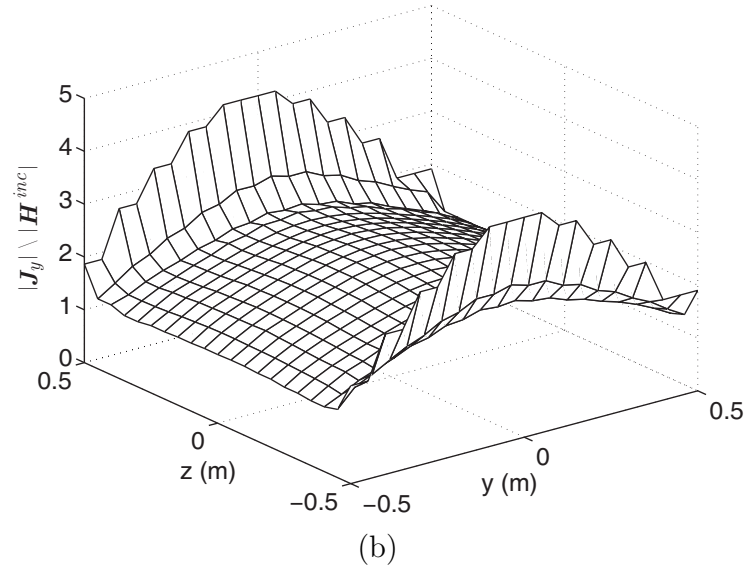
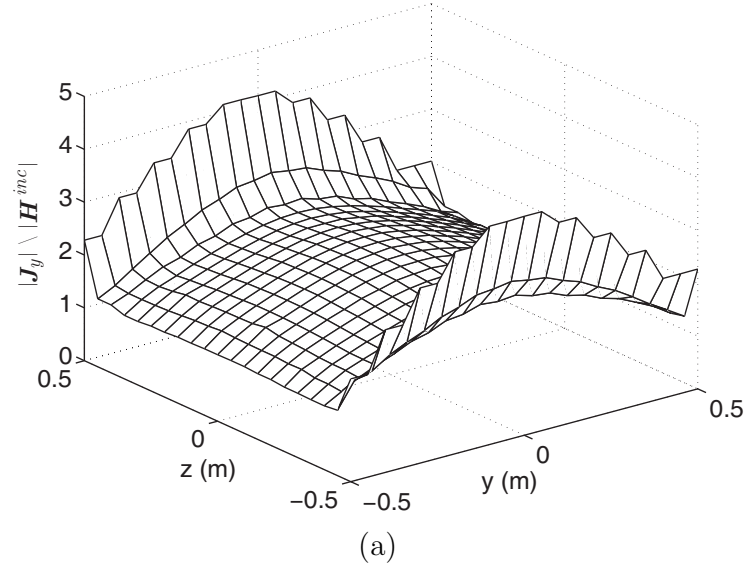


Figure 3.37: Magnitude of the y component of the normalized surface current ($|\mathbf{J}_y|/|\mathbf{H}^{inc}|$) induced on the front surface (at $x = 0.5$ m) of a PEC cube with edges of λ , obtained by using MFIE with (a) LL and (b) $\hat{\mathbf{n}} \times \text{LL}$.

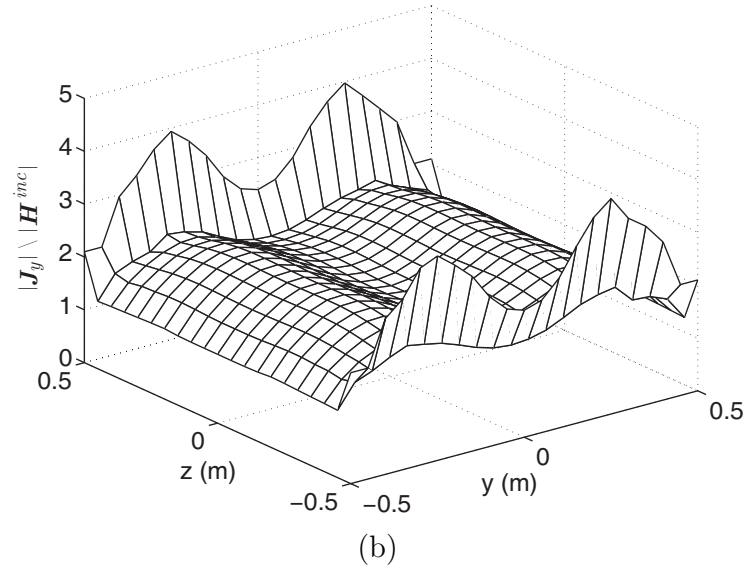
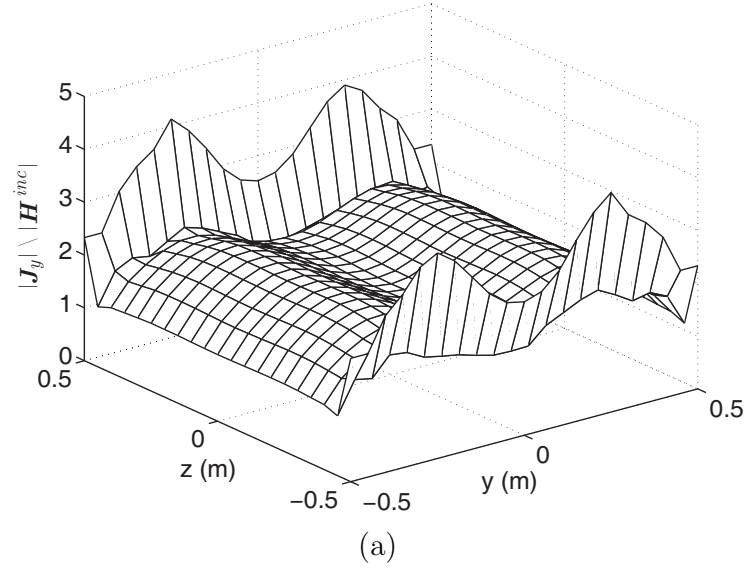
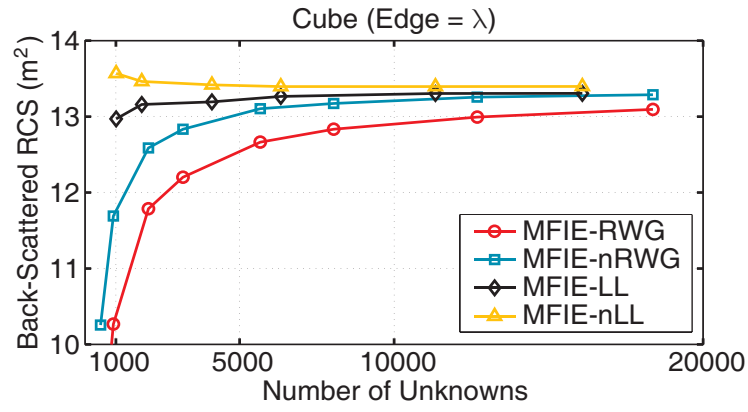
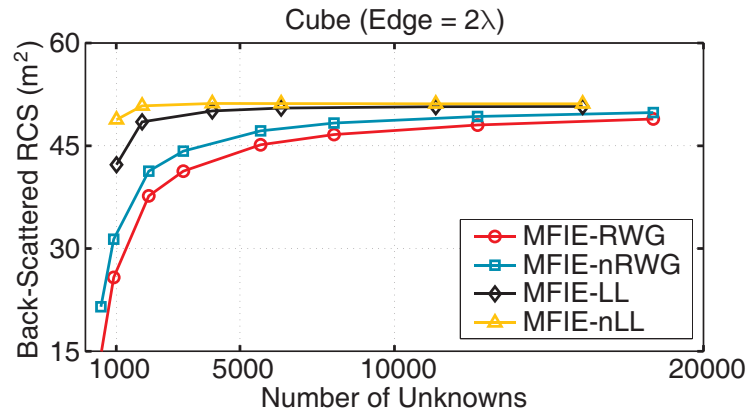


Figure 3.38: Magnitude of the y component of the normalized surface current ($|\mathbf{J}_y|/|\mathbf{H}^{inc}|$) induced on the front surface (at $x = 0.5$ m) of a PEC cube with edges of 2λ , obtained by using MFIE with (a) LL and (b) $\hat{\mathbf{n}} \times \text{LL}$.

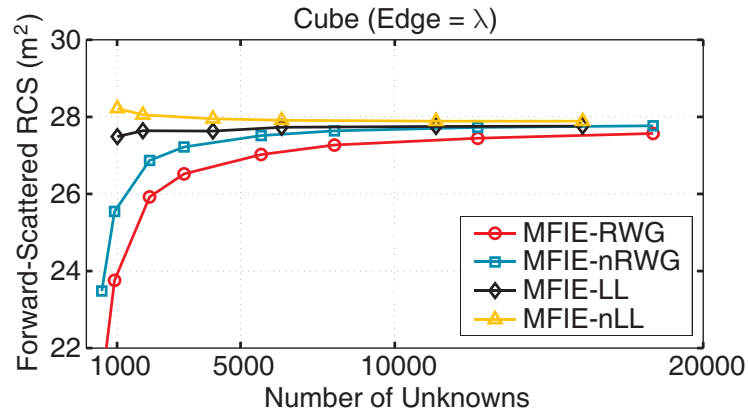


(a)

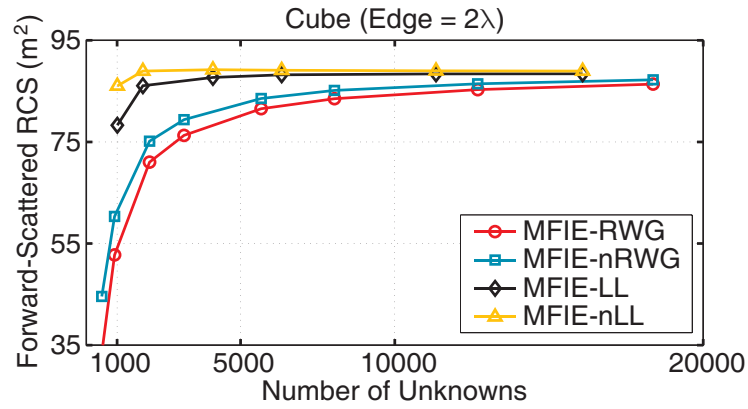


(b)

Figure 3.39: Back-scattered RCS (m^2) of a PEC cube with edges of (a) λ and (b) 2λ as a function of the number of unknowns.



(a)



(b)

Figure 3.40: Forward-scattered RCS (m^2) of a PEC cube with edges of (a) λ and (b) 2λ as a function of the number of unknowns.

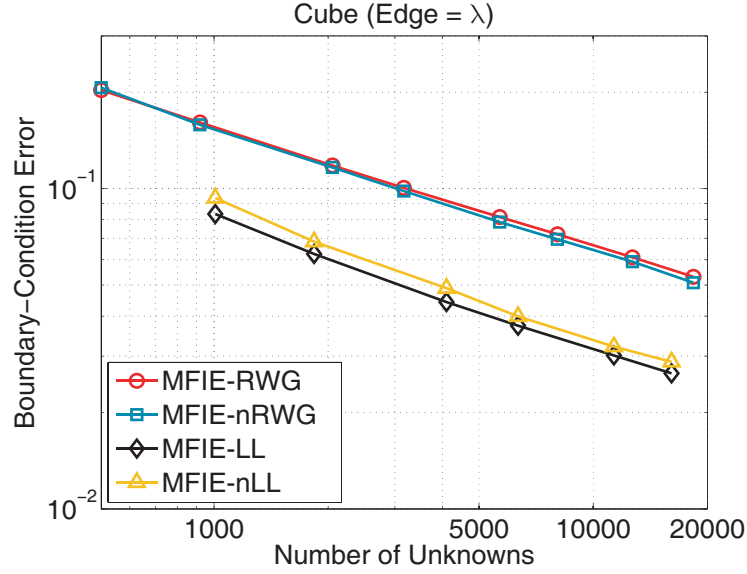


Figure 3.41: BCE defined in (3.104) in the solution of a scattering problem involving a $\lambda \times \lambda \times \lambda$ conducting box formulated with MFIE.

3.7.4 Numerical Results (MFIE)

To compare basis functions in terms of how they affect the accuracy of MFIE implementations, we first present results of scattering problems involving a PEC sphere with a radius of a , when $a = \lambda$, $a = 1.5\lambda$, and $a = 2\lambda$. Discretizations of the sphere with $\lambda/10$ triangles lead to 3723, 8364, and 14,871 edges. The sphere is illuminated by a plane wave and scattering problems are solved with FMM and MLFMA implementations employing the RWG, $\hat{\mathbf{n}} \times \text{RWG}$, LL, and $\hat{\mathbf{n}} \times \text{LL}$ functions. Figures 3.35 and 3.36 display normalized back-scattered and forward-scattered RCS ($\text{RCS}/\pi a^2$, where a is the radius of the sphere), respectively. RCS values are plotted with respect to the number of unknowns, which corresponds to the number of edges for the RWG and the $\hat{\mathbf{n}} \times \text{RWG}$ functions, but twice that for the LL and the $\hat{\mathbf{n}} \times \text{LL}$ functions. To compare performances of functions, convergence to the analytically calculated value is investigated. It can be observed that the convergence is significantly faster for LN-LT type functions, namely, the divergence-conforming LL and the curl-conforming $\hat{\mathbf{n}} \times \text{LL}$ functions.

In other words, for a given number of unknowns, MFIE solutions obtained by these functions are more accurate than those obtained by the RWG and the $\hat{\mathbf{n}} \times \text{RWG}$ functions. We also note that the $\hat{\mathbf{n}} \times \text{RWG}$ functions give more accurate RCS values compared to the RWG functions, but the accuracy is relatively poor compared to the LL and the $\hat{\mathbf{n}} \times \text{LL}$ functions.

RCS values in Figures 3.35 and 3.36 are plotted with respect to the number of unknowns so that different basis functions can be compared fairly. However, since the LL and the $\hat{\mathbf{n}} \times \text{LL}$ functions are defined in pairs at the edges, their triangulation is different from the triangulation of the RWG and the $\hat{\mathbf{n}} \times \text{RWG}$ functions to keep the number of unknowns fixed. In other words, a coarser triangulation is employed for the LL and the $\hat{\mathbf{n}} \times \text{LL}$ functions. Employing these functions for a given number of unknowns, the number of triangles to model the geometry is half of that for the RWG and the $\hat{\mathbf{n}} \times \text{RWG}$ functions. For curved geometries, a coarser triangulation leads to a larger deviation from the actual geometry. Therefore, in the example of the sphere problem, we keep in mind that the LL and the $\hat{\mathbf{n}} \times \text{LL}$ functions improve the accuracy in spite of the decreased quality of the geometry discretization.

To demonstrate the effect of using different basis functions on the modelling of the induced current, we consider scattering problems involving a PEC cube with edges of 1 m, when the frequency is 300 MHz and 600 MHz. The cube is located at the origin and it is illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. Figures 3.37 and 3.38 show the magnitude of the y component of the normalized surface current ($|\mathbf{J}_y|/|\mathbf{H}^{inc}|$) induced on the front surface of the cube when the frequency is 300 MHz and 600 MHz, respectively. The modeled current is obtained for LN-LT type functions and triangulations with mesh size of about $\lambda/10$. Comparing Figures 3.37 and 3.38 with Figures 3.24, 3.25, and 3.30, we observe that the

modelling is significantly improved with LN-LT type functions compared to the CN-LT type RWG functions and the LN-CT type $\hat{\mathbf{n}} \times \text{RWG}$ functions, especially for the representation of singular current at edges. We also note that features and the quality of the current modelling are very similar for both the divergence-conforming LL functions and the curl-conforming $\hat{\mathbf{n}} \times \text{LL}$ functions. Figures 3.39 and 3.40 display back-scattered and forward-scattered RCS (m^2) of PEC cubes, plotted again with respect to the number of unknowns. As in sphere problems, we observe faster convergence for the LL and the $\hat{\mathbf{n}} \times \text{LL}$ functions, while the RWG functions give the worst accuracy for a given number of unknowns in MFIE.

Finally, we consider the boundary-condition error (BCE) in the solution of a scattering problem involving a $\lambda \times \lambda \times \lambda$ conducting box located at the origin. The box is illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. In order to measure the accuracy of numerical solutions, we sample the magnetic field and test the boundary condition point-wise on the surface of the box. A relative error

$$\Delta_R(\mathbf{r}) = \frac{|\mathbf{J}(\mathbf{r}) - \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r})|}{|\hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r})|} \quad (3.103)$$

is calculated at various observation points $\mathbf{r} \in S$. Then, BCE is defined as the integral of the relative error (3.103) over the entire surface, i.e.,

$$\Delta_{BCE}(\mathbf{r}) = \int_S d\mathbf{r} \Delta_R(\mathbf{r}). \quad (3.104)$$

Figure 3.41 presents BCE as a function of the number of unknowns and for various discretizations of the problem with different basis functions. We observe that BCE for the LN-LT type functions is approximately half of that for the RWG and the $\hat{\mathbf{n}} \times \text{RWG}$ functions. In other words, the boundary condition for the magnetic field is satisfied better by using LN-LT type functions. This explains why LN-LT type functions produce more accurate RCS results than the RWG and the $\hat{\mathbf{n}} \times \text{RWG}$ functions.

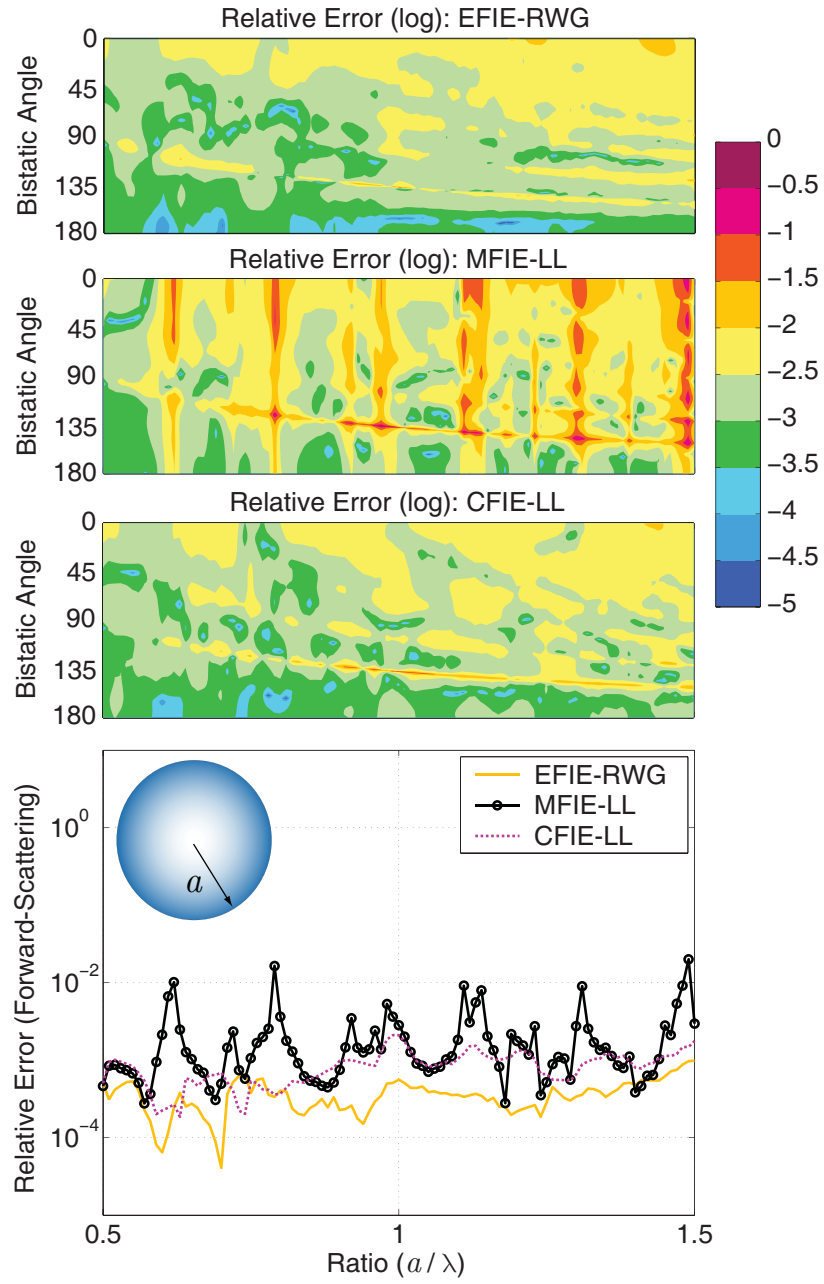
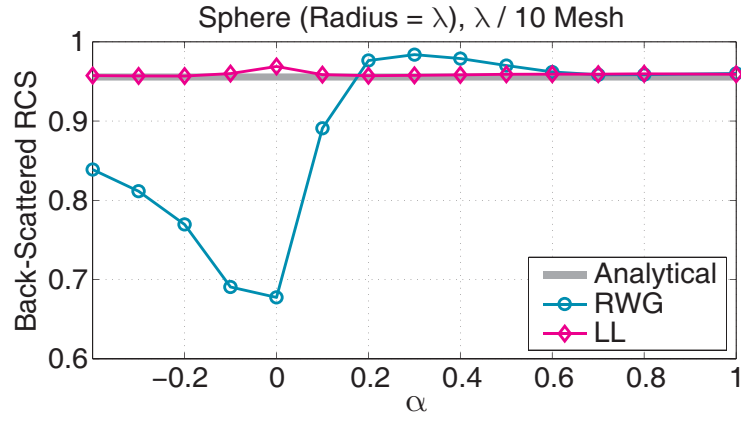
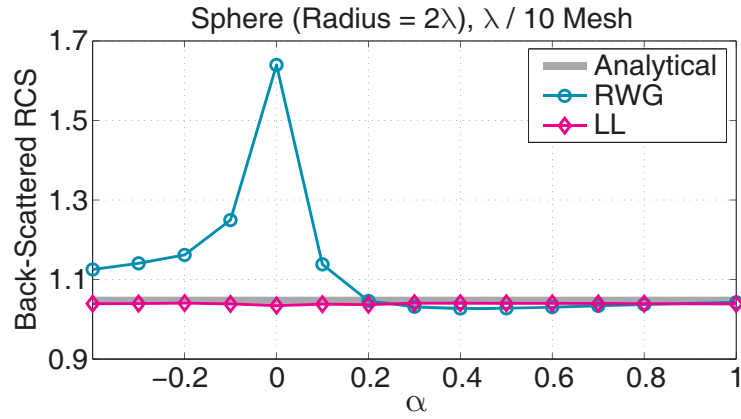


Figure 3.42: Relative error in the bistatic RCS of a PEC sphere with a radius of a , when a changes from 0.5λ to 1.5λ . Computational results are obtained by using EFIE discretized with the RWG functions, as well as MFIE and CFIE discretized with the LL functions.

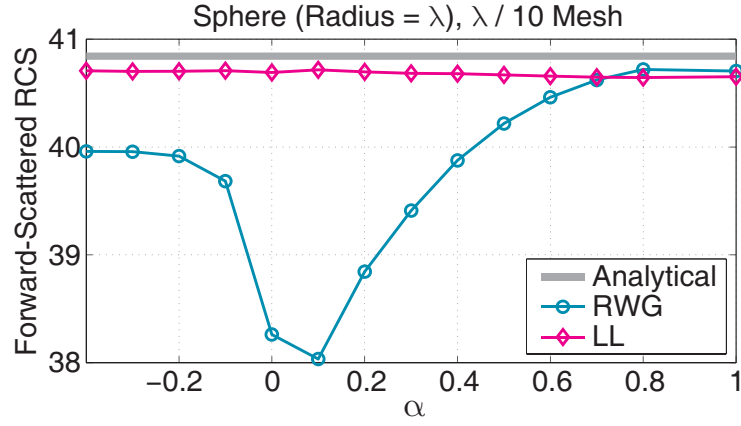


(a)

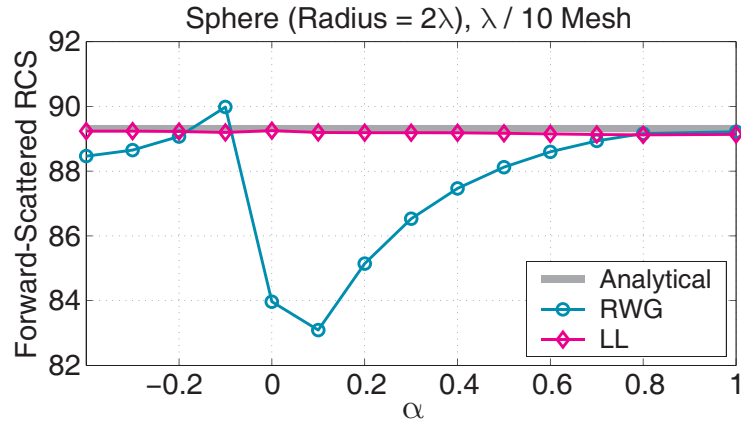


(b)

Figure 3.43: Normalized back-scattered RCS ($\text{RCS}/\pi a^2$) of a PEC sphere with a radius of (a) λ and (b) 2λ as a function of α in CFIE. The number of unknowns is (a) 3723 for RWG and 7446 for LL, (b) 8364 for RWG and 16728 for LL.

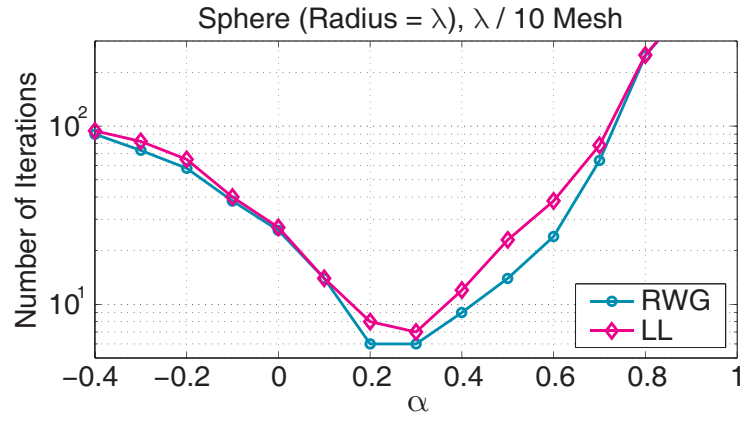


(a)

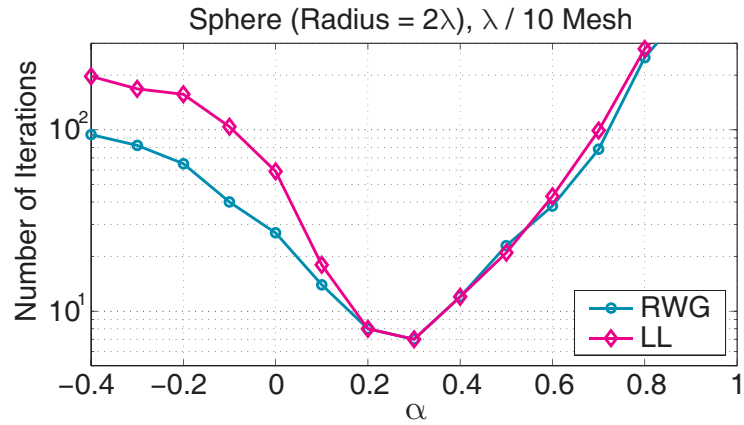


(b)

Figure 3.44: Normalized forward-scattered RCS ($\text{RCS}/\pi a^2$) of a PEC sphere with a radius of (a) λ and (b) 2λ as a function of α in CFIE. The number of unknowns is (a) 3723 for RWG and 7446 for LL, (b) 8364 for RWG and 16728 for LL.

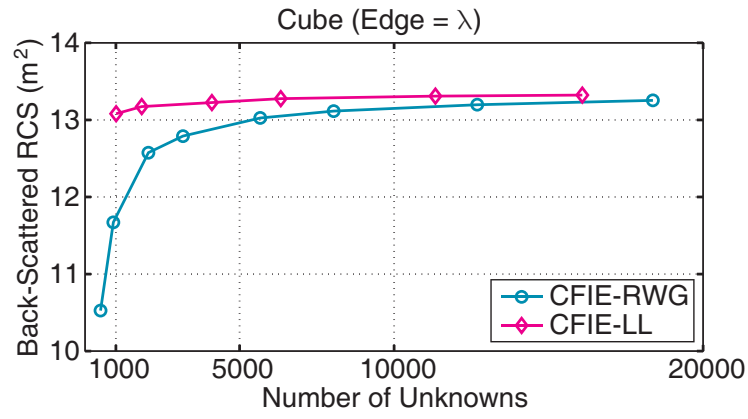


(a)

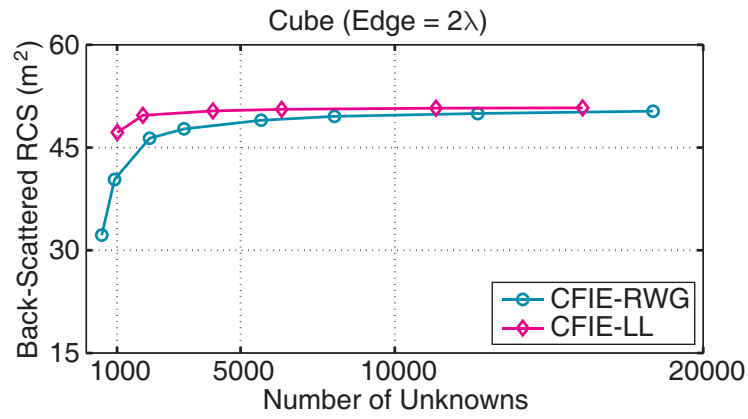


(b)

Figure 3.45: Iteration counts (10^{-3} residual error) for the solution of scattering problems involving a sphere with a radius of (a) λ (3723 edges) and (b) 1.5λ (8364 edges) as a function of α in CFIE.

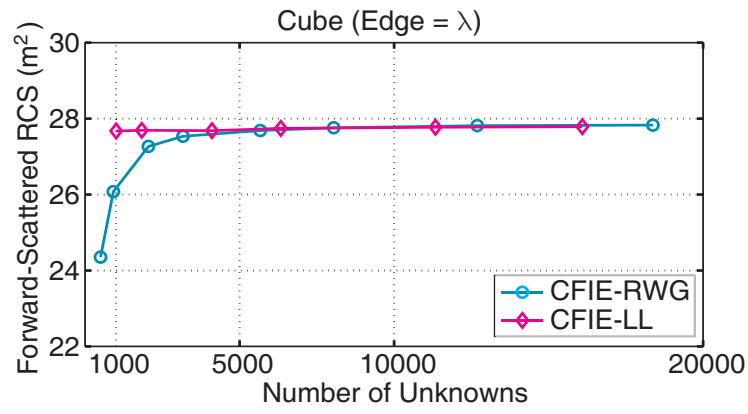


(a)

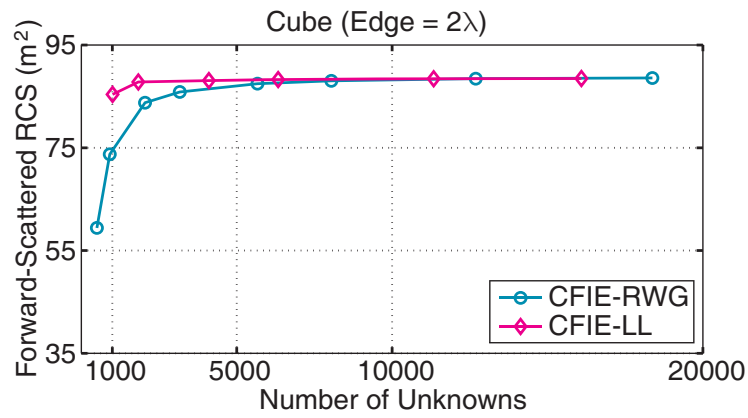


(b)

Figure 3.46: Back-scattered RCS (m^2) of a PEC cube with edges of (a) λ and (b) 2λ as a function of the number of unknowns.



(a)



(b)

Figure 3.47: Forward-scattered RCS (m^2) of a PEC cube with edges of (a) λ and (b) 2λ as a function of the number of unknowns.

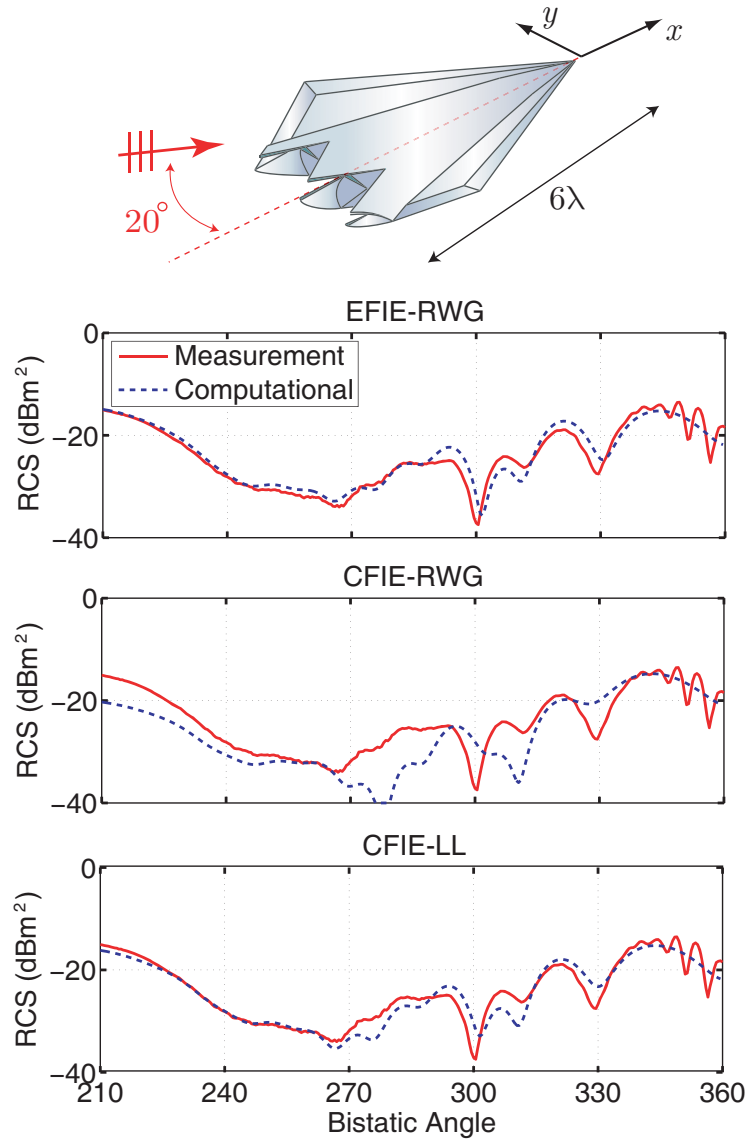


Figure 3.48: Co-polar bistatic RCS (in dBm²) of the stealth airborne target Flamme at 3 GHz. The maximum dimension of the scaled geometry is 0.6 m, corresponding to 6λ at 3 GHz. The target is illuminated by a plane wave propagating in the x - y plane at 200° angle from the x axis, with the electric field polarized in θ direction (vertical polarization).

3.7.5 Numerical Results (CFIE)

The divergence-conforming LL functions can also be used to improve the accuracy of CFIE, similar to MFIE. As an example, we consider scattering problems involving a sphere with a radius of a , when a changes from 0.5λ to 1.5λ . In Figure 3.42, the relative error in bistatic RCS values obtained with EFIE, MFIE, and CFIE ($\alpha = 0.2$) formulations is plotted as a function of bistatic angle and frequency. We also focus on the relative error in the forward-scattering direction as a function of frequency. EFIE is discretized with the RWG functions, whereas MFIE and CFIE are discretized with the LL functions. Comparing results in Figure 3.42 with the previous results in Figure 3.23, we observe that MFIE and CFIE become as accurate as the reference EFIE with the RWG functions. In MFIE solutions, there are two consequences when the RWG functions are replaced by the LL functions. First, effects of internal resonances are narrower as a function of frequency for the LL functions. Second, the excessive error of MFIE with the RWG functions disappears and the accuracy is improved for all frequencies. Using the LL functions instead of the RWG functions, accuracy of CFIE is also improved to levels of EFIE.

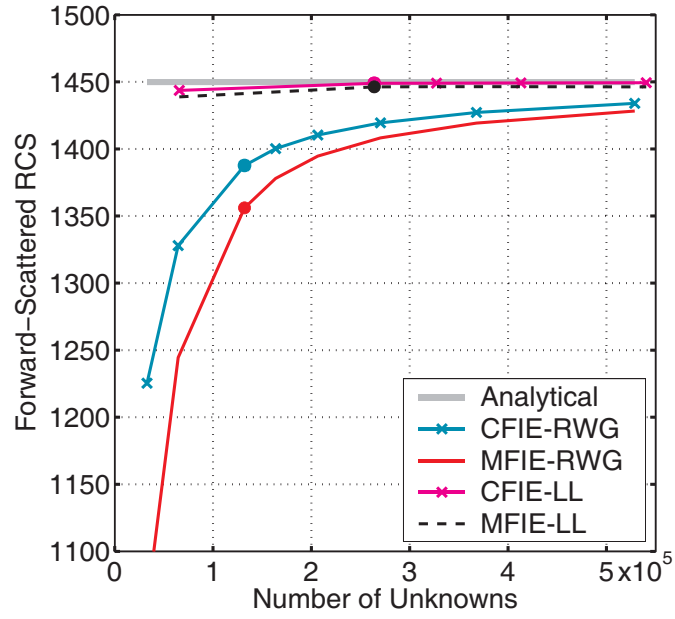
To further compare the RWG and the LL functions in terms of the accuracy of CFIE implementations, Figures 3.43 and 3.44 present back-scattered and forward-scattered RCS, respectively, of a PEC sphere with a radius of a , when $a = \lambda$ and $a = 1.5\lambda$. For each case, the sphere is discretized with $\lambda/10$ triangles. RCS values are calculated for different values of the coupling parameter α . Values at $\alpha = 0$ and $\alpha = 1$ correspond to those obtained by MFIE and EFIE, respectively. Besides the overall improvement in accuracy by the use of the LL functions, Figures 3.43 and 3.44 demonstrate that the significant and undesired variance in accuracy with the use of the RWG functions is dramatically reduced with the use of the LL functions. In other words, accuracy is stabilized with respect to the variable α and becomes weakly dependent on the choice of this

variable.

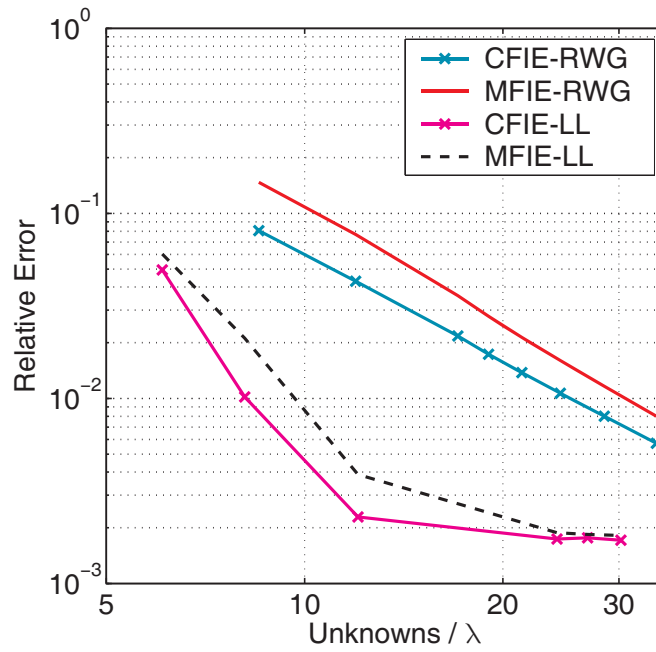
Although there is no optimum α for the accuracy of the LL functions, Figure 3.45 shows that the conditioning of the matrix equation depends strongly on this variable. In this figure, CGS iteration counts required to reach 10^{-3} residual error for FMM solutions of sphere problems are depicted. FMM employing the CGS solver is accelerated by BDP obtained by keeping self-interactions of boxes in the FMM structure. It is observed that the approximate range of α from 0.2 to 0.3 is optimum for both the RWG and the LL functions.

Figures 3.46 and 3.47 present back-scattered and forward-scattered RCS of a cube with edges of 1 m, when the frequency is 300 MHz and 600 MHz. CFIE formulations ($\alpha = 0.2$) employing the RWG and the LL functions are compared. In this convergence analysis, we also observe a significant improvement in the accuracy of CFIE by using the LL functions.

Finally, Figure 3.48 presents the solution of a scattering problem involving a stealth airborne target Flamme [81]. The scaled size of the Flamme is 0.6 m corresponding to 6λ at 3 GHz, and it is illuminated by a plane wave propagating in the x - y plane at $\phi = 200^\circ$ angle from the nose. The electric field is polarized in the θ direction (vertical polarization). Figure 3.48 presents the co-polar bistatic RCS (in dBm^2) on the x - y plane as a function of the bistatic angle ϕ from 210° to 360° . We observe that computational results provided by EFIE employing the RWG functions agree well with measurements. However, we cannot obtain the same agreement when using CFIE discretized with the RWG functions. We also observe that the accuracy of CFIE can be improved to levels of EFIE by employing the LL functions instead of the RWG functions.

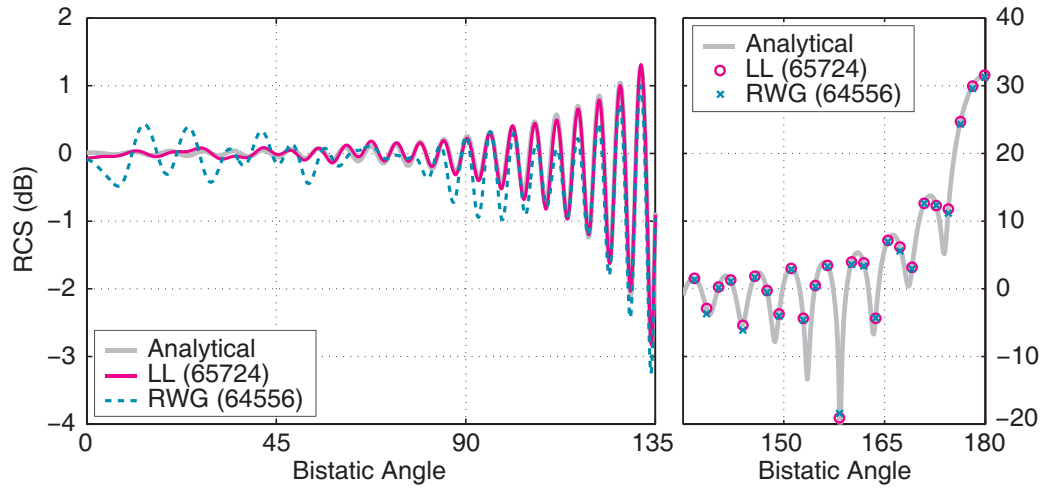


(a)

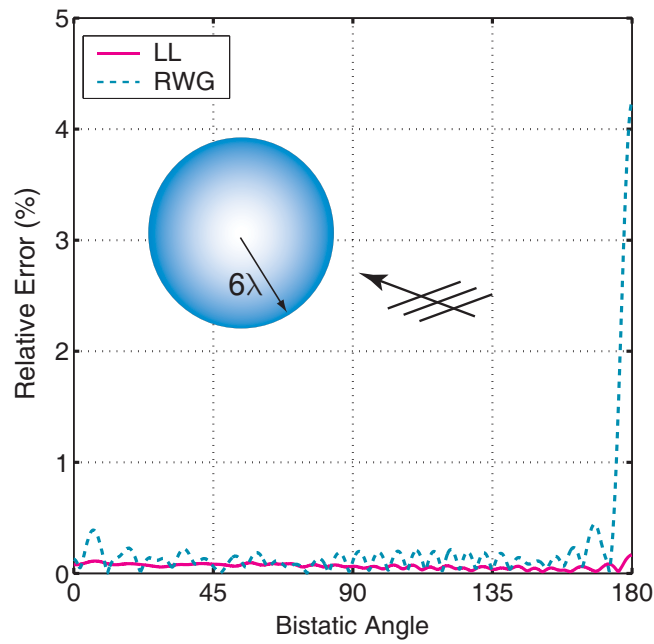


(b)

Figure 3.49: (a) Normalized forward-scattered RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 6λ as a function of the number of unknowns. The dots on the curves correspond to the $\lambda/10$ discretization. (b) Error in the scattered field as a function of the number of linear unknowns per wavelength.



(a)



(b)

Figure 3.50: (a) Normalized bistatic RCS ($RCS/\pi a^2$) of a sphere with a radius of 6λ as a function of bistatic angle from 0° (back-scattering) to 180° (forward-scattering). (b) Relative error as defined in (2.172) for the bistatic RCS depicted in Figure 3.50(a). The error for the RWG functions is significantly larger than the error for the LL functions, especially in the forward-scattering direction, where the RWG error is above 4%.

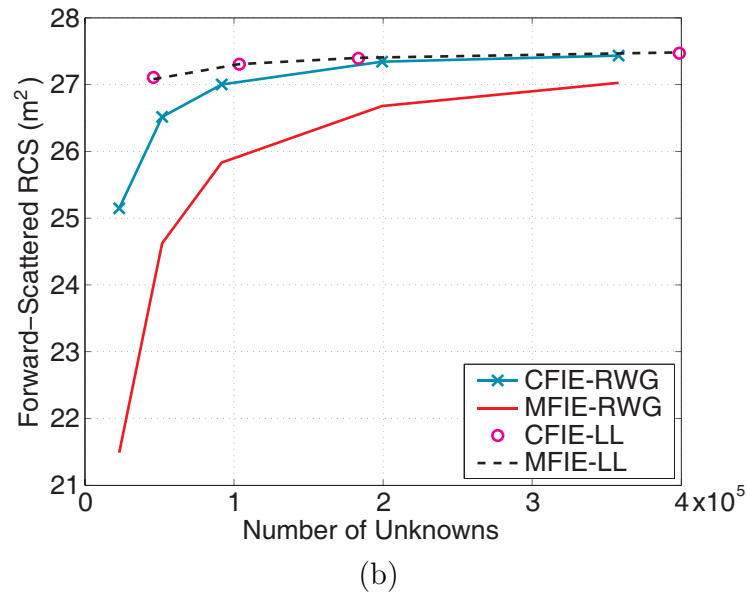
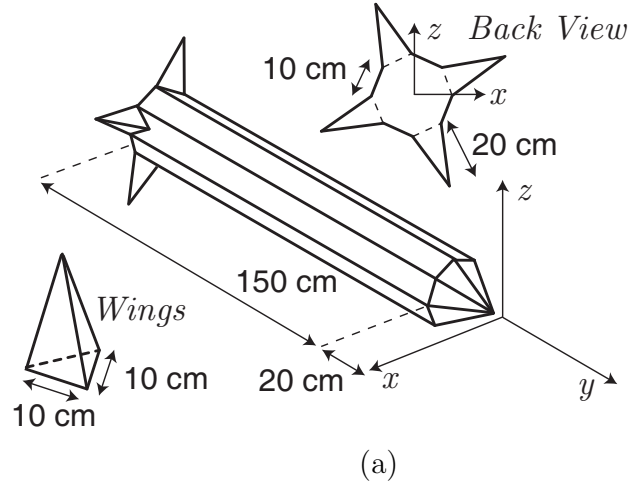


Figure 3.51: (a) A scatterer composed of planar surfaces connected with sharp edges and corners. (b) Forward-scattered RCS (m^2) with respect to the number of unknowns for the scatterer in Figure 3.51(a) illuminated by a plane wave propagating in the $-y$ direction at 4 GHz. Curves are obtained with CFIE and MFIE implementations employing the RWG and the LL functions.

3.7.6 Numerical Results for Large-Scale Problems

Replacing the RWG functions with the LL functions significantly improves the accuracy of MFIE and CFIE, even for large and smooth geometries. As an example, we consider scattering problems involving a large sphere with a radius of 6λ . The sphere is illuminated by a plane wave and scattering problems discretized by various triangulations are solved by different MLFMA implementations involving MFIE and CFIE formulations. In Figure 3.49(a), normalized forward-scattered RCS is plotted with respect to the number of unknowns. Since the size of the scatterer is fixed with respect to the wavelength, the number of unknowns is controlled by changing the fineness of the triangulation. As the number of unknowns increases from about 50,000 to 500,000, all computational curves approach the analytical value obtained by a Mie-series solution. Convergence is significantly faster for both MFIE and CFIE formulations with the LL functions, and MFIE with the RWG functions has the slowest convergence. We also observe the poor performance of CFIE with the RWG functions. Since computational resources required for the RWG and the LL functions are similar for a fixed number of unknowns, the LL functions are superior to RWG functions because of the significant boost in the accuracy.

Figure 3.49(b) offers another view for the accuracy of RCS results obtained with the RWG and the LL functions. The RCS error, which is defined as the error in the scattered field in reference to an exact Mie-series solution, is plotted on a log-log scale as a function of the number of linear unknowns per wavelength. Then, the slope of the error curve signifies the order of convergence of the numerical solution. It is seen in Figure 3.49(b) that the error decreases with higher slopes for the LL functions than for the RWG functions, demonstrating that the LL functions provide higher-order convergence. Note that slopes of the error curves for the LL functions are not constant, as opposed to slopes of the error curves for the RWG functions. This is because the LL functions reduce the error

so effectively that the residual error is mainly due to the discretization, i.e., the error in the geometrical representation of the sphere.

In Figure 3.50(a), we present the normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 6λ . The sphere is discretized with mesh sizes of about $\lambda/8$ for RWG and $\lambda/5$ for LL. Then, the number of unknowns is approximately the same for RWG and LL cases although the triangulation is coarser for the LL case. In Figure 3.50(a), computational results are obtained with the CFIE formulation, and analytical results are also displayed. The plot is divided into two parts with different scales of the y axis for clarity of details. For bistatic directions from 0° (back-scattering) to 135° , the LL functions unambiguously provide more accurate results. In the range from 135° to 180° (forward-scattering), the improvement is not evident in the dB scale. On the other hand, we plot the relative error (2.172) in Figure 3.50(b), where the large error in the forward-scattering direction is visible. The error with the LL functions is consistently lower and under 0.2%. The error with the RWG functions peaks in the forward-scattering direction, where it is above 4%. This error is not acceptable since near-field and far-field interactions of the matrix equation are calculated with a maximum error of 1% and 0.1%, respectively. Furthermore, the relative residual error in the iterative solution is 10^{-6} so that we expect the total computational error to be 1% at most. In the RWG case, the error due to the simultaneous discretization of the geometry and the current distribution is significant and dominates the computational error. As Figure 3.50(b) suggests, the excessive error due to the discretization with the RWG functions can be easily prevented by replacing them with the LL functions.

Finally, we present results of scattering problems involving a relatively long object with sharp edges and corners as described in Figure 3.51(a). The scatterer is composed of planar surfaces only, and therefore, its triangulation does not lead to a deviation from the actual geometry, as opposed to the sphere. The object

is illuminated by a plane wave propagating in the $-y$ direction and we calculate the forward-scattered RCS at 4 GHz by employing MFIE and CFIE formulations with the RWG and the LL functions. We apply different discretizations with the triangulation size varying from 1.5 cm to 0.375 cm, leading to total of 15,340 to 238,672 triangles, respectively. We also note that $\lambda/10$ triangulation corresponds to 91,782 edges on the object. Figure 3.51(b) presents the forward-scattered RCS values with respect to number of unknowns; we observe that all four curves tend to converge to each other as the mesh gets denser, the number of unknowns increases, and computations become more accurate. However, the convergence rate depends on the type of the formulation and the basis function. MFIE with the RWG functions has the slowest convergence while CFIE with the RWG functions shows relatively poor performance compared to MFIE and CFIE formulations employing the LL functions. As in the case of the large sphere problem, we observe that the LL functions provide significantly improved accuracy for both MFIE and CFIE solutions of the problem.

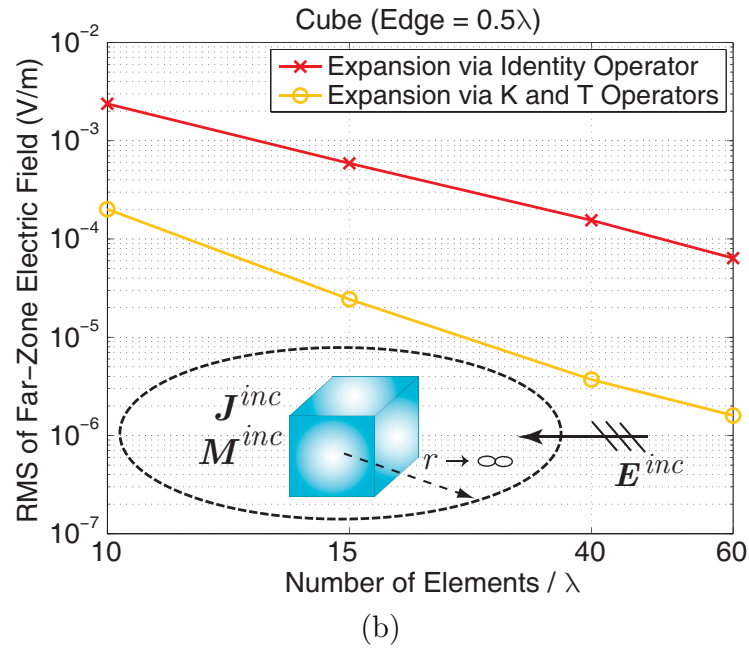
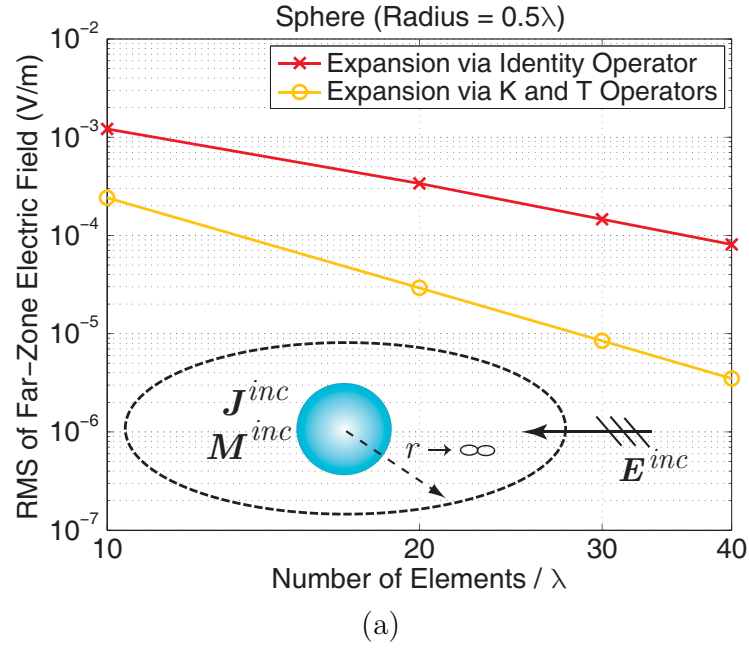


Figure 3.52: RMS of the far-zone electric field due to nonradiating currents $\{\mathbf{J}^{inc}(\mathbf{r}), \mathbf{M}^{inc}(\mathbf{r})\}$ expanded on the surface of (a) a sphere with a radius of 0.5λ and (b) a cube with 0.5λ edges. Both surfaces are illuminated by a plane wave with unit amplitude propagating in free space.

3.8 Excessive Discretization Error Due to the Identity Operator

In this section, we show that the identity operator is a major error source in surface integral equations. This may explain why normal and mixed formulations involving well-tested identity operators are significantly inaccurate compared to tangential formulations.

Discretization of the well-tested identity operator is trivial. The integral

$$\bar{\mathbf{I}}^T[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{b}_n(\mathbf{r}) \quad (3.105)$$

does not contain any singularity and it can be evaluated accurately (even exactly, if desired) by using a low-order Gaussian quadrature rule. On the other hand, the identity operator behaves like an operator with a highly-singular kernel [53],[65]. This alternative interpretation can be understood when (3.105) is rewritten as a double integral over testing and basis functions as

$$\bar{\mathbf{I}}^T[m, n] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' \delta(\mathbf{r}, \mathbf{r}') \mathbf{b}_n(\mathbf{r}'), \quad (3.106)$$

where $\delta(\mathbf{r}, \mathbf{r}')$ is a Dirac delta function as the kernel of the operator. Therefore, the discretization of the identity operator may cause an unexpectedly large error, even though the discretization itself involves very small or no error.

To present the excessive error due to the discretization of the identity operator, we consider the expansion of nonradiating currents $\{\mathbf{J}^{inc}(\mathbf{r}), \mathbf{M}^{inc}(\mathbf{r})\}$ in a series of RWG functions, as in (2.209). For an arbitrary surface, unknown coefficients $\mathbf{a}_J^{inc}$ and $\mathbf{a}_M^{inc}$ can be calculated by using two different methods:

- The matrix equation in (2.226) is solved. This method involves only integro-differential operators and does not contain any (well-tested) identity operator.

- We consider the identity equation in (2.223), which is remarkably simpler than the first one, as it does not involve the discretization of any integro-differential operators.

We note that the two expansion methods are analogous to solutions of electromagnetics problems using tangential and normal/mixed formulations, respectively. In fact, solutions with surface integral equations can be interpreted as the expansion of total currents in a series of basis functions.

We perform experiments on two different geometries: a sphere with a radius of 0.5λ and a cube with edges of 0.5λ . Both objects are illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. Amplitude of the plane wave is unity. Nonradiating currents $\{\mathbf{J}^{inc}(\mathbf{r}), \mathbf{M}^{inc}(\mathbf{r})\}$ are expanded in a series of RWG functions on objects using the two methods described above, i.e., using tangentially-tested $\mathcal{K}$ and $\mathcal{T}$ operators and using the identity operator. Expansion coefficients are calculated and used to compute radiated fields in the far zone. The far-zone electric field is sampled at 360 points as

$$\mathbf{E}_{\{\theta,\phi\}}^{\infty}[n] = \lim_{r \rightarrow \infty} \left\{ r E_{\{\theta,\phi\}}(r, \theta = \pi/2, \phi_n) \right\}, \quad (3.107)$$

where $\mathbf{E}_{\theta}$ and $\mathbf{E}_{\phi}$ are arrays containing θ and ϕ components of the radiated electric field, and $\phi_n = (n-1)\pi/180$ for $n = 1, 2, \dots, 360$. Then, we compute the root-mean-square (RMS) of the total electric field as

$$f_{\text{RMS}} = \sqrt{\frac{1}{360} \sum_{n=1}^{360} \left(|\mathbf{E}_{\theta}^{\infty}[n]|^2 + |\mathbf{E}_{\phi}^{\infty}[n]|^2 \right)}. \quad (3.108)$$

Since expanded currents $\{\mathbf{J}^{inc}(\mathbf{r}), \mathbf{M}^{inc}(\mathbf{r})\}$ should not radiate, RMS value in (3.108) directly corresponds to the error in calculations. Using the RWG functions defined on planar triangles, we compute far-zone fields analytically; hence, the error is only due to the discretization of currents.

Figures 3.52(a) and 3.52(b) present the RMS value in (3.108) for sphere and cube problems, respectively, as a function of the number of elements per wavelength. In both cases, we observe that the RMS value decreases as the triangulation becomes finer. However, there exists a significant discrepancy between results obtained with the two discretization methods. Given a mesh size, the RMS value is consistently smaller with the first method using integro-differential operators $\mathcal{K}$ and $\mathcal{T}$ compared to the second method using the well-tested identity operator. In other words, the discretization of the well-tested identity operator generates larger error compared to the discretization of directly-tested $\mathcal{K}$ and $\mathcal{T}$ operators, even when we use the same set of RWG functions.

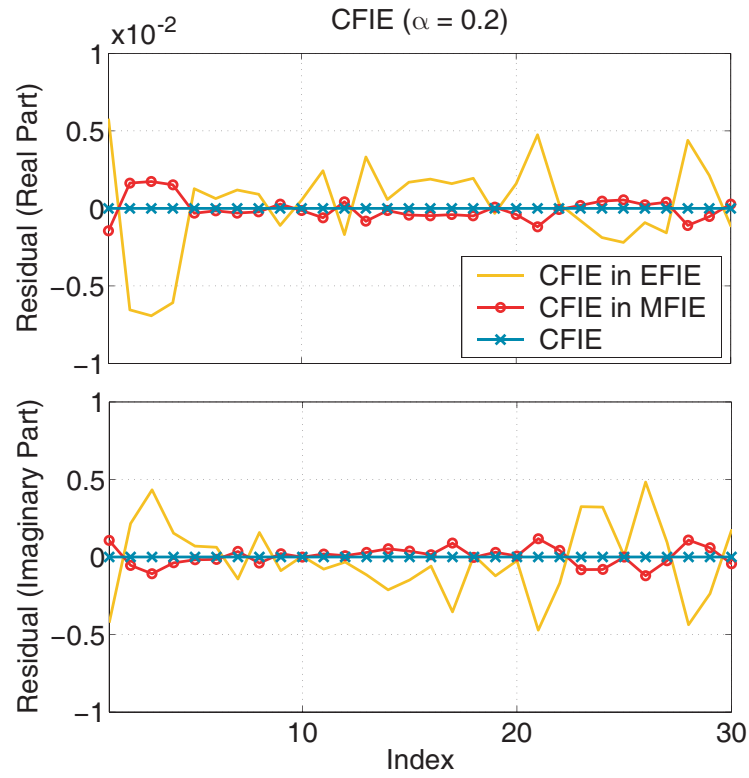


Figure 3.53: First 30 elements of residual vectors $\mathbf{r}_C$ (CFIE), $\mathbf{r}_{C \rightarrow E}$ (CFIE in EFIE), and $\mathbf{r}_{C \rightarrow M}$ (CFIE in MFIE) for the iterative CFIE ($\alpha = 0.2$) solution of a radiation problem involving a $\lambda \times \lambda \times \lambda$ conducting box.

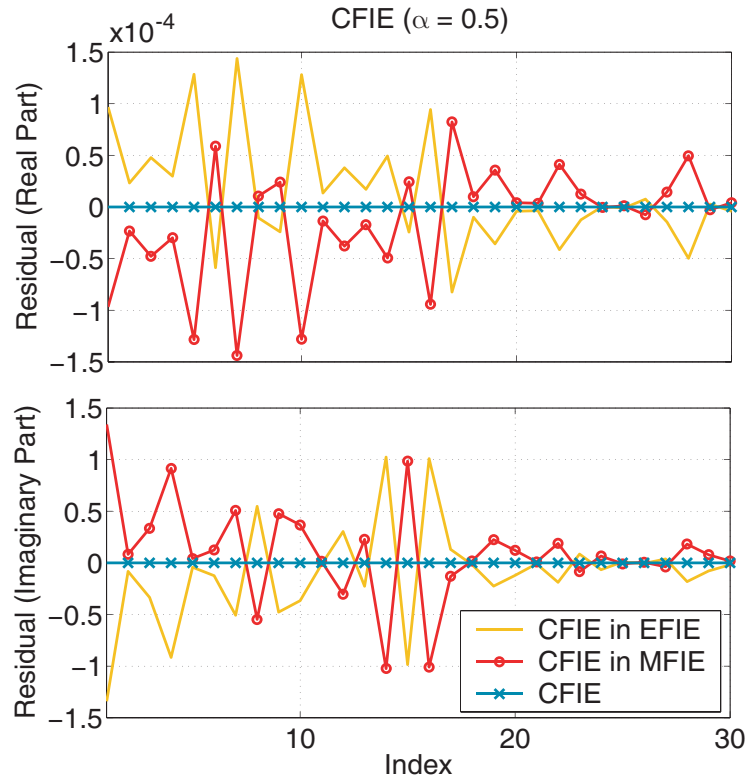


Figure 3.54: First 30 elements of residual vectors $\mathbf{r}_C$ (CFIE), $\mathbf{r}_{C \rightarrow E}$ (CFIE in EFIE), and $\mathbf{r}_{C \rightarrow M}$ (CFIE in MFIE) for the iterative CFIE ($\alpha = 0.5$) solution of a scattering problem involving a $\lambda \times \lambda \times \lambda$ conducting box.

3.9 Compatibility of Surface Integral Equations

Due to the excessive error in MFIE, matrix equations obtained with EFIE and MFIE are incompatible, i.e., a solution vector calculated by using MFIE does not satisfy EFIE with the desired level of accuracy, and vice versa. In fact, for a given problem, coefficient vectors obtained with EFIE and MFIE can be significantly different, i.e.,

$$(\bar{\mathbf{Z}}_E)^{-1} \cdot \mathbf{w}_E = \mathbf{a}_E \neq \mathbf{a}_M = (\bar{\mathbf{Z}}_M)^{-1} \cdot \mathbf{w}_M, \quad (3.109)$$

even when EFIE and MFIE solutions are not contaminated with internal resonances. Being a convex combination of EFIE and MFIE, CFIE solutions are contaminated with the MFIE inaccuracy. Therefore, coefficients obtained with CFIE are also incompatible with EFIE and MFIE systems.

Since EFIE is more accurate than MFIE, one can write

$$\mathbf{a}_M = \mathbf{a}_E + \Delta \mathbf{a}_{ME}, \quad (3.110)$$

where the discrepancy between coefficients obtained with EFIE and MFIE is interpreted as the error in MFIE. The coefficient vector obtained with CFIE is related to those obtained with EFIE and MFIE as

$$\mathbf{a}_C = \alpha(\bar{\mathbf{Z}}_C)^{-1} \cdot \bar{\mathbf{Z}}_E \cdot \mathbf{a}_E + (1 - \alpha)(\bar{\mathbf{Z}}_C)^{-1} \cdot \bar{\mathbf{Z}}_M \cdot \mathbf{a}_M. \quad (3.111)$$

Using (3.110) in (3.111), we arrive at

$$\mathbf{a}_C = \mathbf{a}_E + \Delta \mathbf{a}_{CE}, \quad (3.112)$$

where

$$\Delta \mathbf{a}_{CE} = (1 - \alpha)(\bar{\mathbf{Z}}_C)^{-1} \cdot \bar{\mathbf{Z}}_M \cdot \Delta \mathbf{a}_{ME}, \quad (3.113)$$

which can be interpreted as the CFIE solution being contaminated with the inaccuracy of MFIE.

When the solution of CFIE is obtained by using an iterative algorithm, the residual vector

$$\mathbf{r}_C = \mathbf{w}_C - \bar{\mathbf{Z}}_C \cdot \mathbf{a}_C \quad (3.114)$$

is minimized. Using the definition of CFIE in (3.3), the residual vector for CFIE can be written as

$$\mathbf{r}_C = \alpha \mathbf{r}_{C \rightarrow E} + (1 - \alpha) \mathbf{r}_{C \rightarrow M}, \quad (3.115)$$

where

$$\mathbf{r}_{C \rightarrow E} = \mathbf{w}_E - \bar{\mathbf{Z}}_E \cdot \mathbf{a}_C \quad (3.116)$$

$$\mathbf{r}_{C \rightarrow M} = \mathbf{w}_M - \bar{\mathbf{Z}}_M \cdot \mathbf{a}_C \quad (3.117)$$

are residual vectors obtained by testing the CFIE solution in EFIE and MFIE systems, respectively.

An important observation is that the minimization of the norm of $\mathbf{r}_C$ does not require the minimization of norms of $\mathbf{r}_{C \rightarrow E}$ and $\mathbf{r}_{C \rightarrow M}$. As an example, we consider the solution of radiation and scattering problems involving a $\lambda \times \lambda \times \lambda$ conducting box, when problems are formulated with CFIE. In the radiation problem, the box is excited with a Hertzian dipole located inside the box at $x = 0.1\lambda$. In the scattering problem, the box is illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. Discretizations of problems with the RWG functions on $\lambda/10$ triangles lead to 2052×2052 matrix equations. Matrix elements are calculated with a maximum of 1% error, and iterative solutions are performed by using the BiCGStab algorithm. Iterations are stopped when the residual error

$$\Delta_{res}^C = \frac{\|\mathbf{r}_C\|_2}{\|\mathbf{w}_C\|_2}, \quad (3.118)$$

where $\|\cdot\|_2$ represents the 2-norm of a vector, is reduced to below 10^{-6} .

Figure 3.53 presents the first 30 elements of residual vectors $\mathbf{r}_C$, $\mathbf{r}_{C \rightarrow E}$, and $\mathbf{r}_{C \rightarrow M}$, denoted by “CFIE”, “CFIE in EFIE”, and “CFIE in MFIE”, respectively, when $\alpha = 0.2$ in the CFIE solution of the radiation problem involving a $\lambda \times \lambda \times \lambda$ conducting box. We observe that $\mathbf{r}_{C \rightarrow E}$ and $\mathbf{r}_{C \rightarrow M}$ involve significantly larger elements than $\mathbf{r}_C$. Furthermore, elements of $\mathbf{r}_{C \rightarrow E}$ are exactly four times those of $\mathbf{r}_{C \rightarrow M}$. This is because, as the norm of $\mathbf{r}_C$ is minimized, elements of $\mathbf{r}_{C \rightarrow E}$ and $\mathbf{r}_{C \rightarrow M}$ are scaled with respect to each other, i.e.,

$$\mathbf{r}_{C \rightarrow E} \approx -\frac{(1-\alpha)}{\alpha} \mathbf{r}_{C \rightarrow M}, \quad (3.119)$$

instead of converging to zero. For $\alpha = 0.2$, the ratio of $-(1-\alpha)/\alpha = -4$ is manifested in Figure 3.53. More precisely, the error criteria for the iterative solution, i.e.,

$$\Delta_{res}^C = \frac{\|\mathbf{r}_C\|_2}{\|\mathbf{w}_C\|_2} = \frac{\|\alpha \mathbf{r}_{C \rightarrow E} + (1-\alpha) \mathbf{r}_{C \rightarrow M}\|_2}{\|\mathbf{w}_C\|_2} \leq 10^{-6}, \quad (3.120)$$

is achieved without the minimization of $\|\mathbf{r}_{C \rightarrow E}\|_2$ and $\|\mathbf{r}_{C \rightarrow M}\|_2$.

Figure 3.54 presents the first 30 elements of residual vectors $\mathbf{r}_C$, $\mathbf{r}_{C \rightarrow E}$, and $\mathbf{r}_{C \rightarrow M}$, when $\alpha = 0.5$ in the CFIE solution of the scattering problem involving a $\lambda \times \lambda \times \lambda$ conducting box. Similar to the previous example, $\mathbf{r}_{C \rightarrow E}$ and $\mathbf{r}_{C \rightarrow M}$, denoted by “CFIE in EFIE” and “CFIE in MFIE”, respectively, involve significantly larger elements than $\mathbf{r}_C$, denoted by “CFIE”. As opposed to the previous solution, however, $\mathbf{r}_{C \rightarrow E} = -\mathbf{r}_{C \rightarrow M}$ since $-(1-\alpha)/\alpha = -1$. Depending on the value of α , compatibility of the CFIE solution with EFIE and MFIE systems varies, but the CFIE solution cannot satisfy both systems at the same time with the desired level of accuracy.

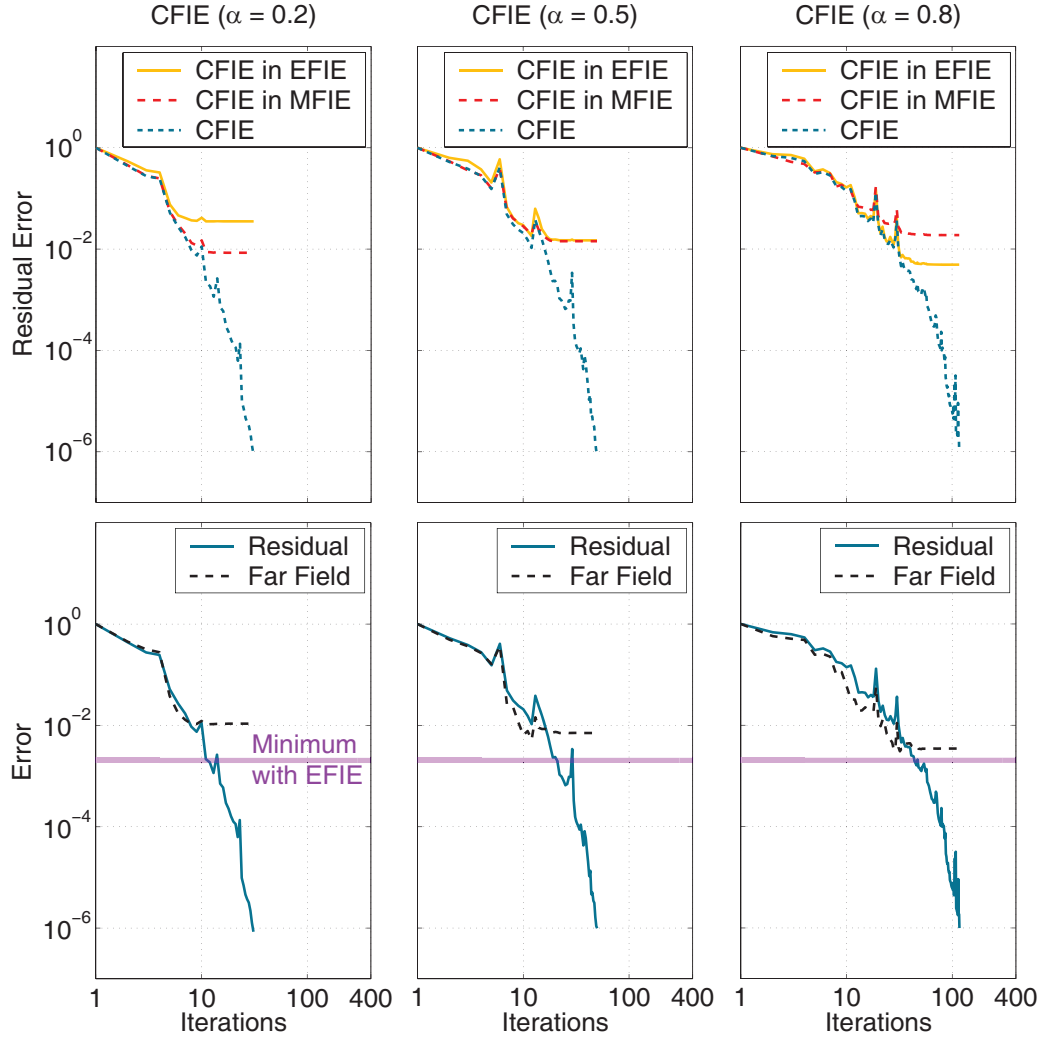


Figure 3.55: Iterative solutions of a radiation problem involving a $\lambda \times \lambda \times \lambda$ conducting box excited with a Hertzian dipole located inside the box at $x = 0.1\lambda$. Residual errors and the far-field error defined in (3.122) are plotted as a function BiCGStab iterations for CFIE when $\alpha=0.2$, 0.5 , and 0.8 .

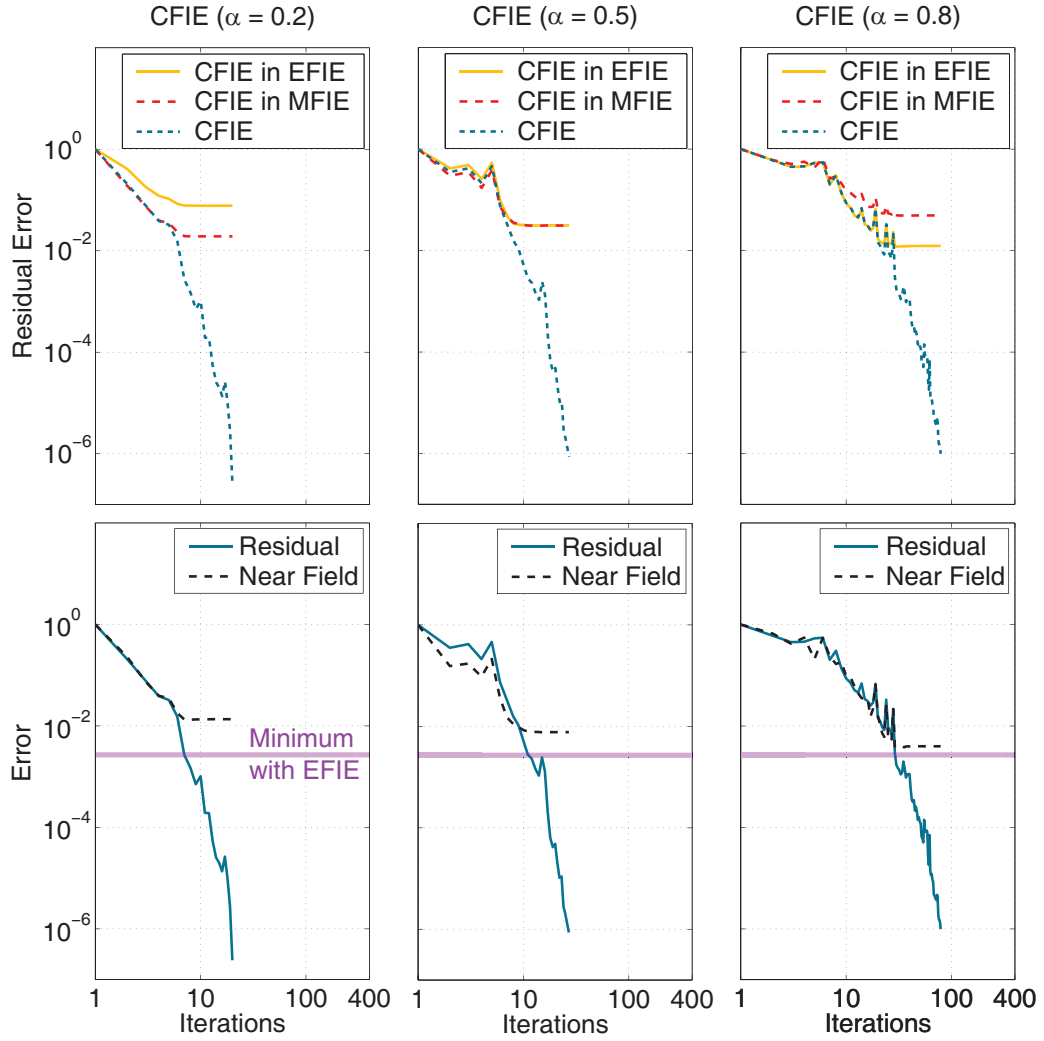


Figure 3.56: Iterative solutions of a scattering problem involving a $\lambda \times \lambda \times \lambda$ conducting box illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. Residual errors and the near-field error defined in (3.124) are plotted as a function BiCGStab iterations for CFIE when $\alpha=0.2$, 0.5 , and 0.8 .

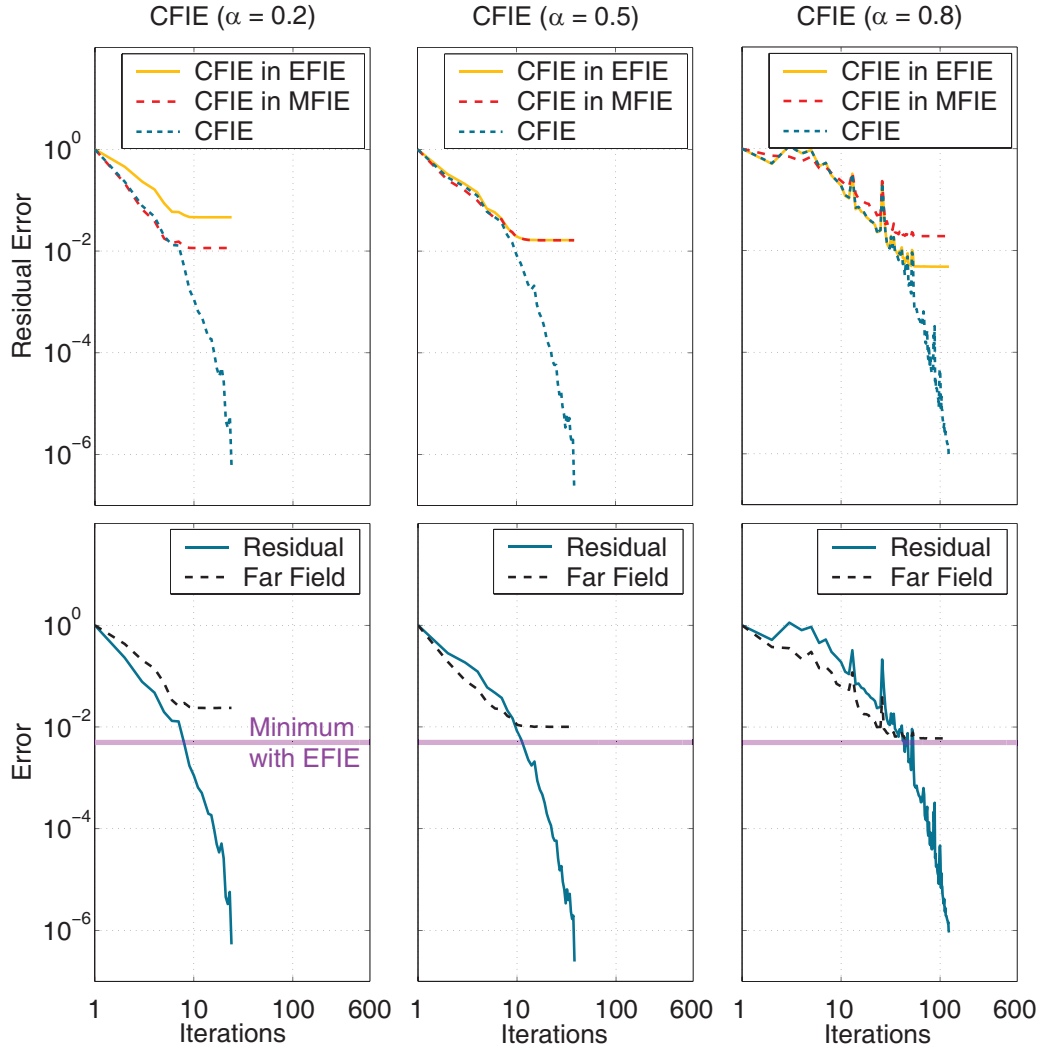


Figure 3.57: Iterative solutions of a scattering problem involving a conducting sphere with a radius of 1.5λ illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. Residual errors and the far-field error defined in (3.125) are plotted as a function BiCGStab iterations for CFIE when $\alpha=0.2, 0.5$, and 0.8 .

3.9.1 Convergence to Minimum Achievable Error

In an iterative solution of a matrix equation obtained from the discretization of an integral-equation formulation, accuracy of the solution is saturated and the solution error cannot be reduced any further after a number of iterations. This is because there are various error sources, such as the numerical calculation of matrix elements, and the overall error cannot be minimized by only reducing the residual error. Then, in an iterative solution, a further reduction of the residual error becomes meaningless after the minimum solution error is achieved.

A CFIE solution can be significantly incompatible with EFIE and MFIE systems, even when the residual error is small. In fact, an iterative solution of CFIE involves a breakpoint, where a further reduction of the residual error does not improve the solution in terms of compatibility with EFIE and MFIE. Since EFIE is more accurate than CFIE, this breakpoint, where the compatibility of the CFIE solution with the EFIE system is maximized, corresponds to the last useful iteration to obtain the highest possible accuracy with CFIE. We note that the minimum achievable error in an EFIE solution depends on various error sources, and not only on the reduction of the residual error. In the case of CFIE, however, the excessive discretization error in the MFIE part becomes a major error source, and the minimum achievable error is closely related to the compatibility of the CFIE solution with the EFIE system.

As an example, Figure 3.55 presents iterative solutions of a radiation problem involving a $\lambda \times \lambda \times \lambda$ conducting box excited with a Hertzian dipole located inside the box at $x = 0.1\lambda$. The radiation problem is formulated with CFIE when $\alpha = 0.2, 0.5$, and 0.8 . The residual error in (3.118) is plotted with respect to BiCGStab iterations. We also plot

$$\Delta_{res}^{C \rightarrow E} = \frac{\|\mathbf{r}_{C \rightarrow E}\|_2}{\|\mathbf{w}_E\|_2} \quad \text{and} \quad \Delta_{res}^{C \rightarrow M} = \frac{\|\mathbf{r}_{C \rightarrow M}\|_2}{\|\mathbf{w}_M\|_2} \quad (3.121)$$

denoted by “CFIE in EFIE” and “CFIE in MFIE”, respectively, in the first row of Figure 3.55. Our observations are as follows:

- The number of iterations is 31, 50, and 117 when $\alpha = 0.2$, 0.5, and 0.8, respectively.
- Compatibility of the CFIE solution with EFIE and MFIE systems is saturated at about the 10th, 20th, and 50th iterations, respectively. Values of $\Delta_{res}^{C \rightarrow E}$ and $\Delta_{res}^{C \rightarrow M}$, which are obtained by testing the coefficient vector in EFIE and MFIE systems, respectively, become steady after these breakpoints, and they do not drop further as iterations continue.
- The maximum compatibility of the CFIE solution with the EFIE or the MFIE system depends on the value of α .

For the radiation problem, we also calculate the error in the far-zone electric field at each iteration in addition to the residual error. Incident and total electric fields are sampled in the far-zone on the x - y plane at $(r = \infty, \theta = \pi/2, \phi_n)$, where $\phi_n = (n - 1)\pi/180$ for $n = 1, 2, \dots, 360$. The total electric field is obtained by combining the incident field radiated by the Hertzian dipole and the secondary field due to the induced electric current on the surface of the cube. Coefficients for the discretized induced current are provided by the iterative algorithm at each iteration. Theoretically, the total electric field should be zero outside the cube, because the source is shielded by a PEC closed surface. Then, the far-field error is defined as

$$\Delta_{FF} = \frac{\|\mathbf{E}_{tot}^\infty\|_2}{\|\mathbf{E}_{inc}^\infty\|_2}, \quad (3.122)$$

where $\mathbf{E}_{tot}^\infty$ and $\mathbf{E}_{inc}^\infty$ are arrays of 360 elements containing total and incident electric fields in the far zone. In the second row of Figure 3.55, the far-field error (3.122) in CFIE solutions is plotted with respect to iterations. The minimum achievable error provided by EFIE is also indicated in each plot. Our observations can be listed as follows:

- Accuracy of CFIE solutions is saturated and cannot be improved any further after some numbers of iterations.
- The last useful iteration and the minimum achievable accuracy in a CFIE solution depends on the value of α .
- For each value of α , the last useful iteration in terms of the solution accuracy corresponds to the breakpoint, where the compatibility of the CFIE solution with the EFIE system is saturated.

As a result, the compatibility of the CFIE solution with the EFIE system is an important indicator in determining the last useful iteration for the highest possible accuracy with CFIE.

Figure 3.56 presents iterative solutions of a scattering problem involving a $\lambda \times \lambda \times \lambda$ conducting box illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. Similar to the previous example, the residual error is plotted as a function of BiCGStab iterations for CFIE with $\alpha = 0.2, 0.5$, and 0.8 . In the first row of Figure 3.56, we again plot $\Delta_{res}^{C \rightarrow E}$ and $\Delta_{res}^{C \rightarrow M}$ to demonstrate the compatibility of CFIE solutions with EFIE and MFIE systems. For this problem, saturation occurs at the 7th, 10th, and 30th iterations, respectively, when $\alpha = 0.2, 0.5$, and 0.8 . During solutions, we also calculate the error in the near-zone electric field at each iteration. In this case, we sample the total electric field inside the cube at $19 \times 19 = 361$ regularly-spaced points, i.e.,

$$E_{tot}[n] = \sqrt{|E_{tot,x}(x_n, y_n, z)|^2 + |E_{tot,y}(x_n, y_n, z)|^2 + |E_{tot,z}(x_n, y_n, z)|^2}, \quad (3.123)$$

where $E_{tot,x}$, $E_{tot,y}$, and $E_{tot,z}$ are x , y , and z components of the total electric field, $z = 0$, and $-\lambda/2 \leq x_n, y_n \leq \lambda/2$ for $n = 1, 2, \dots, 361$. Because of the shielding effect of the PEC closed surface, the total electric field, which is obtained by combining the incident plane wave and the secondary field due to the induced

electric current, should be zero inside the cube. Then, we define the near-field error as

$$\Delta_{NF} = \frac{\|\mathbf{E}_{tot}\|_2}{\|\mathbf{E}_{inc}\|_2}, \quad (3.124)$$

where $\mathbf{E}_{tot}$ and $\mathbf{E}_{inc}$ are arrays of 361 elements containing total and incident electric fields at sample points. As depicted in the second row of Figure 3.56, the last useful iteration for the minimum possible error with CFIE coincides with the breakpoint for the compatibility of the solution with the EFIE system.

Finally, Figure 3.57 presents the iterative solutions of a scattering problem involving a conducting sphere with a radius of 1.5λ . The sphere is illuminated by a plane wave propagating in the $-z$ direction with the electric field polarized in the x direction. Discretization of the problem with the RWG functions on $\lambda/10$ triangles lead to 8364×8364 matrix equations. Iterative solutions are again performed by using the BiCGStab algorithm. As depicted in the first row of Figure 3.57, the compatibility of the CFIE solutions with the EFIE system saturates at the 9th, 13th, and 56th iterations, respectively, when $\alpha = 0.2, 0.5$, and 0.8 . During iterative solutions of this problem, the far-field error is calculated [similar to (2.229)] as

$$\Delta_R = \frac{\|\mathbf{E}_{C,\theta}^\infty - \mathbf{E}_{A,\theta}^\infty\|_2}{\|\mathbf{E}_{A,\theta}^\infty\|_2}, \quad (3.125)$$

where $\mathbf{E}_{C,\theta}^\infty$ and $\mathbf{E}_{A,\theta}^\infty$ are arrays of 360 elements containing computational and analytical values for the co-polar electric field on the x - z plane. Figure 3.57 confirms the previous results that the minimum achievable error and the last useful iteration in CFIE solutions are closely related to the compatibility of solutions with the EFIE system.

Chapter 4

Efficient Solutions of Electromagnetics Problems with the Multilevel Fast Multipole Algorithm

In this chapter, we present efficient implementations of MLFMA for the solution of large-scale electromagnetics problems. We investigate MLFMA in detail by considering its major stages, and we focus on interpolations and antinterpolations during aggregation and disaggregation stages. Efficient and accurate interpolation of translation operators is also discussed. Finally, we present LF-MLFMA for the efficient solution of electromagnetics problems involving small details with respect to wavelength. For the sake of brevity, we consider only PEC objects. But, as discussed in Chapter 2, MLFMA and all procedures presented in this chapter can easily be applied to electromagnetics problems involving dielectric, multiple dielectric, and dielectric-metallic objects.

4.1 Multilevel Fast Multipole Algorithm

Matrix equations obtained by the discretization of integral-equation formulations can be solved iteratively, where required MVMs are accelerated by MLFMA. For an $N \times N$ dense matrix equation, MLFMA reduces the complexity of an MVM from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$. In general, MLFMA splits MVMs as

$$\bar{\mathbf{Z}} \cdot \mathbf{x} = \bar{\mathbf{Z}}_{NF} \cdot \mathbf{x} + \bar{\mathbf{Z}}_{FF} \cdot \mathbf{x}, \quad (4.1)$$

where near-field interactions denoted by $\bar{\mathbf{Z}}_{NF}$ are calculated directly and stored in memory to perform partial multiplications $\bar{\mathbf{Z}}_{NF} \cdot \mathbf{x}$, while multiplications involving far-field interactions, i.e., $\bar{\mathbf{Z}}_{FF} \cdot \mathbf{x}$, are performed efficiently.

4.1.1 Recursive Clustering

Without losing generality, we consider a smooth object with an electrical dimension of kD , where $k = 2\pi/\lambda$ is the wavenumber. Discretization (triangulation) of the object with $\lambda/10$ mesh size leads to N unknowns, where $N = \mathcal{O}(k^2 D^2)$. The fundamental idea in MLFMA is to replace interactions of basis and testing functions with interactions of groups of functions. Specifically, far-field interactions between distant basis and testing functions are calculated efficiently in a group-by-group manner. In addition, in order to calculate far-field interactions in a multilevel scheme, a tree structure is constructed by placing the object in a cubic box and recursively dividing the computational domain into subdomains, until the box size is about 0.25λ . A multilevel tree structure with $L = \mathcal{O}(\log(kD)) = \mathcal{O}(\log N)$ levels is obtained by considering nonempty boxes. Although the actual number of levels is $(L + 2)$, the highest two levels are not used directly in MLFMA. At level l from 1 to L , the number of boxes can be approximated as

$$N_l \approx 4^{(1-l)} N_1, \quad (4.2)$$

where $N_1 = \mathcal{O}(N)$. In other words, the number of boxes decreases approximately by a factor of four from a level to the next higher level.

The tree structure in MLFMA can be constructed by using a top-down or a bottom-up scheme. In the top-down scheme, the size of the largest cube enclosing the object is minimized, while the size of the smallest boxes at the lowest level depends on the size of the object and the number of levels. In the bottom-up scheme, however, the size of the smallest boxes is fixed to some value (such as 0.25λ), and box sizes at higher levels are recursively doubled until the whole object is enclosed by the largest box. For a given problem, one of the two strategies can be preferable in terms of efficiency and accuracy.

4.1.2 Far-Field Interactions

In MLFMA, far-field interactions are calculated in a group-by-group manner using the factorization and diagonalization of the homogeneous-space Green's function [79],[80]. In each MVM, three main stages, i.e., aggregation, translation, and disaggregation, are performed as described below.

Aggregation

In this stage, radiated fields of boxes are calculated from the bottom of the tree structure to the highest level ($l = L$). At the lowest level, radiation patterns of basis functions, which are calculated and stored during the setup stage, are multiplied with coefficients provided by the iterative solver and combined to obtain radiated fields of the smallest boxes. The radiated field of a box C at a reference point $\mathbf{r}_C$ can be calculated as

$$\mathbf{S}_C(\mathbf{r}_C, \mathbf{k}) = \sum_{n \in C} \mathbf{x}[n] \mathbf{S}_n(\mathbf{r}_C, \mathbf{k}), \quad (4.3)$$

where $\mathbf{S}_n$ represents the radiation pattern of the n th basis function inside the box, and $\mathbf{x}[n]$ represents coefficients provided by the iterative solver. In (4.3), radiation functions $\mathbf{S}_C$ and $\mathbf{S}_n$ have only θ and ϕ components, and they are functions of the angular direction $\hat{\mathbf{k}}$. Radiated fields of boxes at higher levels ($l > 1$) are obtained by shifting and combining radiated fields of boxes at lower levels, i.e.,

$$\mathbf{S}_C(\mathbf{r}_C, \mathbf{k}) = \sum_{C' \in C} \beta(\mathbf{k}, \mathbf{r}_C - \mathbf{r}_{C'}) \mathbf{S}_{C'}(\mathbf{r}_{C'}, \mathbf{k}), \quad (4.4)$$

where

$$\beta(\mathbf{k}, \mathbf{r}_C - \mathbf{r}_{C'}) = \exp [i\mathbf{k} \cdot (\mathbf{r}_C - \mathbf{r}_{C'})]. \quad (4.5)$$

Translation

In the translation stage of MLFMA, radiated fields of boxes are translated into incoming fields for other boxes. The incoming field to the center of a box C (due to translations) is calculated as

$$\mathbf{G}_C(\mathbf{r}_C, \mathbf{k}) = \sum_{C' \in \mathcal{F}\{C\}} \alpha_{\tau(l)}(\mathbf{k}, \mathbf{r}_C - \mathbf{r}_{C'}) \mathbf{S}_{C'}(\mathbf{r}_{C'}, \mathbf{k}), \quad (4.6)$$

where $\mathcal{F}\{C\}$ represents boxes in the far-field list of C , and $\alpha_{\tau(l)}$ is the diagonal translation operator, as defined in (2.134).

Disaggregation

In the disaggregation stage, total incoming fields at box centers are calculated from the top of the tree structure to the lowest level. At the highest level, the total incoming field for a box is obtained by the combination of incoming fields due to translations. At lower levels, however, the incoming field to the center of a box involves a contribution from the incoming field to the center of its parent box. For a box C at a level $l < L$, the total incoming field is calculated as

$$\mathbf{G}_C^+(\mathbf{r}_C, \mathbf{k}) = \mathbf{G}_C(\mathbf{r}_C, \mathbf{k}) + \beta(\mathbf{k}, \mathbf{r}_C - \mathbf{r}_{C'}) \mathbf{G}_{C'}^+(\mathbf{r}_{C'}, \mathbf{k}), \quad (4.7)$$

where $\mathbf{G}_C$ is the incoming field due to translations, and C' is the parent box of C , i.e., $C \in C'$. Following the disaggregation operations at the lowest level, incoming fields are received by testing functions via an angular integration as

$$\sum_{n=1}^N \bar{\mathbf{Z}}_{FF}[m, n] \mathbf{a}[n] = \left(\frac{ik}{4\pi} \right)^2 \int d^2 \hat{\mathbf{k}} \mathbf{F}_m(\mathbf{r}_C, \mathbf{k}) \cdot \mathbf{G}_C^+(\mathbf{r}_C, \mathbf{k}), \quad (4.8)$$

where $\mathbf{F}_m$ represents the receiving pattern of the m th testing function inside a box C at the lowest level. Similar to radiation patterns of basis functions, receiving patterns of testing functions are also calculated during the setup of MLFMA.

4.1.3 Near-Field Interactions

In MLFMA, there are also $\mathcal{O}(N^2/N_1) = \mathcal{O}(N)$ near-field interactions, which are calculated directly in the setup stage of the program and stored in memory to be used multiple times during the iterative solution. These interactions are between basis and testing functions that are located in the same box or in two touching boxes at the lowest level. We note that the near-field matrix $\bar{\mathbf{Z}}_{NF}$ has a sparsity of $\mathcal{O}(1/N_1) = \mathcal{O}(1/N)$.

4.1.4 Sampling

For each box in the tree structure, radiated and incoming fields are sampled on the unit sphere. We choose samples regularly spaced in the ϕ direction and use the Gauss-Legendre quadrature in the θ direction [80]. For level $l = 1, 2, \dots, L$, the number of samples is $S_l^\theta = [\tau(l) + 1]$ and $S_l^\phi = 2[\tau(l) + 1]$ along θ and ϕ directions, respectively, where $\tau(l)$ is the truncation number for the series in (2.134) to calculate translation operators. For each level l , the truncation number is determined by the excess bandwidth formula [85] considering the worst-case

scenario [97], i.e.,

$$\tau(l) \approx 1.73ka_l + 2.16(d_0)^{2/3}(ka_l)^{1/3}, \quad (4.9)$$

where a_l is the box size and d_0 is the desired digits of accuracy. We note that the sampling rate depends on the box size as measured by the wavelength ($ka_l = 2\pi a_l/\lambda$), and the total number of samples can be approximated as

$$S_l = S_l^\theta S_l^\phi \approx 2^{(l-1)} S_1^\theta \times 2^{(l-1)} S_1^\phi = 4^{(l-1)} S_1, \quad (4.10)$$

where $S_1 = \mathcal{O}(1)$.

Considering the sampling along the angular directions θ and ϕ , aggregations at the lowest level can be rewritten as

$$\begin{bmatrix} \mathbf{s}_C^\theta & \mathbf{s}_C^\phi \end{bmatrix}_{S_1 \times 2} = \sum_{n \in C} \mathbf{x}[n] \begin{bmatrix} \mathbf{s}_n^\theta & \mathbf{s}_n^\phi \end{bmatrix}_{S_1 \times 2}, \quad (4.11)$$

where $\mathbf{s}_n^\theta$ and $\mathbf{s}_n^\phi$ are arrays of S_1 elements containing θ and ϕ components of the radiation pattern of the n th basis function. Similarly, $\mathbf{s}_C^\theta$ and $\mathbf{s}_C^\phi$ in (4.11) are arrays of S_1 elements containing θ and ϕ components of the radiated field of a box C at the lowest level. For a box C at a higher level ($l > 1$),

$$\begin{bmatrix} \mathbf{s}_C^\theta & \mathbf{s}_C^\phi \end{bmatrix}_{S_l \times 2} = \sum_{C' \in C} [\bar{\boldsymbol{\beta}}_{C' \rightarrow C}^\diamond]_{S_l \times S_l} \cdot [\bar{\boldsymbol{\Gamma}}_{(l-1) \rightarrow l}]_{S_l \times S_{l-1}} \cdot \begin{bmatrix} \mathbf{s}_{C'}^\theta & \mathbf{s}_{C'}^\phi \end{bmatrix}_{S_{l-1} \times 2}, \quad (4.12)$$

where $\bar{\boldsymbol{\beta}}_{C' \rightarrow C}^\diamond$ is a diagonal matrix containing shift functions in (4.5) and $\bar{\boldsymbol{\Gamma}}_{(l-1) \rightarrow l}$ is a sparse interpolation (upsampling) matrix. We use Lagrange interpolation to match different sampling rates of consecutive levels [1].

Similar to aggregations, translation operations can be written in the matrix form as

$$\begin{bmatrix} \mathbf{g}_C^\theta & \mathbf{g}_C^\phi \end{bmatrix}_{S_l \times 2} = \sum_{C' \in \mathcal{F}\{C\}} [\bar{\boldsymbol{\alpha}}_{C' \rightarrow C}^\diamond]_{S_l \times S_l} \cdot \begin{bmatrix} \mathbf{s}_{C'}^\theta & \mathbf{s}_{C'}^\phi \end{bmatrix}_{S_l \times 2}, \quad (4.13)$$

where $\mathbf{g}_C^\theta$ and $\mathbf{g}_C^\phi$ are arrays of S_l elements containing θ and ϕ components of the incoming field to the center of a box C , and $\bar{\boldsymbol{\alpha}}_{C' \rightarrow C}^\diamond$ is a diagonal translation

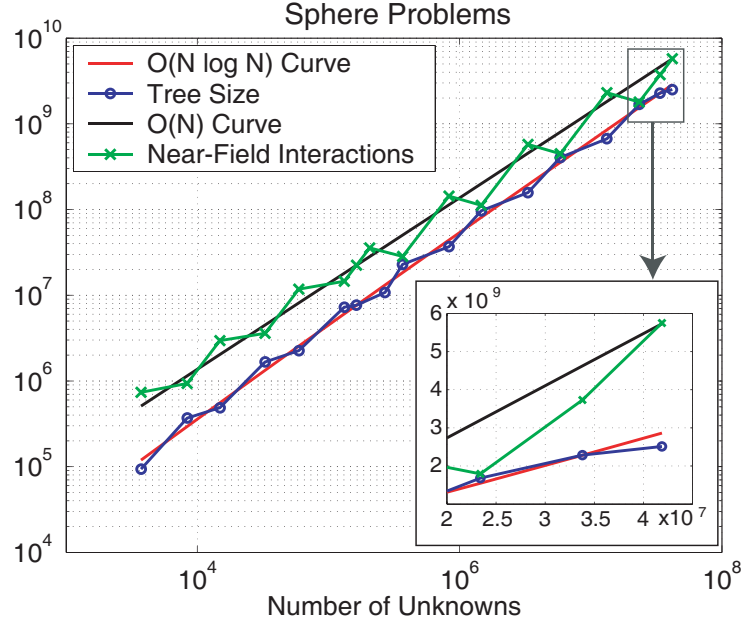


Figure 4.1: Tree size and the number of near-field interactions for the solution of sphere problems using top-down scheme to construct the multilevel tree.

matrix. Then, for a box $C \in C'$ at a level $l < L$, the total incoming field is calculated as

$$\begin{aligned} \begin{bmatrix} \mathbf{g}_C^{+, \theta} & \mathbf{g}_C^{+, \phi} \end{bmatrix}_{S_l \times 2} &= \begin{bmatrix} \mathbf{g}_C^\theta & \mathbf{g}_C^\phi \end{bmatrix}_{S_l \times 2} \\ &+ [\bar{\mathbf{\Gamma}}_{(l+1) \rightarrow l}]_{S_l \times S_{l+1}} \cdot [\bar{\boldsymbol{\beta}}_{C' \rightarrow C}^\diamond]_{S_{l+1} \times S_{l+1}} \cdot \begin{bmatrix} \mathbf{g}_{C'}^{+, \theta} & \mathbf{g}_{C'}^{+, \phi} \end{bmatrix}_{S_{l+1} \times 2}, \end{aligned} \quad (4.14)$$

where $\bar{\mathbf{\Gamma}}_{(l+1) \rightarrow l}$ is a sparse interpolation (downsampling) matrix. Finally, at the lowest level, incoming fields are received by testing functions as

$$\begin{aligned} \sum_{n=1}^N \bar{\mathbf{Z}}_{FF}[m, n] \mathbf{a}[n] &\approx \left(\frac{ik}{4\pi} \right)^2 \left\{ [\mathbf{f}_m^\theta]_{1 \times S_1} \cdot [\bar{\mathbf{W}}]_{S_1 \times S_1} \cdot [\mathbf{g}_C^{+, \theta}]_{S_1 \times 1} \right. \\ &\quad \left. + [\mathbf{f}_m^\phi]_{1 \times S_1} \cdot [\bar{\mathbf{W}}]_{S_1 \times S_1} \cdot [\mathbf{g}_C^{+, \phi}]_{S_1 \times 1} \right\} \end{aligned} \quad (4.15)$$

for $m \in C$. In (4.15), $\bar{\mathbf{W}}$ is a diagonal matrix involving weights of the angular integration in (4.8), while $\mathbf{f}_m^\theta$ and $\mathbf{f}_m^\phi$ are arrays of S_1 elements containing θ and ϕ components of the receiving pattern of the m th testing function.

4.1.5 Computational Requirements

When MLFMA is used, the memory requirement for a MVM (Θ^{mem}) is proportional to the tree size S_T , i.e.,

$$\Theta^{\text{mem}} \propto S_T = \sum_{l=1}^L N_l S_l = \sum_{l=1}^L 2N_l[\tau(l) + 1]^2. \quad (4.16)$$

The processing time (Θ^{time}) is also related to the tree size as

$$\Theta^{\text{time}} \propto \sum_{l=1}^L c_l N_l S_l = \sum_{l=1}^L 2c_l N_l[\tau(l) + 1]^2, \quad (4.17)$$

where c_l represents relative weights for levels $l = 1, 2, \dots, L$. Using $N_l \approx 4^{(1-l)} N_1$ and $S_l \approx 4^{(l-1)} S_1$,

$$N_l S_l \approx N_1 S_1 = \mathcal{O}(N) \quad (4.18)$$

and all levels of MLFMA have equal importance with $\mathcal{O}(N)$ complexity in terms of processing time and memory. Then,

$$\Theta^{\text{mem}} = \mathcal{O}(L)\mathcal{O}(N) = \mathcal{O}(N \log N) \quad (4.19)$$

$$\Theta^{\text{time}} = \mathcal{O}(L)\mathcal{O}(N) = \mathcal{O}(N \log N), \quad (4.20)$$

i.e., the complexity of MLFMA is $\mathcal{O}(N \log N)$. Although this is true in general, measurements may present deviations from the ideal case depending on the construction technique for the tree structure, even when N is very large. For example, in Figure 4.1, tree size (S_T) is plotted as a function of the number of unknowns for the solution of scattering problems involving sphere geometries of various sizes, when the top-down scheme is used to construct the multilevel tree and the number of accurate digits d_0 is 2. The radius of the sphere changes from λ to 110λ corresponding to 3723 and 41,883,648 unknowns, respectively, using $\lambda/10$ triangulation. We observe that the tree size oscillates around the $\mathcal{O}(N \log N)$ curve. Due to such local variations, processing time and memory requirement for MVMs with respect to N cannot be strictly proportional to $N \log N$. As an example, the tree size grows only by 50% when the number of

unknowns increases from 23,405,664 to 41,883,648. Then, the memory required for MVMs increases by about 50%, which is below the asymptotical estimation of 85%.

Processing time for the initial setup of MLFMA (prior to the iterative solution) is dominated by calculating near-field interactions and it is proportional to N^2/N_1 . The amount of memory to store near-field interactions is also significant. Asymptotically, $N_1 = \mathcal{O}(N)$ and near-field interactions has a complexity of $\mathcal{O}(N)$. However, similar to MVMs, local variations in the processing time and memory requirement for near-field interactions may exhibit behavior different than the asymptotical estimation. This is because, as depicted in Figure 4.1, the number of near-field interactions oscillates around the $\mathcal{O}(N)$ curve when a top-down scheme is used to construct the tree structure. Consequently, variation in processing time and memory with respect to N can be higher or lower than the asymptotically linear estimate.

Radiation and receiving patterns of basis and testing functions are sampled according to the sampling rate of boxes at the lowest level. These patterns are calculated analytically and stored in memory before iterative solutions. We reduce the number of samples to $S_1 = [\tau(1)/2 + 1] \times [2\tau(1) + 2]$ using the symmetry of patterns. Although the processing time to calculate radiation and receiving patterns is negligible, significant amount of memory is required to store them, which is comparable to the memory used for near-field interactions.

Similar to radiation and receiving patterns, translation operators are also calculated and stored in memory before iterations. For each box at any level, there are $\mathcal{O}(1)$ boxes at the same level to translate the radiated field to. In addition, using the symmetry of cubic (identical) boxes, the number of different translation operators is $7^3 - 3^3 = 316$, independent of the level [84]. Although using

Table 4.1: Major Parts of MLFMA and Their Computational Requirements

	Memory	
Part	Proportional to	Complexity
MVM	$\sum_{l=1}^L N_l[\tau(l) + 1]^2$	$\mathcal{O}(N \log N)$
Radiation and Receiving Patterns	$N[\tau(1) + 1]^2$	$\mathcal{O}(N)$
Translation Operators	$\sum_{l=1}^L d_l[\tau(l) + 1]^2$	$\mathcal{O}(N)$
Near-Field Interactions	N^2/N_1	$\mathcal{O}(N)$
	Processing Time	
Part	Proportional to	Complexity
MVM	$\sum_{l=1}^L c_l N_l[\tau(l) + 1]^2$	$\mathcal{O}(N \log N)$
Radiation and Receiving Patterns	$N[\tau(1) + 1]^2$	$\mathcal{O}(N)$
Translation Operators	$\sum_{l=1}^L d_l[\tau(l) + 1]^2$	$\mathcal{O}(N)$
Near-Field Interactions	N^2/N_1	$\mathcal{O}(N)$
Note: c_l and d_l represent relative weights for levels $l = 1, 2, \dots, L$.		

the symmetry reduces the number of translation operators significantly, we also need interpolation methods to calculate these operators in $\mathcal{O}(N)$ time during the setup stage [90]. With the optimization of interpolations, both calculation time and memory for translation operators are insignificant compared to other parts of the implementation, especially when the problem size is large.

As a summary, Table 4.3 lists major parts of MLFMA and their computation requirements for the solution of large problems.

4.1.6 Anterpolation

Interpolations during the disaggregation stage can be replaced with anterpolations (transpose interpolations) to improve the accuracy of MLFMA [86]. Consider the interaction of the n th basis function and the m th testing function via a translation at level $l \neq 1$. Then, the corresponding matrix element can be

written as

$$\bar{\mathbf{Z}}[m, n] = \left(\frac{ik}{4\pi} \right)^2 \int d^2 \hat{\mathbf{k}} \mathbf{F}_m(\mathbf{r}_C, \mathbf{k}) \cdot \mathbf{G}_n(\mathbf{r}_C, \mathbf{k}), \quad (4.21)$$

where $\mathbf{G}_n$ is the incoming field due to the n th basis function. Equation (4.21) represents the calculation of a single interaction in MLFMA. (We note that this expression is not used explicitly, because interactions are actually calculated in a group-by-group manner.) Since $\tau_l > \tau_1$ for $l > 1$, the sampling rate for $\mathbf{G}_n$ is larger than the sampling rate for $\mathbf{F}_m$. An accurate way to calculate (4.21) could be to interpolate $\mathbf{F}_m$ and to perform the numerical integration in the fine grid. This leads to

$$\begin{aligned} \bar{\mathbf{Z}}[m, n] \approx \left(\frac{ik}{4\pi} \right)^2 \bigg\{ & [\mathbf{g}_n^\theta]_{1 \times S_l} \cdot [\bar{\mathbf{W}}]_{S_l \times S_l} \cdot [\bar{\Gamma}_{1 \rightarrow l}]_{S_l \times S_1} \cdot [\mathbf{f}_m^\theta]_{S_1 \times 1} \\ & + [\mathbf{g}_n^\phi]_{1 \times S_l} \cdot [\bar{\mathbf{W}}]_{S_l \times S_l} \cdot [\bar{\Gamma}_{1 \rightarrow l}]_{S_l \times S_1} \cdot [\mathbf{f}_m^\phi]_{S_1 \times 1} \bigg\}, \end{aligned} \quad (4.22)$$

where $\mathbf{g}_n^\theta$ and $\mathbf{g}_n^\phi$ are arrays of S_l elements containing θ and ϕ components of the incoming field due to the n th basis function. On the other hand, the procedure in (4.22) contradicts with the structure of MLFMA, where the disaggregation is performed downward from top to bottom of the tree. As given in (4.15), incoming fields must be received at the lowest level, i.e.,

$$\begin{aligned} \bar{\mathbf{Z}}[m, n] \approx \left(\frac{ik}{4\pi} \right)^2 \bigg\{ & [\mathbf{f}_m^\theta]_{1 \times S_1} \cdot [\bar{\mathbf{W}}]_{S_1 \times S_l} \cdot [\bar{\Gamma}_{l \rightarrow 1}]_{S_1 \times S_l} \cdot [\mathbf{g}_n^\theta]_{S_l \times 1} \\ & + [\mathbf{f}_m^\phi]_{1 \times S_1} \cdot [\bar{\mathbf{W}}]_{S_1 \times S_l} \cdot [\bar{\Gamma}_{l \rightarrow 1}]_{S_1 \times S_l} \cdot [\mathbf{g}_n^\phi]_{S_l \times 1} \bigg\}. \end{aligned} \quad (4.23)$$

The procedure in (4.23) fits into the MLFMA structure, but it is inaccurate since the integration is performed in the coarse grid.

In MLFMA, we use another procedure, which is obtained by taking the transpose of (4.22), i.e.,

$$\begin{aligned} \bar{\mathbf{Z}}[m, n] \approx \left(\frac{ik}{4\pi} \right)^2 \bigg\{ & [\mathbf{f}_m^\theta]_{1 \times S_1} \cdot [(\bar{\Gamma}_{1 \rightarrow l})^\dagger]_{S_1 \times S_l} \cdot [\bar{\mathbf{W}}]_{S_l \times S_l} \cdot [\mathbf{g}_n^\theta]_{S_l \times 1} \\ & + [\mathbf{f}_m^\phi]_{1 \times S_1} \cdot [(\bar{\Gamma}_{1 \rightarrow l})^\dagger]_{S_1 \times S_l} \cdot [\bar{\mathbf{W}}]_{S_l \times S_l} \cdot [\mathbf{g}_n^\phi]_{S_l \times 1} \bigg\}, \end{aligned} \quad (4.24)$$

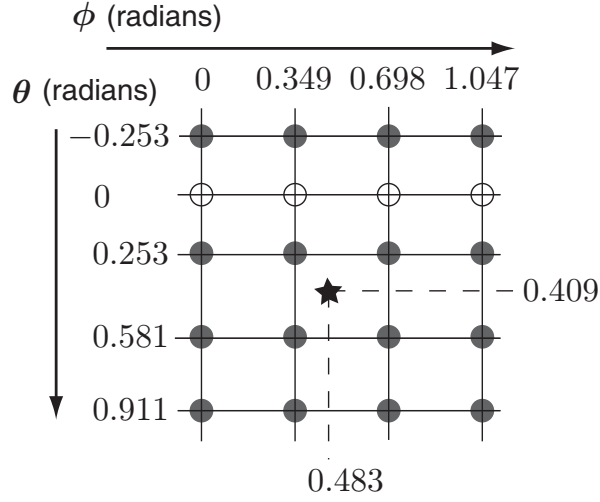


Figure 4.2: Lagrange interpolation employing 4×4 points (shaded circles) located in the coarse grid to evaluate the function at a point (star) located in the fine grid. Sampling values of θ and ϕ are specified in radians and selected from a practical case.

where $(\bar{\mathbf{I}}_{1 \rightarrow l})^\dagger$ is a transpose interpolation (anterpolation) matrix. This procedure is suitable for MLFMA and its accuracy is exactly the same as the accuracy of the hypothetical procedure in (4.22). We note that integration weights related to level l also need to be anterpolated; hence, it is appropriate to introduce them during translations when radiated fields are converted into incoming fields.

4.2 Lagrange Interpolation and Anterpolation

In our MLFMA implementations, we use local Lagrange interpolation, which is preferable due to its favorable computing cost and controllable error [1]. From the local neighborhood of each target point in the fine grid, where the field is to be obtained by interpolation, $2p \times 2p$ points in the coarse grid are selected. This is illustrated in Figure 4.2 for $p = 2$, where values of the field at 4×4 points (shaded circles) are employed to compute the value at the target point (star). Let $f(\theta, \phi)$ be a scalar function representing a radiated or incoming field in MLFMA.

Using a two-dimensional Lagrange interpolation, the value of the function at a target point (θ, ϕ) in the fine grid is obtained by using $2p \times 2p$ samples in the coarse grid, i.e.,

$$f(\theta, \phi) \approx \sum_{j=s+1-p}^{s+p} w_j(\phi) \sum_{i=t+1-p}^{t+p} v_i(\theta) f(\theta_i, \phi_j), \quad (4.25)$$

where $w_j(\phi)$ and $v_i(\theta)$ represent interpolation weights derived as

$$w_j(\phi) = \prod_{\substack{m=s+1-p \\ m \neq j}}^{s+p} \frac{\phi - \phi_m}{\phi_j - \phi_m} \quad (4.26)$$

for the ϕ direction, and

$$v_i(\theta) = \prod_{\substack{n=t+1-p \\ n \neq i}}^{t+p} \frac{\theta - \theta_n}{\theta_i - \theta_n} \quad (4.27)$$

for the θ direction, respectively. We note that reference indices s and t in (4.25)–(4.27) are determined by the location of the target point (θ, ϕ) , with respect to samples in the coarse grid.

Since sampling rates for fields are determined by considering their spectral contents, we keep the number of interpolation points $4p^2$ fixed for entire aggregation and disaggregation stages. In other words, the same number of interpolation points are employed at each level of MLFMA. Although functions being interpolated get richer in terms of harmonic content for higher levels, the sampling rate is also increased so that a fixed number of interpolation points is sufficient to obtain the same level of accuracy in all levels.

Interpolation of a function at $S_l^\theta \times S_l^\phi$ points requires

$$\Theta^{\text{time}} = 4p^2 S_l^\theta S_l^\phi = 8p^2 [\tau(l) + 1]^2 \quad (4.28)$$

operations. If weights in θ and ϕ directions are combined, interpolation in (4.25) can be expressed as a MVM, i.e.,

$$[\mathbf{f}]_{S_l \times 1} \approx [\bar{\mathbf{\Gamma}}_{(l-1) \rightarrow l}]_{S_l \times S_{l-1}} \cdot [\mathbf{f}]_{S_{l-1} \times 1}, \quad (4.29)$$

where $\mathbf{f}$ represents one-dimensional arrays involving samples of f . For interpolations from level $(l-1)$ to l , there are $4p^2$ nonzero elements per row in $\bar{\mathbf{\Gamma}}_{(l-1) \rightarrow l}$. The matrix representation in (4.29) is preferred due to its simplicity, and it is very useful for an easy implementation of antepolations in MLFMA. However, the amount of memory required for the interpolation matrix is proportional to

$$\Theta_{\text{matrix}}^{\text{memory}} = 4p^2 S_l^\theta S_l^\phi = 8p^2 [\tau(l) + 1]^2, \quad (4.30)$$

which can be significant for large problems. Considering the original form in (4.25), it is possible to store interpolation weights along θ and ϕ directions separately, in two arrays of sizes $2p[\tau(l) + 1]$ and $4p[\tau(l) + 1]$, respectively. Then, the total memory used for interpolations from level $(l-1)$ to l becomes

$$\Theta_{\text{array}}^{\text{memory}} = (2p + 4p)[\tau(l) + 1] = 6p[\tau(l) + 1], \quad (4.31)$$

without any change in the number of operations and processing time. The reduction in memory by using the array representation instead of the matrix representation is

$$\frac{\Theta_{\text{array}}^{\text{memory}}}{\Theta_{\text{matrix}}^{\text{memory}}} = \frac{6p[\tau(l) + 1]}{8p^2[\tau(l) + 1]^2} = \frac{3}{4p[\tau(l) + 1]}, \quad (4.32)$$

which is especially significant for higher levels.

4.2.1 Improving the Efficiency of Interpolations and Antepolations via a Two-Step Method

In general, interpolations and antepolations constitute the major computational bulk of MLFMA. Therefore, to obtain an efficient solver, it is extremely important to optimize the interpolation/anteppolation routines in MLFMA. The number of operations required for the conventional (one-step) Lagrange interpolation from level $(l-1)$ to l is $8p^2[\tau(l)+1]^2$. This is because there are $[\tau(l)+1] \times 2[\tau(l)+1]$ points in the fine grid (samples for level l) and each of these points has $4p^2$ contributions from the coarse grid. On the other hand, locations of sampling points

in θ and ϕ directions are independent of each other. Therefore, interpolations along the two directions can be performed consecutively, as follows:

- Perform an interpolation along the θ direction as

$$f(\theta, \phi_j) \approx \sum_{i=t+1-p}^{t+p} v_i(\theta) f(\theta_i, \phi_j), \quad (4.33)$$

which requires $2pS_l^\theta S_{l-1}^\phi$ operations.

- Perform an interpolation along the ϕ direction using the result of the first step, i.e.,

$$f(\theta, \phi) \approx \sum_{j=s+1-p}^{s+p} w_j(\phi) f(\theta, \phi_j). \quad (4.34)$$

This step requires $2pS_l^\theta S_l^\phi$ operations.

Consequently, using the two-step method, the processing time required to interpolate the function at $S_l^\theta \times S_l^\phi$ points is

$$\Theta_{\text{two-step}}^{\text{time}} = 2pS_l^\theta (S_{l-1}^\phi + S_l^\phi) = 4p[\tau(l) + 1][\tau(l) + \tau(l-1) + 2]. \quad (4.35)$$

Comparing processing times required for one-step and two-step interpolation methods,

$$\begin{aligned} \frac{\Theta_{\text{two-step}}^{\text{time}}}{\Theta_{\text{one-step}}^{\text{time}}} &= \frac{4p[\tau(l) + 1][\tau(l) + \tau(l-1) + 2]}{8p^2[\tau(l) + 1]^2} \\ &< \frac{8p[\tau(l) + 1][\tau(l) + 1]}{8p^2[\tau(l) + 1]^2} = \frac{1}{p} \end{aligned} \quad (4.36)$$

since $\tau(l-1) < \tau(l)$. Therefore, the two-step method is always faster than the one-step method. To store the intermediate array between steps, the two-step method requires a bit more memory than is used in the one-step method. Nevertheless, the speedup in the two-step method more than compensates for the small increase in memory.

The two-step interpolation method is easily implemented by decomposing the

conventional one-step method into two successive parts. To demonstrate the acceleration provided by the two-step method, we present the solution of scattering problems involving PEC spheres of various radii from 6λ to 48λ illuminated by a plane wave. Problems are formulated by CFIE and discretized with the RWG functions. Triangulations with $\lambda/10$ mesh size lead to large matrix equations involving 132,003 to 8,447,808 unknowns. Problems are solved iteratively, with MVMs performed efficiently by MLFMA. Solutions are parallelized into 16 processes on a cluster of AMD Opteron 870 processors. The hierarchical partitioning strategy is used for the efficient parallelization of MLFMA. Far-field interactions are calculated with two digits of accuracy and interpolation/interpolation operations are performed using 6×6 stencils ($p = 3$). Table 4.2 lists the processing time required for the aggregation stage, in addition to the speedup offered by the two-step interpolation method. Compared to the conventional one-step method, the two-step method reduces the processing time of the aggregation stage by about 45%. To demonstrate the overall improvement, Table 4.2 also presents processing times required for MVMs, which are reduced by 25-30% with the two-step interpolation method.

4.2.2 Improving the Accuracy of Interpolations and Anterpolations Using Samples at Poles

As the problem size grows and more levels are required to construct the tree structure of MLFMA, it becomes critical to minimize the interpolation error. This is because interpolation and anterpolation operations are performed between all consecutive levels and the overall error is accumulated during aggregation and disaggregation stages. In this section, we introduce a method to reduce the error around the two poles of the sphere, i.e., the north pole ($\theta = 0$) and the south pole ($\theta = \pi$). This is essential since the error in these regions is usually larger than the error in other regions.

Table 4.2: Processing Time Required for an Aggregation Stage and for an MVM when Interpolation/Anterpolation Operations are Performed by using One-Step and Two-Step Interpolation Methods

Sphere Problems		Aggregation		
Radius (λ)	Unknowns	One-Step (s)	Two-Step (s)	Reduction
6	132,000	0.646	0.360	44%
7.5	206,499	0.844	0.470	44%
10	367,821	2.18	1.19	45%
15	829,881	3.86	2.14	45%
20	1,462,854	9.47	5.28	44%
30	3,319,524	17.1	9.43	45%
40	5,851,416	40.7	22.3	45%
48	8,447,808	54.7	30.8	44%
Sphere Problems		MVM		
Radius (λ)	Unknowns	One-Step (s)	Two-Step (s)	Reduction
6	132,000	1.98	1.47	26%
7.5	206,499	2.52	1.86	26%
10	367,821	6.21	4.46	28%
15	829,881	11.1	8.00	28%
20	1,462,854	27.0	19.6	27%
30	3,319,524	47.0	33.4	29%
40	5,851,416	112	79.2	29%
48	8,447,808	152	109	28%

Figure 4.2 demonstrates a practical case, where the value of the field is to be obtained at a point (star) with spherical coordinates $(\theta, \phi) = (0.409, 0.483)$ specified in radians. The figure is related to an aggregation step from a level with a box size of 0.25λ to the next level with a box size of 0.5λ . Using the excess bandwidth formula for three digits of accuracy, the truncation number is 8 and 12 for the lower and the higher levels, respectively. Therefore, the number of samples in the θ (ϕ) direction is 9 (18) for the coarse grid and 13 (26) for the fine grid. The interpolation parameter, p , is set to 2 so that 4×4 points (shaded circles) are employed in the interpolation. According to the conventional sampling scheme [80], samples are regularly spaced in the ϕ direction while they are chosen as the Gauss-Legendre points in the θ direction.

In Figure 4.2, there is only one sample in the θ direction on the coarse grid ($\theta = 0.253$) above the target point up to the north pole ($\theta = 0$). Therefore, four points for the interpolation are provided from the region on the other side of the pole. Considering the next sample in the decreasing θ direction, these are points with $\theta = -0.253$. Although this is the best choice, there exists a wide gap in the θ direction from $\theta = -0.253$ to $\theta = 0.253$. These wide gaps created near the poles in all levels lead to larger interpolation errors compared to other regions far from the poles.

To reduce the interpolation error described above, we employ the poles by sampling fields at $\theta = 0$ and $\theta = \pi$. Although radiated and incoming fields in MLFMA are vectors with two spherical components represented as

$$\mathbf{f}(\theta, \phi) = \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \cdot \mathbf{f}(\theta, \phi) + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \cdot \mathbf{f}(\theta, \phi), \quad (4.37)$$

we evaluate and store fields at the poles in x and y directions. As an example, for the north pole, θ and ϕ components are extracted as

$$\hat{\boldsymbol{\theta}} \cdot \mathbf{f}(0, \phi) = [\cos \phi f_x(\theta = 0) + \sin \phi f_y(\theta = 0)] \quad (4.38)$$

and

$$\hat{\phi} \cdot \mathbf{f}(0, \phi) = [\cos \phi f_y(\theta = 0) - \sin \phi f_x(\theta = 0)] \quad (4.39)$$

whenever required for the interpolation. Consequently, independent of the value of ϕ , all interpolations performed near the poles are improved by this technique using (4.38) and (4.39) without having to store values of $\mathbf{f}(\theta = 0, \phi)$ for each sample of ϕ . This is illustrated in Figure 4.2, where samples computed with (4.38) and (4.39) are represented by circles located at $\theta = 0$.

To include calculations related to the poles, four more complex numbers are required (two for each pole) for each basis and testing functions. The extra memory and the increase in the processing time are negligible compared to base requirements, since the data size for each basis and testing function is raised from $4[\tau(l) + 1]^2$ to $4[\tau(l) + 1]^2 + 4$, where $\tau(l)$ is at least 7 for two digits of accuracy. During aggregation and disaggregation processes, values at the poles are calculated and stored for each box to improve the interpolation and the antepolation at all levels. However, the poles do not contribute to the angular integration.

As a numerical example, Figure 4.3 depicts the relative interpolation error related to a basis box C_1 with the size of 2λ at the fourth level from the bottom of the tree structure. For the field of the box, the number of angular samples in θ and ϕ directions are 33 and 66, respectively, determined by the excess bandwidth formula for three digits of accuracy. By the row-wise arrangement of the θ - ϕ space, the interpolation error is plotted with respect to samples. Only the error in the θ component of the field, which is the dominant component in this example, is plotted. The relative interpolation error is defined as

$$\Delta_R[n] = \frac{|\tilde{\mathbf{f}}[n] - \mathbf{f}[n]|}{\max |\mathbf{f}|}, \quad (4.40)$$

where $\mathbf{f}$ is the exact field, $\tilde{\mathbf{f}}$ is the perturbed field obtained via interpolations, and $n = 1, 2, \dots, 2178$ is the sample index. The exact data is calculated by evaluating the field for each basis function inside the cube with the sample rate defined for

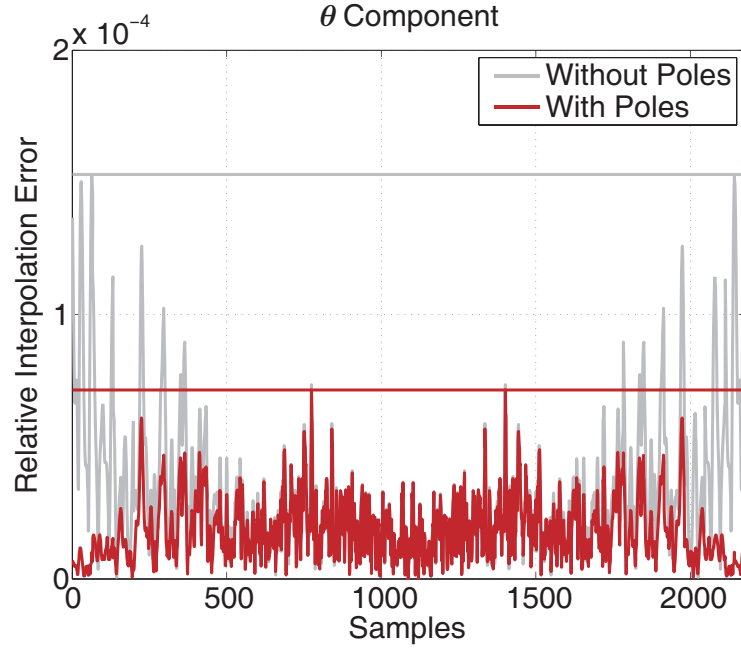


Figure 4.3: Relative interpolation error defined in (4.40) with respect to samples on a 33×66 grid converted into one-dimensional data by a row-wise arrangement of the θ - ϕ space. The reference data is obtained without interpolation. To obtain the interpolated data, aggregation is performed from the lowest (first) level to the fourth level by employing interpolations with (red) and without (gray) poles.

the fourth level so that it is obtained without any interpolation. However, in the case of the perturbed data, fields of basis functions are sampled according to the smallest box size, which is 0.25λ . Then, three aggregation steps are performed from the lowest (first) level to the fourth level. Consequently, the perturbed data is the practical case, where the interpolation error is introduced at three passages between levels, i.e., from first to second, second to third, and third to fourth.

In Figure 4.3, interpolation error is plotted when the poles are not employed as in the conventional case (gray) and when they are employed (red). For a clear comparison, maximum errors are also indicated in the plot with horizontal lines. Employing the poles decreases errors and the maximum relative error is reduced approximately to its half from 1.530×10^{-4} to 7.147×10^{-5} . The reduction in the error is clearly visible at the two ends of the horizontal axis corresponding to

points located near the poles. However, the improvement extends beyond these narrow polar regions towards the middle of the horizontal axis corresponding to points located around the equator. This is due to the fact that an interpolation error made in the first steps of the aggregation is propagated towards the equator region in the next steps. Consequently, the use of the poles improves the interpolation accuracy also for points located far from the poles. In general, improvements obtained by adding the poles in the Lagrange interpolation become more significant as the problem size grows and more levels are required in the MLFMA tree.

Demonstration of the antepolation error is not as straightforward as the interpolation error since the antepolated function is different from the original function. Therefore, to prove that employing poles also increases the accuracy of the antepolation in MLFMA, we present in Figure 4.4 errors in partial MVMs. After translating the radiation of C_1 into an incoming wave for a testing box C_2 , we perform disaggregation steps from the fourth level to the first level. This way, as presented in Figure 4.4(a), we obtain the error in the partial MVM related to boxes C_1 and C_2 with respect to the index of testing functions in C_2 . Similar to (4.40), the relative error is defined as

$$\Delta_R[m] = \frac{|\tilde{\mathbf{y}}[m] - \mathbf{y}[m]|}{\max |\mathbf{y}|}, \quad (4.41)$$

where

$$\mathbf{y}[m] = \sum_{n=1}^{N(C_1)} \bar{\mathbf{Z}}[m, n] \mathbf{x}[n] \quad \left(m = 1, 2, \dots, N(C_2) \right) \quad (4.42)$$

is the result of the partial multiplication without interpolation and antepolation errors. This reference data is obtained by evaluating radiation and receiving patterns of basis and testing functions inside boxes C_1 and C_2 , respectively, with the sampling rate defined for the fourth level so that interpolation and antepolation are not involved. In (4.42), $\mathbf{x}[n]$ represents coefficients of basis functions inside C_1 for $n = 1, 2, \dots, N(C_1)$, $\bar{\mathbf{Z}}[m, n]$ represents matrix elements calculated

by MLFMA, and m is the testing index from 1 to $N(C_2) = 704$. The perturbed data in (4.41) is obtained by performing usual aggregation and disaggregation steps so that values in Figure 4.4(a) contain both interpolation and ant interpolation errors. Figure 4.4(a) displays a significant improvement in the accuracy obtained by using the poles. Next, we eliminate the interpolation error by computing incoming waves to the center of the testing box C_2 without employing interpolation. The result is depicted in Figure 4.4(b), where the error in the partial MVM is due to only the ant interpolation. Therefore, Figure 4.4(b) clearly demonstrates that employing the poles enhances the accuracy of ant interpolations, similar to the improvement shown in Figure 4.3 for interpolations.

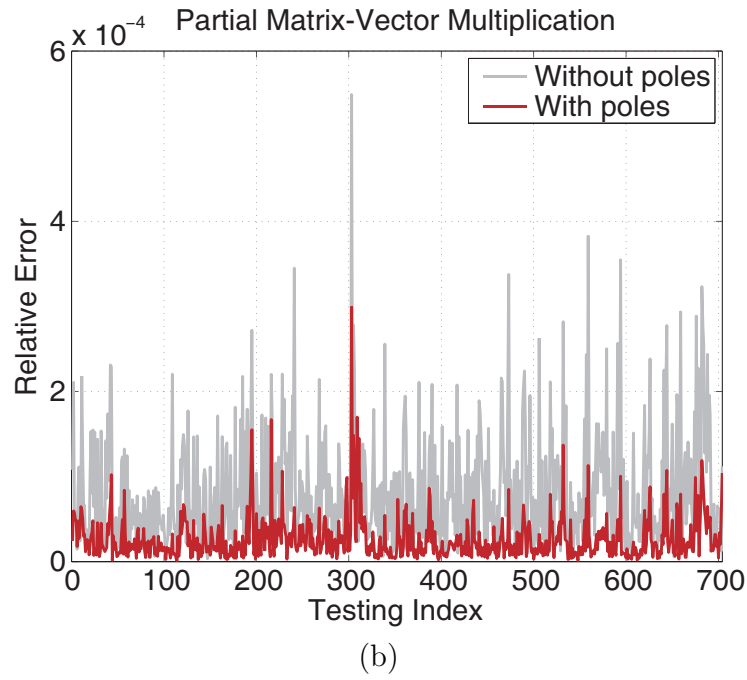
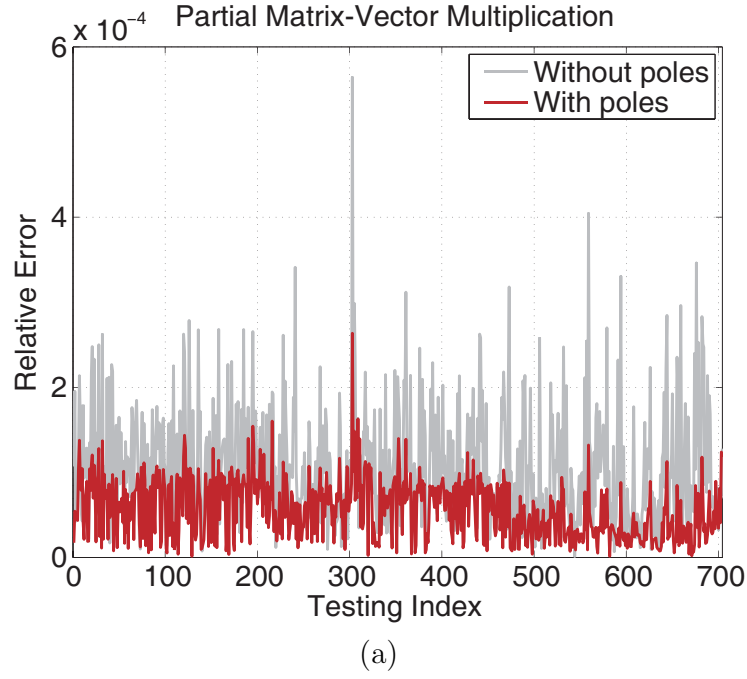


Figure 4.4: Relative errors defined in (4.41) in partial MVMs related to two boxes C_1 and C_2 in the fourth level with respect to the index of the testing function located in C_2 . The reference data is obtained without interpolation and antepolation. (a) Relative error when interpolation and antepolation are employed. (b) Relative error when interpolation is eliminated and the error is only due to the antepolation.

4.3 Optimal Interpolation of Translation Operators

Direct calculation of translation operators requires $\mathcal{O}(N^{3/2})$ operations, and the processing time for their setup becomes substantial as the problem size grows. In [90], Lagrange interpolation was proposed to efficiently fill translation matrices for large problems. However, parameters of the interpolation, namely, the number of interpolation points and the oversampling factor, were fixed. With parameters fixed, interpolation error is not controllable and the processing time is not minimized. In this section, we revisit the Lagrange interpolation of translation operators and optimize parameters of the interpolation to obtain the desired level of the accuracy with the minimum processing time. The optimal parameters are compared to fixed parameters to demonstrate the improvement obtained with the optimization.

4.3.1 Lagrange Interpolation of Translation Operators

A three-dimensional translation operator between a pair of basis and testing boxes is given in (2.134). For level l , the summation in (2.134) is truncated at $\tau(l)$, which is determined by the excess bandwidth formula in (4.9). In Figure 4.5(a), the truncation number $\tau(l)$ is plotted with respect to the number of accurate digits d_0 and for different values of the box size a_l increasing by a factor of two from 0.25λ to 64λ . Figure 4.5(a) demonstrates that L grows rapidly as the box size increases. For a fixed a_l , however, $\tau(l)$ increases gradually with respect to d_0 and the variation is small for large a_l .

Processing time required to calculate the translation operator is measured on a 1.8 GHz Opteron 244 processor. In Figure 4.5(b), the processing time is plotted with respect to same parameters as in Figure 4.5(a). Values are given

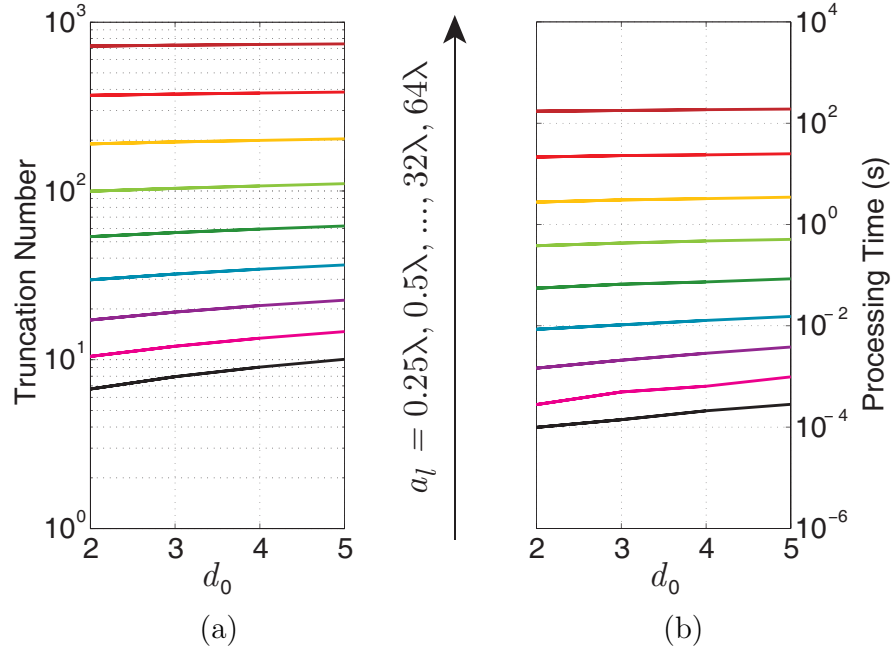


Figure 4.5: (a) Truncation number as a function of d_0 and the box size a_l . (b) Processing time to compute the translation function for a single box-box interaction. In both figures, there are 9 curves for different values of the box size increasing by a factor of two from 0.25λ to 64λ . The lowest and the highest curves correspond to 0.25λ and 64λ , respectively.

for a single interaction between a pair of basis and testing boxes while a typical problem requires the calculation of numerous box-box interactions. Since $\tau(l) = \mathcal{O}(ka_l)$, the processing time to evaluate (2.134) for a fixed $\hat{\mathbf{k}}$ is $\mathcal{O}(ka_l)$. In addition, the number of angular directions $\hat{\mathbf{k}}$ is $\mathcal{O}(\tau^2(l))$ and the processing time to evaluate (2.134) becomes $\mathcal{O}(k^3a_l^3)$ for a box-box interaction. For the lowest level of MLFMA, $\mathcal{O}(k^3a_l^3) = \mathcal{O}(1)$, which is acceptable although the number of boxes at this level is $\mathcal{O}(N)$. However, for the largest boxes of a problem, $\mathcal{O}(ka_l) = \mathcal{O}(N^{1/2})$ and $\mathcal{O}(k^3a_l^3) = \mathcal{O}(N^{3/2})$, which is significant. In fact, as N becomes large, the processing time required to calculate translation operators for a problem is dominated by evaluations for higher levels, although there are a few boxes at those levels. In addition, the setup time for the translation matrix becomes dominant compared to the time required for other parts of MLFMA, even MVMs that can be performed in $\mathcal{O}(N \log N)$ time.

Defining a variable $\varphi = \cos^{-1}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}_{CC'})$, the translation operator can be expressed as a band-limited function of φ [90] as

$$\alpha_{\tau(l)}(k, R_{CC'}, \varphi) = \sum_{t=0}^{\tau(l)} i^t (2t+1) h_t^{(1)}(k R_{CC'}) P_t(\cos \varphi). \quad (4.43)$$

Choosing an oversampling factor q and sampling the operator along φ from 0 to π at $\lfloor q\tau(l) \rfloor$ equally-spaced points ($\lfloor \cdot \rfloor$ represents the floor operation), i.e., at $\varphi_i = \pi(i-1)/(\lfloor q\tau(l) \rfloor - 1)$ and $i = 1, \dots, \lfloor q\tau(l) \rfloor$, the translation operator can be obtained by Lagrange interpolation at any point as

$$\tilde{\alpha}_{\tau(l)}(k, R_{CC'}, \varphi) = \sum_{i=t+1-p}^{t+p} \alpha_{\tau(l)}(k, R_{CC'}, \varphi_i) w_i(\varphi), \quad (4.44)$$

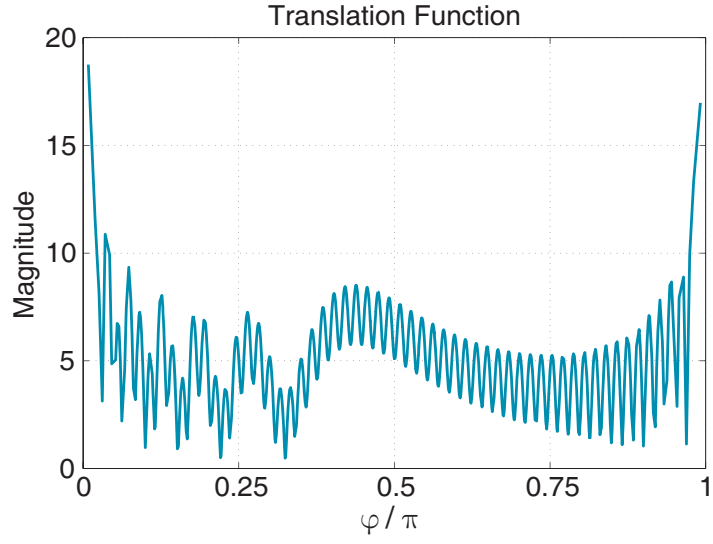
where $\tilde{\alpha}$ represents the translation function perturbed by the interpolation error,

$$t = \left\lfloor \frac{\varphi(\lfloor q\tau(l) \rfloor - 1)}{\pi} + 1 \right\rfloor, \quad (4.45)$$

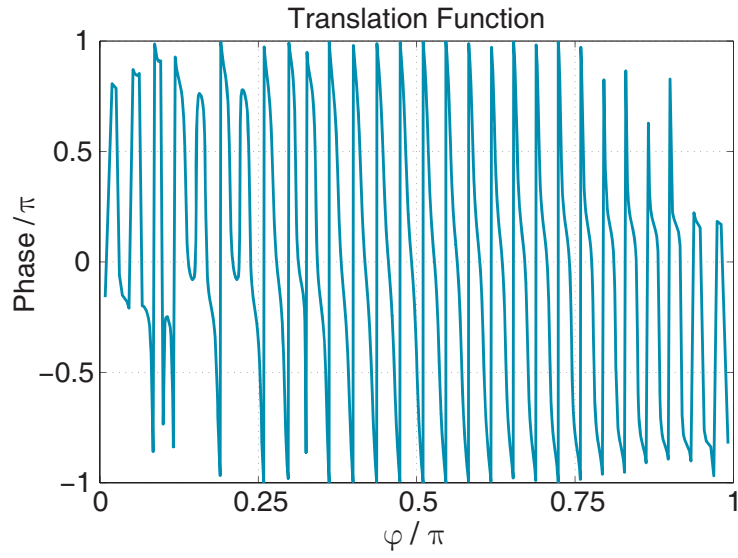
and

$$w_i(\varphi) = \prod_{\substack{j=t+1-p \\ j \neq i}}^{t+p} \frac{\varphi - \varphi_j}{\varphi_i - \varphi_j}. \quad (4.46)$$

In (4.44) and (4.46), p is the number of interpolation points employed at each side of the target location φ .

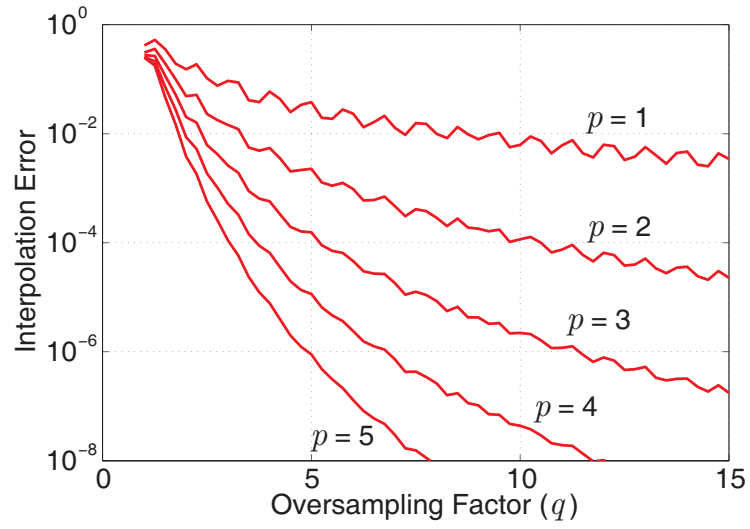


(a)

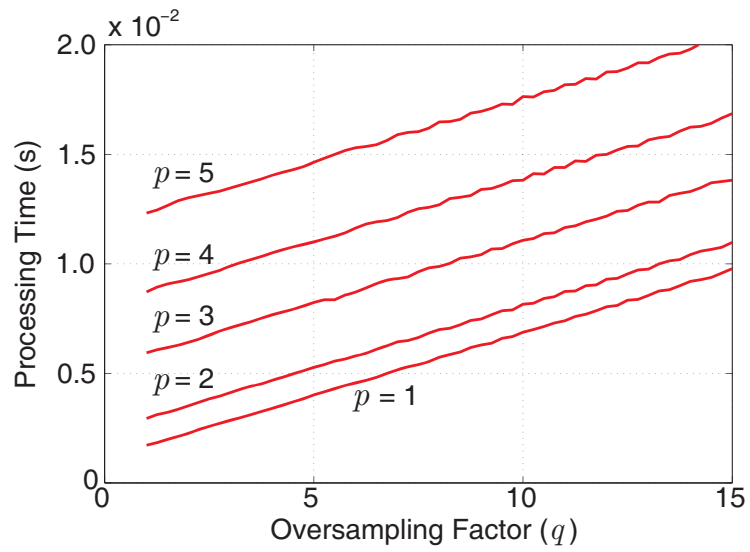


(b)

Figure 4.6: (a) Magnitude and (b) phase of the translation function with respect to φ for the case of $a_l = 4\lambda$, $d_0 = 3$, and $\mathbf{r}_{CC'} = \hat{\mathbf{x}}2a$.



(a)



(b)

Figure 4.7: (a) Interpolation error and (b) processing time with respect to interpolation parameters p and q for the translation function in Figure 4.6.

4.3.2 Optimal Interpolation

Figures 4.6(a) and 4.6(b) depict the magnitude and phase of the translation operator, respectively, for two boxes separated by $\mathbf{r}_{CC'} = \hat{\mathbf{x}}2a_l$, where $a_l = 4\lambda$. The number of accurate digits d_0 is 3 and $\tau(l) = 57$. We perform the direct calculation of the translation operator, where the function is evaluated at the required points by using (4.43). In the ϕ direction, there are $2[\tau(l) + 1] = 116$ samples that are equally spaced from 0 to 2π . In the θ direction, there are $[\tau(l) + 1] = 58$ samples (zeros of the Legendre polynomial) and they are not equally spaced. Then, there are a total of $2[\tau(l) + 1]^2 = 6728$ distinct $\hat{\mathbf{k}}$ directions to evaluate the translation operator. It should be noted that the transform from (2.134) to (4.43) not only depends on $\tau(l)$, but also on relative positions of boxes, i.e., it also depends on $\mathbf{r}_{CC'}$.

Before the translation matrix is filled via Lagrange interpolation, parameters q and p must be determined. For fixed values of d_0 and a_l , we perform a scan over q and p parameters to find their optimal values. Figure 4.7(a) demonstrates the interpolation error with respect to q and p for the case in Figure 4.6. Interpolation error is defined as

$$\Delta_I = \max_n \left\{ \frac{|\tilde{\alpha}_{\tau(l)}(\varphi[n]) - \alpha_{\tau(l)}(\varphi[n])|}{\max_{\varphi} \{|\alpha_{\tau(l)}(\varphi)|\}} \right\}, \quad (4.47)$$

where $n = 1, \dots, 2[\tau(l) + 1]^2$ and $\varphi[n]$ represents sampling points. Interpolation error decreases when q or p is increased. In this case $d_0 = 3$, which means that MLFMA computes interactions with three digits of accuracy. Thus (p, q) pairs leading to larger than 10^{-3} error are not allowable. In other words, the error introduced by the interpolation of the translation operator should be adjusted according to the desired level of accuracy.

This strategy yields a set of (p, q) pairs satisfying the error criterion. Optimization is completed by choosing the (p, q) pair with the minimum processing time.

Table 4.3: Speedup Obtained by Using the Optimal (p, q) Pair for $a_l \geq 4\lambda$

d_0	(p, q)	$a_l = 4\lambda$	$a_l = 8\lambda$	$a_l = 16\lambda$	$a_l = 32\lambda$	$a_l = 64\lambda$
2	(2,3.5)	14.0	27.5	54.3	108.3	216.0
3	(2,6.5)	10.8	20.2	40.0	77.0	151.9
4	(3,6.0)	7.9	15.0	28.9	56.9	113.7
5	(3,8.5)	7.1	13.0	24.7	48.4	96.6

As shown in Figure 4.7(b), processing time (measured on a 1.8 GHz Opteron 244 processor) to evaluate the translation operator increases as p or q is increased. Then, there exists an optimal (p, q) pair satisfying the desired level of accuracy with the minimum processing time. We scan parameters p and q for various values of a_l and d_0 . All possible values of $\mathbf{r}_{CC'}$ according to the one-box-buffer scheme [85] are also checked. In the end, we obtain optimal values listed in Table 4.3 with the corresponding speedup compared to the direct calculation. We note that values presented in Table 4.3 do not depend on the computer platform. The optimal (p, q) pairs are valid for $a_l \geq 4\lambda$ and they are found to be independent of $\mathbf{r}_{CC'}$. For smaller boxes such as $a_l = \lambda$ or 2λ , the interpolation does not lead to a significant speedup, and therefore, we prefer to calculate these translations directly. In the case of much smaller boxes such as $a_l = 0.25\lambda$ or 0.5λ , direct calculation is faster than the interpolation for any (p, q) pair satisfying the desired accuracy. Figures 4.8(a) and 4.8(b) compare optimal (p, q) pairs to fixed $p = 3, q = 5.0$ values suggested in [90]. In Figure 4.8(a), interpolation error is plotted with respect to the box size a_l from 4λ to 64λ and for different levels of accuracy, i.e., for $d_0 = 2, 3, 4$, and 5 corresponding to $10^{-2}, 10^{-3}, 10^{-4}$, and 10^{-5} relative errors, respectively. In the optimized case, error is always below the desired level of accuracy. However, with fixed parameters, error is not controllable and is localized around 10^{-4} . The corresponding speedup is plotted in Figure 4.8(b), where it increases with increasing box size and decreases with increasing number of accurate digits in the optimized case. This relationship

is also evident in Table 4.3. Comparing Figures 4.8(a) and (b), the following observations can be made:

- For $d_0 = 2$ and 3, fixed $p = 3, q = 5.0$ satisfies the desired level of accuracy but optimal (p, q) pairs provide higher speedup.
- For $d_0 = 4$ and 5, the fixed $p = 3, q = 5.0$ seems to give higher speedup compared to optimal (p, q) pairs, however, the accuracy is not satisfied with fixed parameters.

Based on these observations, we conclude that optimization is essential to improve the interpolation of the translation operator.

To demonstrate the overall improvement obtained with interpolation, we present results of a scattering problem involving a conducting sphere of radius 20λ . This is a 1,462,854-unknown problem solved by a parallel MLFMA implementation with seven levels. The problem is solved on a cluster of 32 processors (2.6 GHz Pentium 4 Celeron). The box size is 0.25λ for the lowest level and 16λ for the highest level. As an example, if the number of accurate digits d_0 is set to 3, then $\tau(l)$ takes values from 8 to 195. We use the one-box-buffer scheme and reduce the number of translations by exploiting the symmetry [84]. During the setup phase of the program, each processor checks all of its box-box interactions to eliminate the unneeded translations. In Figure 4.9(a), processing time for the calculation of translation operators is plotted with respect to d_0 . For both types of calculations (direct and interpolated), the maximum is chosen among processing times spent by 32 processors. In Figure 4.9(b), the speedup obtained by the interpolation method over direct calculation is plotted as a function of d_0 . The speedup is over 14 up to $d_0 = 5$.

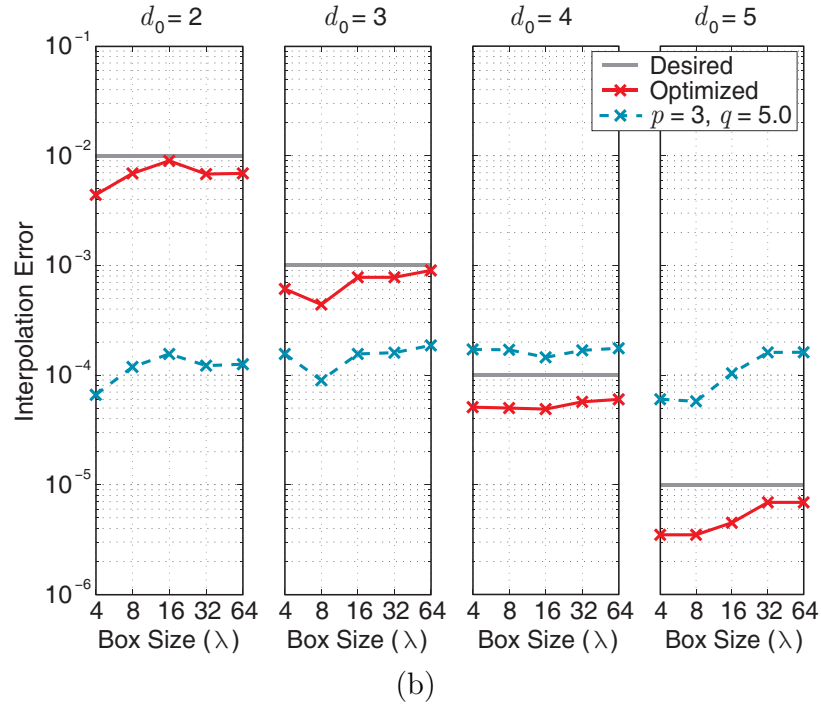
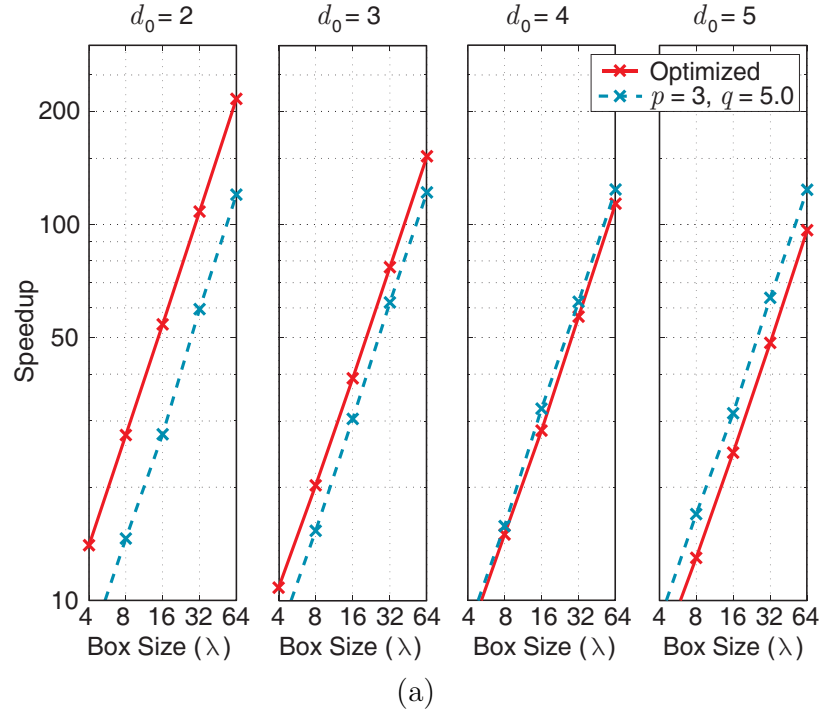
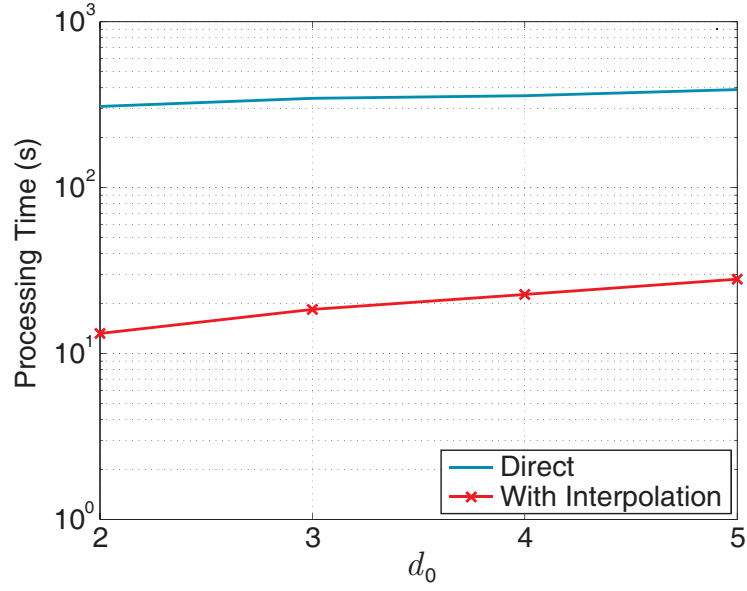
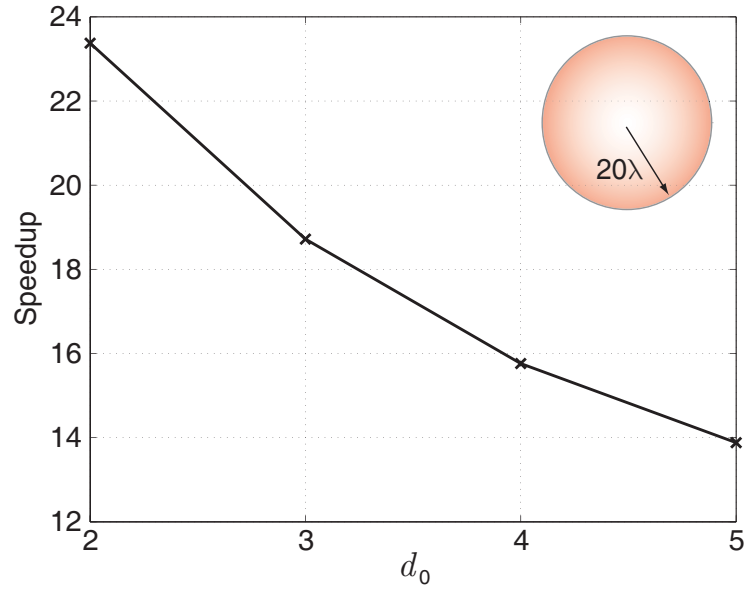


Figure 4.8: (a) Interpolation error and (b) corresponding speedup for different box sizes from 4λ to 64λ and for $d_0 = 2, 3, 4, 5$. ($r_{CC'} = \hat{x}2a$)



(a)



(b)

Figure 4.9: (a) Processing time to compute translation operators for a 1,462,854-unknown sphere problem. (b) Speedup obtained with optimal interpolation compared to direct calculation of translation operators.

4.4 Multilevel Fast Multipole Algorithm for Low-Frequency Problems

MLFMA is a powerful method, which provides fast and accurate solutions of scattering and radiation problems in electromagnetics. Unfortunately, it suffers from a low-frequency breakdown, which limits its applicability to those problems involving small details with respect to wavelength. A major limitation is that the box size in MLFMA cannot be very small compared to wavelength since the plane wave expansion becomes invalid for short distances. Hence, when an ordinary MLFMA is applied to low-frequency problems discretized with very small elements, the lowest-level boxes may involve many discretization elements. As a result, the complexity of the MLFMA implementation can be more than $\mathcal{O}(N \log N)$ due to excessively large numbers of near-field interactions that must be calculated directly.

In this section, we present an efficient implementation of LF-MLFMA based on multipoles [103]–[108]. Fields are expanded in a series multipoles, but they are not converted into plane waves to avoid the low-frequency breakdown. This way, the box size is not restricted, and the object can be partitioned into sub-boxes that can be much smaller than the wavelength. Without the plane wave expansion, translations are not diagonalized and involve dense MVMs, which may decrease the efficiency. On the other hand, the number of multipoles does not depend on the box size for small boxes, and the resulting LF-MLFMA implementation has $\mathcal{O}(N)$ complexity, provided that the object is not very large. Nevertheless, since the multipole expansion becomes inefficient for very large boxes, we also combine LF-MLFMA and the ordinary MLFMA to obtain a broadband implementation for dense discretizations of very large objects.

4.4.1 Addition Theorem

Consider the decomposition of an arbitrary vector $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j = (r_{ij}, \theta_{ij}, \phi_{ij})$ into three vectors, i.e.,

$$\mathbf{r}_{ij} = \mathbf{r}_{iI} + \mathbf{r}_{IJ} + \mathbf{r}_{Jj}, \quad (4.48)$$

where

$$\mathbf{r}_{IJ} = \mathbf{r}_I - \mathbf{r}_J = (r_{IJ}, \theta_{IJ}, \phi_{IJ}) \quad (4.49)$$

$$\mathbf{r}_{iI} = \mathbf{r}_i - \mathbf{r}_I = (r_{iI}, \theta_{iI}, \phi_{iI}) \quad (4.50)$$

$$\mathbf{r}_{Jj} = \mathbf{r}_J - \mathbf{r}_j = (r_{Jj}, \theta_{Jj}, \phi_{Jj}) \quad (4.51)$$

and $|\mathbf{r}_{IJ}| > |\mathbf{r}_{iI} + \mathbf{r}_{Jj}|$. The three-dimensional addition theorem can be written as

$$\Psi_{t,u}(k, \mathbf{r}_{ij}) = \sum_{t'=0}^{\infty} \sum_{u'=-t'}^{t'} \Re\{\Psi_{t',u'}(k, \mathbf{r}_{iI} + \mathbf{r}_{Jj})\} \alpha_{t',u',t,u}(k, \mathbf{r}_{IJ}). \quad (4.52)$$

In (4.52),

$$\Psi_{t,u}(k, \mathbf{r}) = h_t^{(1)}(kr) Y_{t,u}(\theta, \phi) \quad (4.53)$$

and

$$\Re\{\Psi_{t,u}(k, \mathbf{r})\} = j_t(kr) Y_{t,u}(\theta, \phi) \quad (4.54)$$

represent wave functions (multipole fields), where $h_t^{(1)}$ is the spherical Hankel function of the first kind, j_t is the spherical Bessel function of the first kind,

$$Y_{t,u}(\theta, \phi) = \sqrt{\frac{(2t+1)(t-u)!}{4\pi(t+u)!}} P_t^u(\cos \theta) \exp(iu\phi) \quad (4.55)$$

represents spherical harmonics, and P_t^u is the associated Legendre function. The multipole-to-multipole translation function in (4.52) can be derived as

$$\alpha_{t',u',t,u}(k, \mathbf{r}) = \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} 4\pi i^{t'+t''-t} \Psi_{t'',u''}(k, \mathbf{r}) G_{t,u,t',u'',t'',u''}, \quad (4.56)$$

where

$$G_{t,u,t',u',t'',u''} = \int d^2\hat{\mathbf{k}} Y_{t,u}(\theta, \phi) Y_{t',u'}^*(\theta, \phi) Y_{t'',u''}^*(\theta, \phi) \quad (4.57)$$

are the Gaunt coefficients. Using

$$\int d^2\hat{\mathbf{k}} Y_{t,u}^*(\theta, \phi) Y_{t',u'}(\theta, \phi) = \delta_{tt'} \delta_{uu'}, \quad (4.58)$$

$$Y_{t,u}(\theta, \phi) = (-1)^u Y_{t,-u}^*(\theta, \phi), \quad (4.59)$$

and $Y_{0,0}(\theta, \phi) = 1/\sqrt{4\pi}$, we obtain

$$\begin{aligned} G_{0,0,t',u',t'',u''} &= \int d^2\hat{\mathbf{k}} Y_{0,0}(\theta, \phi) Y_{t',u'}^*(\theta, \phi) Y_{t'',u''}^*(\theta, \phi) \\ &= \frac{(-1)^{u''}}{\sqrt{4\pi}} \int d^2\hat{\mathbf{k}} Y_{t',u'}^*(\theta, \phi) Y_{t'',-u''}(\theta, \phi) \\ &= \frac{(-1)^{u''}}{\sqrt{4\pi}} \delta_{t't''} \delta_{u'(-u'')} \end{aligned} \quad (4.60)$$

$$G_{t,u,0,0,t'',u''} = \frac{1}{\sqrt{4\pi}} \delta_{tt''} \delta_{uu''} \quad (4.61)$$

$$G_{t,u,t',u',0,0} = \frac{1}{\sqrt{4\pi}} \delta_{tt'} \delta_{uu'}. \quad (4.62)$$

Setting $t = u = 0$ in (4.56) and using (4.60), we derive the monopole-to-multipole translation function as

$$\begin{aligned} \alpha_{t',u',0,0}(k, \mathbf{r}) &= \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} 4\pi i^{t'+t''} \Psi_{t'',u''}(k, \mathbf{r}) G_{0,0,t',u',t'',u''} \\ &= \frac{1}{\sqrt{4\pi}} \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} 4\pi i^{t'+t''} \Psi_{t'',u''}(k, \mathbf{r}) (-1)^{u''} \delta_{t't''} \delta_{u'(-u'')} \\ &= \sqrt{4\pi} (-1)^{t'+u'} \Psi_{t',-u'}(k, \mathbf{r}) \\ &= \sqrt{4\pi} (-1)^{t'+u'} h_{t'}^{(1)}(kr) Y_{t',-u'}(\theta, \phi). \end{aligned} \quad (4.63)$$

Similarly, setting $t' = u' = 0$ in (4.56) and using (4.61), we derive the multipole-to-monopole translation function as

$$\alpha_{0,0,t,u}(k, \mathbf{r}) = \sqrt{4\pi} \Psi_{t,u}(k, \mathbf{r}) = \sqrt{4\pi} h_t^{(1)}(kr) Y_{t,u}(\theta, \phi). \quad (4.64)$$

Finally, setting $t' = u' = 0$ in (4.63) or $t = u = 0$ in (4.64), the monopole-to-monopole translation function can be derived as

$$\alpha_{0,0,0,0}(k, \mathbf{r}) = \sqrt{4\pi} h_0^{(1)}(kr) Y_{0,0}(\theta, \phi) = h_0^{(1)}(r) = \frac{\exp(ikr)}{ikr}, \quad (4.65)$$

which is simply a scaled version of the homogeneous-space Green's function.

It is remarkable that most of the Gaunt coefficients are identically zero. Considering only nonzero terms, the multipole-to-multipole translation function can be written as

$$\alpha_{t',u',t,u}(k, \mathbf{r}) = \sum_{t''=|t-t'|}^{|t+t'|} 4\pi i^{t'+t''-t} \Psi_{t'',u-u'}(k, \mathbf{r}) G_{t,u,t',u',t'',u-u'}. \quad (4.66)$$

Using (4.66) and (4.60)–(4.61), it is trivial to derive the monopole-to-multipole and multipole-to-monopole translation functions as

$$\begin{aligned} \alpha_{t',u',0,0}(k, \mathbf{r}) &= \sum_{t''=|-t'|}^{|t'|} 4\pi i^{t'+t''} \Psi_{t'',-u'}(k, \mathbf{r}) G_{0,0,t',u',t'',-u'} \\ &= 4\pi (-1)^{t'} \Psi_{t',-u'}(k, \mathbf{r}) \frac{(-1)^{-u'}}{\sqrt{4\pi}} \delta_{t't'} \delta_{u'u'} \\ &= \sqrt{4\pi} (-1)^{t'+u'} \Psi_{t',-u'}(k, \mathbf{r}) \end{aligned} \quad (4.67)$$

and

$$\begin{aligned} \alpha_{0,0,t,u}(k, \mathbf{r}) &= \sum_{t''=|t|}^{|t|} 4\pi i^{t''-t} \Psi_{t,u}(k, \mathbf{r}) G_{t,u,0,0,t,u} \\ &= 4\pi \Psi_{t,u}(k, \mathbf{r}) \delta_{t't''} \delta_{uu} = \sqrt{4\pi} \Psi_{t,u}(k, \mathbf{r}), \end{aligned} \quad (4.68)$$

respectively.

The addition theorem can be used to factorize the homogeneous-space Green's function, i.e.,

$$\begin{aligned} g(\mathbf{r}_i, \mathbf{r}_j) &= \frac{ik}{\sqrt{4\pi}} \Psi_{0,0}(k, \mathbf{r}_{ij}) \\ &= \sum_{t'=0}^{\infty} \sum_{u'=-t'}^{t'} \Re\{\Psi_{t',u'}(k, \mathbf{r}_{iI} + \mathbf{r}_{jJ})\} \alpha_{t',u',0,0}(k, \mathbf{r}_{IJ}). \end{aligned} \quad (4.69)$$

Using (4.53), (4.54), (4.59), and (4.67) in (4.69), we obtain

$$\begin{aligned}
g(\mathbf{r}_i, \mathbf{r}_j) &= ik \sum_{t'=0}^{\infty} \sum_{u'=-t'}^{t'} \Re\{\Psi_{t',u'}(k, \mathbf{r}_{iI} + \mathbf{r}_{jJ})\} (-1)^{t'+u'} \Psi_{t',-u'}(k, \mathbf{r}_{IJ}) \\
&= ik \sum_{t'=0}^{\infty} \sum_{u'=-t'}^{t'} (-1)^{t'} j_{t'}(kr_{iIJj}) Y_{t',u'}(\theta_{iIJj}, \phi_{iIJj}) h_{t'}^{(1)}(kr_{IJ}) Y_{t',u'}^*(\theta_{IJ}, \phi_{IJ}) \\
&= ik \sum_{t'=0}^{\infty} (-1)^{t'} j_{t'}(kr_{iIJj}) h_{t'}^{(1)}(kr_{IJ}) \sum_{u'=-t'}^{t'} Y_{t',u'}(\theta_{iIJj}, \phi_{iIJj}) Y_{t',u'}^*(\theta_{IJ}, \phi_{IJ}),
\end{aligned} \tag{4.70}$$

where

$$\mathbf{r}_{iIJj} = (r_{iIJj}, \theta_{iIJj}, \phi_{iIJj}) = \mathbf{r}_{iI} + \mathbf{r}_{jJ}. \tag{4.71}$$

The inner summation in (4.70) can be evaluated as

$$\sum_{u'=-t'}^{t'} Y_{t',u'}(\theta_{iIJj}, \phi_{iIJj}) Y_{t',u'}^*(\theta_{IJ}, \phi_{IJ}) = \frac{(2t'+1)}{4\pi} P_{t'}(\hat{\mathbf{r}}_{IJ} \cdot \hat{\mathbf{r}}_{iIJj}), \tag{4.72}$$

where $P_{t'}$ is the Legendre polynomial. Inserting (4.72) in (4.70), we obtain

$$g(\mathbf{r}_i, \mathbf{r}_j) = \frac{ik}{4\pi} \sum_{t'=0}^{\infty} (-1)^{t'} (2t'+1) j_{t'}(kr_{iIJj}) h_{t'}^{(1)}(kr_{IJ}) P_{t'}(\hat{\mathbf{r}}_{IJ} \cdot \hat{\mathbf{r}}_{iIJj}), \tag{4.73}$$

which is called the Gegenbauer's addition theorem.

4.4.2 Factorization of Translation Functions

Consider a multipole-to-multipole translation from $\mathbf{r}_i$ to $\mathbf{r}_j$, i.e.,

$$\alpha_{t',u',t,u}(k, \mathbf{r}_{ij}) = \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} 4\pi i^{t'+t''-t} \Psi_{t'',u''}(k, \mathbf{r}_{ij}) G_{t,u,t',u'',t'',u''}. \tag{4.74}$$

Using the addition theorem (4.52) in (4.74) and changing the order of summations, we obtain

$$\begin{aligned}
\alpha_{t',u',t,u}(k, \mathbf{r}_{ij}) &= \sum_{t_1=0}^{\infty} \sum_{u_1=-t_1}^{t_1} 4\pi i^{t'-t} \Re\{\Psi_{t_1,u_1}(k, \mathbf{r}_{iIJj})\} \\
&\quad \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} i^{t''} G_{t,u,t',u'',t'',u''} \alpha_{t_1,u_1,t'',u''}(k, \mathbf{r}_{IJ}).
\end{aligned} \tag{4.75}$$

Inner summations can be modified by using (4.56) as

$$\begin{aligned}
& \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} i^{t''} G_{t,u,t',u',t'',u''} \alpha_{t_1,u_1,t'',u''}(k, \mathbf{r}_{IJ}) \\
&= \sum_{t_2=0}^{\infty} \sum_{u_2=-t_2}^{t_2} 4\pi i^{t_1+t_2} \Psi_{t_2,u_2}(k, \mathbf{r}_{IJ}) \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} G_{t,u,t',u',t'',u''} G_{t'',u'',t_1,u_1,t_2,u_2} \\
&= \sum_{t_2=0}^{\infty} \sum_{u_2=-t_2}^{t_2} 4\pi i^{t_1+t_2} \Psi_{t_2,u_2}(k, \mathbf{r}_{IJ}) \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} G_{t,u,t_1,u_1,t'',u''} G_{t'',u'',t',u',t_2,u_2} \\
&= \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} i^{t''+t_1-t'} G_{t,u,t_1,u_1,t'',u''} \alpha_{t',u',t'',u''}(k, \mathbf{r}_{IJ}) \tag{4.76}
\end{aligned}$$

$$= \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} i^{t''+t_1-t'} G_{t,u,t'',u'',t_1,u_1} \alpha_{t',u',t'',u''}(k, \mathbf{r}_{IJ}) \tag{4.77}$$

leading to

$$\begin{aligned}
\alpha_{t',u',t,u}(k, \mathbf{r}_{ij}) &= \sum_{t_1=0}^{\infty} \sum_{u_1=-t_1}^{t_1} 4\pi i^{t'-t} \Re\{\Psi_{t_1,u_1}(k, \mathbf{r}_{iIJj})\} \\
&\quad \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} i^{t''+t_1-t'} G_{t,u,t'',u'',t_1,u_1} \alpha_{t',u',t'',u''}(k, \mathbf{r}_{IJ}) \\
&= \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} \alpha_{t',u',t'',u''}(k, \mathbf{r}_{IJ}) \beta_{t'',u'',t,u}(k, \mathbf{r}_{iIJj}), \tag{4.78}
\end{aligned}$$

where

$$\beta_{t',u',t,u}(k, \mathbf{r}) = \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} 4\pi i^{t'+t''-t} \Re\{\Psi_{t'',u''}(k, \mathbf{r})\} G_{t,u,t',u',t'',u''} \tag{4.79}$$

is the multipole-to-multipole shift function. By performing similar derivations, one can also show that

$$\alpha_{t',u',t,u}(k, \mathbf{r}_{ij}) = \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} \beta_{t',u',t'',u''}(k, \mathbf{r}_{iIJj}) \alpha_{t'',u'',t,u}(k, \mathbf{r}_{IJ}), \tag{4.80}$$

$$\beta_{t',u',t,u}(k, \mathbf{r}_{iIJj}) = \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} \beta_{t',u',t'',u''}(k, \mathbf{r}_{iI}) \beta_{t'',u'',t,u}(k, \mathbf{r}_{Jj}), \tag{4.81}$$

and

$$\begin{aligned}
\alpha_{t',u',t,u}(k, \mathbf{r}_{ij}) &= \sum_{t_1=0}^{\infty} \sum_{u_1=-t_1}^{t_1} \sum_{t_2=0}^{\infty} \sum_{u_2=-t_2}^{t_2} \beta_{t',u',t_1,u_1}(k, \mathbf{r}_{iI}) \\
&\quad \alpha_{t_1,u_1,t_2,u_2}(k, \mathbf{r}_{IJ}) \beta_{t_2,u_2,t,u}(k, \mathbf{r}_{Jj}). \tag{4.82}
\end{aligned}$$

We note that the multipole-to-multipole shift function in (4.79) is the regular part of the multipole-to-multipole translation function in (4.66). Considering only nonzero Gaunt coefficients,

$$\beta_{t',u',t,u}(k, \mathbf{r}) = \sum_{t''=|t-t'|}^{|t+t'|} 4\pi i^{t'+t''-t} \Re\{\Psi_{t'',u-u'}(k, \mathbf{r})\} G_{t,u,t',u',t'',u-u'}. \quad (4.83)$$

Setting $t = u = 0$ in (4.83) and using (4.60), we derive the monopole-to-multipole shift function as

$$\begin{aligned} \beta_{t',u',0,0}(k, \mathbf{r}) &= \sum_{t''=|-t'|}^{|t'|} 4\pi i^{t'+t''} \Re\{\Psi_{t'',-u'}(k, \mathbf{r})\} G_{0,0,t',u',t'',-u'} \\ &= 4\pi (-1)^{t'} \Re\{\Psi_{t',-u'}(k, \mathbf{r})\} \frac{(-1)^{-u'}}{\sqrt{4\pi}} \delta_{t't'} \delta_{u'u'} \\ &= \sqrt{4\pi} (-1)^{t'+u'} \Re\{\Psi_{t',-u'}(k, \mathbf{r})\} \\ &= \sqrt{4\pi} (-1)^{t'+u'} j_{t'}(kr) Y_{t',-u'}(\theta, \phi). \end{aligned} \quad (4.84)$$

Similarly, setting $t' = u' = 0$ in (4.83) and using (4.61), we derive the multipole-to-monopole shift function as

$$\begin{aligned} \beta_{0,0,t,u}(k, \mathbf{r}) &= \sum_{t''=|t|}^{|t|} 4\pi i^{t''-t} \Re\{\Psi_{t,u}(k, \mathbf{r}_{ij})\} G_{t,u,0,0,t,u} \\ &= 4\pi \Re\{\Psi_{t,u}(k, \mathbf{r})\} \frac{1}{\sqrt{4\pi}} \delta_{tt} \delta_{uu} \\ &= \sqrt{4\pi} \Re\{\Psi_{t,u}(k, \mathbf{r})\} = \sqrt{4\pi} j_t(kr) Y_{t,u}(\theta, \phi). \end{aligned} \quad (4.85)$$

Finally, setting $t' = u' = 0$ in (4.84) or $t = u = 0$ in (4.85), the monopole-to-monopole shift function can be derived as

$$\beta_{0,0,0,0}(k, \mathbf{r}) = \sqrt{4\pi} j_0(kr) Y_{0,0}(\theta, \phi) = j_0(kr) = \frac{\sin(kr)}{kr}. \quad (4.86)$$

Consider the application of (4.80) to the monopole-to-monopole translation function, i.e.,

$$\alpha_{0,0,0,0}(k, \mathbf{r}_{ij}) = \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} \beta_{0,0,t'',u''}(k, \mathbf{r}_{iIJj}) \alpha_{t'',u'',0,0}(k, \mathbf{r}_{IJ}). \quad (4.87)$$

Using (4.59), (4.63), (4.72), and (4.85) in (4.87), we obtain the Gegenbauer's addition theorem, i.e.,

$$\begin{aligned} g(\mathbf{r}_i, \mathbf{r}_j) &= \frac{ik}{4\pi} \alpha_{0,0,0,0}(k, \mathbf{r}_{ij}) \\ &= ik \sum_{t''=0}^{\infty} (-1)^{t''} j_{t''}(kr_{iIJj}) h_{t''}^{(1)}(kr_{IJ}) \\ &\quad \sum_{u''=-t''}^{t''} (-1)^{u''} Y_{t'',u''}(\theta_{iIJj}, \phi_{iIJj}) Y_{t'',-u''}(\theta_{IJ}, \phi_{IJ}) \\ &= \frac{ik}{4\pi} \sum_{t''=0}^{\infty} (-1)^{t''} (2t''+1) j_{t''}(kr_{iIJj}) h_{t''}^{(1)}(kr_{IJ}) P_{t''}(\hat{\mathbf{r}}_{IJ} \cdot \hat{\mathbf{r}}_{iIJj}). \end{aligned} \quad (4.88)$$

4.4.3 Expansions

Consider a vector $\mathbf{X} = \hat{\mathbf{k}}X$, where $X \gg r = |\mathbf{r}|$ for an observation point $\mathbf{r}$. Then,

$$\frac{\exp(ik|\mathbf{X} + \mathbf{r}|)}{4\pi|\mathbf{X} + \mathbf{r}|} \approx \frac{\exp(ikX)}{4\pi X} \exp(i\mathbf{k} \cdot \mathbf{r}). \quad (4.89)$$

Using the Gegenbauer's addition theorem and the large-argument form of the spherical Hankel function, we obtain

$$\exp(i\mathbf{k} \cdot \mathbf{r}) = \sum_{t=0}^{\infty} i^t (2t+1) j_t(kr) P_t(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}), \quad (4.90)$$

which can be interpreted as the expansion of a plane wave in a series of spherical waves. In order to derive an expression for the expansion of a spherical wave in a series of plane waves, we multiply both sides of (4.90) with $P_{t''}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}'')$, where

$\hat{\mathbf{r}}''$ is an arbitrary direction. Performing an angular integration on both sides,

$$\begin{aligned}
& \int d^2 \hat{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{r}) P_{t''}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}'') \\
&= \int d^2 \hat{\mathbf{k}} \sum_{t=0}^{\infty} i^t (2t+1) j_t(kr) P_t(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) P_{t''}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}'') \\
&= \sum_{t=0}^{\infty} i^t (2t+1) j_t(kr) \int d^2 \hat{\mathbf{k}} P_t(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) P_{t''}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}'') \\
&= \sum_{t=0}^{\infty} i^t (2t+1) j_t(kr) \frac{4\pi}{(2t''+1)} P_t(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'') \delta_{tt''} \\
&= 4\pi i^{t''} j_{t''}(kr) P_{t''}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'').
\end{aligned} \tag{4.91}$$

Finally, choosing $\hat{\mathbf{r}}'' = \hat{\mathbf{r}}$, we obtain

$$j_{t''}(kr) = \frac{(-i)^{t''}}{4\pi} \int d^2 \hat{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{r}) P_{t''}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}). \tag{4.92}$$

4.4.4 Diagonalization

FMM and MLFMA are based on the diagonalization of the Gegenbauer's addition theorem. Using (4.91) in (4.88) for $\mathbf{r} = \mathbf{r}_{IJj}$ and $\mathbf{r}'' = \mathbf{r}_{IJ}$, we obtain

$$\begin{aligned}
g(\mathbf{r}_i, \mathbf{r}_j) &= \frac{ik}{(4\pi)^2} \sum_{t''=0}^{\infty} i^{t''} (2t''+1) h_{t''}^{(1)}(kr_{IJ}) \\
&\quad \int d^2 \hat{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{r}_{IJj}) P_{t''}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}_{IJ}).
\end{aligned} \tag{4.93}$$

Changing the order of the summation and the angular integration, the homogeneous-space Green's function can be written as

$$g(\mathbf{r}_i, \mathbf{r}_j) = \frac{ik}{(4\pi)^2} \int d^2 \hat{\mathbf{k}} \beta(\mathbf{k}, \mathbf{r}_{IJj}) \alpha(\mathbf{k}, \mathbf{r}_{IJ}), \tag{4.94}$$

where β is the diagonal (plane-wave-to-plane-wave) shift function defined in (4.5) and

$$\alpha(\mathbf{k}, \mathbf{r}) = \sum_{t''=0}^{\infty} i^{t''} (2t''+1) h_{t''}^{(1)}(kr) P_{t''}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) \tag{4.95}$$

is the diagonal (plane-wave-to-plane-wave) translation function. In practice, the summation in (4.95) is truncated as given in (2.134). We note that, using (4.94), derivatives of the Green's functions can also be written in diagonal forms as

$$\{\nabla g\}(\mathbf{r}_i, \mathbf{r}_j) = \frac{ik}{(4\pi)^2} \int d^2\hat{\mathbf{k}}(i\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r}_{IJj}) \alpha(\mathbf{k}, \mathbf{r}_{IJ}) \quad (4.96)$$

$$\{\nabla\nabla' g\}(\mathbf{r}_i, \mathbf{r}_j) = \frac{ik}{(4\pi)^2} \int d^2\hat{\mathbf{k}}(\mathbf{k}\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r}_{IJj}) \alpha(\mathbf{k}, \mathbf{r}_{IJ}). \quad (4.97)$$

In general, diagonalization can be applied to all multipole-to-multipole shift and translation functions. For example, consider a multipole-to-multipole shift from $\mathbf{r}_i$ to $\mathbf{r}_j$, i.e.,

$$\begin{aligned} \beta_{t',u',t,u}(k, \mathbf{r}_{ij}) &= \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} 4\pi i^{t'+t''-t} \Re\{\Psi_{t'',u''}(k, \mathbf{r}_{ij})\} G_{t,u,t',u',t'',u''} \\ &= \sum_{t''=0}^{\infty} \sum_{u''=-t''}^{t''} 4\pi i^{t'+t''-t} j_{t''}(kr_{ij}) Y_{t'',u''}(\theta_{ij}, \phi_{ij}) \\ &\quad \int d^2\hat{\mathbf{k}} Y_{t,u}(\theta, \phi) Y_{t',u'}^*(\theta, \phi) Y_{t'',u''}^*(\theta, \phi). \end{aligned} \quad (4.98)$$

Changing the order of summations and the angular integration, and using (4.72) and (4.90), we obtain

$$\begin{aligned} \beta_{t',u',t,u}(k, \mathbf{r}_{ij}) &= 4\pi \int d^2\hat{\mathbf{k}} [i^{t'} Y_{t',u'}^*(\theta, \phi)] [i^{-t} Y_{t,u}(\theta, \phi)] \\ &\quad \sum_{t''=0}^{\infty} i^{t''} j_{t''}(kr_{ij}) \sum_{u''=-t''}^{t''} Y_{t'',u''}(\theta_{ij}, \phi_{ij}) Y_{t'',u''}^*(\theta, \phi) \\ &= \int d^2\hat{\mathbf{k}} [i^{t'} Y_{t',u'}^*(\theta, \phi)] [i^{-t} Y_{t,u}(\theta, \phi)] \\ &\quad \sum_{t''=0}^{\infty} i^{t''} (2t'' + 1) j_{t''}(kr_{ij}) P_{t''}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}_{ij}) \\ &= \int d^2\hat{\mathbf{k}} [i^{t'} Y_{t',u'}^*(\theta, \phi)] \exp(i\mathbf{k} \cdot \mathbf{r}_{ij}) [i^{-t} Y_{t,u}(\theta, \phi)] \\ &= \int d^2\hat{\mathbf{k}} T_{t',u'}^{P \rightarrow M}(\theta, \phi) \beta(\mathbf{k}, \mathbf{r}_{ij}) T_{t,u}^{M \rightarrow P}(\theta, \phi), \end{aligned} \quad (4.99)$$

where

$$T_{t,u}^{M \rightarrow P}(\theta, \phi) = i^{-t} Y_{t,u}(\theta, \phi) \quad (4.100)$$

is the multipole-to-plane-wave transformer and

$$T_{t',u'}^{P \rightarrow M}(\theta, \phi) = i^{t'} Y_{t',u'}^*(\theta, \phi) \quad (4.101)$$

is the plane-wave-to-multipole transformer. Applying a similar procedure, one can derive the diagonalization of the multipole-to-multipole translation function as

$$\alpha_{t',u',t,u}(k, \mathbf{r}_{ij}) = \int d^2 \hat{\mathbf{k}} T_{t',u'}^{P \rightarrow M}(\theta, \phi) \beta(\mathbf{k}, \mathbf{r}_{iI}) \alpha(\mathbf{k}, \mathbf{r}_{IJ}) \beta(\mathbf{k}, \mathbf{r}_{Jj}) T_{t,u}^{M \rightarrow P}(\theta, \phi). \quad (4.102)$$

We note that (4.102) reduces to (4.94) when $t = u = t' = u' = 0$.

4.4.5 Factorization of Matrix Elements

Low-frequency problems can be solved efficiently by using the factorization of the Green's function without diagonalization. Consider the solution of electromagnetics problems involving metallic objects discretized with the RWG functions. Matrix elements related to far-field interactions can be written as

$$\begin{aligned} \bar{\mathbf{Z}}_{FF,E}[m, n] &= ik \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{b}_n(\mathbf{r}') \\ &\quad - \frac{i}{k} \int_{S_m} d\mathbf{r} \nabla \cdot \mathbf{t}_m(\mathbf{r}) \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \end{aligned} \quad (4.103)$$

$$= \bar{\mathbf{Z}}_{FF,E}^V[m, n] + \bar{\mathbf{Z}}_{FF,E}^S[m, n] \quad (4.104)$$

and

$$\bar{\mathbf{Z}}_{FF,M}[m, n] = \int_{S_m} d\mathbf{r} (\mathbf{t}_m(\mathbf{r}) \times \hat{\mathbf{n}}) \cdot \int_{PV,S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \times \nabla' g(\mathbf{r}, \mathbf{r}') \quad (4.105)$$

for EFIE and MFIE, respectively. Using (4.82), the Green's function can be factorized as

$$\begin{aligned} g(\mathbf{r}, \mathbf{r}') &= \frac{ik}{4\pi} \sum_{t_1=0}^{\infty} \sum_{u_1=-t_1}^{t_1} \sum_{t_2=0}^{\infty} \sum_{u_2=-t_2}^{t_2} \beta_{0,0,t_1,u_1}(k, \mathbf{r} - \mathbf{r}_C) \\ &\quad \alpha_{t_1,u_1,t_2,u_2}(k, \mathbf{r}_{CC'}) \beta_{t_2,u_2,0,0}(k, \mathbf{r}_{C'} - \mathbf{r}'), \end{aligned} \quad (4.106)$$

where $|\mathbf{r}_{CC'}| = |\mathbf{r}_C - \mathbf{r}_{C'}| > |\mathbf{r} - \mathbf{r}' - \mathbf{r}_C + \mathbf{r}_{C'}|$. Using (4.106) in (4.103)–(4.105), we obtain

$$\begin{aligned} \bar{\mathbf{Z}}_{FF,E}^V[m, n] &= \left(\frac{ik}{4\pi}\right)^2 \sum_{t_1=0}^{\infty} \sum_{u_1=-t_1}^{t_1} \sum_{t_2=0}^{\infty} \sum_{u_2=-t_2}^{t_2} \mathbf{F}_{m,t_1,u_1}^{E,V}(\mathbf{r}_C, k) \\ &\quad \cdot \alpha_{t_1,u_1,t_2,u_2}(k, \mathbf{r}_{CC'}) \mathbf{S}_{n,t_2,u_2}^{E,V}(\mathbf{r}_{C'}, k) \end{aligned} \quad (4.107)$$

$$\begin{aligned} \bar{\mathbf{Z}}_{FF,E}^S[m, n] &= \left(\frac{ik}{4\pi}\right)^2 \sum_{t_1=0}^{\infty} \sum_{u_1=-t_1}^{t_1} \sum_{t_2=0}^{\infty} \sum_{u_2=-t_2}^{t_2} F_{m,t_1,u_1}^{E,S}(\mathbf{r}_C, k) \\ &\quad \alpha_{t_1,u_1,t_2,u_2}(k, \mathbf{r}_{CC'}) S_{n,t_2,u_2}^{E,S}(\mathbf{r}_{C'}, k) \end{aligned} \quad (4.108)$$

$$\begin{aligned} \bar{\mathbf{Z}}_{FF,M}[m, n] &= \left(\frac{ik}{4\pi}\right)^2 \sum_{t_1=0}^{\infty} \sum_{u_1=-t_1}^{t_1} \sum_{t_2=0}^{\infty} \sum_{u_2=-t_2}^{t_2} \mathbf{F}_{m,t_1,u_1}^M(\mathbf{r}_C, k) \\ &\quad \cdot \alpha_{t_1,u_1,t_2,u_2}(k, \mathbf{r}_{CC'}) \mathbf{S}_{n,t_2,u_2}^M(\mathbf{r}_{C'}, k), \end{aligned} \quad (4.109)$$

where

$$\mathbf{S}_{n,t,u}^{E,V}(\mathbf{r}_{C'}, k) = \mathbf{S}_{n,t,u}^M(\mathbf{r}_{C'}, k) = \sqrt{4\pi} \int_{S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \beta_{t,u,0,0}(k, \mathbf{r}_{C'} - \mathbf{r}') \quad (4.110)$$

$$S_{n,t,u}^{E,S}(\mathbf{r}_{C'}, k) = \frac{\sqrt{4\pi}}{ik} \int_{S_n} d\mathbf{r}' \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \beta_{t,u,0,0}(k, \mathbf{r}_{C'} - \mathbf{r}') \quad (4.111)$$

represent the radiation pattern of the n th basis function and

$$\mathbf{F}_{m,t,u}^{E,V}(\mathbf{r}_C, k) = \sqrt{4\pi} \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \beta_{0,0,t,u}(k, \mathbf{r} - \mathbf{r}_C) \quad (4.112)$$

$$F_{m,t,u}^{E,S}(\mathbf{r}_C, k) = \frac{\sqrt{4\pi}}{ik} \int_{S_m} d\mathbf{r} \nabla \cdot \mathbf{t}_m(\mathbf{r}) \beta_{0,0,t,u}(k, \mathbf{r} - \mathbf{r}_C) \quad (4.113)$$

$$\mathbf{F}_{m,t,u}^M(\mathbf{r}_C, k) = \frac{\sqrt{4\pi}}{ik} \int_{S_m} d\mathbf{r} (\mathbf{t}_m(\mathbf{r}) \times \hat{\mathbf{n}}) \times \nabla \beta_{0,0,t,u}(k, \mathbf{r} - \mathbf{r}_C) \quad (4.114)$$

represent the receiving pattern of the m th testing function. Using a Galerkin scheme,

$$\mathbf{F}_{m,t,u}^{E,V}(\mathbf{r}_C, k) = \{\mathbf{S}_{m,t,u}^{E,V}(\mathbf{r}_C, \mathbf{k})\}^* \quad (4.115)$$

$$F_{m,t,u}^{E,S}(\mathbf{r}_C, k) = -\{S_{m,t,u}^{E,S}(\mathbf{r}_C, \mathbf{k})\}^* \quad (4.116)$$

since

$$\beta_{t',u',t,u}(k, \mathbf{r}) = \beta_{t,u,t',u'}^*(k, -\mathbf{r}). \quad (4.117)$$

In (4.107)–(4.109), matrix elements are calculated by a direct translation between box centers $\mathbf{r}_{C'}$ and $\mathbf{r}_C$, where radiation and receiving patterns are evaluated. However, in practice, multilevel interactions are required by performing several shifts before and after translations. For example, let a multipole-to-multipole translation be performed between $\mathbf{r}_D$ and $\mathbf{r}_{D'}$, which are close to receiving and radiation centers, i.e., $\mathbf{r}_C$ and $\mathbf{r}_{C'}$, respectively. Then, the translation function in (4.107)–(4.109) is further factorized as

$$\alpha_{t_1, u_1, t_2, u_2}(k, \mathbf{r}_{CC'}) = \sum_{t_3=0}^{\infty} \sum_{u_3=-t_3}^{t_3} \sum_{t_4=0}^{\infty} \sum_{u_4=-t_4}^{t_4} \beta_{t_1, u_1, t_3, u_3}(k, \mathbf{r}_{CD}) \alpha_{t_3, u_3, t_4, u_4}(k, \mathbf{r}_{DD'}) \beta_{t_4, u_4, t_2, u_2}(k, \mathbf{r}_{D'C'}), \quad (4.118)$$

where $\mathbf{r}_{CD} = \mathbf{r}_C - \mathbf{r}_D$ and $\mathbf{r}_{D'C'} = \mathbf{r}_{D'} - \mathbf{r}_{C'}$.

In practice, summations in (4.107)–(4.109) and (4.118), as well as those in translation and shift functions, must be truncated at $t = \tau$. We perform a worst-case analysis in order to determine the truncation number τ required for a desired level of accuracy. Consider two identical boxes with edges of $a_l = 2^{l-2}\lambda$, where l changes from -7 to 4 , i.e., $\lambda/512 \leq a_l \leq 4\lambda$. The “source” box is centered at $(x, y, z) = (0, 0, 0)$ and the “observation” box is centered at $(x, y, z) = (2a_l, 0, 0)$, according to a one-box-buffer scheme. We test a total of 64 cases, when source and observation points are located at box corners, i.e., $\mathbf{r}' = (\pm a_l/2, \pm a_l/2, \pm a_l/2)$ and $\mathbf{r} = (2a_l \pm a_l/2, \pm a_l/2, \pm a_l/2)$. The Green’s function is calculated by using the factorization in (4.106) with multipole-to-multipole translations between box centers. The relative error of the factorization compared to the analytical evaluation of the Green’s function is obtained for different values of the truncation number τ . Considering 64 cases, the maximum error is selected to determine the number of accurate digits for a given truncation number. As a result, we obtain truncation numbers required for various box sizes and different numbers

Table 4.4: Truncation Numbers Obtained by a Worst-Case Analysis of the Factorization of the Green's Function in (4.106)

l	Box Size	Number of Accurate Digits				
		$d_0 = 1$	$d_0 = 2$	$d_0 = 3$	$d_0 = 4$	$d_0 = 5$
4	4λ	25	28	31	33	36
3	2λ	14	16	19	21	27
2	λ	8	10	12	19	25
1	$\lambda/2$	5	8	12	19	24
0	$\lambda/4$	4	7	11	17	24
-1	$\lambda/8$	3	7	11	17	24
-2	$\lambda/16$	3	7	11	17	24
-3	$\lambda/32$	3	7	11	17	24
-4	$\lambda/64$	3	7	11	17	24
-5	$\lambda/128$	3	7	11	17	24
-6	$\lambda/256$	3	7	11	17	24
-7	$\lambda/512$	3	7	11	17	24

of accurate digits (d_0) from 1 to 5, as listed in Table 4.4. We observe that, for a desired level of accuracy, the truncation number is almost constant when the box size is small. For very large boxes, however, it grows rapidly as the box size increases; this limits the applicability of multipole-to-multipole interactions for large boxes.

Next, we test truncation numbers listed in Table 4.4 for two-level interactions. We again consider two $a_l \times a_l \times a_l$ boxes centered at $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (2a_l, 0, 0)$. This time, each box contains a smaller box with edges of $a_l/2$. Source and observation points are located at corners of small boxes. We test $8 \times 8 \times 8 \times 8 = 4096$ cases by considering eight different locations for source and observation points, as well as eight different placement of small boxes inside large boxes. In each case, the homogeneous-space Green's function is calculated by using a factorization involving two shifts on the source side (from the source point to the center of the small box and from the center of the small box to the center of the large box), a multipole-to-multipole translation between centers of large boxes, and two shifts on the observation side (from the center

Table 4.5: Maximum Relative Error in a Two-Level Calculation of the Green's Function Using the Truncation Numbers in Table 4.4

Box Size		Maximum Relative Error		
Large	Small	$d_0 = 2$	$d_0 = 3$	$d_0 = 4$
4λ	2λ	1.1882E-2	6.2268E-4	8.2273E-5
2λ	λ	5.1073E-3	4.3117E-4	8.3946E-5
λ	$\lambda/2$	3.9394E-3	8.3753E-4	1.3247E-4
$\lambda/2$	$\lambda/4$	4.5135E-3	1.0285E-3	1.0108E-4
$\lambda/4$	$\lambda/8$	7.8453E-3	1.0573E-3	9.7895E-5
$\lambda/8$	$\lambda/16$	7.1366E-3	1.0489E-3	9.6000E-5
$\lambda/16$	$\lambda/32$	6.9732E-3	1.0468E-3	9.5533E-5
$\lambda/32$	$\lambda/64$	6.9331E-3	1.0463E-3	9.5417E-5
$\lambda/64$	$\lambda/128$	6.9232E-3	1.0461E-3	9.5388E-5
$\lambda/128$	$\lambda/256$	6.9207E-3	1.0461E-3	9.5380E-5
$\lambda/256$	$\lambda/512$	6.9201E-3	1.0461E-3	9.5379E-5

of the large box to the center of the small box and from the center of the small box to the observation point). Table 4.5 lists the maximum relative error of the factorization compared to the analytical evaluation of the Green's function for different box sizes and desired digits of accuracy. We observe that, in some cases, the error is larger than desired. This is because truncation numbers listed in Table 4.4 are obtained for single-level interactions and extra shifts in multilevel interactions deteriorate the accuracy of the factorization. Nevertheless, relative errors listed in Table 4.5 are obtained for the worst case scenarios and they are not significantly larger than target errors.

4.4.6 Low-Frequency Multilevel Fast Multipole Algorithm

LF-MLFMA can be implemented by using the factorization of matrix elements in (4.107)–(4.109). Recursive clustering and construction of the tree structure in LF-MLFMA are very similar to those in the ordinary MLFMA. However, as

opposed to MLFMA, there is no limitation for the box size. Without considering the electrical size of the object, a tree structure of $L = \mathcal{O}(\log N)$ levels is constructed by placing the object in a cubic box and recursively dividing the computational domain into subdomains. Then, far-field interactions are calculated in a group-by-group manner via three main stages, i.e., aggregation, translation, and disaggregation. For level $l = 1, 2, \dots, L$, the number of multipoles to represent fields is $M_l = [\tau(l) + 1]^2$.

In the aggregation stage, radiated fields of boxes are calculated in terms of multipoles from the bottom of the tree structure to the highest level ($l = L$). Aggregations at the lowest level can be written as

$$\begin{bmatrix} \mathbf{s}_C^x & \mathbf{s}_C^y & \mathbf{s}_C^z & \mathbf{s}_C^s \end{bmatrix}_{M_1 \times 4} = \sum_{n \in C} \mathbf{x}[n] \begin{bmatrix} \mathbf{s}_n^x & \mathbf{s}_n^y & \mathbf{s}_n^z & \mathbf{s}_n^s \end{bmatrix}_{M_1 \times 4}, \quad (4.119)$$

where $\mathbf{x}[n]$ represent coefficients provided by the iterative solver, $\{\mathbf{s}_n^x, \mathbf{s}_n^y, \mathbf{s}_n^z, \mathbf{s}_n^s\}$ are arrays of M_1 elements containing the radiation pattern of the n th basis function, and $\{\mathbf{s}_C^x, \mathbf{s}_C^y, \mathbf{s}_C^z, \mathbf{s}_C^s\}$ are arrays of M_1 elements containing the radiated field of a box C at the lowest level. We note that $\mathbf{s}_n^s$ and $\mathbf{s}_C^s$, which are related to scalar electric potential, are not required for MFIE. For a box C at a higher level ($l > 1$),

$$\begin{bmatrix} \mathbf{s}_C^x & \mathbf{s}_C^y & \mathbf{s}_C^z & \mathbf{s}_C^s \end{bmatrix}_{M_l \times 4} = \sum_{C' \in C} [\bar{\beta}_{C' \rightarrow C}]_{M_l \times M_{l-1}} \cdot \begin{bmatrix} \mathbf{s}_{C'}^x & \mathbf{s}_{C'}^y & \mathbf{s}_{C'}^z & \mathbf{s}_{C'}^s \end{bmatrix}_{M_{l-1} \times 4}, \quad (4.120)$$

where $\bar{\beta}_{C' \rightarrow C}$ is an $M_l \times M_{l-1}$ dense matrix containing multipole-to-multipole shift functions $\beta_{t', u', t, u}(k, \mathbf{r}_C - \mathbf{r}_{C'})$ for $t' = 1, 2, \dots, \tau(l)$ and $t = 1, 2, \dots, \tau(l-1)$.

In the translation stage, radiated fields of boxes are translated into incoming fields for other boxes. The incoming field to the center of a box C due to boxes

in the far-field of C , i.e., $\mathcal{F}\{C\}$, can be calculated as

$$\begin{bmatrix} \mathbf{g}_C^x & \mathbf{g}_C^y & \mathbf{g}_C^z & \mathbf{g}_C^s \end{bmatrix}_{M_l \times 4} = \sum_{C' \in \mathcal{F}\{C\}} [\bar{\alpha}_{C' \rightarrow C}]_{M_l \times M_l} \cdot \begin{bmatrix} \mathbf{s}_{C'}^x & \mathbf{s}_{C'}^y & \mathbf{s}_{C'}^z & \mathbf{s}_{C'}^s \end{bmatrix}_{M_l \times 4}, \quad (4.121)$$

where $\{\mathbf{g}_n^x, \mathbf{g}_n^y, \mathbf{g}_n^z, \mathbf{g}_n^s\}$ are arrays of M_l elements containing the incoming field to the center of a box C and $\bar{\alpha}_{C' \rightarrow C}$ is an $M_l \times M_l$ dense matrix containing multipole-to-multipole translation functions $\alpha_{t', u', t, u}(k, \mathbf{r}_C - \mathbf{r}_{C'})$ for $t' = 1, 2, \dots, \tau(l)$ and $t = 1, 2, \dots, \tau(l)$.

Finally, in the disaggregation stage, total incoming fields at box centers are calculated from the top of the tree structure to the lowest level. For a box C at a level $l < L$, the total incoming field is calculated as

$$\begin{aligned} \begin{bmatrix} \mathbf{g}_C^{+,x} & \mathbf{g}_C^{+,y} & \mathbf{g}_C^{+,z} & \mathbf{g}_C^{+,s} \end{bmatrix}_{M_l \times 4} &= \begin{bmatrix} \mathbf{g}_C^x & \mathbf{g}_C^y & \mathbf{g}_C^z & \mathbf{g}_C^s \end{bmatrix}_{M_l \times 4} \\ &+ [\bar{\beta}_{C' \rightarrow C}]_{M_l \times M_{l+1}} \cdot \begin{bmatrix} \mathbf{g}_{C'}^{+,x} & \mathbf{g}_{C'}^{+,y} & \mathbf{g}_{C'}^{+,z} & \mathbf{g}_{C'}^{+,s} \end{bmatrix}_{M_{l+1} \times 4} \end{aligned} \quad (4.122)$$

by combining contributions from translations and from the incoming field to the center of the parent box C' . In (4.122), $\bar{\beta}_{C' \rightarrow C}$ is an $M_l \times M_{l+1}$ dense matrix containing multipole-to-multipole shift functions $\beta_{t', u', t, u}(k, \mathbf{r}_C - \mathbf{r}_{C'})$ for $t' = 1, 2, \dots, \tau(l-1)$ and $t = 1, 2, \dots, \tau(l)$. We note that

$$[\bar{\beta}_{C' \rightarrow C}]_{M_l \times M_{l+1}} = \left\{ [\bar{\beta}_{C \rightarrow C'}]_{M_{l+1} \times M_l} \right\}^H \quad (4.123)$$

since $\beta_{t', u', t, u}(k, \mathbf{r}_C - \mathbf{r}_{C'}) = \beta_{t, u, t', u'}^*(k, \mathbf{r}_{C'} - \mathbf{r}_C)$. At the lowest level, total incoming fields are received by testing functions as

$$\begin{aligned} \sum_{n=1}^N \bar{\mathbf{Z}}_{FF}[m, n] \mathbf{a}[n] &\approx \left(\frac{ik}{4\pi} \right)^2 \left\{ [\mathbf{f}_m^x]_{1 \times M_1} \cdot [\mathbf{g}_C^{+,x}]_{M_1 \times 1} + [\mathbf{f}_m^y]_{1 \times M_1} \cdot [\mathbf{g}_C^{+,y}]_{M_1 \times 1} \right. \\ &\quad \left. + [\mathbf{f}_m^z]_{1 \times M_1} \cdot [\mathbf{g}_C^{+,z}]_{M_1 \times 1} + [\mathbf{f}_m^s]_{1 \times M_1} \cdot [\mathbf{g}_C^{+,s}]_{M_1 \times 1} \right\} \end{aligned} \quad (4.124)$$

for $m \in C$, where $\{\mathbf{f}_m^x, \mathbf{f}_m^y, \mathbf{f}_m^z, \mathbf{f}_m^s\}$ are arrays of M_1 elements containing the receiving pattern of the m th testing function.

Using multipoles, the processing time and the memory requirement for aggregation, translation, and disaggregation stages at level l are proportional to $N_l M_l^4$, where $N_l \approx 4^{(1-l)} N_1$ is the number of boxes. For small objects, $M_l = \mathcal{O}(1)$, which is independent of N , and LF-MLFMA is dominated by computations at the lowest level with $\mathcal{O}(N)$ complexity. On the other hand, LF-MLFMA is not appropriate for large-scale problems since the number of multipoles can be prohibitively large at higher levels. As a remedy, we employ the ordinary MLFMA at higher levels, where the multipole representation becomes inefficient. The resulting broadband implementation of MLFMA provides efficient solutions of large-scale problems involving small details with respect to wavelength [108].

In the broadband MLFMA, radiated fields represented by multipoles at level l are converted into plane waves as

$$\begin{bmatrix} \mathbf{s}_C^x & \mathbf{s}_C^y & \mathbf{s}_C^z & \mathbf{s}_C^s \end{bmatrix}_{S_l \times 4} = \bar{\mathbf{T}}_{S_l \times M_l}^{M \rightarrow P} \cdot \begin{bmatrix} \mathbf{s}_C^x & \mathbf{s}_C^y & \mathbf{s}_C^z & \mathbf{s}_C^s \end{bmatrix}_{M_l \times 4}, \quad (4.125)$$

where $\bar{\mathbf{T}}^{M \rightarrow P}$ is an $S_l \times M_l$ transformer matrix containing multipole-to-plane-wave transformers in (4.100). We note that only θ and ϕ components of radiated fields are required using the plane-wave representation. They can be obtained as

$$\begin{aligned} \mathbf{s}_C^\theta[n] &= \cos(\theta[n]) \cos(\phi[n]) \mathbf{s}_C^x[n] \\ &\quad + \cos(\theta[n]) \sin(\phi[n]) \mathbf{s}_C^y[n] - \sin(\theta[n]) \mathbf{s}_C^z[n] \end{aligned} \quad (4.126)$$

$$\mathbf{s}_C^\phi[n] = -\sin(\theta[n]) \sin(\phi[n]) \mathbf{s}_C^x[n] + \sin(\theta[n]) \cos(\phi[n]) \mathbf{s}_C^y[n] \quad (4.127)$$

for $n = 1, \dots, S_l$, where $\theta[n]$ and $\phi[n]$ represent spherical coordinates of samples on the unit sphere. When fields are converted into plane waves, aggregation, translation, and disaggregation stages at level l and higher levels can be performed as in the ordinary MLFMA. After disaggregation operations from the highest level to level l , incoming fields represented by plane waves are transformed back to cartesian system and converted into multipoles as

$$\begin{bmatrix} \mathbf{s}_C^x & \mathbf{s}_C^y & \mathbf{s}_C^z & \mathbf{s}_C^s \end{bmatrix}_{M_l \times 4} = \bar{\mathbf{T}}_{M_l \times S_l}^{P \rightarrow M} \cdot \begin{bmatrix} \mathbf{s}_C^x & \mathbf{s}_C^y & \mathbf{s}_C^z & \mathbf{s}_C^s \end{bmatrix}_{S_l \times 4}, \quad (4.128)$$

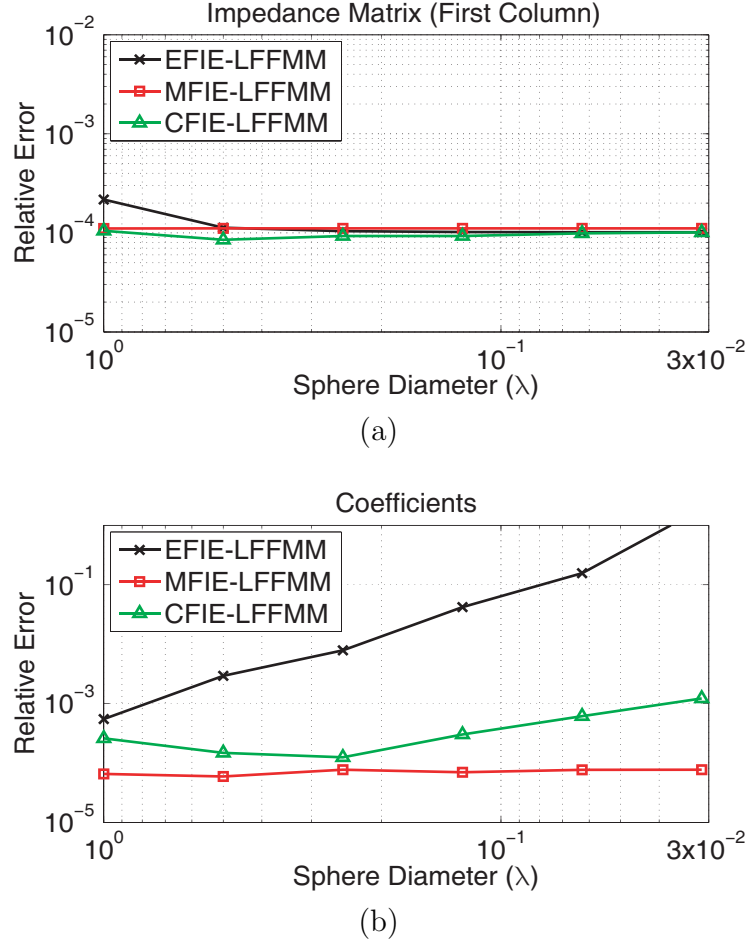


Figure 4.10: Relative error (a) in the first column of the impedance matrix and (b) in the expansion coefficients obtained by using LF-MLFMA compared to a very accurate MOM implementation for the solution of scattering problems involving a PEC sphere.

where $\bar{\mathbf{T}}^{P \rightarrow M}$ is an $M_l \times S_l$ transformer matrix containing plane-wave-to-multipole transformers in (4.101).

4.4.7 Numerical Results

In order to test the accuracy of the LF-MLFMA implementation, we consider the solution of scattering problems involving a PEC sphere of radius 0.3 m discretized with 930 RWG functions. The sphere is illuminated by a plane-wave

propagating in the $-z$ direction with the electric field polarized in the x direction in free space. We consider various frequencies in the range from 15.625 MHz to 500 MHz. The diameter of the sphere changes from $\lambda/32$ to λ , whereas the mesh size changes from $\lambda/320$ to $\lambda/10$. Problems are formulated by EFIE, MFIE, and CFIE. Solutions are performed by using a very accurate MOM implementation and LF-MLFMA with three digits of accuracy. Figure 4.10(a) presents the relative error in the first column of the impedance matrix as a function of sphere size with respect to wavelength. The relative error is defined as

$$\Delta_R = \frac{\max_m \left\{ \left| \bar{\mathbf{Z}}_{\text{LF-MLFMA}}[m, 1] - \bar{\mathbf{Z}}_{\text{MOM}}[m, 1] \right| \right\}}{\max_m \left\{ \left| \bar{\mathbf{Z}}_{\text{MOM}}[m, 1] \right| \right\}}, \quad (4.129)$$

where $\bar{\mathbf{Z}}_{\text{LF-MLFMA}}$ and $\bar{\mathbf{Z}}_{\text{MOM}}$ are matrices obtained by using LF-MLFMA and the reference MOM implementation, respectively. We observe that matrix elements can be computed accurately with the desired level of accuracy using LF-MLFMA. The relative error is less than 10^{-3} for all formulations and at all frequencies. On the other hand, accurate computations of matrix elements do not necessarily lead to accurate solutions. Figure 4.10(b) presents the relative error (defined with respect to the reference MOM solution as in (4.129)) in the coefficients to expand the electric current obtained by using LF-MLFMA. In the case of EFIE, the relative error increases as the mesh size decreases. This is because EFIE suffers from a low-frequency breakdown and three digits of accuracy provided by LF-MLFMA becomes insufficient when the condition number is very large. We note that CFIE is also affected by the low-frequency breakdown problem, but MFIE provides stable solutions in the entire frequency range.

In order to demonstrate the efficiency of LF-MLFMA, we consider the solution of real-life electromagnetics problems involving metamaterials. Figure 4.11 depicts a 1-layer metamaterial wall, which is constructed by periodically arranging 18×11 SRRs. A single SRR has dimensions in the order of microns and resonates at about 100 GHz when embedded into a homogeneous host medium

with a relative permittivity of 4.8 [12]. The incident field is generated by a Hertzian dipole located at $x = -1.2$ mm. For numerical solutions, surfaces are discretized with $\lambda/100$ triangles, where λ is the wavelength in the host medium. Such a dense discretization is required for accurate modelling of SRRs that involve small details with respect to wavelength. Problems are formulated with EFIE and matrix equations involving 16,236 unknowns are solved iteratively by the GMRES method without restart. MVMs are performed by MLFMA and LF-MLFMA with 2 digits of accuracy. Iterative solutions are also accelerated by the SAI preconditioner.

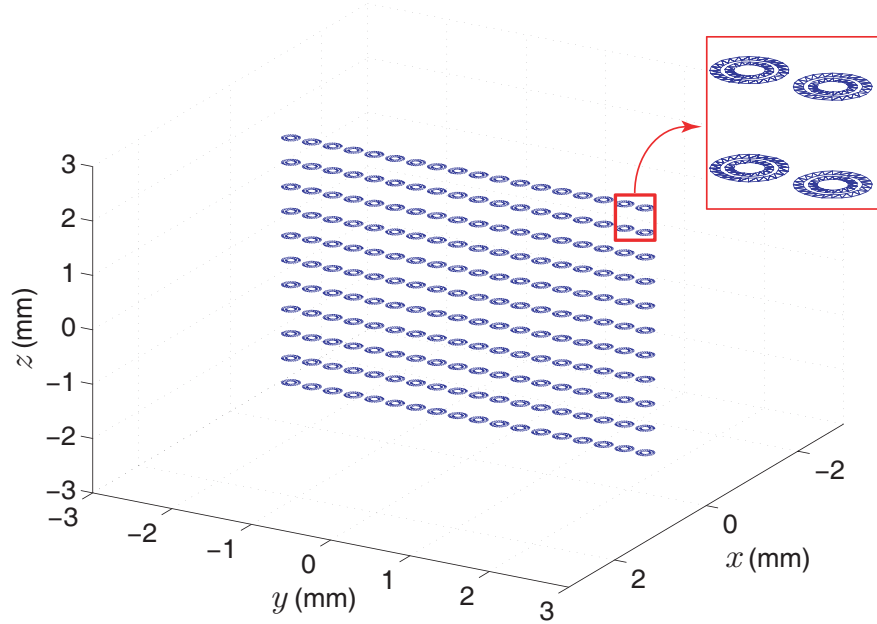


Figure 4.11: 1-layer metamaterial wall involving 18×11 SRRs.

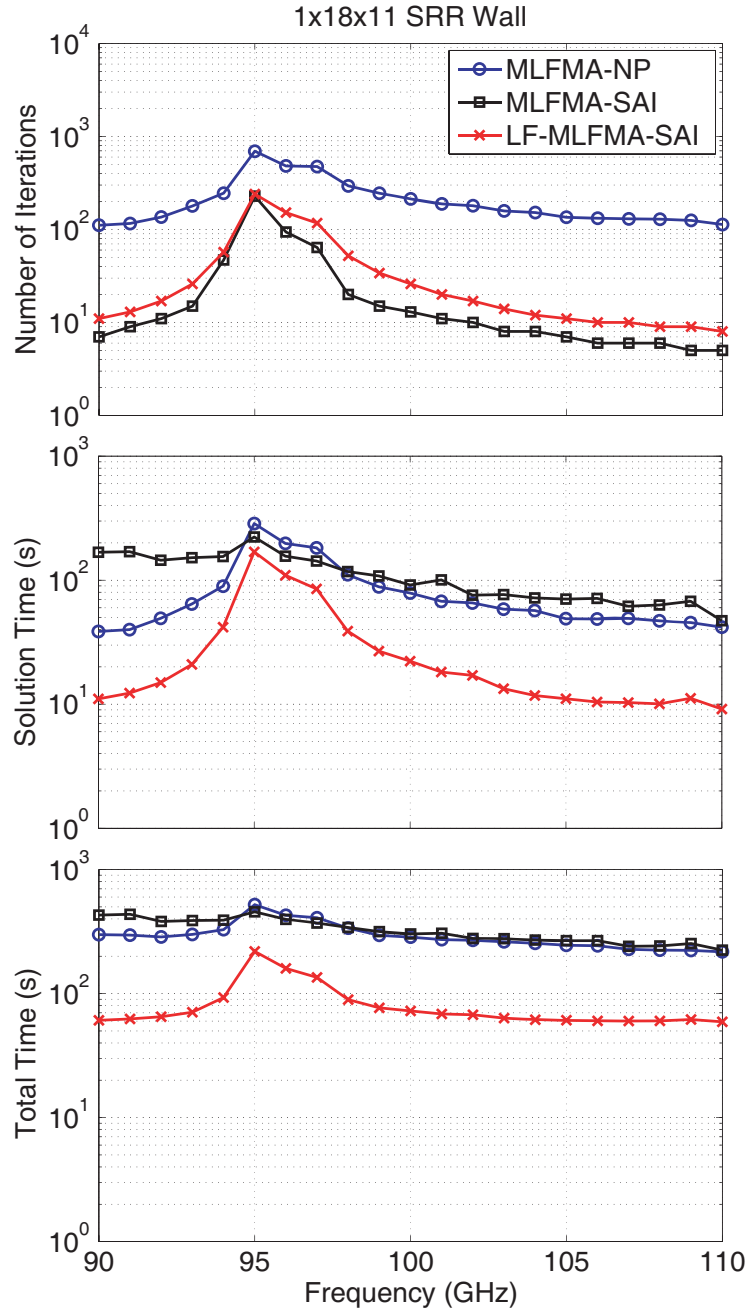


Figure 4.12: Solutions of electromagnetics problems involving a 1-layer SRR wall discretized with 16,236 unknowns. The number of GMRES iterations (10^{-3} residual error), the solution time, and the total time including setup and solution times are plotted with respect to frequency.

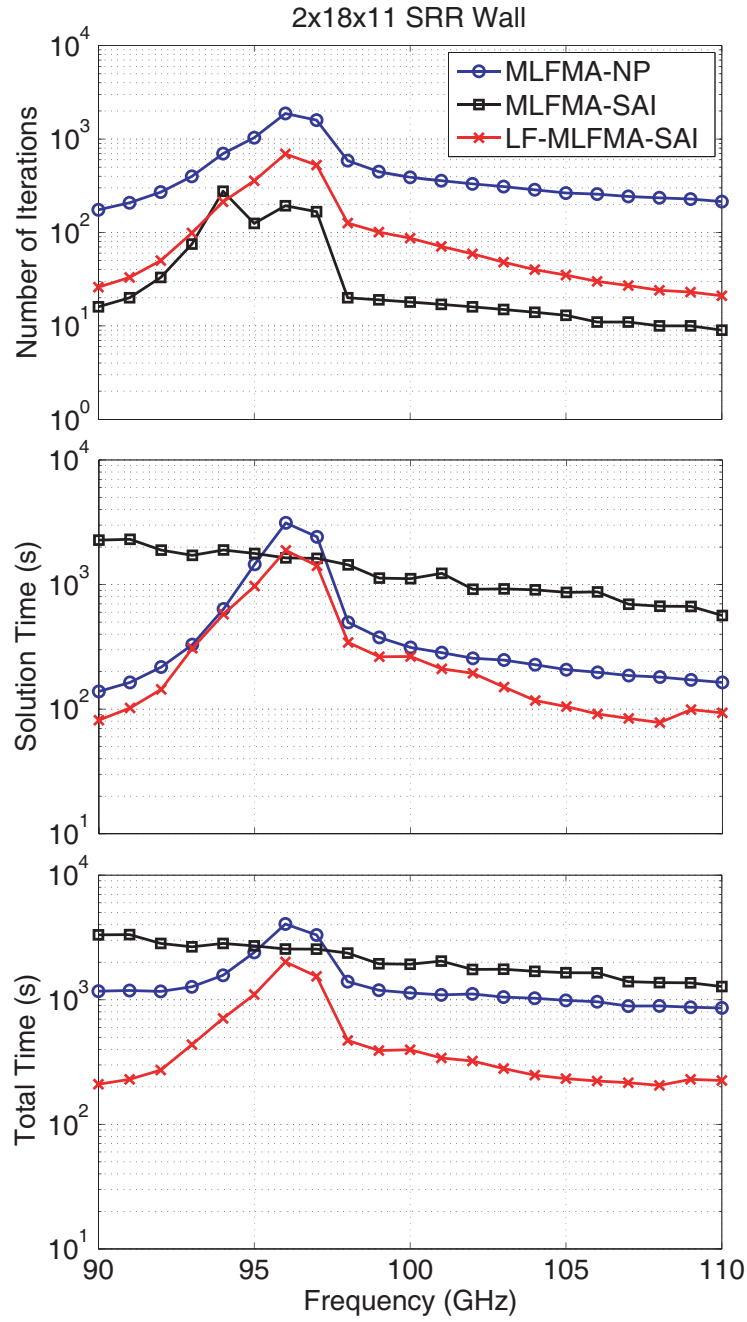


Figure 4.13: Solutions of electromagnetics problems involving a 2-layer SRR wall discretized with 32,472 unknowns. The number of GMRES iterations (10^{-3} residual error), the solution time, and the total time including setup and solution times are plotted with respect to frequency.

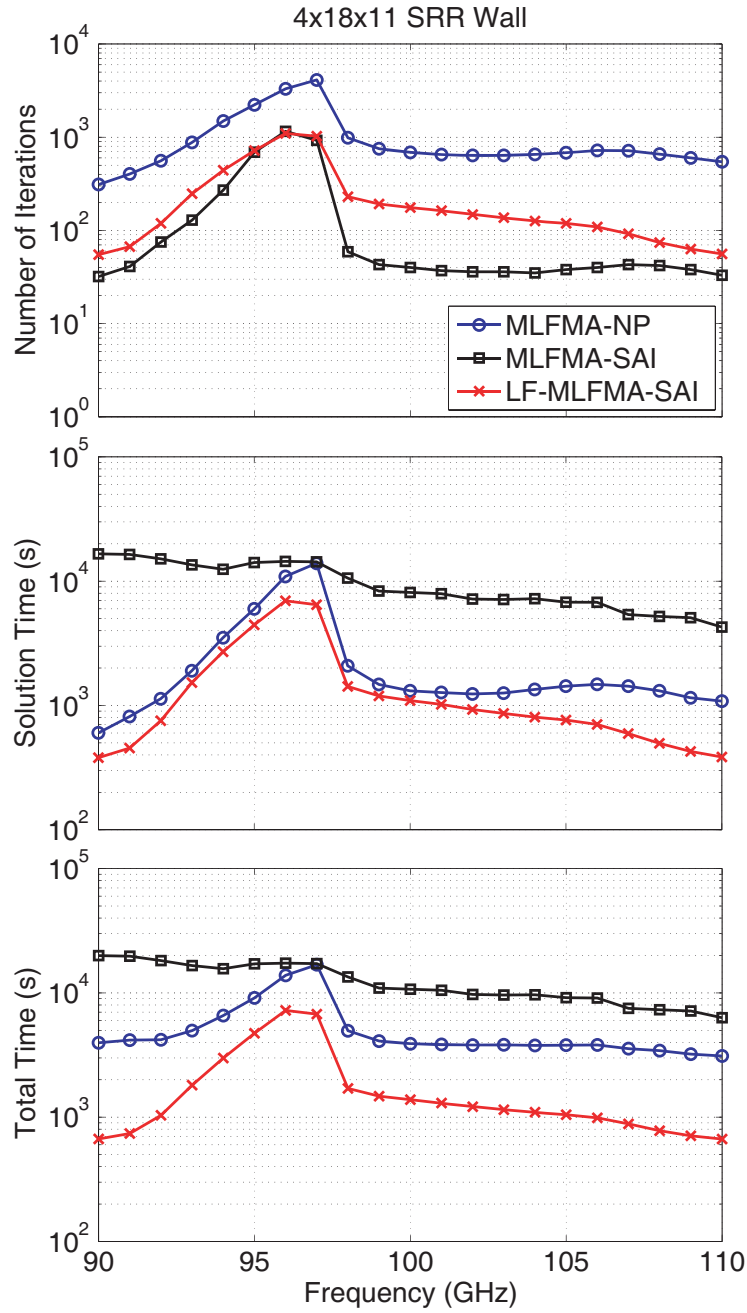


Figure 4.14: Solutions of electromagnetics problems involving a 4-layer SRR wall discretized with 64,944 unknowns. The number of GMRES iterations (10^{-3} residual error), the solution time, and the total time including setup and solution times are plotted with respect to frequency.

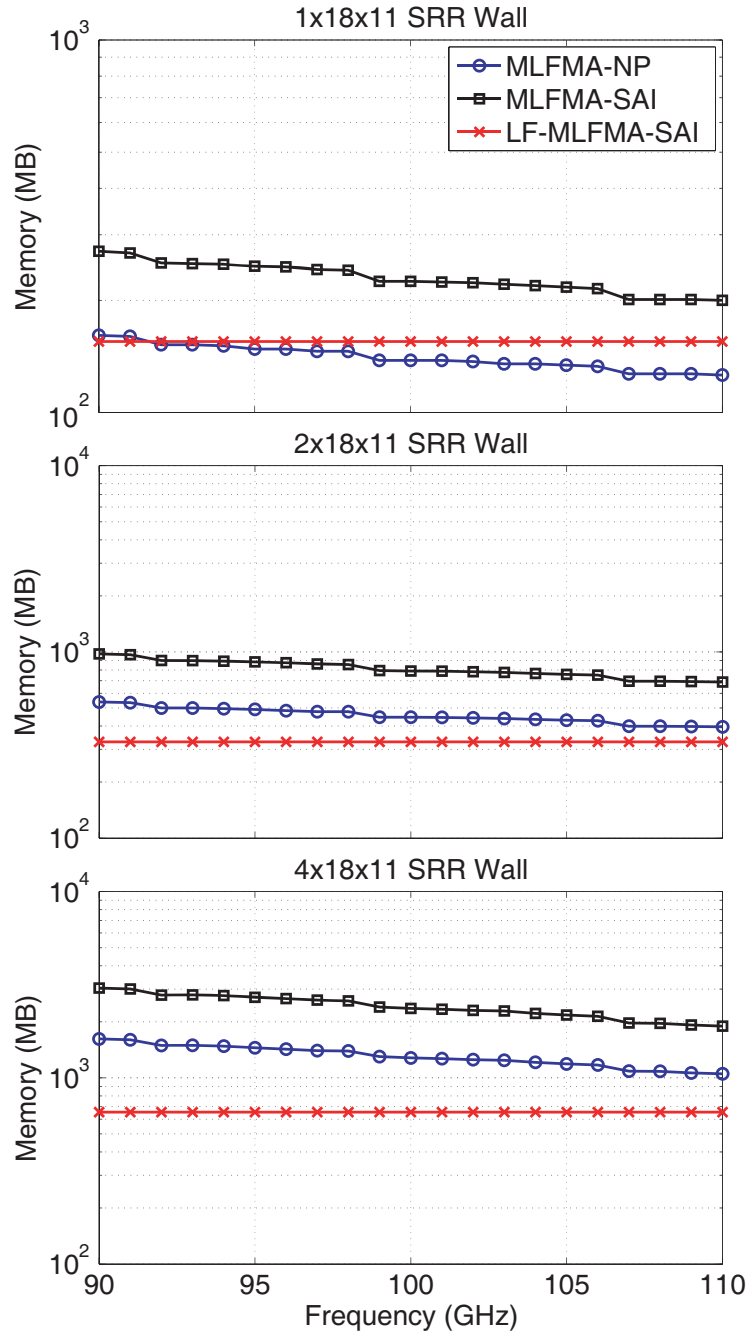


Figure 4.15: Memory required for the solution of electromagnetics problems involving 1-layer, 2-layer, and 4-layer SRR walls discretized with 16,236, 32,472, and 64,944 unknowns, respectively.

Figure 4.12 presents the solution of electromagnetics problems involving the 1-layer SRR wall depicted in Figure 4.11. The number of iterations for 10^{-3} residual error, the solution time, and the total time are plotted with respect to frequency from 90 GHz to 110 GHz. The number of iterations become maximum at 95 GHz due to a numerical resonance. Considering only MLFMA, the SAI preconditioner reduces the number of iterations significantly compared to the NP case. On the other hand, the solution time is not reduced due to the additional factorization cost of the preconditioner. In other words, reducing the number of iterations does not necessarily accelerate solutions. We also observe that using LF-MLFMA instead of MLFMA increases the number of iterations since the number of near-field interactions used to construct the preconditioner is smaller in LF-MLFMA, compared to MLFMA. On the other hand, the most efficient solutions are provided by LF-MLFMA. Superior performance of LF-MLFMA is more apparent in terms of the total time, which includes the setup time dominated by the calculation of near-field interactions in addition to the solution time.

We also consider 2-layer and 4-layer SRR walls involving $2 \times 18 \times 11$ and $4 \times 18 \times 11$ SRRs, respectively. Figures 4.13 and 4.14 present the number of iterations, the solution time, and the total time as a function of frequency. We observe that LF-MLFMA provides more efficient solutions than MLFMA. Finally, Figure 4.15 presents the memory required for the solution of 1-layer, 2-layer, and 4-layer SRR walls. In addition to faster solutions, LF-MLFMA requires less memory than MLFMA.

Chapter 5

Efficient Parallelization of the Multilevel Fast Multipole Algorithm for the Solution of Large-Scale Electromagnetics Problems

MLFMA is a powerful method for the fast and efficient solution of scattering and radiation problems in electromagnetics. However, many real-life problems require discretizations with millions of unknowns, which may not be handled easily with sequential implementations of MLFMA running on a single processor. In order to achieve the solution of such large problems, MLFMA can be parallelized and employed on a cluster of computers. In this chapter, we present our efforts for the efficient solution of large-scale electromagnetics problems using MLFMA on parallel computers. We introduce a hierarchical partitioning strategy, which improves the parallelization efficiency significantly, compared to the previous parallelization approaches. We demonstrate the effectiveness of our

algorithm by solving very large electromagnetics problems involving as many as 200 million unknowns on relatively inexpensive computing platforms with distributed-memory architectures.

Problems considered in this chapter involve closed conducting surfaces, which can be formulated with CFIE. As discussed in Chapter 3, CFIE produces well-conditioned matrix equations that are easy to solve iteratively. In all solutions, problems are discretized with about $\lambda/10$ mesh size, and near-field interactions are calculated with maximum 1% error. For relatively small problems involving less than 20 million unknowns, tree structures are constructed by using a bottom-up scheme with 0.25λ smallest box size, and far-field interactions are calculated with three digits of accuracy. For larger problems, we use either a top-down scheme or a bottom-up scheme with 0.175λ smallest box size. For those problems, far-field interactions are calculated with two digits of accuracy, unless stated otherwise. During the aggregation stage, interpolations from a level to the next higher level are performed using 6×6 samples in the coarse grid for each sample in the fine grid. Finally, iterative solutions are performed by using BiCGStab accelerated with BDP, and the residual error for the iterative convergence is set to 10^{-6} and 10^{-3} for small and large problems, respectively.

5.1 On the Parallelization of the Multilevel Fast Multipole Algorithm

Because of its complicated structure, parallelization of MLFMA is not trivial and several issues must be carefully considered to obtain an efficient parallelization of MLFMA.

- Partitioning: The main task in the parallelization of MLFMA is partitioning the multilevel tree structure among processors. For high efficiency, it is essential to distribute the tree structure with minimal duplication.
- Load-balancing: Parallelization cannot be achieved efficiently without distributing tasks equally among processors. Load-balancing algorithms must be used for each part of the implementation to obtain a sufficiently high parallelization efficiency.
- Communications: In parallel MLFMA, processors need to communicate with each other to transfer data. Using appropriate partitioning schemes and load-balancing algorithms significantly reduces the data traffic. However, remaining communications must be organized carefully. For high efficiency, it is also essential to use high-speed networks to connect processors.

5.2 Parallel Computing Platforms

We have been using the following parallel computers for the solution of large-scale problems in electromagnetics. Each computer involves 16 computing nodes.

- Clovertown Cluster (Swindon): Each node has 16 GB of memory and two quad-core Intel Xeon Clovertown processors with 2.66 GHz clock rate.
- Tigerton Cluster (Swindon): Each node has 32 GB of memory and two quad-core Intel Xeon Tigerton processors with 2.93 GHz clock rate.
- Harpertown Cluster (BiLCEM): Each node has 32 GB of memory and two quad-core Intel Xeon Harpertown processors with 3.00 GHz clock rate.
- Dunnington Cluster (Swindon): Each node has 48 GB of memory and four six-core Intel Xeon Dunnington processors with 2.40 GHz clock rate.

In all clusters, memory in a node is available to all cores in the node. Nodes are connected via Infiniband networks, while processors in a node are connected through high-speed mainboard buses. In the context of parallelization, we use the terms “processor” and “core” synonymously. For a solution on p processors, we use the maximum number of nodes available, i.e., the number of processes per node is minimized. In other words, if a code is parallelized into 2^i processes, and if $2^i \leq 16$, then we use 2^i nodes, each running only one process. When $2^i > 16$, however, the solution is parallelized over 16 nodes, and $2^i/16$ processors are employed per node.

5.3 Simple Parallelization

A simple parallelization of MLFMA is based on distributing boxes among processors at all levels. We apply load-balancing for both near-field and far-field interactions. In the following subsections, we provide details of the simple parallelization of MLFMA.

5.3.1 Near-Field Interactions

Near-field interactions should be distributed among processors using a load-balancing algorithm. Considering the sparse near-field matrix, the rows $m = 1, 2, \dots, N$ are assigned to processors in such a way that all processors have approximately equal numbers of near-field interactions (near-field partitioning). Distributing rows equally among processors usually fails to provide good load-balancing, even for the solution of problems involving symmetrical geometries, such as a sphere. After distribution, near-field interactions are calculated in each processor without any communication.

5.3.2 Far-Field Interactions

Calculation of far-field interactions is organized according to the partitioning of the tree structure (far-field partitioning). We choose a level of distribution (LoD) to distribute boxes among processors using a load-balancing algorithm that considers the combined load of all descendants (children, grandchildren, etc.) of each box at LoD. The combined load for a box is the size of the subtree attached to the box; we account for all descendants, each weighted by the number of field samples. The load-balancing algorithm assigns the whole branch of the tree starting at an LoD box to the same processor. This way, aggregation and disaggregation stages up to LoD can be performed without any communication. During the setup stage, each processor calculates and stores radiation and receiving patterns of basis and testing functions included in its local tree according to the far-field partitioning.

Using the simple parallelization technique, there can be two different schemes for the parallelization of higher levels above LoD. In the first scheme, boxes are still distributed among processors equally as much as possible. Then, a box and its parent box may belong to different processors. As a consequence, aggregation and disaggregation stages involve dense communications for higher levels of the tree structure. Rather than this scheme, we prefer the second one, where boxes at levels above LoD are allowed to be duplicated in different processors, if required. This way, all aggregation and disaggregation stages can be performed without any communication. We note that, when radiated fields are translated into incoming fields, contributions from different processors are combined automatically for duplicated boxes.

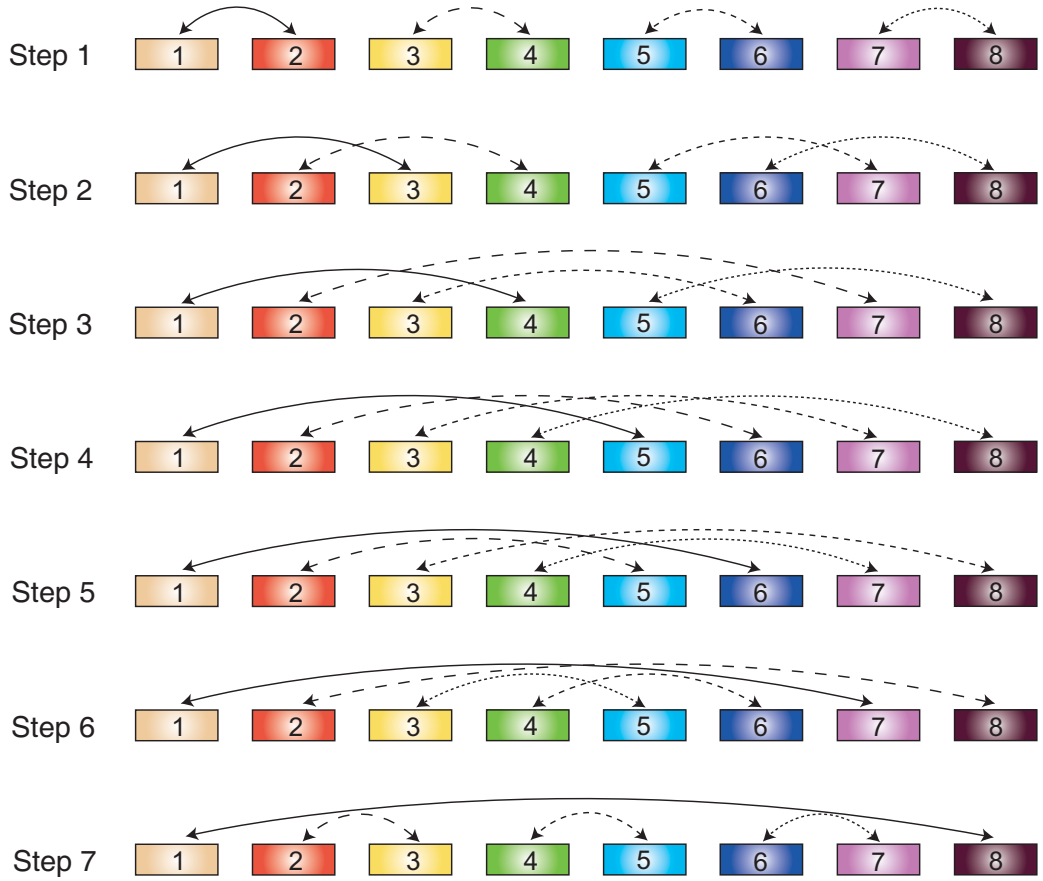


Figure 5.1: Processor pairing for translations.

The translation stage is one of the most critical parts in terms of the efficiency of the simple parallelization technique. This is because dense one-to-one communications are required between processors. In general, each processor sends some data (radiated fields) to all other processors. We organize these communications using a communication map, which consists of interaction layers to match processors. For p processors, it can be shown that communications can be achieved in $(p - 1)$ steps, as depicted in Figure 5.1 for a 8-processor case. After processors are paired, the following operations are performed on the receiver and sender sides:

- The sender and receiver determine box-box interactions involving basis boxes on the sender side and testing boxes on the receiver side.

- Radiated fields of basis boxes are sent one by one.
- When the radiated field of a basis box is received by the receiver, all of translations involving this basis box and testing boxes owned by the receiver are performed. This ensures that the same data is not transferred more than once.

To further improve the efficiency of translations, we use nonblocking send and receive operations of MPI to transfer the data. In addition to inter-processor translations, there are also intra-processor translations that are related to boxes located in the same processor. These translations can be performed independently in each processor, without any communication.

In the setup of MLFMA, each processor is tasked with calculating a set of translation operators that will be required during MVMs. For a given translation, the operator is calculated by the processor working on the testing box. Due to symmetry, a translation operator can be used for many interactions at a level. Therefore, some of translation operators are duplicated and included in more than one processor. Those duplications may decrease the efficiency of the parallelization, especially due to increasing cost of operators at higher levels.

Finally, to match near-field and far-field partitioning schemes during MVMs, all-to-one and one-to-all communications are required, as depicted in Figure 5.2. After near-field computations are performed, the partitioning of the output vector is modified for far-field computations. We note that the iterative solver is also parallelized according to the far-field partitioning. The processing time for these communications is usually negligible. Similar all-to-all communications are also required during solutions of the preconditioner system.

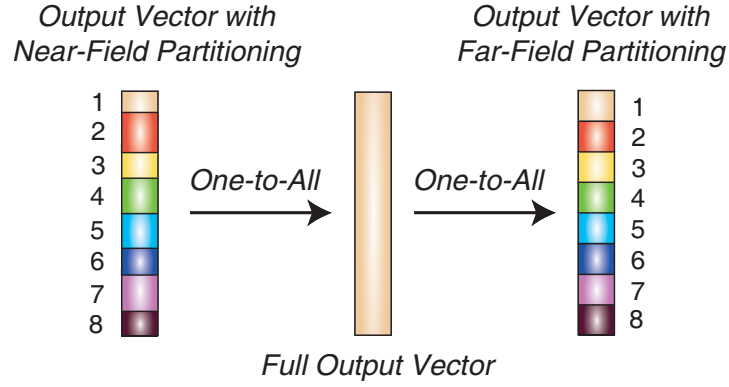


Figure 5.2: Communications performed in each MVM to match near-field and far-field partitioning schemes.

5.4 Hybrid Parallelization

The simple parallelization technique, which is based on assigning each box to a processor, usually fails to provide efficient solutions. This is mainly due to dense communications between processors, duplication of computations, and unbalanced distribution of the workload among processors. Since such problems arise mostly at higher levels of MLFMA, a hybrid parallelization technique, which applies different partitioning strategies for lower and higher levels, is developed to improve the efficiency [134]. In this technique, processor assignments are made on the basis of fields of boxes at higher levels. In other words, each box at higher levels is shared by all processors, while each processor is assigned to the same portion of fields for all boxes.

As an example, Figure 5.3 depicts a 4-level tree structure ($L = 4$), where levels are represented by two-dimensional rectangles. Horizontal and vertical dimensions of rectangles correspond to boxes and θ samples of fields, respectively. At lower levels (distributed levels), there are many boxes with small numbers of samples for radiated and incoming fields. Therefore, it is appropriate to distribute boxes at these levels by assigning each of them to a single processor. At higher levels

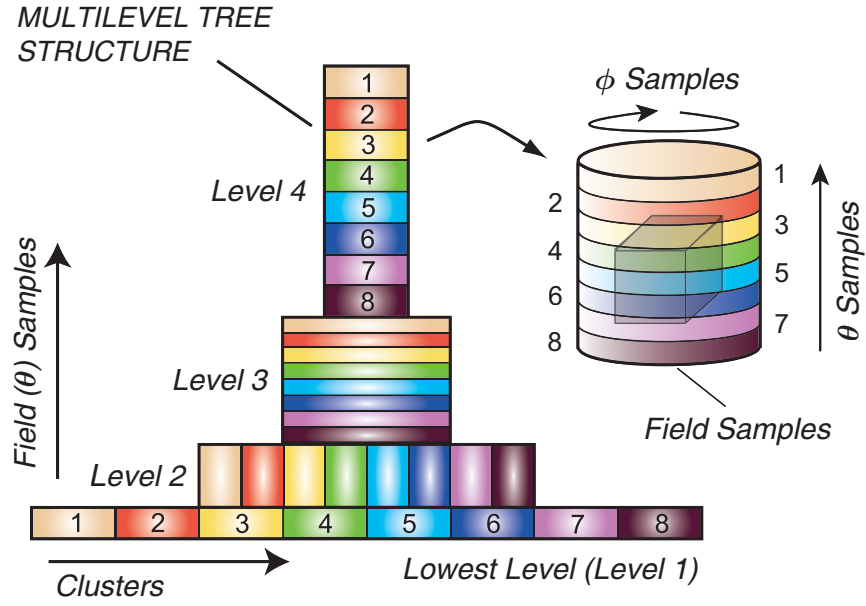


Figure 5.3: Distribution of a 4-level tree structure among eight processors using the hybrid partitioning strategy.

(shared levels), however, it is easier to distribute field samples among processors by assigning each box to all processors, since there are a few boxes at these levels with large numbers of samples. We note that field samples are partitioned only along the θ direction for an easy implementation of interpolation/interpolation operations.

Distributing field samples instead of boxes at higher levels of MLFMA has two major consequences:

- Translations can be performed without any communication at higher levels, whereas aggregation and disaggregation stages require one-to-one communications.
- Load-balancing is improved since the distribution of samples is easier than the distribution of boxes at higher levels.

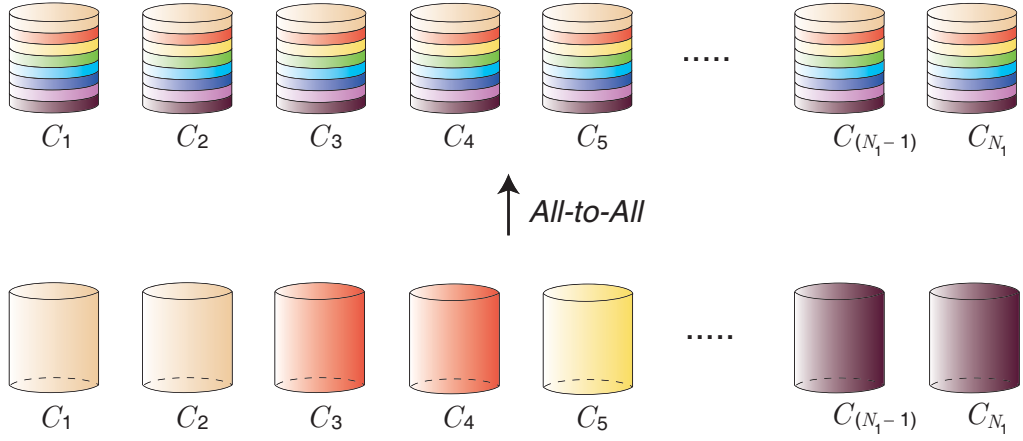


Figure 5.4: All-to-all communications performed at LoD to change the far-field partitioning scheme from distributed levels to shared levels.

In the following subsections, we provide details of the hybrid parallelization of MLFMA by emphasizing major differences compared to the simple parallelization technique.

5.4.1 Aggregation Stage

Using the hybrid technique, the partitioning scheme is changed during the aggregation stage at LoD by employing an all-to-all communication, as shown in Figure 5.4. For each box, samples of the radiated field stored in a processor is distributed among all processors. From LoD to the highest level L , the aggregation stage involves one-to-one communications that are required for the interpolation of fields. This is illustrated in Figure 5.5, where an interpolation is performed on samples of a box C located at level 3 of the tree structure in Figure 5.3. As an example, only the interpolation in processor 5 is depicted although similar operations are also performed in other processors. To compute the data at each sample in the fine grid, a set of samples are used in the coarse grid. Even though a local interpolation method is used, some of those coarse samples may be located in other processors. Therefore, one-to-one communications are performed

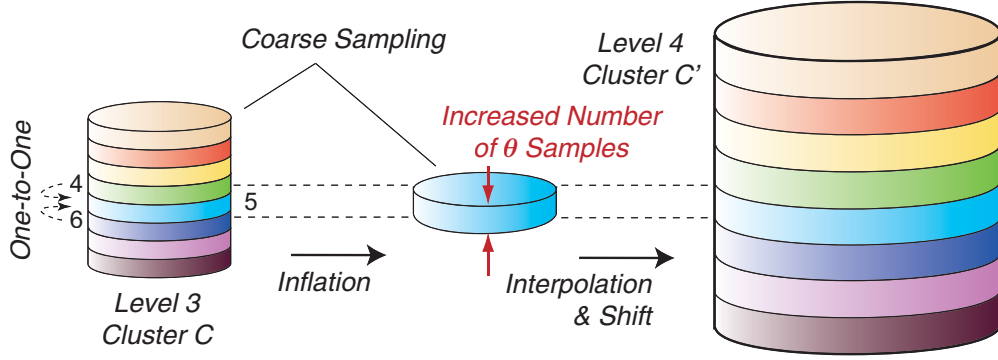


Figure 5.5: Interpolations at shared levels involving one-to-one communications.

to provide the required data (inflation). After the data is prepared, interpolation and shifting operations are performed to include the contribution of the box C in the radiated field of its parent box C' .

We note that communications at shared levels are mainly required between processors located close to each other. In other words, the processor with index n requires data from its neighbors, i.e., $(n - 1)$ and $(n + 1)$. On the other hand, depending on the partitioning and the number of interpolation points, more data might be required from other processors next to neighbors. We apply a load-balancing algorithm to distribute fields appropriately so that the amount of the data transferred among all processors is minimized. However, as the number of processes increases and fields are distributed over many processors, dense one-to-one communications cannot be avoided; this may reduce the efficiency of the parallelization.

For each problem, we carefully choose the number of distributed and shared levels by an optimization. For this purpose, we assign LoD to a series of possible levels and monitor the distribution of boxes and their fields. For some of levels (higher levels), distribution of fields is better than the distribution of boxes,

i.e., field samples can be partitioned evenly among processors, but not boxes. For others (lower levels), however, boxes can be partitioned easily, while it is difficult to partition field samples among processors. Then, we choose LoD such that distributing fields (boxes) is more preferable for all levels above (below) LoD. The choice of LoD depends on the tree structure (hence the geometry of the target) as well as the number of processors. However, our measurements show that, for a given problem, LoD is insensitive to the latter parameter if only a small number (e.g., 2 to 16) of processors are employed.

5.4.2 Translation Stage

Using the hybrid technique, all translations at shared levels can be performed without any communication since field samples are distributed among processors and a processor is assigned to the same portion of radiated and incoming fields for all boxes. In addition, translation operators are not duplicated at shared levels, since they are also partitioned among processors. At distributed levels, however, some translations are needed between boxes located in different processors. Those inter-processor translations are performed via one-to-one communications using a communication map, as in the simple parallelization of MLFMA.

5.4.3 Disaggregation Stage

Similar to the aggregation stage, one-to-one communications are required during the disaggregation stage at shared levels. Since the antinterpolation is the transpose of the interpolation, some of samples obtained from an antinterpolation operation should be sent to other processors. This is because interpolations during the aggregation stage are performed using the inflated data prepared by one-to-one communications among processors. As the reverse of this process, antinterpolations produce inflated data, which must be deflated via one-to-one communications.

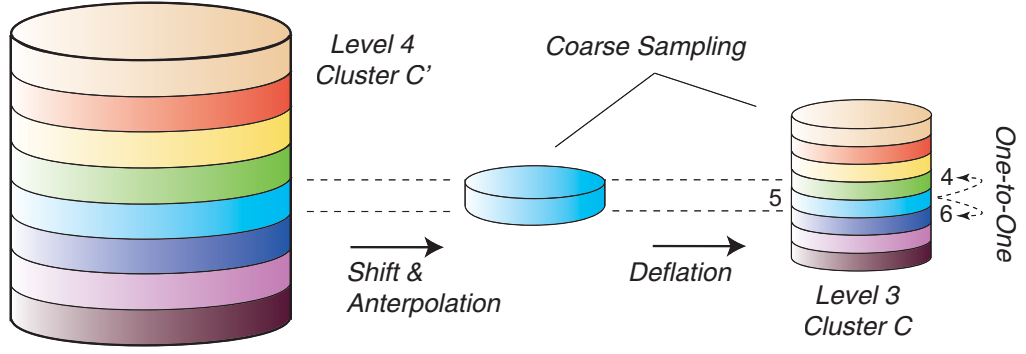


Figure 5.6: Anteropolarizations (transpose interpolations) in shared levels involving one-to-one communications.

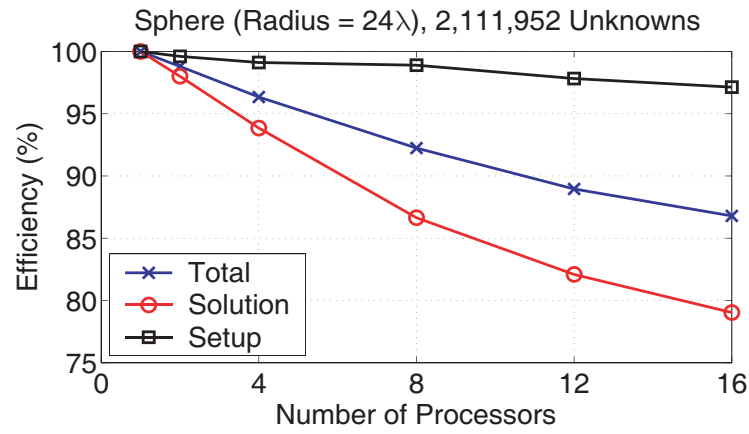
This is illustrated in Figure 5.6, where processor 5 performs the anteropolarization operation on samples of box C' for its sub-boxes C . Some of the resulting data in the coarse grid is used locally, while the rest are sent to other processors (deflation), i.e., exactly the reverse of the interpolation. As the disaggregation operation proceeds down to LoD, the partitioning is changed via an all-to-all communication. Then, the disaggregation is performed from LoD to the lowest level without any communication. Finally, at the lowest level, each processor performs angular integrations and produces a partition of the output vector.

5.4.4 Communications

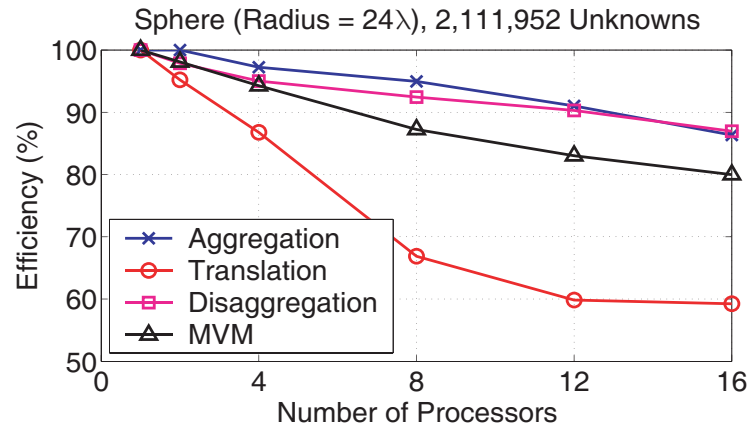
As a summary, Table 5.1 lists communications required during MVMs using the hybrid parallelization technique. For distributed and shared levels, communications are required in translation and aggregation/disaggregation stages, respectively. In addition to those one-to-one communications, the hybrid parallelization involves collective all-to-one, one-to-all, and all-to-all communications.

Table 5.1: Communications Required During MVMs Using the Hybrid Parallelization Technique

Part	Communication
Near-Field Interactions	All-to-One and One-to-All
Distributed Aggregation/Disaggregation	None
Aggregation/Disaggregation at LoD	All-to-All
Shared Aggregation/Disaggregation	One-to-One
Distributed Translation	One-to-One
Shared Translation	None



(a)



(b)

Figure 5.7: Parallelization efficiency for the solution of a scattering problem involving a sphere of radius 24λ discretized with 2,111,952 unknowns.

5.5 Numerical Results for the Hybrid Parallelization Technique

To demonstrate the efficiency provided by the hybrid parallelization technique, we consider the solution of a scattering problem involving a sphere with a radius of 24λ . The problem is discretized with 2,111,952 unknowns and solved on the Clovertown cluster. Figure 5.7 depicts the efficiency (with respect to the solution with a single processor) when the solution is parallelized into 2, 4, 8, 12, and 16 processes. The parallelization efficiency is defined as

$$\varepsilon_p = \frac{T_1}{pT_p}, \quad (5.1)$$

where T_p is the processing time of the solution using p processors. Figure 5.7(a) shows that the overall efficiency (setup and iterative solution) is above 85% when the number of processors is 16. In this case, efficiency ratios for the setup and the solution parts are about 97% and 80%, respectively. We observe in Figure 5.7(a) that the setup part is parallelized very efficiently, since this part is communication-free and computations (especially near-field interactions) are perfectly distributed among processors using load-balancing algorithms.

In Figure 5.7(b), we present the parallelization efficiency for aggregation, translation, and disaggregation stages, in addition to the overall efficiency for MVMs. We observe that aggregation and disaggregation stages are parallelized with about 87% efficiency, while the efficiency for the translation stage is 59% for the 16-processor case. To further investigate the parallelization, Figure 5.8 presents processing time and efficiency (with respect to the solution with two processors) for various categorized parts of MVMs. Our observations are as follows:

- Aggregation and disaggregation stages at distributed levels ($l = 1, 2, 3, 4$ for this problem) constitute the significant part of the processing time of

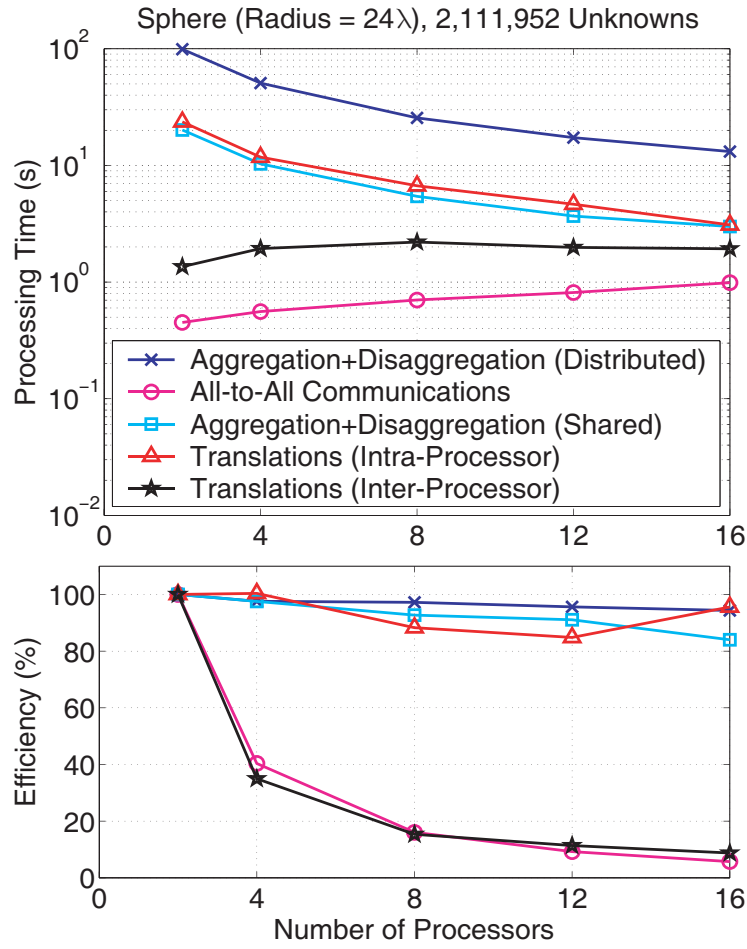


Figure 5.8: Processing time and parallelization efficiency for various categorized parts of MVMs for the solution of a scattering problem involving a sphere of radius 24λ discretized with 2,111,952 unknowns.

MVM. These stages are perfectly parallelized, thanks to the load-balancing algorithm for distributed levels.

- The parallelization efficiency of aggregation and disaggregation stages at shared levels (from $l = 5$ to $l = L = 7$ in this problem) is also quite high. However, the efficiency drops to about 80% for the 16-processor case. This is due to the increasing amount of one-to-one communications for interpolations and antepolations.

- Parallelization efficiency of communication-free (intra-processor) translations is in the 80%–100% range. All of translations at shared levels and some of those at distributed levels are communication-free.
- Translations that are performed with communications (inter-processor translations) and all-to-all communications performed at LoD exhibit reduced efficiency as the number of processes increases. Since they take longer processing time, inter-processor translations affect the overall efficiency more than all-to-all communications.

In general, inter-processor translations are the bottleneck of the parallelization. Since these translations are performed at distributed levels, their negative contributions can be minimized by increasing the number of shared levels. However, aggregation and disaggregation at low levels cannot be performed efficiently by partitioning coarsely-sampled fields. In fact, we carefully determine the number of distributed and shared levels to optimize the parallelization efficiency for the solution of each problem. Results presented in Figures 5.7 and 5.8 are obtained with the best possible choice for the number of distributed/shared levels in terms of the overall parallelization efficiency.

Even though the hybrid parallelization technique increases the parallelization efficiency significantly, compared to simple parallelization approaches, the improvement can be insufficient, especially when the number of processors is large. As a remedy, we developed a hierarchical partitioning technique that is well suited for the multilevel structure of MLFMA. With the enhanced load-balancing offered by the hierarchical technique, parallelization of MLFMA can be improved significantly.

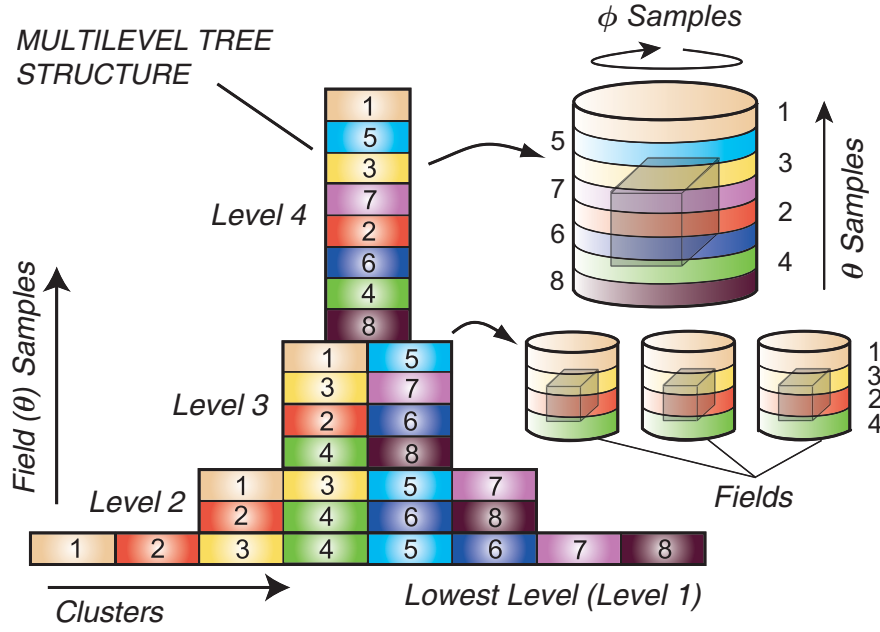


Figure 5.9: Distribution of a 4-level tree structure among eight processors using the hierarchical partitioning strategy.

5.6 Hierarchical Parallelization

The hierarchical parallelization is based on the simultaneous partitioning of boxes and their fields at all levels. We adjust the partitioning in both directions (boxes and samples of fields) appropriately by considering the number of boxes and the number of samples at each level. As an example, we again consider a 4-level tree structure ($L = 4$), where each level is represented by a rectangular box. Hierarchical partitioning of the tree structure among eight processors is depicted in Figure 5.9. At the lowest level, boxes are distributed among eight processors, and each box is assigned to a single processor, without any partitioning of field samples. Then, at the next level ($l = 2$), field samples are partitioned among two groups of processors, i.e., (1,3,5,7) and (2,4,6,8), while the number of box partitions is reduced to four. At this level, samples of each box are shared by two processors. As we proceed to higher levels, the number of partitions for boxes

and samples of fields are systematically decreased and increased, respectively. In the following subsections, we present the hierarchical parallelization of MLFMA in detail by considering the main stages of the algorithm.

5.6.1 Partitioning of the Tree Structure

We consider the parallelization of MLFMA on a cluster of p processors, where $p = 2^i$ for some integer i . Using the hierarchical partitioning technique, the number of partitions for boxes at level l is chosen as

$$p_{l,c} = \max \left\{ \frac{p}{2^{(l-1)}}, 1 \right\} = \max \{p 2^{(1-l)}, 1\}. \quad (5.2)$$

We note that boxes are not partitioned for levels $l > \log_2(p)$, if such a level exists.

The number of boxes assigned to each processor can be approximated as

$$N_l^p \approx \frac{N_l}{p_{l,c}} \approx \begin{cases} 2^{(1-l)} N_1/p, & l \leq \log_2(p) \\ 4^{(1-l)} N_1, & l > \log_2(p) \end{cases}. \quad (5.3)$$

In addition, samples of fields are divided into

$$p_{l,s} = \frac{p}{p_{l,c}} = \min \{2^{(l-1)}, p\} \quad (5.4)$$

partitions along the θ direction for level l . The number of θ samples assigned to each processor is

$$S_l^{\theta,p} \approx \frac{S_l^\theta}{p_{l,s}} \approx \begin{cases} S_1^\theta, & l \leq \log_2(p) \\ 2^{(l-1)} S_1^\theta/p, & l > \log_2(p) \end{cases}. \quad (5.5)$$

Also considering the sampling in the ϕ direction, the total number of samples per processor can be written as

$$S_l^p = S_l^{\theta,p} S_l^\phi \approx \begin{cases} 2^{(l-1)} S_1, & l \leq \log_2(p) \\ 4^{(l-1)} S_1/p, & l > \log_2(p) \end{cases}. \quad (5.6)$$

Finally, the size of the local data at each processor is

$$N_l^p S_l^p \approx N_1 S_1/p \quad (5.7)$$

for $l = 1, 2, \dots, L$.

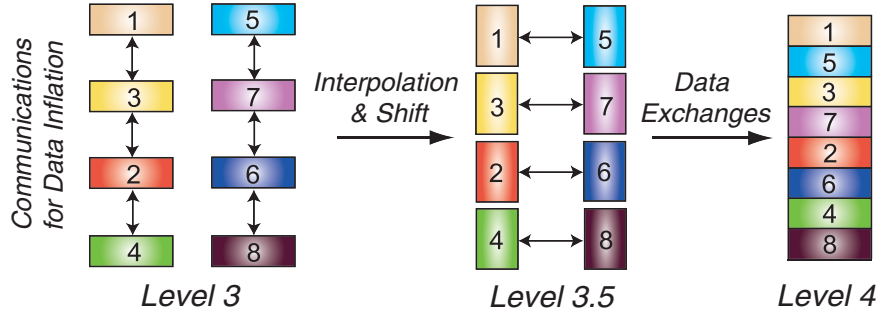


Figure 5.10: Aggregation operations from level 3 to level 4 for the partitioned tree structure in Figure 5.9.

5.6.2 Aggregation Stage

For the partitioned tree structure in Figure 5.9, aggregation operations from level 3 to level 4 are depicted in Figure 5.10. Since field samples are partitioned among processors, one-to-one communications are required to inflate the local data in accordance with the interpolation requirements. As shown in Figure 5.10, one-to-one communications occur within two separate groups of processors that are located in the same columns, i.e., (1,2,3,4) and (5,6,7,8). As an important advantage of the hierarchical partitioning technique, distribution of θ samples into large numbers of partitions is avoided. Therefore, communications are required only between pairs of processors located next to each other. For example, processor 3 communicates with processors 1 and 2, but not with processor 4. Processors 3 and 4 would need to communicate with each other if the number of the θ samples required for the interpolation is larger than the number of the θ samples per processor. However, using the hierarchical technique, the number of partitions along the θ direction, hence the number of θ samples per processor, can be adjusted such that those secondary communications between distant processors are avoided.

When the required data is prepared by one-to-one communications for a box,

its radiated field is interpolated and shifted to the center of its parent box. Temporary levels involving parent boxes and field samples after interpolation and shifting operations are denoted as intermediate levels. As an example, for the partitioned tree structure in Figure 5.9, level 3.5 is depicted in Figure 5.10. Following the interpolation, the number of samples along the θ direction assigned to each processor is doubled, i.e.,

$$S_{l+1/2}^{\theta,p} \approx 2S_l^{\theta,p} \approx \begin{cases} 2S_1^\theta, & l \leq \log_2(p) \\ 2^l S_1^\theta/p, & l > \log_2(p) \end{cases}. \quad (5.8)$$

At the same time, the number of boxes in each processor can be written as

$$N_{l+1/2}^p \approx \frac{N_l^p}{4} \approx \begin{cases} 2^{-(l+1)} N_1/p, & l \leq \log_2(p) \\ 4^{(-l)} N_1, & l > \log_2(p) \end{cases}. \quad (5.9)$$

Intermediate levels are defined temporarily and used to arrange the data in each processor, before communications are performed to modify the partitioning according to the hierarchical technique.

From an intermediate level $(l + 1/2)$ to the next level $(l + 1)$, data is exchanged among processors, if $l \leq \log_2(p)$. As depicted in Figure 5.10, processors are paired according to their positions in the partitioning scheme. Each processor performs the following communications.

- Send half of field samples of each box at the intermediate level.
- Receive the complementary data, which involves field samples of some boxes, from the associated processor.

With data exchanges, the number of boxes in each processor is doubled with respect to the number of boxes at the intermediate level, while the number of samples along the θ direction is halved. Then, we have

$$N_{l+1}^p \approx \begin{cases} 2^{(-l)} N_1/p, & (l + 1) \leq \log_2(p) \\ 4^{(-l)} N_1, & (l + 1) > \log_2(p) \end{cases} \quad (5.10)$$

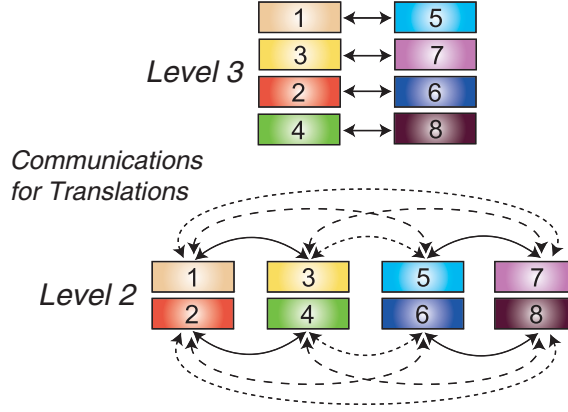


Figure 5.11: One-to-one communications during the translation stage at levels 2 and 3 of the partitioned tree structure in Figure 5.9.

and

$$S_{l+1}^{\theta,p} \approx \left\{ \begin{array}{ll} S_1^\theta, & (l+1) \leq \log_2(p) \\ 2^l S_1^\theta / p, & (l+1) > \log_2(p) \end{array} \right\}, \quad (5.11)$$

which agree with expressions in (5.3) and (5.5), respectively.

5.6.3 Translation Stage

Using the hierarchical partitioning technique, one-to-one communications are required during the translation stage for levels $l \leq \log_2(p)$, since boxes are partitioned and some translations are needed between boxes located in different processors. As in the simple parallelization technique, those communications are achieved by pairing processors and transferring radiated fields of boxes between pairs. On the other hand, using the hierarchical partitioning strategy, communications are required only among processors that are located in the same row of the partitioning. For example, as depicted in Figure 5.11, communications at level 2 are performed within two separate groups of processors, i.e., (1,3,5,7) and (2,4,6,8). In general, for inter-processor translations at level l , each processor is

paired one by one with the other $(p_{l,c} - 1)$ processors. Once a pairing is established, radiated fields of boxes are transferred, and translations are performed by the receiver processor.

5.6.4 Disaggregation Stage

As usual, the parallelization of the disaggregation stage is very similar to the parallelization of the aggregation stage using the hierarchical partitioning strategy. When incoming fields are calculated at box centers at level l , partitioning is modified via data exchanges among processors, if $l \leq \log_2(p) + 1$. This way, the partitioning at level $(l - 1/2)$ is generated as required for antepolation and shifting operations. Then, incoming fields at box centers are antepolated and shifted to centers of their sub-boxes at level $(l - 1)$. Following an antepolation operation, some of the resulting data are used locally, while the rest are sent to other processors. Similar to communications during interpolations, data are transferred among neighboring processors in the same column of the partitioning scheme.

5.6.5 Communications in the Hierarchical Parallelization

Using the hierarchical partitioning strategy, computations on the tree structure are distributed among processors with improved load-balancing, compared to the previous strategies based on partitioning with respect to only boxes or only samples of fields. However, there are still unavoidable communications among processors, which may reduce the efficiency of the parallelization. In this section, we investigate these communications in detail.

Communications in Aggregation and Disaggregation Stages

During an interpolation operation in a processor at level $l = 2, 3, \dots, (L - 1)$, the amount of data required from other processors for each box is proportional to the number of samples in the ϕ direction. Considering also the number of boxes per processor, the communication volume for interpolations at level l can be written as

$$\begin{aligned} v_{int,l} &\propto 2^{(l-1)} S_1^\phi \begin{cases} 2^{(1-l)} N_1/p, & l \leq \log_2(p) \\ 4^{(1-l)} N_1, & l > \log_2(p) \end{cases} \\ &= \begin{cases} N_1 S_1^\phi/p, & l \leq \log_2(p) \\ N_1 S_1^\phi/2^{(l-1)}, & l > \log_2(p) \end{cases} \leq N_1 S_1^\phi/p \end{aligned} \quad (5.12)$$

for $l = 2, 3, \dots, (L - 1)$. In general, the communication volume $v_{int,l}$ tends to decrease with the increasing number of processors p . We also note that aggregations at the lowest level do not involve one-to-one communications to inflate data, since samples are not partitioned at this level.

For levels $l \leq \log_2(p)$, half of the produced data is exchanged between pairs of processors to modify the partitioning in accordance with the hierarchical partitioning strategy. The volume of these communications can be expressed as

$$v_{exc,l} \propto \begin{cases} N_1 S_1/p, & l \leq \log_2(p) \\ 0, & l > \log_2(p) \end{cases} \leq N_1 S_1/p \quad (5.13)$$

for $l = 1, 2, \dots, (L - 1)$, where the upper bound is again inversely proportional to the number of processors p . The communication volume for the disaggregation stage is the same as the communication volume for the aggregation stage.

Communications in the Translation Stage

During the translation stage, each processor is paired one by one with $(p_{l,c} - 1) = \max\{p 2^{(1-l)} - 1, 0\}$ processors to perform inter-processor translations. For each

pair, the number of box-box interactions required to be performed is proportional to the number of boxes per processor. In addition, the size of the data transferred in each interaction is proportional to the number of local samples per box, i.e., S_l^p . Therefore, the communication volume for translations can be written as

$$v_{trans,l} \propto \max \{p 2^{(1-l)} - 1, 0\} N_l^p S_l^p \leq N_1 S_1 / 2^{(l-1)} \quad (5.14)$$

for $l = 1, 2, \dots, L$. We note that the upper bound for the communication volume in (5.14) depends on the level. Specifically, communications during translations may become significant for lower levels, i.e., when l is small. In fact, due to these communications, the translation stage is the major bottleneck in the hierarchical parallelization of MLFMA.

5.6.6 Improved Partitioning of the Tree Structure

In the previous sections, we assume that the partitioning changes regularly on the tree structure, i.e., the number of partitions of boxes decreases from a level to the next upper level and the number of partitions of field samples is adjusted accordingly. In fact, for a given problem, we choose the number of partitions at each level carefully. For example, in some cases, the number of partitions do not change from a level to the next level. This is required especially to avoid excessive communications during aggregation and disaggregation stages.

5.6.7 Comparisons with the Previous Parallelization Techniques

The hierarchical parallelization technique is compared with two previous approaches, namely, the simple and the hybrid parallelization techniques. As detailed in Section 5.3, the simple parallelization of MLFMA is based on the distribution of boxes among processors at all levels. A major disadvantage of this

technique is the difficulty in distributing small numbers of boxes at higher levels of the tree structure. When the number of processors is large, those boxes must be duplicated over multiple processors. Otherwise, a large amount of data is required to be communicated during aggregation and disaggregation stages. In addition, when using the simple parallelization technique, translations involve dense communications among processors.

The hybrid parallelization technique was successfully developed to improve the parallelization of MLFMA [134]. As detailed in Section 5.4, lower (distributed) levels of MLFMA are partitioned as in the simple technique, i.e., boxes are distributed among processors. At higher (shared) levels, however, samples of fields are distributed, instead of boxes. Unlike the hierarchical parallelization, samples at a shared level are distributed among all processors, without any partitioning of boxes. Distributing samples provides improved load-balancing and communication-free translations for higher levels of the tree structure. On the other hand, problems arise for some levels at the middle of the tree, where it is not efficient to distribute either fields or boxes among processors.

The hierarchical parallelization technique provides two important advantages, compared to simple and hybrid techniques.

- Improved Load-Balancing: Partitioning both boxes and samples of fields leads to an improved load-balancing of the workload among processors at each level. Consequently, duplication of computations, which may occur in the simple parallelization, and waits for the synchronization of processors are minimized.
- Reduced Communications: Although the hierarchical partitioning increases types of communications, compared to simple and hybrid approaches, the amount of data transferred is decreased. In addition, due to the improved

load-balancing, the average package size is enlarged, and the number of communication events is reduced. As a result, the communication time is significantly shortened.

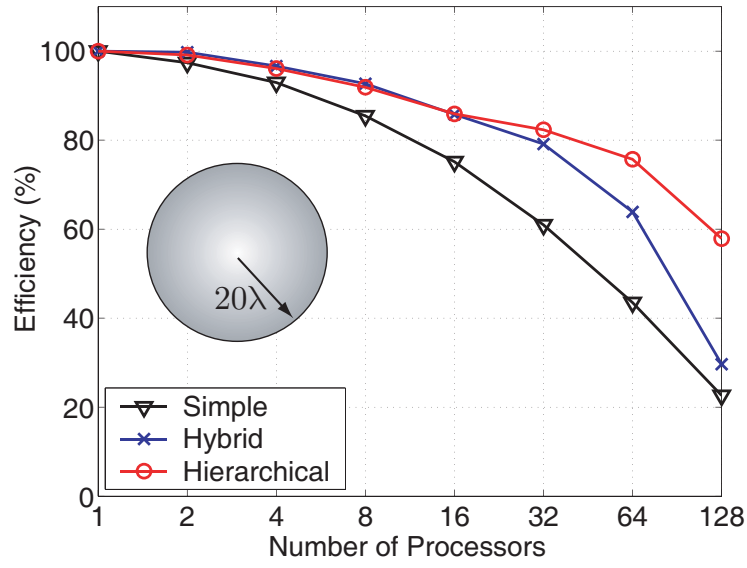
To demonstrate reduced communications with the hierarchical technique, we consider the solution of a scattering problem involving a sphere of radius 20λ discretized with 1,462,854 unknowns. The problem is solved by a 7-level MLFMA parallelized with hybrid and hierarchical techniques on 64 processors. The hybrid technique involves three shared and 4 distributed levels. The number of box partitions in the hierarchical technique is 64, 64, 32, 16, 8, 4, and 2 from the lowest level to the highest level. We note that field samples are not distributed for levels $l = 1$ and $l = 2$. Table 5.2 lists the number of communications and the amount of data communicated during a MVM. We observe that the total amount of communications is reduced by 31% using the hierarchical partitioning strategy compared to the hybrid strategy. In addition, the average size of packages is increased by 51%, and the number of communication events is reduced by 54%. These results are also confirmed with other experiments involving different numbers of processors and objects with diverse geometries.

Table 5.2: Communications During a MVM for the Solution of a Scattering Problem Involving a Sphere Discretized with 1,462,854 Unknowns on 64 Processors

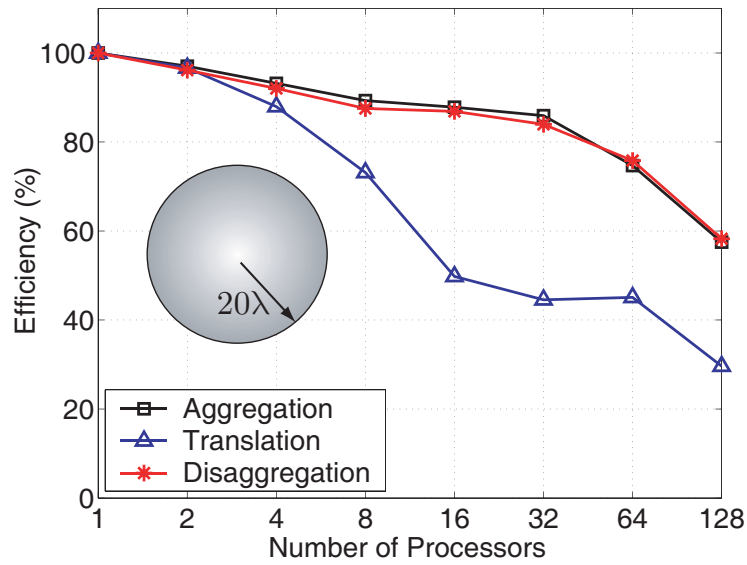
	Hybrid		Hierarchical	
	Number of Events	Amount (Bytes)	Number of Events	Amount (Bytes)
Interpolation at $l = 6$	0	0	1170	182,520
Interpolation at $l = 5$	0	0	590	148,680
Interpolation at $l = 4$	6960	1,030,080	292	108,040
Interpolation at $l = 3$	1360	646,272	136	107,712
Data Exchange at $l = 7$	0	0	2	213,304
Data Exchange at $l = 6$	0	0	2	122,640
Data Exchange at $l = 5$	0	0	2	106,560
Data Exchange at $l = 4$	0	0	2	73,920
Data Exchange at $l = 3$	0	0	2	55,552
Change Partitioning	1160	171,680	0	0
Aggregation Total	9480	1,848,032	2198	1,118,928
Aggregation Ratio	100%	100%	23%	61%
Disaggregation Total	9480	1,848,032	2198	1,118,928
Disaggregation Ratio	100%	100%	23%	61%
Translation at $l = 7$	0	0	56	111,104
Translation at $l = 6$	0	0	148	156,288
Translation at $l = 5$	0	0	235	139,120
Translation at $l = 4$	425	749,700	427	179,340
Translation at $l = 3$	910	615,160	1005	313,560
Translation at $l = 2$	1812	587,088	1861	602,964
Translation at $l = 1$	3228	464,832	3483	501,552
Translation Total	6375	2,416,780	7215	2,003,928
Translation Ratio	100%	100%	113%	83%
MVM	25,335	6,112,844	11,611	4,241,784
MVM Ratio	100%	100%	46%	69%
Average Package Size	241 Bytes (100%)		365 Bytes (151%)	

Finally, another important advantage of the hierarchical parallelization technique appears when MLFMA is employed on a cluster with multiprocessor nodes. Most of the mainboards built recently have multiple processors connected via high-speed buses. Then, parallel computers are constructed by clustering a number of multiprocessor computing units (nodes), instead of processors. Resulting parallel computers are highly nonuniform, since communications among processors in the same node are significantly faster than those among processors located in different nodes. Using multicore processors further complicates the situation, since communications within nodes also have diverse rates, depending on whether inter-core communications are taking place in the same processor or between two processors in the same node. The hierarchical parallelization technique is suitable for such parallel platforms.

As an example, let the tree structure in Figure 5.9 be partitioned among two nodes, each having four processors, i.e., processors 1–4 and processors 5–8 are located in two different nodes. Then, all communications during aggregation and disaggregation stages from level 1 to level 3 are performed inside nodes. Inter-node communications are required only for translations and for data exchanges during aggregation and disaggregation stages between level 3 and level 4. In general, the hierarchical partitioning strategy facilitates processor arrangements in nonuniform platforms to minimize inter-node communications. However, in this study, we do not use this advantage directly; hence, the improved efficiency obtained with the hierarchical parallelization is general and valid for all types of distributed-memory architectures.

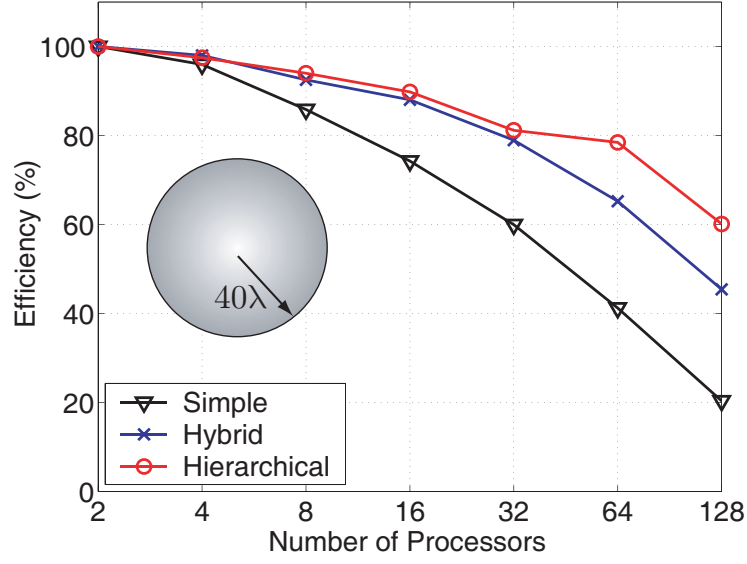


(a)

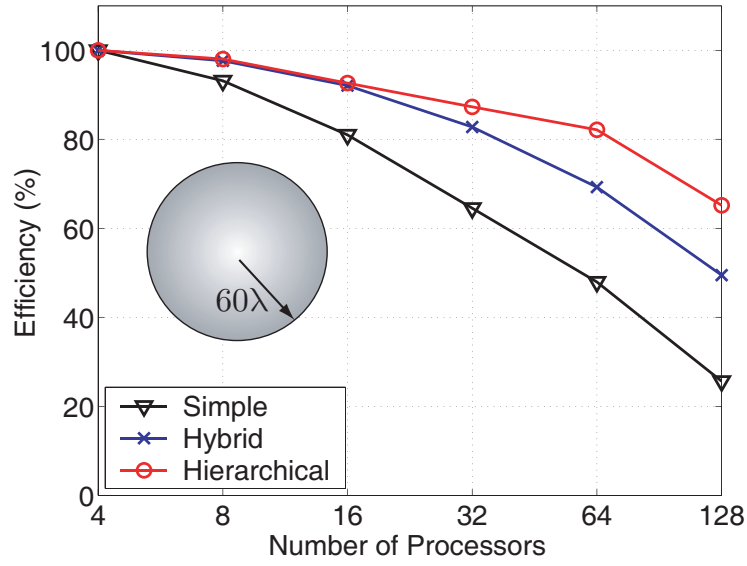


(b)

Figure 5.12: Parallelization efficiency with respect to the number of processors for the solution of a scattering problem involving a sphere of radius 20λ discretized with 1,462,854 unknowns. (a) Overall efficiency including setup and iterations, when the solution is parallelized by using simple, hybrid, and hierarchical techniques. (b) Efficiencies for MLFMA stages, i.e., aggregation, translation, and disaggregation, using the hierarchical technique.



(a)



(b)

Figure 5.13: Parallelization efficiency for the solution of scattering problems involving (a) a sphere of radius 40λ discretized with 5,851,416 unknowns and (b) a sphere of radius 60λ discretized with 13,278,096 unknowns. Parallel efficiency is defined with respect to two and four processors, respectively.

5.7 Numerical Results for the Hierarchical Parallelization Technique

We demonstrate the improved efficiency provided by the hierarchical parallelization technique, compared to the previous parallelization approaches, on scattering problems involving spheres of various sizes discretized with millions of unknowns. Figure 5.12 presents the parallelization efficiency obtained for the solution of a scattering problem involving a sphere of radius 20λ discretized with 1,462,854 unknowns. Figure 5.12(a) depicts the efficiency for the total time (including the setup and iterations), when the solution is parallelized on 2, 4, 8, 16, 32, 64, and 128 processors of the Tigerton cluster. We observe that the hierarchical technique improves the parallelization efficiency significantly, compared to simple and hybrid approaches, especially when the number of processors is large. The hybrid parallelization performs better than the simple parallelization; however, its efficiency drops rapidly for $p > 32$, and it becomes inefficient, compared to the hierarchical parallelization. Using 128 processors, the hierarchical parallelization technique provides 58% efficiency, corresponding to a 74-fold speedup with respect to the single-processor solution.

Figure 5.12(b) presents the parallelization efficiency for the three main stages of MVMs, i.e., aggregation, translation, and disaggregation, using the hierarchical parallelization technique. We observe that the translation stage becomes a major bottleneck when the number of processors increases. For a solution on 128 processors, the parallelization efficiency of translations drops below 30%. This is because the communication volume for translations, given in (5.14), does not scale with the number of processors p , unlike the communication volume for aggregation and disaggregation stages. In addition, many communications required for inter-processor translations occur among processors located in different nodes. Then, the rate of communications during the translation stage is

mostly limited by the speed of the Infiniband network. Nevertheless, even the parallelization of translations is improved with the hierarchical technique, and the overall efficiency provided by the hierarchical technique is consistently higher than those obtained with simple and hybrid approaches.

Figure 5.13 presents the parallelization efficiency for the solution of scattering problems involving spheres of radii 40λ and 60λ discretized with 5,851,416 and 13,278,096 unknowns, respectively, where the efficiency is defined with respect to solutions with two and four processors. Similar to the previous results, the parallelization efficiency is increased significantly by using the hierarchical parallelization technique.

Even though Figures 5.12 and 5.13 compare relative performances of different parallelization techniques, we emphasize that they do not provide the complete information for the efficiency of solutions. In general, one should also consider the following factors.

- Clock Rate of the Processors: Although using faster processors leads to faster solutions, the parallelization efficiency can be degraded as the computation time is reduced. This is because the communication time becomes more significant when the processing time for computations is small.
- Efficient Implementation of the Algorithm: We carefully implement MLFMA by minimizing the processing time, which may also have an adverse effect on the parallelization efficiency. For example, as opposed to common implementations of MLFMA, we calculate and store radiation and receiving patterns of basis and testing functions during the setup of the program, and we use them efficiently during iterations. Calculating patterns on the fly in each MVM without storing them is also a common

Table 5.3: Solutions of Sphere Problems Using the Hierarchical Parallelization Technique

Sphere Radius	20λ	40λ	60λ
Unknowns	1,462,854	5,851,416	13,278,096
Number of Iterations	27	30	43
Setup Time (s)			
Simple	184	1065	2630
Hybrid	89	356	812
Hierarchical	88	348	797
Solution Time (s)			
Simple	512	2467	5400
Hybrid	442	1162	3122
Hierarchical	184	804	2257

practice for low-memory implementations. That would increase the processing time, but the parallelization efficiency would also increase, since those calculations can be parallelized very efficiently.

- Accuracy Parameters: The accuracy of solutions also affects the parallelization efficiency. For example, most of communications during aggregation and disaggregation stages could be avoided by reducing the number of interpolation points. This would increase the parallelization efficiency, but the accuracy of solutions would deteriorate.

We note that parallel-efficiency results presented in Figures 5.12 and 5.13 are obtained under strict circumstances, using an *efficient* and *accurate* implementation of MLFMA on a cluster of processors with a relatively *high clock rate*. To quantify the efficiency of solutions, Table 5.3 lists processing times, when the three problems are solved on 128 processors. Using the hierarchical parallelization technique, the largest problem with 13,278,096 unknowns is solved in less than one hour.

Table 5.4: Major Versions of Parallel MLFMA

Version	Details
<i>v-54</i>	Hybrid parallelization
<i>v-74</i>	Hierarchical parallelization
<i>v-85</i>	Hierarchical parallelization, memory revised
<i>v-96</i>	Hierarchical parallelization, memory revised
<i>v-101</i>	Hierarchical parallelization, memory revised

5.8 Solutions of Large-Scale Electromagnetics Problems

In this section, we present the solution of increasingly large electromagnetics problems via parallel MLFMA. By developing sophisticated implementations of the hierarchical partitioning strategy, we are able to solve problems involving hundreds of millions of unknowns. All solutions presented in this section could be relaxed by sacrificing the accuracy, which would enable the solution of even larger problems. In addition, the problem size could be enlarged at the cost of increasing processing time by performing some of computations on the fly and reducing memory requirements. However, we do not follow those tricks since the major purpose of this study is solving very large electromagnetics problems both *fast* and *accurately*.

Table 5.4 lists major versions of parallel MLFMA implemented during a 1.5-year period from March 2007 to September 2008. Version 54 involves the hybrid parallelization technique optimized with load-balancing algorithms. The hierarchical technique is implemented and optimized for the first time in version 74. By increasing the efficiency in terms of processing time and memory, the hierarchical technique provides faster solutions of larger problems, compared to the hybrid technique. However, as the number of processors increases, memory can easily become a bottleneck that prevents the solution of larger problems. From

Table 5.5: Solutions of Large-Scale Electromagnetics Problems with MLFMA Parallelized on the Clovertown Cluster

Problem	Sphere	Sphere	Sphere	Flamme	Flamme
Size	160λ	192λ	220λ	240λ	320λ
Unknowns	23,405,664	33,791,232	41,883,648	14,326,512	24,782,400
Solver	<i>v</i> -54	<i>v</i> -54	<i>v</i> -54	<i>v</i> -54	<i>v</i> -54
Levels	9	9	9	9	10
Box Size	0.156λ	0.188λ	0.215λ	0.234λ	0.156λ
Processors	16	16	16	16	16
Excitation	H (0°)	H (0°)	H (0°)	H (0°)	H (0°)
Iterations	17	21	19	31	36
Processing Time (minutes)					
Setup	94	183	274	160	104
Solution	155	264	290	99	241
Total	252	451	570	261	349

version 74 to 101, we systematically revise the implementation by identifying and eliminating the major obstacles in terms of memory. At the end, version 101 is developed and used to solve problems involving more than 200 million unknowns. These are the largest integral-equation problems ever solved in the literature.

5.8.1 Solutions on the Clovertown Cluster

In Table 5.5, we present the solution of large-scale electromagnetics problems on 16 processors of the Clovertown cluster. First three problems involve spheres of radii 80λ , 96λ , and 110λ , which are discretized with 23,405,664, 33,791,232, and 41,883,648 unknowns, respectively. We also consider two scattering problems involving the stealth airborne target Flamme at 120 GHz and 160 GHz. The size of the Flamme corresponds to 240λ and 320λ , and discretizations with $\lambda/10$ triangles lead to matrix equations with 14,326,512 and 24,782,400 unknowns, respectively. All solutions are parallelized by using the hybrid technique. We observe that each problem is solved in less than 10 hours.

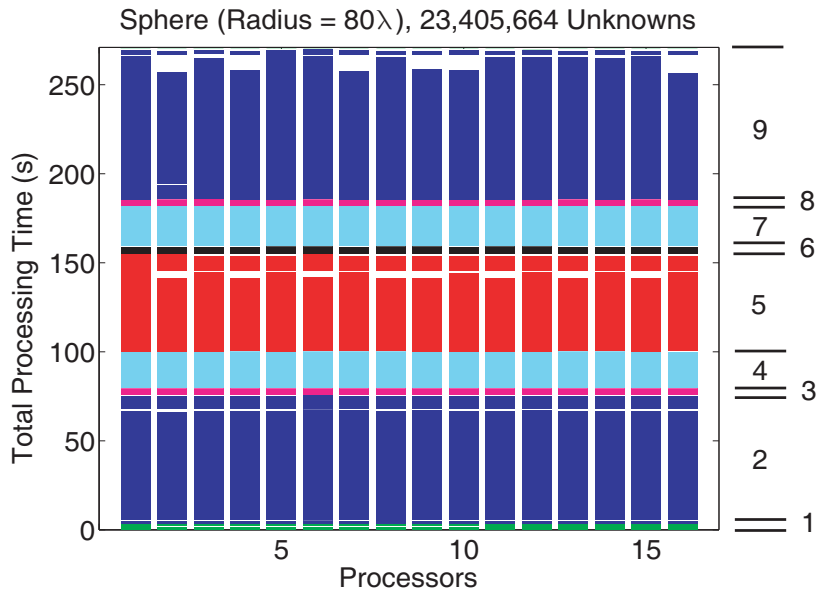
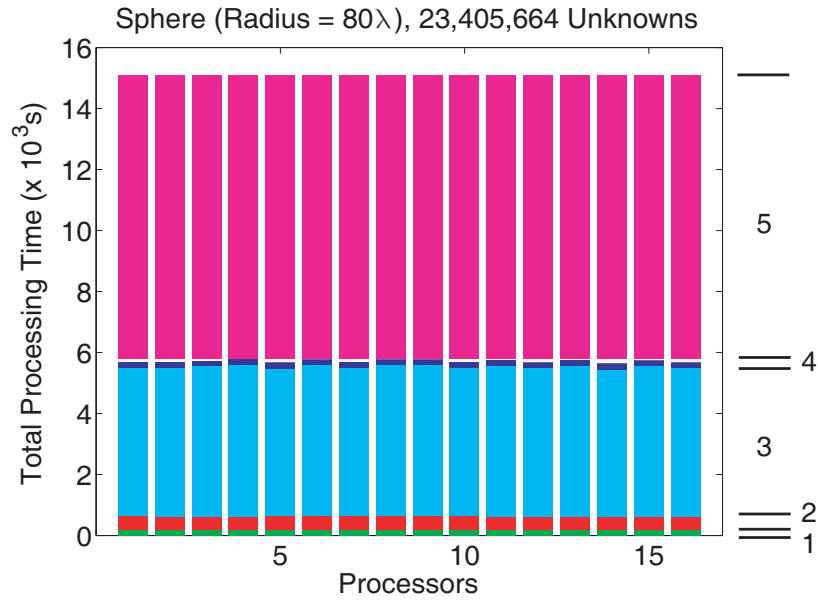


Figure 5.14: Time diagrams for the solution of a scattering problem involving a sphere of radius 80λ discretized with 23,405,664 unknowns using the hybrid parallelization technique on 16 processors of the Clovertown cluster. (a) Overall time and (b) single MVM. In diagrams, white areas correspond to waits before operations that require synchronization.

In Figure 5.14, we further present details of the solution of the 23-million-unknown problem involving a sphere with a radius of 80λ . In Figure 5.14(a), the total processing time is depicted for all processors from 1 to 16. After the input and the recursive clustering part⁽¹⁾, computations of translation operators⁽²⁾ and radiation/receiving patterns⁽⁴⁾ require negligible time. Calculation of near-field interactions⁽³⁾ dominates the setup time, which is about 94 minutes. Then the solution part⁽⁵⁾, involving a total of 34 MVMs, is performed in about 155 minutes. The processing time for a MVM is depicted in 5.14(b), including the near-field stage⁽¹⁾, aggregation/disaggregation in distributed levels^(2,9), all-to-all communications^(3,8), aggregation/disaggregation in shared levels^(4,7), communication-free (intra-processor) translations⁽⁵⁾, and inter-processor translations⁽⁶⁾. The most problematic parts in terms of parallelization efficiency, i.e., all-to-all communications and inter-processor translations, require negligible time compared to other parts of the MVM. This is commonly observed with large-sized problems and supports the conclusion that the parallelization efficiency for a fixed number of processors usually increases as the problem size grows.

To present the accuracy of solutions, Figures 5.15, 5.16, and 5.17 depict the normalized bistatic RCS ($\text{RCS}/\pi a^2$, where a is the radius of the sphere) values for spheres of radii 80λ , 96λ , and 110λ , respectively. Analytical values obtained by Mie-series solutions are also plotted from 172° to 180° , where 180° corresponds to the forward-scattering direction. Figures 5.15–5.17 show that computational values sampled at 0.1° are in agreement with analytical values.

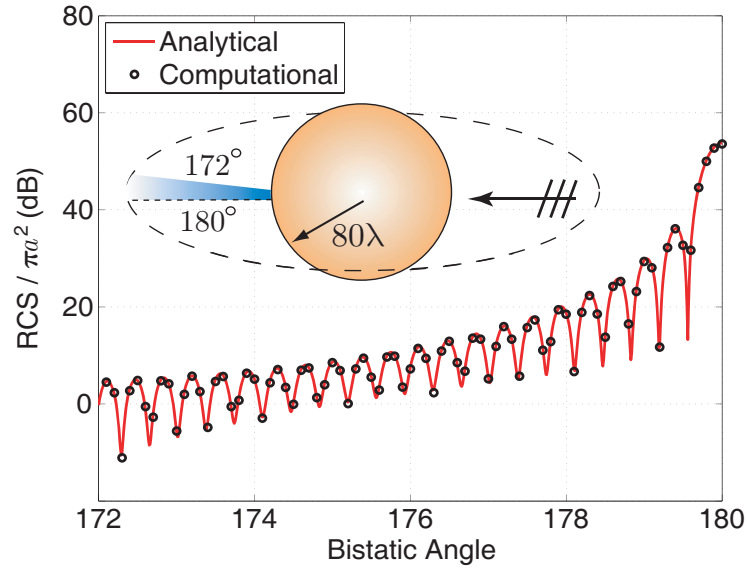


Figure 5.15: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 80λ discretized with 23,405,664 unknowns from 172° to 180° , where 180° corresponds to the forward-scattering direction.

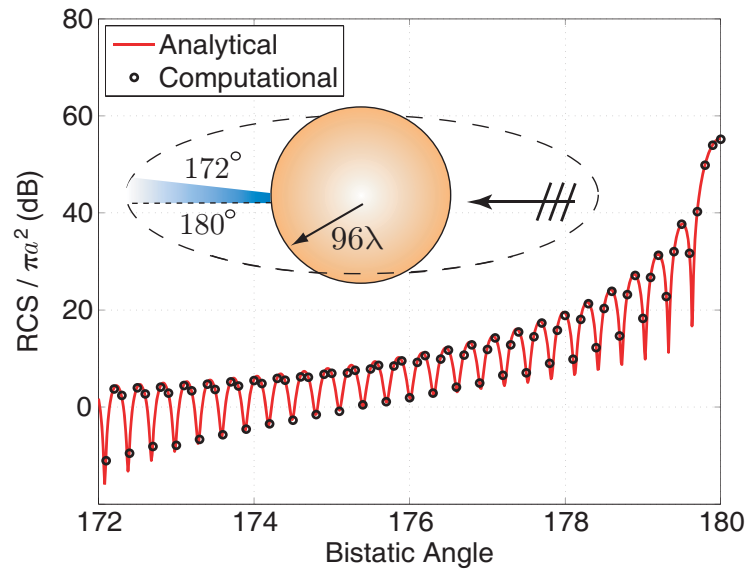


Figure 5.16: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 96λ discretized with 33,791,232 unknowns from 172° to 180° , where 180° corresponds to the forward-scattering direction.

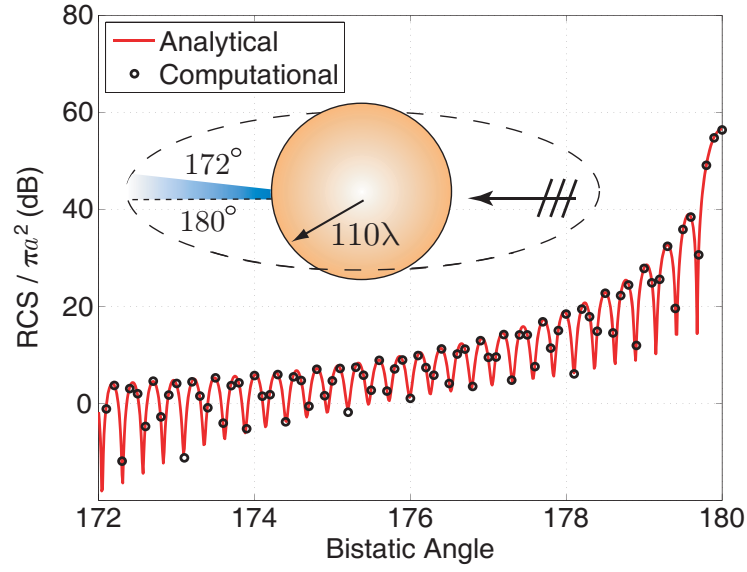
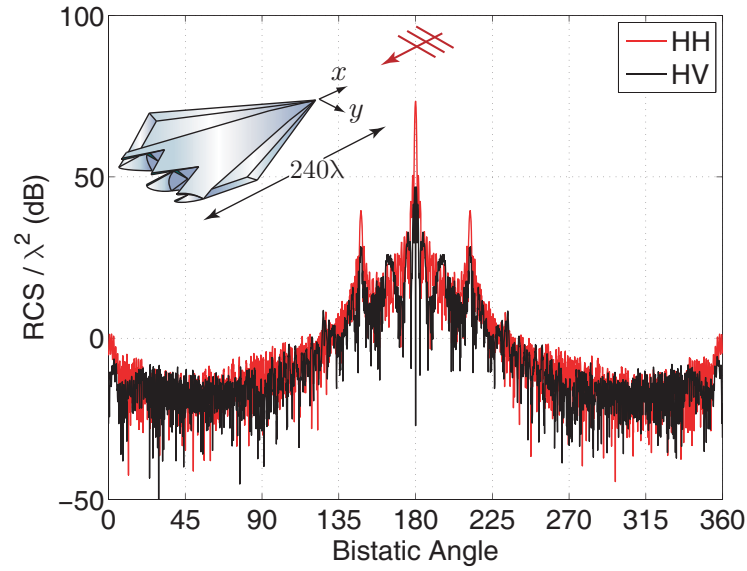
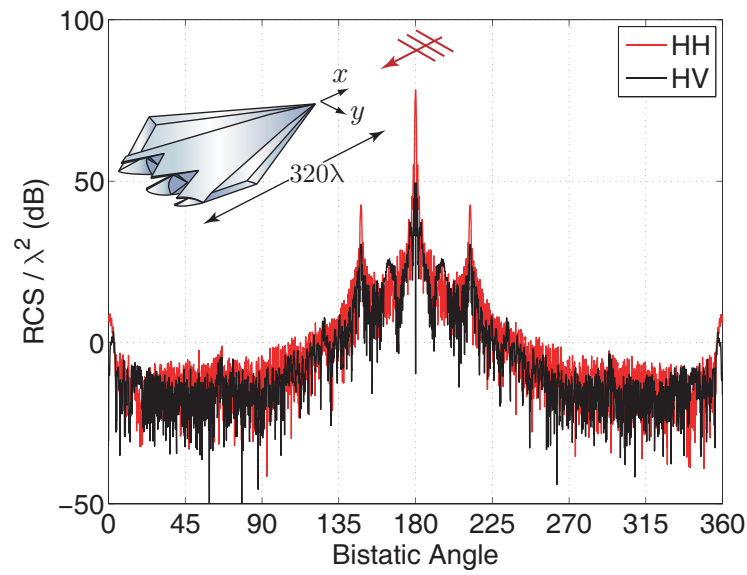


Figure 5.17: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 110λ discretized with 41,883,648 unknowns from 172° to 180° , where 180° corresponds to the forward-scattering direction.

Finally, Figure 5.18 presents the normalized bistatic RCS (RCS/λ^2) of the stealth airborne target Flamme at 120 GHz and 160 GHz, when the target is illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction (H polarization). RCS values are plotted on the x - y plane as a function of the bistatic angle ϕ , where 0° and 180° correspond to back-scattering and forward-scattering directions, respectively. In both cases, we observe that the back-scattered RCS of the Flamme is very low and it is 70 dB less than the forward-scattered RCS.



(a)



(b)

Figure 5.18: Normalized bistatic RCS (RCS/λ^2) of the stealth airborne target Flamme at (a) 120 GHz and (b) 160 GHz. The target is illuminated by a plane wave propagating in the $-x$ direction with the electric field in polarized in the y direction.

Table 5.6: Solutions of Large-Scale Electromagnetics Problems with MLFMA Parallelized on the Tigerton Cluster

Problem	Sphere	Sphere	Box	Flamme	Almond
Size	240λ	300λ	200λ	480λ	539λ
Unknowns	53,112,384	85,148,160	75,276,288	57,306,048	61,367,808
Solver	v -74	v -74	v -74	v -74	v -74
Levels	9	10	9	10	10
Box Size	0.235λ	0.147λ	0.196λ	0.235λ	0.263λ
Processors	16	16	16	16	16
Excitation	H (0°)	H (0°)	Dipole	H&V (30°)	H&V (30°)
Iterations	21	23	11	43&34	18&16
Processing Time (minutes)					
Setup	361	294	188	536	462
Solution	176	398	89	481	282
Total	547	710	291	1028	755

5.8.2 Solutions on the Tigerton Cluster

Table 5.6 presents the solution of large-scale electromagnetics problems on the Tigerton cluster. In general, it is possible to solve larger problems on the Tigerton cluster with 512 GB memory, compared to the Clovertown cluster with 256 GB memory. In addition, the hierarchical technique is used to increase the parallelization efficiency. As shown in Table 5.6, a scattering problem involving a sphere of radius 150λ discretized with 85,148,160 unknowns is solved in less than 12 hours. Figures 5.19 and 5.20 present the normalized bistatic RCS ($\text{RCS}/\pi a^2$) values for spheres of radii 120λ and 150λ , respectively. We observe that computational values agree well with analytical values obtained by Mie-series solutions.

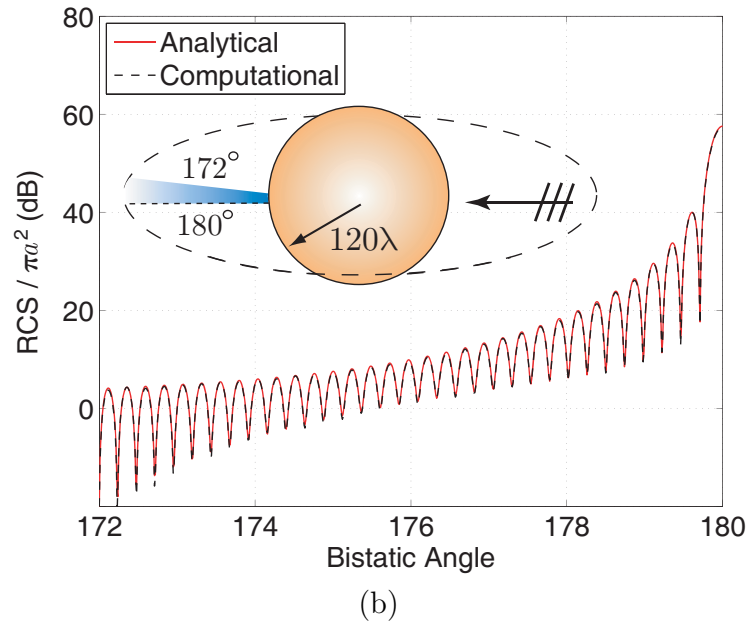
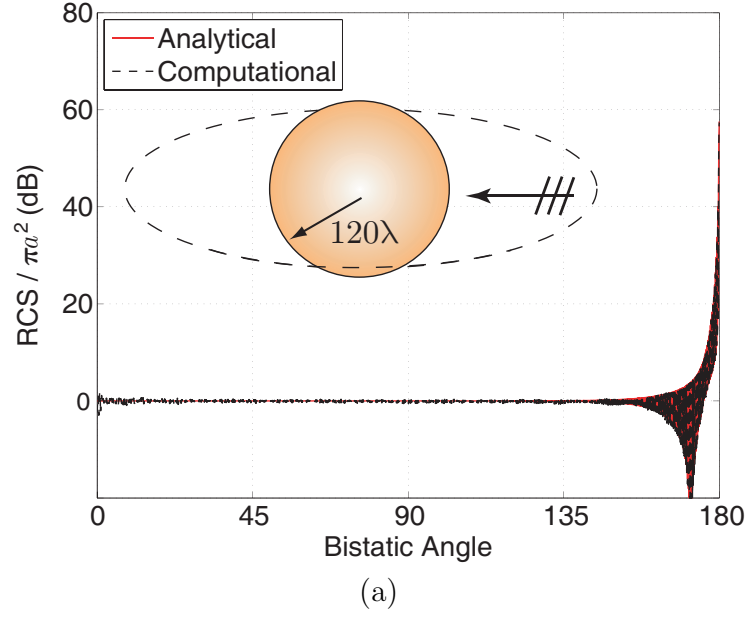


Figure 5.19: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 120λ discretized with 53,112,384 unknowns (a) from 0° to 180° and (b) from 172° to 180° , where 180° corresponds to the forward-scattering direction.

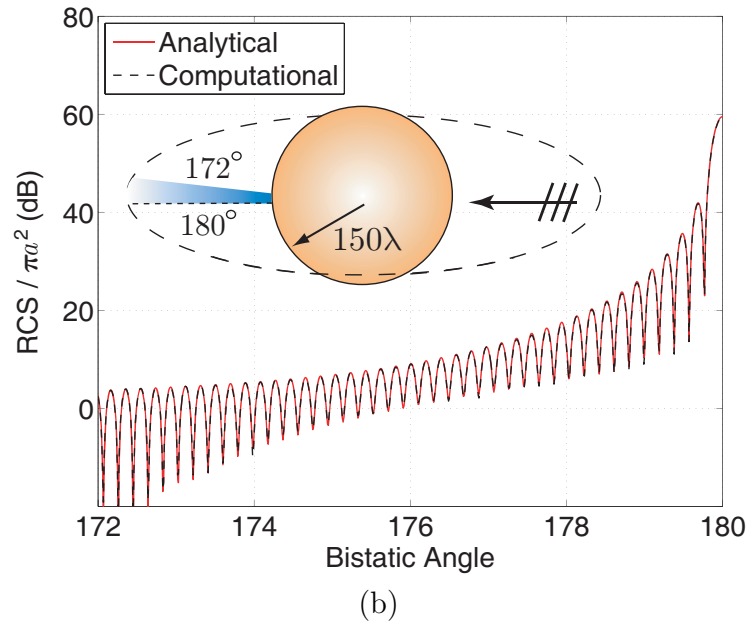
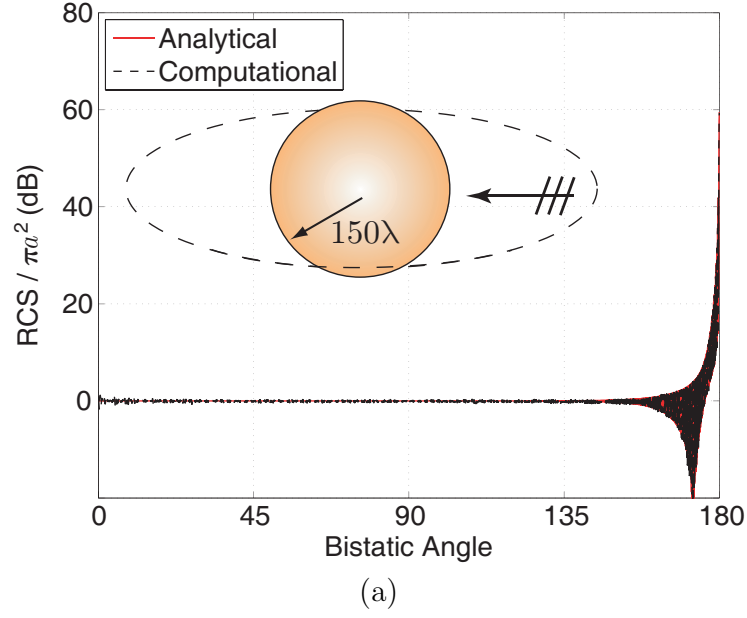
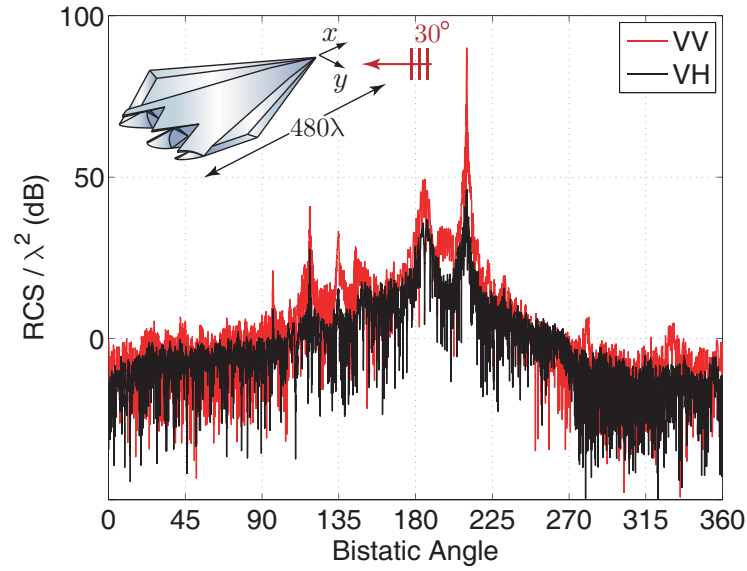
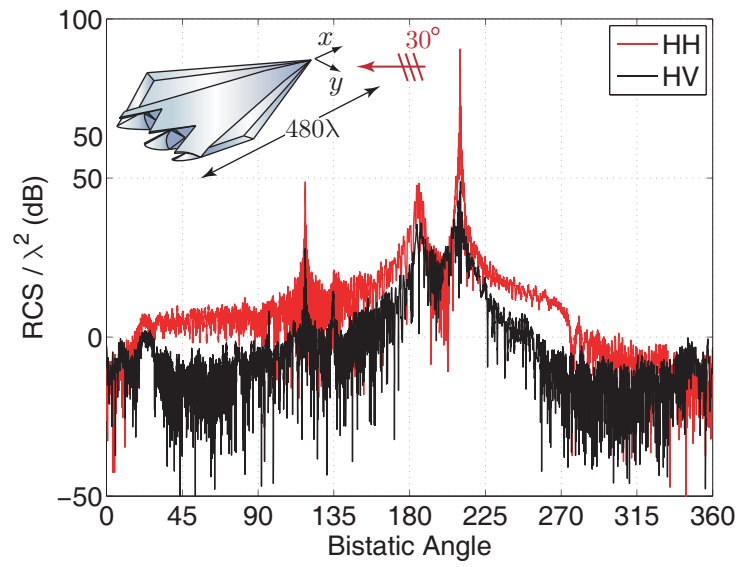


Figure 5.20: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 150λ discretized with 85,148,160 unknowns (a) from 0° to 180° and (b) from 172° to 180° , where 180° corresponds to the forward-scattering direction.

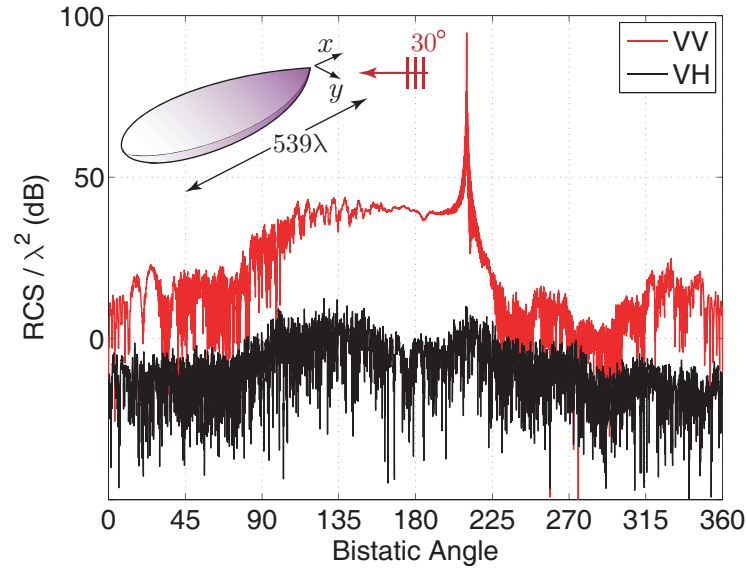


(a)

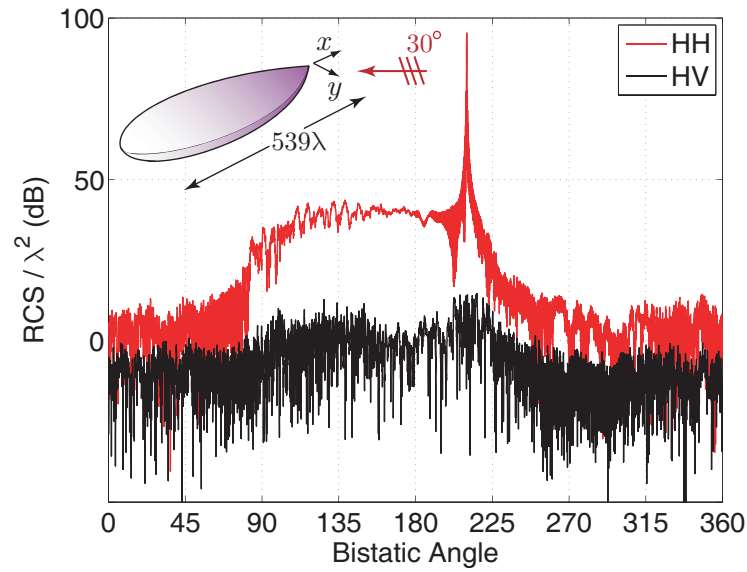


(b)

Figure 5.21: Normalized bistatic RCS (RCS/λ^2) of the stealth airborne target Flamme at 240 GHz. The target is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field in polarized in (a) z direction and (b) ϕ direction.



(a)



(b)

Figure 5.22: Normalized bistatic RCS (RCS/λ^2) of the NASA Almond at 640 GHz. The target is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field in polarized in (a) z direction and (b) ϕ direction.

In addition to canonical problems, we consider the solution of scattering problems involving the Flamme at 240 GHz and the NASA Almond at 640 GHz, discretized with 57,306,048 and 61,367,808 unknowns, respectively. The size of the Flamme corresponds to 480λ at 240 GHz. The size of the NASA Almond is 25.27 cm, corresponding to 539λ at 640 GHz. Both targets are illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis (from $\phi = 30^\circ$). Both vertical and horizontal polarizations are considered, i.e., each problem is solved twice after the setup of MLFMA. Figures 5.21 and 5.22 present the normalized bistatic RCS (RCS/λ^2) of the Flamme and the NASA Almond, respectively, on the x - y plane as a function of the bistatic angle ϕ . We note that the back-scattered RCS (at 30°) is low for both targets, but their scattering characteristics are significantly different, especially from 90° to the forward-scattering direction (210°). In this range, RCS of the Flamme makes peaks at several angles, but RCS of the NASA Almond is generally higher than RCS of the Flamme.

Finally, we consider the solution of a radiation problem involving a $200\lambda \times 200\lambda \times 200\lambda$ conducting box discretized with 75,276,288 unknowns. A Hertzian dipole oriented in the z direction is located on the top of the box. Figure 5.23 presents the z component of the normalized electric field in the far zone on the x - y plane. In addition to the incident field radiated by the Hertzian dipole, we consider the secondary field due to the induced electric current on the surface of the cube. We observe that the secondary field is 30–40 dB less than the incident field, and the incident field is very close to the total field. In fact, the secondary field can be interpreted as the noise created by the box to the radiation of the Hertzian dipole. Since the box is very large, it behaves like an infinite ground plane and it does not affect the radiation of the dipole significantly.

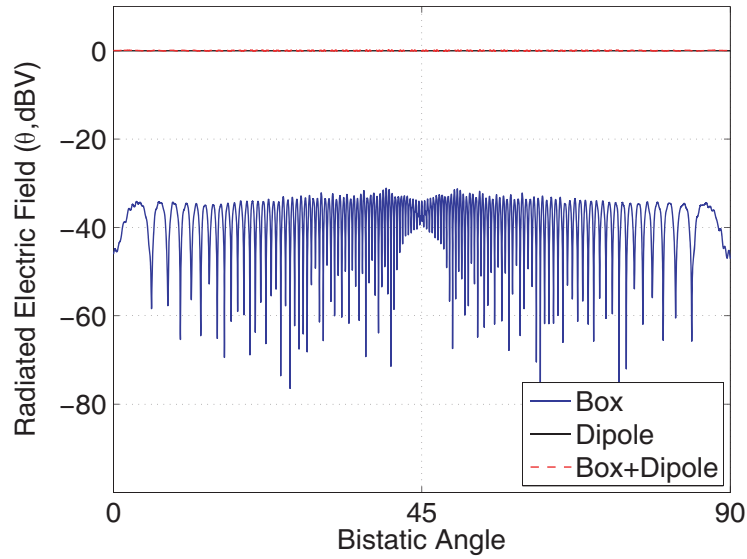


Figure 5.23: Radiated electric field (dBV) in the far zone on the x - y plane due to a Hertzian dipole located on a $200\lambda \times 200\lambda \times 200\lambda$ conducting box.

Table 5.7: Solutions of a Scattering Problem Involving a Sphere of Radius 110λ with Parallel MLFMA (v -74) on 16 Processors of the Tigerton and the Harpertown Clusters

	Swindon Tigerton	BiLCEM Harpertown	Ratio
Setup (minutes)	312	233	0.75
Solution (minutes)	175	169	0.97
Total (minutes)	496	410	0.83

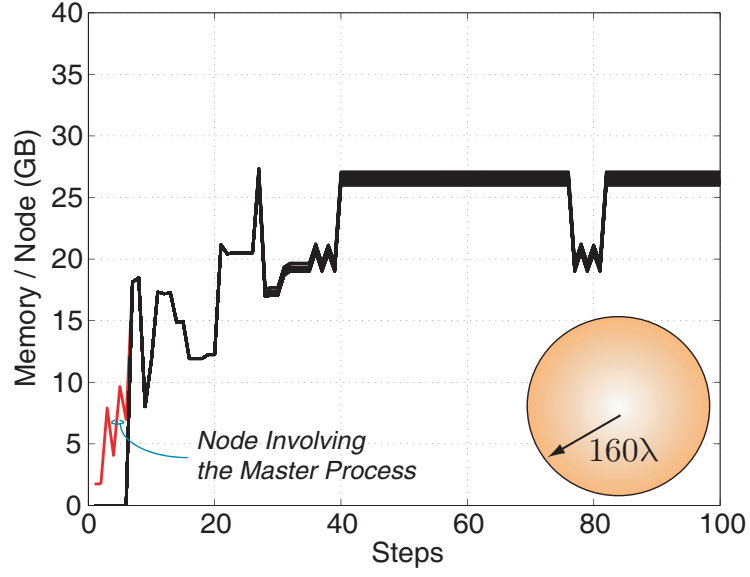
5.8.3 Solutions on the Harpertown Cluster

In Table 5.7, we consider the solution of a scattering problem involving a sphere of radius 110λ discretized with 41,883,648 unknowns and we compare the Tigerton and the Harpertown clusters in terms of processing time. MLFMA with three digits of accuracy is parallelized on 16 processors using the hierarchical technique. We observe that the Harpertown cluster is significantly faster than the Tigerton cluster for the setup part, even though the clock rate increases only from 2.93 GHz to 3.00 GHz. For the solution part, however, performances of the two clusters become close to each other.

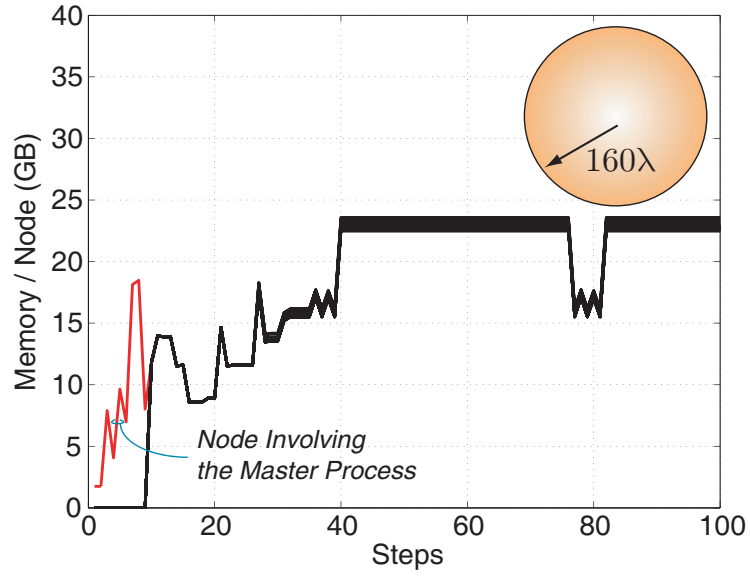
Table 5.8: Solutions of Large-Scale Electromagnetics Problems with MLFMA Parallelized on the Harpertown Cluster

Problem	Sphere	Sphere	Sphere	Sphere	Flamme
Size	320λ	320λ	320λ	320λ	640λ
Unknowns	93,622,656	93,622,656	93,622,656	93,622,656	99,129,600
Solver	<i>v</i> -74	<i>v</i> -85	<i>v</i> -85	<i>v</i> -96	<i>v</i> -96
Levels	10	10	10	10	11
Box Size	0.156λ	0.156λ	0.156λ	0.156λ	0.156λ
Processors	16	16	32	64	32
Excitation	H (0°)	H (0°)	H (0°)	H (0°)	V (30°)
Iterations	24	24	24	24	50
Processing Time (minutes)					
Setup	263	261	132	72	166
Solution	454	450	243	144	368
Total	751	712	377	217	534

Table 5.8 presents the solution of large-scale electromagnetics problems on the Harpertown cluster. A scattering problem involving a sphere of radius 160λ discretized with 93,622,656 unknowns is solved by using different implementations of the hierarchical parallelization technique. From *v*-74 to *v*-85 and *v*-85 to *v*-96, we systematically decrease the memory requirement by eliminating sequential (non-parallel) input/output arrays. We note that major arrays containing near-field interactions, radiation and receiving patterns of basis and testing functions, translation operators, and radiated/incoming fields of boxes are parallelized in all implementations. However, *v*-74 involves some input/output arrays that are not parallelized for simplicity. Although those arrays are negligible for relatively small problems, they easily become a bottleneck in terms of memory as the problem size grows.

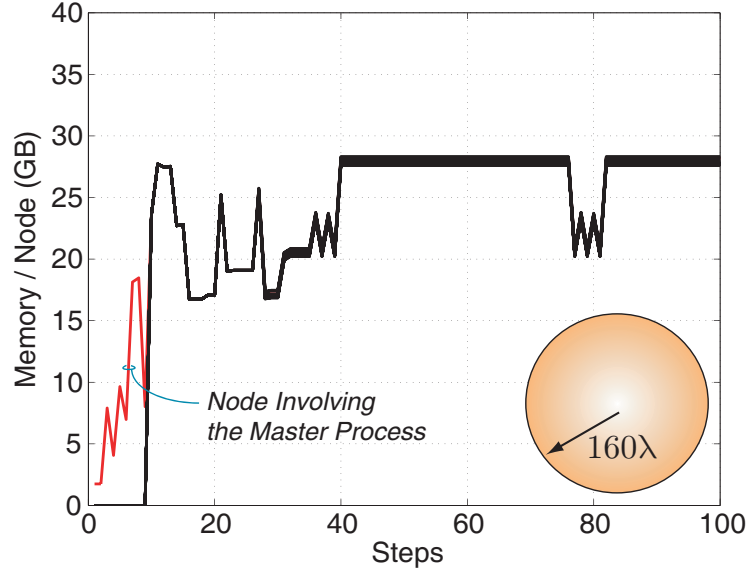


(a)

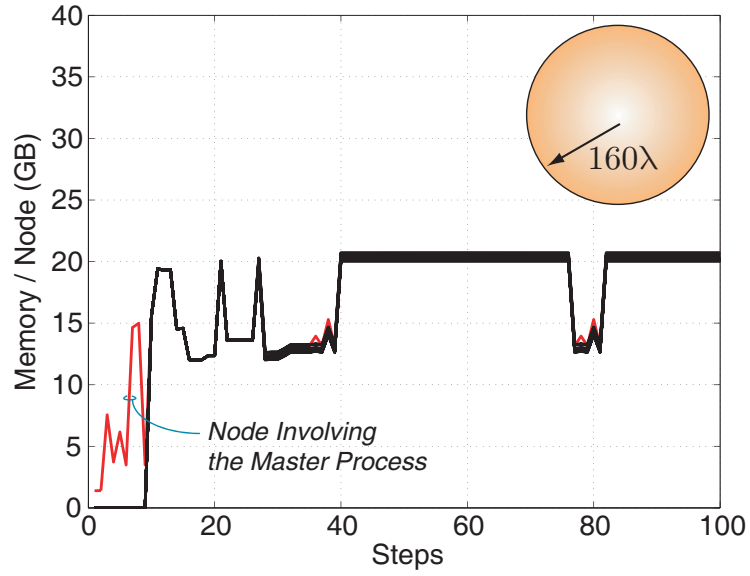


(b)

Figure 5.24: Memory required by each node for the solution of a scattering problem involving a sphere of radius 160λ discretized with 93,622,656 unknowns, when the solution is performed by using (a) v -74 and (b) v -85 implementations parallelized on 16 processors.

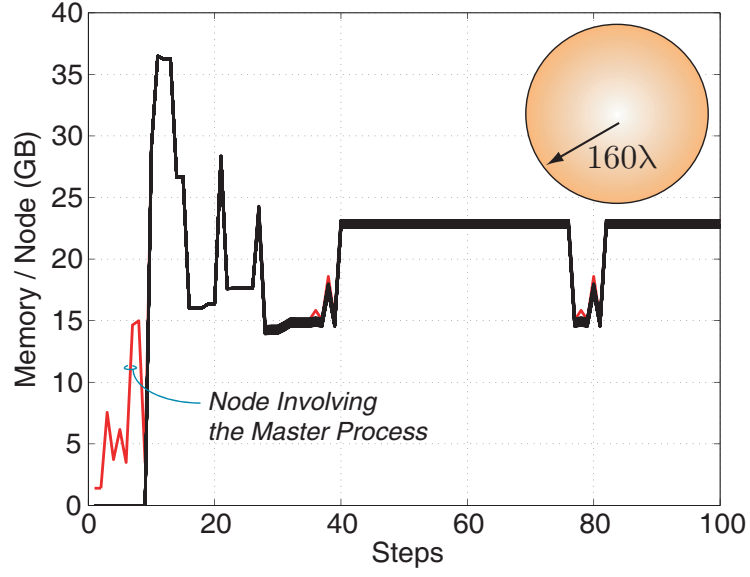


(a)

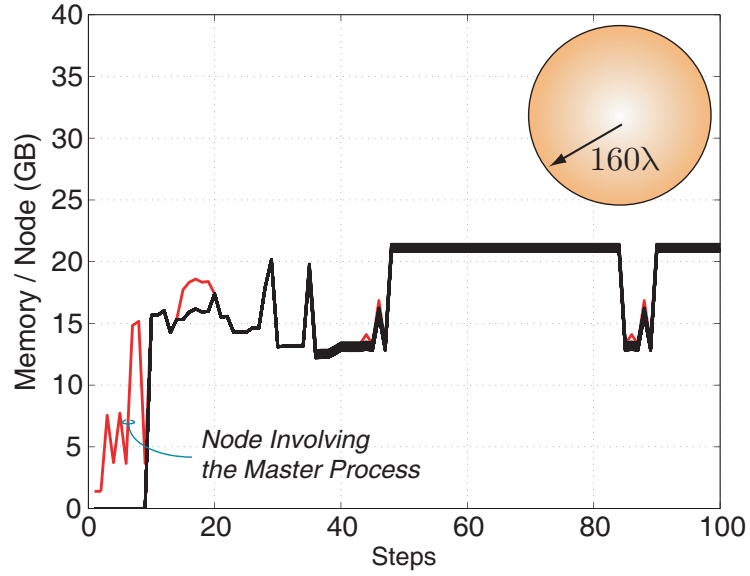


(b)

Figure 5.25: Memory required by each node for the solution of a scattering problem involving a sphere of radius 160λ discretized with 93,622,656 unknowns, when the solution is performed by using (a) v -85 and (b) v -96 implementations parallelized on 32 processors.



(a)



(b)

Figure 5.26: Memory required by each node for the solution of a scattering problem involving a sphere of radius 160λ discretized with 93,622,656 unknowns, when the solution is performed by using (a) v -96 and (b) v -101 implementations parallelized on 64 processors.

Figure 5.24 depicts memory required by *v*-74 and *v*-85 implementations for the solution of a sphere problem discretized with 93,622,656 unknowns on 16 processors. Memory used by each node (hence each processor) is plotted as a function of time steps. We note that the amount of memory does not increase monotonically since we immediately deallocate arrays that are not used in the next stages of the program. Figure 5.24 shows that the memory requirement is reduced significantly by parallelizing some of sequential arrays in *v*-74. In Figure 5.25, we compare memory required by *v*-85 and *v*-96 implementations for the solution of the same problem on 32 processors. We further eliminate bottlenecks in *v*-85 and reduce memory used by each node (two processors) to about 20 GB. Finally, in Figure 5.26, we compare *v*-96 with the final version of the parallel MLFMA, i.e., *v*-101, for the solution of the same problem on 64 processors. We note that it was impossible to solve this problem on 64 processors using *v*-85 and earlier implementations due to the duplication of sequential arrays on multiple processors. Using *v*-101, however, the maximum memory used by each node (four processors) is only about 21 GB, which is much smaller than 32 GB, the maximum allowable memory. As shown in Table 5.8, this problem is solved in less than four hours on 64 processors.

Figure 5.27 depicts the normalized bistatic RCS ($\text{RCS}/\pi a^2$) values for a sphere of radius 160λ . As in the previous examples, we observe a good agreement between computational and analytical solutions. In Figure 5.28, we present the solution of another scattering problem involving the Flamme discretized with 99,129,600 unknowns. The frequency is 320 GHz and the size of the Flamme corresponds to 640λ . As shown in Table 5.8, this problem is solved in less than nine hours on 32 processors for a single illumination.

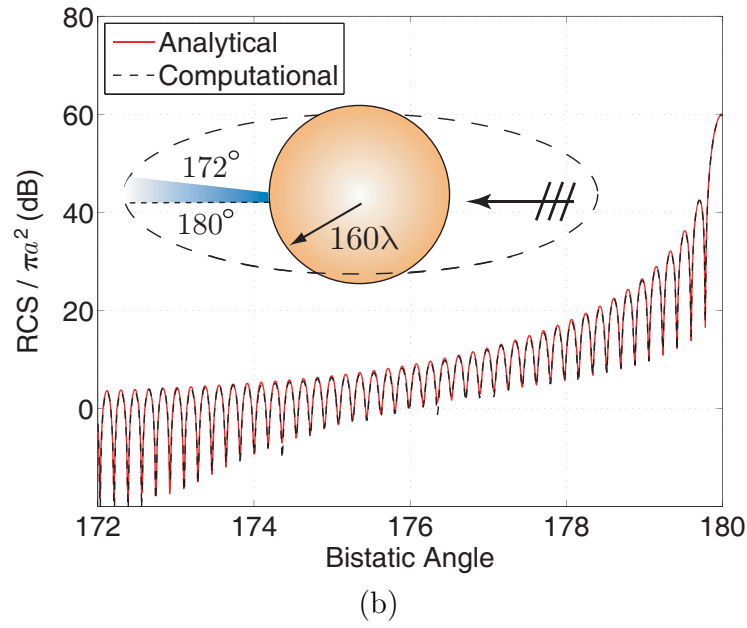
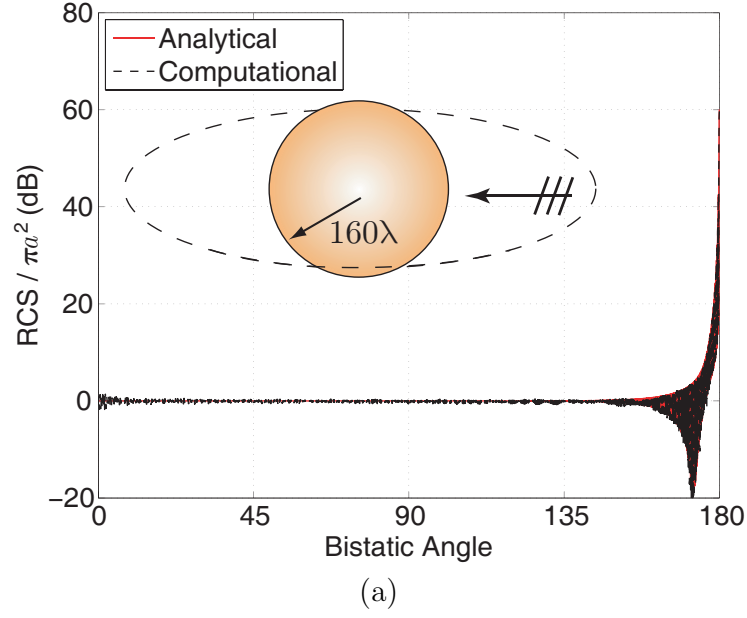


Figure 5.27: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 160λ discretized with 93,622,656 unknowns (a) from 0° to 180° and (b) from 172° to 180° , where 180° corresponds to the forward-scattering direction.

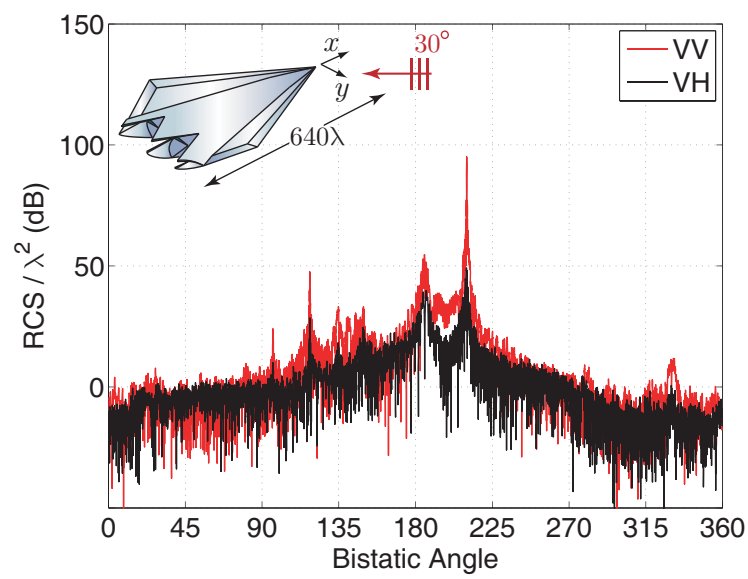


Figure 5.28: Normalized bistatic RCS (RCS/λ^2) of the stealth airborne target Flamme at 320 GHz. The target is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field in polarized in the z direction.

Table 5.9: Solutions of Large-Scale Electromagnetics Problems Involving More Than 100 Million Unknowns with MLFMA Parallelized on the Harpertown Cluster

Problem	Sphere	Flamme	Almond	Wing
Size	360 λ	720 λ	715 λ	400 λ
Unknowns	135,164,928	134,741,760	125,167,104	121,896,960
Solver	<i>v</i> -101	<i>v</i> -101	<i>v</i> -101	<i>v</i> -101
Levels	10	11	11	10
Box Size	0.176 λ	0.176 λ	0.175 λ	0.195 λ
Excitation	H (0°)	V (30°)	V (30°)	V (30°)
Iterations	23	44	20	17
Processing Time for 16 Processors				
Setup (minutes)	480	537	435	366
Solution (minutes)	489	644	330	177
Total (minutes)	975	1186	769	546
MVM (s)	618	425	449	285
Aggregation (s)	200	138	154	98
Translation (s)	194	129	128	69
Disaggregation (s)	209	144	155	106
MVM Rest	15	14	12	12
Processing Time (minutes) for 32 Processors				
Setup (minutes)	243 (99%)	272 (99%)	219 (99%)	186 (98%)
Solution (minutes)	263 (93%)	369 (87%)	171 (97%)	99 (89%)
Total (minutes)	511 (95%)	646 (92%)	394 (98%)	289 (95%)
MVM (s)	327 (95%)	234 (91%)	239 (94%)	155 (92%)
Aggregation (s)	103 (97%)	72 (96%)	77 (100%)	51 (96%)
Translation (s)	105 (92%)	75 (86%)	72 (89%)	40 (86%)
Disaggregation (s)	106 (99%)	76 (95%)	80 (97%)	54 (98%)
MVM Rest (s)	13	11	10	10
Processing Time (minutes) for 64 Processors				
Setup (minutes)	126 (95%)	136 (99%)	109 (100%)	96 (95%)
Solution (minutes)	161 (76%)	203 (79%)	101 (82%)	62 (71%)
Total (minutes)	292 (84%)	345 (86%)	215 (89%)	163 (84%)
MVM (s)	194 (80%)	128 (83%)	137 (82%)	92 (78%)
Aggregation (s)	58 (86%)	37 (93%)	43 (90%)	28 (88%)
Translation (s)	64 (76%)	41 (79%)	39 (82%)	21 (82%)
Disaggregation (s)	60 (87%)	38 (95%)	45 (86%)	30 (88%)
MVM Rest (s)	12	12	10	13

Table 5.10: Peak Memory (GB) Required for the Solutions in Table 5.9

Problem	Sphere	Flamme	Almond	Wing
Size	360λ	720λ	715λ	400λ
Unknowns	135,164,928	134,741,760	125,167,104	121,896,960
16 Processors	424	427	385	431
32 Processors	438 (97%)	455 (94%)	414 (93%)	443 (97%)
64 Processors	467 (91%)	513 (83%)	471 (82%)	500 (86%)

5.8.4 Further Solutions on the Harpertown Cluster

Table 5.9 presents the solution of large-scale problems discretized with more than 100 million unknowns on the Harpertown cluster. A scattering problem involving a sphere of radius 180λ discretized with 135,164,928 unknowns is solved by a 10-level MLFMA. We also consider the Flamme at 360 GHz, the NASA Almond at 850 GHz, and a wing-shaped object having a maximum dimension of 400λ at 120 GHz. The sphere is illuminated by a plane wave propagating in the $-x$ direction with the electric field polarized in the y direction. Other three objects are illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field polarized in the z direction (V polarization). Solutions are performed on 16, 32, and 64 processors of the Harpertown cluster. Table 5.9 lists processing times and the parallelization efficiency obtained for 32 and 64 processors with respect to 16 processors. In addition to setup and solution parts, processing times for aggregation, translation, and disaggregation stages are also listed separately. We note that “MVM Rest” in Table 5.9 accounts for near-field interactions, all-to-all communications to change the partitioning of the output vector before far-field interactions, and input/output operations. Table 5.9 shows that, using 64 processors, parallelization efficiency for the total time is more than 80% for all problems. Due to this relatively high efficiency provided by the hierarchical partitioning strategy, we are able to solve each problem in three to six hours.

Table 5.10 lists the total memory required for the solution of problems in Table 5.9. Increasing the number of processors raises the memory requirement since the efficiency of the parallelization is not perfectly 100%. Nevertheless, using 64 processors, the parallelization efficiency in terms of memory usage is also more than 80% for all problems.

Figure 5.29 presents details of the processing time for 64-processor solutions of scattering problems in Table 5.9. We observe that the calculation of near-field interactions and the iterative solution are dominant parts. The processing time for near-field interactions is perfectly balanced among processors for all cases, except for the wing-shaped object. We use load-balancing algorithms in order to distribute near-field interactions evenly among processors. For the wing-shaped object, however, this strategy does not lead a perfect parallelization in terms of processing time. Because, we calculate all interactions accurately with 1% error by using adaptive integration methods, and the number of integration points, thus the processing time, for an interaction depends on relative positions of basis and testing functions. On the discretized wing-shaped object, there are very difficult interactions that require longer processing times, compared to other interactions. Using a load-balancing algorithm without considering the computation time, those difficult interactions are accumulated in some processors, leading to unequal processing times for the near-field stage. Nevertheless, we still prefer distributing near-field interactions equally among processors, since this is the best choice in terms of memory and it is difficult to guess the computation time for each interaction a priori. For 64-processor solutions, Figure 5.30 presents details of MVMs, which involve near-field, aggregation, intra-processor translation, inter-processor translation, and disaggregation stages.

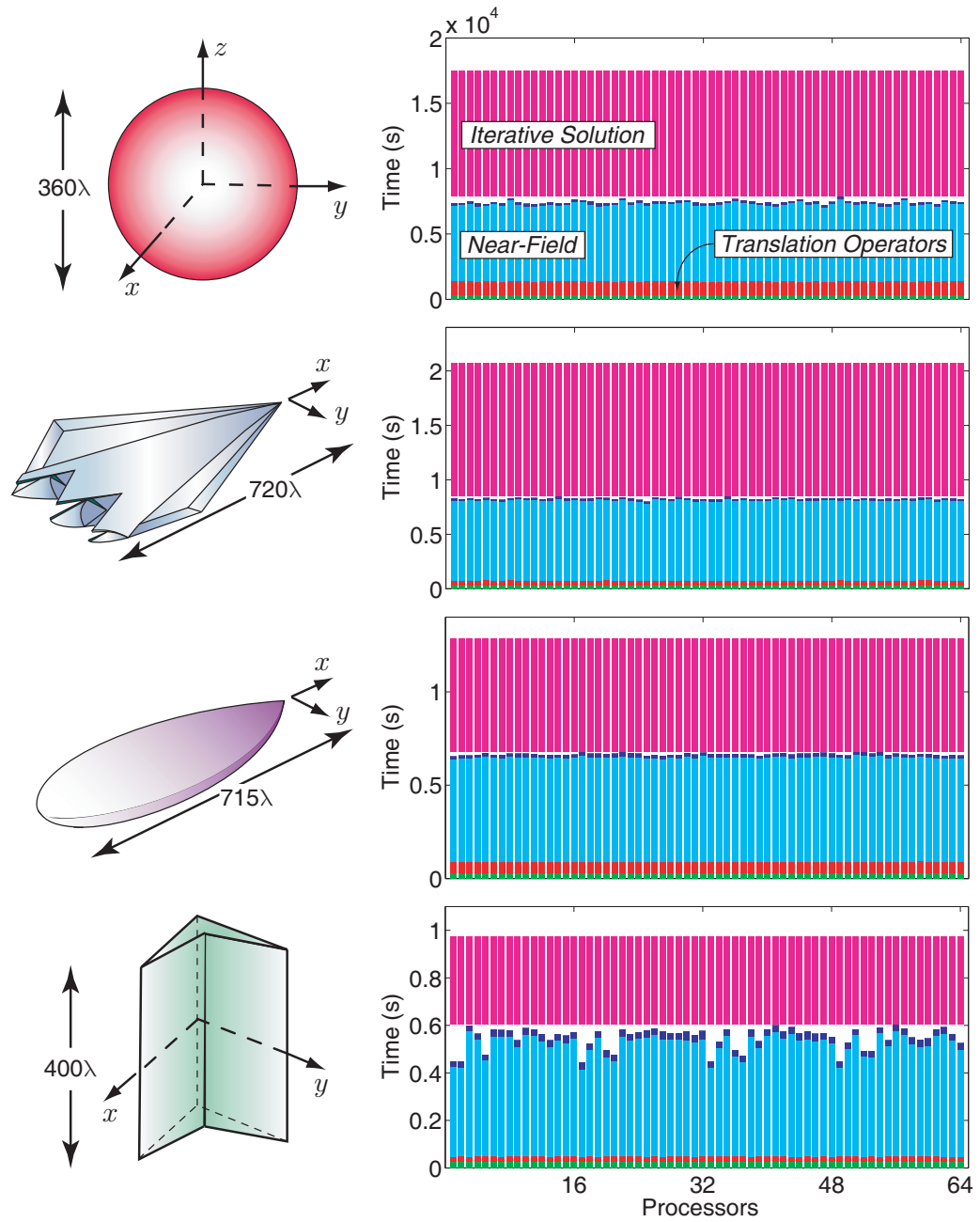


Figure 5.29: Details of the processing time for the solution of large scattering problems on 64 processors of the Harpertown cluster.

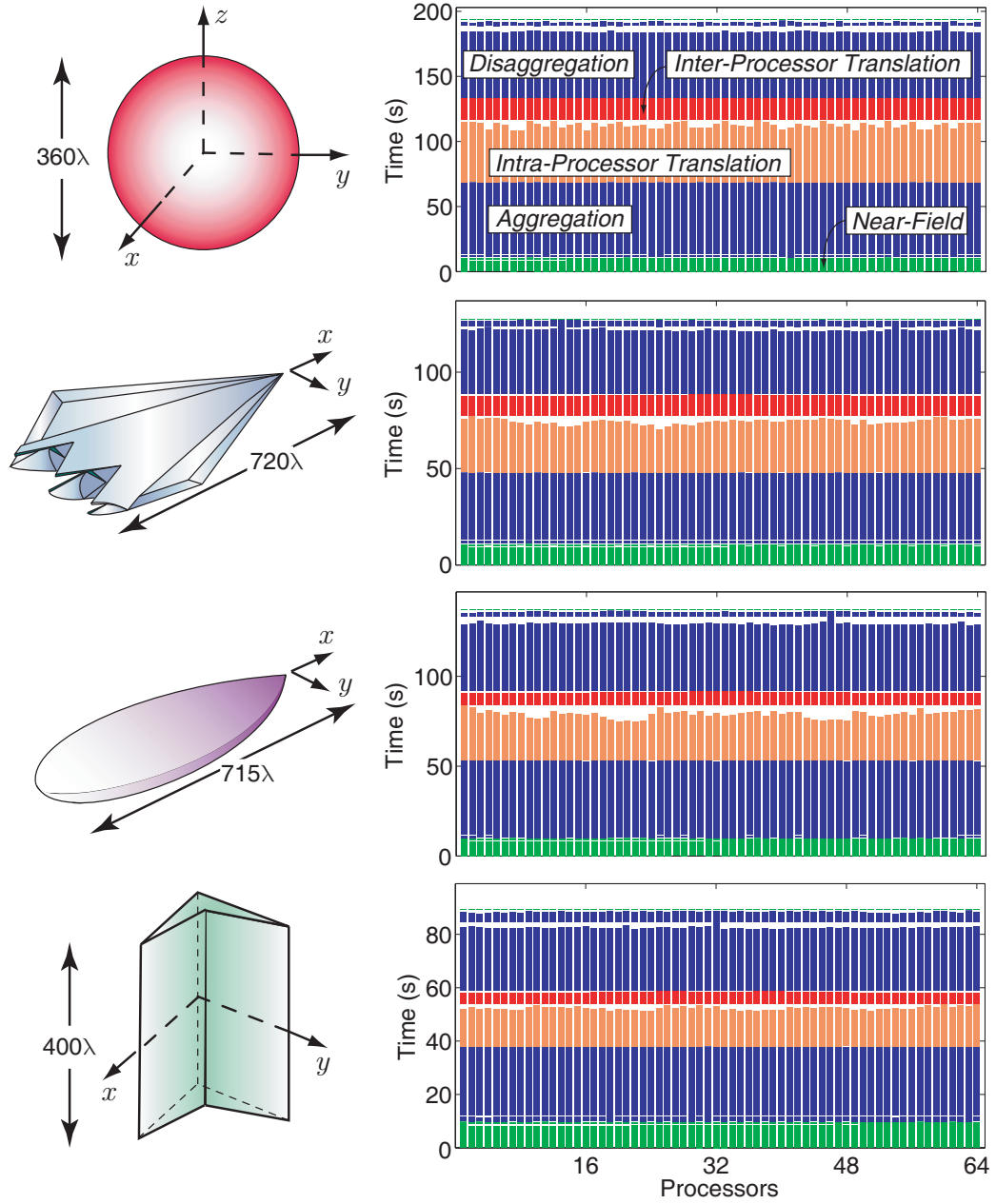


Figure 5.30: Details of MVMs for the solution of large scattering problems on 64 processors of the Harpertown cluster.

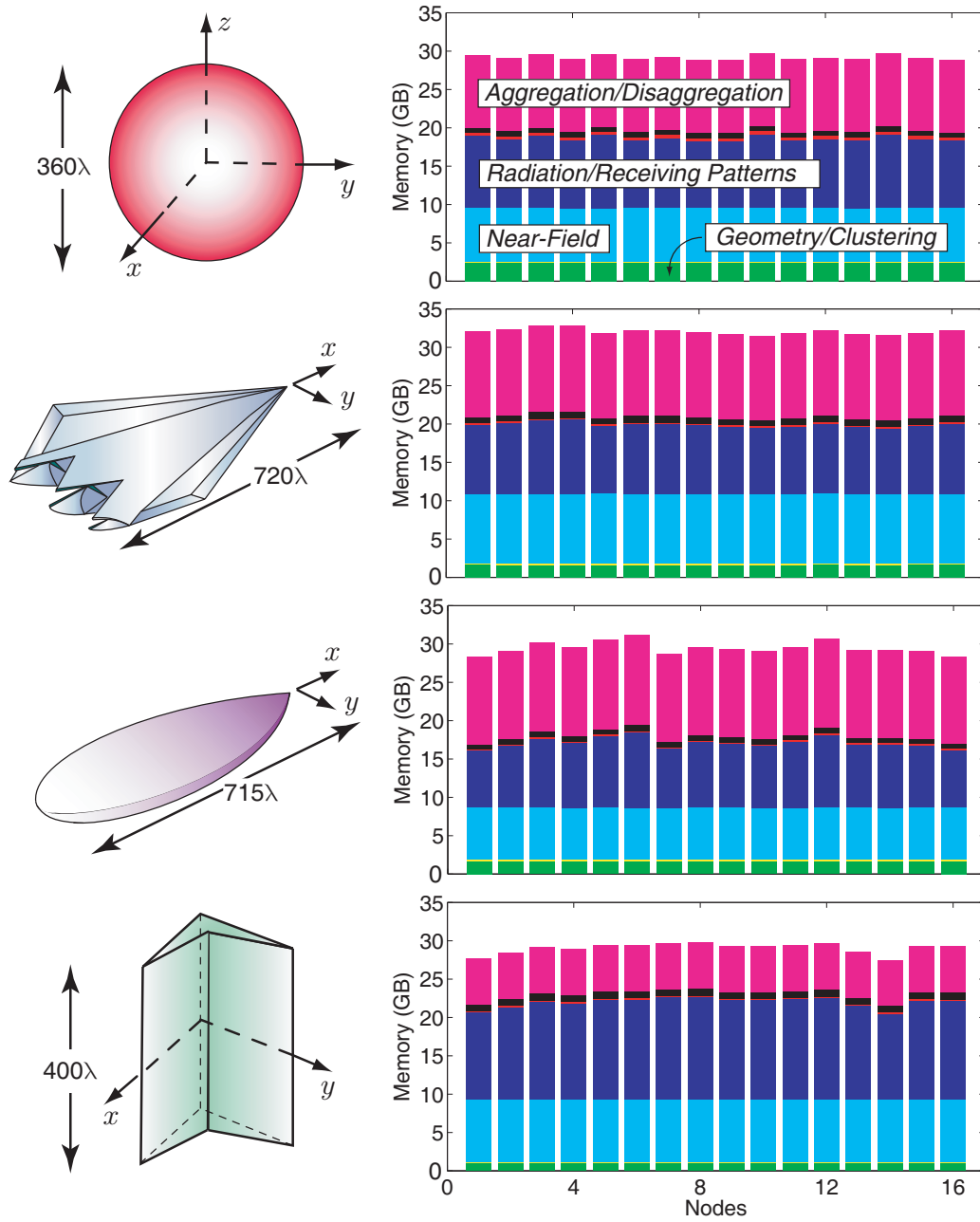


Figure 5.31: Details of memory used for the solution of large scattering problems on 64 processors of the Harpertown cluster.

Figure 5.31 presents details of the memory usage for 64-processor solutions. We consider memory used in each computing node from 1 to 16. A majority of memory is required to store near-field interactions and radiation/receiving patterns of basis/testing functions. Another significant contribution is due to aggregation and disaggregation arrays, which contain radiated and incoming fields of boxes calculated during MVMs. Memory used in computing nodes are not equal, mostly due to different amounts of radiation and receiving patterns of basis and testing functions assigned to processors. Those patterns are distributed according to the far-field partitioning of the tree structure. In general, boxes at the lowest level are distributed equally among processors. However, populations of boxes, i.e., numbers of basis and testing functions in boxes, may vary significantly. A load-balancing algorithm, which accounts for populations of boxes, could be used to improve the distribution of radiation and receiving patterns. However, this method would deteriorate the load-balancing of MVMs in terms of both memory and processing time.

Figure 5.32 depicts the normalized bistatic RCS ($\text{RCS}/\pi a^2$) values for a sphere with a radius of 180λ . We observe that computational and analytical results agree well with each other. In Figures 5.32–5.35, we present RCS values for the Flamme, the NASA Almond, and the wing-shaped object. The normalized RCS (RCS/λ^2) is plotted on the x - y plane as a function of the bistatic angle, where 30° and 210° correspond to back-scattering and forward-scattering directions, respectively. The back-scattered RCS of the Flamme is extremely low; it is 100 dB less than the forward-scattered RCS. For the NASA Almond, the back-scattered RCS is again very small, but we observe relatively high co-polar RCS values from 90° to 210° . Finally, the back-scattered RCS of the wing-shaped object is quite large, and we observe peaks at diverse angles due to strong reflections from this target.

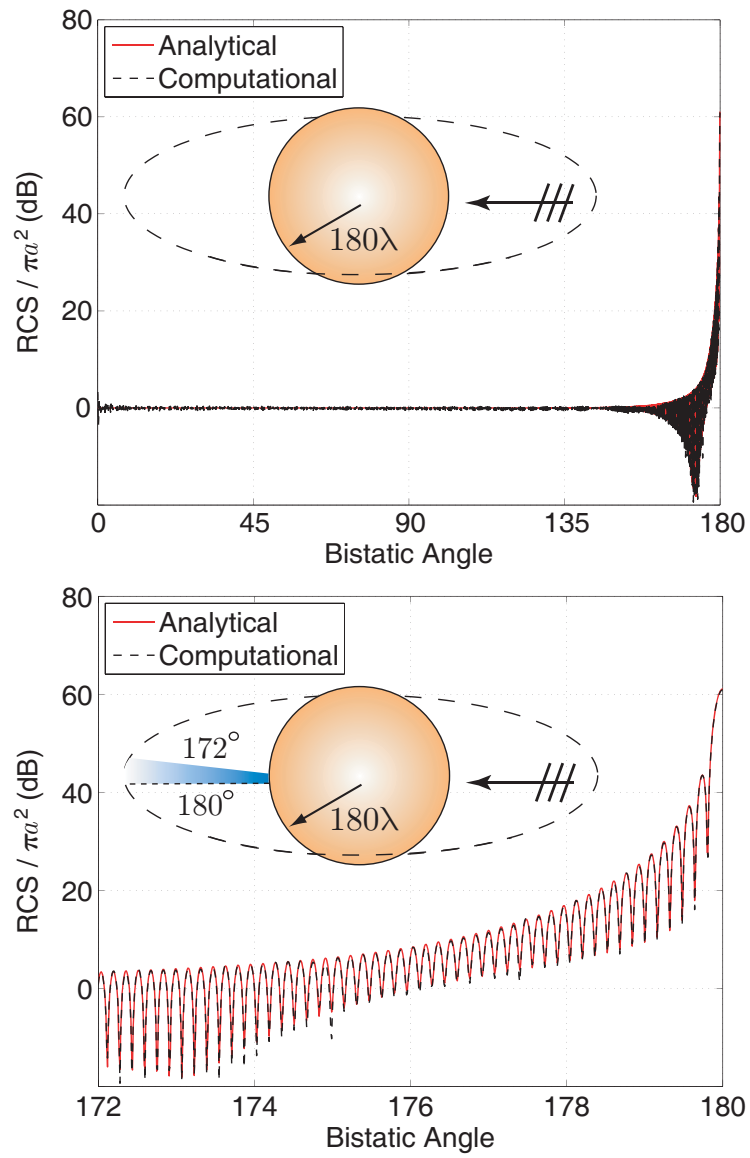


Figure 5.32: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 180λ discretized with 135,164,928 unknowns (a) from 0° to 180° and (b) from 172° to 180° , where 180° corresponds to the forward-scattering direction.

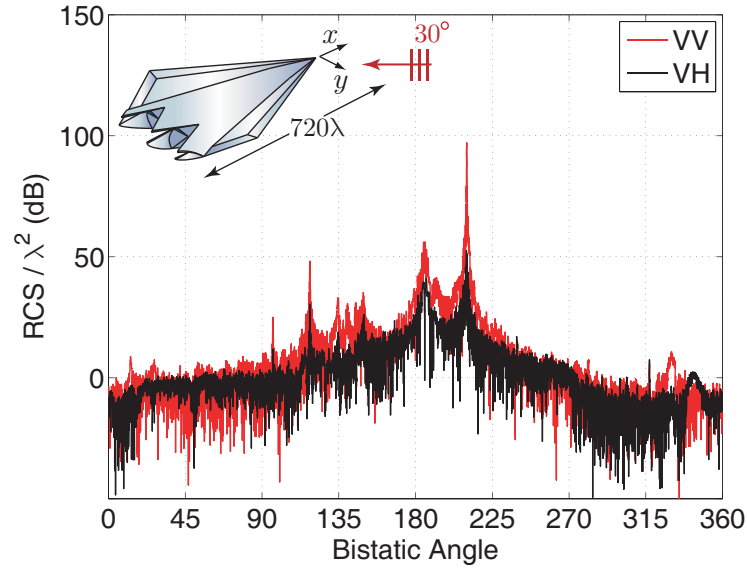


Figure 5.33: Normalized bistatic RCS (RCS/λ^2) of the stealth airborne target Flamme at 360 GHz. The target is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field in polarized in the z direction.

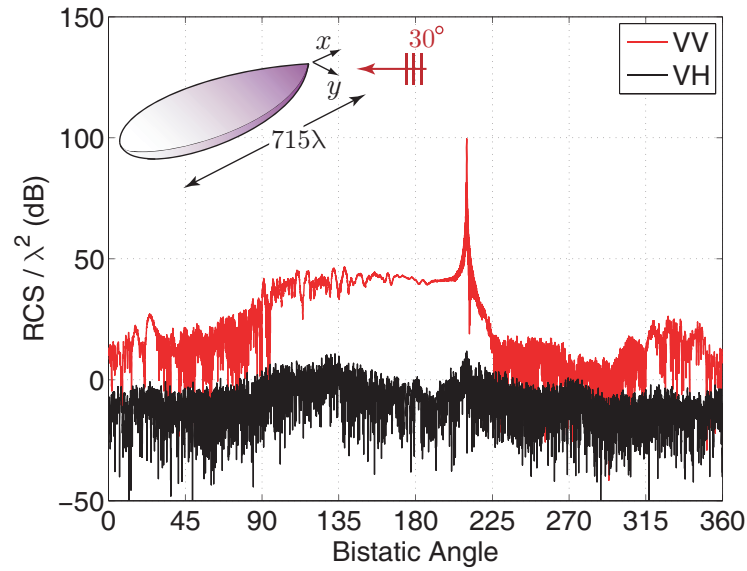


Figure 5.34: Normalized bistatic RCS (RCS/λ^2) of the NASA Almond at 850 GHz. The target is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field in polarized in the z direction.

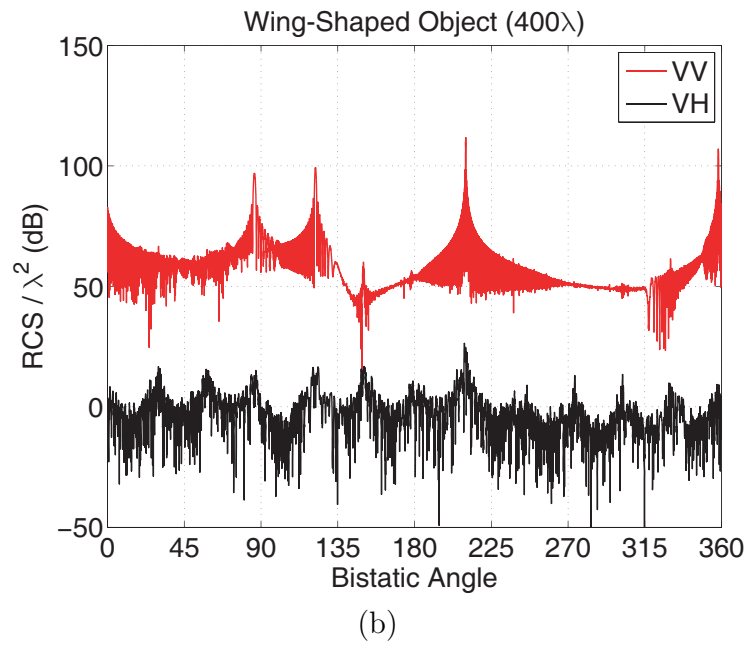
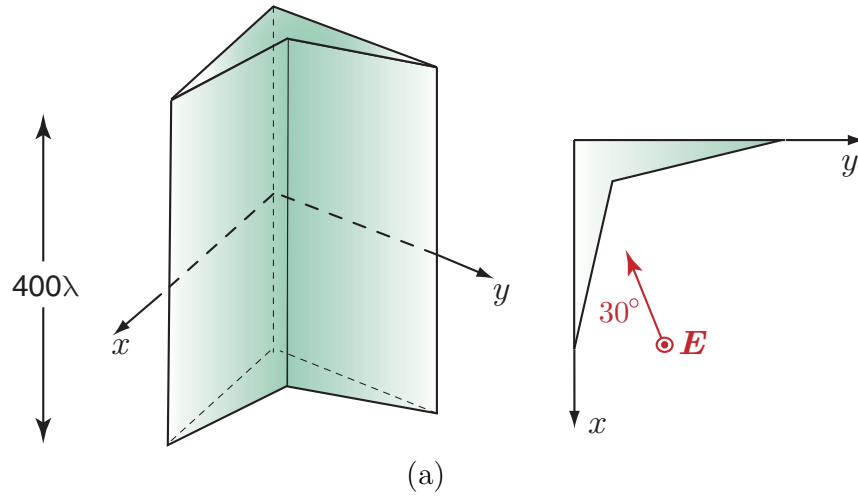


Figure 5.35: (a) A wing-shaped object having a maximum dimension of 400λ and illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field in polarized in the z direction. (b) Normalized bistatic RCS (RCS/λ^2) of the wing-shaped object.

Table 5.11: Solutions of a Scattering Problem Involving a Sphere of Radius 180λ (135,164,928 Unknowns) with Parallel MLFMA (v -101) on 32 and 64 Processors of the Harpertown and the Dunnington Clusters

	BiLCEM Harpertown		Swindon Dunnington		Ratio	
	32	64	32	64	32	64
Setup (minutes)	243	126	312	157	1.28	1.25
Solution (minutes)	263	161	343	222	1.30	1.38
Total (minutes)	511	292	661	386	1.29	1.32
MVM (s)	327	194	430	274	1.32	1.41
Aggregation (s)	103	58	132	66	1.28	1.14
Translation (s)	105	64	141	121	1.34	1.89
Disaggregation (s)	106	60	133	68	1.25	1.13
MVM Others (s)	13	12	24	12	1.84	1.00

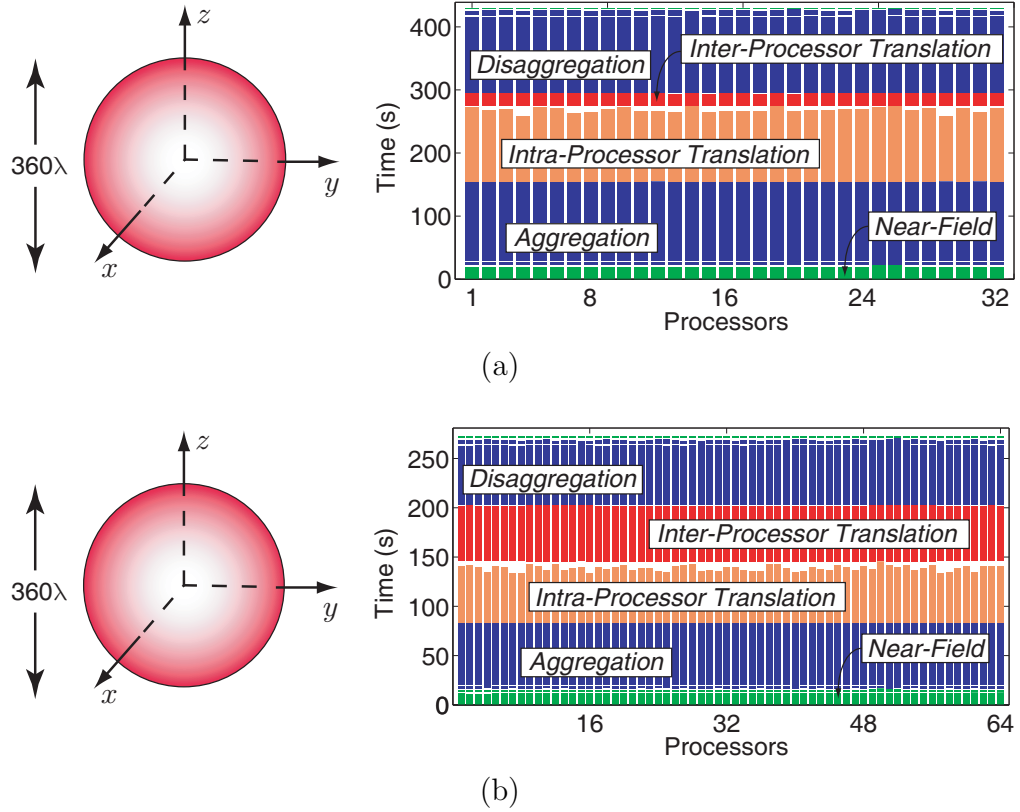


Figure 5.36: Details of MVMs for the solution of a scattering problem involving a sphere of radius 180λ discretized with 135,164,928 unknowns on (a) 32 and (b) 64 processors of the Dunnington cluster.

Table 5.12: Solutions of Large-Scale Electromagnetics Problems with MLFMA Parallelized on the Dunnington Cluster

Geometry	Size	Unknowns	Number of Levels / Box Size	Number of Processors	Excitation / Iterations	Setup (minutes)	Solution (minutes)	Total (minutes)
Sphere	380 λ	167,534,592	10 / 0.186 λ	16	H (0°) 27	899	846	1752
Sphere	380 λ	167,534,592	10 / 0.186 λ	32	H (0°) 27	455 (99%)	481 (88%)	944 (93%)
Sphere	380 λ	167,534,592	10 / 0.186 λ	64	H (0°) 27	228 (99%)	278 (76%)	514 (85%)
Sphere	380 λ	167,534,592	11 / 0.175 λ	16	H (0°) 25	951	904	1864
Sphere	380 λ	167,534,592	11 / 0.175 λ	32	H (0°) 25	477 (100%)	460 (98%)	944 (99%)
Sphere	380 λ	167,534,592	11 / 0.175 λ	64	H (0°) 25	243 (98%)	291 (78%)	541 (86%)
Sphere	400 λ	182,102,016	11 / 0.175 λ	64	H (0°) 26	247	362	618
Sphere	420 λ	204,823,296	11 / 0.175 λ	64	H (0°) 25	278	357	645
Flamme	880 λ	204,664,320	12 / 0.175 λ	64	H (0°) 40	265	654	928
Flamme	880 λ	204,664,320	12 / 0.175 λ	64	H (30°) 38	266	616	891
Flamme	880 λ	204,664,320	12 / 0.175 λ	64	H (60°) 42	265	678	952
Almond	925 λ	203,476,224	12 / 0.175 λ	64	H (30°) 20	223	217	450

Table 5.13: Parameters Related to the Solution of Scattering Problems Discretized with More Than 200 Million Unknowns

Problem	Sphere	Flamme	Almond
Size	420λ	880λ	925λ
Unknowns	204,823,296	204,664,320	203,476,224
Levels	11	12	12
Boxes	32,838,414	24,513,819	28,704,346
Lowest-Level Boxes	24,201,112	18,228,249	21,301,667
Near-Field Interactions	24.6×10^9	29.4×10^9	23.0×10^9
Near-Field Sparsity	5.86×10^{-7}	7.02×10^{-7}	5.56×10^{-7}
Truncation Numbers	5–1986	5–3945	5–3945

5.8.5 Solutions on the Dunnington Cluster

In Table 5.11, we consider the solution of a scattering problem involving a sphere of radius 180λ discretized with 135,164,928 unknowns and we compare the Harpertown and the Dunnington clusters in terms of processing time. MLFMA is parallelized on 32 and 64 processors using the hierarchical technique. For 32 processors, the Harpertown cluster with 3.00 GHz clock rate is approximately 1.30 times faster than the Dunnington cluster with 2.40 GHz clock rate. This ratio is observed for both setup and solution parts, as well as aggregation, translation, and disaggregation stages of MLFMA. For 64 processors, however, the Harpertown cluster is 1.38 times faster the Dunnington cluster. We also observe that the Dunnington cluster is slower than the Harpertown cluster particularly for the translation stage. As depicted in Figure 5.36, the relative duration of the inter-processor translations compared to other stages of MLFMA increases substantially on 64 processors of the Dunnington cluster. Due to those relatively slow translations, the processing time increases more than expected from the clock rate. Nevertheless, the Dunnington cluster has a total of 768 GB memory and it allows for the solution of larger electromagnetics problems compared to the Harpertown cluster.

Table 5.12 presents the solution of large-scale problems on the Dunnington cluster. A scattering problem involving a sphere of radius 190λ discretized with 167,534,592 unknowns is solved on 16, 32, and 64 processors using 10-level (top-down scheme) and 11-level (bottom-up scheme) MLFMA. In both cases, the problem is solved in about nine hours on 64 processors. We also consider the solution of larger problems involving a sphere of radius 200λ and a sphere of radius 210λ discretized with 182,102,016 and 204,823,296 unknowns, respectively. Both two problems are solved in less than 11 hours on 64 processors.

Table 5.12 also presents the solution of scattering problems involving the NASA Almond at 1100 GHz and the Flamme at 440 GHz. The NASA Almond is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field polarized in the ϕ direction (H polarization). This problem is solved in about 7.5 hours. For the Flamme, we consider three different illuminations involving plane waves propagating in the x - y plane at 0° , 30° , and 60° angles from the x axis. Solution of each problem requires 15–16 hours.

Table 5.13 lists important parameters related to the solution of scattering problems involving more than 200 million unknowns. Tree structures have 24–33 million boxes and most of them are used at the lowest level. The number of near-field interactions is 23–30 billion and the sparsity of the near-field matrix is less than 10^{-6} for all problems. The truncation numbers are in range from 5 to 1986 for the sphere problem and from 5 to 3945 for the others.

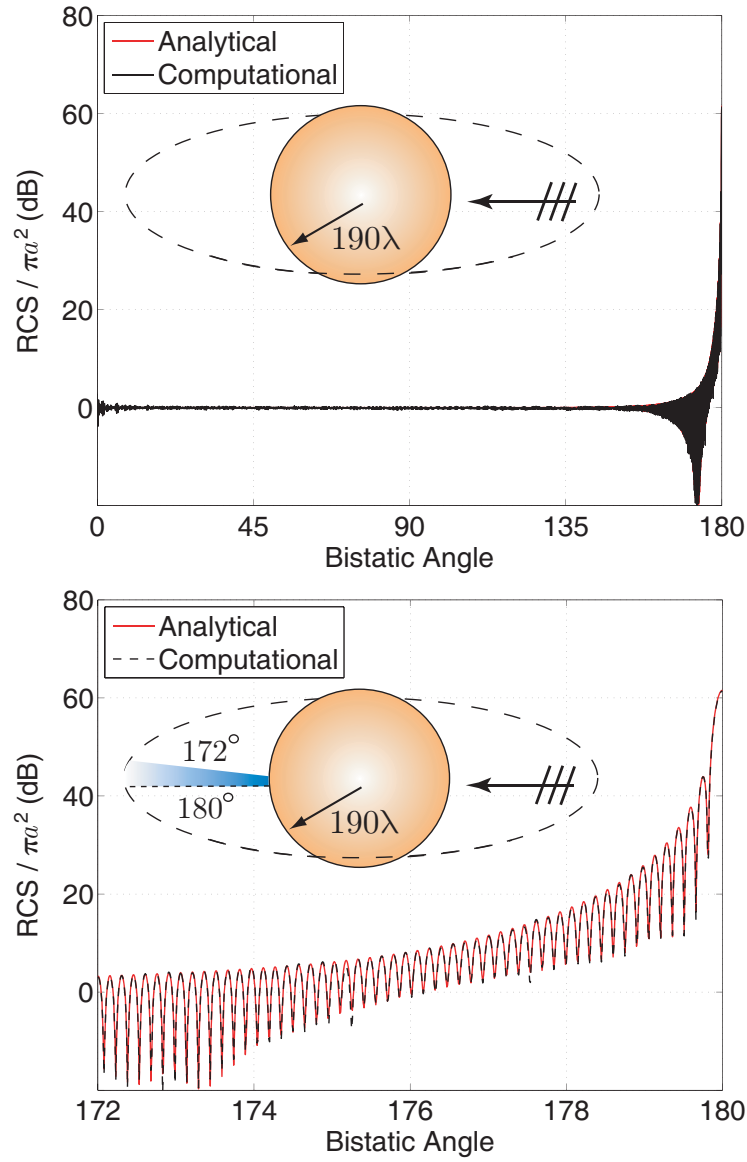


Figure 5.37: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 190λ discretized with 167,534,592 unknowns (a) from 0° to 180° and (b) from 172° to 180° , where 180° corresponds to the forward-scattering direction.

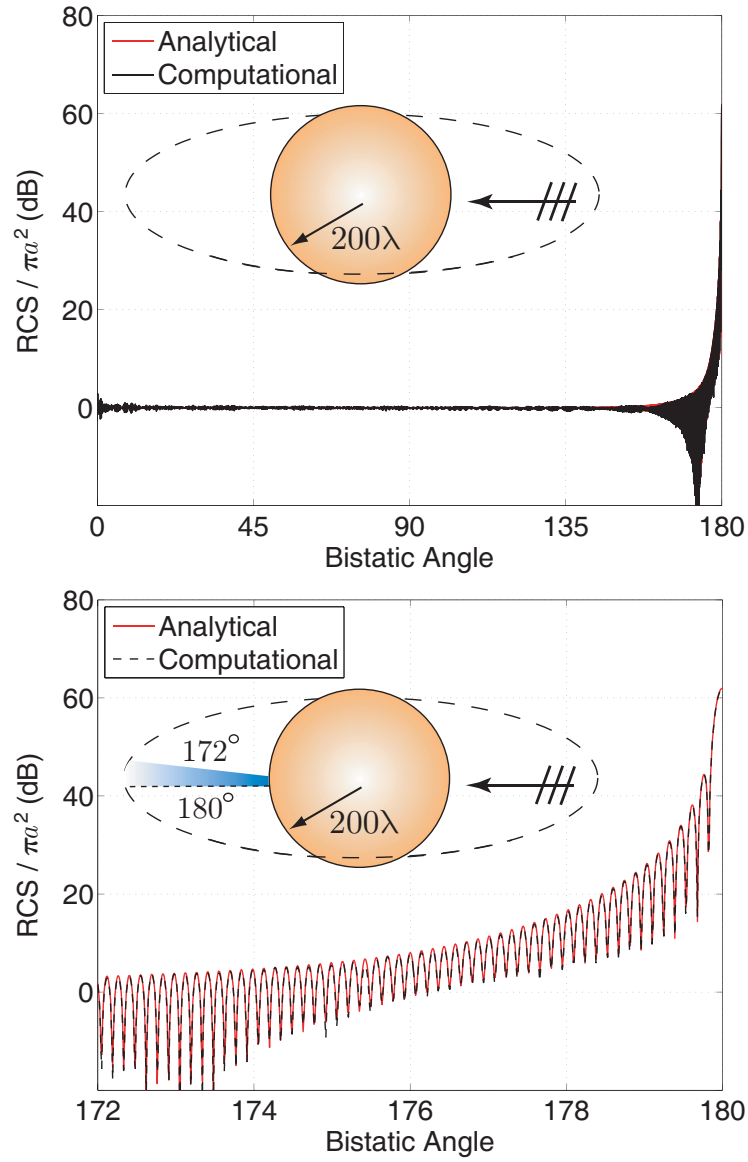


Figure 5.38: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 200λ discretized with 182,102,016 unknowns (a) from 0° to 180° and (b) from 172° to 180° , where 180° corresponds to the forward-scattering direction.

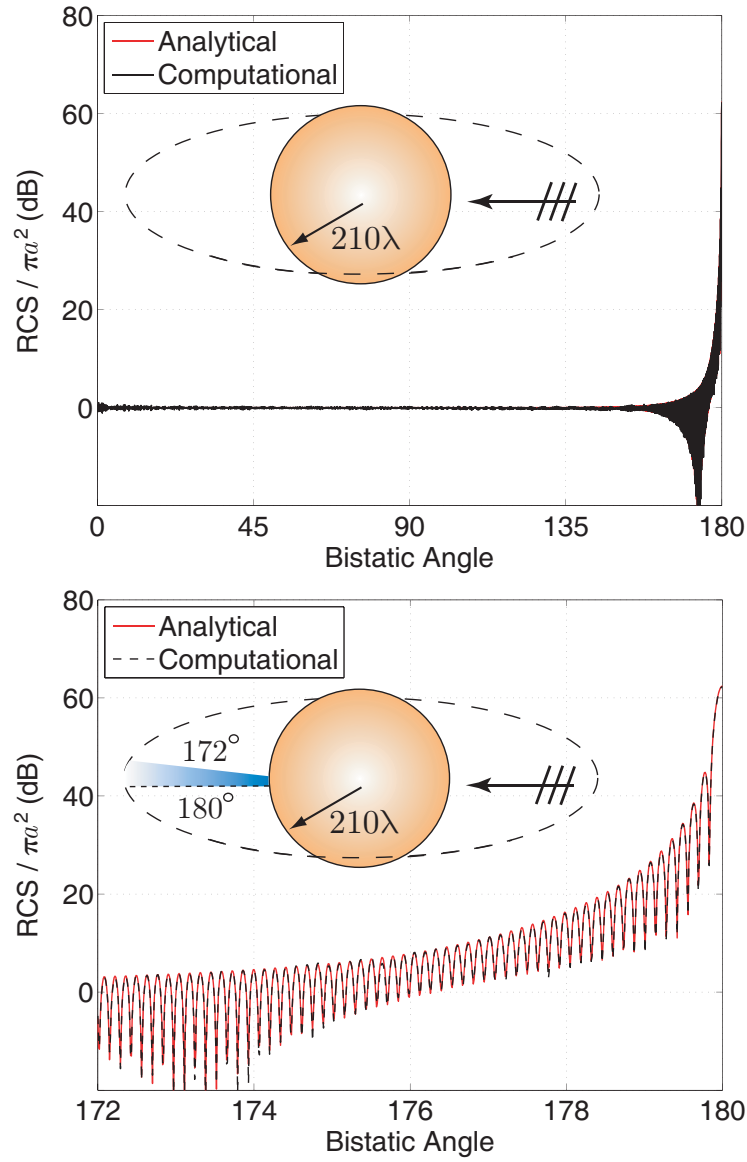


Figure 5.39: Normalized bistatic RCS ($\text{RCS}/\pi a^2$) of a sphere with a radius of 210λ discretized with 204,823,296 unknowns (a) from 0° to 180° and (b) from 172° to 180° , where 180° corresponds to the forward-scattering direction.

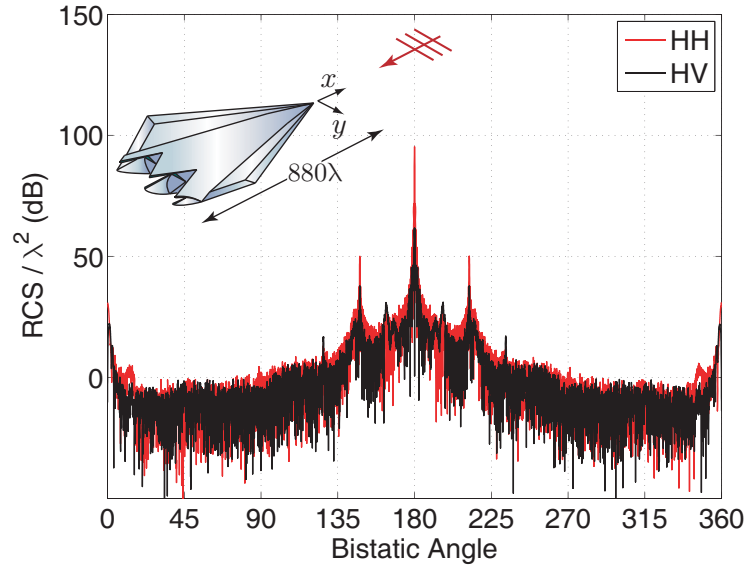


Figure 5.40: Normalized bistatic RCS (RCS/λ^2) of the stealth airborne target Flamme at 440 GHz. The target is illuminated by a plane wave propagating in $-x$ direction with the electric field in polarized in the y direction.

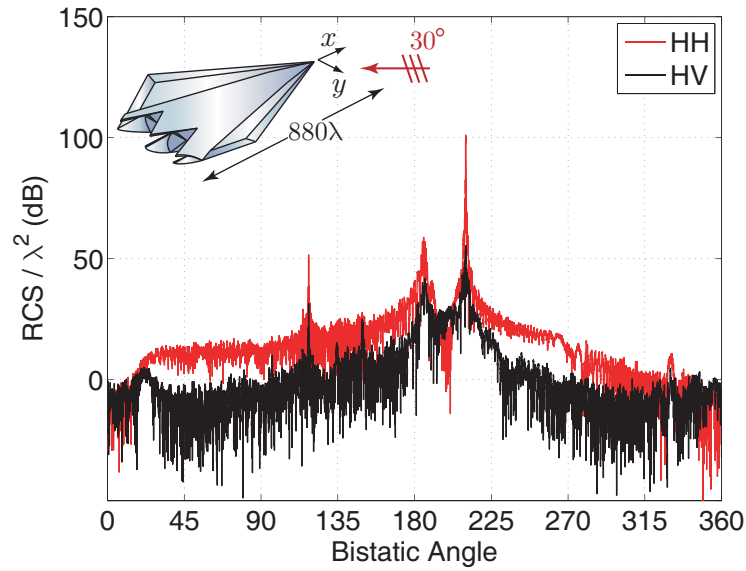


Figure 5.41: Normalized bistatic RCS (RCS/λ^2) of the stealth airborne target Flamme at 440 GHz. The target is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field in polarized in the ϕ direction.

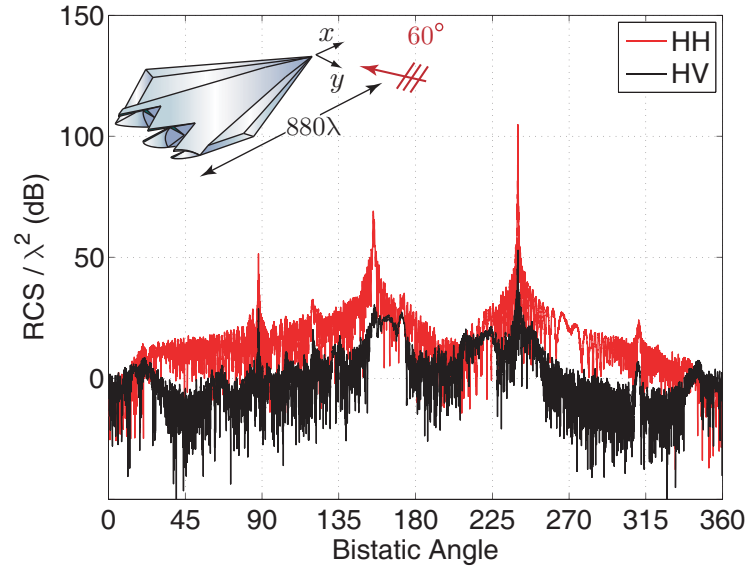


Figure 5.42: Normalized bistatic RCS (RCS/λ^2) of the stealth airborne target Flamme at 440 GHz. The target is illuminated by a plane wave propagating in the x - y plane at a 60° angle from the x axis with the electric field in polarized in the ϕ direction.

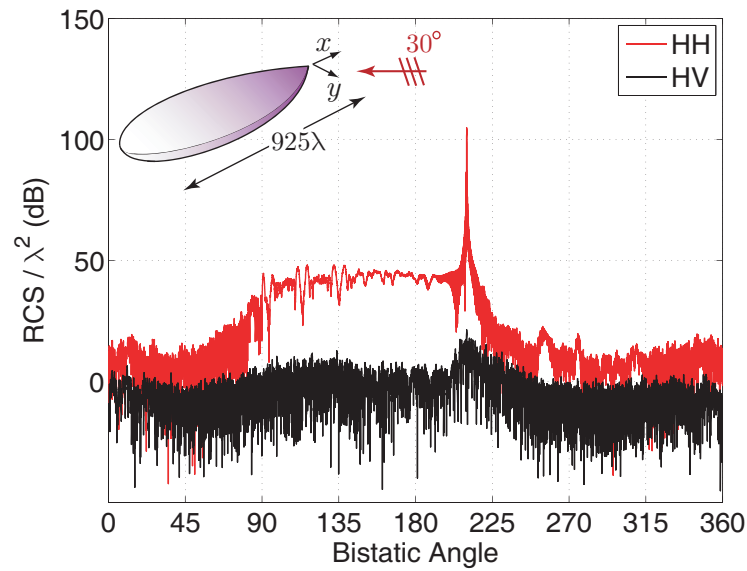


Figure 5.43: Normalized bistatic RCS (RCS/λ^2) of the NASA Almond at 1100 GHz. The target is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field in polarized in the ϕ direction.

Figures 5.37–5.39 present normalized bistatic RCS ($\text{RCS}/\pi a^2$) values for spheres of radii 190λ , 200λ , and 210λ , respectively. We observe that computational values are in agreement with analytical values obtained by Mie-series solutions. Figures 5.40–5.42 present the normalized RCS (RCS/λ^2) of the Flamme when it is illuminated as listed in Table 5.12. For all illuminations, the back-scattered RCS of the Flamme is very low, as usual. Finally, Figure 5.43 depicts the normalized RCS (RCS/λ^2) of the NASA Almond, which is also low in the back-scattering direction but has relatively large values from 90° to 270° .

Chapter 6

Applications and Post Processing

Numerical solutions of real-life electromagnetics problems are usually followed by various post-processing procedures, such as computation of fields at various locations, depending on the application. In this chapter, we provide an overview for calculating near-zone and far-zone fields from coefficients obtained by MLFMA implementations. We also present various examples to applications that we have investigated via MLFMA. Specifically, we consider scattering from the stealth airborne target Flamme, transmission through frequency-selective metamaterials and periodic dielectric structures, and radiation from LP antennas. We present the effect of external resonances on scattering characteristics of the Flamme. Unusual electromagnetic properties of metamaterials are demonstrated on various metamaterial walls involving SRRs and TWs. We also present frequency-selective characteristics of dielectric structures involving periodical arrangements of rectangular slabs. Finally, we present accurate simulations of trapezoidal-tooth LP antennas and their circular arrays using our sophisticated simulation environment based on MLFMA.

6.1 Near-Zone Fields

Numerical solutions of a scattering or radiation problem involving a PEC object provide expansion coefficients of the electric current on the surface of the object. Finding coefficients $\mathbf{a}_J[n]$ for $n = 1, 2, \dots, N$, we calculate secondary (scattered or radiated) electric and magnetic fields at any point $\mathbf{r}$ as

$$\mathbf{E}^{sec}(\mathbf{r}) = ik\eta \sum_{n=1}^N \mathbf{a}_J[n] \int_{S_n} d\mathbf{r}' \left[\mathbf{b}_n(\mathbf{r}') + \frac{1}{k^2} \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \nabla \right] g(\mathbf{r}, \mathbf{r}') \quad (6.1)$$

$$\mathbf{H}^{sec}(\mathbf{r}) = \sum_{n=1}^N \mathbf{a}_J[n] \int_{S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \times \nabla' g(\mathbf{r}, \mathbf{r}'). \quad (6.2)$$

If the observation point $\mathbf{r}$ is close to the object, singularity extraction methods are required to evaluate integrals in (6.1) and (6.2) accurately. On the other hand, singularity extractions must be avoided when the observation point is far from the object and $|\mathbf{r}|$ is very large.

For dielectric objects, expansion coefficients of both electric and magnetic currents must be considered to compute electric and magnetic fields. If the observation point $\mathbf{r}$ is outside the object,

$$\begin{aligned} \mathbf{E}^{sec}(\mathbf{r}) = & ik_o \eta_o \sum_{n=1}^N \mathbf{a}_J[n] \int_{S_n} d\mathbf{r}' \left[\mathbf{b}_n(\mathbf{r}') + \frac{1}{k_o^2} \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \nabla \right] g_o(\mathbf{r}, \mathbf{r}') \\ & - \sum_{n=1}^N \mathbf{a}_M[n] \int_{S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \times \nabla' g_o(\mathbf{r}, \mathbf{r}') \end{aligned} \quad (6.3)$$

$$\begin{aligned} \mathbf{H}^{sec}(\mathbf{r}) = & \sum_{n=1}^N \mathbf{a}_J[n] \int_{S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \times \nabla' g_o(\mathbf{r}, \mathbf{r}') \\ & + \frac{ik_o}{\eta_o} \sum_{n=1}^N \mathbf{a}_M[n] \int_{S_n} d\mathbf{r}' \left[\mathbf{b}_n(\mathbf{r}') + \frac{1}{k_o^2} \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \nabla \right] g_o(\mathbf{r}, \mathbf{r}'), \end{aligned} \quad (6.4)$$

where we use electrical parameters of the host medium. On the other hand, if the observation point is inside the object,

$$\begin{aligned} \mathbf{E}^{sec}(\mathbf{r}) = & -ik_i\eta_i \sum_{n=1}^N \mathbf{a}_J[n] \int_{S_n} d\mathbf{r}' \left[\mathbf{b}_n(\mathbf{r}') + \frac{1}{k_i^2} \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \nabla \right] g_i(\mathbf{r}, \mathbf{r}') \\ & + \sum_{n=1}^N \mathbf{a}_M[n] \int_{S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \times \nabla' g_i(\mathbf{r}, \mathbf{r}') \end{aligned} \quad (6.5)$$

$$\begin{aligned} \mathbf{H}^{sec}(\mathbf{r}) = & - \sum_{n=1}^N \mathbf{a}_J[n] \int_{S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \times \nabla' g_i(\mathbf{r}, \mathbf{r}') \\ & - \frac{ik_i}{\eta_i} \sum_{n=1}^N \mathbf{a}_M[n] \int_{S_n} d\mathbf{r}' \left[\mathbf{b}_n(\mathbf{r}') + \frac{1}{k_i^2} \nabla' \cdot \mathbf{b}_n(\mathbf{r}') \nabla \right] g_i(\mathbf{r}, \mathbf{r}'), \end{aligned} \quad (6.6)$$

where electrical parameters of the object is used. In our typical computations involving many observation points, we do not check whether a given observation point is located inside or outside the object. Instead, we compute both (6.3)–(6.4) and (6.5)–(6.6), and superpose results to obtain secondary fields at all observation points. This is legitimate since expressions in (6.3)–(6.4) and (6.5)–(6.6) are related to outer and inner problems and they do not contribute to electromagnetic fields at observation points inside and outside the object, respectively. A similar procedure is applied for composite objects involving multiple dielectric and metallic parts. Without checking positions of observation points, we allow the radiation of electric and magnetic currents using electrical parameters of each medium.

As an example, we consider the solution of a scattering problem involving a dielectric sphere of radius $0.5\lambda_o$ coated with a dielectric shell of radius λ_o located in free space. The spherical object is illuminated by a plane wave propagating in the $-z$ direction with the electric field polarized in the x direction. Relative permittivities of the core and the shell are 4.0 and 2.0, respectively. The problem is formulated by JMCFIE and solved via MLFMA. Figures 6.1(a) and 6.1(b) present the radial (ρ) and the angular (ϕ) components of the total electric field in a $2\lambda_o \times 2\lambda_o$ area on the x - y plane. We observe that the angular component of the electric field is continuous across dielectric interfaces.

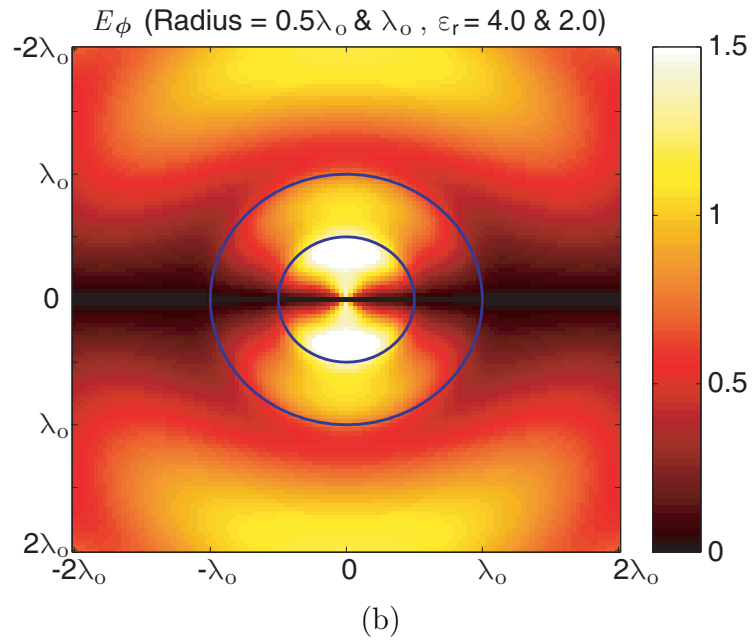
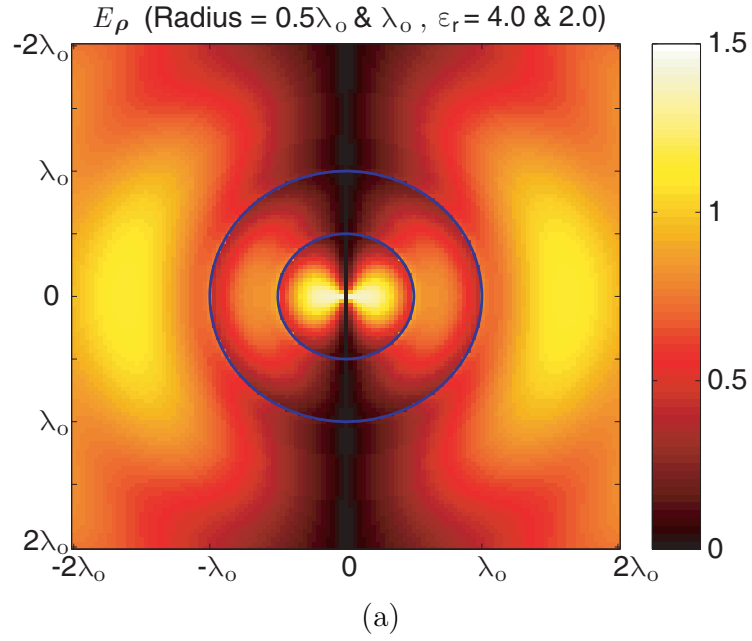


Figure 6.1: (a) Radial and (b) angular components of the total electric field in a $2\lambda_o \times 2\lambda_o$ area around a dielectric sphere of radius $0.5\lambda_o$ coated with a dielectric shell of radius λ_o . The spherical object is illuminated by a plane wave and relative permittivities of the core and the shell are 4.0 and 2.0, respectively.

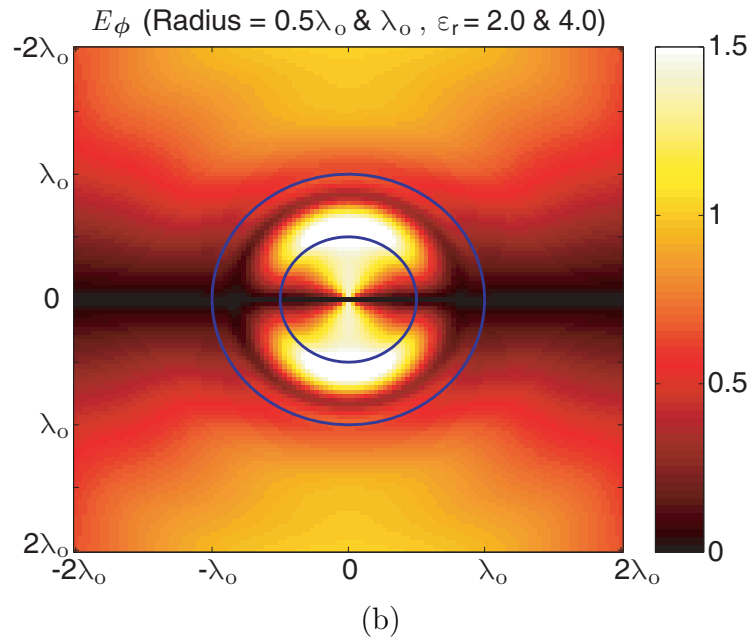
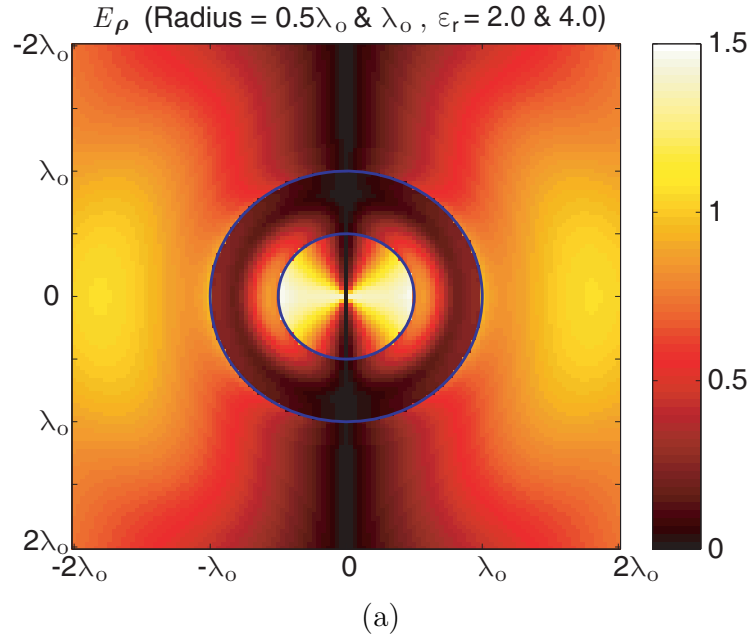


Figure 6.2: (a) Radial and (b) angular components of the total electric field in a $2\lambda_o \times 2\lambda_o$ area around a dielectric sphere of radius $0.5\lambda_o$ coated with a dielectric shell of radius λ_o . The spherical object is illuminated by a plane wave and relative permittivities of the core and the shell are 2.0 and 4.0, respectively.

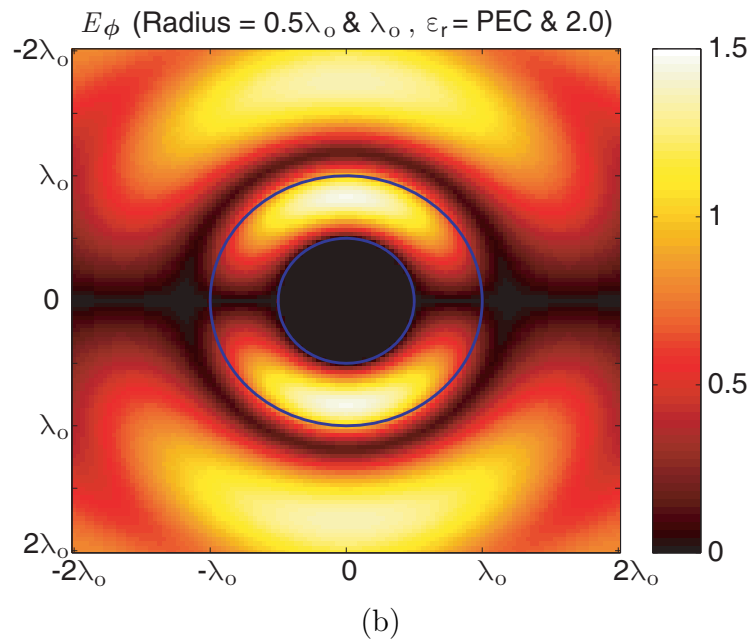
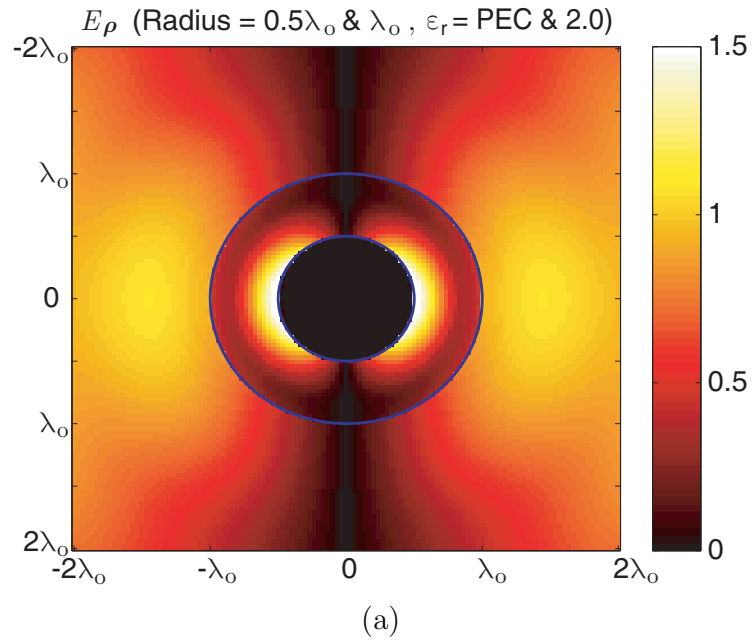


Figure 6.3: (a) Radial and (b) angular components of the total electric field in a $2\lambda_o \times 2\lambda_o$ area around a PEC sphere of radius $0.5\lambda_o$ coated with a dielectric shell of radius λ_o . The spherical object is illuminated by a plane wave and the relative permittivity of the shell is 2.0.

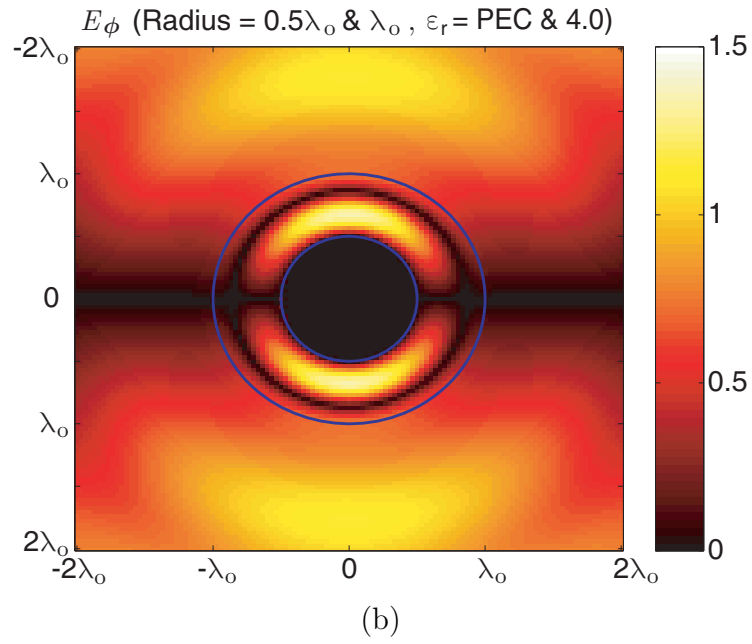
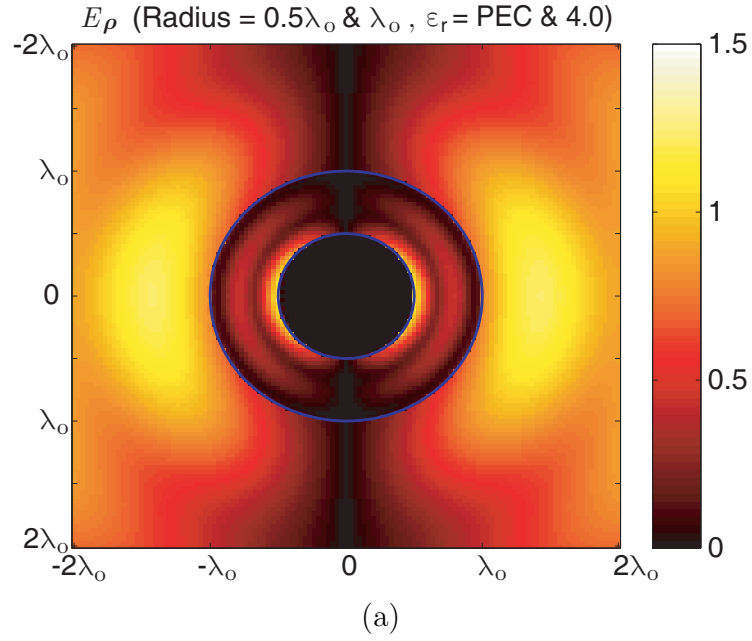


Figure 6.4: (a) Radial and (b) angular components of the total electric field in a $2\lambda_o \times 2\lambda_o$ area around a PEC sphere of radius $0.5\lambda_o$ coated with a dielectric shell of radius λ_o . The spherical object is illuminated by a plane wave and the relative permittivity of the shell is 4.0.

Figures 6.2–6.4 present similar results when the material properties of the core and the shell are changed. In Figure 6.2, relative permittivities of the core and the shell are 2.0 and 4.0, respectively. In Figure 6.3, the core is PEC and the relative permittivity of the shell is 2.0. Finally, in Figure 6.4, the core is again PEC, but the relative permittivity of the shell is 4.0. We note that the magnitude of the electric field becomes very small inside PEC spheres. Although this is a trivial result and expected due the shielding by PEC surfaces, we emphasize that vanishingly small fields inside PEC spheres present the high accuracy of solutions.

6.2 Far-Zone Fields

In the far zone, electric and magnetic fields decay with $1/|\mathbf{r}|$ and they become zero when $|\mathbf{r}|$ is infinite. In those cases, we are interested in normalized field quantities, i.e.,

$$\begin{aligned} \mathbf{E}^{sec,\infty}(\theta, \phi) &= \exp(-ik_o|\mathbf{r}|) \lim_{|\mathbf{r}| \rightarrow \infty} \{|\mathbf{r}| \mathbf{E}^{sec}(\mathbf{r})\} \\ &= \frac{ik_o\eta_o}{4\pi} \sum_{n=1}^N \mathbf{a}_J[n] (\bar{\mathbf{I}}^{3 \times 3} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \mathbf{V}_n(\theta, \phi) \\ &\quad - \frac{ik_o}{4\pi} \sum_{n=1}^N \mathbf{a}_M[n] \hat{\mathbf{k}} \times \mathbf{V}_n(\theta, \phi) \end{aligned} \quad (6.7)$$

and

$$\begin{aligned} \mathbf{H}^{sec,\infty}(\theta, \phi) &= \exp(-ik_o|\mathbf{r}|) \lim_{|\mathbf{r}| \rightarrow \infty} \{|\mathbf{r}| \mathbf{H}^{sec}(\mathbf{r})\} \\ &= \frac{1}{\eta_o} \hat{\mathbf{k}} \times \mathbf{E}^{sec,\infty}(\theta, \phi), \end{aligned} \quad (6.8)$$

where

$$\mathbf{V}_n(\theta, \phi) = \int_{S_n} d\mathbf{r}' \mathbf{b}_n(\mathbf{r}') \exp(-ik_o\hat{\mathbf{r}} \cdot \mathbf{r}') \quad (6.9)$$

is the vector current moment of the n th basis function.

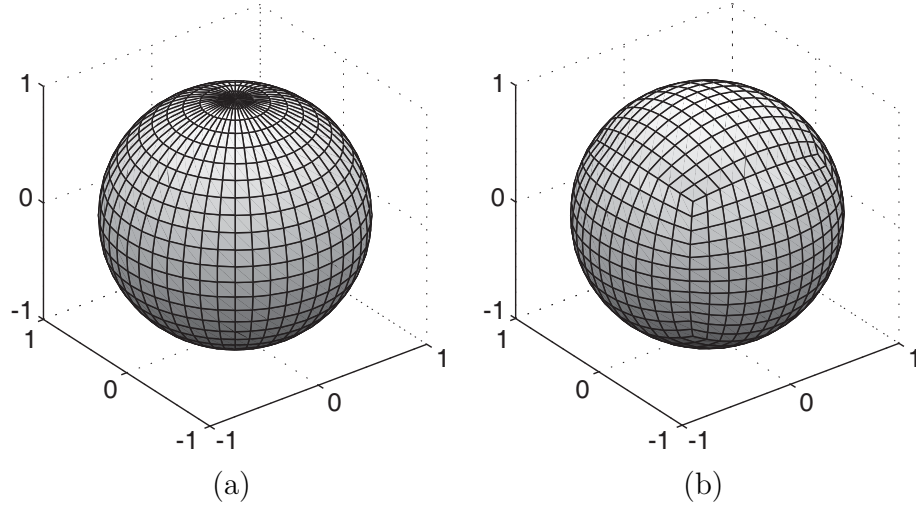


Figure 6.5: Discretization of the unit sphere: (a) Sampling with equally-spaced points in the θ - ϕ domain and (b) uniform sampling of six biquadratic patches in the u - v domain.

In many cases, far-zone electromagnetic fields need to be calculated at various directions as a function of angular parameters θ and ϕ . If the unit sphere is sampled with equally-spaced points in the θ - ϕ domain, the resulting discretization is illustrated in Figure 6.5(a), where an unnecessary crowding of sampling points around the north pole and the south pole is observed. Since far-zone fields are completely arbitrary, there are no preferred directions requiring denser sampling. Thus, not only is the dense sampling around poles redundant and inefficient (a significant amount of computation is performed for each direction), but also generates misleading (more difficult to read) three dimensional plots. As an alternative, the unit sphere can be modeled with six biquadratic patches, as shown in Figure 6.5(b). Each biquadratic patch has the general representation of

$$\mathbf{r}(u, v) = \sum_{m=0}^2 \sum_{n=0}^2 \mathbf{a}_{m,n} u^m v^n \quad (6.10)$$

for $0 \leq u, v \leq 1$, where $\mathbf{a}_{m,n}$ represents vector coefficients and $\mathbf{r}(u, v)$ is the position vector from the origin pointing to the surface. Then, sampling points can be obtained on each patch by a uniform distribution in the u - v domain.

Figure 6.5(b) illustrates a discretization with 13×13 points on each patch, producing a total of 864 directions on the unit sphere. In contrast, the discretization in Figure 6.5(a), which has comparable fidelity around the equator, contains 33% more sampling directions.

Far-zone scattering properties of targets are usually characterized by RCS defined as

$$\text{RCS}(\theta, \phi) = \frac{4\pi}{\eta_o} |E_{\theta}^{sec, \infty}(\theta, \phi)|^2 + \frac{4\pi}{\eta_o} |E_{\phi}^{sec, \infty}(\theta, \phi)|^2, \quad (6.11)$$

which depends on the frequency, the shape and material properties of the object, the excitation (illumination), and the observation angle. For a given target and excitation, we usually investigate RCS as a function of the frequency and the observation angle (bistatic). As an example, in Figures 6.6 and 6.7, we present the bistatic co-polar RCS of the stealth airborne target Flamme located on the x - y plane. In Figure 6.6, the target is illuminated by a plane wave propagating at 60° angle from the nose. Both horizontal (H) and vertical (V) polarizations are considered. In polar plots, radial and angular directions represent the frequency and the observation angle, respectively. The bistatic RCS is plotted as a function of frequency from 2 GHz to 11 GHz and as a function of observation angle from 0° to 360° . We note that the RCS of the Flamme is very low in the back-scattering direction (0°) due to the stealth property of the target. For the horizontal polarization, a strong reflection from the target at about 330° is also remarkable. Figure 6.7 presents similar results when the Flamme is illuminated by a plane wave propagating at 120° angle from the nose. For this excitation, the back-scattered RCS of the Flamme is usually at low levels, but it increases at various frequencies, particularly for the horizontal polarization.

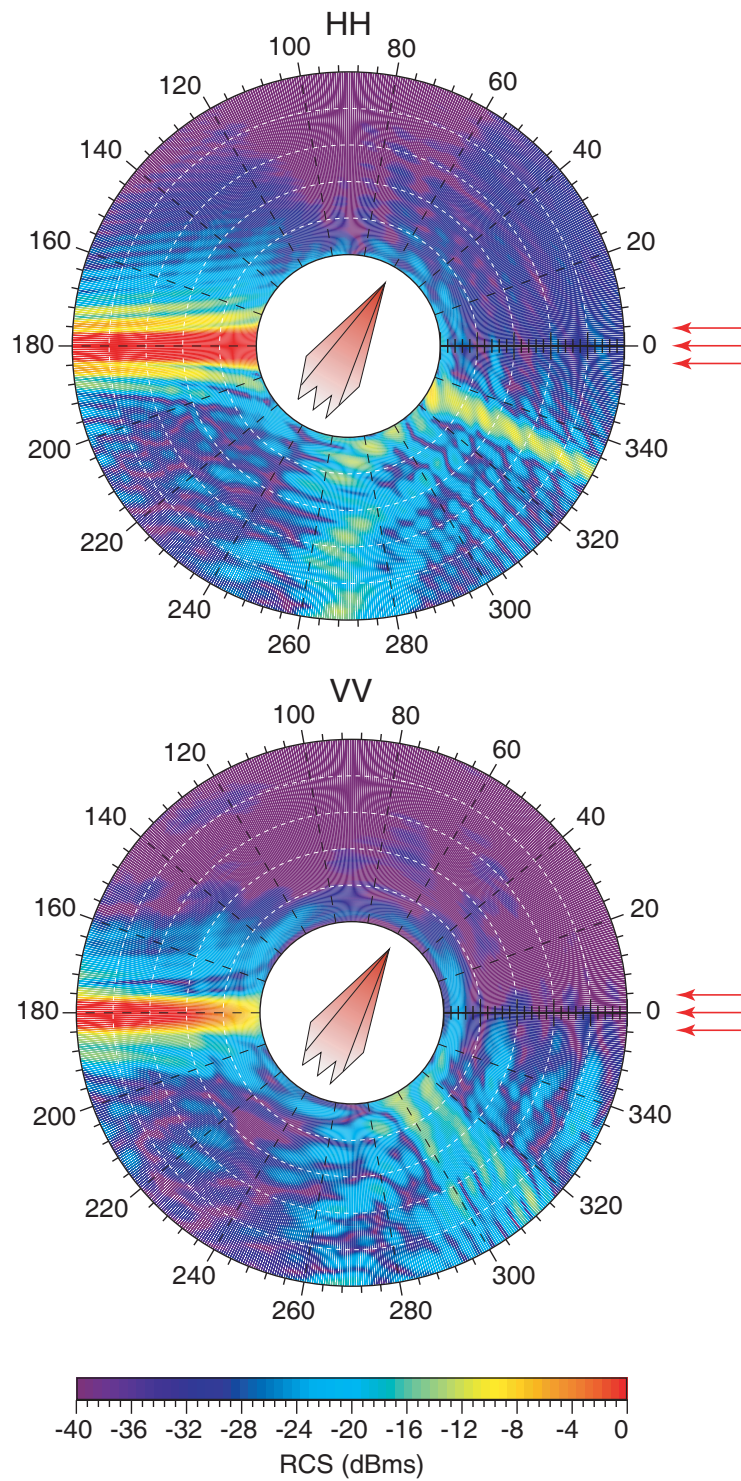


Figure 6.6: Co-polar bistatic RCS of the Flamme from 2 GHz to 11 GHz. The target is illuminated by a plane wave propagating at 60° angle from the nose with the electric field polarized in horizontal and vertical directions.

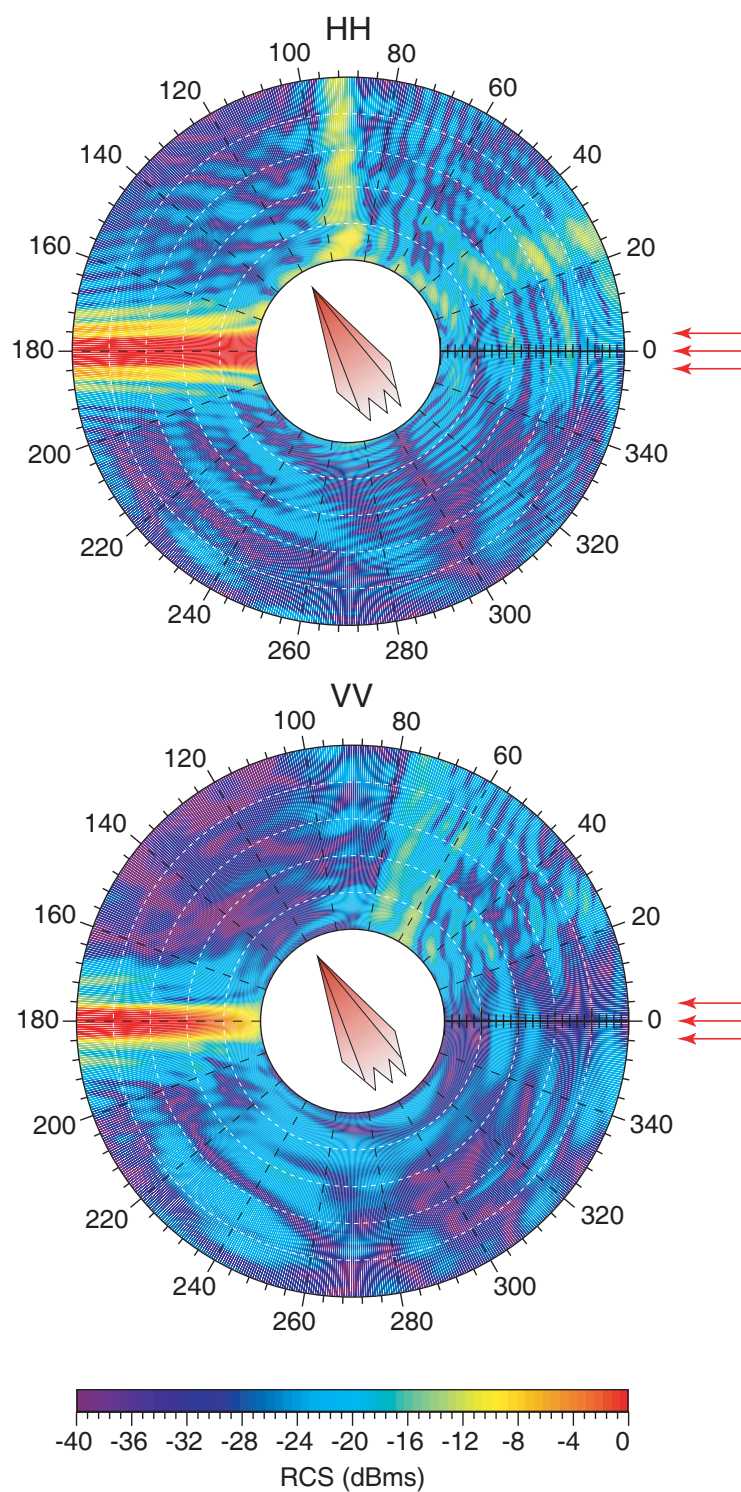


Figure 6.7: Co-polar bistatic RCS of the Flamme from 2 GHz to 11 GHz. The target is illuminated by a plane wave propagating at 120° angle from the nose with the electric field polarized in horizontal and vertical directions.

A major parameter to measure the radiation quality of an antenna is the directive gain in a desired direction (θ_0, ϕ_0) , i.e.,

$$D(\theta_0, \phi_0) = 4\pi P^\infty(\theta_0, \phi_0) \left[\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta P^\infty(\theta, \phi) \right]^{-1}, \quad (6.12)$$

where

$$P^\infty(\theta, \phi) = \frac{1}{\eta_o} |E_\theta^{sec, \infty}(\theta, \phi)|^2 + \frac{1}{\eta_o} |E_\phi^{sec, \infty}(\theta, \phi)|^2 \quad (6.13)$$

is the radiation intensity. Using the sampling scheme depicted in 6.5(b), the integral in (6.12) can be expressed in the u - v domain for each biquadratic surface as

$$I = \int_0^1 du \int_0^1 dv \left[|E_\theta^{sec, \infty}(u, v)|^2 + |E_\phi^{sec, \infty}(u, v)|^2 \right] \varrho(u, v). \quad (6.14)$$

In (6.14),

$$\varrho(u, v) = \sqrt{j_{uu}(u, v)j_{vv}(u, v) - j_{uv}^2(u, v)} \quad (6.15)$$

is the Jacobian, where

$$\begin{aligned} j_{uu}(u, v) &= \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} \\ &= \sum_{m_1=1}^2 \sum_{m_2=1}^2 \sum_{n_1=0}^2 \sum_{n_2=0}^2 \mathbf{a}_{m_1, n_1} \cdot \mathbf{a}_{m_2, n_2} m_1 m_2 u^{m_1+m_2-2} v^{n_1+n_2} \end{aligned} \quad (6.16)$$

$$\begin{aligned} j_{vv}(u, v) &= \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v} \\ &= \sum_{m_1=0}^2 \sum_{m_2=0}^2 \sum_{n_1=1}^2 \sum_{n_2=1}^2 \mathbf{a}_{m_1, n_1} \cdot \mathbf{a}_{m_2, n_2} n_1 n_2 u^{m_1+m_2} v^{n_1+n_2-2} \end{aligned} \quad (6.17)$$

$$\begin{aligned} j_{uv}(u, v) &= \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \\ &= \sum_{m_1=1}^2 \sum_{m_2=0}^2 \sum_{n_1=0}^2 \sum_{n_2=1}^2 \mathbf{a}_{m_1, n_1} \cdot \mathbf{a}_{m_2, n_2} m_1 n_2 u^{m_1+m_2-1} v^{n_1+n_2-1}. \end{aligned} \quad (6.18)$$

The directive gain is extensively used in Section 6.6 to compare various designs of LP antennas and their arrays.

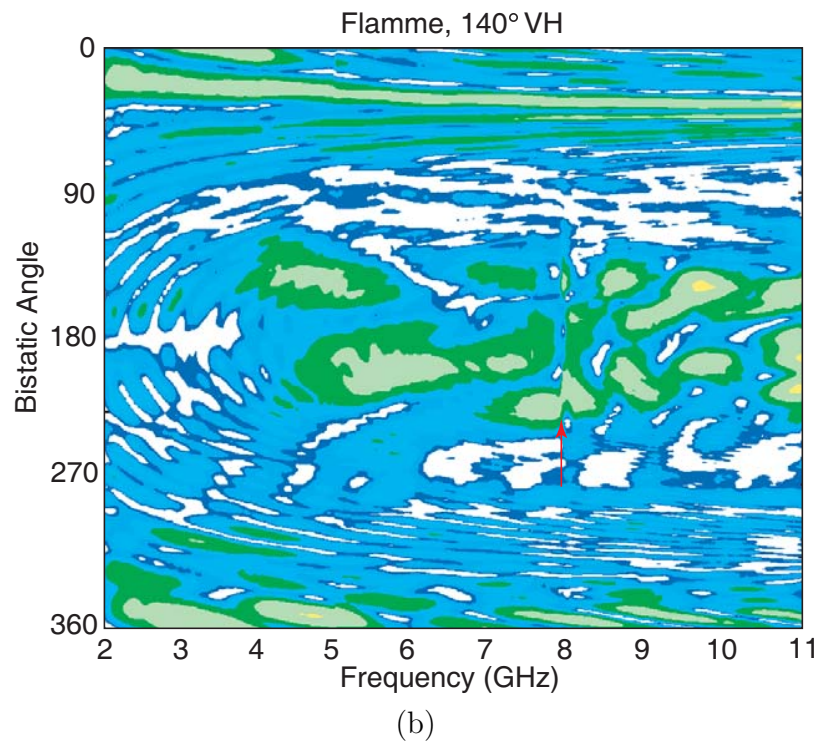
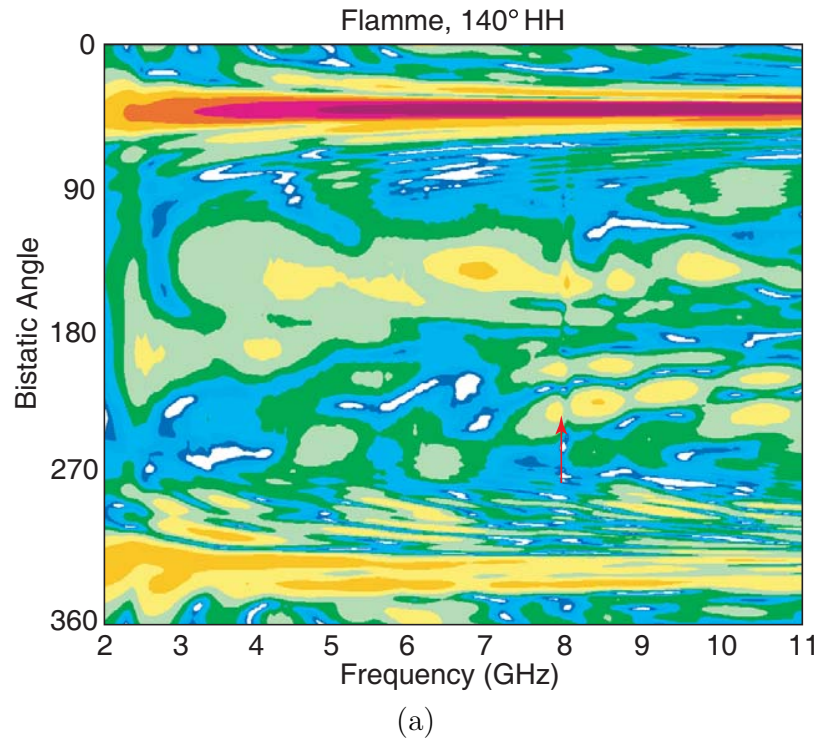
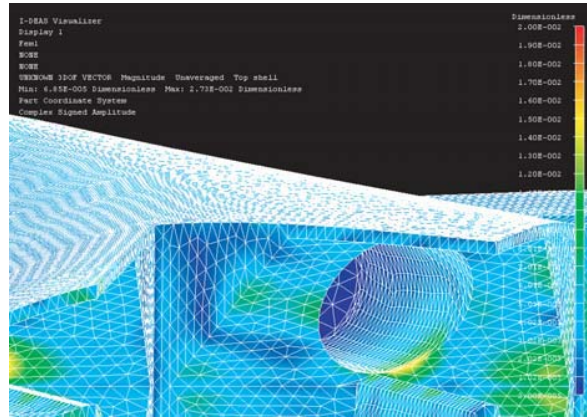


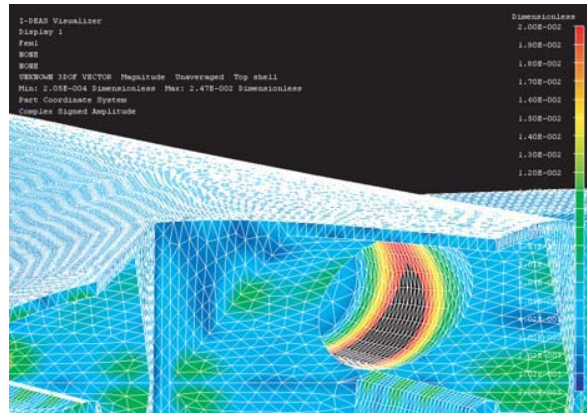
Figure 6.8: (a) Co-polar and (b) cross-polar bistatic RCS of the Flamme from 2 GHz to 11 GHz. The target is illuminated by a plane wave propagating at 140° angle from the nose with the electric field polarized in the vertical direction.

Flamme, 140°V, 7.5 GHz



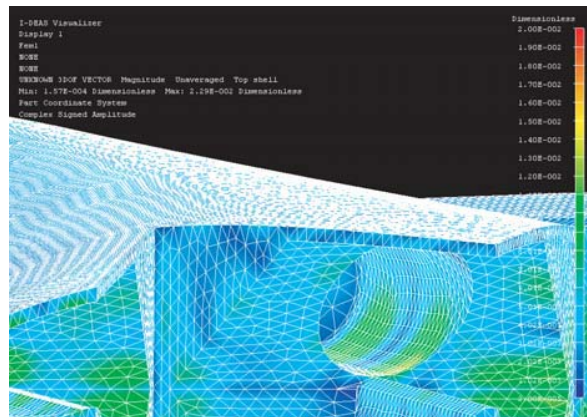
(a)

Flamme, 140°V, 7.9 GHz



(b)

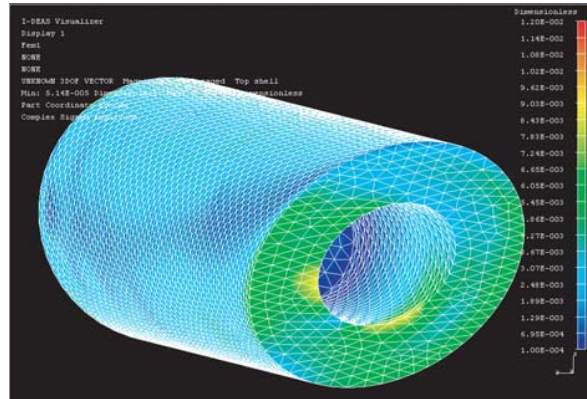
Flamme, 140°V, 8.4 GHz



(c)

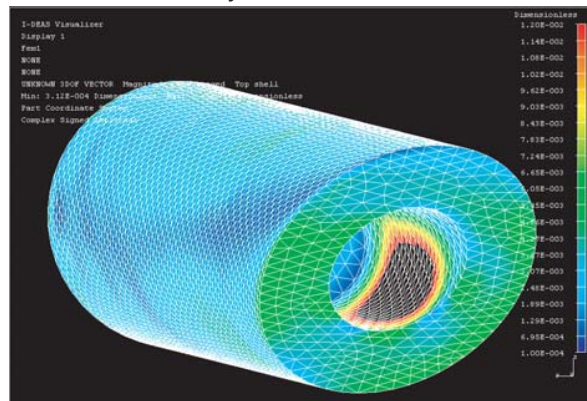
Figure 6.9: Induced electric current on the surface of the Flamme at (a) 7.5 GHz, (b) 7.9 GHz, and (c) 8.4 GHz. The target is illuminated by a plane wave propagating at 140° angle from the nose with the electric field polarized in the vertical direction.

Cavity, 140°V, 7.5 GHz



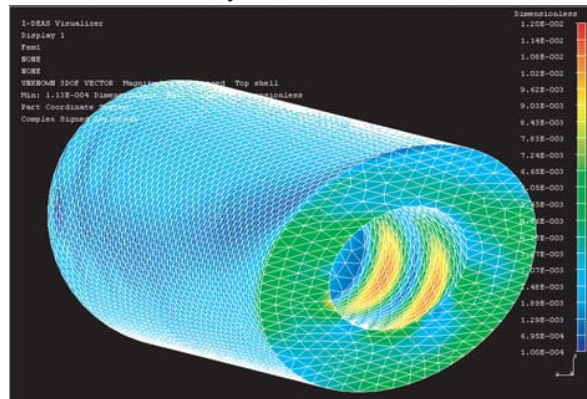
(a)

Cavity, 140°V, 7.9 GHz



(b)

Cavity, 140°V, 8.4 GHz



(c)

Figure 6.10: Induced electric current on the surface of the cavity of the Flamme at (a) 7.5 GHz, (b) 7.9 GHz, and (c) 8.4 GHz. The cavity is isolated and illuminated by a plane wave similar to the Flamme.

6.3 External Resonances (Flamme)

External resonances can be encountered when a scatterer involves a cavity and the cavity is excited appropriately. Those physical resonances lead to rapid changes in scattering characteristics of the target, which can be visible when electromagnetic fields are investigated as a function of frequency. For example, Figure 6.8 presents the bistatic RCS of the Flamme from 2 GHz to 11 GHz when the target is illuminated by a plane wave propagating at 140° angle from the nose. The plane wave is vertically polarized and both co-polar (VV) and cross-polar (VH) RCS is considered. We observe that RCS values change rapidly around 7.9 GHz due to the resonance of the cavity located at the back of the Flamme.

To further investigate external resonances of the Flamme, Figure 6.9 presents the electric current density induced on the surface of the target at three different frequencies, i.e., 7.5 GHz, 7.9 GHz, and 8.4 GHz. Comparing Figures 6.9(a) and 6.9(b), we observe that the value of the current density inside the cavity at the resonance frequency (7.9 GHz) is significantly larger than the value of the current density at the ordinary frequency (7.5 GHz). As depicted in Figure 6.9(c), another external resonance occurs at 8.4 GHz. However, at this frequency, the value of the current density inside the cavity does not increase significantly and the resonance is not visible in RCS results depicted in Figure 6.8.

Finally, in order to verify that the cavity of the Flamme resonates at 7.9 GHz and 8.4 GHz, Figure 6.10 presents the solution of scattering problems involving the cavity alone. The cavity is isolated from the rest of the target and illuminated by the same plane wave at 7.5 GHz, 7.9 GHz, and 8.4 GHz. We observe that the value of the current density inside the cavity becomes large at 7.9 GHz and 8.4 GHz due to resonances.

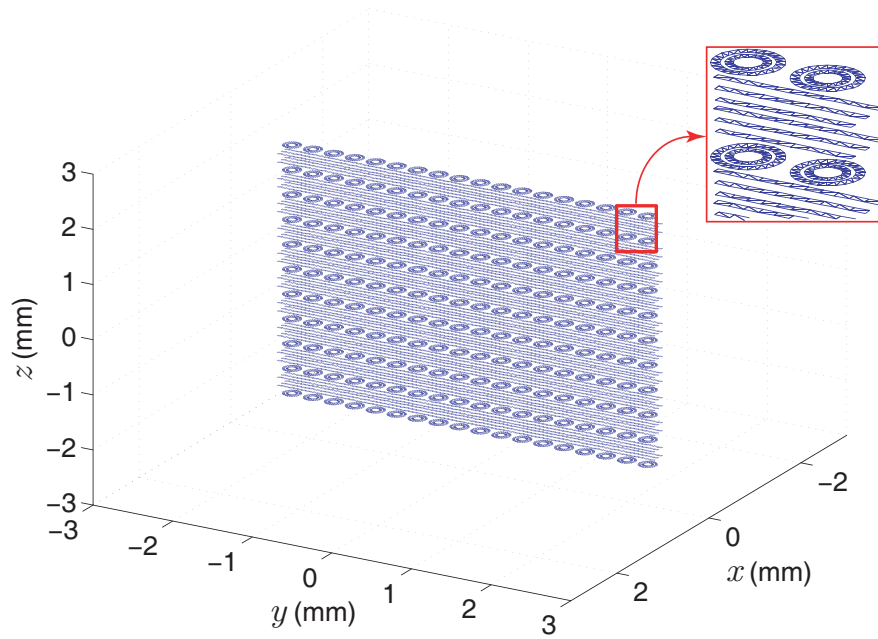


Figure 6.11: 1-layer CMM wall involving 18×11 SRRs combined with TWs.

6.4 Metamaterials

Metamaterials are artificial structures that are constructed by periodically arranging unit cells, such as SRRs and TWs. Due to the resonant nature of those cells, electromagnetic properties of the host medium, i.e., permittivity, permeability, or both, can effectively become negative for some frequencies, which can be useful in diverse applications, such as sub-wavelength focusing, cloaking, and designing improved antennas. At a resonance frequency, even a single SRR is able to block the transmission of waves, which can be observed as a shadowing effect. This unusual behavior can be explained with the induced negative effective permeability in the medium. Due to the complex effective wavenumber, the propagating power decays rapidly in the transmission region behind the SRR. It is well known that periodical arrangements of TWs can also be used to block electromagnetic waves, if the spacing between TWs is small compared to the wavelength and if TWs are aligned parallel to the electric field of the incident

wave. As opposed to SRRs, however, the shadowing effect can be obtained with TWs over a wide range of frequencies. In the context of metamaterials, the shielding property of TWs can be explained as the induction of negative effective permittivity in the medium.

Figure 6.11 depicts a 1-layer composite metamaterial (CMM) wall constructed by combining the 18×11 SRR array in Figure 4.11 with TWs. Dimensions of TWs are compatible with dimensions of SRRs, and they exhibit negative effective permittivity in a wide range of frequencies, including the resonance frequency of SRRs, i.e., 100 GHz. By arranging SRRs and TWs in the same medium, both effective permittivity and effective permeability can be simultaneously negative for some frequencies. When SRRs do not resonate, CMM structures are opaque due to the negative effective permittivity induced by TWs. At resonance frequencies of SRRs, however, CMMs are unexpectedly transparent, which can be explained by the induced double negativity [9].

Figure 6.12 presents the solution of metamaterial problems involving 1-layer SRR and CMM walls depicted in Figures 4.11 and 6.11, respectively. Both walls are embedded into a homogeneous host medium with a relative permittivity of 4.8 and located at $x = 0$. A y -directed Hertzian dipole is radiating from $x = 1.2$ mm. Problems are formulated with EFIE, discretized with the RWG functions, and solved iteratively using MLFMA. We compute the total electric and magnetic fields ($\mathbf{E}^{tot}$ and $\mathbf{H}^{tot}$) at various points on the $z = 0$ plane. Then, the power transmission at an observation point $\mathbf{r} = (x, y, 0)$ is defined as

$$\mathcal{T}(\mathbf{r}) = \frac{|\mathbf{P}_{av}^{tot}(\mathbf{r})|}{|\mathbf{P}_{av}^{inc}(\mathbf{r})|}, \quad (6.19)$$

where $\mathbf{P}_{av}^{tot}$ and $\mathbf{P}_{av}^{inc}$ represent total and incident time-average power densities, i.e.,

$$\mathbf{P}_{av}^{tot} = \frac{1}{2} \text{Re} \{ \mathbf{E}^{tot}(\mathbf{r}) \times [\mathbf{H}^{tot}(\mathbf{r})]^* \} \quad (6.20)$$

$$\mathbf{P}_{av}^{inc} = \frac{1}{2} \text{Re} \{ \mathbf{E}^{inc}(\mathbf{r}) \times [\mathbf{H}^{inc}(\mathbf{r})]^* \}. \quad (6.21)$$

In Figure 6.12, we observe that the power transmission to the left of the SRR wall is almost unity (0 dB) at 95 GHz. Then, at 100 GHz, the transmission drops drastically, and a shadowing effect is observed. At this frequency, the SRR wall is opaque, due to the negative effective permeability stimulated in the medium. The wall becomes transparent again at 105 GHz, although the transmitted power is less than 0 dB. The CMM structure constructed by combining SRRs and TWs is opaque at 95 GHz. At this frequency, the transmission property of the CMM is dominated by the negative effective permittivity dictated by TWs. At 100 GHz, however, the structure becomes transparent, and we observe a relatively high power transmission to the left of the array. This unusual behavior of the CMM is a result of double negativity; since SRRs resonate at 100 GHz, both effective permittivity and permeability of the medium become negative at this frequency. The transparency of the CMM wall tapers down at 105 GHz. We note that all computational results in Figure 6.12 agree well with the experimental results in the literature [9]–[13].

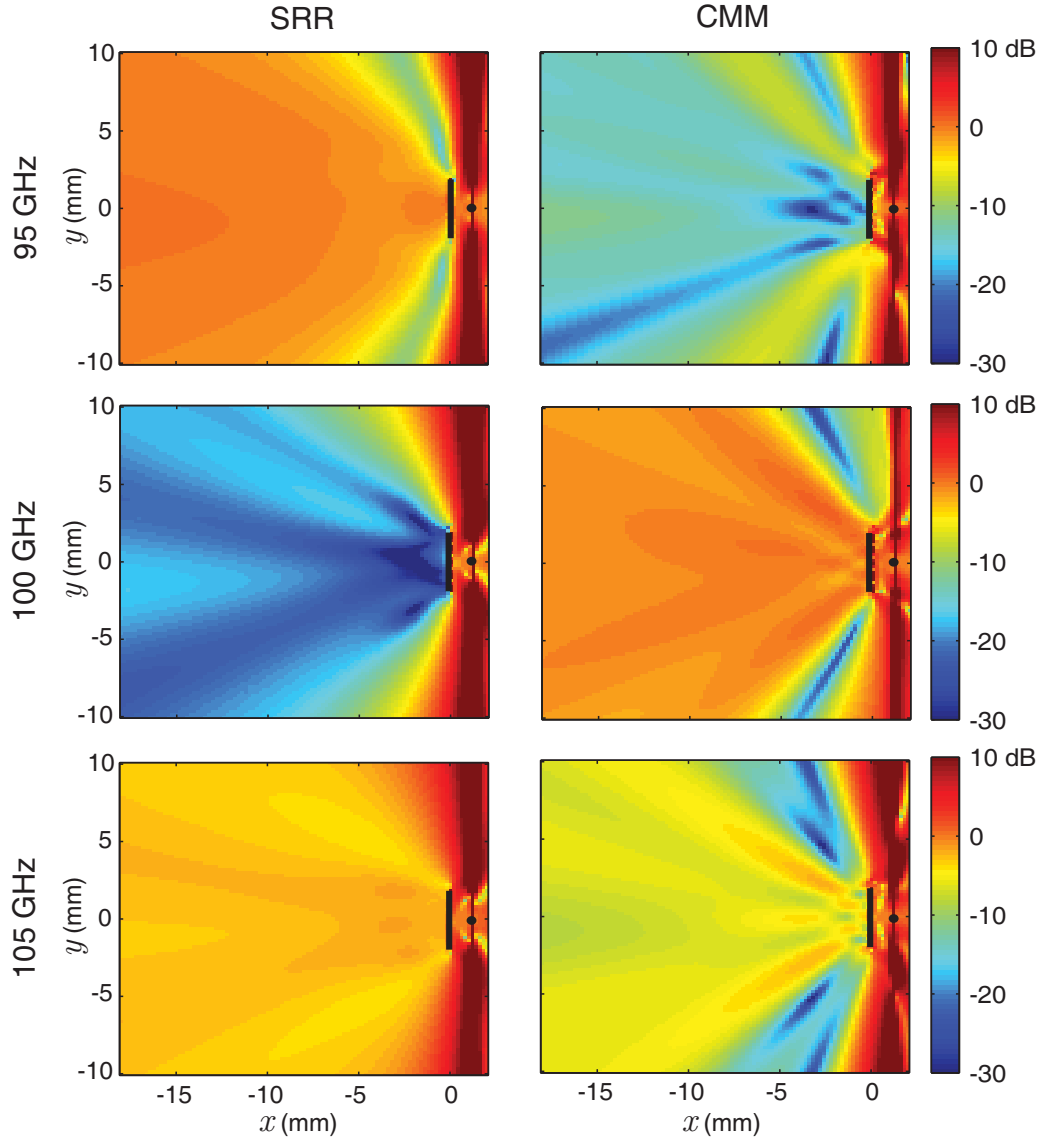


Figure 6.12: Power transmission on the $z = 0$ plane for 1-layer SRR and CMM walls at 95 GHz, 100 GHz, and 105 GHz. A y -directed Hertzian dipole is radiating from $x = 1.2$ mm.

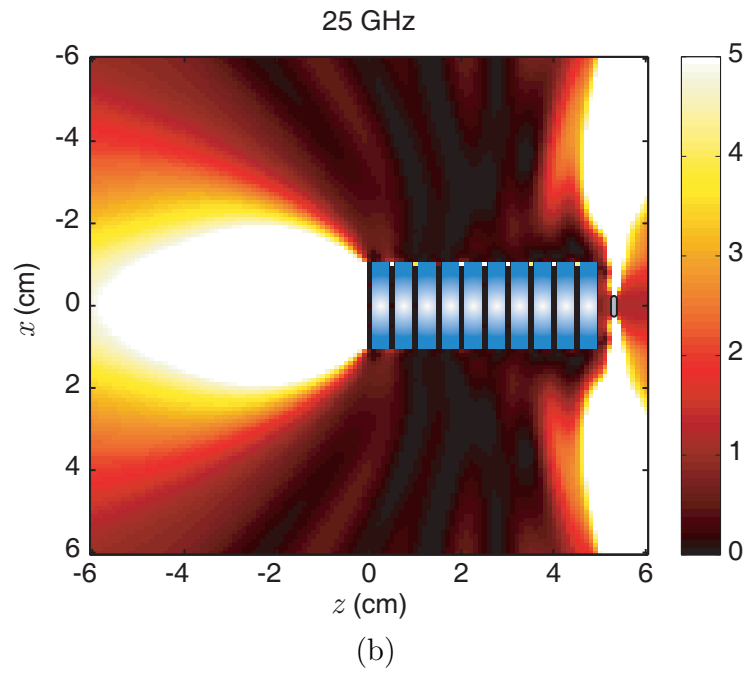
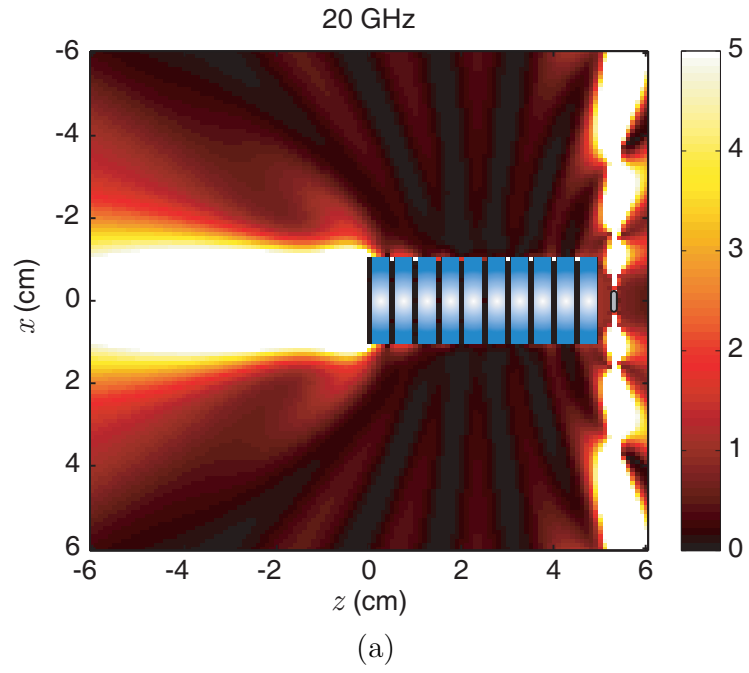
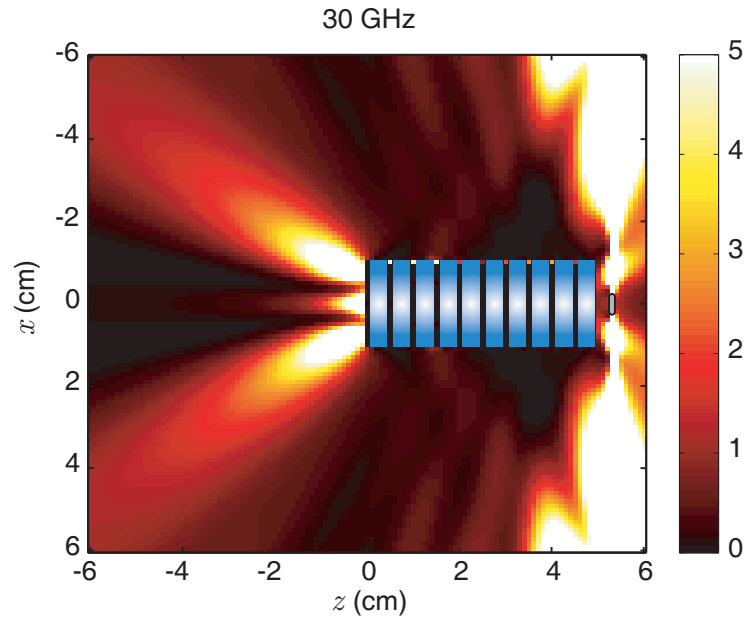
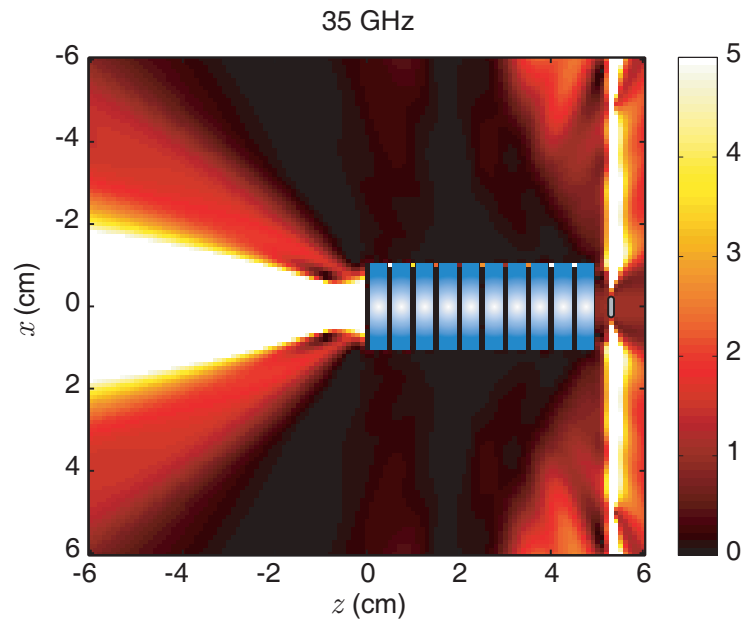


Figure 6.13: Power transmission on the $y = 0$ plane for a 10-layer dielectric structure illuminated by an x -directed Hertzian dipole at (a) 20 GHz and (b) 25 GHz.



(a)



(b)

Figure 6.14: Power transmission on the $y = 0$ plane for a 10-layer dielectric structure illuminated by an x -directed Hertzian dipole at (a) 30 GHz and (b) 35 GHz.

6.5 Periodic Dielectric Structures

Periodic dielectric structures are useful in many applications due to their frequency-selective electromagnetic characteristics. For example, Figures 6.13 and 6.14 present the solution of transmission problems involving a periodical arrangement of $0.41 \text{ cm} \times 2 \text{ cm} \times 2 \text{ cm}$ dielectric slabs with a relative permittivity of 4.8. The structure involving 10 identical slabs is illuminated by an x -directed Hertzian dipole at 20 GHz, 25 GHz, 30 GHz, and 35 GHz. Problems are formulated with JMCFIE and discretized with the RWG functions. Matrix equations involving 77,400 unknowns are solved iteratively via MLFMA. Iterative solutions are also accelerated by using the SAI preconditioner. The power transmission defined in (6.19) is computed at various locations around the structure on the z - x plane.

Figure 6.13 shows that the dielectric structure behaves like a waveguide at 25 GHz and 30 GHz. At those frequencies, power radiated by the Hertzian dipole is transmitted to the region on the LHS of the structure. At 35 GHz, however, the structure becomes opaque and does not transmit the power, as depicted in Figure 6.14(a). Since the structure is constructed by using finite-size slabs, shadowing is not observed for all locations in the transmission region. Specifically, we note high transmission values even at 35 GHz, especially around the edges of the structure. Nevertheless, the electromagnetic response of the structure at 35 GHz is completely different from its waveguide-like response at 20 GHz and 25 GHz. Finally, as depicted in Figure 6.14(b), the power transmission rises again when the frequency is increased to 35 GHz.

6.6 Log-Periodic Antennas and Arrays

LP antennas are important with their ability to display nearly frequency-independent characteristics over wide bands of frequency [2]–[4]. Unlike LP dipole arrays, which have well-known and time-tested theoretical design recipes, more complicated LP antennas require some degree of correction following their theoretical design according to log-periodic rules. In this section, nonplanar trapezoidal-tooth LP antennas and their circular arrays are considered. We demonstrate that accurate numerical simulations can bridge the gap between approximate designs of LP structures obtained with theoretical design recipes and more precise performance requirements.

Radiation problems involving LP structures are formulated with EFIE, discretized with the RWG functions, and solved iteratively using MLFMA. Although the number of unknowns is relatively low (1000–10,000) and those problems could be solved with MOM, acceleration provided by MLFMA is necessary since multiple solutions are required at several frequencies. In many cases, we use high sampling rate in frequency sweeps so that we can properly observe rapid oscillations in radiation patterns and directive gains of LP antennas and their array configurations. This leads to about 100 to 1000 solutions per problem, requiring an efficient algorithm, such as MLFMA, to reach solutions quickly without sacrificing accuracy.

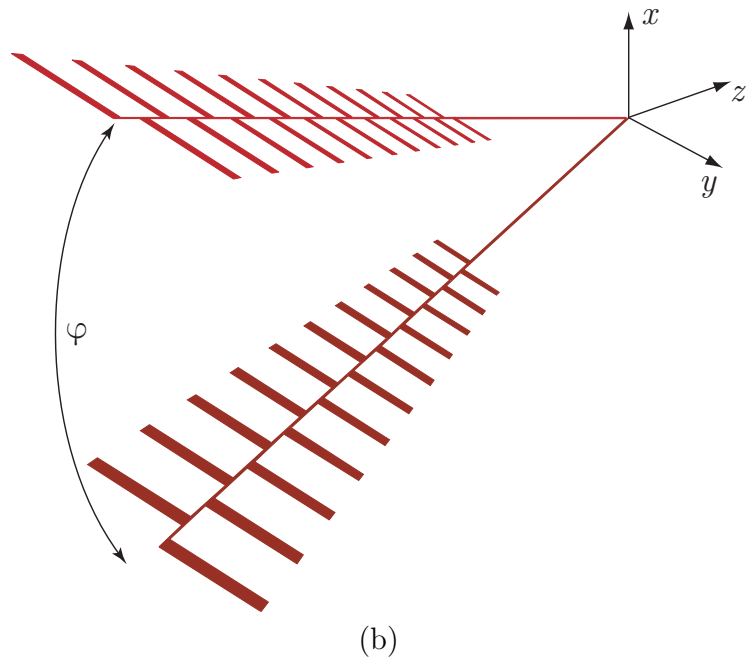
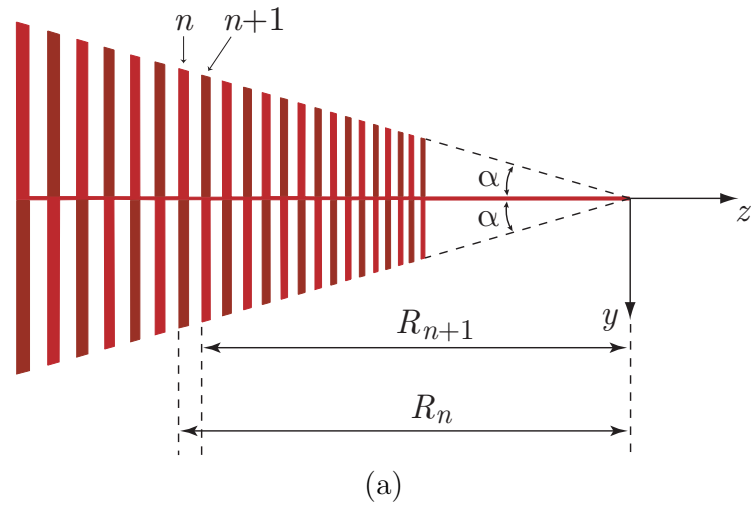


Figure 6.15: A nonplanar trapezoidal-tooth LP antenna design; (a) top view and (b) three-dimensional view.

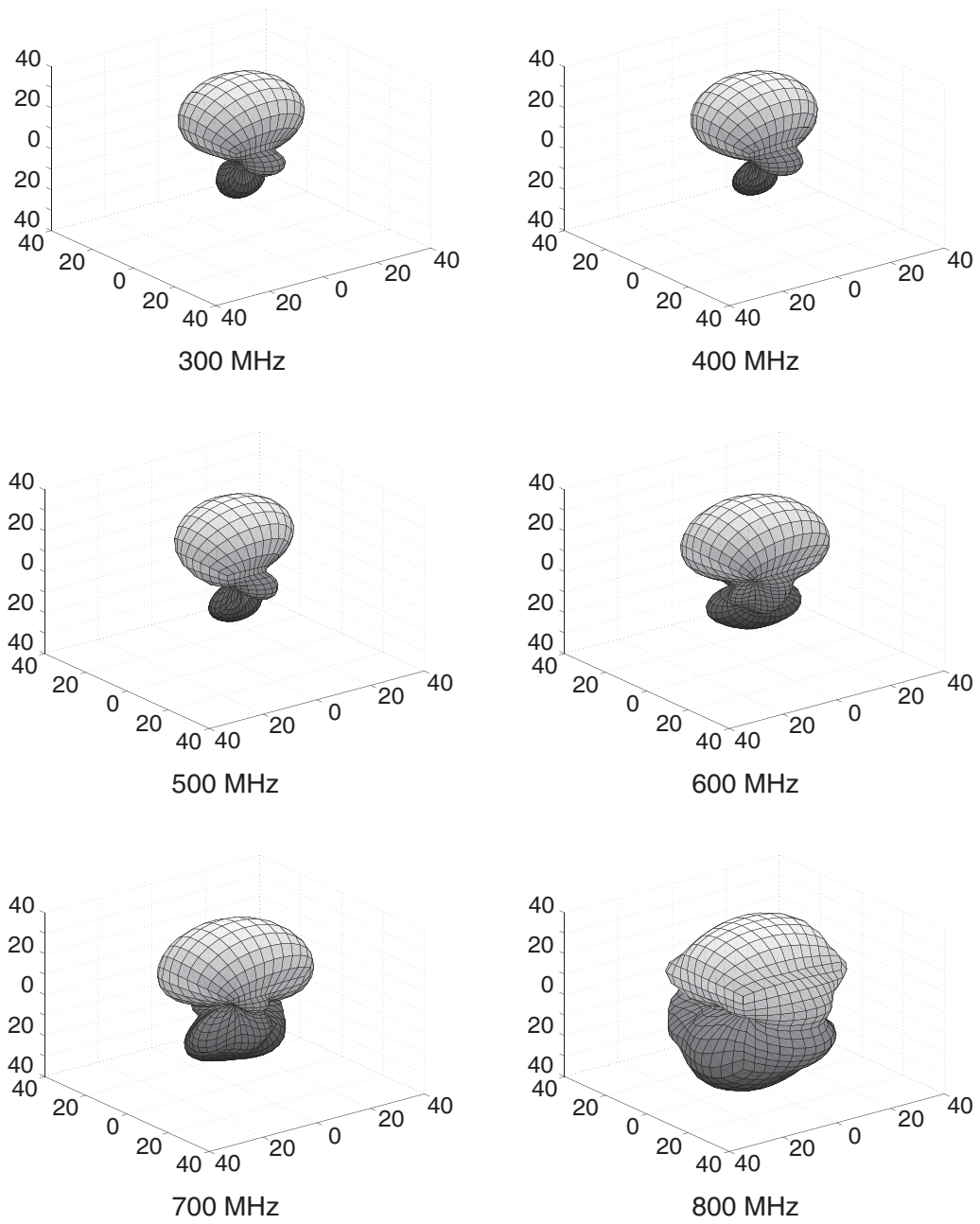


Figure 6.16: Far-zone radiation pattern of the LP antenna design shown in Figure 6.15 for various frequencies in the 300-800 MHz range. Normalized radiated power is presented in the logarithmic scale.

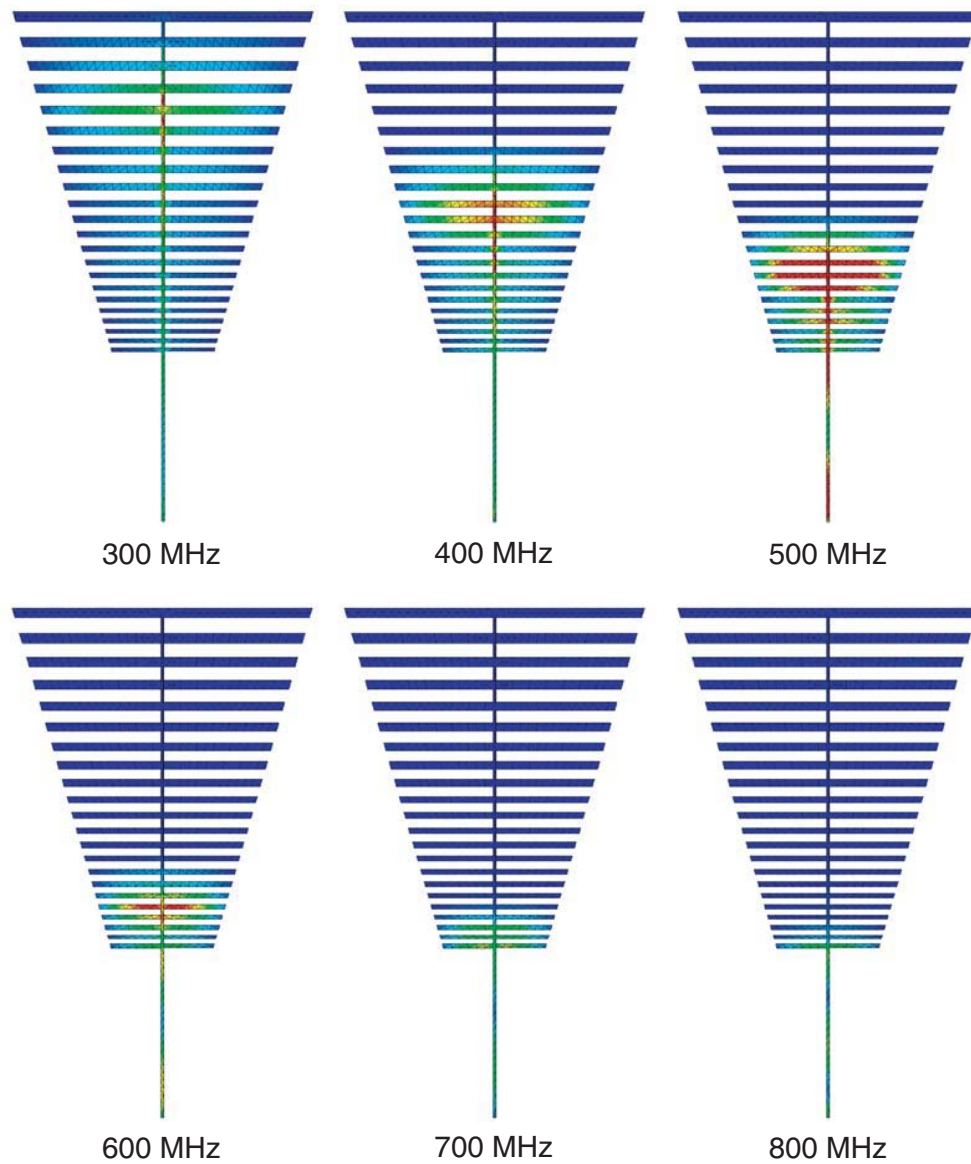


Figure 6.17: Current distribution on the LP antenna design shown in Figure 6.15 at different frequencies. Active region is seen to shift as a function of the frequency and spills out of the antenna at high frequencies.

6.6.1 Nonplanar Trapezoidal-Tooth Log-Periodic Antennas

In Figure 6.15, a nonplanar antenna design comprising two planar LP arms with narrow trapezoidal-tooth elements is shown with its apex located at the origin. On each arm, the antenna has 22 teeth, alternating from side to side of the constant-width feed-line, whose lengths, widths, and distances to the source location are determined according to log-periodic rules. The tooth sequence does not continue up to the apex location, where the source is located, establishing a toothless part of the feed-line to supply the electrical connection. The geometric growth factor

$$\tau = \frac{R_{n+1}}{R_n} \quad (6.22)$$

and the tooth-width ratio

$$\varepsilon = \frac{R_n - w_n}{R_n} \quad (6.23)$$

are selected as 0.95 and 0.98, respectively, where R_n is the distance between the source location and the n th tooth, whereas w_n is the width of the n th tooth. The angle α is 15° so that arms expand from the source location with an angle of 30° , and teeth are attached to center strips with constant widths. If the length of arms is kept constant, increasing or decreasing the angle α shifts the frequency range of operation to lower or higher frequencies, respectively. On the other hand, if it is desirable to keep the frequency range unchanged, then varying the angle α affects the length of the antenna arm.

For the LP antenna in Figure 6.15, the angle between the two arms (φ) is chosen as 45° , which affects the directive gain of the antenna. Specifically, higher values of φ reduce the width of the main beam, but increase the side-lobe level. Lower values of φ suppress side lobes, but increase the beamwidth. Values of φ in the 30° – 45° range are observed to provide a trade-off for high directive gain.

The smallest and the largest teeth of the antenna in Figure 6.15 are approximately 9 cm and 27 cm, respectively. A significant majority of the current on an LP antenna exists in the active region, which resides on teeth that are about quarter-wavelength long so that pairs of quasi-symmetric teeth form half-wavelength long radiating elements. We note that the location of the active region depends on the frequency, i.e., for low and high frequencies of the operation range, the active region resides on large and small elements, respectively, and partially spills out of the antenna. At intermediate frequencies, for which the active region is completely accommodated on the surface of the antenna, frequency independence is satisfied. In this case, the antenna behaves as if it has an infinite length, since the current out of the active region is negligible. In other words, truncation of the element sequence at both ends of the antenna does not make a difference compared to a theoretical infinite structure. On the other hand, as the active region reaches the end of the antenna and begins to overflow, the antenna becomes dysfunctional and the frequency independence collapses. If the small current outside the active region is ignored, the antenna in Figure 6.15 is expected to operate nearly frequency independent in the 300–800 MHz range. Nevertheless, this simple guess does not consider the overflow of the active region due to its finite but unknown width, which could easily be included in the analysis by using the available information if the design was an LP dipole array [2]. For the design under consideration, this valuable information can only be obtained through numerical simulations.

In the operation frequency range, variation in radiation characteristics with respect to the frequency is directly related to the density of elements on an LP antenna. If τ is close to unity, which means that elements are closely spaced, the active region moves on the antenna smoothly. As a consequence, variation in the radiation characteristics becomes small in the interval $[f_i, f_{i+1}]$, where $f_i = \tau f_{i+1}$. On the other hand, for lower values of τ , frequency independence deteriorates

due to the large separation between elements. We note that choosing a small τ value makes it easier to construct the antenna with fewer elements for the same frequency range of operation.

Figure 6.16 shows normalized plots of the three-dimensional far-zone radiation pattern of the LP antenna in Figure 6.15 at various frequencies in the 300–800 MHz range. The main beam points in the z direction, along the apex of the antenna, due to backfire radiation. For low frequencies, the radiation pattern seems to be nearly frequency independent. However, the antenna fails to maintain the same radiation pattern for frequencies above 500 MHz. In other words, the antenna does not operate frequency independently in the frequency band it is designed for. Since the radiation pattern of the antenna deteriorates towards the higher end of the frequency band, it is easy to guess that the trouble is associated with the smaller teeth of the antenna. Indeed, plots of the current distribution at various frequencies shown in Figure 6.17 demonstrate that the active region spills out of the antenna for higher frequencies. Both the observation and the diagnosis of this problem, which is not predicted by theoretical design rules, are performed in the simulation environment. The solution is also monitored by simulations: extra teeth are added one at a time towards the source end of the antenna until the frequency-independence property of the antenna is found satisfactory over the entire 300–800 GHz frequency range.

The corrected design with 2×38 teeth is shown in Figure 6.18, while radiation-pattern and current-distribution plots for this design are shown in Figures 6.19 and 6.20, respectively. Far-field plots in Figure 6.19 are nearly frequency independent in the desired frequency range. Plots of the current distribution in Figure 6.20 show that the active region is successfully located on the antenna for all frequencies in the desired band, as opposed to the high-frequency spillover demonstrated in Figure 6.17. Finally, for a frequency-independent antenna, the

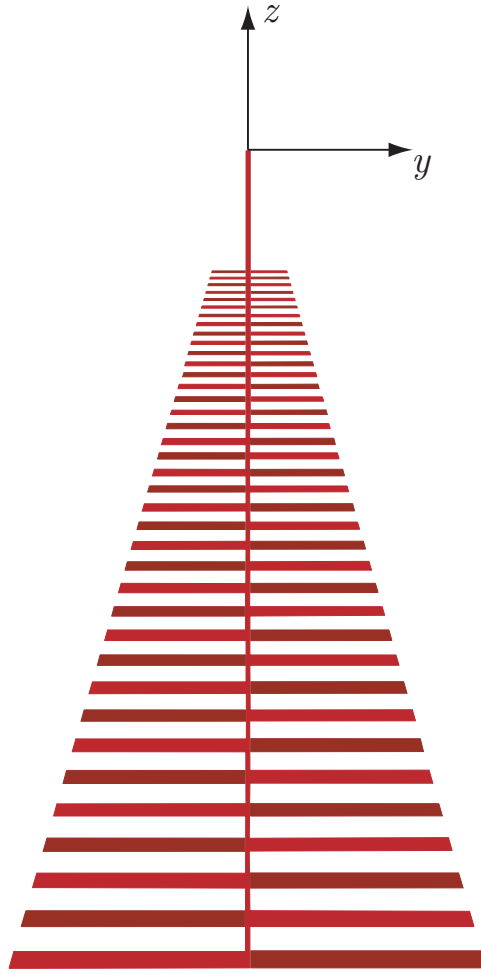


Figure 6.18: Top view of a nonplanar trapezoidal-tooth LP antenna design with 16 teeth added to each arm of the original antenna design shown in Figure 6.15.

directive gain is expected to remain relatively unchanged over the frequency band of interest. The directive-gain curves in Figure 6.21 clearly demonstrate the improvement in the operation of the modified antenna.

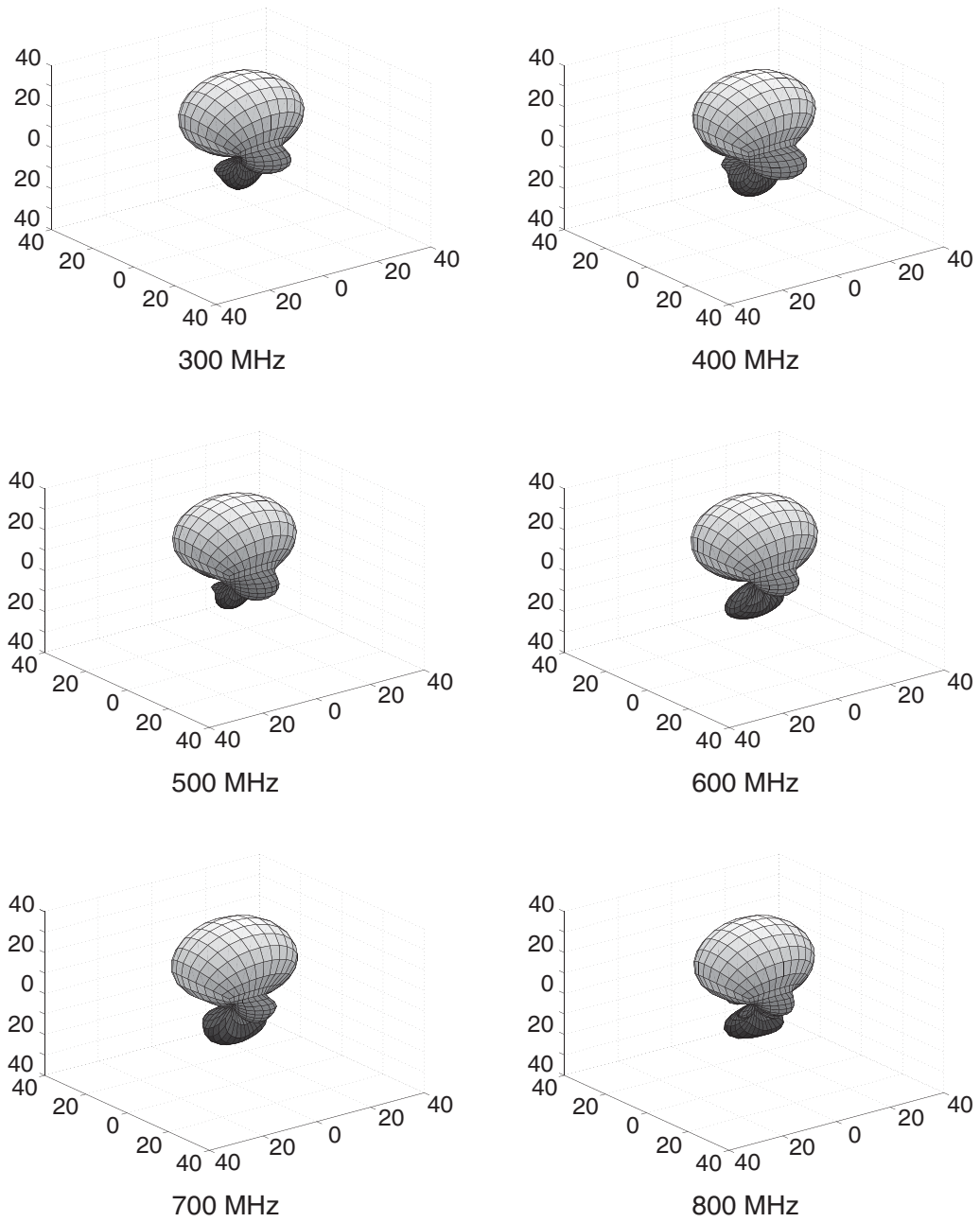


Figure 6.19: Far-zone radiation pattern of the corrected LP antenna design shown in Figure 6.18.

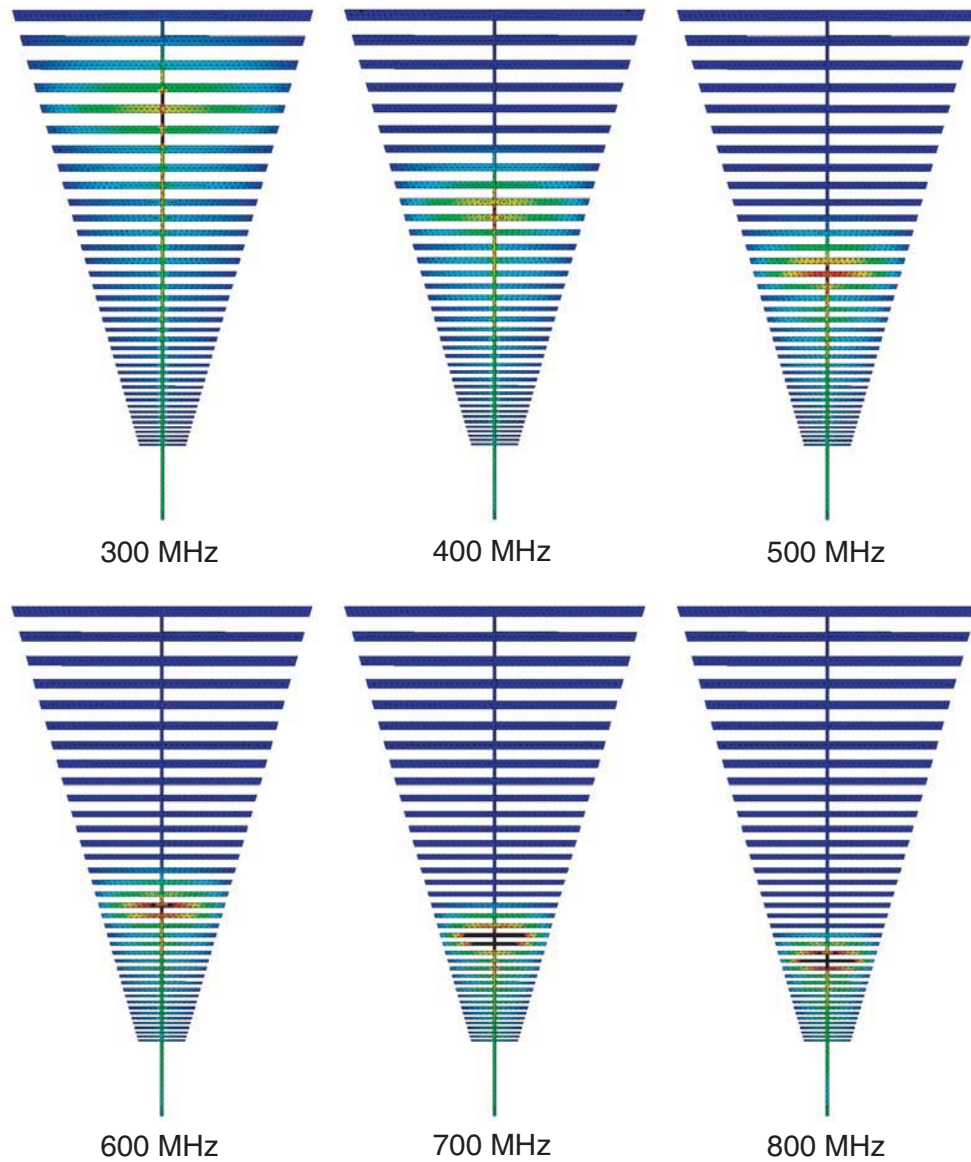


Figure 6.20: Current distribution on the corrected LP antenna design shown in Figure 6.18 at different frequencies. Active region is seen to be successfully contained on the antenna for all frequencies.

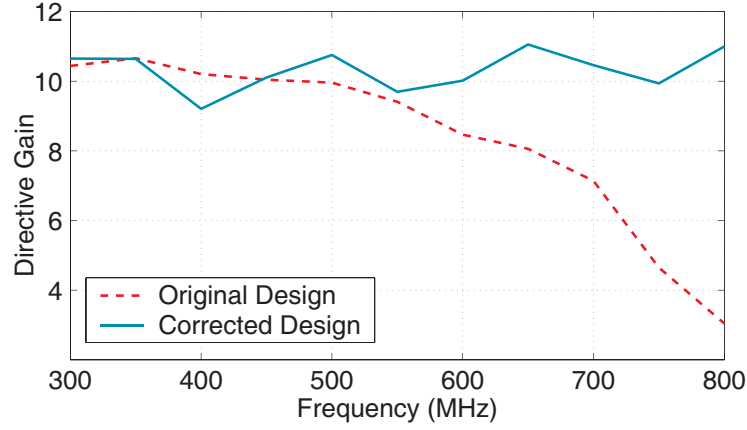


Figure 6.21: Directive gain in the z direction for the original and the corrected LP antenna designs shown in Figures 6.15 and 6.18, respectively.

6.6.2 Circular Arrays of Log-Periodic Antennas

We employ LP antennas to construct circular arrays with regularly-spaced elements, as depicted in Figure 6.22. To preserve frequency independence, the antennas are arranged in a circular form. Theoretically, the radiation intensity of a circular array of identical elements can be calculated as [2]

$$P_a^\infty(\theta, \phi) = f_{AF}(\theta, \phi) P_e^\infty(\theta, \phi), \quad (6.24)$$

where P_e^∞ and P_a^∞ represent radiation intensities of a single element and the array, respectively. This theoretical scenario is depicted in Figure 6.23, where the elements are represented by the dots on the circle. The array factor f_{AF} in (6.24) can be calculated as

$$f_{AF}(\theta, \phi) = \sum_{p=1}^P I_p \exp(-ika \sin \theta \cos(\phi - \phi_p)), \quad (6.25)$$

where P is the number of antennas, k is the wavenumber, a is the radius of the circle, I_p is the complex excitation coefficient of the p th element, and ϕ_p is the angle between the p th element and the x axis. At this stage, we assume that dots in Figure 6.23 represent the center of active regions located on LP antennas. Since the active region appears on those elements that are about a

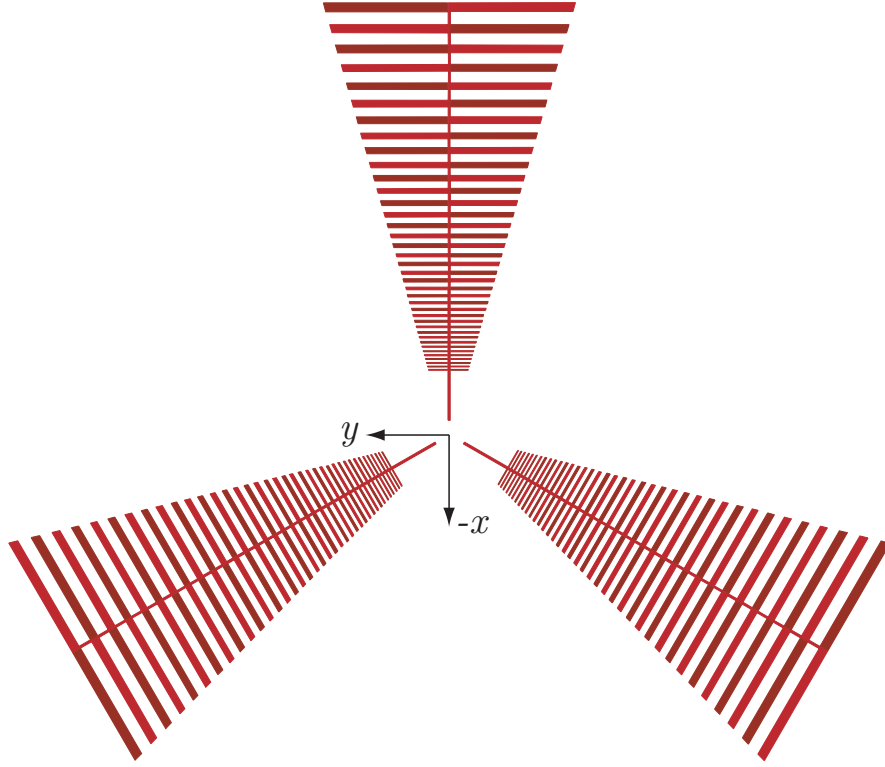


Figure 6.22: Circular array of LP antennas constructed by employing three identical trapezoidal-tooth antennas depicted in Figure 6.18. Antennas are regularly spaced.

quarter-wavelength long, the circle in Figure 6.23 has a radius that is inversely proportional to k . Consequently, f_{AF} becomes independent of frequency with the elimination of the k factor in the exponential term of (6.25). Finally, if f_e is almost frequency-independent, we conclude that the radiation intensity of the array f_a should also be independent of frequency to the same degree.

Despite the simple discussion above, simulations show that we cannot ignore mutual couplings between LP antennas that are located close together, as in Figure 6.22. To prove this, we compare radiations from the single antenna in Figure 6.18 with the array in Figure 6.22, where only the antenna on the x axis is active. Considering (6.24) and (6.25) in this case, I_p is zero for the unexcited elements, and the radiation of the array should be the same as the radiation of

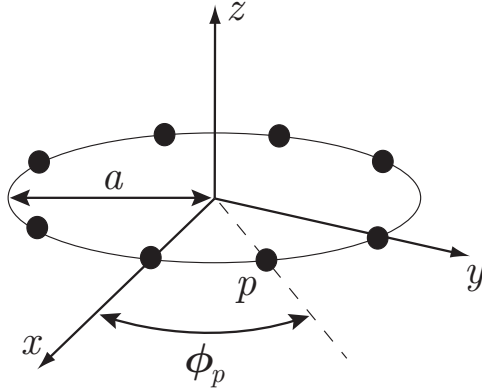
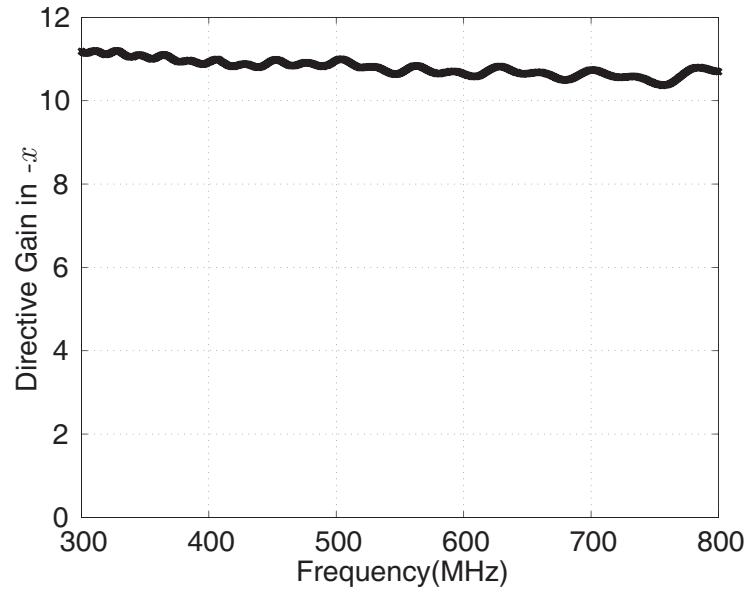


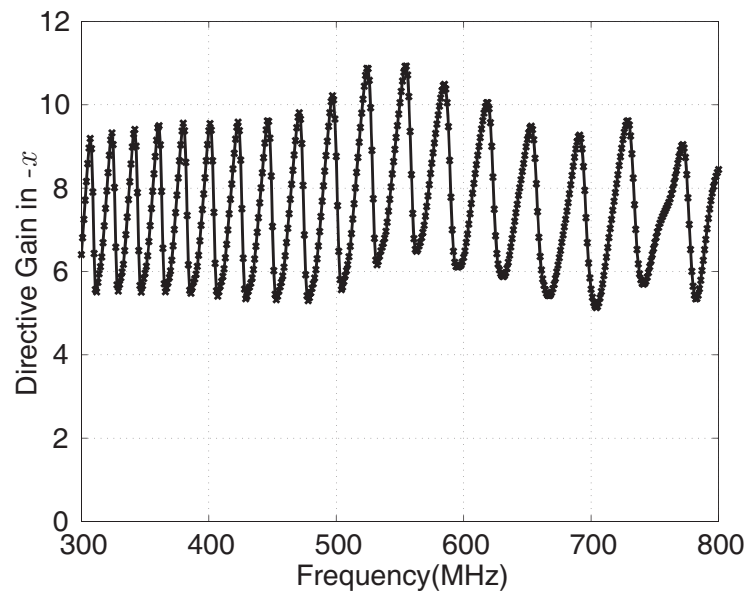
Figure 6.23: Configuration for a circular array, where array elements are represented by dots on a circle.

the single antenna. On the other hand, in Figure 6.24, we plotted the directive gain in the $-x$ direction with respect to frequency, and it is completely different for the array and the single antenna. Due to mutual couplings, unexcited antennas in the array significantly modify the total radiation. The directive gain of the array oscillates much more than the single element, demonstrating a drop in the quality of frequency independence.

We have confirmed that oscillations in Figure 6.24(b) are directly related to the geometric growth factor (τ) of the LP array. Let f_i and f_{i+1} be two consecutive frequencies, at which the directive gain makes a peak. Then, $f_i/f_{i+1} \approx \tau$, which is equal to 0.95 in our case. Figures 6.25(a) and (b) show the directive gain when τ becomes 0.98 and 0.85, respectively. It is clearly observed that the oscillation rate depends on τ . In fact, this relation also exists in the case of the single antenna in Figure 6.24(a). However, the active region on the single antenna moves so smoothly that radiation characteristics change little in intervals $[f_i, f_{i+1}]$ and the oscillation is less visible. When we examine array configurations, we note that mutual couplings between antennas disturb the localization of the active region, and we see a large effect on radiation characteristics.

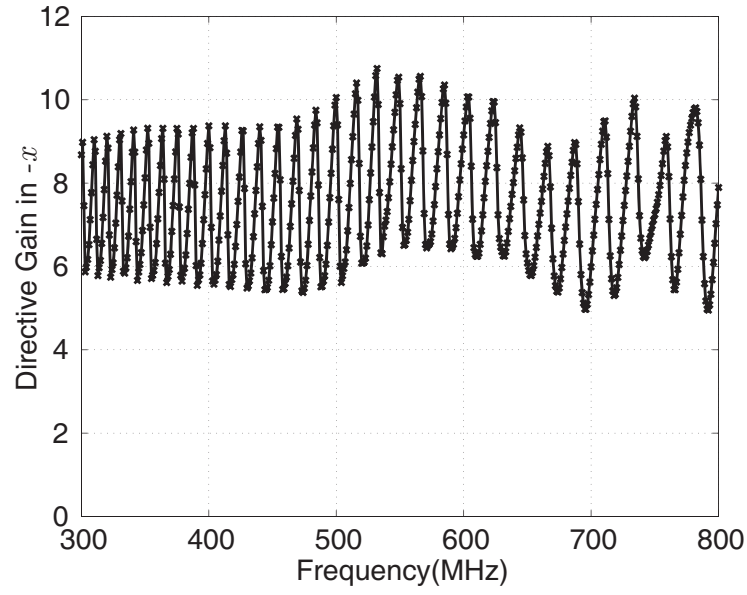


(a)

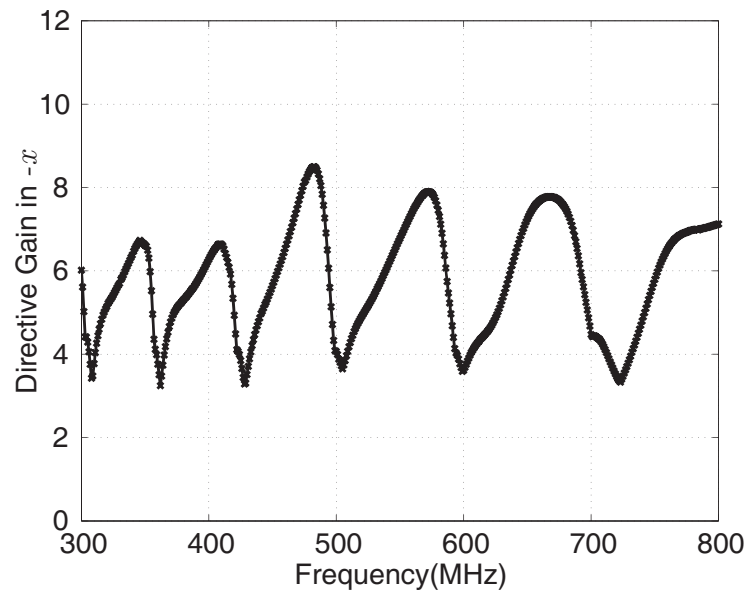


(b)

Figure 6.24: Directive gain in the $-x$ direction sampled with 1 MHz resolution for (a) the trapezoidal-tooth LP antenna in Figure 6.18 and (b) the circular array in Figure 6.22 with only the antenna on the x axis active.



(a)



(b)

Figure 6.25: Directive gain in the $-x$ direction sampled with 1 MHz resolution for the configuration in Figure 6.22 with only the antenna on the x axis active for (a) $\tau = 0.98$ and (b) $\tau = 0.85$.

As discussed in Section 6.6.1, frequency independence of an LP structure can be controlled by the geometric growth factor τ . However, as depicted in Figures 6.24 and 6.25, it becomes increasingly difficult to control the variation in radiation characteristics as multiple LP antennas are coalesced in an array configuration. In other words, it might be impossible to obtain the desired level of frequency independence with the highest allowable τ , which must be lower than unity because of nonzero widths of elements. This is mainly due to the complicated mutual couplings among antennas. In order to improve frequency-independence properties of LP arrays, we optimize excitations of array elements via genetic algorithms (GAs).

GAs have been successfully employed in many computational-electromagnetics applications. They are especially useful when the optimization space is large and it is difficult to derive an analytical expression for the cost function of the optimization. We therefore employ GAs to improve the frequency independency of LP arrays. The optimization procedure can also be used to provide beam-steering ability to those arrays. Since it is desired to point the main beam in a specific direction, choosing the directive gain (6.12) as the cost function of the optimization is logical. Specifically, we employ GAs to optimize excitations of array elements that maximize the directive gain in some given direction (θ, ϕ) .

Consider the three-element LP array in Figure 6.22. GAs work on a pool of individuals (citizens), each of which represents a trial combination of optimization variables. An individual suggests values for antenna excitations $I_1 = A_1 \exp \varphi_1$, $I_2 = A_2 \exp \varphi_2$, and $I_3 = A_3 \exp \varphi_3$, where $A_{1,2,3}$ and $\varphi_{1,2,3}$ represent the amplitude and the phase, respectively. Without loss of generality, we take $\varphi_3 = 0$ and there are five variables to be optimized. Parameter values are selected from the optimization space formed by sampling $A_{1,2,3}$ and $\varphi_{1,2}$ in $[0,1]$ and $[0^\circ, 360^\circ]$ ranges, respectively. We observe that uniform samplings with intervals of 0.1 in

amplitude and 36° in phase are sufficient, leading to 10 samples for each variable. We employ a one-to-one map to convert values represented by each individual into a single lengthy word of binary numbers, called the chromosome. Each individual also has a degree of success, which is simply the value of the cost function of the optimization, i.e., the directive gain at the optimization angle. We define the successful individual as the set of excitations I_1 , I_2 , and I_3 , which results in high directive gain. In the beginning of the optimization, individuals are created randomly. The optimization is then continued as new generations are formed and the pool is modified progressively. There are three important operations to produce a new generation from the old one.

- Crossover: Two successful individuals are selected to exchange some bits of their chromosomes randomly and generate two new individuals called children. There are various crossover schemes to perform exchanges.
- Mutation: Some of binary numbers in chromosomes are modified randomly, i.e., 1 changes into 0 and 0 changes into 1.
- Elitism: One or two most successful individuals are preserved in the pool without any modification.

Heuristically, as new generations are produced and the pool evolves, the overall success of the population increases. In the extreme case, all individuals are the same with the highest possible success after a number of generations. Then, the optimization is completed and any individual gives optimal values via an inverse mapping from the chromosome to excitations. However, it usually is sufficient to interrupt iterations after a number of generations and select the most successful individual in the pool as the optimization result.

Prior to performing an optimization for the LP array, we carefully adjust parameters of GAs by examining results obtained at some frequencies. We especially

consider different values for the size of the pool, the mutation rate, and the number of generations. By checking final results and convergence characteristics of GAs, parameters are selected and fixed, i.e., the same set of parameters is used for all computations at different frequencies. For optimizations of LP arrays, we usually use pools with 20–30 individuals and keep the mutation rate for each digit of chromosomes at about 5%. We apply elitism only for the most successful individual. The limit for the number of generations is selected to be 50, i.e., we stop the iterations after the 50th generation. Therefore, the number of trials to complete the optimization at a single frequency is about 1000–1500. A brute-force approach to check all possible combinations of the five optimization variables with the same sampling would require 10^5 trials, which is 100 times larger than that of GAs. With numerous experiments, we have confirmed that genetic algorithms converge to optimized values that are very close to those found by the brute-force approach.

Although GAs reduce the number of trials significantly compared to a brute-force optimization, the calculation of the success for each individual has to be performed efficiently. We therefore employ a superposition technique. For any LP array, the number of MLFMA solutions can be kept as low as the number of elements in the array. In each solution, we feed only one antenna with the excitation strength of unity. The radiated field is then calculated and stored in the memory. For the rotationally-symmetric array in Figure 6.22, the number of MLFMA solutions per frequency is further reduced to one using the symmetry. We note that the solution by exciting an antenna in the array is different from the solution of an antenna alone, since the former includes mutual couplings between antennas. Whenever it is required to test a set of values for excitations, radiated fields are multiplied by corresponding coefficients and superposed to obtain the total radiation intensity and the directive gain of the whole array.

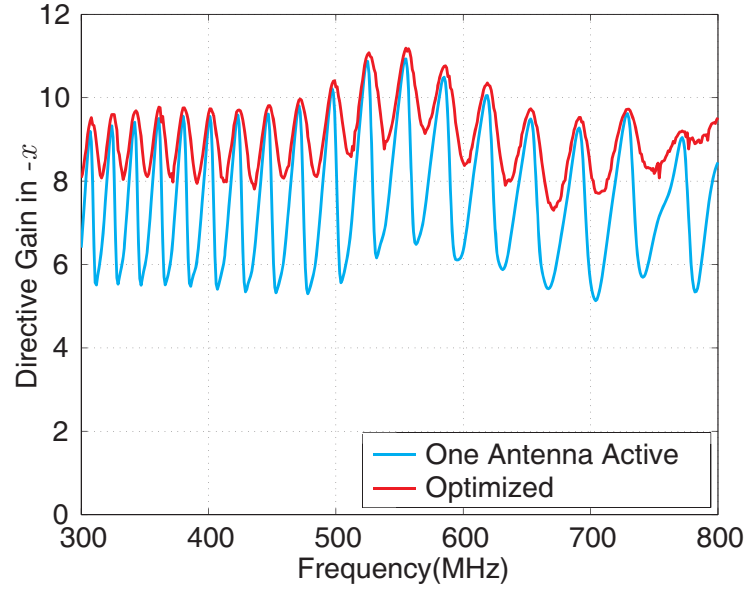


Figure 6.26: Directive gain in the $-x$ direction sampled with 1 MHz resolution for the array in Figure 6.22 when (a) only the antenna on the x axis is active, and (b) three antennas are active with optimized excitations.

Figure 6.26 presents the result of the optimization by GAs for the array in Figure 6.22. The directive gain in the $-x$ direction is plotted with respect to frequency for two cases; when only the antenna on the x axis is excited, and when all antennas are active with optimized excitations. For any frequency, optimization is expected to provide increased directive gain. This is in fact the case; however optimization is even more effective at those frequencies, where the directive gain is low. Therefore, the optimization by GAs reduces the variation in the directive gain with respect to frequency, and the array becomes more frequency-independent.

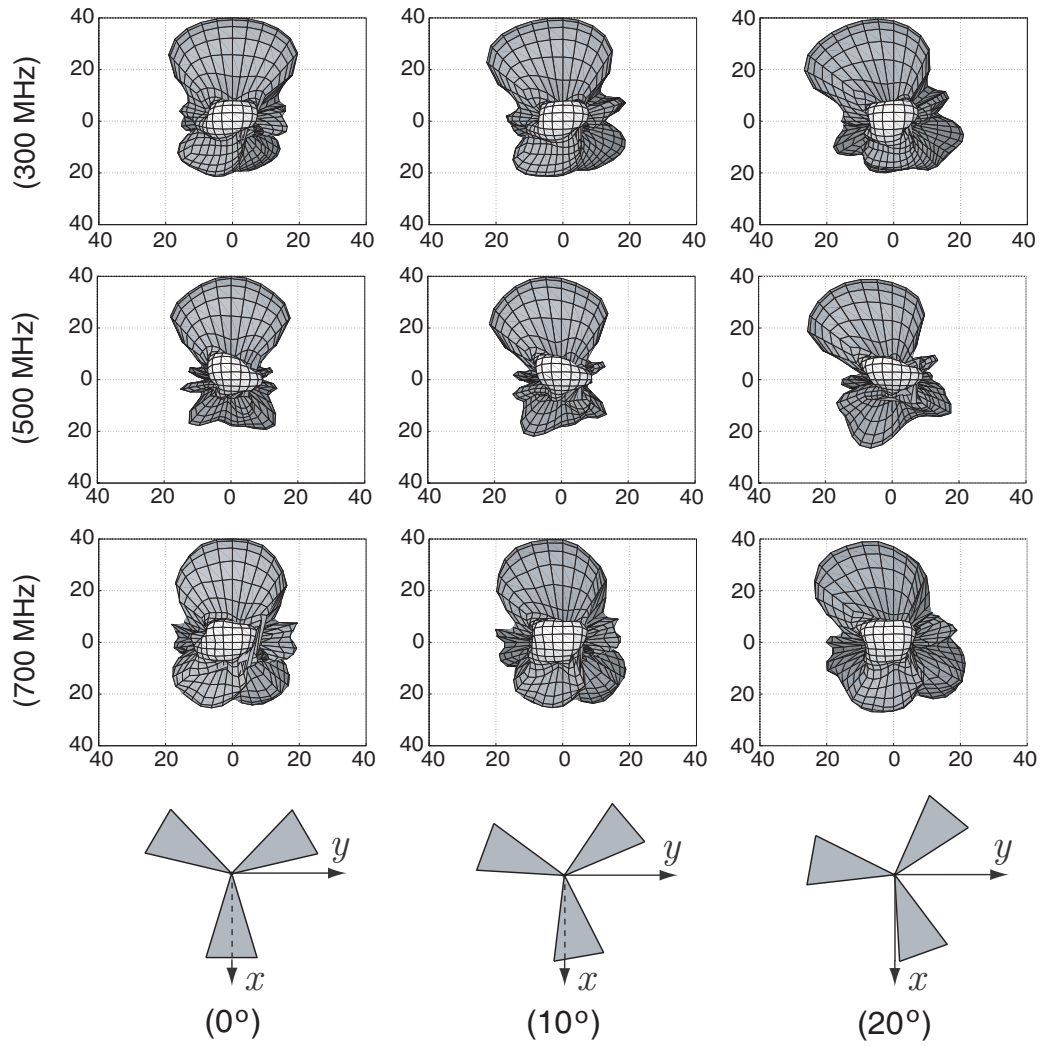


Figure 6.27: Far-zone radiation pattern of the array in Figure 6.22 for various frequencies and alignments. Directive gain is optimized in the $-x$ direction.

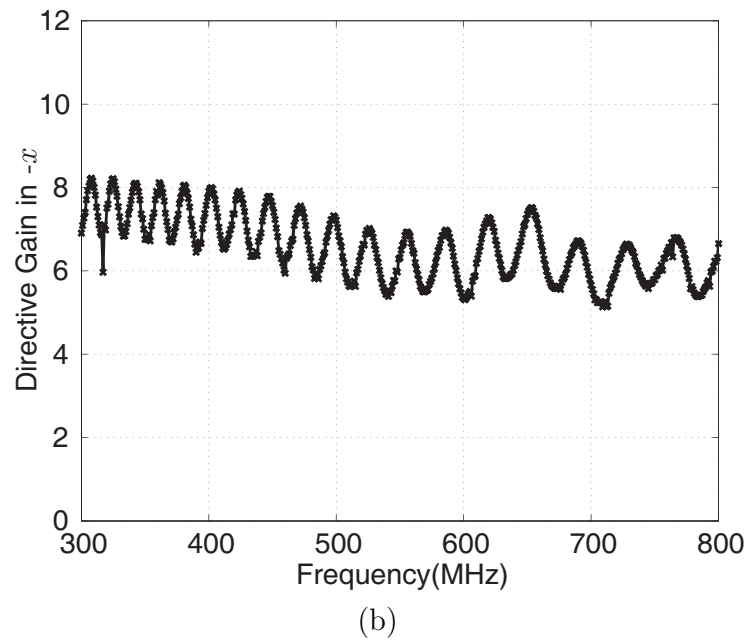
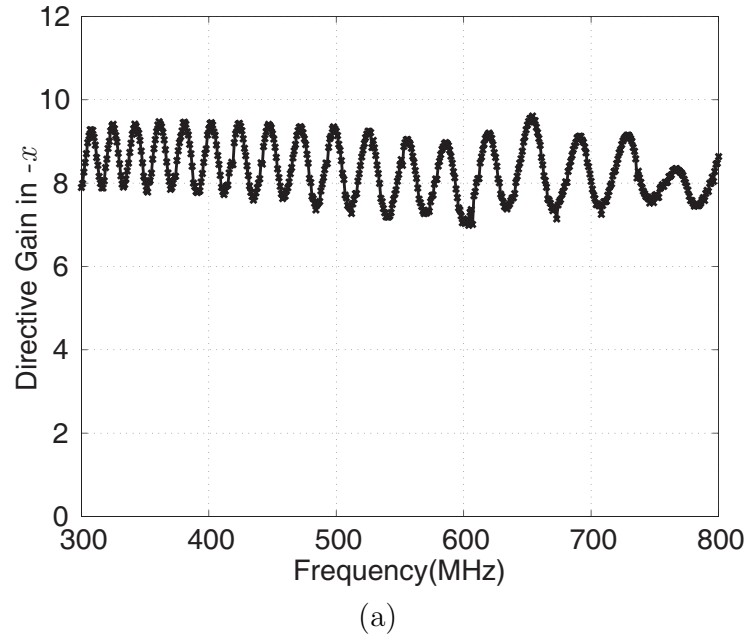


Figure 6.28: Optimized directive gain in the $-x$ direction sampled with 1 MHz resolution when the array in Figure 6.22 is rotated in the ϕ direction for (a) 10° and (b) 20° .

Since the array in Figure 6.22 is symmetric, i.e., its elements are regularly spaced, it already has some beam-steering ability. Considering the optimization of the directive gain in the $-x$ ($\phi = 180^\circ$) direction, the same directive gain can also be obtained at $\phi = 60^\circ$ and $\phi = 300^\circ$ by simply exchanging excitations among antennas. However, it is also desirable to steer the main beam in any direction within a sector between two antennas. This goal can again be achieved via optimization with genetic algorithms. Figures 6.27 and 6.28 show the results of the optimization. Since it is easier to rotate the antenna than to change the optimization direction, we use different configurations shown at the bottom of Figure 6.27; the optimization angle is fixed at $\phi = 180^\circ$ for all configurations. The array is rotated 10° and 20° in the ϕ direction to test the beam-steering ability in a sector of $2 \times 20^\circ = 40^\circ$. We confirmed that a rotation in the negative direction is not required; due to symmetry, results of an optimization in the $-\phi_0$ direction can be approximated from the optimization in the $+\phi_0$ direction. In Figure 6.28, we observe that the optimized directive gain in the $-x$ direction drops as the antenna is rotated; this is because the main beam is not totally controllable, and it is difficult to maintain the directive gain at high levels. This is also evident in Figure 6.27, where the far-zone radiation pattern of the array is plotted for some selected frequencies and for different alignments of the array. The maximum radiation cannot be kept in the $-x$ direction when the antenna is rotated 20° . Nevertheless, the directive gain is larger than five up to 20° . This means that the array in Figure 6.22 can provide a directive gain larger than five in three distinct sectors, each of which has an extent of 40° . We also note that this is valid for a wide frequency range from 300 MHz to 800 MHz.

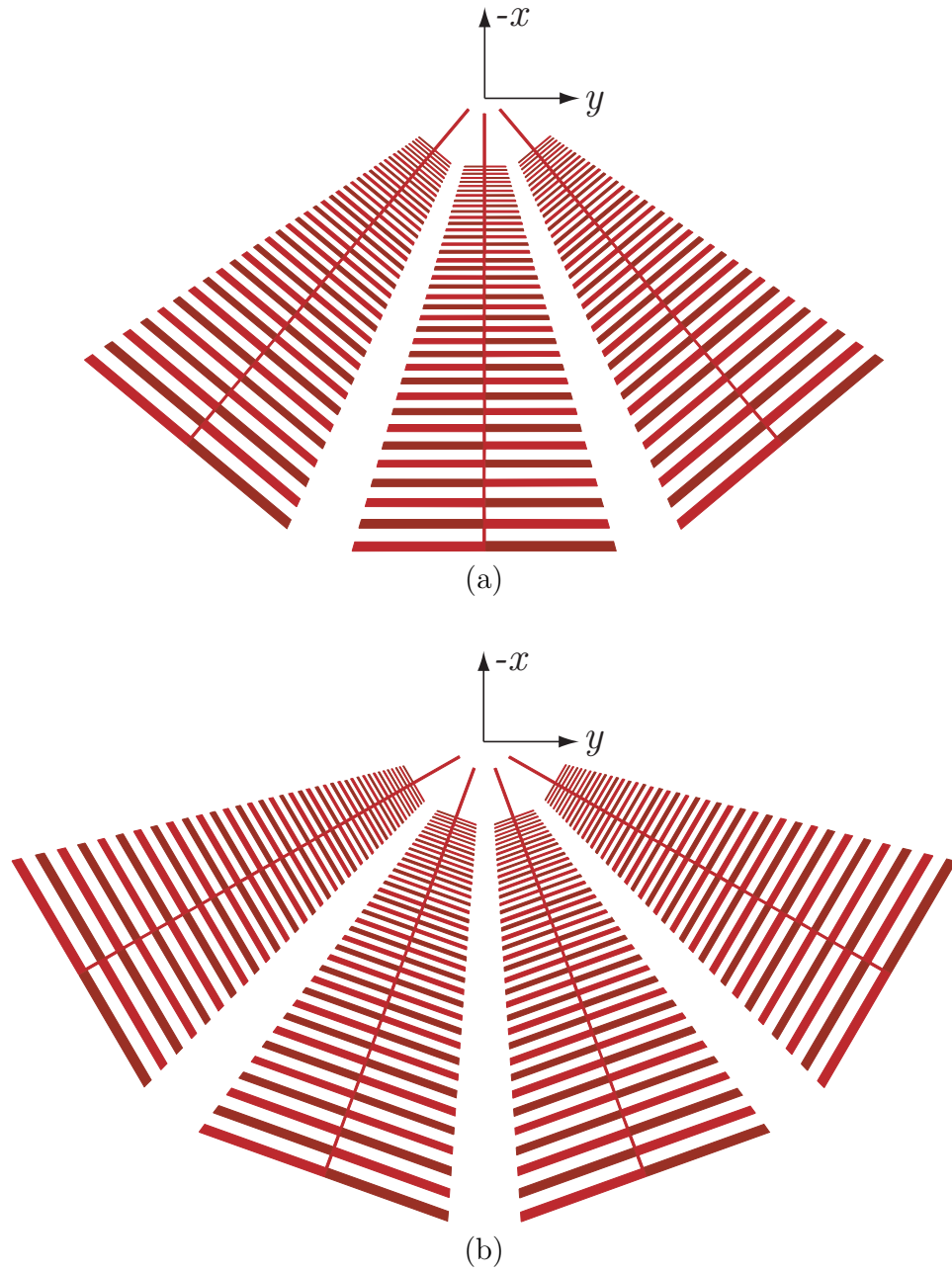


Figure 6.29: Circular-sectoral arrays of closely spaced LP antennas with (a) three elements and (b) four elements.

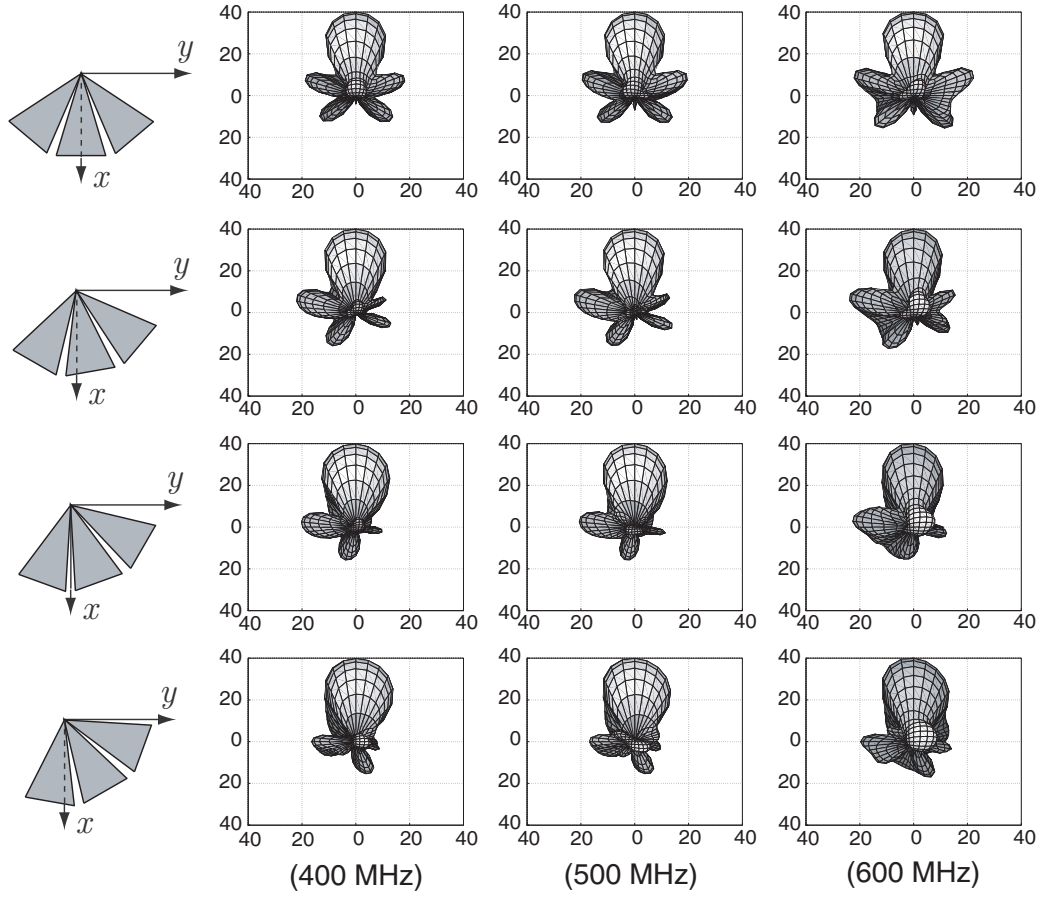


Figure 6.30: Far-zone radiation pattern of the array in Figure 6.29(a) for various frequencies and alignments. Directive gain is optimized in the $-x$ direction.

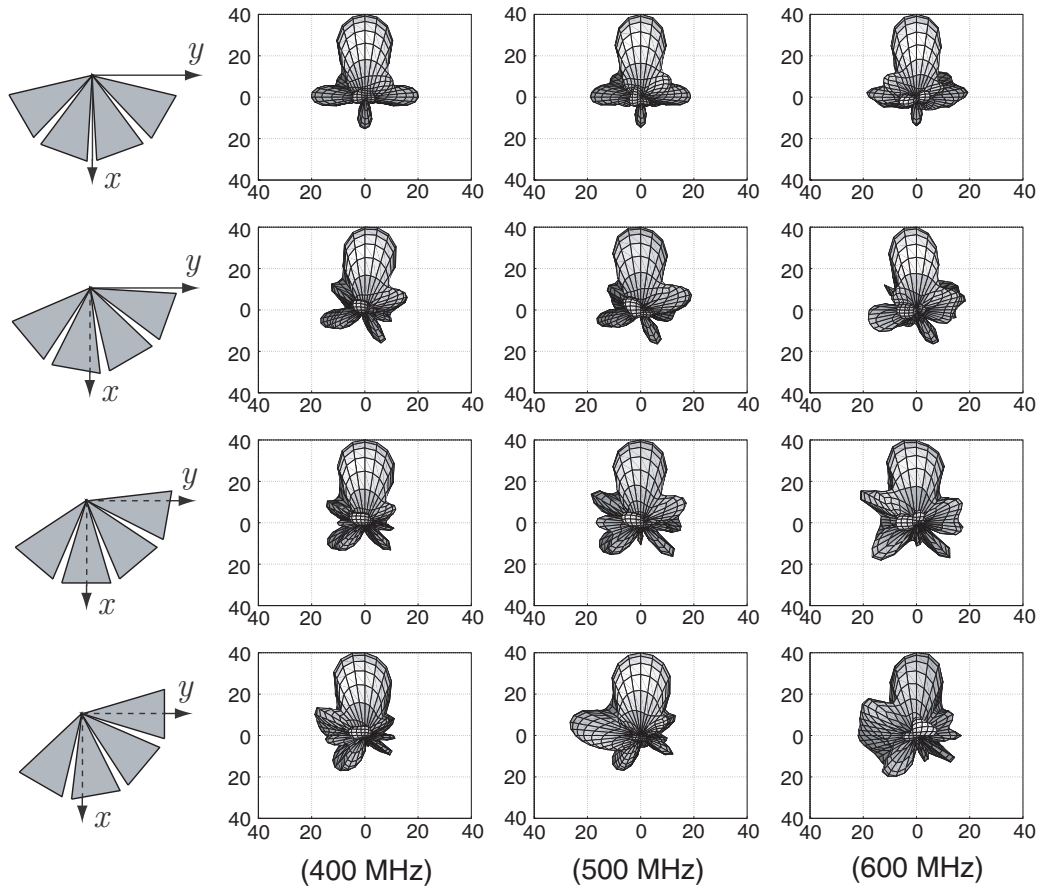
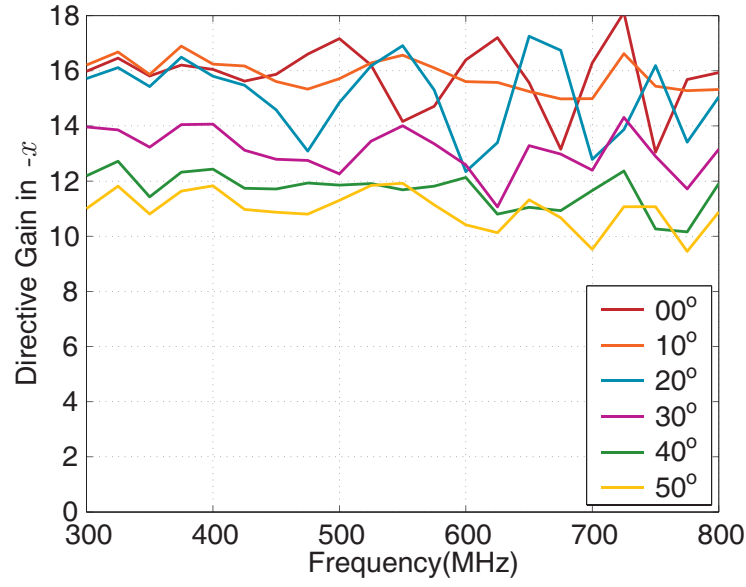
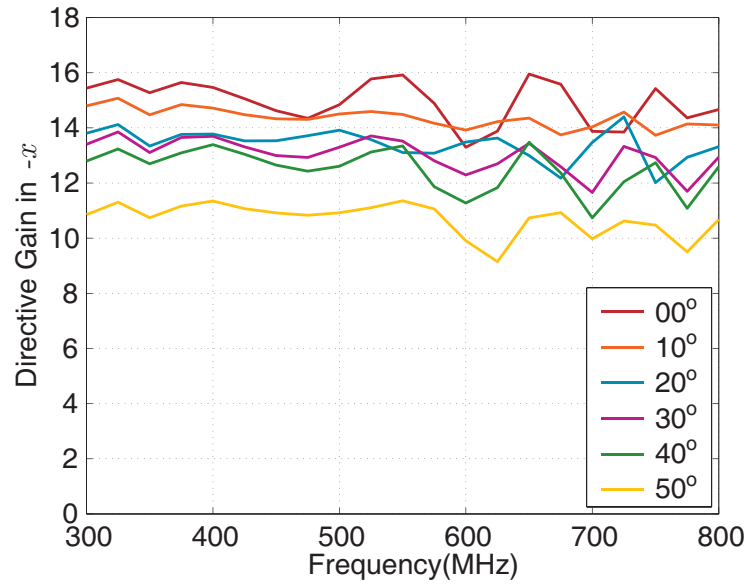


Figure 6.31: Far-zone radiation pattern of the array in Figure 6.29(b) for various frequencies and alignments. Directive gain is optimized in the $-x$ direction.



(a)



(b)

Figure 6.32: Directive gain in the $-x$ direction obtained by the genetic optimization for (a) the three-element array in Figure 6.29(a) and (b) the four-element array in Figure 6.29(b). Arrays are rotated for different angles from 0° to 50° to test the beam-steering ability in a sector of 100° .

6.6.3 Circular-Sectoral Arrays of Log-Periodic Antennas

Finally, we focus on sectoral arrays of LP antennas, where elements are placed side by side in a circular arrangement, as depicted in Figure 6.29. Such an array with closely spaced elements is observed to provide a wider scanning range and higher directive gain compared to full-circular arrays, e.g., depicted in Figure 6.22. Therefore, sectoral arrays might be preferable depending on the application. On the other hand, sectoral arrays with closely localized elements do not eliminate the need for full-circular arrays, where regularly spaced elements provide narrower but multiple scanning ranges.

Figures 6.30 and 6.31 present far-field radiation patterns for three-element and four-element arrays depicted in Figure 6.29, demonstrating beam-steering abilities of arrays for 0° , 10° , 20° , and 30° on the azimuth plane. The directive gain is optimized in the $-x$ direction while rotating arrays from 0° to 30° as depicted in the first column of each figure. Since steering the beam at -10° , -20° , and -30° can also be realized by symmetry, a total scan range of 60° can be achieved. Due to the frequency independence, radiation patterns do not change significantly with respect to the frequency in the range from 400 MHz to 700 MHz. When the array is rotated from 0° to 30° , the main beam is in the $-x$ direction due to the optimization by GAs. However, as the rotation angle is increased, it becomes difficult to keep the main beam in the $-x$ direction. Especially with the further increase of the rotation angle up to 60° – 70° (not shown here), we observe that the main beam is not in the $-x$ direction anymore. In other words, although GAs attempt to maximize the directive gain in the $-x$ direction, the main beam cannot be pointed in the $-x$ direction.

For more quantitative information, Figures 6.32(a) and 6.32(b) present the directive gain in the $-x$ direction as a function of the frequency with 25 MHz intervals and for different orientations of three-element and four-element arrays,

respectively. Results of optimizations for 40° and 50° orientations are also included. It can be observed that both arrays in Figure 6.29 provide directive gain over 9 in angular sectors of 100° . This is achieved in the frequency range from 300 MHz to 800 MHz and this range can be extended by adding more teeth to LP antennas. On the other hand, Figure 6.32 also shows that the directive gain of the four-element array has larger fluctuations compared to the directive gain of the three-element array. This is due to increasing mutual couplings among antennas when the array becomes more populous. In other words, mutual couplings among antennas tend to deteriorate the frequency independence and this becomes more significant as the number of elements in the array is increased.

Chapter 7

Concluding Remarks

Finally, we list major conclusions of this Ph.D. study.

1. JMCFIE is a suitable formulation for the efficient solution of large-scale dielectric problems using MLFMA. This formulation is also applicable to composite problems involving multiple dielectric and dielectric/metallic regions.
2. For accurate solutions of electromagnetics problems involving low-contrast dielectric objects, conventional surface formulations can be stabilized by extracting nonradiating currents. A stabilization procedure based on fictitious incident fields enables accurate solutions even for extremely low contrasts.
3. Although they lead to better-conditioned matrix equations, normal and mixed formulations are noticeably and consistently more inaccurate than tangential formulations. Computational experiments show that the source of the excessive error in normal and mixed formulations is well-tested identity operators.

4. Accuracy of normal and mixed formulations can be improved to levels of tangential formulations by employing more appropriate basis functions, such as the LL functions, instead of the conventional RWG functions.
5. Composite structures with coexisting open and closed metallic parts can be formulated with HIEs, which are based on flexible combinations of EFIE and MFIE depending on the testing point. HIEs lead to well-conditioned matrix equations that are easy to solve iteratively.
6. Accuracy and efficiency of MLFMA implementations can be improved significantly via optimizations and simple modifications on conventional schemes of interpolations.
7. A broadband MLFMA implementation based on using multipoles for small clusters is appropriate for the efficient solution of multi-scale problems.
8. Hierarchical partitioning strategy provides efficient parallelization of MLFMA and enables the solution of large-scale problems discretized with hundreds of millions of unknowns on distributed-memory architectures.

As presented in this thesis, sophisticated implementations of MLFMA can provide fast and accurate solutions of real-life electromagnetics problems, such as scattering from airborne targets, radiation from antennas, and transmission through metamaterials.

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Appendix A

Basics

A.1 Solutions of Maxwell's Equations for Homogeneous Media

Faraday's law of induction:

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \quad (\text{A.1})$$

$$\oint_C \mathbf{E}(\mathbf{r}, t) \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot d\mathbf{a} \quad (\text{A.2})$$

Ampere's circuital law with Maxwell's extension:

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t) \quad (\text{A.3})$$

$$\oint_C \mathbf{H}(\mathbf{r}, t) \cdot d\mathbf{l} = \int_S \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \cdot d\mathbf{a} + \int_S \mathbf{J}(\mathbf{r}, t) \cdot d\mathbf{a} \quad (\text{A.4})$$

Gauss's law:

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho_e(\mathbf{r}, t) \quad (\text{A.5})$$

$$\oint_S \mathbf{D}(\mathbf{r}, t) \cdot d\mathbf{a} = \int_V \rho_e(\mathbf{r}, t) dV \quad (\text{A.6})$$

Gauss's Law for magnetism:

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \quad (\text{A.7})$$

$$\oint_S \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{a} = 0 \quad (\text{A.8})$$

Continuity equation:

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\frac{\partial \rho_e(\mathbf{r}, t)}{\partial t} \quad (\text{A.9})$$

$$\oint_S \mathbf{J}(\mathbf{r}, t) \cdot d\mathbf{a} = -\int_V \frac{\partial \rho_e(\mathbf{r}, t)}{\partial t} dV \quad (\text{A.10})$$

Constitutive relations:

$$\mathbf{D}(\mathbf{r}, t) = \bar{\epsilon}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}, t) \quad (\text{A.11})$$

$$\mathbf{B}(\mathbf{r}, t) = \bar{\mu}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r}, t) \quad (\text{A.12})$$

Constitutive relations for isotropic medium:

$$\mathbf{D}(\mathbf{r}, t) = \epsilon(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) \quad (\text{A.13})$$

$$\mathbf{B}(\mathbf{r}, t) = \mu(\mathbf{r}) \mathbf{H}(\mathbf{r}, t) \quad (\text{A.14})$$

Constitutive relations for isotropic and homogeneous medium:

$$\mathbf{D}(\mathbf{r}, t) = \epsilon \mathbf{E}(\mathbf{r}, t) \quad (\text{A.15})$$

$$\mathbf{B}(\mathbf{r}, t) = \mu \mathbf{H}(\mathbf{r}, t) \quad (\text{A.16})$$

Maxwell's equations for isotropic and homogeneous (simple) medium:

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} \quad (\text{A.17})$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \epsilon \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t) \quad (\text{A.18})$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{1}{\epsilon} \rho_e(\mathbf{r}, t) \quad (\text{A.19})$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0 \quad (\text{A.20})$$

Phasor notation with $e^{-i\omega t}$ convention:

$$y(\mathbf{r}, t) = \text{Re}\{y(\mathbf{r}) \exp(-i\omega t)\} \quad (\text{A.21})$$

$$\frac{\partial y(\mathbf{r}, t)}{\partial t} = -i\omega \text{Re}\{y(\mathbf{r}) \exp(-i\omega t)\} \quad (\text{A.22})$$

$$\mathbf{x}(\mathbf{r}, t) = \text{Re}\{\mathbf{x}(\mathbf{r}) \exp(-i\omega t)\} \quad (\text{A.23})$$

Maxwell's equations for simple medium in phasor notation with $e^{-i\omega t}$ convention:

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu\mathbf{H}(\mathbf{r}) \quad (\text{A.24})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\epsilon\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \quad (\text{A.25})$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon}\rho_e(\mathbf{r}) \quad (\text{A.26})$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}) = 0 \quad (\text{A.27})$$

Continuity equation for simple medium in phasor notation with $e^{-i\omega t}$ convention:

$$\nabla \cdot \mathbf{J}(\mathbf{r}) = i\omega\rho_e(\mathbf{r}) \quad (\text{A.28})$$

Linear dependency of Maxwell's equations:

$$\nabla \cdot \nabla \times \mathbf{E}(\mathbf{r}) = 0 = i\omega\mu\nabla \cdot \mathbf{H}(\mathbf{r}) \rightarrow \nabla \cdot \mathbf{H}(\mathbf{r}) = 0 \quad (\text{A.29})$$

$$\begin{aligned} \nabla \cdot \nabla \times \mathbf{H}(\mathbf{r}) &= 0 = -i\omega\epsilon\nabla \cdot \mathbf{E}(\mathbf{r}) + \nabla \cdot \mathbf{J}(\mathbf{r}) \\ &= -i\omega\epsilon\nabla \cdot \mathbf{E}(\mathbf{r}) + i\omega\rho_e(\mathbf{r}) \rightarrow \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon}\rho_e(\mathbf{r}) \end{aligned} \quad (\text{A.30})$$

Boundary conditions ($\hat{\mathbf{n}}$ is directed into the medium 1):

$$\hat{\mathbf{n}} \times [\mathbf{E}_1(\mathbf{r}) - \mathbf{E}_2(\mathbf{r})] = 0 \quad (\text{A.31})$$

$$\hat{\mathbf{n}} \times [\mathbf{H}_1(\mathbf{r}) - \mathbf{H}_2(\mathbf{r})] = \mathbf{J}(\mathbf{r}) \quad (\text{A.32})$$

$$\hat{\mathbf{n}} \cdot [\mathbf{D}_1(\mathbf{r}) - \mathbf{D}_2(\mathbf{r})] = \rho_e(\mathbf{r}) \quad (\text{A.33})$$

$$\hat{\mathbf{n}} \cdot [\mathbf{B}_1(\mathbf{r}) - \mathbf{B}_2(\mathbf{r})] = 0 \quad (\text{A.34})$$

Boundary conditions for a PEC surface:

$$\hat{\mathbf{n}} \times \mathbf{E}_1(\mathbf{r}) = 0 \quad (\text{A.35})$$

$$\hat{\mathbf{n}} \times \mathbf{H}_1(\mathbf{r}) = \mathbf{J}_s(\mathbf{r}) \quad (\text{A.36})$$

$$\hat{\mathbf{n}} \cdot \mathbf{E}_1(\mathbf{r}) = \frac{1}{\epsilon} \rho_{es}(\mathbf{r}) \quad (\text{A.37})$$

$$\hat{\mathbf{n}} \cdot \mathbf{H}_1(\mathbf{r}) = 0 \quad (\text{A.38})$$

Helmholtz equation for the electric field:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu \nabla \times \mathbf{H}(\mathbf{r}) \quad (\text{A.39})$$

$$\nabla \nabla \cdot \mathbf{E}(\mathbf{r}) - \nabla^2 \mathbf{E}(\mathbf{r}) = i\omega\mu \left[-i\omega\epsilon \mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \right] \quad (\text{A.40})$$

$$\frac{1}{\epsilon} \nabla \rho_e(\mathbf{r}) - \nabla^2 \mathbf{E}(\mathbf{r}) = w^2 \mu \epsilon \mathbf{E}(\mathbf{r}) + i\omega\mu \mathbf{J}(\mathbf{r}) \quad (\text{A.41})$$

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = -i\omega\mu \mathbf{J}(\mathbf{r}) + \frac{1}{\epsilon} \nabla \rho_e(\mathbf{r}) \quad (k = \omega\sqrt{\epsilon\mu}) \quad (\text{A.42})$$

Helmholtz equation for the magnetic field:

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\epsilon \nabla \times \mathbf{E}(\mathbf{r}) + \nabla \times \mathbf{J}(\mathbf{r}) \quad (\text{A.43})$$

$$\nabla \nabla \cdot \mathbf{H}(\mathbf{r}) - \nabla^2 \mathbf{H}(\mathbf{r}) = -i\omega\epsilon [i\omega\mu \mathbf{H}(\mathbf{r})] + \nabla \times \mathbf{J}(\mathbf{r}) \quad (\text{A.44})$$

$$-\nabla^2 \mathbf{H}(\mathbf{r}) = w^2 \mu \epsilon \mathbf{H}(\mathbf{r}) + \nabla \times \mathbf{J}(\mathbf{r}) \quad (\text{A.45})$$

$$\nabla^2 \mathbf{H}(\mathbf{r}) + k^2 \mathbf{H}(\mathbf{r}) = -\nabla \times \mathbf{J}(\mathbf{r}) \quad (\text{A.46})$$

Potentials:

$$\nabla \cdot \mathbf{H}(\mathbf{r}) = 0 \rightarrow \mathbf{H}(\mathbf{r}) = \frac{1}{\mu} \nabla \times \mathbf{A}_m(\mathbf{r}) = \frac{1}{i\omega\mu} \nabla \times \mathbf{E}(\mathbf{r}) \quad (\text{A.47})$$

$$\nabla \times [\mathbf{E}(\mathbf{r}) - i\omega \mathbf{A}_m(\mathbf{r})] = 0 \quad (\text{A.48})$$

$$\mathbf{E}(\mathbf{r}) = i\omega \mathbf{A}_m(\mathbf{r}) - \nabla \phi_e(\mathbf{r}) \quad (\text{A.49})$$

Lorentz gauge and Helmholtz equation for potentials:

$$\nabla^2 \mathbf{A}_m(\mathbf{r}) + k^2 \mathbf{A}_m(\mathbf{r}) = -\mu \mathbf{J}(\mathbf{r}) + \nabla \left(\nabla \cdot \mathbf{A}_m(\mathbf{r}) - i\omega\epsilon\mu\phi_e(\mathbf{r}) \right) \quad (\text{A.50})$$

$$\nabla \cdot \mathbf{A}_m(\mathbf{r}) = i\omega\epsilon\mu\phi_e(\mathbf{r}) \quad (\text{A.51})$$

$$\nabla^2 \mathbf{A}_m(\mathbf{r}) + k^2 \mathbf{A}_m(\mathbf{r}) = -\mu \mathbf{J}(\mathbf{r}) \quad (\text{A.52})$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = i\omega \nabla \cdot \mathbf{A}_m(\mathbf{r}) - \nabla \cdot \nabla \phi_e(\mathbf{r}) = \frac{1}{\epsilon} \rho_e(\mathbf{r}) \quad (\text{A.53})$$

$$\nabla^2 \phi_e(\mathbf{r}) + k^2 \phi_e(\mathbf{r}) = -\frac{1}{\epsilon} \rho_e(\mathbf{r}) \quad (\text{A.54})$$

Green's function (kernel of the free-space Helmholtz equation):

$$\left(\nabla^2 + k^2 \right) g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r}, \mathbf{r}') \quad (\text{A.55})$$

$$g(\mathbf{r}, \mathbf{r}') = \frac{\exp(ikR)}{4\pi R} \quad (\text{A.56})$$

$$R = |\mathbf{r} - \mathbf{r}'| \quad (\text{A.57})$$

Potentials due to arbitrary charge and current distributions:

$$\phi_e(\mathbf{r}) = \frac{1}{\epsilon} \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \rho_e(\mathbf{r}') \quad (\text{A.58})$$

$$\mathbf{A}_m(\mathbf{r}) = \mu \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') \quad (\text{A.59})$$

Electric and magnetic fields due to arbitrary charge and current distributions:

$$\mathbf{E}(\mathbf{r}) = i\omega \mathbf{A}_m(\mathbf{r}) - \nabla \phi_e(\mathbf{r}) \quad (\text{A.60})$$

$$= ik\eta \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') - \frac{1}{\epsilon} \nabla \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \rho_e(\mathbf{r}') \quad (\text{A.61})$$

$$= ik\eta \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') - \frac{1}{i\omega\epsilon} \nabla \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{J}(\mathbf{r}') \quad (\text{A.62})$$

$$= ik\eta \left\{ \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') + \frac{1}{k^2} \int_S d\mathbf{r}' \nabla g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{J}(\mathbf{r}') \right\} \quad (\text{A.63})$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu} \nabla \times \mathbf{A}_m(\mathbf{r}) = \nabla \times \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') \quad (\text{A.64})$$

$$= \int_S d\mathbf{r}' \nabla g(\mathbf{r}, \mathbf{r}') \times \mathbf{J}(\mathbf{r}') \quad (\text{A.65})$$

Maxwell's equations with Dirac ansatz:

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu\mathbf{H}(\mathbf{r}) - \mathbf{M}(\mathbf{r}) \quad (\text{A.66})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\epsilon\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \quad (\text{A.67})$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon} \rho_e(\mathbf{r}) \quad (\text{A.68})$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}) = \frac{1}{\mu} \rho_m(\mathbf{r}) \quad (\text{A.69})$$

Continuity equation for magnetic current and charge:

$$\nabla \cdot \mathbf{M}(\mathbf{r}) = i\omega\rho_m(\mathbf{r}) \quad (\text{A.70})$$

Electromagnetic fields with Dirac ansatz:

$$\mathbf{E}(\mathbf{r}) = i\omega\mathbf{A}_m(\mathbf{r}) - \nabla\phi_e(\mathbf{r}) - \frac{1}{\epsilon} \nabla \times \mathbf{A}_e(\mathbf{r}) \quad (\text{A.71})$$

$$\mathbf{H}(\mathbf{r}) = i\omega\mathbf{A}_e(\mathbf{r}) - \nabla\phi_m(\mathbf{r}) + \frac{1}{\mu} \nabla \times \mathbf{A}_m(\mathbf{r}) \quad (\text{A.72})$$

Potentials:

$$\phi_m(\mathbf{r}) = \frac{1}{\mu} \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \rho_m(\mathbf{r}') \quad (\text{A.73})$$

$$\mathbf{A}_e(\mathbf{r}) = \epsilon \int_S d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \mathbf{M}(\mathbf{r}') \quad (\text{A.74})$$

$$\nabla \cdot \mathbf{A}_e(\mathbf{r}) = i\omega\mu\epsilon\phi_m(\mathbf{r}) \quad (\text{A.75})$$

$$\nabla^2 \mathbf{A}_e(\mathbf{r}) + k^2 \mathbf{A}_e(\mathbf{r}) = -\epsilon \mathbf{M}(\mathbf{r}) \quad (\text{A.76})$$

$$\nabla^2 \phi_m(\mathbf{r}) + k^2 \phi_m(\mathbf{r}) = -\frac{1}{\mu} \rho_m(\mathbf{r}) \quad (\text{A.77})$$

Electric and magnetic fields due to arbitrary charge and current distributions:

$$\mathbf{E}(\mathbf{r}) = \eta \mathcal{T}\{\mathbf{J}\}(\mathbf{r}) - \mathcal{K}\{\mathbf{M}\}(\mathbf{r}) \quad (\text{A.78})$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\eta} \mathcal{T}\{\mathbf{M}\}(\mathbf{r}) + \mathcal{K}\{\mathbf{J}\}(\mathbf{r}) \quad (\text{A.79})$$

Integro-differential operators:

$$\mathcal{T}\{\mathbf{X}\}(\mathbf{r}) = ik \int_S d\mathbf{r}' \left[\mathbf{X}(\mathbf{r}') + \frac{1}{k^2} \nabla' \cdot \mathbf{X}(\mathbf{r}') \nabla \right] g(\mathbf{r}, \mathbf{r}') \quad (\text{A.80})$$

$$\mathcal{K}\{\mathbf{X}\}(\mathbf{r}) = \int_S d\mathbf{r}' \mathbf{X}(\mathbf{r}') \times \nabla' g(\mathbf{r}, \mathbf{r}') \quad (\text{A.81})$$

Complex Poynting vector (power density):

$$\mathbf{P}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r}) \quad (\text{A.82})$$

Real (time-average) power density:

$$\mathbf{P}_{av}(\mathbf{r}) = \langle \mathbf{P}_{av}(\mathbf{r}, t) \rangle = \langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \rangle \quad (\text{A.83})$$

$$= \frac{1}{2} \text{Re}\{\mathbf{P}(\mathbf{r})\} = \frac{1}{2} \text{Re}\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})\} \quad (\text{A.84})$$

A.2 Fields of a Hertzian Dipole

Let $\mathbf{I}_{DM} = \hat{\mathbf{u}}_{DM} I_{DM}$ be dipole moment of a Hertzian dipole located at the origin:

$$\mathbf{A}_m(\mathbf{r}) = \mu \mathbf{I}_{DM} \frac{\exp(ikR)}{4\pi R} \quad (\text{A.85})$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu} \nabla \times \mathbf{A}_m(\mathbf{r}) = \nabla \times \mathbf{I}_{DM} \frac{\exp(ikR)}{4\pi R} - \mathbf{I}_{DM} \times \nabla \left[\frac{\exp(ikR)}{4\pi R} \right] \quad (\text{A.86})$$

$$= -\mathbf{I}_{DM} \times \nabla \left[\frac{\exp(ikR)}{4\pi R} \right] \quad (\text{A.87})$$

$$= -\mathbf{I}_{DM} \times \left(\hat{\mathbf{x}} \frac{\partial R}{\partial x} + \hat{\mathbf{y}} \frac{\partial R}{\partial y} + \hat{\mathbf{z}} \frac{\partial R}{\partial z} \right) \frac{\partial}{\partial R} \left[\frac{\exp(ikR)}{4\pi R} \right] \quad (\text{A.88})$$

$$= -\mathbf{I}_{DM} \times \hat{\mathbf{R}} \frac{\partial}{\partial R} \left[\frac{\exp(ikR)}{4\pi R} \right] \quad (\text{A.89})$$

$$= \mathbf{I}_{DM} \times \hat{\mathbf{R}} \frac{\exp(ikR)}{4\pi R} \left(\frac{1}{R} - ik \right) \quad (\text{A.90})$$

$$\mathbf{E}(\mathbf{r}) = i\omega \mathbf{A}_m(\mathbf{r}) - \nabla \phi_e(\mathbf{r}) \quad (\text{A.91})$$

$$= i\omega \mathbf{A}_m(\mathbf{r}) - \frac{1}{i\omega\epsilon\mu} \nabla \nabla \cdot \mathbf{A}_m(\mathbf{r}) \quad (\text{A.92})$$

$$= i\omega\mu \mathbf{I}_{DM} \frac{\exp(ikR)}{4\pi R} - \frac{1}{i\omega\epsilon} \nabla \nabla \cdot \mathbf{I}_{DM} \frac{\exp(ikR)}{4\pi R} \quad (\text{A.93})$$

$$= i\omega\mu \mathbf{I}_{DM} \frac{\exp(ikR)}{4\pi R} - \frac{1}{i\omega\epsilon} \nabla \left[\mathbf{I}_{DM} \cdot \nabla \left(\frac{\exp(ikR)}{4\pi R} \right) \right] \quad (\text{A.94})$$

$$= i\omega\mu \mathbf{I}_{DM} \frac{\exp(ikR)}{4\pi R} - \frac{1}{i\omega\epsilon} \nabla \left\{ \mathbf{I}_{DM} \cdot \hat{\mathbf{R}} \left[ik \frac{\exp(ikR)}{4\pi R^2} - \frac{\exp(ikR)}{4\pi R^3} \right] \right\} \quad (\text{A.95})$$

$$= i\omega\mu \mathbf{I}_{DM} \frac{\exp(ikR)}{4\pi R} - \frac{1}{i\omega\epsilon} \nabla (\mathbf{I}_{DM} \cdot \hat{\mathbf{R}}) \left[ik \frac{\exp(ikR)}{4\pi R^2} - \frac{\exp(ikR)}{4\pi R^3} \right] - \frac{1}{i\omega\epsilon} (\mathbf{I}_{DM} \cdot \hat{\mathbf{R}}) \nabla \left[ik \frac{\exp(ikR)}{4\pi R^2} - \frac{\exp(ikR)}{4\pi R^3} \right] \quad (\text{A.96})$$

$$\nabla (\mathbf{I}_{DM} \cdot \hat{\mathbf{R}}) = \mathbf{I}_{DM} \quad (\text{A.97})$$

$$\nabla \left[ik \frac{\exp(ikR)}{4\pi R^2} - \frac{\exp(ikR)}{4\pi R^3} \right] = -\hat{\mathbf{R}} \frac{\exp(ikR)}{4\pi R^2} \left(k^2 + \frac{3ik}{R} - \frac{3}{R^2} \right) \quad (\text{A.98})$$

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= iw\mu\mathbf{I}_{DM}\frac{\exp(ikR)}{4\pi R} - \frac{1}{iw\epsilon}\mathbf{I}_{DM}\left[ik\frac{\exp(ikR)}{4\pi R^2} - \frac{\exp(ikR)}{4\pi R^3}\right] \\ &+ \frac{1}{iw\epsilon}(\mathbf{I}_{DM} \cdot \mathbf{R})\hat{\mathbf{R}}\frac{\exp(ikR)}{4\pi R^2}\left(k^2 + \frac{3ik}{R} - \frac{3}{R^2}\right)\end{aligned}\quad (\text{A.99})$$

$$\begin{aligned}&= iw\mu\frac{\exp(ikR)}{4\pi R}\left[\mathbf{I}_{DM}\left(1 + \frac{i}{kR} - \frac{1}{k^2 R^2}\right)\right. \\ &\quad \left. - (\mathbf{I}_{DM} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}}\left(1 + \frac{3i}{kR} - \frac{3}{k^2 R^2}\right)\right]\end{aligned}\quad (\text{A.100})$$

Far-zone fields:

$$\mathbf{H}^{far}(\mathbf{r}) \approx \mathbf{I}_{DM} \times \hat{\mathbf{R}}\frac{\exp(ikR)}{4\pi R}(-ik) \quad (\text{A.101})$$

$$\approx -ik\mathbf{I}_{DM} \times \hat{\mathbf{r}}\frac{\exp(ikr)}{4\pi r}\exp(-ik\hat{\mathbf{r}} \cdot \mathbf{r}') \quad (\text{A.102})$$

$$\mathbf{E}^{far}(\mathbf{r}) \approx iw\mu\frac{\exp(ikR)}{4\pi R}\left[\mathbf{I}_{DM} - (\mathbf{I}_{DM} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}}\right] \quad (\text{A.103})$$

$$\approx ik\eta\frac{\exp(ikr)}{4\pi r}\exp(-ik\hat{\mathbf{r}} \cdot \mathbf{r}')\left[\mathbf{I}_{DM} - (\mathbf{I}_{DM} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\right] \quad (\text{A.104})$$

Far-zone real power density:

$$\mathbf{P}_{av}^{far}(\mathbf{r}) = -\frac{k^2\eta}{32\pi^2 r^2}\left[\mathbf{I}_{DM} - (\mathbf{I}_{DM} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\right] \times (\mathbf{I}_{DM} \times \hat{\mathbf{r}}) \quad (\text{A.105})$$

$$= \frac{k^2\eta}{32\pi^2 r^2}\left[\hat{\mathbf{r}}(\mathbf{I}_{DM} \cdot \mathbf{I}_{DM}) - (\mathbf{I}_{DM} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{I}_{DM})\right] \quad (\text{A.106})$$

$$= \hat{\mathbf{r}}\frac{k^2\eta}{32\pi^2 r^2}|\mathbf{I}_{DM}|^2\left[1 - (\hat{\mathbf{r}} \cdot \hat{\mathbf{u}}_{DM})^2\right] \quad (\text{A.107})$$

A.3 Method of Moments

Let $\mathcal{L}$ be a linear operator applied on an unknown function $\mathbf{f}(x)$:

$$\mathcal{L}\{\mathbf{f}\}(x) = \mathbf{g}(x) \quad (\text{A.108})$$

Expand $\mathbf{f}(x)$ in a series of known basis functions:

$$\mathbf{f}(x') \approx \sum_{n=1}^N \mathbf{a}[n]\mathbf{b}_n(x') \quad (\text{A.109})$$

Test the equation via inner products with testing functions:

$$\int dx \mathbf{t}_m(x) \cdot \sum_{n=1}^N \mathbf{a}[n] \mathcal{L}\{\mathbf{b}_n\}(x) = \int dx \mathbf{t}_m(x) \cdot \mathbf{g}(x) \quad (m = 1, 2, \dots, N) \quad (\text{A.110})$$

Change the order of summation and integration:

$$\sum_{n=1}^N \mathbf{a}[n] \int dx \mathbf{t}_m(x) \cdot \mathcal{L}\{\mathbf{b}_n\}(x) = \int dx \mathbf{t}_m(x) \cdot \mathbf{g}(x) \quad (m = 1, 2, \dots, N) \quad (\text{A.111})$$

Construct an $N \times N$ matrix equation:

$$\sum_{n=1}^N \mathbf{a}[n] \bar{\mathbf{Z}}[m, n] = \mathbf{v}[m] \quad (m = 1, 2, \dots, N) \quad (\text{A.112})$$

$$\bar{\mathbf{Z}}[m, n] = \int dx \mathbf{t}_m(x) \cdot \mathcal{L}\{\mathbf{b}_n\}(x) \quad (\text{A.113})$$

$$\mathbf{v}[m] = \int dx \mathbf{t}_m(x) \cdot \mathbf{g}(x) \quad (\text{A.114})$$

A.4 Gaussian Quadratures

Numerical integrations of nonsingular functions on triangles can be performed accurately and efficiently by using Gaussian quadratures [96]. A symmetrical quadrature rule on a triangle with a spatial support of S_m can be written as

$$\int_{S_m} d\mathbf{r} f(\mathbf{r}) \approx A_m \sum_{q=1}^Q w_q f(\mathbf{r}_{qm}), \quad (\text{A.115})$$

where A_m is the area of the triangle, Q is the number of quadrature points, and w_q represents the integration weight for q th sample point. Locations of sample points are determined as

$$\mathbf{r}_{qm} = \zeta_{q1} \mathbf{r}_{m1} + \zeta_{q2} \mathbf{r}_{m2} + \zeta_{q3} \mathbf{r}_{m3}, \quad (\text{A.116})$$

where $\mathbf{r}_{m1}$, $\mathbf{r}_{m2}$, and $\mathbf{r}_{m3}$ are node coordinates of the triangle, whereas ζ_{q1} , ζ_{q2} , and ζ_{q3} are simplex coordinates satisfying $\zeta_{q1} + \zeta_{q2} + \zeta_{q3} = 1$.

A.5 Electromagnetic Excitation

Let $\bar{\mathbf{Z}} \cdot \mathbf{a} = \mathbf{w}$ be an $N \times N$ matrix equation constructed by using MOM. The excitation vector $\mathbf{w}$ is obtained by testing incident electric and magnetic fields, i.e.,

$$\mathbf{w}[m] = \int_{S_m} d\mathbf{r} \begin{Bmatrix} \mathbf{t}_m(\mathbf{r}) \cdot \\ \mathbf{t}_m(\mathbf{r}) \cdot \hat{\mathbf{n}} \times \end{Bmatrix} \begin{Bmatrix} \eta^{-1} \mathbf{E}^{inc}(\mathbf{r}) \\ \mathbf{H}^{inc}(\mathbf{r}) \end{Bmatrix}, \quad (\text{A.117})$$

where S_m is the spatial support of the testing function $\mathbf{t}_m$ for $m = 1, 2, \dots, N$. We have been using the following types of excitations in our numerical simulations.

Plane-Wave Excitation

A plane wave propagating in the $\hat{\mathbf{k}}$ direction with the electric field polarized in the $\hat{\mathbf{e}}$ direction ($\hat{\mathbf{e}} \perp \hat{\mathbf{k}}$) can be written as

$$\mathbf{E}^{inc}(\mathbf{r}) = \hat{\mathbf{e}} E_a \exp(ik\hat{\mathbf{k}} \cdot \mathbf{r}) \quad (\text{A.118})$$

$$\mathbf{H}^{inc}(\mathbf{r}) = \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E}^{inc}(\mathbf{r}) = \hat{\mathbf{k}} \times \hat{\mathbf{e}} \frac{E_a}{\eta} \exp(ik\hat{\mathbf{k}} \cdot \mathbf{r}), \quad (\text{A.119})$$

where E_a is the amplitude of the plane wave.

Hertzian Dipole

Electric and magnetic fields of a Hertzian (ideal) dipole with dipole moment $\mathbf{I}_{DM}$ located at $\mathbf{r}_d$ can be written as

$$\begin{aligned} \mathbf{E}^{inc}(\mathbf{r}) = iw\mu \frac{\exp(ik|\mathbf{r} - \mathbf{r}_d|)}{4\pi|\mathbf{r} - \mathbf{r}_d|} \left[\mathbf{I}_{DM} \left(1 + \frac{i}{k|\mathbf{r} - \mathbf{r}_d|} - \frac{1}{k^2|\mathbf{r} - \mathbf{r}_d|^2} \right) \right. \\ \left. - (\mathbf{r} - \mathbf{r}_d) \frac{\mathbf{I}_{DM} \cdot (\mathbf{r} - \mathbf{r}_d)}{|\mathbf{r} - \mathbf{r}_d|^2} \left(1 + \frac{3i}{k|\mathbf{r} - \mathbf{r}_d|} - \frac{3}{k^2|\mathbf{r} - \mathbf{r}_d|^2} \right) \right] \end{aligned} \quad (\text{A.120})$$

and

$$\mathbf{H}^{inc}(\mathbf{r}) = \mathbf{I}_{DM} \times (\mathbf{r} - \mathbf{r}_d) \frac{\exp(ik|\mathbf{r} - \mathbf{r}_d|)}{4\pi|\mathbf{r} - \mathbf{r}_d|^2} \left(\frac{1}{|\mathbf{r} - \mathbf{r}_d|} - ik \right), \quad (\text{A.121})$$

respectively. Singularity extraction methods are required to evaluate integrals in (A.117) when the dipole is close to the observation point and $|\mathbf{r} - \mathbf{r}_d|$ is small.

Complex-Source-Point Excitation

In order to obtain an electromagnetic beam, which satisfies the Maxwell's equations exactly, an ideal dipole can be located at $\mathbf{r}_d = \mathbf{r}_{d,R} + i\mathbf{r}_{d,I}$ in complex coordinates [41]. Then, expressions in (A.120) and (A.121) can be used to compute electric and magnetic fields at any point $\mathbf{r}$, where

$$|\mathbf{r} - \mathbf{r}_d| = (\mathbf{r} - \mathbf{r}_d) \cdot (\mathbf{r} - \mathbf{r}_d)^*. \quad (\text{A.122})$$

The real part of the source location, i.e., $\mathbf{r}_{d,R}$, determines the position of the beam in real coordinates, whereas the imaginary part, i.e., $\mathbf{r}_{d,I}$, determines the direction of the beam and the width of the beam waist.

As an example, we consider various complex-source-point excitations obtained by placing a Hertzian dipole with $\bar{\mathbf{I}}_{DM} = \hat{\mathbf{z}}$ in complex coordinates. Figure A.1 presents a normalized electric field, i.e., $|\mathbf{E}(\mathbf{r})| \exp(-ik|\mathbf{r}_{d,I}|)$, on the x - y plane, when $\mathbf{r}_d = i\lambda\hat{\mathbf{x}}$ and $\mathbf{r}_d = i\lambda\hat{\mathbf{y}}$. Since $\mathbf{r}_{d,R} = 0$, both beams are located at the origin. However, the beam propagates in the positive x and y directions when $\mathbf{r}_{d,I} = \lambda\hat{\mathbf{x}}$ and $\mathbf{r}_{d,I} = \lambda\hat{\mathbf{y}}$, respectively. In Figure A.2, we consider another two beams in the x direction with $\mathbf{r}_d = 2i\lambda\hat{\mathbf{x}}$ and $\mathbf{r}_d = 10i\lambda\hat{\mathbf{x}}$. We note that increasing the value of $|\mathbf{r}_{d,I}|$ increases the width of the beam waist.

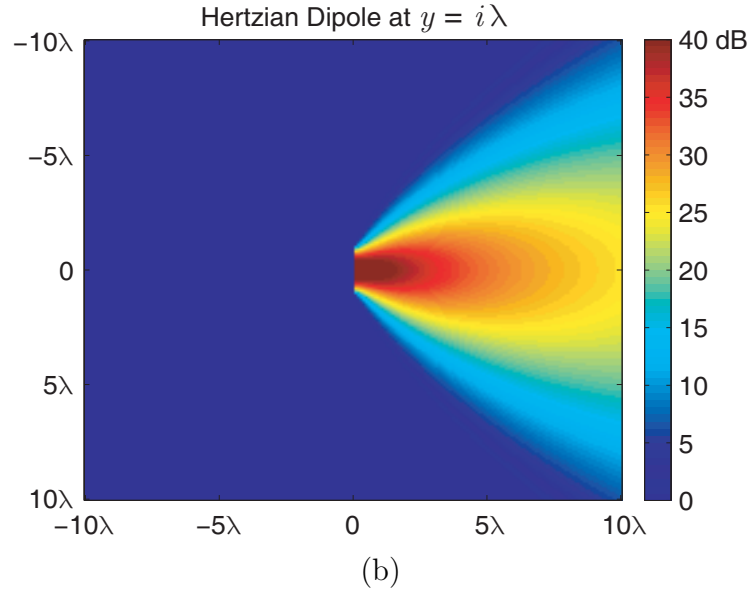
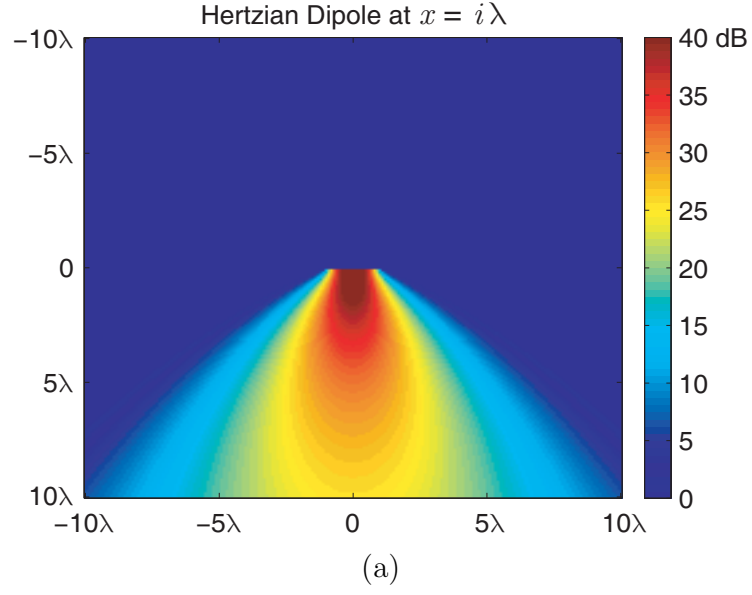


Figure A.1: Normalized electric field on the x - y plane when a Hertzian dipole with $\mathbf{I}_{DM} = \hat{\mathbf{z}}$ is located at (a) $\mathbf{r}_d = i\lambda\hat{\mathbf{x}}$ and (b) $\mathbf{r}_d = i\lambda\hat{\mathbf{y}}$.

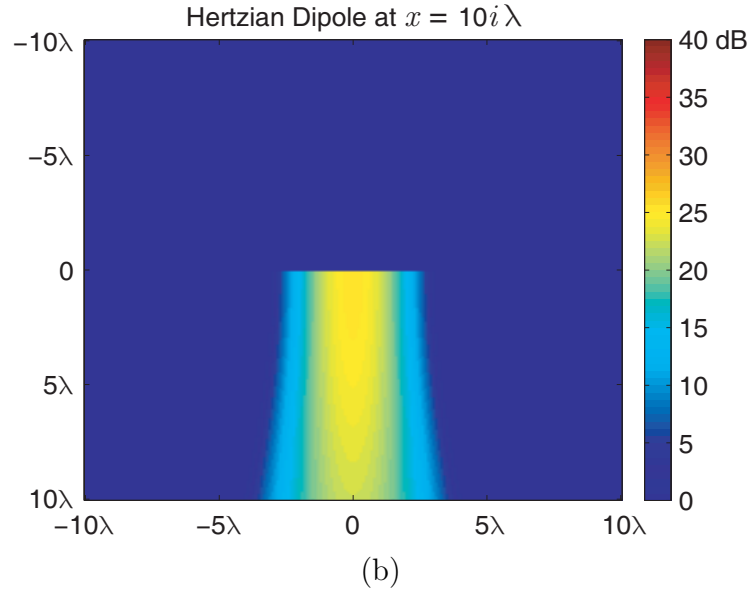
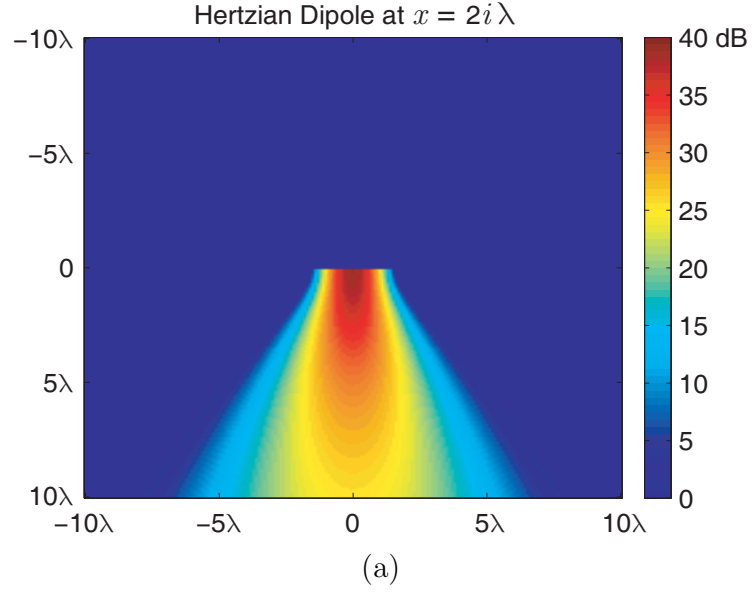


Figure A.2: Normalized electric field on the x - y plane when a Hertzian dipole with $\mathbf{I}_{DM} = \hat{\mathbf{z}}$ is located at (a) $\mathbf{r}_d = 2i\lambda\hat{\mathbf{x}}$ and (b) $\mathbf{r}_d = 10i\lambda\hat{\mathbf{x}}$.

Delta-Gap Excitation

A voltage source can be placed on a metallic surface by using a delta-gap excitation. A local electric field is defined inside an infinitely narrow opening between two triangles of the discretization, i.e.,

$$\mathbf{E}^{inc}(\mathbf{r}) = I_e \lim_{d \rightarrow 0} \hat{\mathbf{u}} \delta(\mathbf{r}, \mathbf{r}_e) / d, \quad (\text{A.123})$$

where e is the index of the edge, $\mathbf{r}_e$ represents any point at the edge, $\hat{\mathbf{u}}$ is the unit vector on the surface perpendicular to the edge, I_e is a complex coefficient related to the strength of the feed, and d is the width of the theoretical gap. In the limit $d \rightarrow 0$, the gap shrinks to the edge e . The dirac delta function $\delta(\mathbf{r}, \mathbf{r}_e)$ in (A.123) indicates that $\mathbf{E}^{inc}(\mathbf{r})$ is zero outside the small gap.

As an example, consider T-EFIE and let a delta-gap excitation at e th edge be tested by the RWG functions. Using (A.123) in (A.117), we obtain

$$\mathbf{w}_E[m] = \int_{S_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \eta^{-1} \mathbf{E}^{inc}(\mathbf{r}) = \pm \frac{I_e l_e}{\eta} \delta[m, e], \quad (\text{A.124})$$

where l_e represents the length of the edge and

$$\delta[m, e] = \begin{cases} 1, & m = e \\ 0, & m \neq e. \end{cases} \quad (\text{A.125})$$

We note that only one element ($m = e$) of the excitation vector is nonzero.

Current-Source Excitation

Similar to the voltage source, a current source can be used to excite a metallic structure involving open surfaces. A current source and a corresponding sink can be placed at two different edges and an electrical connection is established mathematically. As opposed to the delta-gap excitation, a current source can be placed at a physical edge, which is not shared by two triangles.

As an example, consider again T-EFIE discretized with the RWG functions, and let a current source and a current sink be placed at two physical edges e_1 and e_2 , respectively. Then, we define two half RWG functions at e_1 and e_2 . The resulting matrix equation can be written as

$$\sum_{n=1}^{N+2} \bar{\mathbf{Z}}_E[m, n] \mathbf{a}[n] = \mathbf{w}_E[m] = 0 \quad (m = 1, \dots, N), \quad (\text{A.126})$$

where the dimension of the equation is increased to $N \times (N+2)$ due to extra half functions. We note that $\mathbf{w}_E[m] = 0$ since there is no incident-field excitation in this case and the integral for the RHS in (A.117) evaluates to zero. On the other hand, coefficients corresponding to e_1 and e_2 are fixed, i.e.,

$$\mathbf{a}[e_1] = I_e \quad (\text{A.127})$$

$$\mathbf{a}[e_2] = -I_e. \quad (\text{A.128})$$

In other words, the expansion coefficients $\mathbf{a}[e_1]$ and $\mathbf{a}[e_2]$ are forced to be $\pm I_e$ to simulate the source and the sink. By setting the two coefficients as above, we solve the system

$$\sum_{\substack{n=1 \\ n \neq e_1, e_2}}^{N+2} \bar{\mathbf{Z}}_E[m, n] \mathbf{a}[n] = -I_e \bar{\mathbf{Z}}_E[m, e_1] + I_e \bar{\mathbf{Z}}_E[m, e_2] \quad (m = 1, \dots, N), \quad (\text{A.129})$$

to determine the coefficients $\mathbf{a}[n]$ for $n \neq e_1, e_2$.

A.6 Krylov-Subspace Iterative Methods

Krylov-subspace iterative methods can be investigated in the context of diagonalization processes.

Symmetric Lanczos Process

Consider a matrix equation

$$\bar{\mathbf{Z}} \cdot \mathbf{a} = \mathbf{w}, \quad (\text{A.130})$$

where $\bar{\mathbf{Z}}$ is an $N \times N$ symmetric matrix. We define

$$\beta^{(1)} \mathbf{v}^{(1)} = \mathbf{w} \quad (\text{A.131})$$

and construct

$$\beta^{(k+1)} \mathbf{v}^{(k+1)} = \bar{\mathbf{Z}} \cdot \mathbf{v}^{(k)} - \alpha^{(k)} \mathbf{v}^{(k)} - \beta^{(k)} \mathbf{v}^{(k-1)}, \quad (\text{A.132})$$

where

$$\alpha^{(k)} = [\mathbf{v}^{(k)}]^H \cdot \bar{\mathbf{Z}} \cdot \mathbf{v}^{(k)} - [\mathbf{v}^{(k)}]^H \cdot \beta^{(k)} \mathbf{v}^{(k-1)} \quad (\text{A.133})$$

and $\beta^{(k)} \geq 0$ are chosen appropriately such that $\|\mathbf{v}^{(k)}\|_2 = 1$. The recursion can be rewritten as

$$\begin{aligned} [\bar{\mathbf{Z}}]_{N \times N} \cdot [\bar{\mathbf{V}}]_{N \times k} &= [\bar{\mathbf{V}}]_{N \times k} \cdot [\bar{\Delta}]_{k \times k} \\ &\quad + \beta^{(k+1)} [0 \quad 0 \quad \cdots \quad 0 \quad \mathbf{v}^{(k+1)}]_{N \times k}, \end{aligned} \quad (\text{A.134})$$

where

$$\bar{\mathbf{V}} = [\mathbf{v}^{(1)} \quad \mathbf{v}^{(2)} \quad \cdots \quad \mathbf{v}^{(k)}]_{N \times k} \quad (\text{A.135})$$

is a unitary matrix, i.e.,

$$\bar{\mathbf{V}}^H \cdot \bar{\mathbf{V}} = \bar{\mathbf{I}}. \quad (\text{A.136})$$

This process is called tridiagonalization since

$$\bar{\Delta} = \begin{bmatrix} \alpha^{(1)} & \beta^{(2)} & & & & \\ \beta^{(2)} & \alpha^{(2)} & \beta^{(3)} & & & \\ & \beta^{(3)} & \alpha^{(3)} & \beta^{(4)} & & \\ & & \beta^{(4)} & \alpha^{(4)} & \ddots & \\ & & & \ddots & \ddots & \beta^{(k-1)} \\ & & & & \beta^{(k-1)} & \alpha^{(k-1)} & \beta^{(k)} \\ & & & & & \beta^{(k)} & \alpha^{(k)} \end{bmatrix}_{k \times k} \quad (\text{A.137})$$

is tridiagonal.

Iterative Methods Based on the Symmetric Lanczos Process

If the matrix $\bar{\mathbf{Z}}$ is positive definite, the CG method can be applied to solve the matrix equation in (A.130). In this method, a subproblem

$$[\bar{\Delta}]_{k \times k} \cdot \mathbf{y}^{(k)} = \begin{bmatrix} \beta^{(1)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{k \times 1} \quad (\text{A.138})$$

is solved, and the original solution can be obtained as $\mathbf{a}^{(k)} = \bar{\mathbf{V}} \cdot \mathbf{y}^{(k)}$. For the solution of the subproblem, a Cholesky factorization is used, i.e.,

$$[\bar{\Delta}]_{k \times k} = [\bar{\mathbf{L}}]_{k \times k} \cdot [\bar{\mathbf{D}}]_{k \times k} \cdot [\bar{\mathbf{L}}^H]_{k \times k}, \quad (\text{A.139})$$

where $\bar{\mathbf{L}}$ is lower triangular and $\bar{\mathbf{D}}$ is diagonal.

For a general symmetric $\bar{\mathbf{Z}}$ matrix, which is not positive definite, the MINRES method can be applied by minimizing the norm of

$$\begin{bmatrix} & & \bar{\Delta} & \\ 0 & \dots & 0 & \beta^{(k)} \end{bmatrix}_{(k+1) \times k} \cdot \mathbf{y}^{(k)} - \begin{bmatrix} \beta^{(1)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(k+1) \times 1} \quad (\text{A.140})$$

to obtain $\mathbf{y}^{(k)}$ and $\mathbf{a}^{(k)} = \bar{\mathbf{V}} \cdot \mathbf{y}^{(k)}$. For the solution of the subproblem in (A.140), a QR factorization is used as

$$\begin{bmatrix} & & \bar{\Delta} & \\ 0 & \dots & 0 & \beta^{(k)} \end{bmatrix}_{(k+1) \times k} = [\bar{\mathbf{Q}}]_{(k+1) \times (k+1)} \cdot \begin{bmatrix} \bar{\mathbf{R}} \\ 0 \end{bmatrix}_{(k+1) \times k}, \quad (\text{A.141})$$

where $\bar{\mathbf{Q}}$ is unitary and $\bar{\mathbf{R}}$ is upper triangular.

Nonsymmetric Lanczos Process

Consider a matrix equation in the form of (A.130), where $\bar{\mathbf{Z}}$ is an $N \times N$ non-symmetric matrix. We define

$$\beta^{(1)} \mathbf{v}^{(1)} = \mathbf{w} \quad (\text{A.142})$$

and $\mathbf{u}^{(1)} \cdot \mathbf{v}^{(1)} = 1$ to construct

$$\beta^{(k+1)} \mathbf{u}^{(k+1)} = \bar{\mathbf{Z}} \cdot \mathbf{u}^{(k)} - \alpha^{(k)} \mathbf{u}^{(k)} - \delta^{(k)} \mathbf{u}^{(k-1)} \quad (\text{A.143})$$

$$\delta^{(k+1)} \mathbf{v}^{(k+1)} = \bar{\mathbf{Z}} \cdot \mathbf{v}^{(k)} - \alpha^{(k)} \mathbf{v}^{(k)} - \beta^{(k)} \mathbf{v}^{(k-1)}. \quad (\text{A.144})$$

In (A.143) and (A.144),

$$\alpha^{(k)} = [\mathbf{u}^{(k)}]^H \cdot \bar{\mathbf{Z}} \cdot \mathbf{v}^{(k)}, \quad (\text{A.145})$$

$\beta^{(k)} \geq 0$, and $\delta^{(k)} \geq 0$ are chosen appropriately such that $\|\mathbf{v}^{(k)}\|_2 = 1$ and $\|\mathbf{u}^{(k)}\|_2 = 1$. The recursion can be rewritten as

$$\begin{aligned} [\bar{\mathbf{Z}}]_{N \times N} \cdot [\bar{\mathbf{V}}]_{N \times k} &= [\bar{\mathbf{V}}]_{N \times k} \cdot [\bar{\Delta}]_{k \times k} \\ &\quad + \delta^{(k+1)} [0 \quad 0 \quad \dots \quad 0 \quad \mathbf{v}^{(k+1)}]_{N \times k} \end{aligned} \quad (\text{A.146})$$

$$\begin{aligned} [\bar{\mathbf{Z}}]_{N \times N} \cdot [\bar{\mathbf{U}}]_{N \times k} &= [\bar{\mathbf{U}}]_{N \times k} \cdot [\bar{\Delta}]_{k \times k} \\ &\quad + \beta^{(k+1)} [0 \quad 0 \quad \dots \quad 0 \quad \mathbf{v}^{(k+1)}]_{N \times k}, \end{aligned} \quad (\text{A.147})$$

where

$$\bar{\mathbf{V}} = [\mathbf{v}^{(1)} \quad \mathbf{v}^{(2)} \quad \dots \quad \mathbf{v}^{(k)}]_{N \times k} \quad (\text{A.148})$$

and

$$\bar{\mathbf{U}} = [\mathbf{u}^{(1)} \quad \mathbf{u}^{(2)} \quad \dots \quad \mathbf{u}^{(k)}]_{N \times k} \quad (\text{A.149})$$

satisfy

$$\bar{\mathbf{U}}^H \cdot \bar{\mathbf{V}} = \bar{\mathbf{I}}. \quad (\text{A.150})$$

This process is also called tridiagonalization, since

$$\bar{\Delta} = \begin{bmatrix} \alpha^{(1)} & \beta^{(2)} & & & & \\ \delta^{(2)} & \alpha^{(2)} & \beta^{(3)} & & & \\ & \delta^{(3)} & \alpha^{(3)} & \beta^{(4)} & & \\ & & \delta^{(4)} & \alpha^{(4)} & \ddots & \\ & & & \ddots & \ddots & \beta^{(k-1)} \\ & & & & \delta^{(k-1)} & \alpha^{(k-1)} & \beta^{(k)} \\ & & & & & \delta^{(k)} & \alpha^{(k)} \end{bmatrix}_{k \times k} \quad (\text{A.151})$$

is tridiagonal.

Iterative Methods Based on the Nonsymmetric Lanczos Process

BiCG and QMR methods, as well as their transpose-free variants, i.e., BiCGStab, CGS, TFQMR, are based on the nonsymmetric Lanczos process. In BiCG, a subproblem similar to (A.138) is solved via an LU factorization, i.e.,

$$\bar{\Delta}_{k \times k} = \bar{L}_{k \times k} \cdot \bar{D}_{k \times k} \cdot \bar{R}_{k \times k}. \quad (\text{A.152})$$

QMR is similar to MINRES, and the norm of

$$\begin{bmatrix} & & \bar{\Delta} & \\ 0 & \dots & 0 & \delta^{(k)} \end{bmatrix}_{(k+1) \times k} \cdot \mathbf{y}^{(k)} - \begin{bmatrix} \delta^{(1)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(k+1) \times 1} \quad (\text{A.153})$$

is minimized by using a QR factorization.

Arnoldi Process

Arnoldi process involves a recursion in the form of

$$\begin{aligned} [\bar{\mathbf{Z}}]_{N \times N} \cdot [\bar{\mathbf{V}}]_{N \times k} &= [\bar{\mathbf{V}}]_{N \times k} \cdot [\bar{\mathbf{H}}]_{k \times k} \\ &+ \beta^{(k+1)} [0 \quad 0 \quad \dots \quad 0 \quad \mathbf{v}^{(k+1)}]_{N \times k}, \end{aligned} \quad (\text{A.154})$$

$$\begin{aligned}
[\bar{\mathbf{Z}}]_{N \times N} \cdot [\bar{\mathbf{V}}]_{N \times k} &= [\bar{\mathbf{V}}]_{N \times k} \cdot [\bar{\mathbf{H}}]_{k \times k} \\
&+ \beta^{(k+1)} [0 \quad 0 \quad \cdots \quad 0 \quad \mathbf{v}^{(k+1)}]_{N \times k},
\end{aligned} \tag{A.155}$$

where $\bar{\mathbf{V}} = [\mathbf{v}_n^{(1)} \quad \mathbf{v}_n^{(2)} \quad \cdots \quad \mathbf{v}_n^{(k)}]$ is again a unitary matrix. As opposed to the matrix obtained by a tridiagonalization, $\bar{\mathbf{H}}$ is a Hessian matrix. A subproblem similar to (A.138) is considered for the solution. The efficient GMRES method is based on the Arnoldi process.

Golub-Kahan Process

For a general $N \times N$ matrix equation in the form of (A.130), Golub-Kahan process starts with

$$\beta^{(1)} \mathbf{x}^{(1)} = \mathbf{w} \tag{A.156}$$

$$\alpha^{(1)} \mathbf{v}^{(1)} = \bar{\mathbf{Z}}^H \cdot \mathbf{x}^{(1)} \tag{A.157}$$

and construct a recursion as

$$\beta^{(k+1)} \mathbf{x}^{(k+1)} = \bar{\mathbf{Z}} \cdot \mathbf{v}^{(k)} - \alpha^{(k)} \mathbf{x}^{(k)} \tag{A.158}$$

$$\alpha^{(k+1)} \mathbf{v}^{(k+1)} = \bar{\mathbf{Z}}^H \cdot \mathbf{x}^{(k+1)} - \beta^{(k+1)} \mathbf{v}^{(k)}, \tag{A.159}$$

where $\alpha^{(k)} \geq 0$ and $\beta^{(k)} \geq 0$ are chosen appropriately such that $\|\mathbf{x}^{(k)}\|_2 = 1$ and $\|\mathbf{v}^{(k)}\|_2 = 1$. The recursion can be rewritten as

$$[\bar{\mathbf{Z}}]_{N \times N} \cdot [\bar{\mathbf{V}}]_{N \times k} = [\bar{\mathbf{X}}]_{N \times (k+1)} \cdot [\bar{\mathbf{\Lambda}}]_{(k+1) \times k} \tag{A.160}$$

$$\begin{aligned}
[\bar{\mathbf{Z}}^H]_{N \times N} \cdot [\bar{\mathbf{X}}]_{N \times (k+1)} &= [\bar{\mathbf{V}}]_{N \times k} \cdot [\bar{\mathbf{\Lambda}}^H]_{k \times (k+1)} \\
&+ \alpha^{(k+1)} [0 \quad 0 \quad \cdots \quad 0 \quad \mathbf{v}^{(k+1)}]_{N \times (k+1)},
\end{aligned} \tag{A.161}$$

where $\bar{\mathbf{A}}$ is lower-bidiagonal, i.e.,

$$\bar{\mathbf{A}} = \begin{bmatrix} \alpha^{(1)} & & & & & \\ \beta^{(2)} & \alpha^{(2)} & & & & \\ & \beta^{(3)} & \alpha^{(3)} & & & \\ & & \beta^{(4)} & \alpha^{(4)} & & \\ & & & \ddots & \ddots & \\ & & & & \beta^{(k-1)} & \alpha^{(k-1)} \\ & & & & & \beta^{(k)} & \alpha^{(k)} \\ & & & & & & \beta^{(k+1)} \end{bmatrix}_{(k+1) \times k}. \quad (\text{A.162})$$

The LSQR method is based on the Golub-Kahan process, where a subproblem similar to (A.140) is solved via a QR factorization.

A.7 Least-Squares QR Method

The LSQR method is different from many other Krylov-subspace algorithms by providing the solution of different kinds of matrix equations, such as rectangular and rank-deficient square systems, in addition to square systems with full rank. In this section, we focus on LSQR and present major stages of this algorithm by using various examples.

Bidiagonalization

The core of LSQR, i.e., the Golub-Kahan process, can also be called bidiagonalization, since the resulting matrix in (A.162) is bidiagonal. As an example, we

consider a 4×4 matrix equation, i.e.,

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \cdot \mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}. \quad (\text{A.163})$$

Using (A.156)–(A.159), the first and the second iterations can be written as

$$[\bar{\mathbf{Z}}]_{4 \times 4} \cdot \begin{bmatrix} 0.42 \\ 0.47 \\ 0.52 \\ 0.57 \end{bmatrix} = \begin{bmatrix} 0.18 & -0.63 \\ 0.37 & -0.59 \\ 0.55 & -0.05 \\ 0.73 & 0.49 \end{bmatrix} \cdot \begin{bmatrix} 35.2 \\ 2.80 \end{bmatrix} \quad (\text{A.164})$$

and

$$[\bar{\mathbf{Z}}]_{4 \times 4} \cdot \begin{bmatrix} 0.42 & 0.78 \\ 0.47 & 0.23 \\ 0.52 & -0.32 \\ 0.57 & -0.48 \end{bmatrix} = \begin{bmatrix} 0.18 & -0.63 & -0.75 \\ 0.37 & -0.59 & 0.59 \\ 0.55 & -0.05 & 0.18 \\ 0.73 & 0.49 & -0.24 \end{bmatrix} \cdot \begin{bmatrix} 35.2 & 0 \\ 2.80 & 1.69 \\ 0 & 0.09 \end{bmatrix}. \quad (\text{A.165})$$

We note that

$$\begin{bmatrix} 0.42 & 0.47 & 0.52 & 0.57 \\ 0.78 & 0.23 & -0.32 & -0.48 \end{bmatrix} \cdot \begin{bmatrix} 0.42 & 0.78 \\ 0.47 & 0.23 \\ 0.52 & -0.32 \\ 0.57 & -0.48 \end{bmatrix} = \bar{\mathbf{I}}_{2 \times 2}, \quad (\text{A.166})$$

$$\begin{bmatrix} 0.18 & 0.37 & 0.55 & 0.73 \\ -0.63 & -0.59 & -0.05 & 0.49 \\ -0.75 & 0.59 & 0.18 & -0.24 \end{bmatrix} \cdot \begin{bmatrix} 0.18 & -0.63 & -0.75 \\ 0.37 & -0.59 & 0.59 \\ 0.55 & -0.05 & 0.18 \\ 0.73 & 0.49 & -0.24 \end{bmatrix} = \bar{\mathbf{I}}_{3 \times 3}, \quad (\text{A.167})$$

and

$$\begin{bmatrix} 35.2 & 0 \\ 2.80 & 1.69 \\ 0 & 0.09 \end{bmatrix} \quad (\text{A.168})$$

is bidiagonal. Recursions in (A.156)–(A.159) can be performed one more time as a third iteration, but this leads to $\beta^{(4)} = 0$ since the rank of $\bar{\mathbf{Z}}$ is only three. Therefore, in this example, the bidiagonalization process must stop after two iterations.

Next, we consider a matrix equation involving a 4×4 matrix with full rank, i.e.,

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 3 \\ 8 & 9 & 10 & 5 \\ 12 & 6 & 14 & 15 \end{bmatrix} \cdot \mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}. \quad (\text{A.169})$$

In this case, the bidiagonalization process involves three iterations and we obtain

$$[\bar{\mathbf{Z}}]_{4 \times 4} \cdot \begin{bmatrix} 0.49 & 0.14 & -0.75 \\ 0.38 & -0.69 & -0.21 \\ 0.61 & -0.23 & 0.61 \\ 0.50 & 0.68 & 0.15 \end{bmatrix} = \begin{bmatrix} 0.18 & -0.55 & 0.44 & 0.69 \\ 0.37 & -0.58 & 0.21 & -0.70 \\ 0.55 & -0.19 & -0.79 & 0.20 \\ 0.73 & 0.57 & 0.38 & 0.03 \end{bmatrix} \cdot \begin{bmatrix} 30.3 & 0 & 0 \\ 3.53 & 5.27 & 0 \\ 0 & 3.82 & 1.37 \\ 0 & 0 & 0.32 \end{bmatrix}. \quad (\text{A.170})$$

Finally, we consider an over-determined matrix equation involving a 4×3 matrix,

i.e.,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 12 & 6 & 14 \end{bmatrix} \cdot \mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}. \quad (\text{A.171})$$

After two iterations, the bidiagonalization process leads to

$$[\bar{\mathbf{Z}}]_{4 \times 3} \cdot \begin{bmatrix} 0.56 & 0.62 \\ 0.43 & -0.78 \\ 0.71 & -0.02 \end{bmatrix} = \begin{bmatrix} 0.18 & -0.76 & 0.48 \\ 0.37 & -0.43 & -0.12 \\ 0.55 & -0.10 & -0.72 \\ 0.73 & 0.48 & 0.48 \end{bmatrix} \cdot \begin{bmatrix} 25.4 & 0 \\ 1.44 & 2.85 \\ 0 & 2.42 \end{bmatrix}. \quad (\text{A.172})$$

Generating Subproblems

Consider the iterative solution of a least-squares problem

$$\min_{\mathbf{a}} \|\mathbf{r}\|_2 = \min_{\mathbf{a}} \|\mathbf{w} - \bar{\mathbf{Z}}_{M \times N} \cdot \mathbf{a}\|_2, \quad (\text{A.173})$$

where, in general, $M \neq N$. For k th iteration, we use (A.156) and (A.160) to arrive at

$$\begin{aligned} \mathbf{r} &= \beta^{(1)} \mathbf{x}^{(1)} - [\bar{\mathbf{X}}]_{M \times (k+1)} \cdot [\bar{\mathbf{\Lambda}}]_{(k+1) \times k} \cdot [\bar{\mathbf{V}}^H]_{k \times N} \cdot \mathbf{a}^{(k)} \\ &= [\bar{\mathbf{X}}]_{M \times (k+1)} \cdot \left\{ \begin{bmatrix} \beta^{(1)} & 0 & \dots & 0 \end{bmatrix}^\dagger - [\bar{\mathbf{\Lambda}}]_{(k+1) \times k} \cdot [\bar{\mathbf{V}}^H]_{k \times N} \cdot \mathbf{a}^{(k)} \right\} \\ &= [\bar{\mathbf{X}}]_{M \times (k+1)} \cdot \left\{ \begin{bmatrix} \beta^{(1)} & 0 & \dots & 0 \end{bmatrix}^\dagger - [\bar{\mathbf{\Lambda}}]_{(k+1) \times k} \cdot \mathbf{y}^{(k)} \right\}, \end{aligned} \quad (\text{A.174})$$

where

$$\mathbf{y}^{(k)} = [\bar{\mathbf{V}}^H]_{k \times N} \cdot \mathbf{a}^{(k)} \quad (\text{A.175})$$

or

$$\mathbf{a}^{(k)} = [\bar{\mathbf{V}}]_{N \times k} \cdot \mathbf{y}^{(k)}. \quad (\text{A.176})$$

Then instead of solving the original problem in (A.173), we consider the subproblem

$$\min_{\mathbf{y}^{(k)}} \|\mathbf{r}_{sub}\|_2 = \min_{\mathbf{y}^{(k)}} \left\| \begin{bmatrix} \beta^{(1)} & 0 & \dots & 0 \end{bmatrix}^\dagger - [\bar{\mathbf{A}}]_{(k+1) \times k} \cdot \mathbf{y}^{(k)} \right\|_2 \quad (\text{A.177})$$

and obtain the actual solution via (A.176). We note that the solution of the subproblem in (A.177) is significantly easier than the solution of the original problem in (A.173) since $\bar{\mathbf{A}}$ is a bidiagonal matrix. However, (A.177) should be solved at each iteration k of the bidiagonalization process.

As an example, we consider the solution the problem in (A.163). For $k = 2$,

$$[\bar{\mathbf{A}}]_{(k+1) \times k} = \begin{bmatrix} 35.2 & 0 \\ 2.80 & 1.69 \\ 0 & 0.09 \end{bmatrix} \quad (\text{A.178})$$

and $\beta^{(1)} = 5.48$. Then, the subproblem is minimizing the 2-norm of

$$\begin{bmatrix} 35.2 & 0 \\ 2.80 & 1.69 \\ 0 & 0.09 \end{bmatrix} \cdot \mathbf{y}^{(k)} = \begin{bmatrix} 5.48 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{A.179})$$

which gives $\mathbf{y}^{(k)} = [0.16 \quad -0.26]^\dagger$. Finally, we obtain the solution for the original problem as

$$\mathbf{a}^{(k)} = \begin{bmatrix} 0.42 & 0.78 \\ 0.47 & 0.23 \\ 0.52 & -0.32 \\ 0.57 & -0.48 \end{bmatrix} \cdot \mathbf{y}^{(k)} = \begin{bmatrix} -0.14 \\ 0.01 \\ 0.16 \\ 0.22 \end{bmatrix}. \quad (\text{A.180})$$

QR Factorization

Solution of the subproblem in (A.177) can be performed by using a QR factorization, i.e.,

$$[\bar{\mathbf{Q}}]_{(k+1) \times (k+1)} \cdot \begin{bmatrix} \bar{\mathbf{R}} \\ 0 \end{bmatrix}_{(k+1) \times k} \cdot \mathbf{y}^{(k)} = \begin{bmatrix} \beta^{(1)} & 0 & \dots & 0 \end{bmatrix}^\dagger \quad (\text{A.181})$$

or

$$\begin{bmatrix} \bar{\mathbf{R}} \\ 0 \end{bmatrix}_{(k+1) \times k} \cdot \mathbf{y}^{(k)} = [\bar{\mathbf{Q}}^H]_{(k+1) \times (k+1)} \cdot \begin{bmatrix} \beta^{(1)} & 0 & \dots & 0 \end{bmatrix}^\dagger. \quad (\text{A.182})$$

We note that $\beta^{(1)} \bar{\mathbf{Q}}^*[1, k+1]$ should be zero for consistency. Otherwise, the system is inconsistent and it can be solved only in the least-squares sense.

Following the QR factorization, we obtain a sparse matrix equation in the form of

$$\begin{bmatrix} R_{11} & R_{12} & & & \\ & R_{22} & R_{23} & & \\ & & R_{33} & R_{34} & \\ & & & \ddots & \ddots \\ & & & & \ddots & R_{(k-1)k} \\ & & & & & R_{kk} \end{bmatrix}_{k \times k} \cdot \mathbf{y}^{(k)} = \beta_1 \begin{bmatrix} Q_{11}^* \\ Q_{12}^* \\ \vdots \\ \vdots \\ Q_{1k}^* \end{bmatrix}, \quad (\text{A.183})$$

where $R_{mn} = \mathbf{R}[m, n]$ and $Q_{mn} = \bar{\mathbf{Q}}[m, n]$. We eliminate $\mathbf{y}^{(k)}$ by using (A.176)

as

$$\mathbf{a}^{(k)} = \beta_1 [\bar{\mathbf{V}}]_{N \times k} \cdot \begin{bmatrix} R_{11} & R_{12} & & & \\ & R_{22} & R_{23} & & \\ & & R_{33} & R_{34} & \\ & & & \ddots & \ddots \\ & & & & \ddots & R_{(k-1)k} \\ & & & & & R_{kk} \end{bmatrix}_{k \times k}^{-1} \cdot \begin{bmatrix} Q_{11}^* \\ Q_{12}^* \\ \vdots \\ \vdots \\ Q_{1k}^* \end{bmatrix}. \quad (\text{A.184})$$

Finally, the expression in (A.184) can be used to derive a recursive relation as

$$\mathbf{a}^{(k)} = \mathbf{a}^{(k-1)} + \beta_1 Q_{1k}^* \mathbf{z}^{(k)} \quad (\text{A.185})$$

$$\mathbf{z}^{(k)} = \frac{1}{R_{kk}} \mathbf{v}^{(k)} - \frac{R_{(k-1)k}}{R_{kk}} \mathbf{z}^{(k-1)}. \quad (\text{A.186})$$

We note that the subproblem in (A.177) is never solved directly. Elements of the QR factorization, i.e., R_{kk} , $R_{(k-1)k}$, and Q_{1k} , are used in (A.185) and (A.186) to obtain $\mathbf{a}^{(k)}$. In order to calculate those elements, the required QR factorization in the k th iteration is not performed from scratch. Instead, we use recursive relations, as detailed in [113].

A.8 Rao-Wilton-Glisson and Linear-Linear Functions

In this section, we use the notation in [57] to derive the LL functions from the RWG functions by a decomposition. The zeroth-order divergence-conforming bases for triangular elements, such as the RWG functions, can be written as

$$\mathbf{f}_\beta(\mathbf{r}) = \frac{1}{J_\beta} [\zeta_{\beta+1}(\mathbf{r}) \boldsymbol{\ell}_{\beta-1} - \zeta_{\beta-1}(\mathbf{r}) \boldsymbol{\ell}_{\beta+1}] \quad (\beta = 1, 2, 3), \quad (\text{A.187})$$

where J_β indicates the Jacobian derived for the RWG functions as

$$J_\beta = \frac{2A}{|\boldsymbol{\ell}_\beta|} \quad (\text{A.188})$$

and A is the area of the triangle. In (A.187), $\zeta_i(\mathbf{r})$ for $i = 1, 2, 3$ represent parent coordinates with the dependency relationship

$$\zeta_1(\mathbf{r}) + \zeta_2(\mathbf{r}) + \zeta_3(\mathbf{r}) = 1 \quad (\text{A.189})$$

when $\mathbf{r}$ is on the triangle. In addition, $\boldsymbol{\ell}_i$ in (A.187) for $i = 1, 2, 3$ represent edge vectors with the dependence

$$\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 + \boldsymbol{\ell}_3 = 0. \quad (\text{A.190})$$

The RWG functions, which provide three degrees of freedom per triangle, are zeroth-order complete since they can represent any constant vector function on a triangle and their surface divergence is also constant. Given the three RWG functions corresponding to the three edges of a triangle as in (A.187), the set of LL functions associated with the same triangle can be derived as

$$\mathbf{f}_\beta^{(1)}(\mathbf{r}) = \frac{1}{J_\beta} \zeta_{\beta+1}(\mathbf{r}) \boldsymbol{\ell}_{\beta-1} \quad (\text{A.191})$$

$$\mathbf{f}_\beta^{(2)}(\mathbf{r}) = -\frac{1}{J_\beta} \zeta_{\beta-1}(\mathbf{r}) \boldsymbol{\ell}_{\beta+1} \quad (\text{A.192})$$

with the decomposition

$$\mathbf{f}_\beta(\mathbf{r}) = \mathbf{f}_\beta^{(1)}(\mathbf{r}) + \mathbf{f}_\beta^{(2)}(\mathbf{r}) \quad (\beta = 1, 2, 3). \quad (\text{A.193})$$

We note that both of the distributions in (A.187) and (A.191)–(A.192) provide a linear variation for parallel components along the edges β . On the other hand,

$$\nabla \zeta_\beta(\mathbf{r})|_{\zeta_\beta(\mathbf{r})=0} \cdot \mathbf{f}_\beta^{(1)}(\mathbf{r}) = \zeta_{\beta+1}(\mathbf{r}) \quad (\text{A.194})$$

$$\nabla \zeta_\beta(\mathbf{r})|_{\zeta_\beta(\mathbf{r})=0} \cdot \mathbf{f}_\beta^{(2)}(\mathbf{r}) = \zeta_{\beta-1}(\mathbf{r}) \quad (\text{A.195})$$

so that normal components of the LL functions in (A.191)–(A.192) have also linear variation along edges β , while they are constant for the RWG functions in (A.187), i.e.,

$$\nabla \zeta_\beta(\mathbf{r})|_{\zeta_\beta(\mathbf{r})=0} \cdot \mathbf{f}_\beta(\mathbf{r}) = \zeta_{\beta+1}(\mathbf{r}) + \zeta_{\beta-1}(\mathbf{r}) = 1 - \zeta_\beta(\mathbf{r}) = 1. \quad (\text{A.196})$$

Any linearly varying vector function on the triangle can be represented by a

combination of the LL functions since

$$\frac{\mathbf{f}_2^{(1)}(\mathbf{r})}{|\boldsymbol{\ell}_2|} + \frac{\mathbf{f}_2^{(2)}(\mathbf{r})}{|\boldsymbol{\ell}_2|} - \frac{\mathbf{f}_3^{(1)}(\mathbf{r})}{|\boldsymbol{\ell}_3|} - \frac{\mathbf{f}_3^{(2)}(\mathbf{r})}{|\boldsymbol{\ell}_3|} = \frac{\boldsymbol{\ell}_1}{2A} \quad (\text{A.197})$$

$$\frac{\mathbf{f}_3^{(1)}(\mathbf{r})}{|\boldsymbol{\ell}_3|} + \frac{\mathbf{f}_3^{(2)}(\mathbf{r})}{|\boldsymbol{\ell}_3|} - \frac{\mathbf{f}_1^{(1)}(\mathbf{r})}{|\boldsymbol{\ell}_1|} - \frac{\mathbf{f}_1^{(2)}(\mathbf{r})}{|\boldsymbol{\ell}_1|} = \frac{\boldsymbol{\ell}_2}{2A} \quad (\text{A.198})$$

$$\frac{\mathbf{f}_2^{(2)}(\mathbf{r})}{|\boldsymbol{\ell}_2|} - \frac{\mathbf{f}_3^{(1)}(\mathbf{r})}{|\boldsymbol{\ell}_3|} = \frac{\zeta_1(\mathbf{r})\boldsymbol{\ell}_1}{2A} \quad (\text{A.199})$$

$$\frac{\mathbf{f}_3^{(2)}(\mathbf{r})}{|\boldsymbol{\ell}_3|} - \frac{\mathbf{f}_1^{(1)}(\mathbf{r})}{|\boldsymbol{\ell}_1|} = \frac{\zeta_2(\mathbf{r})\boldsymbol{\ell}_2}{2A} \quad (\text{A.200})$$

$$- \frac{\mathbf{f}_3^{(2)}(\mathbf{r})}{|\boldsymbol{\ell}_3|} = \frac{\zeta_2(\mathbf{r})\boldsymbol{\ell}_1}{2A} \quad (\text{A.201})$$

$$\frac{\mathbf{f}_3^{(1)}(\mathbf{r})}{|\boldsymbol{\ell}_3|} = \frac{\zeta_1(\mathbf{r})\boldsymbol{\ell}_2}{2A}. \quad (\text{A.202})$$

On the other hand, the LL functions are not strictly first-order complete since their surface divergences

$$\nabla_S \cdot \mathbf{f}_\beta^{(1)}(\mathbf{r}) = \frac{1}{J_\beta} \boldsymbol{\ell}_{\beta-1} \cdot \nabla \zeta_{\beta+1}(\mathbf{r}) = \frac{1}{J_\beta} = \frac{|\boldsymbol{\ell}_\beta|}{2A} \quad (\text{A.203})$$

$$\nabla_S \cdot \mathbf{f}_\beta^{(2)}(\mathbf{r}) = -\frac{1}{J_\beta}(\mathbf{r})\boldsymbol{\ell}_{\beta+1} \cdot \nabla \zeta_{\beta-1}(\mathbf{r}) = -\frac{1}{J_\beta} = -\frac{|\boldsymbol{\ell}_\beta|}{2A} \quad (\text{A.204})$$

for $\beta = 1, 2$, or 3 are constants. As indicated in [57], a strictly first-order complete basis requires eight degrees of freedom on the triangle, whereas the LL functions have six degrees of freedom. Nevertheless, LL functions have the advantage that their implementations do not require higher-order techniques while providing significantly higher accuracy than the RWG functions.

○

Appendix B

Extras

B.1 Mie-Series Solutions

Mie-series solutions of scattering problems involving spherical objects are known for about 100 years [14]. The solution procedure is based on a purely analytical application of the Maxwell's equations on spherical objects that are illuminated by electromagnetic waves. Although the solution method is named after its developer German physicist Gustav Mie, it is known that the basic theory was presented previously by Ludvig Valentin Lorenz in 1890. Since then, many variations of the theory have been published in journal papers and text books. Mie-series solutions are still used in electromagnetics, especially to test the accuracy of computational solvers.

Mie-series solutions involve the expansion of electromagnetic fields in terms of spherical functions. Among various types of procedures, we follow a rigorous derivation based on Debye potentials. We consider time-harmonic scattering from spherical objects located at the origin in a homogeneous medium extending to infinity. Angular frequency is ω . Total electric and magnetic fields in each

homogeneous medium are separated into known incident and unknown scattered fields. By applying boundary conditions on spherical surfaces, unknown coefficients for scattered fields can be determined analytically. In theory, electromagnetic fields can only be expanded by infinite numbers of terms. Therefore, an exact solution cannot be generated by using a computer program. However, a finite number of terms is usually sufficient to obtain the solution with a desired level of accuracy.

B.1.1 Definitions

Some of spherical functions, which are frequently used in Mie-series solutions, are defined as

$$\omega_n(\theta) = \frac{n}{\sin \theta} \left[\cos \theta P_n(\cos \theta) - P_{n-1}(\cos \theta) \right] \quad (\text{B.1})$$

$$\Omega_n(\theta) = \sin \theta \frac{\partial}{\partial \theta} \omega_n(\theta) \quad (\text{B.2})$$

$$= \frac{n}{\sin \theta} \left[(n \cos^2 \theta - 1) P_n(\cos \theta) - 2(n-1) \cos \theta P_{n-1}(\cos \theta) + (n-1) P_{n-2}(\cos \theta) \right] \quad (\text{B.3})$$

$$\gamma_n(kr) = kr h_n^{(1)}(kr) \quad (\text{B.4})$$

$$\Gamma_n(kr) = \frac{\partial}{\partial r} \left[r h_n^{(1)}(kr) \right] = \left[kr h_{n-1}^{(1)}(kr) - n h_n^{(1)}(kr) \right] \quad (\text{B.5})$$

$$v_n(kr) = kr j_n(kr) \quad (\text{B.6})$$

$$\Upsilon_n(kr) = \frac{\partial}{\partial r} \left[r j_n(kr) \right] = \left[kr j_{n-1}(kr) - n j_n(kr) \right]. \quad (\text{B.7})$$

B.1.2 Debye Potentials

Debye potentials at $\mathbf{r} = (r, \theta, \phi)$ are defined as

$$\pi_e(\mathbf{r}) = \frac{\cos \phi}{\eta} \sum_{n=1}^{\infty} a_n \nu(kr) \omega_n(\theta) \quad (\text{B.8})$$

$$\pi_m(\mathbf{r}) = -\sin \phi \sum_{n=1}^{\infty} b_n \nu(kr) \omega_n(\theta), \quad (\text{B.9})$$

where ν is a spherical Bessel or spherical Hankel function. In (B.8) and (B.9), a_n and b_n are coefficients determined by boundary conditions. Derivatives of Debye potentials are derived as

$$\frac{\partial}{\partial \theta} \pi_e(\mathbf{r}) = \frac{\cos \phi}{\eta \sin \theta} \sum_{n=1}^{\infty} a_n \nu_n(kr) \Omega_n(\theta) \quad (\text{B.10})$$

$$\frac{\partial}{\partial \phi} \pi_e(\mathbf{r}) = -\frac{\sin \phi}{\eta} \sum_{n=1}^{\infty} a_n \nu_n(kr) \omega_n(\theta) \quad (\text{B.11})$$

$$\frac{\partial}{\partial \theta} \pi_m(\mathbf{r}) = -\frac{\sin \phi}{\sin \theta} \sum_{n=1}^{\infty} b_n \nu_n(kr) \Omega_n(\theta) \quad (\text{B.12})$$

$$\frac{\partial}{\partial \phi} \pi_m(\mathbf{r}) = -\cos \phi \sum_{n=1}^{\infty} b_n \nu_n(kr) \omega_n(\theta). \quad (\text{B.13})$$

B.1.3 Electric and Magnetic Fields

Electromagnetic fields can be obtained from Debye potentials as

$$E_\theta(\mathbf{r}) = \frac{i}{\omega \epsilon r} \frac{\partial^2}{\partial r \partial \theta} r \pi_e(\mathbf{r}) + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \pi_m(\mathbf{r}) \quad (\text{B.14})$$

$$E_\phi(\mathbf{r}) = \frac{i}{\omega \epsilon r \sin \theta} \frac{\partial^2}{\partial r \partial \phi} r \pi_e(\mathbf{r}) - \frac{\partial}{\partial \theta} \pi_m(\mathbf{r}) \quad (\text{B.15})$$

$$H_\theta(\mathbf{r}) = -\frac{i}{\omega \mu r} \frac{\partial^2}{\partial r \partial \theta} r \pi_m(\mathbf{r}) + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \pi_e(\mathbf{r}) \quad (\text{B.16})$$

$$H_\phi(\mathbf{r}) = -\frac{i}{\omega \mu r \sin \theta} \frac{\partial^2}{\partial r \partial \phi} r \pi_m(\mathbf{r}) - \frac{\partial}{\partial \theta} \pi_e(\mathbf{r}). \quad (\text{B.17})$$

B.1.4 Incident Fields

We assume that spherical objects are located in a homogeneous space with electromagnetic parameters (ϵ_o, μ_o) and they are illuminated by a plane wave propagating in the z direction with the electric field polarized in the x direction, i.e.,

$$E_\theta^{inc}(\mathbf{r}) = \cos \theta \cos \phi \exp(ik_o r \cos \theta) \quad (\text{B.18})$$

$$E_\phi^{inc}(\mathbf{r}) = -\sin \phi \exp(ik_o r \cos \theta). \quad (\text{B.19})$$

Incident electric and magnetic fields can be expanded in a series of spherical functions at $\mathbf{r} = (r, \theta, \phi)$ as

$$E_{\theta}^{inc}(\mathbf{r}) = -\frac{\cos \phi}{k_o r \sin \theta} \sum_{n=1}^{\infty} R_n^{(0)} \left[v_n(k_o r) \omega_n(\theta) - i \Upsilon_n(k_o r) \Omega_n(\theta) \right] \quad (\text{B.20})$$

$$E_{\phi}^{inc}(\mathbf{r}) = \frac{\sin \phi}{k_o r \sin \theta} \sum_{n=1}^{\infty} R_n^{(0)} \left[v_n(k_o r) \Omega_n(\theta) - i \Upsilon_n(k_o r) \omega_n(\theta) \right] \quad (\text{B.21})$$

$$H_{\theta}^{inc}(\mathbf{r}) = \frac{i \sin \phi}{k_o \eta_o r \sin \theta} \sum_{n=1}^{\infty} R_n^{(0)} \left[\Upsilon_n(k_o r) \Omega_n(\theta) + i v_n(k_o r) \omega_n(\theta) \right] \quad (\text{B.22})$$

$$H_{\phi}^{inc}(\mathbf{r}) = \frac{i \cos \phi}{k_o \eta_o r \sin \theta} \sum_{n=1}^{\infty} R_n^{(0)} \left[\Upsilon_n(k_o r) \omega_n(\theta) + i v_n(k_o r) \Omega_n(\theta) \right], \quad (\text{B.23})$$

where

$$R_n^{(0)} = \frac{(-i)^{-n} (2n+1)}{n(n+1)}. \quad (\text{B.24})$$

B.1.5 Perfectly-Conducting Sphere

Consider a PEC sphere with a radius of r_o . Scattered electric and magnetic fields outside the sphere at $\mathbf{r} = (r, \theta, \phi)$ can be derived as

$$E_{\theta}^{sca,o}(\mathbf{r}) = -\frac{\cos \phi}{k_o r \sin \theta} \sum_{n=1}^{\infty} \left[b_n \gamma_n(k_o r) \omega_n(\theta) - i a_n \Gamma_n(k_o r) \Omega_n(\theta) \right] \quad (\text{B.25})$$

$$E_{\phi}^{sca,o}(\mathbf{r}) = \frac{\sin \phi}{k_o r \sin \theta} \sum_{n=1}^{\infty} \left[b_n \gamma_n(k_o r) \Omega_n(\theta) - i a_n \Gamma_n(k_o r) \omega_n(\theta) \right] \quad (\text{B.26})$$

$$H_{\theta}^{sca,o}(\mathbf{r}) = \frac{i \sin \phi}{k_o \eta_o r \sin \theta} \sum_{n=1}^{\infty} \left[b_n \Gamma_n(k_o r) \Omega_n(\theta) + i a_n \gamma_n(k_o r) \omega_n(\theta) \right] \quad (\text{B.27})$$

$$H_{\phi}^{sca,o}(\mathbf{r}) = \frac{i \cos \phi}{k_o \eta_o r \sin \theta} \sum_{n=1}^{\infty} \left[b_n \Gamma_n(k_o r) \omega_n(\theta) + i a_n \gamma_n(k_o r) \Omega_n(\theta) \right]. \quad (\text{B.28})$$

Applying boundary conditions on the PEC surface, we obtain

$$a_n = -R_n^{(0)} \frac{\Upsilon_n(k_o r_o)}{\Gamma_n(k_o r_o)} \quad (\text{B.29})$$

$$b_n = -R_n^{(0)} \frac{v_n(k_o r_o)}{\gamma_n(k_o r_o)}. \quad (\text{B.30})$$

B.1.6 Dielectric Sphere

Consider a lossless dielectric sphere with a radius of r_o and with electromagnetic parameters (ϵ_i, μ_i) . Electric and magnetic fields inside the sphere can be derived as

$$E_\theta^{sca,i}(\mathbf{r}) = -\frac{\cos \phi}{k_i r \sin \theta} \sum_{n=1}^{\infty} \left[d_n v_n(k_i r) \omega_n(\theta) - i c_n \Upsilon_n(k_i r) \Omega_n(\theta) \right] \quad (\text{B.31})$$

$$E_\phi^{sca,i}(\mathbf{r}) = \frac{\sin \phi}{k_i r \sin \theta} \sum_{n=1}^{\infty} \left[d_n v_n(k_i r) \Omega_n(\theta) - i c_n \Upsilon_n(k_i r) \omega_n(\theta) \right] \quad (\text{B.32})$$

$$H_\theta^{sca,i}(\mathbf{r}) = \frac{i \sin \phi}{k_i \eta_i r \sin \theta} \sum_{n=1}^{\infty} \left[d_n \Upsilon_n(k_i r) \Omega_n(\theta) + i c_n v_n(k_i r) \omega_n(\theta) \right] \quad (\text{B.33})$$

$$H_\phi^{sca,i}(\mathbf{r}) = \frac{i \cos \phi}{k_i \eta_i r \sin \theta} \sum_{n=1}^{\infty} \left[d_n \Upsilon_n(k_i r) \omega_n(\theta) + i c_n v_n(k_i r) \Omega_n(\theta) \right]. \quad (\text{B.34})$$

Equations (B.20)–(B.28) can be used to derive electric and magnetic fields outside the sphere. Applying boundary conditions and using

$$v_n(x) \Gamma_n(x) - \gamma_n(x) \Upsilon_n(x) = i, \quad (\text{B.35})$$

we obtain

$$a_n = R_n^{(0)} \frac{\eta_o^{-1} v_n(k_o r_o) \Upsilon_n(k_i r_o) - \eta_i^{-1} v_n(k_i r_o) \Upsilon_n(k_o r_o)}{\eta_i^{-1} v_n(k_i r_o) \Gamma_n(k_o r_o) - \eta_o^{-1} \gamma_n(k_o r_o) \Upsilon_n(k_i r_o)} \quad (\text{B.36})$$

$$b_n = R_n^{(0)} \frac{\eta_i^{-1} v_n(k_o r_o) \Upsilon_n(k_i r_o) - \eta_o^{-1} v_n(k_i r_o) \Upsilon_n(k_o r_o)}{\eta_o^{-1} v_n(k_i r_o) \Gamma_n(k_o r_o) - \eta_i^{-1} \gamma_n(k_o r_o) \Upsilon_n(k_i r_o)} \quad (\text{B.37})$$

$$c_n = R_n^{(0)} \left(\frac{k_i}{k_o \eta_o} \right) \frac{i}{\eta_i^{-1} v_n(k_i r_o) \Gamma_n(k_o r_o) - \eta_o^{-1} \gamma_n(k_o r_o) \Upsilon_n(k_i r_o)} \quad (\text{B.38})$$

$$d_n = R_n^{(0)} \left(\frac{k_i}{k_o \eta_o} \right) \frac{i}{\eta_o^{-1} v_n(k_i r_o) \Gamma_n(k_o r_o) - \eta_i^{-1} \gamma_n(k_o r_o) \Upsilon_n(k_i r_o)}. \quad (\text{B.39})$$

B.1.7 Coated Perfectly-Conducting Sphere

Consider a PEC sphere with a radius of r_s coated with a dielectric shell with a radius of r_o . The dielectric shell is lossless with electromagnetic parameters

(ϵ_s, μ_s) . Electric and magnetic fields inside the shell can be derived as

$$E_\theta^{sca,s}(\mathbf{r}) = -\frac{\cos \phi}{k_s r \sin \theta} \sum_{n=1}^{\infty} \left[f_n^+ \gamma_n(k_s r) \omega_n(\theta) - i e_n^+ \Gamma_n(k_s r) \Omega_n(\theta) \right] \\ - \frac{\cos \phi}{k_s r \sin \theta} \sum_{n=1}^{\infty} \left[f_n^- v_n(k_s r) \omega_n(\theta) - i e_n^- \Upsilon_n(k_s r) \Omega_n(\theta) \right] \quad (\text{B.40})$$

$$E_\phi^{sca,s}(\mathbf{r}) = \frac{\sin \phi}{k_s r \sin \theta} \sum_{n=1}^{\infty} \left[f_n^+ \gamma_n(k_s r) \Omega_n(\theta) - i e_n^+ \Gamma_n(k_s r) \omega_n(\theta) \right] \\ + \frac{\sin \phi}{k_s r \sin \theta} \sum_{n=1}^{\infty} \left[f_n^- v_n(k_s r) \Omega_n(\cos \theta) - i e_n^- \Upsilon_n(k_s r) \omega_n(\theta) \right] \quad (\text{B.41})$$

$$H_\theta^{sca,s}(\mathbf{r}) = \frac{i \sin \phi}{k_s \eta_s r \sin \theta} \sum_{n=1}^{\infty} \left[f_n^+ \Gamma_n(k_s r) \Omega_n(\theta) + i e_n^+ \gamma_n(k_s r) \omega_n(\theta) \right] \\ + \frac{i \sin \phi}{k_s \eta_s r \sin \theta} \sum_{n=1}^{\infty} \left[f_n^- \Upsilon_n(k_s r) \Omega_n(\theta) + i e_n^- v_n(k_s r) \omega_n(\theta) \right] \quad (\text{B.42})$$

$$H_\phi^{sca,s}(\mathbf{r}) = \frac{i \cos \phi}{k_s \eta_s r \sin \theta} \sum_{n=1}^{\infty} \left[f_n^+ \Gamma_n(k_s r) \omega_n(\theta) + i e_n^+ \gamma_n(k_s r) \Omega_n(\theta) \right] \\ + \frac{i \cos \phi}{k_s \eta_s r \sin \theta} \sum_{n=1}^{\infty} \left[f_n^- \Upsilon_n(k_s r) \omega_n(\theta) + i e_n^- v_n(k_s r) \Omega_n(\theta) \right]. \quad (\text{B.43})$$

Equations (B.20)–(B.28) can be used to derive electric and magnetic fields outside the object. Applying boundary conditions, we obtain

$$a_n = R_n^{(0)} \frac{1 + R_n^{(1)} R_n^{(3)}}{R_n^{(5)} + R_n^{(1)} R_n^{(4)}} \quad (\text{B.44})$$

$$b_n = R_n^{(0)} \frac{1 + R_n^{(2)} R_n^{(6)}}{R_n^{(8)} + R_n^{(2)} R_n^{(7)}}, \quad (\text{B.45})$$

where

$$R_n^{(1)} = -\frac{\Upsilon_n(k_s r_s)}{\Gamma_n(k_s r_s)} \quad (\text{B.46})$$

$$R_n^{(2)} = -\frac{v_n(k_s r_s)}{\gamma_n(k_s r_s)} \quad (\text{B.47})$$

$$R_n^{(3)} = \frac{\eta_s^{-1} \gamma_n(k_s r_o) \Upsilon_n(k_o r_o) - \eta_o^{-1} v_n(k_o r_o) \Gamma_n(k_s r_o)}{\eta_s^{-1} v_n(k_s r_o) \Upsilon_n(k_o r_o) - \eta_o^{-1} v_n(k_o r_o) \Upsilon_n(k_s r_o)} \quad (\text{B.48})$$

$$R_n^{(4)} = \frac{\eta_o^{-1} \gamma_n(k_o r_o) \Gamma_n(k_s r_o) - \eta_s^{-1} \gamma_n(k_s r_o) \Gamma_n(k_o r_o)}{\eta_s^{-1} v_n(k_s r_o) \Upsilon_n(k_o r_o) - \eta_o^{-1} v_n(k_o r_o) \Upsilon_n(k_s r_o)} \quad (\text{B.49})$$

$$R_n^{(5)} = \frac{\eta_o^{-1}\gamma_n(k_or_o)\Upsilon_n(k_sr_o) - \eta_s^{-1}\nu_n(k_sr_o)\Gamma_n(k_or_o)}{\eta_s^{-1}\nu_n(k_sr_o)\Upsilon_n(k_or_o) - \eta_o^{-1}\nu_n(k_or_o)\Upsilon_n(k_sr_o)} \quad (\text{B.50})$$

$$R_n^{(6)} = \frac{\eta_o^{-1}\gamma_n(k_sr_o)\Upsilon_n(k_or_o) - \eta_s^{-1}\nu_n(k_or_o)\Gamma_n(k_sr_o)}{\eta_o^{-1}\nu_n(k_sr_o)\Upsilon_n(k_or_o) - \eta_s^{-1}\nu_n(k_or_o)\Upsilon_n(k_sr_o)} \quad (\text{B.51})$$

$$R_n^{(7)} = \frac{\eta_s^{-1}\gamma_n(k_or_o)\Gamma_n(k_sr_o) - \eta_o^{-1}\gamma_n(k_sr_o)\Gamma_n(k_or_o)}{\eta_o^{-1}\nu_n(k_sr_o)\Upsilon_n(k_or_o) - \eta_s^{-1}\nu_n(k_or_o)\Upsilon_n(k_sr_o)} \quad (\text{B.52})$$

$$R_n^{(8)} = \frac{\eta_s^{-1}\gamma_n(k_or_o)\Upsilon_n(k_sr_o) - \eta_o^{-1}\nu_n(k_sr_o)\Gamma_n(k_or_o)}{\eta_o^{-1}\nu_n(k_sr_o)\Upsilon_n(k_or_o) - \eta_s^{-1}\nu_n(k_or_o)\Upsilon_n(k_sr_o)}. \quad (\text{B.53})$$

Let $r_s \neq r_o$, $\epsilon_s = \epsilon_o$, and $\mu_s = \mu_o$. Then,

$$a_n \rightarrow R_n^{(0)} \frac{R_n^{(1)} R_n^{(3)}}{R_n^{(5)}} = -R_n^{(0)} \frac{\Upsilon_n(k_or_o)}{\Gamma_n(k_or_o)} \quad (\text{B.54})$$

$$b_n \rightarrow R_n^{(0)} \frac{R_n^{(2)} R_n^{(6)}}{R_n^{(8)}} = -R_n^{(0)} \frac{\nu_n(k_or_o)}{\gamma_n(k_or_o)}, \quad (\text{B.55})$$

which are in agreement with expressions in (B.29) and (B.30) for a PEC sphere in a homogeneous space.

B.1.8 Coated Dielectric Sphere

Finally, consider a dielectric sphere with a radius of r_s coated with a dielectric shell with a radius of r_o . Both the core and the shell are lossless with electromagnetic parameters (ϵ_i, μ_i) and (ϵ_s, μ_s) , respectively. Once again, (B.20)–(B.28) can be used to derive electric and magnetic fields outside the sphere. Similarly, equations (B.31)–(B.34) and (B.40)–(B.43) can be used to derive electromagnetic fields inside the core and the shell, respectively. Using boundary conditions, we obtain

$$a_n = R_n^{(0)} \frac{1 + R_n^{(1)} R_n^{(3)}}{R_n^{(5)} + R_n^{(1)} R_n^{(4)}} \quad (\text{B.56})$$

$$b_n = R_n^{(0)} \frac{1 + R_n^{(2)} R_n^{(6)}}{R_n^{(8)} + R_n^{(2)} R_n^{(7)}}, \quad (\text{B.57})$$

where

$$R_n^{(1)} = \frac{\eta_s^{-1}\gamma_n(k_s r_s)\Upsilon_n(k_i r_s) - \eta_i^{-1}\nu_n(k_i r_s)\Gamma_n(k_s r_s)}{\eta_i^{-1}\nu_n(k_i r_s)\Upsilon_n(k_s r_s) - \eta_s^{-1}\nu_n(k_s r_s)\Upsilon_n(k_i r_s)} \quad (\text{B.58})$$

$$R_n^{(2)} = \frac{\eta_i^{-1}\gamma_n(k_s r_s)\Upsilon_n(k_i r_s) - \eta_s^{-1}\nu_n(k_i r_s)\Gamma_n(k_s r_s)}{\eta_s^{-1}\nu_n(k_i r_s)\Upsilon_n(k_s r_s) - \eta_i^{-1}\nu_n(k_s r_s)\Upsilon_n(k_i r_s)} \quad (\text{B.59})$$

$$R_n^{(3)} = \frac{\eta_s^{-1}\nu_n(k_s r_o)\Upsilon_n(k_o r_o) - \eta_o^{-1}\nu_n(k_o r_o)\Upsilon_n(k_s r_o)}{\eta_s^{-1}\gamma_n(k_s r_o)\Upsilon_n(k_o r_o) - \eta_o^{-1}\nu_n(k_o r_o)\Gamma_n(k_s r_o)} \quad (\text{B.60})$$

$$R_n^{(4)} = \frac{\eta_o^{-1}\gamma_n(k_o r_o)\Upsilon_n(k_s r_o) - \eta_s^{-1}\nu_n(k_s r_o)\Gamma_n(k_o r_o)}{\eta_s^{-1}\gamma_n(k_s r_o)\Upsilon_n(k_o r_o) - \eta_o^{-1}\nu_n(k_o r_o)\Gamma_n(k_s r_o)} \quad (\text{B.61})$$

$$R_n^{(5)} = \frac{\eta_o^{-1}\gamma_n(k_o r_o)\Gamma_n(k_s r_o) - \eta_s^{-1}\gamma_n(k_s r_o)\Gamma_n(k_o r_o)}{\eta_s^{-1}\gamma_n(k_s r_o)\Upsilon_n(k_o r_o) - \eta_o^{-1}\nu_n(k_o r_o)\Gamma_n(k_s r_o)} \quad (\text{B.62})$$

$$R_n^{(6)} = \frac{\eta_o^{-1}\nu_n(k_s r_o)\Upsilon_n(k_o r_o) - \eta_s^{-1}\nu_n(k_o r_o)\Upsilon_n(k_s r_o)}{\eta_o^{-1}\gamma_n(k_s r_o)\Upsilon_n(k_o r_o) - \eta_s^{-1}\nu_n(k_o r_o)\Gamma_n(k_s r_o)} \quad (\text{B.63})$$

$$R_n^{(7)} = \frac{\eta_s^{-1}\gamma_n(k_o r_o)\Upsilon_n(k_s r_o) - \eta_o^{-1}\nu_n(k_s r_o)\Gamma_n(k_o r_o)}{\eta_o^{-1}\gamma_n(k_s r_o)\Upsilon_n(k_o r_o) - \eta_s^{-1}\nu_n(k_o r_o)\Gamma_n(k_s r_o)} \quad (\text{B.64})$$

$$R_n^{(8)} = \frac{\eta_s^{-1}\gamma_n(k_o r_o)\Gamma_n(k_s r_o) - \eta_o^{-1}\gamma_n(k_s r_o)\Gamma_n(k_o r_o)}{\eta_o^{-1}\gamma_n(k_s r_o)\Upsilon_n(k_o r_o) - \eta_s^{-1}\nu_n(k_o r_o)\Gamma_n(k_s r_o)}. \quad (\text{B.65})$$

Let $r_s \neq r_o$, $\epsilon_s = \epsilon_o \neq \epsilon_i$, and $\mu_s = \mu_o \neq \mu_i$. Then,

$$a_n \rightarrow \frac{R_n^{(0)}}{R_n^{(1)} R_n^{(4)}} = R_n^{(0)} \frac{\eta_i^{-1}\nu_n(k_i r_s)\Upsilon_n(k_o r_s) - \eta_o^{-1}\nu_n(k_o r_s)\Upsilon_n(k_i r_s)}{\eta_o^{-1}\gamma_n(k_o r_s)\Upsilon_n(k_i r_s) - \eta_i^{-1}\nu_n(k_i r_s)\Gamma_n(k_o r_s)} \quad (\text{B.66})$$

$$b_n \rightarrow \frac{R_n^{(0)}}{R_n^{(2)} R_n^{(7)}} = R_n^{(0)} \frac{\eta_o^{-1}\nu_n(k_i r_s)\Upsilon_n(k_o r_s) - \eta_i^{-1}\nu_n(k_o r_s)\Upsilon_n(k_i r_s)}{\eta_i^{-1}\gamma_n(k_o r_s)\Upsilon_n(k_i r_s) - \eta_o^{-1}\nu_n(k_i r_s)\Gamma_n(k_o r_s)}, \quad (\text{B.67})$$

which are in agreement with expressions in (B.36) and (B.37) for a dielectric sphere in a homogeneous domain.

B.1.9 Far-Field Expressions

Using the large-argument form of the spherical Hankel function, i.e.,

$$\lim_{x \rightarrow \infty} h_n^{(1)}(x) \approx (-i)^{n+1} \frac{\exp(ix)}{x} \quad (\text{B.68})$$

$$\lim_{x \rightarrow \infty} \Gamma_n(x) \approx \left[(-i)^n \exp(ix) - n(-i)^{n+1} \frac{\exp(ix)}{x} \right] \approx (-i)^n \exp(ix), \quad (\text{B.69})$$

we obtain

$$\begin{aligned} E_\theta^\infty(\theta, \phi) &= \lim_{r \rightarrow \infty} [r \exp(-ik_o r)] E_\theta^{sca,o} \\ &= \frac{i \cos \phi}{k_o \sin \theta} \sum_{n=1}^{\infty} (-i)^n \left[a_n \Omega_n(\theta) + b_n \omega_n(\theta) \right] \end{aligned} \quad (\text{B.70})$$

$$\begin{aligned} E_\phi^\infty(\theta, \phi) &= \lim_{r \rightarrow \infty} [r \exp(-ik_o r)] E_\phi^{sca,o} \\ &= -\frac{i \sin \phi}{k_o \sin \theta} \sum_{n=1}^{\infty} (-i)^n \left[a_n \omega_n(\theta) + b_n \Omega_n(\theta) \right] \end{aligned} \quad (\text{B.71})$$

$$\text{RCS}(\theta, \phi) = 4\pi \left[|E_\theta^\infty(\theta, \phi)|^2 + |E_\phi^\infty(\theta, \phi)|^2 \right]. \quad (\text{B.72})$$

When $\theta = 0^\circ$ (back-scattering direction),

$$\frac{\omega_n(\theta)}{\sin \theta} \rightarrow -\frac{n(n+1)}{2} \quad (\text{B.73})$$

$$\frac{\Omega_n(\theta)}{\sin \theta} \rightarrow -\frac{n(n+1)}{2} \quad (\text{B.74})$$

$$E_\theta^\infty(\theta, \phi) \rightarrow -\frac{i \cos \phi}{k_o} \sum_{n=1}^{\infty} \frac{n(n+1)}{2} (-i)^n (a_n + b_n) \quad (\text{B.75})$$

$$E_\phi^\infty(\theta, \phi) \rightarrow \frac{i \sin \phi}{k_o} \sum_{n=1}^{\infty} \frac{n(n+1)}{2} (-i)^n (a_n + b_n), \quad (\text{B.76})$$

and

$$\text{RCS}(\theta = 0^\circ) = \frac{\pi}{k_o^2} \left| \sum_{n=1}^{\infty} n(n+1) (-i)^n (a_n + b_n) \right|^2. \quad (\text{B.77})$$

Similarly, when $\theta = 180^\circ$ (forward-scattering direction),

$$\frac{\omega_n(\theta)}{\sin \theta} \rightarrow \frac{n(n+1)}{2} (-1)^n \quad (\text{B.78})$$

$$\frac{\Omega_n(\theta)}{\sin \theta} \rightarrow -\frac{n(n+1)}{2} (-1)^n \quad (\text{B.79})$$

$$E_\theta^\infty(\theta, \phi) \rightarrow -\frac{i \cos \phi}{k_o} \sum_{n=1}^{\infty} \frac{n(n+1)}{2} (i)^n (a_n - b_n) \quad (\text{B.80})$$

$$E_\phi^\infty(\theta, \phi) \rightarrow -\frac{i \sin \phi}{k_o} \sum_{n=1}^{\infty} \frac{n(n+1)}{2} (i)^n (a_n - b_n), \quad (\text{B.81})$$

and

$$\text{RCS}(\theta = 180^\circ) = \frac{\pi}{k_o^2} \left| \sum_{n=1}^{\infty} n(n+1) (i)^n (a_n - b_n) \right|^2. \quad (\text{B.82})$$

B.2 Other Formulations

B.2.1 Electric-Field Volume Integral Equation

Volume integral equations can be used to formulate electromagnetics problems involving dielectric objects. Major advantages and disadvantages of volume formulations compared to surface formulations can be listed as follows:

- For thin dielectric objects, volume integral equations produce better-conditioned matrix equations than surface integral equations.
- Volume integral equations does not breakdown for low-contrast objects as opposed to conventional surface integral equations.
- Heterogeneous dielectric objects can be modeled easily with volume integral equations, which allow different electrical parameters for discretization elements.
- Volume integral equations usually require more numbers of unknowns than surface integral equations for a given problem.

In this section, we briefly present a volume integral equation, namely, EFVIE, derived for non-magnetic dielectric objects.

We consider a dielectric object with a relative permittivity of ϵ_r located in free space. Equivalent electric current can be defined as

$$\mathbf{J}(\mathbf{r}) = i\omega\kappa(\mathbf{r})\mathbf{D}(\mathbf{r}) = i\omega\kappa(\mathbf{r})\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E}(\mathbf{r}), \quad (\text{B.83})$$

where ϵ_0 is the permittivity of free space and

$$\kappa(\mathbf{r}) = \frac{1 - \epsilon_r(\mathbf{r})}{\epsilon_r(\mathbf{r})} \quad (\text{B.84})$$

is the contrast at the observation point $\mathbf{r}$. When $\mathbf{r}$ is located outside the object, $\kappa(\mathbf{r}) = 0$, hence $\mathbf{J}(\mathbf{r}) = 0$. The total electric field in (B.83) can be written as

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \mathbf{E}^{inc}(\mathbf{r}) \\ &+ ik_0\eta_0 \int_D d\mathbf{r}' \mathbf{J}(\mathbf{r}') g_0(\mathbf{r}, \mathbf{r}') - \frac{i\eta_0}{k_0} \nabla \int_D d\mathbf{r}' \mathbf{J}(\mathbf{r}') \cdot \nabla' g_0(\mathbf{r}, \mathbf{r}'), \end{aligned} \quad (\text{B.85})$$

where $\mathbf{E}^{inc}$ is the incident electric field created by external sources, $k_0 = 2\pi/\lambda_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the wave number, and η_0 is the wave impedance in free space. Using (B.85) in (B.83), one can obtain [48]

$$\begin{aligned} \frac{\mathbf{D}(\mathbf{r})}{\epsilon_0\epsilon_r(\mathbf{r})} - ik_0\eta_0 \int_D d\mathbf{r}' i\omega\kappa(\mathbf{r}') \mathbf{D}(\mathbf{r}') g_0(\mathbf{r}, \mathbf{r}') \\ + \frac{i\eta_0}{k_0} \nabla \int_D d\mathbf{r}' i\omega\kappa(\mathbf{r}') \mathbf{D}(\mathbf{r}') \cdot \nabla' g_0(\mathbf{r}, \mathbf{r}') = \mathbf{E}^{inc}(\mathbf{r}), \end{aligned} \quad (\text{B.86})$$

which is called EFVIE.

For numerical solutions of EFVIE, we expand $i\omega\mathbf{D}$ in a series of basis functions, i.e.,

$$i\omega\mathbf{D}(\mathbf{r}) = \sum_{n=1}^N a_n \mathbf{b}_n(\mathbf{r}), \quad (\text{B.87})$$

and test the equation by using a set of testing functions $\mathbf{t}_m$ for $m = 1, 2, \dots, N$.

We obtain $N \times N$ dense matrix equations in the form of

$$\bar{\mathbf{Z}}_{EV} \cdot \mathbf{a} = \mathbf{w}_{EV}, \quad (\text{B.88})$$

where matrix elements and elements of the RHS vector are derived as

$$\begin{aligned} \bar{\mathbf{Z}}_{EV}[m, n] &= \frac{1}{i\omega\epsilon_0} \int_{D_m} d\mathbf{r} \frac{1}{\epsilon_r(\mathbf{r})} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{b}_n(\mathbf{r}) \\ &- i\omega\mu_0 \int_{D_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \int_{D_n} d\mathbf{r}' \kappa(\mathbf{r}') \mathbf{b}_n(\mathbf{r}') g_0(\mathbf{r}, \mathbf{r}') \\ &- \frac{1}{i\omega\epsilon_0} \int_{D_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \nabla \int_{D_n} d\mathbf{r}' \kappa(\mathbf{r}') \mathbf{b}_n(\mathbf{r}') \cdot \nabla' g_0(\mathbf{r}, \mathbf{r}') \end{aligned} \quad (\text{B.89})$$

and

$$\mathbf{w}_{EV}[m] = \int_{D_m} d\mathbf{r} \mathbf{t}_m(\mathbf{r}) \cdot \mathbf{E}^{inc}(\mathbf{r}), \quad (\text{B.90})$$

respectively.

Similar to surface formulations, we use the RWG functions for the discretization of EFVIE. Each RWG function is defined in a pair of tetrahedra sharing a common surface. The n th RWG function with a spatial support of $D_n = D_{n1} + D_{n2}$ can be written as

$$\mathbf{f}_n^{\text{RWG,V}}(\mathbf{r}) = \begin{cases} \frac{A_n}{3V_{n1}}(\mathbf{r} - \mathbf{r}_{n1}), & \mathbf{r} \in D_{n1} \\ \frac{A_n}{3V_{n2}}(\mathbf{r}_{n2} - \mathbf{r}), & \mathbf{r} \in D_{n2} \\ 0, & \mathbf{r} \notin D_n, \end{cases} \quad (\text{B.91})$$

where A_n represents the area of the common surface, whereas V_{n1} and V_{n2} are volumes of the first (D_{n1}) and the second (D_{n2}) tetrahedron, respectively. In (B.91), $\mathbf{r}_{n1}$ and $\mathbf{r}_{n2}$ are coordinates of the two vertices opposite of the common surface. We note that the charge density represented by the n th RWG function can be derived as

$$\nabla \cdot \mathbf{f}_n^{\text{RWG,V}}(\mathbf{r}) = \begin{cases} \frac{A_n}{V_{n1}}, & \mathbf{r} \in D_{n1} \\ -\frac{A_n}{V_{n2}}, & \mathbf{r} \in D_{n2} \\ 0, & \mathbf{r} \notin D_n. \end{cases} \quad (\text{B.92})$$

Consider the interaction of two half RWG functions $\mathbf{t}_{ma}^{\text{RWG,V}}$ and $\mathbf{b}_{nb}^{\text{RWG,V}}$ associated with the a th tetrahedron of the m th surface and b th tetrahedron of the n th surface, respectively. We obtain

$$\begin{aligned} \bar{\mathbf{Z}}_{EV}[m, n, a, b] = & \frac{\mathcal{C}_{ma,nb}}{i\omega\epsilon_0} \int_{D_{ma}} d\mathbf{r} \frac{1}{\epsilon_r(\mathbf{r})} (\mathbf{r} - \mathbf{r}_{ma}) \cdot (\mathbf{r} - \mathbf{r}_{nb}) \\ & - i\omega\mu_0 \mathcal{C}_{ma,nb} \int_{D_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \int_{D_{nb}} d\mathbf{r}' \kappa(\mathbf{r}') (\mathbf{r}' - \mathbf{r}_{nb}) g_0(\mathbf{r}, \mathbf{r}') \\ & - \frac{\mathcal{C}_{ma,nb}}{i\omega\epsilon_0} \int_{D_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \nabla \int_{D_{nb}} d\mathbf{r}' \kappa(\mathbf{r}') (\mathbf{r}' - \mathbf{r}_{nb}) \cdot \nabla' g_0(\mathbf{r}, \mathbf{r}'), \end{aligned} \quad (\text{B.93})$$

where

$$\mathcal{C}_{ma,nb} = \frac{A_m A_n}{9V_{ma}V_{nb}} \gamma_{ma} \gamma_{nb} \quad (\text{B.94})$$

and $\gamma_{nb}, \gamma_{ma} = \pm 1$, depending on the direction of basis and testing functions in tetrahedra.

At this stage, we assume that the permittivity is constant inside each tetrahedron. Then,

$$\begin{aligned} \bar{\mathbf{Z}}_{EV}[m, n, a, b] &= \frac{\mathcal{C}_{ma,nb}}{i\omega\epsilon_0\epsilon_{r,nb}} \int_{D_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot (\mathbf{r} - \mathbf{r}_{nb}) \\ &\quad - i\omega\mu_0\kappa_{nb}\mathcal{C}_{ma,nb} \int_{D_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \int_{D_{nb}} d\mathbf{r}' (\mathbf{r}' - \mathbf{r}_{nb}) g_0(\mathbf{r}, \mathbf{r}') \\ &\quad - \frac{\mathcal{C}_{ma,nb}\kappa_{nb}}{i\omega\epsilon_0} \int_{D_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \nabla \int_{D_{nb}} d\mathbf{r}' (\mathbf{r}' - \mathbf{r}_{nb}) \cdot \nabla' g_0(\mathbf{r}, \mathbf{r}'). \end{aligned} \quad (\text{B.95})$$

For the third term in (B.95), we use the divergence theorem and properties of the RWG functions to arrive at

$$\int_{D_{nb}} d\mathbf{r}' (\mathbf{r}' - \mathbf{r}_{nb}) \cdot \nabla' g_0(\mathbf{r}, \mathbf{r}') = \frac{3V_{nb}}{A_n} \int_{S_n} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}') - 3 \int_{D_{nb}} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}'). \quad (\text{B.96})$$

Inserting (B.96) in (B.95), and using the divergence theorem again, we obtain

$$\begin{aligned} \bar{\mathbf{Z}}_{EV}[m, n, a, b] &= \frac{\mathcal{C}_{ma,nb}}{i\omega\epsilon_0\epsilon_{r,nb}} \int_{D_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot (\mathbf{r} - \mathbf{r}_{nb}) \\ &\quad - i\omega\mu_0\kappa_{nb}\mathcal{C}_{ma,nb} \int_{D_{ma}} d\mathbf{r} (\mathbf{r} - \mathbf{r}_{ma}) \cdot \int_{D_{nb}} d\mathbf{r}' (\mathbf{r}' - \mathbf{r}_{nb}) g_0(\mathbf{r}, \mathbf{r}') \\ &\quad - \frac{9\mathcal{C}_{ma,nb}\kappa_{nb}}{i\omega\epsilon_0} \int_{D_{ma}} d\mathbf{r} \int_{D_{nb}} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}') \\ &\quad + \frac{A_n\kappa_{nb}\gamma_{nb}\gamma_{ma}}{i\omega\epsilon_0 V_{nb}} \int_{S_m} d\mathbf{r} \int_{D_{nb}} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}') \\ &\quad + \frac{A_m\kappa_{nb}\gamma_{nb}\gamma_{ma}}{i\omega\epsilon_0 V_{ma}} \int_{D_{ma}} d\mathbf{r} \int_{S_n} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}') \\ &\quad - \frac{\kappa_{nb}\gamma_{nb}\gamma_{ma}}{i\omega\epsilon_0} \int_{S_m} d\mathbf{r} \int_{S_n} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}'). \end{aligned} \quad (\text{B.97})$$

Finally, (B.97) can be written in terms of basic integrals as

$$\begin{aligned}
\bar{\mathbf{Z}}_{EV}[m, n, a, b] = & \frac{\mathcal{C}_{ma,nb}}{i\omega\epsilon_0\epsilon_{r,nb}} \left\{ (x_{ma}x_{nb} + y_{ma}y_{nb} + z_{ma}z_{nb})V_{ma} + I_{17} + I_{18} + I_{19} \right. \\
& - (x_{ma} + x_{nb})I_{14} - (y_{ma} + y_{nb})I_{15} - (z_{ma} + z_{nb})I_{16} \left. \right\} \\
& - i\omega\mu_0\kappa_{nb}\mathcal{C}_{ma,nb} \left\{ (x_{ma}x_{nb} + y_{ma}y_{nb} + z_{ma}z_{nb})I_1 + I_8 + I_9 + I_{10} \right. \\
& - x_{nb}I_2 - y_{nb}I_3 - z_{nb}I_4 - x_{ma}I_5 - y_{ma}I_6 - z_{ma}I_7 \left. \right\} \\
& - \frac{9\mathcal{C}_{ma,nb}\kappa_{nb}}{i\omega\epsilon_0} \left\{ I_1 - \frac{V_{ma}}{A_m}I_{11} - \frac{V_{nb}}{A_n}I_{12} + \frac{V_{ma}V_{nb}}{A_mA_n}I_{13} \right\}, \quad (\text{B.98})
\end{aligned}$$

where

$$I_{\{1,2,3,4\}} = \int_{V_{ma}} d\mathbf{r} \{1, x, y, z\} \int_{V_{nb}} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}') \quad (\text{B.99})$$

$$I_{\{5,6,7\}} = \int_{V_{ma}} d\mathbf{r} \int_{V_{nb}} d\mathbf{r}' \{x', y', z'\} g_0(\mathbf{r}, \mathbf{r}') \quad (\text{B.100})$$

$$I_8 = \int_{V_{ma}} d\mathbf{r} x \int_{V_{nb}} d\mathbf{r}' x' g_0(\mathbf{r}, \mathbf{r}') \quad (\text{B.101})$$

$$I_9 = \int_{V_{ma}} d\mathbf{r} y \int_{V_{nb}} d\mathbf{r}' y' g_0(\mathbf{r}, \mathbf{r}') \quad (\text{B.102})$$

$$I_{10} = \int_{V_{ma}} d\mathbf{r} z \int_{V_{nb}} d\mathbf{r}' z' g_0(\mathbf{r}, \mathbf{r}') \quad (\text{B.103})$$

$$I_{11} = \int_{S_m} d\mathbf{r} \int_{D_{nb}} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}') \quad (\text{B.104})$$

$$I_{12} = \int_{D_{ma}} d\mathbf{r} \int_{S_n} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}') \quad (\text{B.105})$$

$$I_{13} = \int_{S_m} d\mathbf{r} \int_{S_n} d\mathbf{r}' g_0(\mathbf{r}, \mathbf{r}') \quad (\text{B.106})$$

$$I_{\{14,15,16\}} = \int_{V_{ma}} d\mathbf{r} \{x, y, z\} \quad (\text{B.107})$$

$$I_{\{17,18,19\}} = \int_{V_{ma}} d\mathbf{r} \{x^2, y^2, z^2\}. \quad (\text{B.108})$$

Inner integrals in (B.99)–(B.108) can be calculated accurately by using singularity extraction methods [91]. We note that surface integrals in I_{11} , I_{12} , and I_{13} can be omitted when electrical parameters do not change across S_m or S_n . This is because contributions from two tetrahedra on the two sides of the surface cancel each other.

B.2.2 Born Approximation

When the contrast of the object is low, i.e., $\kappa(\mathbf{r}) \approx 0$, the equivalent volume electric current can be approximated as

$$\mathbf{J}(\mathbf{r}) = i\omega\kappa(\mathbf{r})\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E}^{inc}(\mathbf{r}) = i\omega[1 - \epsilon_r(\mathbf{r})]\epsilon_0\mathbf{E}^{inc}(\mathbf{r}), \quad (\text{B.109})$$

by replacing the total electric field with the incident field, which is called the Born approximation.

B.2.3 Impedance Boundary Conditions

A commonly used approximation for a smooth object with high conductivity is using an impedance boundary condition (IBC), such as

$$\mathbf{M}(\mathbf{r}) = \mathcal{Z}_s \hat{\mathbf{n}} \times \mathbf{J}(\mathbf{r}), \quad (\text{B.110})$$

where $\mathcal{Z}_s$ is the impedance of the surface. For example, using (B.110) in T-EFIE, one can obtain

$$\eta \hat{\mathbf{t}} \cdot \mathcal{T}\{\mathbf{J}\}(\mathbf{r}) - \mathcal{Z}_s \hat{\mathbf{t}} \cdot \mathcal{K}_{PV}\{\hat{\mathbf{n}} \times \mathbf{J}\}(\mathbf{r}) - \frac{\Omega_o \mathcal{Z}_s}{4\pi} \hat{\mathbf{t}} \cdot \mathbf{J}(\mathbf{r}) = -\hat{\mathbf{t}} \cdot \mathbf{E}^{inc}(\mathbf{r}) \quad (\text{B.111})$$

or

$$\begin{aligned} & \hat{\mathbf{t}} \cdot ik\eta \int_S d\mathbf{r}' \left[\mathbf{J}(\mathbf{r}') + \frac{1}{k^2} \nabla' \cdot \mathbf{J}(\mathbf{r}') \nabla \right] g(\mathbf{r}, \mathbf{r}') \\ & - \mathcal{Z}_s \hat{\mathbf{t}} \cdot \int_S d\mathbf{r}' \hat{\mathbf{n}}' \times \mathbf{J}(\mathbf{r}') \times \nabla' g(\mathbf{r}, \mathbf{r}') - \frac{\Omega_o \mathcal{Z}_s}{4\pi} \hat{\mathbf{t}} \cdot \mathbf{J}(\mathbf{r}) = -\hat{\mathbf{t}} \cdot \mathbf{E}^{inc}(\mathbf{r}). \end{aligned} \quad (\text{B.112})$$

EFIE in the form of (B.112) can be discretized and solved numerically. However, it should be emphasized that IBCs are valid and provide accurate solutions under two strict circumstances:

- The object must be smooth and much thicker than the skin depth.
- Magnitude of the refractive index of the object must be significantly larger than the unity.

B.3 Solutions of Large Problems

As demonstrated in Chapter 4, parallelization of MLFMA provides the solution of electromagnetics problems discretized with hundreds of millions of unknowns. In this section, we discuss some important details that must be considered for efficient and accurate solutions of those large-scale problems.

B.3.1 Mesh Refinement

Discretizations involving large numbers of triangles can be performed efficiently via a mesh-refinement algorithm. For example, a coarse discretization with T triangles provided by a commercial program can be used to generate a fine discretization with $4T$ triangles by dividing each original triangle into four sub-triangles. This refinement procedure can be applied repeatedly in order to decrease the mesh size, as required. On the other hand, although the mesh size can be appropriate for accurate solutions, a fine discretization obtained via the refinement algorithm does not provide a better modelling of curved surfaces compared to the coarse kernel discretization. Hence, the mesh-refinement algorithm may not be preferred if a similar discretization with the same mesh size can be obtained via a commercial program.

As an example, Figure B.1 presents the solution of a scattering problem involving the stealth airborne target Flamme at 160 GHz. The size of the Flamme corresponds to 320λ , and it is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field polarized in the z direction. Discretizations of the target with $\lambda/10$ triangles using a commercial program and the mesh-refinement algorithm lead to matrix equations involving 24,386,412 and 24,782,400 unknowns, respectively. The kernel discretization for the mesh refinement involves only 258,150 triangles. The problem is solved by a

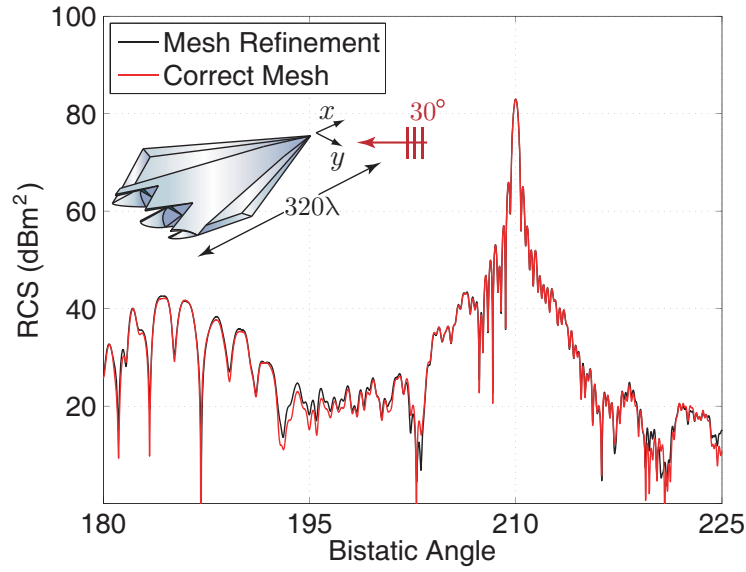


Figure B.1: Bistatic RCS (dBm²) of the stealth airborne target Flamme at 160 GHz. The target is illuminated by a plane wave propagating in the x - y plane at a 30° angle from the x axis with the electric field in polarized in the z direction.

10-level MLFMA with three digits of accuracy. Figure B.1 presents the bistatic RCS values on the x - y plane as a function of the bistatic angle ϕ from 180° to 225° , where 210° corresponds to the forward-scattering direction. We observe that RCS values obtained by employing the refined mesh are very close to those obtained by employing the “correct” mesh of the commercial program. Small discrepancies between two results are due to different levels of accuracy offered by the mesh-refinement algorithm and the commercial program for modelling curved surfaces of the Flamme.

B.3.2 Mesh Quality

In addition to the mesh size, the quality of the discretization is essential to obtain accurate solutions of electromagnetics problems. On the other hand, it becomes increasingly difficult to detect errors in the discretization as the problem size gets larger and the discretization involves many triangles.

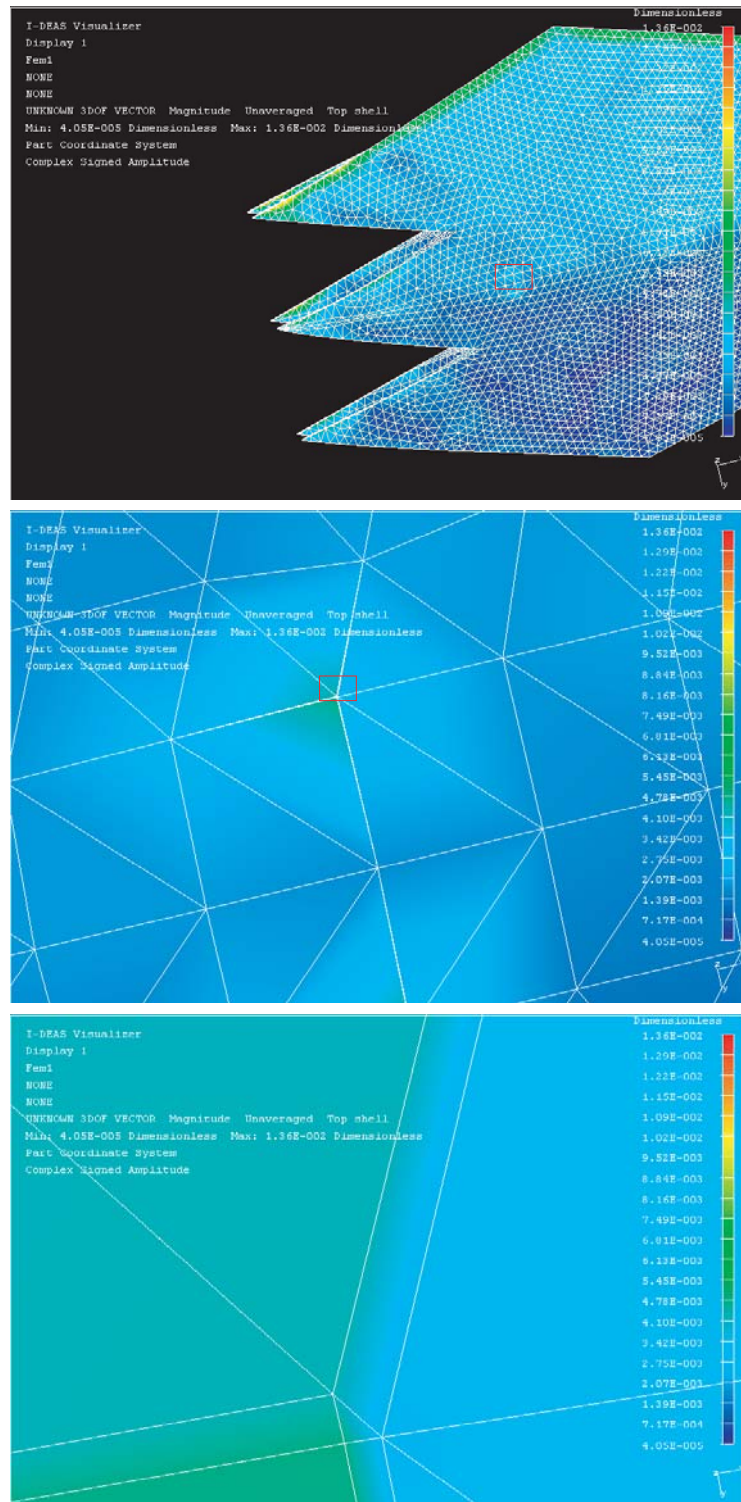


Figure B.2: Induced electric current on the surface of the Flamme. Unusual currents due to triangles with high aspect ratios are clearly visible.

As an example to errors encountered during discretizations, Figure B.2 presents the solution of a scattering problem involving the Flamme. The electric current induced on the surface of the target is investigated. We observe that current values become unusually high at two symmetric locations. By zooming into those locations, extremely narrow triangles with very large aspect ratios are observed. We note that contamination of the accuracy of the solution due to distorted discretization elements, such as triangles with large aspect ratios, can be difficult to detect without investigating the induced current on each and every portion of the surface.

B.3.3 Recursive Clustering

In MLFMA, a multilevel tree structure must be constructed with less than $\mathcal{O}(N \log N)$ complexity. Otherwise, the recursive clustering part can easily become a bottleneck of the implementation. We apply following steps to construct a tree structure for a given discretized object.

- Determine the number of levels L and the box size at each level. Identify boxes from the top to the bottom of the tree structure for each unknown. At this stage, boxes are indexed according to their locations in oct-tree.
- Generate a real array of size N . The value of the n th element of the array is $C + n/N$, where C is the index of the box at the lowest level including n th unknown.
- Sort the real-type array by using a quick-sort algorithm.
- By tracing the sorted array once, find the number of boxes and the number of unknowns in each box.
- Rename each box by considering only non-empty boxes in the tree structure.

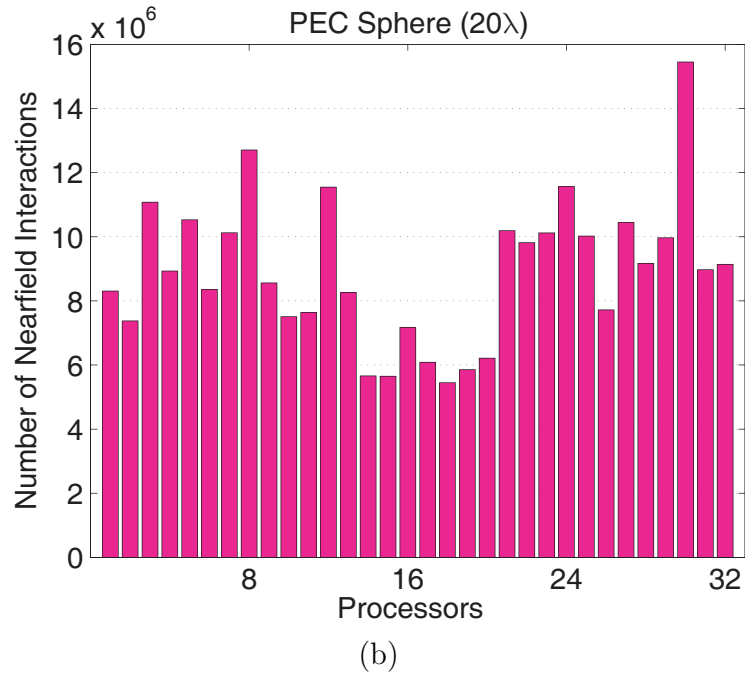
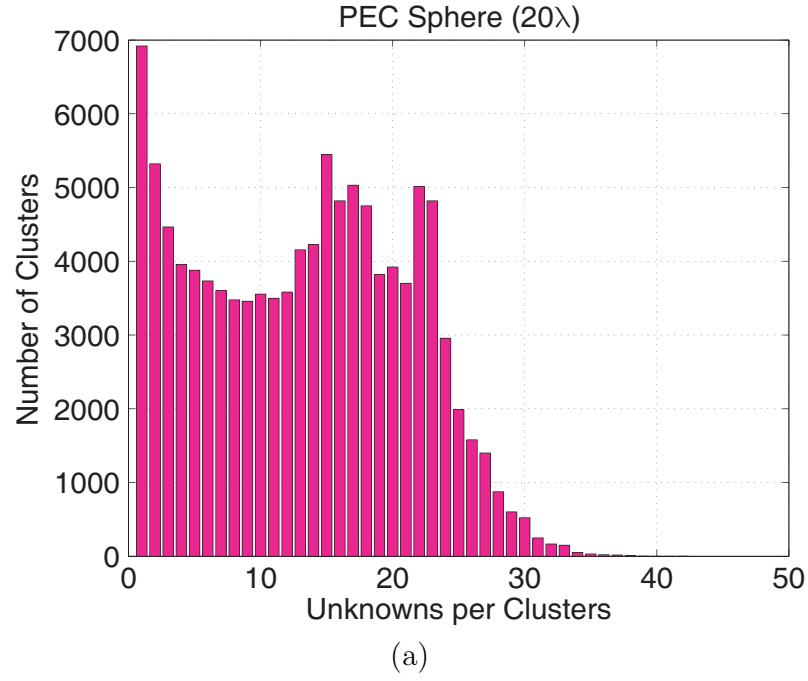
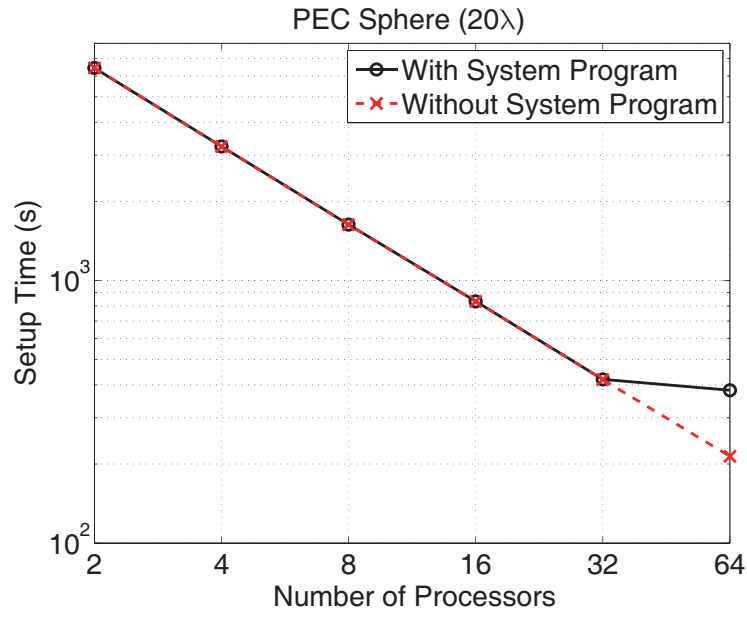


Figure B.3: Solution of a scattering problem involving a perfectly-conducting sphere discretized with 1,462,854 unknowns. (a) Histogram of the lowest-level boxes in terms of the number of unknowns. (b) Number of near-field interactions assigned to each processor when rows of the matrix equation are distributed equally among 32 processors.

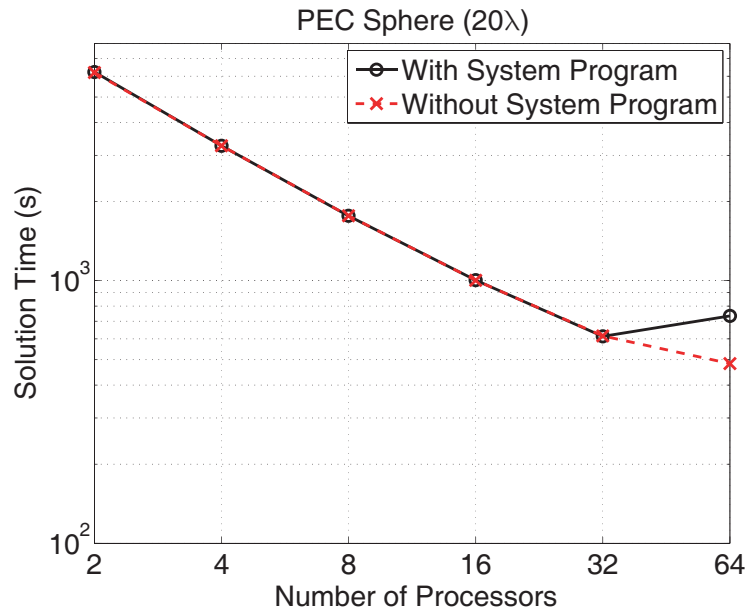
B.3.4 Distribution of Near-Field Interactions

Using cubic boxes in MLFMA provides many computational advantages due to the symmetry. On the other hand, since the box size is fixed, basis and testing functions are distributed unequally among the lowest-level boxes. Hence, during the parallelization of MLFMA, load-balancing algorithms are required at each part of the algorithm, even for near-field interactions. As an example, Figure B.3 presents the solution of a scattering problem involving a perfectly-conducting sphere of radius 20λ discretized with 1,462,854 unknowns. The problem is solved by a 7-level MLFMA using a bottom-up scheme. Figure B.3(a) depicts an histogram of the lowest-level boxes in terms of the number of unknowns. We observe that the number of unknowns per box changes in a wide range from 1 to 42. We also note that about 7000 out of 109,790 boxes involve only one unknown.

Figure B.3(b) depicts the number of near-field interactions assigned to each processor when the solution is parallelized on a cluster of 32 processors and rows of the matrix equation are distributed equally among processors. Since the number of unknowns per box vary significantly, the number of near-field interactions per row also vary significantly, even for the simple sphere geometry. Hence, equal distribution of rows leads to unequal distribution of unknowns among processors. As a remedy, we use a load-balancing algorithm in order to distribute near-field interactions (instead of rows) equally among processors, which leads to the near-field partitioning. A load-balancing of far-field interactions leads to the far-field partitioning of the matrix equation, which can be significantly different from the near-field partitioning.



(a)



(b)

Figure B.4: Processing time for the solution of a scattering problem involving a perfectly-conducting sphere of radius 20λ discretized with 1,462,854 unknowns. (a) Setup time and (b) solution time are investigated with and without a system program employed on the BiLCEM Clovertown cluster.

B.3.5 Optimization of Parallel Solutions

Solutions via parallel MLFMA cannot be performed efficiently without optimization. In this subsection, we summarize some of our observations during our studies on the parallelization of MLFMA.

Effects of Other Running Programs:

Figure B.4 presents the solution of a scattering problem involving a perfectly-conducting sphere of radius 20λ discretized with 1,462,854 unknowns. The problem is solved by a 7-level MLFMA using the hybrid strategy and a bottom-up scheme on the BiLCEM Clovertown cluster. The cluster involves eight nodes and each node has two quad-core Intel Xeon Clovertown processors with 2.33 GHz clock rate. Two different cases are considered:

1. A system program requiring a significant processing load is employed on a single processor of a node, in addition to the parallel MLFMA.
2. All 64 processors are reserved for the parallel MLFMA.

Figures B.4(a) and B.4(b) depict the processing time for setup and solution parts, respectively, as a function of the number of processors. We observe that the processing time decreases regularly as the number of processors increases up to 32 in both cases, i.e., with and without the system program. When the number of processors is 64, however, the existence of the system program affects the processing time significantly, even though the program is employed on a single processor. In fact, for the solution part, the processing time increases from 32 processors to 64 processors. This experiment shows the necessity of a master node to host system programs without affecting the parallelization of MLFMA.

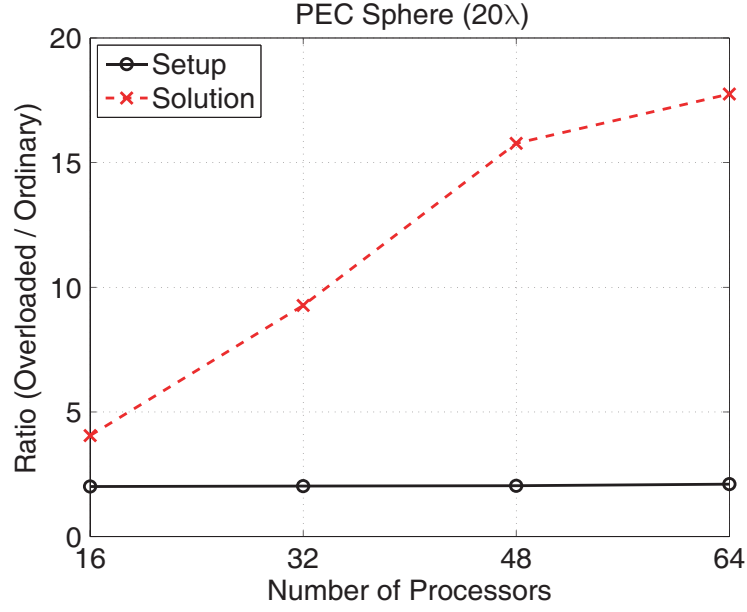


Figure B.5: Solutions of a scattering problem involving a perfectly-conducting sphere of radius 20λ discretized with 1,462,854 unknowns on the BiLCEM Clovertown cluster. Processing times are compared when the parallel computer is employed as usual and when it is overloaded.

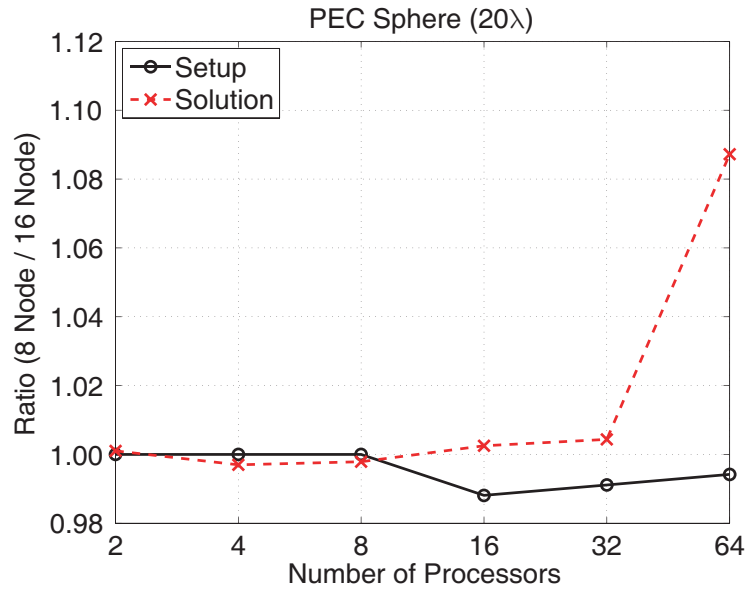
In Figure B.5, we again consider the solution of the sphere problem involving 1,462,854 unknowns on the BiLCEM Clovertown cluster. In addition to ordinary solutions on 16–64 processors, we consider overloaded solutions, i.e., when dummy programs are employed at the same time on all 64 processors of the cluster. In Figure B.5, the ratio of processing times required for overloaded and ordinary solutions is plotted as a function of the number of processors. We observe that the setup time is doubled since each processor is used with 50% efficiency. On the other hand, overloading significantly affects the solution part, which involves many communications between processors. Using 32 and 64 processors, overloading leads to more than 15-fold deceleration for the solution part.

As discussed in Chapter 4, using slower processors may increase the parallelization efficiency since the computation time increases and dominates the communication time. This observation does not contradict with our observations in

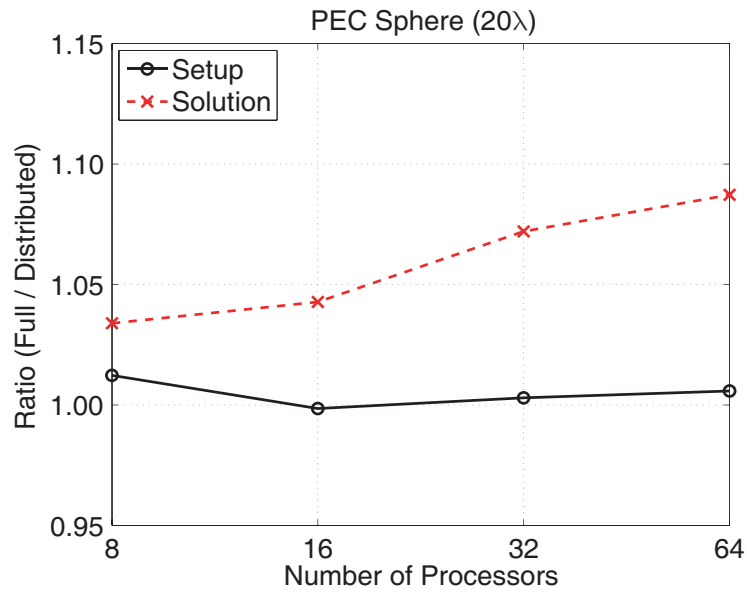
Figure B.5. Specifically, overloading in Figure B.5 does not only slow down processors but also decreases the performance of the whole system, including communications. The setup part, which involves only computations, is affected by 50% due to the reduced effective clock rate. The solution part, however, is affected more than 50% since it requires synchronization between processors and delays in different processors are accumulated.

Distribution of Processes:

Figure B.6 presents solutions of a scattering problem involving a perfectly-conducting sphere of radius 20λ discretized with 1,462,854 unknowns on the Tigerton cluster in Swindon. Solutions are parallelized among various numbers of processors. In Figure B.6(a), we compare two different cases, i.e., when 8 and 16 nodes of the cluster are employed. In both cases, the number of nodes is maximized for all solutions, as usual. We observe that the ratio of processing times required on 8 and 16 nodes is usually close to one, except for the 64-processor case. In this case, using 8 nodes instead of 16 nodes increases the solution time by 8%. This experiment shows that using all processors in a node may decrease the performance of the parallel computer, even though a sufficient number of processors is actually available. Figure B.6(b) presents the solution of the same problem on 16 nodes of the Tigerton cluster when the number of nodes is maximized (distributed) and when the number of nodes is minimized (full). For example, 16 processors are selected from 16 nodes (1 processor per each node) and 2 nodes (8 processors per each node) in distributed and full schemes, respectively. Figure B.6(b) confirms that using all processors in a node increases the solution time. This is not observed for the setup part, which only involves computations and does not require any communication between processors.

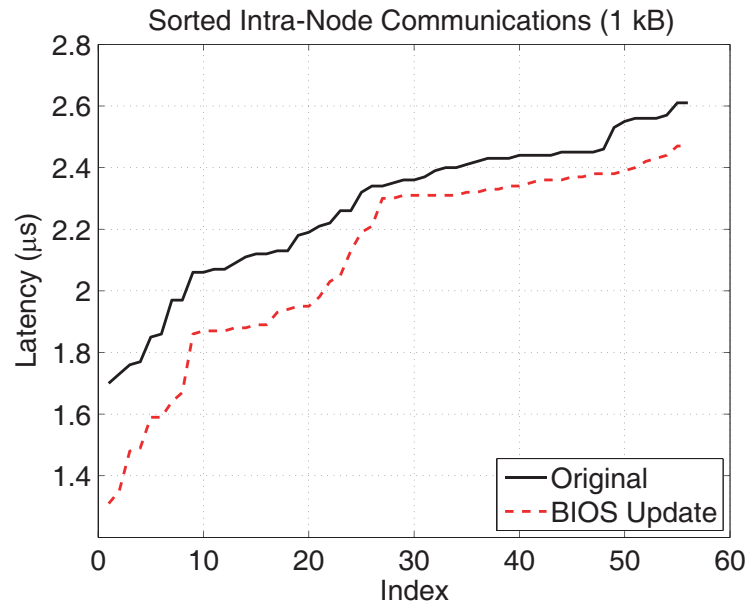


(a)

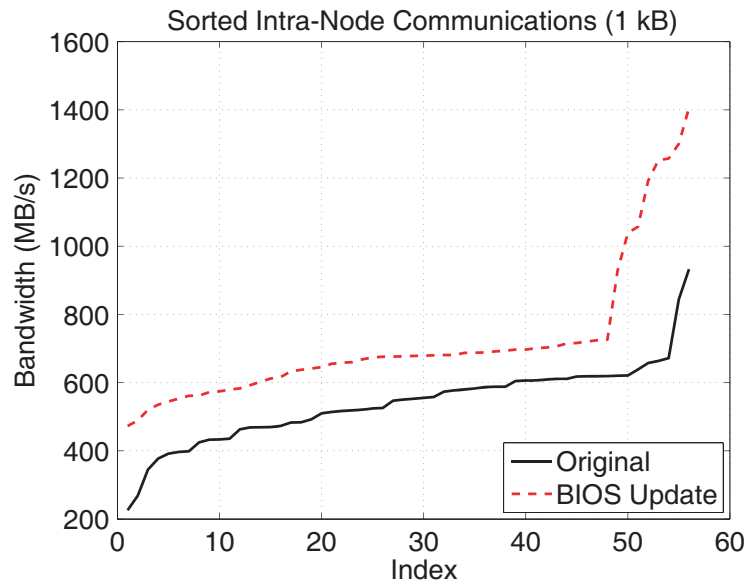


(b)

Figure B.6: Solutions of a scattering problem involving a perfectly-conducting sphere of radius 20λ discretized with 1,462,854 unknowns on the Tigerton cluster. Processing times are compared when (a) different numbers of nodes are employed and (b) processors are distributed among nodes in different ways.



(a)



(b)

Figure B.7: Optimization of intra-node communications in the BiLCEM Clover-town cluster via a BIOS update. (a) Sorted latency (μs) and (b) sorted bandwidth (MB/s).

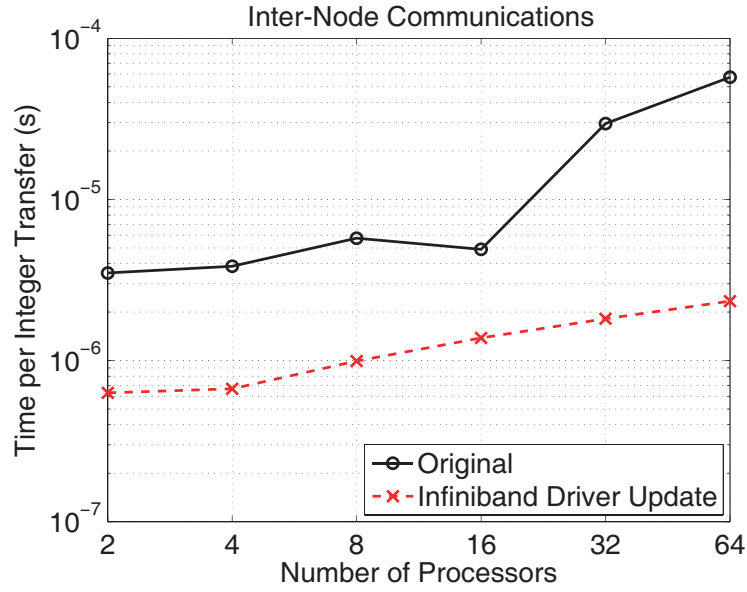


Figure B.8: Optimization of inter-node communications in the BiLCEM Clovertown cluster via an Infiniband driver update.

Optimization of the Parallel Environment

The parallel environment should also be optimized for the best possible performance of the parallel MLFMA. As an example, Figure B.7 presents the optimization of intra-node communications between eight processors in the same node of the BiLCEM Clovertown cluster. A simple update of the basic input/output system (BIOS) significantly accelerates intra-node communications. Specifically, the latency decreases and the bandwidth increases, as depicted in Figures B.7(a) and B.7(b), respectively.

In Figure B.8, we present the acceleration of inter-node communications by updating the Infiniband driver. The processing time required for an integer transfer is plotted when processors located in different nodes perform communications with each other. If the number of processors is larger than two, multiple communications are performed at the same time. Hence, the communication time increases as the number of processors increases. Figure B.8 shows that a simple

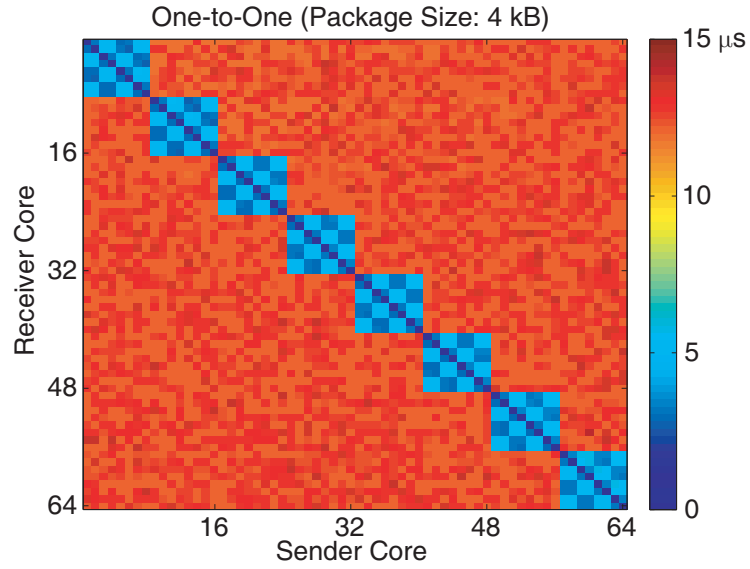
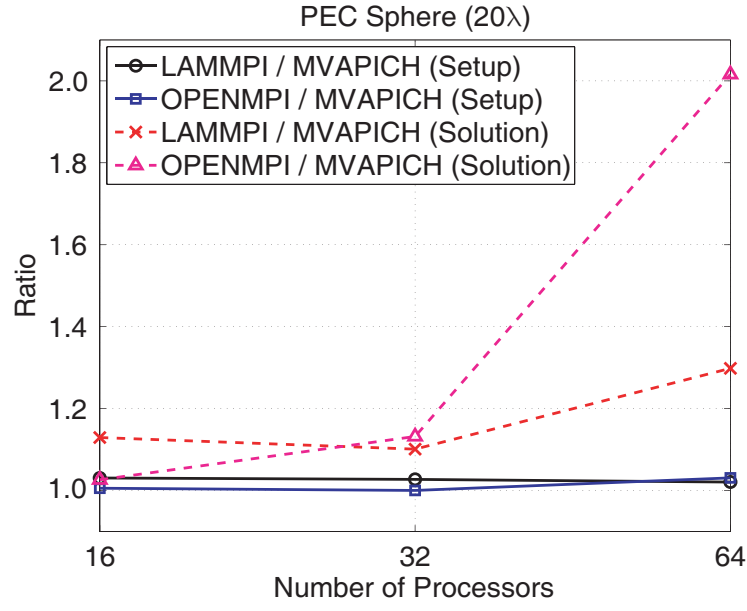


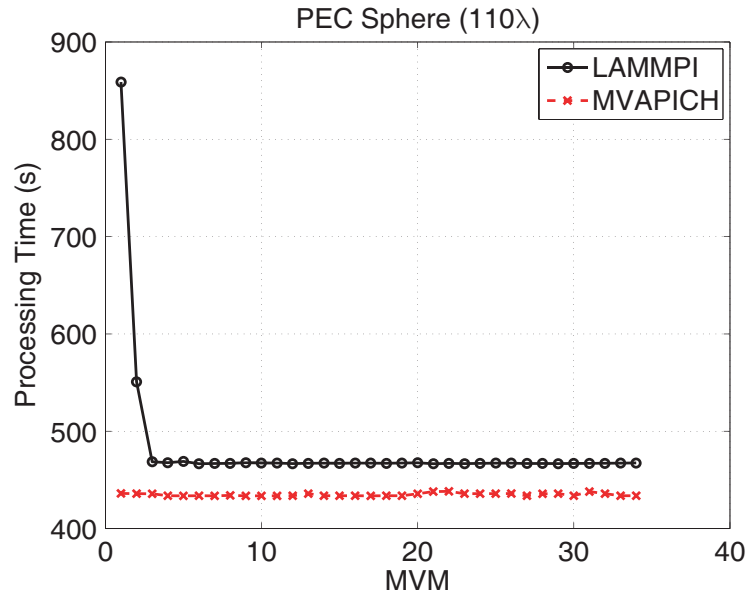
Figure B.9: Time required for one-to-one communications in the BiLCEM Clovertown cluster after optimizations. A data package of size 4 kB is communicated between all processors (cores) of the cluster.

update of the Infiniband driver significantly accelerates inter-node communications.

Figure B.9 presents the time required for one-to-one communications in the BiLCEM Clovertown cluster after the parallel environment is optimized. A single data package of 4 kB is communicated between all pairs of processors (cores) of the cluster. We observe large discrepancies between the time required for intra-node and inter-communications. Specifically, the package can be communicated in more than 10 μs when cores are located on different nodes, whereas it takes less than 6 μs when cores are located in the same node. In addition, Figure B.9 shows that the speed of intra-node communications also depends on the location of cores in the node, e.g., communications are faster when cores are located in the same processor package.



(a)



(b)

Figure B.10: Comparisons of MPI programs for solutions of scattering problems involving (a) a perfectly-conducting sphere of radius 20λ discretized with 1,462,854 unknowns and (b) a perfectly-conducting sphere of radius 110λ discretized with 41,883,648 unknowns.

Finally, the choice of the MPI program is another major parameter, which affects the parallelization of MLFMA. Among several choices, one program may perform better than others for a given parallel computer. As an example, MVAPICH usually leads to more efficient solutions than LAMMPI and OPENMPI on the BiLCEM Clovertown cluster. Figure B.10(a) presents a comparison of different MPI programs for the solution of a scattering problem involving a perfectly-conducting sphere of radius 20λ discretized with 1,462,854 unknowns. We observe that MVAPICH leads to faster iterative solutions than LAMMPI and OPENMPI, especially when the number of processors is large. It is remarkable that the setup time does not depend on the MPI program, but the solution time does. Using 64 processors, OPENMPI and LAMMPI are 100% and 30% slower than MVAPICH for the solution part.

On the BiLCEM Clovertown cluster, MVAPICH is also more stable than other MPI programs for the solution of large-scale problems. As an example, Figure B.10(b) presents the solution of a scattering problem involving a sphere of radius 110λ discretized with 41,883,648 unknowns. The processing time required for each MVM is plotted. We observe that first two MVMs require significantly more time than others using LAMMPI. This undesired event is not observed when using MVAPICH, which provides more stable solutions than LAMMPI. It should be emphasized that our findings about MPI programs are based on the latest versions of those programs and they are measured on the BiLCEM Clovertown cluster, i.e., they may not be generalized to other parallel computers. On the other hand, results in Figure B.10 shows that the choice of the MPI program is crucial in terms of the efficiency of parallel solutions.

Efficient Use of the Computational Environment

It is well-known that programming is essential as much as underlying algorithms to obtain efficient solvers. Specifically, programs must be optimized to eliminate all bottlenecks and to avoid stagnation when the problem size is large. As an example to impacts of programming to the efficiency of solutions, we consider a simple experiment involving multiplications of $N \times N$ dense matrices. Figure B.11 presents the processing time as a function of N^3 measured on an AMD Opteron 870 processor. Theoretically, the complexity of matrix-matrix multiplications is $\mathcal{O}(N^3)$. We consider two different schemes to perform multiplications $\bar{\mathbf{V}} = \bar{\mathbf{X}} \cdot \bar{\mathbf{Y}}$.

1. Ordinary: Loops are constructed as

$$\begin{aligned} &\text{for } m = 1, 2, \dots, N \\ &\quad \text{for } n = 1, 2, \dots, N \\ &\quad \quad \text{for } k = 1, 2, \dots, N \\ &\quad \quad \quad \bar{\mathbf{V}}[m, n] = \bar{\mathbf{V}}[m, n] + \bar{\mathbf{X}}[m, k] \bar{\mathbf{Y}}[k, n] \end{aligned}$$

2. Improved: Loops are constructed as

$$\begin{aligned} &\text{for } n = 1, 2, \dots, N \\ &\quad \text{for } k = 1, 2, \dots, N \\ &\quad \quad \text{for } m = 1, 2, \dots, N \\ &\quad \quad \quad \bar{\mathbf{V}}[m, n] = \bar{\mathbf{V}}[m, n] + \bar{\mathbf{X}}[m, k] \bar{\mathbf{Y}}[k, n] \end{aligned}$$

Figure B.11 shows that, using the first (ordinary) scheme, the processing time of matrix-matrix multiplications does not change linearly with respect to N^3 , and the complexity of matrix-matrix multiplications seems larger than $\mathcal{O}(N^3)$. In addition, the processing time is significantly longer for the first (ordinary) scheme compared to the second (improved) case, especially when N is large. A major difference between two schemes is the use of the data stored in memory.

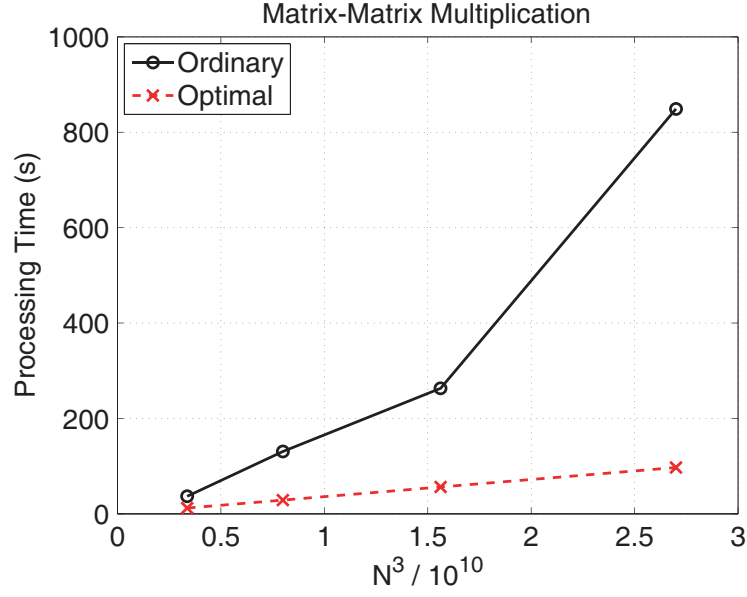


Figure B.11: Processing time required for multiplications of $N \times N$ dense matrices on an AMD Opteron 870 processor.

Using the Fortran programming language, matrices are stored column-wise in memory, and it is always faster to access a local data compared to a distributed data. When loops are constructed by following the ordinary scheme, $\bar{\mathbf{X}}$ and $\bar{\mathbf{V}}$ matrices are not used in the best possible manner. Specifically, both $\bar{\mathbf{X}}$ and $\bar{\mathbf{V}}$ are used row-wise, which leads to significant delays when accessing data in memory. Even the complexity of matrix-matrix multiplications is larger than expected due to the inefficient use of memory. As depicted in Figure B.11, changing the order of loops rectifies the complexity and provides the desired efficiency.

B.4 Strategies for Building Less-Accurate Multilevel Fast Multipole Algorithm

MLFMA can perform a MVM with a desired level of accuracy, which is controlled by the excess bandwidth formula in (4.9). There are many studies to further improve the reliability of implementations by refining formulas for truncation numbers, especially for small boxes [97]. In most cases, the purpose is to obtain accurate results by suppressing the error sources in MLFMA. On the other hand, it is also desirable to build less-accurate forms of MLFMA, which can be more efficient than the original MLFMA. A less-accurate MLFMA can be used to construct a strong preconditioner, where the accuracy is not very critical, but a reasonable approximation is required with high efficiency.

A direct way to construct a less-accurate MLFMA is reducing truncation numbers using (4.9). For example, if the ordinary MLFMA has four digits of accuracy, i.e., $d_0 = 4$, then a less-accurate MLFMA may have one or two digits of accuracy [127]. On the other hand, this strategy has two major disadvantages:

1. A less-accurate MLFMA obtained by decreasing d_0 in (4.9) is not significantly faster than the ordinary MLFMA. Because, the truncation number loosely depends on d_0 for large boxes at higher levels of MLFMA. As an example, let d_0 be reduced from 4 to 1. At the lowest level involving 0.25λ boxes, the truncation number drops significantly by 40% (from 10 to 6). On the other hand, for a higher level with 16λ boxes, the truncation number decreases from 200 to 184, which corresponds to an only 8% reduction. Therefore, reducing the value of d_0 does not provide a significant acceleration, especially when the problem size is large.

2. Extra cost of a less-accurate MLFMA obtained by decreasing d_0 can be significant due to radiation and receiving patterns of basis and testing functions calculated during the setup stage. In addition to ordinary patterns employed by the ordinary MLFMA, a new set of patterns is required for the less-accurate MLFMA with reduced truncation numbers.

As a result, better strategies are required to construct less-accurate and efficient versions of MLFMA.

Another strategy to build a less-accurate MLFMA can be omitting some of far-field interactions. In this case, the number of accurate digits is the same as that for the ordinary MLFMA, but aggregation, translation, disaggregation stages are omitted for a number of higher levels of the tree structure. The resulting less-accurate MLFMA is called the incomplete MLFMA (IMLFMA), which does not require extra computations during the setup stage. In addition, IMLFMA can easily be obtained via minor modifications to the ordinary MLFMA. On the other hand, this strategy fails to provide an acceptable accuracy for a sufficient speedup. For example, half of levels must be ignored to obtain a two-fold speedup with IMLFMA compared to the ordinary MLFMA. However, this would lead to a poor approximation since many interactions (much more than 50%) are ignored.

Alternatively, we propose approximate MLFMA (AMLFMA), which is based on systematically reducing truncation numbers as

$$T_l^r = T_{min} + a_f(T_l - T_{min}), \quad (\text{B.113})$$

where T_{min} is the minimum truncation number and T_l is the ordinary truncation number for level l . In (B.113), a_f represents an approximation factor in the range from 0.0 to 1.0. As a_f is increased from 0.0 to 1.0, AMLFMA becomes more accurate but less efficient, and it corresponds to ordinary MLFMA when $a_f = 1.0$. Since the truncation number at the lowest level is not modified, AMLFMA does

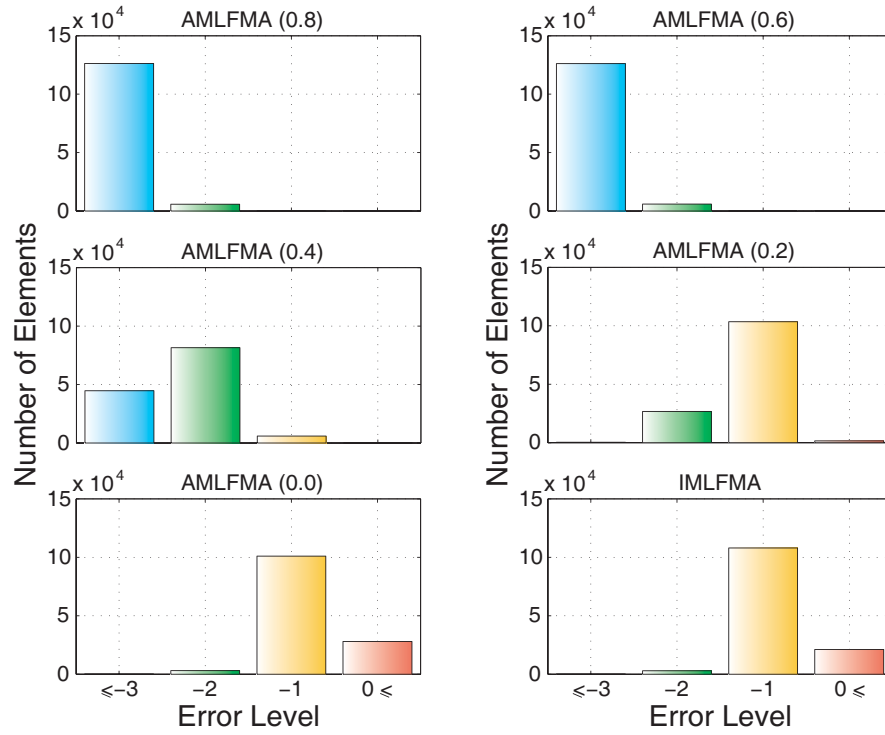


Figure B.12: Errors in MVMs performed by AMLFMA with $a_f = 0.0$ – 0.8 and by IMLFMA (omitting the highest level) for a sphere problem discretized with 132,003 unknowns. The reference data is obtained by using an ordinary MLFMA with three digits of accuracy.

not require extra computations for radiation and receiving patterns of basis and testing functions. Only a new set of translation functions is required, which leads to a negligible extra cost.

As an example, we consider the solution of electromagnetics problems involving a sphere of radius 6λ and a $20\lambda \times 20\lambda$ patch, discretized with 132,003 and 137,792 unknowns, respectively. For both two problems, MVMs are performed by AMLFMA with various values of a_f . In addition to AMLFMA, we also consider IMLFMA, which is constructed by omitting interactions only at the highest level. The number of levels (L) is 4 and 6 for sphere and patch problems, respectively. The input array is filled with ones, and output arrays provided by AMLFMA and IMLFMA are compared with a reference array provided by the

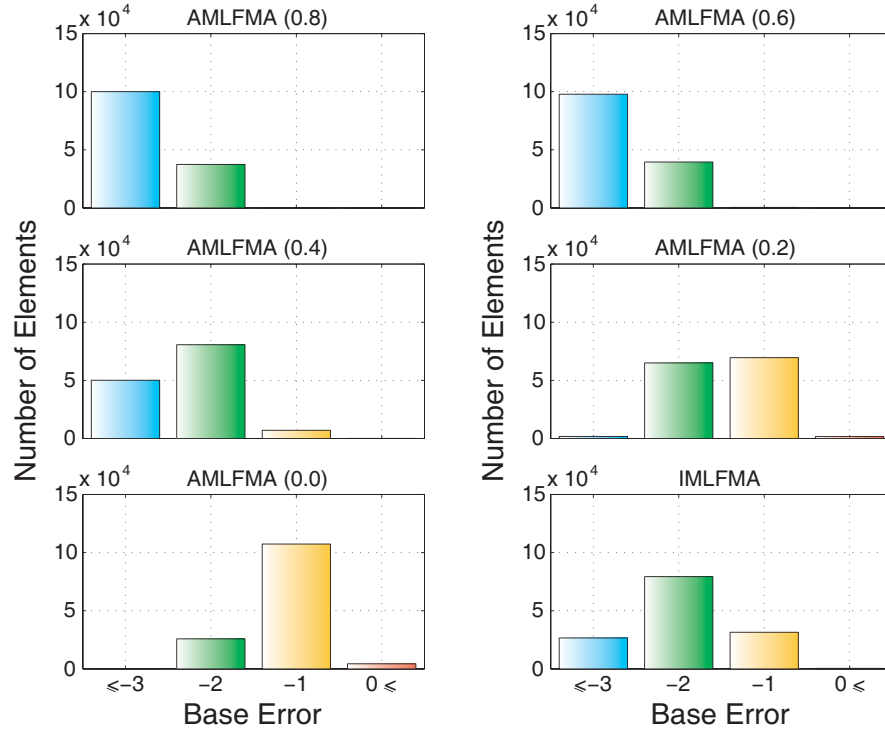


Figure B.13: Errors in MVMs performed by AMLFMA with $a_f = 0.0$ – 0.8 and by IMLFMA (omitting the highest level) for a patch problem discretized with 137,792 unknowns. The reference data is obtained by using an ordinary MLFMA with three digits of accuracy.

ordinary MLFMA with three digits of accuracy. For each element of the output vector $m = 1, 2, \dots, N$, a base error is defined as

$$\Delta_b[m] = \lceil \log_{10}(\Delta_r[m]) \rceil, \quad (\text{B.114})$$

where $\lceil \cdot \rceil$ represents the ceil operation and $\Delta_r[m]$ represents the relative error with respect to the reference value provided by the ordinary MLFMA.

Figures B.12 and B.13 present the number of elements satisfying various base errors for sphere and patch problems, respectively. Figure B.12 shows that the accuracy of the MVM deteriorates slightly when a_f is 0.8 and 0.6 for the sphere problem, i.e., the base error is -3 or below for most of elements. Those elements are calculated with high accuracy, which is even better than acceptable since the ordinary MLFMA has three digits of accuracy. Reducing the value of a_f to 0.4

Table B.1: Processing Time (s) for MVMs via MLFMA and Less-Accurate Implementations of MLFMA.

Problem	Unknowns	MLFMA	AMLFMA					IMLFMA
			0.8	0.6	0.4	0.2	0.0	
Sphere	132,003	34.9	28.3	23.2	19.8	16.0	13.0	32.4
Patch	137,792	24.5	19.4	15.2	12.8	10.1	8.1	21.6

and 0.2, accuracy of AMLFMA decreases and numbers of elements satisfying -2 and -1 base errors increase. Finally, when $a_f = 0.0$, i.e., when the truncation number is T_{min} for all levels, there are many elements with 0 base error, which corresponds to minimum 10% relative error. Figure B.13 presents similar results for a completely different geometry, i.e., $20\lambda \times 20\lambda$ patch. Using AMLFMA, we are able adjust the accuracy of MVMs by choosing the approximation factor a_f from 1.0 to 0.0. On the other hand, IMLFMA, which is obtained by omitting only the highest level, provides inaccurate results compared to AMLFMA with $a_f = 0.2 - 0.4$ for both problems.

Figures B.12 and B.13 show that we obtain relatively accurate MVMs by AMLFMA when using $a_f = 0.2 - 0.4$. This cannot be predicted by the excess bandwidth formula in (4.9), which suggests significantly large truncation numbers. This is because the excess bandwidth formula is based on the worst-case scenario for positions of basis and testing functions inside boxes [97]. In fact, the ordinary MLFMA must use truncation numbers obtained with (4.9) to guarantee the desired level of accuracy. However, for a typical problem, there are many interactions that can be computed accurately using lower truncation numbers. We employ those interactions in AMLFMA, which can be used to construct powerful preconditioners, where the accuracy is not very critical.

Finally, Table B.1 lists the processing time required for a MVM performed by ordinary MLFMA, AMLFMA, and IMLFMA measured on a cluster of 16 AMD Opteron 870 processors. For both sphere and patch problems, AMLFMA with

$a_f = 0.2$ provides a significant acceleration by reducing the processing time more than 50% compared the ordinary MLFMA. We also note that the speedup provided by IMLFMA is less than the speedup provided by AMLFMA, even when $a_f = 0.8$.

B.5 Multilevel Fast Multipole Algorithm for Hermitian Matrix-Vector Multiplications

Some of Krylov subspace algorithms, such as LSQR, requires multiplications with the hermitian of the matrix, in addition to multiplications with the matrix itself. Those hermitian MVMs can be performed efficiently by reversing steps of the ordinary FMM or MLFMA. Consider the interaction of the m th testing function with all basis functions using FMM, i.e.,

$$\begin{aligned} \sum_{n=1}^N \bar{\mathbf{Z}}[m, n] \mathbf{x}[n] &= \sum_{n=1}^N \bar{\mathbf{Z}}_{NF}[m, n] \mathbf{x}[n] \\ &+ \left(\frac{ik}{4\pi} \right)^2 \int d^2 \hat{\mathbf{k}} \mathbf{F}_m(\mathbf{r}_C, \mathbf{k}) \cdot \sum_{C' \in \mathcal{F}\{C\}} \alpha_\tau(\mathbf{k}, \mathbf{r}_C - \mathbf{r}_{C'}) \sum_{n \in C'} \mathbf{S}_n(\mathbf{r}_{C'}, \mathbf{k}) \mathbf{x}[n], \end{aligned} \quad (\text{B.115})$$

where $\mathbf{F}_m$ and $\mathbf{S}_n$ represent receiving and radiation patterns, respectively. Then, one can obtain

$$\begin{aligned} \sum_{n=1}^N \bar{\mathbf{Z}}^*[n, m] \mathbf{x}[n] &= \sum_{n=1}^N \bar{\mathbf{Z}}_{NF}^*[n, m] \mathbf{x}[n] \\ &+ \left(\frac{ik}{4\pi} \right)^2 \int d^2 \hat{\mathbf{k}} \mathbf{S}_m^*(\mathbf{r}_{C'}, \mathbf{k}) \cdot \sum_{C \in \mathcal{F}\{C'\}} \alpha_\tau^*(\mathbf{k}, \mathbf{r}_C - \mathbf{r}_{C'}) \sum_{n \in C} \mathbf{F}_n^*(\mathbf{r}_C, \mathbf{k}) \mathbf{x}[n]. \end{aligned} \quad (\text{B.116})$$

The expression in (B.116) can be interpreted as follows:

- Combine receiving patterns of testing functions in each box.

- Translate receiving patterns of boxes into local receiving patterns for other boxes.
- Receive radiated fields of basis functions locally.

When $\bar{\mathbf{Z}}$ is symmetric,

$$\mathbf{F}_n^*(\mathbf{r}_C, \mathbf{k}) = \mathbf{S}_n(\mathbf{r}_C, \mathbf{k}) \quad (\text{B.117})$$

$$\mathbf{S}_m^*(\mathbf{r}_C, \mathbf{k}) = \mathbf{F}_m(\mathbf{r}_C, \mathbf{k}) \quad (\text{B.118})$$

and

$$\begin{aligned} \sum_{n=1}^N \bar{\mathbf{Z}}^*[n, m] \mathbf{x}[n] &= \sum_{n=1}^N \bar{\mathbf{Z}}_{NF}^*[m, n] \mathbf{x}[n] \\ &+ \left(\frac{ik}{4\pi} \right)^2 \int d^2 \hat{\mathbf{k}} \mathbf{F}_m(\mathbf{r}_{C'}, \mathbf{k}) \cdot \sum_{C \in \mathcal{F}\{C'\}} \alpha_\tau^*(\mathbf{k}, \mathbf{r}_C - \mathbf{r}_{C'}) \sum_{n \in C} \mathbf{S}_n(\mathbf{r}_C, \mathbf{k}) \mathbf{x}[n]. \end{aligned} \quad (\text{B.119})$$

Using

$$\mathbf{F}_m(\mathbf{r}_{C'}, \mathbf{k}) \cdot \alpha_\tau^*(\mathbf{k}, \mathbf{r}_C - \mathbf{r}_{C'}) \mathbf{S}_n(\mathbf{r}_C, \mathbf{k}) = \mathbf{F}_m^*(\mathbf{r}_{C'}, \mathbf{k}) \cdot \alpha_\tau^*(\mathbf{k}, \mathbf{r}_{C'} - \mathbf{r}_C) \mathbf{S}_n^*(\mathbf{r}_C, \mathbf{k}), \quad (\text{B.120})$$

we obtain

$$\sum_{n=1}^N \bar{\mathbf{Z}}^*[n, m] \mathbf{x}[n] = \sum_{n=1}^N \bar{\mathbf{Z}}^*[m, n] \mathbf{x}[n] \quad (\text{B.121})$$

as expected.

B.6 Calculation of Some Special Functions

In this section, we present fast and efficient calculations of some special functions, particularly those required for various implementations of MLFMA.

B.6.1 Spherical Bessel Functions

We calculate spherical Bessel and spherical Hankel functions of order t as

$$j_t(x) = \sqrt{\frac{\pi}{2x}} J_{t+1/2}(x) \quad (\text{B.122})$$

$$h_t^{(1,2)}(x) = \sqrt{\frac{\pi}{2x}} H_{t+1/2}^{(1,2)}(x), \quad (\text{B.123})$$

where $J_{t+1/2}$ and $H_{t+1/2}^{(1,2)}$ are ordinary Bessel and Hankel functions that are available in the AMOS library. We note that a recursive calculation of a Bessel or Hankel function of order t requires the calculation of functions with lower orders $t' < t$. Hence, a given value of t , the AMOS library provides all orders up to t simultaneously.

B.6.2 Legendre Functions

Instead of ordinary Legendre functions, we compute normalized Legendre functions defined as

$$R_t^u(\cos \theta) = \sqrt{\frac{(t-u)!}{(t+u)!}} P_t^u(\cos \theta) \quad (\text{B.124})$$

for $|u| \leq t$. We start with

$$R_0^0(\cos \theta) = 1 \quad (\text{B.125})$$

$$R_1^0(\cos \theta) = \cos \theta \quad (\text{B.126})$$

$$R_1^1(\cos \theta) = -\frac{1}{\sqrt{2}}(1 - \cos^2 \theta)^{1/2} = -\frac{1}{\sqrt{2}} \sin \theta \quad (\text{B.127})$$

$$R_1^{-1}(\cos \theta) = \sqrt{2} P_1^{-1}(\cos \theta) = -\frac{1}{\sqrt{2}} P_1^1(\cos \theta) = -R_1^1(\cos \theta), \quad (\text{B.128})$$

and construct a recursion for $t \geq 2$ as follows:

- $|u| \neq t$

$$\begin{aligned}
R_t^u(\cos \theta) &= \sqrt{\frac{(t-u)!}{(t+u)!}} P_t^u(\cos \theta) \\
&= \cos \theta \frac{(2t-1)}{(t-u)} \sqrt{\frac{(t-u)}{(t+u)}} \sqrt{\frac{(t-1-u)!}{(t-1+u)!}} P_{t-1}^u(\cos \theta) \\
&\quad - \frac{(t+u-1)}{(t-u)} \sqrt{\frac{(t-u)}{(t+u)}} \sqrt{\frac{(t-1-u)}{(t-1+u)}} \sqrt{\frac{(t-2-u)!}{(t-2+u)!}} P_{t-2}^u(\cos \theta) \\
&= \cos \theta \frac{(2t-1)}{\sqrt{(t-u)(t+u)}} R_{t-1}^u(\cos \theta) \\
&\quad - \frac{\sqrt{(t-1+u)(t-1-u)}}{\sqrt{(t-u)(t+u)}} R_{t-2}^u(\cos \theta). \tag{B.129}
\end{aligned}$$

- $u = t$

$$\begin{aligned}
R_t^u(\cos \theta) &= R_t^t(\cos \theta) = \frac{1}{\sqrt{(2t)!}} P_t^t(\cos \theta) \\
&= \frac{1}{\sqrt{(2t)!}} (-1)^t (2t-1)!! (1 - \cos^2 \theta)^{t/2} \\
&= \frac{1}{\sqrt{(2t)!}} (-1)^t \frac{(2t)!}{2^t t!} (1 - \cos^2 \theta)^{t/2} \\
&= -\sqrt{\frac{(2t-1)}{2t}} (1 - \cos^2 \theta)^{1/2} R_{t-1}^{t-1}(\cos \theta). \tag{B.130}
\end{aligned}$$

- $u = -t$

$$\begin{aligned}
R_t^u(\cos \theta) &= R_t^{-t}(\cos \theta) = \sqrt{(2t)!} P_t^{-t}(\cos \theta) = \frac{\sqrt{(2t)!}}{(2t)!} (-1)^t P_t^t(\cos \theta) \\
&= \frac{\sqrt{(2t)!}}{(2t)!} \frac{(2t)!}{2^t t!} (1 - \cos^2 \theta)^{t/2} = \frac{\sqrt{(2t)!}}{2^t t!} (1 - \cos^2 \theta)^{t/2} \\
&= \sqrt{\frac{(2t-1)}{2t}} (1 - \cos^2 \theta)^{1/2} R_{t-1}^{-t+1}(\cos \theta). \tag{B.131}
\end{aligned}$$

In recursive formulas (B.129)–(B.131), we set $R_t^u(\cos \theta) = 0$, when $u > t$ or $u < -t$.

B.6.3 Gradient of Multipole-to-Monopole Shift Functions

LF-MLFMA implementations using MFIE require the gradient of multipole-to-monopole shift functions, which can be calculated as

$$\begin{aligned}
\nabla \beta_{0,0,t,u}(k, \mathbf{r}) &= \sqrt{4\pi} \nabla \left(j_t(kr) Y_{t,u}(\theta, \phi) \right) \\
&= \hat{\mathbf{r}} \sqrt{4\pi} \frac{\partial j_t(kr)}{\partial r} Y_{t,u}(\theta, \phi) + \hat{\boldsymbol{\theta}} \sqrt{4\pi} \frac{j_t(kr)}{r} \frac{\partial Y_{t,u}(\theta, \phi)}{\partial \theta} \\
&\quad + \hat{\boldsymbol{\phi}} \sqrt{4\pi} \frac{j_t(kr)}{r \sin \theta} \frac{\partial Y_{t,u}(\theta, \phi)}{\partial \phi} \\
&= \hat{\mathbf{r}} \sqrt{4\pi} (-1)^t \left\{ \frac{t}{r} j_t(r) - k j_{t+1}(kr) \right\} Y_{t,u}(\theta, \phi) \\
&\quad + \hat{\boldsymbol{\theta}} \sqrt{4\pi} (-1)^t \frac{j_t(kr)}{r} \sqrt{\frac{(2t+1)(t-u)!}{4\pi(t+u)!}} \exp(iu\phi) \frac{\partial P_t^u(\cos \theta)}{\partial \theta} \\
&\quad + \hat{\boldsymbol{\phi}} \sqrt{4\pi} (-1)^t \frac{j_t(kr)}{r \sin \theta} iu Y_{t,u}(\theta, \phi) \\
&= \hat{\mathbf{r}} \sqrt{4\pi} (-1)^t \left\{ \frac{t}{r} j_t(kr) - k j_{t+1}(kr) \right\} Y_{t,u}(\theta, \phi) \\
&\quad + \hat{\boldsymbol{\theta}} \sqrt{4\pi} (-1)^t \frac{j_t(kr)}{r \sin \theta} \sqrt{\frac{(2t+1)(t+1-u)(t+1+u)}{(2t+3)}} Y_{t+1,u}(\cos \theta) \\
&\quad - \hat{\boldsymbol{\theta}} \sqrt{4\pi} (-1)^t \frac{j_t(kr) \cos \theta}{r \sin \theta} (t+1) Y_t^u(\cos \theta) \\
&\quad + \hat{\boldsymbol{\phi}} \sqrt{4\pi} (-1)^t \frac{j_t(kr)}{r \sin \theta} iu Y_{t,u}(\theta, \phi). \tag{B.132}
\end{aligned}$$

B.6.4 Calculation of Gaunt Coefficients

Gaunt coefficients $G_{t,u,t',u',t'',u''}$, which are required in LF-MLFMA implementations, are calculated for $|u| \leq t$, $|u'| \leq t'$, $|u''| \leq t''$, $|t - t'| \leq t'' \leq |t + t'|$, $u'' = u - u'$, and when $t + t' + t''$ is even. They can be derived as

$$\begin{aligned}
G_{t,u,t',u',t'',u''} &= (-1)^u \sqrt{\frac{(2t+1)(2t'+1)(2t''+1)}{4\pi}} \\
&\quad \begin{pmatrix} t & t' & t'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t & t' & t'' \\ -u & u' & u'' \end{pmatrix} \tag{B.133}
\end{aligned}$$

in terms of Wigner-3j symbols. In order to calculate Wigner-3j symbols, we start with

$$\begin{pmatrix} t & t' & t+t' \\ u & u' & u'' \end{pmatrix} = \frac{(-1)^{t-t'-u''} [(2t)!(2t')!(t+t'+u'')!(t+t'-u'')!]^{1/2}}{[(2t+2t'+1)!(t+u)!(t-u)!(t'+u')!(t'-u')!]^{1/2}} \quad (\text{B.134})$$

and construct a recursion for $|t-t'| \leq t'' < |t+t'|$ as

$$\begin{pmatrix} t & t' & t'' \\ u & u' & u'' \end{pmatrix} = -\frac{a}{b} \begin{pmatrix} t & t' & t''+2 \\ u & u' & u'' \end{pmatrix} - \frac{c}{b} \begin{pmatrix} t & t' & t''+1 \\ u & u' & u'' \end{pmatrix}, \quad (\text{B.135})$$

where

$$a = (t''+1) \left\{ [(t''+2)^2 - (t-t')^2] \right. \\ \left. [(t+t'+1)^2 - (t''+2)^2] [(t''+2)^2 - (u'')^2] \right\}^{1/2} \quad (\text{B.136})$$

$$b = (t''+2) \left\{ [(t''+1)^2 - (t-t')^2] \right. \\ \left. [(t+t'+1)^2 - (t''+1)^2] [(t''+1)^2 - (u'')^2] \right\}^{1/2} \quad (\text{B.137})$$

$$c = -(2t''+3) [t(t+1)u'' - t'(t'+1)u'' - (t''+1)(t''+2)(u'-u)]. \quad (\text{B.138})$$

○

Appendix C

Other Works

C.1 Regularization of the Combined-Field Integral Equation

In this section, we present a stable implementation of N-N-CFIE, which is obtained by the convex combination of N-EFIE and N-MFIE. Using a Galerkin scheme, N-N-CFIE is extremely unstable due to N-EFIE involving weakly-tested $\mathcal{T}$ operator. We apply a stabilization procedure introduced in [71] and develop a regularized CFIE (RCFIE), which provides well-conditioned matrix equations that can be solved efficiently via iterative methods.

Consider the solution of smooth PEC objects via N-N-CFIE, i.e.,

$$\mathcal{Z}^{\text{N-N-CFIE}(\alpha)}\{\mathbf{J}\}(\mathbf{r}) = -\alpha\eta_o^{-1}\hat{\mathbf{n}} \times \mathbf{E}^{inc}(\mathbf{r}) - (1 - \alpha)\hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}), \quad (\text{C.1})$$

where

$$\mathcal{Z}^{\text{N-N-CFIE}(\alpha)} = \alpha\hat{\mathbf{n}} \times \mathcal{T} + (1 - \alpha)\hat{\mathbf{n}} \times \mathcal{K} - \frac{1}{2}(1 - \alpha)\mathcal{I}. \quad (\text{C.2})$$

A pseudo-CFIE operator is defined as

$$\begin{aligned}\mathcal{Z}^{P(\alpha)} &= \mathcal{Z}^{N-N-CFIE(\alpha)} - (1 - \alpha)\hat{\mathbf{n}} \times \mathcal{K} + i\alpha\hat{\mathbf{n}} \times \mathcal{K} + (1 - \alpha)\mathcal{I} \\ &= \alpha\hat{\mathbf{n}} \times \mathcal{T} + i\alpha\hat{\mathbf{n}} \times \mathcal{K} + \frac{1}{2}(1 - \alpha)\mathcal{I}.\end{aligned}\quad (\text{C.3})$$

Using Calderon identities, i.e.,

$$(\hat{\mathbf{n}} \times \mathcal{K})(\hat{\mathbf{n}} \times \mathcal{T}) = -(\hat{\mathbf{n}} \times \mathcal{T})(\hat{\mathbf{n}} \times \mathcal{K}) \quad (\text{C.4})$$

$$(\hat{\mathbf{n}} \times \mathcal{K})^2 - (\hat{\mathbf{n}} \times \mathcal{T})^2 = \mathcal{I}/4, \quad (\text{C.5})$$

the pseudo-CFIE operator can be inverted as

$$[\mathcal{Z}^{P(\alpha)}]^{-1} = \frac{1}{\alpha^2 + (1 - \alpha)^2} [2(1 - \alpha)\mathcal{I} - 4i\alpha\hat{\mathbf{n}} \times \mathcal{K} - 4\alpha\hat{\mathbf{n}} \times \mathcal{T}]. \quad (\text{C.6})$$

Finally, using (C.6), N-N-CFIE can be regularized as

$$\begin{aligned}[\mathcal{Z}^{P(\alpha)}]^{-1} \mathcal{Z}^{N-N-CFIE(\alpha)} \{\mathbf{J}\}(\mathbf{r}) &= -\alpha\eta_o^{-1} [\mathcal{Z}^{P(\alpha)}]^{-1} \{\hat{\mathbf{n}} \times \mathbf{E}^{inc}\}(\mathbf{r}) \\ &\quad - (1 - \alpha) [\mathcal{Z}^{P(\alpha)}]^{-1} \{\hat{\mathbf{n}} \times \mathbf{H}^{inc}\}(\mathbf{r})\end{aligned}\quad (\text{C.7})$$

or

$$\mathcal{Z}^{RCFIE} \{\mathbf{J}\}(\mathbf{r}) = -[\mathcal{Z}^{P(\alpha)}]^{-1} \{ \alpha\eta_o^{-1} \hat{\mathbf{n}} \times \mathbf{E}^{inc} + (1 - \alpha)\hat{\mathbf{n}} \times \mathbf{H}^{inc} \}(\mathbf{r}), \quad (\text{C.8})$$

where

$$\begin{aligned}\mathcal{Z}^{RCFIE} &= \mathcal{I} + \frac{2(1 - \alpha)}{(1 - \alpha + i\alpha)} (\hat{\mathbf{n}} \times \mathcal{K}) \\ &\quad - \frac{4\alpha}{(1 - \alpha + i\alpha)} [(\hat{\mathbf{n}} \times \mathcal{T}) + i(\hat{\mathbf{n}} \times \mathcal{K})] (\hat{\mathbf{n}} \times \mathcal{K}).\end{aligned}\quad (\text{C.9})$$

RCFIE in (C.8) can be discretized by using the RWG functions. We obtain $N \times N$ dense matrix equations in the form of

$$\bar{\mathbf{Z}}_{RC} \cdot \mathbf{a} = \mathbf{w}_{RC}, \quad (\text{C.10})$$

where

$$\begin{aligned}\bar{\mathbf{Z}}_{RC} &= \bar{\mathbf{I}}^T + \frac{2(1-\alpha)}{(1-\alpha+i\alpha)}\bar{\mathbf{K}}^N \\ &\quad - \frac{4\alpha}{(1-\alpha+i\alpha)}[\bar{\mathbf{T}}^N + i\bar{\mathbf{K}}^N] \cdot [\bar{\mathbf{I}}^T]^{-1} \cdot \bar{\mathbf{K}}^N\end{aligned}\quad (\text{C.11})$$

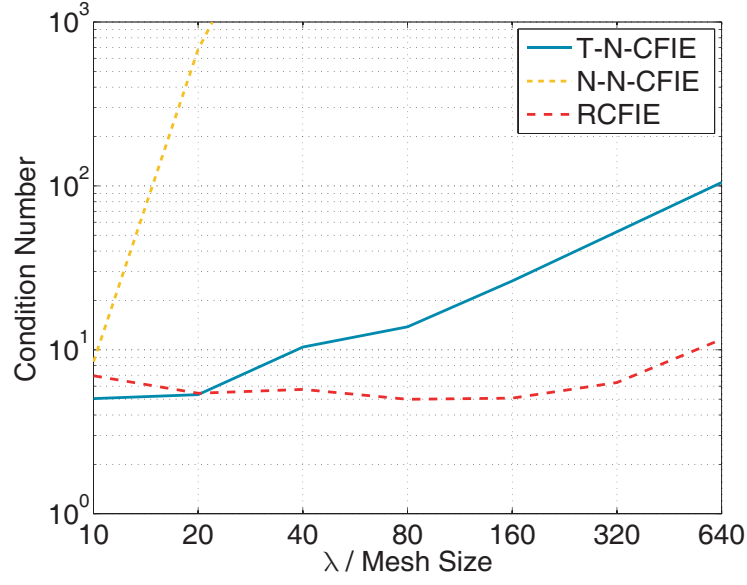
$$\mathbf{w}_{RC} = -\alpha\eta_o^{-1}\bar{\mathbf{Z}}_P^{-1} \cdot [\bar{\mathbf{I}}^T]^{-1} \cdot \mathbf{v}_E^N - (1-\alpha)\bar{\mathbf{Z}}_P^{-1} \cdot [\bar{\mathbf{I}}^T]^{-1} \cdot \mathbf{v}_H^N \quad (\text{C.12})$$

and

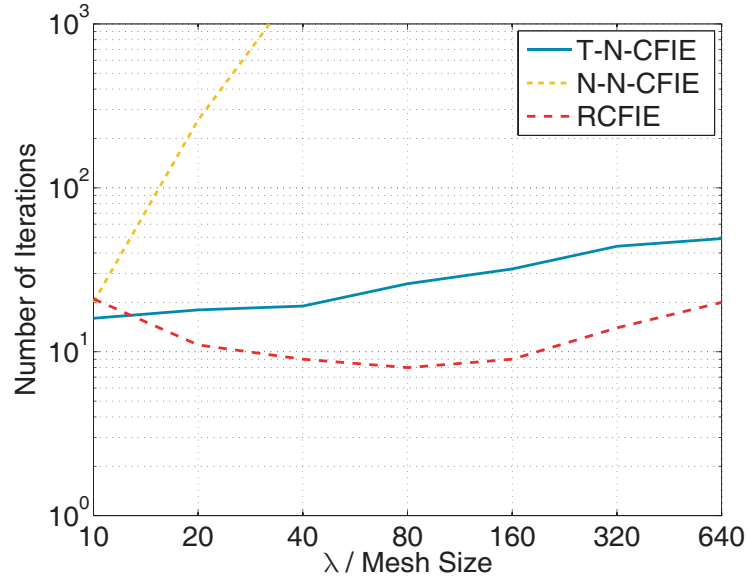
$$\bar{\mathbf{Z}}_P^{-1} = \frac{1}{\alpha^2 + (1-\alpha)^2} [2(1-\alpha)\bar{\mathbf{I}}^T - 4i\alpha\bar{\mathbf{K}}^N - 4\alpha\bar{\mathbf{T}}^N]. \quad (\text{C.13})$$

We note that the inverse of the Gram matrix $\bar{\mathbf{I}}^T$ is used between two discretized operators applied on the coefficient vector or before a discretized operator applied on a RHS vector.

Figure C.1 presents the solution of scattering problems involving a perfectly-conducting sphere of radius 30 cm discretized with 930 RWG functions. The sphere is illuminated by a plane wave at various frequencies decreasing from 500 MHz to 7.8125 MHz. Figure C.1(a) depicts 2-norm condition numbers of constructed matrix equations as a function of the mesh size. We observe that the condition number for the ordinary CFIE, i.e, T-N-CFIE, increases as the mesh size decreases, due to the low-frequency breakdown of T-EFIE. In addition, RCFIE (with $\alpha = 0.5$) provides more stable matrix equations than T-N-CFIE when the mesh size is smaller than $\lambda/20$. It is remarkable that matrix equations obtained from N-N-CFIE have extremely high condition numbers for dense discretizations. Figure C.1(b) also presents favorable properties of RCFIE, which requires fewer iterations than T-N-CFIE when the mesh size smaller than $\lambda/10$.



(a)



(b)

Figure C.1: Solutions of scattering problems involving a perfectly-conducting sphere of radius 30 cm discretized with 930 unknowns at various frequencies. (a) 2-norm condition number and (b) number of CGS iterations (for 10^{-6} residual error) as a function of mesh size in the range from $\lambda/10$ to $\lambda/640$.

Figure C.1 shows that RCFIE provides more efficient solutions than T-N-CFIE for dense discretizations. Nevertheless, for relative small objects with respect to wavelength, the most efficient solutions are usually obtained by using N-MFIE (not shown in Figure C.1). In fact, favorable convergence properties of RCFIE for those problems are mostly due to N-MFIE inside this formulation. We conclude that RCFIE should be tested and compared with other formulations for the solution of large-scale problems (such that MFIE cannot be used due to internal resonances) with dense discretizations. This will be considered elsewhere.

C.2 Using the Linear-Linear Functions for the Identity Operator

When discretized with the RWG functions, normal and mixed formulations are significantly inaccurate compared to tangential formulations due to the excessive discretization error of the identity operator. In addition, the LL functions improve the accuracy of normal and mixed formulations to levels of tangential formulations. Since the source of the error in those formulations is the identity operator, an important question arises: Is it possible to discretize only the identity operator with the LL functions to improve the accuracy without the extra cost of the LL functions compared to the RWG functions. Such a discretization of MFIE using N RWG functions and $2N$ LL functions leads to $2N \times 2N$ matrix equations in the form of

$$-\frac{1}{2}\bar{\mathbf{I}}^{T,LL} \cdot \mathbf{a} + \bar{\mathbf{I}}^{T,LR} \cdot [\bar{\mathbf{I}}^{T,LL}]^{-1} \cdot \bar{\mathbf{K}}_{PV}^{N,RWG} \cdot [\bar{\mathbf{I}}^{T,LL}]^{-1} \cdot \bar{\mathbf{I}}^{T,RL} \cdot \mathbf{a} = -\mathbf{v}_H^{N,LL}, \quad (\text{C.14})$$

where $\bar{\mathbf{I}}^{T,\text{LR}}$ and $\bar{\mathbf{I}}^{T,\text{RL}}$ are $2N \times N$ and $N \times 2N$ sparse matrix equations containing inner products of RWG and LL functions, i.e.,

$$\bar{\mathbf{I}}^{T,\text{LR}}[m, n] = \left\langle \mathbf{t}_m^{\text{LL}}(\mathbf{r}), \mathbf{b}_n^{\text{RWG}}(\mathbf{r}) \right\rangle \quad (\text{C.15})$$

$$\bar{\mathbf{I}}^{T,\text{RL}}[m, n] = \left\langle \mathbf{t}_m^{\text{RWG}}(\mathbf{r}), \mathbf{b}_n^{\text{LL}}(\mathbf{r}) \right\rangle. \quad (\text{C.16})$$

In (C.14), the identity operator is discretized with the LL functions, whereas the integro-differential $\mathcal{K}$ operator is discretized with the RWG functions. Although the mixed discretization in (C.14) is successfully implemented, numerical experiments show that the accuracy of MFIE is not improved. This is because, the implementation using (C.14) still involves the discretization of the identity operator using the RWG functions to perform conversions for the $\mathcal{K}$ operator.

C.3 Rectangular Combined-Field Integral Equation

Ordinary CFIE is a convex combination of EFIE and MFIE. On the other hand, EFIE and MFIE can also be solved in the least-squares sense, instead of adding them into a single equation. Such a simultaneous solution of EFIE and MFIE leads to an $2N \times N$ rectangular matrix equation. As an example, Figure C.2 presents the solution of a scattering problem involving a perfectly-conducting sphere of radius 6λ discretized with 132,003 RWG functions. The problem is formulated by a rectangular CFIE obtained by stacking $\alpha \times \text{EFIE}$ and $(1-\alpha) \times \text{MFIE}$ with $\alpha = 0.33$. The iterative solution is performed by using the LSQR algorithm. Figure C.2 shows that the residual error drops rapidly in the LS sense, but the true residual does not drop beyond a certain value (10^{-2}). This is because EFIE and MFIE do not have exactly the same solution, i.e., they are not compatible, due to the excessive discretization error of the identity operator in MFIE. As depicted in Figure C.2, the CFIE solution is also incompatible with EFIE and

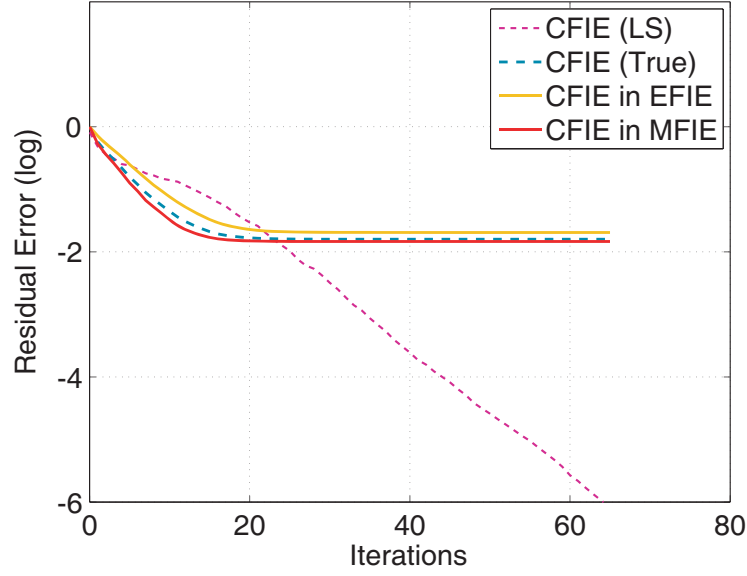


Figure C.2: Solution of a scattering problem involving a perfectly-conducting sphere of radius 6λ formulated with a rectangular CFIE and discretized with 132,003 unknowns.

MFIE, and residual errors obtained by testing the solution in EFIE and MFIE systems also saturate after 20th iteration.

C.4 An Iterative Solution of the Magnetic-Field Integral Equation

Since MFIE is a second-kind integral equation, it can be solved trivially by applying a recursion. Using the expression for MFIE recursively for multiple times, the unknown electric current can be written as

$$\mathbf{J}(\mathbf{r}) = \left(\frac{4\pi}{\Omega_o}\right) \hat{\mathbf{n}} \times \mathcal{K}_{PV,o}\{\mathbf{J}\}(\mathbf{r}) + \left(\frac{4\pi}{\Omega_o}\right) \hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}) \quad (\text{C.17})$$

$$\begin{aligned} &= \left(\frac{4\pi}{\Omega_o}\right)^2 (\hat{\mathbf{n}} \times \mathcal{K}_{PV,o})^2 \{\mathbf{J}\}(\mathbf{r}) + \left(\frac{4\pi}{\Omega_o}\right)^2 \hat{\mathbf{n}} \times \mathcal{K}_{PV,o}\{\hat{\mathbf{n}} \times \mathbf{H}^{inc}\}(\mathbf{r}) \\ &+ \left(\frac{4\pi}{\Omega_o}\right) \hat{\mathbf{n}} \times \mathbf{H}^{inc}(\mathbf{r}), \end{aligned} \quad (\text{C.18})$$

which can be generalized as

$$\mathbf{J}(\mathbf{r}) = \sum_{m=0}^{\infty} \left(\frac{4\pi}{\Omega_o} \right)^{m+1} (\hat{\mathbf{n}} \times \mathcal{K}_{PV,o})^m \{ \hat{\mathbf{n}} \times \mathbf{H}^{inc} \}(\mathbf{r}). \quad (\text{C.19})$$

The expression in (C.19) indicates that the unknown electric current can be computed by applying the $\mathcal{K}$ operator repeatedly on the tangential component of the incident magnetic field. Nevertheless, numerical experiments show that, for a given accuracy, the number of MVMs required in (C.19) is usually higher than the number of MVMs required in ordinary solutions of MFIE via a Krylov-subspace algorithm.

C.5 Phase-Front Extraction

Using a phase-front extraction, we assume that the induced electric current on a large-scale object illuminated by a plane wave is in form of

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}_o(\mathbf{r}) \exp(i\mathbf{k} \cdot \mathbf{r}), \quad (\text{C.20})$$

where $\mathbf{k} = k\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}$ is the direction of the plane wave. By extracting the physical rapid variation $\exp(i\mathbf{k} \cdot \mathbf{r})$, the remaining term $\mathbf{J}_o$, which can be interpreted as the envelope of the current, may vary slowly on the surface of the object. Hence, an accurate discretization of $\mathbf{J}_o$ may require fewer elements than the discretization of $\mathbf{J}$.

As an example, we consider the discretization of EFIE with the RWG functions.

Using a phase-front extraction, the electric current is expanded as

$$\mathbf{J}(\mathbf{r}) \approx \sum_{n=1}^N \mathbf{a}_J[n] \exp(i\mathbf{k} \cdot \mathbf{r}) \mathbf{b}_n^{\text{RWG}}(\mathbf{r}). \quad (\text{C.21})$$

Using RWG testing functions, matrix elements are derived as

$$\begin{aligned}
\bar{\mathbf{Z}}_{E,\text{exp}}^{\text{RWG}}[m, n] &= ik \int_{S_m} d\mathbf{r} \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \cdot \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \exp(i\mathbf{k} \cdot \mathbf{r}') \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \\
&\quad - \frac{i}{k} \int_{S_m} d\mathbf{r} \nabla \cdot \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \left[\exp(i\mathbf{k} \cdot \mathbf{r}') \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \right] \\
&= ik \int_{S_n} d\mathbf{r}' \exp(i\mathbf{k} \cdot \mathbf{r}') \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \cdot \int_{S_m} d\mathbf{r} g(\mathbf{r}, \mathbf{r}') \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \\
&\quad - \frac{i}{k} \int_{S_m} d\mathbf{r} \nabla \cdot \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla' \cdot \left[\exp(i\mathbf{k} \cdot \mathbf{r}') \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \right] \\
&= ik \int_{S_n} d\mathbf{r}' \exp(i\mathbf{k} \cdot \mathbf{r}') \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \cdot \int_{S_m} d\mathbf{r} g(\mathbf{r}, \mathbf{r}') \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \\
&\quad - \frac{i}{k} \int_{S_m} d\mathbf{r} \nabla \cdot \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \exp(i\mathbf{k} \cdot \mathbf{r}') \nabla' \cdot \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \\
&\quad - \frac{i}{k} \int_{S_m} d\mathbf{r} \nabla \cdot \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \int_{S_n} d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') \nabla' \exp(i\mathbf{k} \cdot \mathbf{r}') \cdot \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \\
&= ik \int_{S_n} d\mathbf{r}' \exp(i\mathbf{k} \cdot \mathbf{r}') \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \cdot \int_{S_m} d\mathbf{r} g(\mathbf{r}, \mathbf{r}') \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \\
&\quad - \frac{i}{k} \int_{S_n} d\mathbf{r}' \exp(i\mathbf{k} \cdot \mathbf{r}') \nabla' \cdot \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \int_{S_m} d\mathbf{r} g(\mathbf{r}, \mathbf{r}') \nabla \cdot \mathbf{t}_m^{\text{RWG}}(\mathbf{r}) \\
&\quad - \frac{i}{k} \int_{S_n} d\mathbf{r}' \nabla' \exp(i\mathbf{k} \cdot \mathbf{r}') \cdot \mathbf{b}_n^{\text{RWG}}(\mathbf{r}') \int_{S_m} d\mathbf{r} g(\mathbf{r}, \mathbf{r}') \nabla \cdot \mathbf{t}_m^{\text{RWG}}(\mathbf{r}).
\end{aligned} \tag{C.22}$$

As usual, the final expression in (C.22) is divided into basis integrals and singularity-extraction methods are employed for accurate computations.

The phase-front extraction can be successful only when (C.20) is applicable. Otherwise, the size of discretization elements cannot be reduced and the phase-front extraction may not provide the desired speedup compared to ordinary solutions.

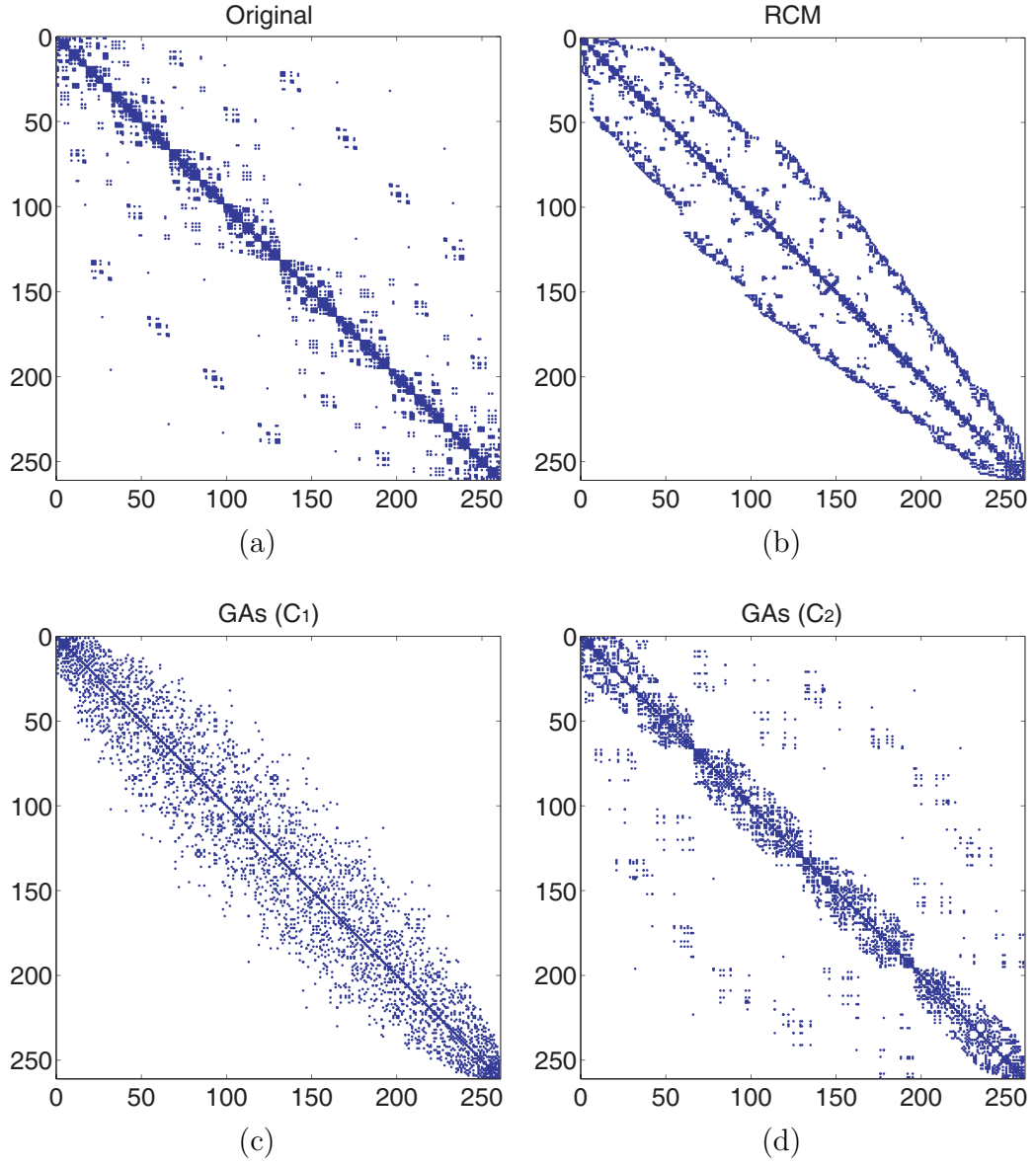


Figure C.3: Distribution of near-field interactions between lowest-level boxes. (a) Original, (b) reordered via an RCM algorithm, (c) reordered via GAs using the cost function defined in (C.23), and (d) reordered via GAs using the cost function defined in (C.24).

C.6 Reordering Sparse Near-Field Matrices

Near-field interactions in MLFMA are essential to construct preconditioners to accelerate iterative solutions. However, using the indexing dictated by the multilevel tree structure, near-field interactions are usually distributed irregularly in the matrix equation. Hence, it can be useful to reorder near-field interactions such that they are located close to the diagonal and they can be used efficiently to construct banded preconditioners.

As an example, Figure C.3 presents the distribution of near-field interactions between lowest-level boxes for a scattering problem involving a sphere of radius λ discretized with 3723 unknowns. Original distribution without reordering is depicted in Figure C.3(a). Applying a reverse-Cuthill-McKee (RCM) algorithm, we obtain the pattern in C.3(b) with a reduced bandwidth of near-field interactions. We also employ GAs in order to collect near-field interactions close to the diagonal as much as possible. Two examples are presented in Figures C.3(c) and C.3(d), which are obtained by using different cost functions for the optimization, i.e.,

$$C_1 = \sum_j |m(j) - n(j)|^2 \quad (\text{C.23})$$

$$C_2 = \sum_j u(|m(j) + n(j)| - 20), \quad (\text{C.24})$$

where m and n represent the row index and the column index of the j th box-box interaction, and $u(x) = 0.5 + 0.5x/|x|$ is the unit step function. In general, Figure C.3 shows that we are able to improve the placement of near-field interactions. However, numerical experiments show that banded preconditioners based on reordered near-field interactions are still inefficient compared to ordinary preconditioners.

Abbreviations

2PBDP: Two-partition block-diagonal preconditioner

4PBDP: Four-partition block-diagonal preconditioner

A: Ampere

AMLFMA: Approximate multilevel fast multipole algorithm

BCE: Boundary-condition error

BDP: Block-diagonal preconditioner

BiCG: Biconjugate gradient (method)

BiCGStab: Biconjugate gradient stabilized (method)

CFIE: Combined-field integral equation

CG: Conjugate gradient

CGS: Conjugate gradient squared (method)

cm: Centimeter

CMM: Composite metamaterial

CNF: Combined normal formulation

CTF: Combined tangential formulation

dB: Decibel

DS-CTF: Double-stabilized combined tangential formulation

EFIE: Electric-field integral equation

EFVIE: Electric-field volume integral equation

FBS-CTF: Field-based-stabilized combined tangential formulation

FFT: Fast Fourier transform

FMM: Fast multipole method
 GA: Genetic algorithms
 GB: Gigabyte
 GHz: Gigahertz
 GMRES: Generalized minimal residual (method)
 HIE: Hybrid integral equation
 IBC: Impedance boundary condition
 ILU: Incomplete LU (method)
 IMLFMA: Incomplete multilevel fast multipole algorithm
 JMCFIE: Electric and magnetic current combined-field integral equation
 kB: Kilobyte
 LF-MLFMA: Low-frequency multilevel fast multipole algorithm
 LHS: Left-hand side
 LL: Linear-linear (functions)
 LoD: Level of distribution
 LP: Log-periodic (antennas)
 LSQR: Least-squares QR (method)
 m: Meter
 MB: Megabyte
 M-CTF: Modified combined tangential formulation
 MFIE: Magnetic-field integral equation
 MHz: Megahertz
 MLFMA: Multilevel fast multipole algorithm
 MNMF: Modified normal Müller formulation
 MOM: Method of moments
 MPI: Message passing interface
 MVM: Matrix-vector multiplication
 NC: No convergence
 N-EFIE: Normal electric-field integral equation

NFP: Near-Field preconditioner
NMF: Normal Müller formulation
N-MFIE: Normal magnetic-field integral equation
NP: No preconditioner (case)
OBSF: Operator-based stabilization formulation
PEC: Perfect electric conductor
PMCHWT: Poggio-Miller-Chang-Harrington-Wu-Tsai (formulation)
PO: Physical optics
PV: Principal value
QMR: Quasi-minimal residual (method)
RCFIE: Regularized CFIE
RCM: Reverse-Cuthill-McKee algorithm
RCS: Radar cross section
REFIE: Regularized EFIE
RHS: Right-hand side
RMS: Root-mean-square
RWG: Rao-Wilton-Glisson (functions)
s: Seconds
SAI: Sparse-approximate-inverse (preconditioner)
S-CNF: Stable combined normal formulation
S-CTF: Stable combined tangential formulation
SRR: Split-ring resonator
T-EFIE: Tangential electric-field integral equation
TFQMR: Transpose-free quasi-minimal residual (method)
T-MFIE: Tangential magnetic-field integral equation
TW: Thin wire
V: Volt
 μ s: Microsecond

Glossary

$\leftarrow^+$: Cumulative addition operation

$\langle \cdot \rangle$: Time-average

$\langle \cdot, \cdot \rangle$: Inner product

$\| \cdot \|_2$: 2-norm

!: Factorial operation

$\dagger$: Transpose operation

H : Hermitian operation

$*$: Complex-conjugate operation

$\perp$: Perpendicular

$\lfloor \cdot \rfloor$: Floor operation

$\lceil \cdot \rceil$: Ceil operation

∇ : Differential operator

a : Box size

$\mathbf{a}[\cdot]$: Expansion coefficients

A : Area

$\mathbf{A}_e(\cdot)$: Electric vector potential

$\mathbf{A}_m(\cdot)$: Magnetic vector potential

$b(\cdot)$, $\mathbf{b}(\cdot)$: Basis function

$\mathbf{B}(\cdot)$, $\mathbf{B}(\cdot, \cdot)$: Magnetic flux density

$\bar{\mathbf{B}}[\cdot, \cdot]$: Block matrix

C : Contour

d_0 : Digits of accuracy
 D : Homogeneous domain
 $D(\cdot, \cdot)$: Directive gain
 $\mathbf{D}(\cdot)$, $\mathbf{D}(\cdot, \cdot)$: Electric flux density
 $\mathbf{D}[\cdot, \cdot]$: Diagonal matrix
 e : Edge
 $\mathbf{E}(\cdot)$, $\mathbf{E}(\cdot, \cdot)$: Electric field intensity
 $f(\cdot)$, $f(\cdot, \cdot)$: Scalar function
 $\mathbf{f}(\cdot)$: Vector function
 $\mathbf{f}[\cdot]$: Array containing samples of a receiving pattern
 $\mathbf{F}(\cdot, \cdot)$, $\mathbf{F}[\cdot, \cdot](\cdot)$: Receiving pattern
 $\mathcal{F}\{\cdot\}$: Far-field list
 $g(\cdot, \cdot)$: Homogeneous-space Green's function
 $\mathbf{g}[\cdot]$: Array containing samples of an incoming field
 G : Gaunt coefficient
 $\mathbf{G}(\cdot, \cdot)$: Incoming field
 $h^{(1)}(\cdot)$: Spherical Hankel function of the first kind
 $h^{(2)}(\cdot)$: Spherical Hankel function of the second kind
 $\mathbf{H}(\cdot)$, $\mathbf{H}(\cdot, \cdot)$: Magnetic field intensity
 I , $\mathbf{I}$: Value of an integral, complex coefficient
 $\mathbf{I}_{DM}$: Dipole moment
 $\bar{\mathbf{I}}[\cdot, \cdot]$, $\bar{\mathbf{I}}[\cdot, \cdot, \cdot, \cdot]$: Identity matrix, discretized identity operator
 $\bar{\mathbf{I}}^{3 \times 3}$: 3×3 unit dyad
 $\mathcal{I}\{\cdot\}(\cdot)$: Identity operator
 $j(\cdot)$: Spherical Bessel function of the first kind
 J : Jacobian
 $\mathbf{J}(\cdot)$, $\mathbf{J}(\cdot, \cdot)$: Electric current density
 k : Wavenumber
 $\hat{\mathbf{k}}$: Angular direction

$\bar{\mathbf{K}}[\cdot, \cdot], \bar{\mathbf{K}}[\cdot, \cdot, \cdot, \cdot]$: Discretized $\mathbf{K}$ operator
 $\mathbf{K}\{\cdot\}(\cdot)$: Integro-differential operator
 l : Length
 $\boldsymbol{\ell}$: Edge vector
 L : Number of levels
 $\bar{\mathbf{L}}[\cdot, \cdot]$: Lower-triangular matrix
 $\mathbf{M}(\cdot)$: Magnetic current density
 $\hat{\mathbf{n}}$: Normal unit vector
 $N, N(\cdot)$: Number of basis/testing functions, triangles, unknowns, or clusters
 $\mathcal{O}(\cdot)$: Order
 p : Number of interpolation points, number of processors, number of partitions
 $P(\cdot)$: Legendre polynomial, associated Legendre function
 $P^\infty(\cdot, \cdot)$: Radiation intensity
 $\mathbf{P}(\cdot)$: Power density
 $\bar{\mathbf{P}}[\cdot, \cdot]$: Preconditioner matrix
 q : Oversampling factor
 $\bar{\mathbf{Q}}[\cdot, \cdot]$: Unitary matrix
 $\text{Re}\{\cdot\}$: Real part
 $\hat{\mathbf{r}}$: Unit vector towards the observation point
 $\mathbf{r}$: Observation point, position
 $\mathbf{r}'$: Source point
 $\mathbf{r}[\cdot]$: Residual vector
 R : Distance
 $\hat{\mathbf{R}}$: Unit vector from source point to observation point
 $\mathbf{R}$: Vector from source point to observation point
 $\bar{\mathbf{R}}[\cdot, \cdot]$: Upper-triangular matrix
 $\mathbf{s}[\cdot]$: Array containing samples of a radiation function
 S : Surface, support, number of samples, size
 $\mathbf{S}(\cdot, \cdot), \mathbf{S}[\cdot, \cdot](\cdot)$: Radiation pattern, radiated field

$\bar{\mathbf{S}}[\cdot, \cdot]$: Schur complement
 t : Processing time, time
 $T(\cdot, \cdot)$: Transformer
 $t(\cdot), \mathbf{t}(\cdot)$: Testing function
 $\hat{\mathbf{t}}$: Tangential unit vector
 $\bar{\mathbf{T}}[\cdot, \cdot], \bar{\mathbf{T}}[\cdot, \cdot, \cdot, \cdot]$: Discretized $\mathcal{T}$ operator, transformer matrix
 $\mathcal{T}(\cdot)$: Power transmission
 $\mathcal{T}\{\cdot\}(\cdot)$: Integro-differential operator
 $u(\cdot)$: Unit step function
 $\bar{\mathbf{W}}$: Diagonal weight matrix
 $v(\cdot), w(\cdot)$: Interpolation weights
 $\mathbf{v}[\cdot], \mathbf{w}[\cdot]$: RHS vectors
 V : Volume
 $\mathcal{V}\{\cdot\}(\cdot), \mathcal{Y}\{\cdot\}(\cdot), \mathcal{Z}\{\cdot\}(\cdot)$: Combined operators
 $\mathbf{V}(\cdot, \cdot)$: Vector current moment
 $\bar{\mathbf{V}}[\cdot, \cdot], \bar{\mathbf{Y}}[\cdot, \cdot], \bar{\mathbf{Z}}[\cdot, \cdot]$: Impedance matrices
 $Y(\cdot, \cdot)$: Spherical harmonics
 $\mathcal{Z}$: Impedance
 $\alpha, \alpha(\cdot)$: Combination parameter
 $\alpha(\cdot, \cdot)$: Translation function
 $\bar{\boldsymbol{\alpha}}[\cdot, \cdot]$: Translation matrix
 $\bar{\boldsymbol{\alpha}}^\diamond[\cdot, \cdot]$: Diagonal translation matrix
 $\beta(\cdot, \cdot)$: Shift function
 $\bar{\boldsymbol{\beta}}[\cdot, \cdot]$: Shift matrix
 $\bar{\boldsymbol{\beta}}^\diamond[\cdot, \cdot]$: Diagonal shift matrix
 γ : Direction of a basis function
 $\gamma(\cdot)$: Normalized spherical Hankel function of the first kind
 $\Gamma(\cdot)$: Derivative of normalized spherical Hankel function of the first kind
 $\bar{\Gamma}$: Sparse interpolation matrix

$\delta, \delta(\cdot), \delta(\cdot, \cdot)$: Kronecker or Dirac delta function
 Δ : Error
 $\bar{\Delta}[\cdot, \cdot]$: Tridiagonal matrix
 $\epsilon, \varepsilon, \epsilon(\cdot), \bar{\epsilon}(\cdot)$: Permittivity
 ε : Efficiency, tooth-width ratio (for LP antennas)
 ζ : Simplex coordinates
 η : Wave impedance
 θ : Angular direction
 Θ : Amount (time, memory)
 κ : Contrast
 λ : Wavelength
 $\bar{\Lambda}$: Lower bidiagonal matrix
 $\mu, \mu(\cdot), \bar{\mu}(\cdot)$: Permeability
 $\nu(\cdot)$: Spherical Bessel or Hankel function
 ξ : Orientation of a basis function
 π : Pi number
 $\pi_E(\cdot), \pi_M(\cdot)$: Debye potentials
 $\rho_e(\cdot), \rho_e(\cdot, \cdot)$: Electric charge density
 $\rho_m(\cdot)$: Magnetic charge density
 $\varrho(\cdot, \cdot)$: Jacobian
 τ : Geometric ratio (for LP antennas)
 $\tau, \tau(\cdot)$: Truncation number
 $v(\cdot)$: Normalized spherical Bessel function
 $\Upsilon(\cdot)$: Derivative of normalized spherical Bessel function
 ϕ : Angular direction
 $\phi_e(\cdot)$: Electric scalar potential
 $\phi_m(\cdot)$: Magnetic scalar potential
 φ : Angular direction, angular function
 $\Psi(\cdot, \cdot)$: Wave function (multipole field)

ω : Angular frequency

$\omega(\cdot)$: Spherical function involving Legendre polynomials

Ω_o, Ω_i : Solid angle

$\Omega(\cdot)$: Spherical function involving Legendre polynomials