PHILOSOPHICAL PERSPECTIVES ON THE PRINCIPLE OF LEAST ACTION

A Master's Thesis

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Ankara
May 2024

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Bilkent University
2024

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The Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

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In Partial Fulfillment of the Requirements for the Degree of
MASTER OF ARTS IN PHILOSOPHY

THE DEPARTMENT OF
PHILOSOPHY
İHSAN DOĞRAMACI BILKENT UNIVERSITY
ANKARA

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By Fathima Afra Mohamed Akram

I certify that I have read this thesis and have found that it is fully adequate, in scope and quality, as a thesis for the degree of Master of Philosophy.

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ABSTRACT

PHILOSOPHICAL PERSPECTIVES ON THE PRINCIPLE OF LEAST ACTION

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May 2024

The principle of least action (PLA) is a pivotal concept in theoretical physics, positing that when an object moves from one point to another, it minimizes a quantity called action. During its inception, the PLA was associated with theological and teleological notions as scholars sought to understand why particles followed such paths. Later, its wide applicability across different theories in physics and its utility in problem-solving led to its recognition as a significant principle.

However, over the past century, due to the PLA yielding the same empirical predictions as the equations of motion (EOMs), the prevailing view in modern physics is that the PLA is merely a mathematical reformulation of the EOMs. In recent decades, philosophers have challenged this view, arguing that while the PLA is empirically equivalent to the EOMs, they are inequivalent in other respects. Some of these philosophers assign physical significance to the quantity of action, while others consider the PLA to be a meta-law.

In this thesis, I will analyze these views. I will use the intriguing case of 'alternative Lagrangians' to argue against assigning physical significance to the action.

I will then argue that the wide applicability of the PLA is not due to its modality as a meta-law but rather to the prevalence of second-order EOMs in the fundamental laws of physics.

Keywords: Alternative Lagrangians, Constraint, Meta-laws, Principle of least action

ÖZET

EN AZ EYLEM İLKESINE İLIŞKIN FELSEFI PERSPEKTIFLER

Mohamed Akram, Fathima Afra

Yüksek Lisans, Felsefe Bölümü

Tez Danışmanı:Dr. Öğr. Üyesi Adriana Monica Solomon

Mayıs 2024

En az eylem ilkesi (EAEL), teorik fizikte çok önemli bir kavramdır. Bu kavram bir nesnenin bir noktadan diğerine hareket ettiğinde eylem adı verilen miktarı en aza indirdiğini öne sürer. Başlangıçta EAEL, nesnelerin neden bu yolları izlediğini anlamaya çalışan bilim insanları tarafından teolojik ve teleolojik kavramlarla ilişkilendirildi. Daha sonra fizikteki farklı teorilere geniş çapta uygulanabilirliği ve problem çözmedeki faydası, onun önemli bir prensip olarak tanınmasına yol açtı.

Bununla birlikte, geçtiğimiz yüzyılda, EAEL'nin hareket denklemleriyle (HD) aynı deneyimsel öndeyileri sağlaması nedeniyle, modern fizikteki hakim görüş, EAEL'nin yalnızca HD'lerinin matematiksel bir yeniden formülasyonu olduğu yönündedir. Fakat son yıllarda felsefeciler, EAEL'nin deneyimsel olarak HD'lerine eşdeğer olmasına rağmen diğer açılardan eşdeğer olmadığını ileri sürerek bu görüşe karşı çıktılar. Bu felsefecilerden bazıları eylemin miktarına fiziksel önem atfederken, diğerleri EAEL'ni bir üstyasa olarak görüyor.

Bu tezimde bu görüşleri analiz edeceğim. Eyleme fiziksel önem atfetmeye karşı çıkmak için ilgi çekici 'alternatif Lagrangianlar' vakasını kullanacağım. Daha sonra, PLA'nın geniş çapta uygulanabilirliğinin, onun bir üstyasa olarak

kipliğinden değil, daha ziyade fiziğin temel yasalarında ikinci dereceden HD'lerinin yaygınlığından kaynaklandığını tartışacağım.

Anahtar Kelimeler: Alternatif Lagrangianlar, Kısıt, Üstyasalar, En az eylem ilkesi

ACKNOWLEDGEMENTS

I would like to thank Monica for giving me hope that I could do better, for granting me the freedom to study the topics I always wanted to, and for guiding me in the most fruitful manner.

I would like to thank Simon for accepting me into the department despite my limited knowledge of philosophy, for supporting me throughout, and for accommodating me when I faced difficulties.

I would also like to thank Alireza for assisting me with various aspects of my academics over the past two years and for encouraging me to think more deeply about my work.

I would also like to express my gratitude to Jonathan and my peers in the department for always maintaining a positive and warm environment, and for their efforts to make our department a socially sensitive space.

Last but certainly not least, I would like to thank my Evrensel for making life sweeter, calmer, and far more beautiful than I could have ever imagined.

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INTRODUCTION

The Principle of Least Action (PLA) serves as a cornerstone in theoretical physics, extending across classical and quantum mechanics, general and special relativity, quantum field theory, and even string theory.[1] It not only provides a foundational framework for various branches of physics but also offers a distinct theoretical lens for their development and expression. At its core, the PLA states that the trajectory a dynamical system follows between two points is the one that minimizes a quantity referred to as action.[2] Since its inception, the PLA has sparked extensive philosophical discourse. Early scholars grappled with the rationale behind particles 'choosing' paths that minimize action during their journeys. Some proposed teleological interpretations, suggesting that particles actively seek such paths, while others attributed it to a reflection of divine wisdom and the economy of nature. Additionally, some consider the PLA to be a remnant of teleology that survived past the Scientific Revolution.[3] Furthermore, the PLA's concise formulation and problem-solving efficacy, coupled with its broad applicability across multiple physics disciplines, have stimulated debates regarding its significance and implications.

Despite the historical fervor that surrounded the PLA, the prevailing perspective in modern physics literature views it as only a mathematical reformulation of the equations of motion (EOM). Henceforth, this viewpoint will be categorized as the *equivalentist* view, as it asserts strict theoretical equivalence between the PLA and the EOMs. However, this perspective is not without its challengers. Some contemporary philosophers argue that the PLA transcends its role as a mere mathematical formulation and holds theoretical implications beyond the mathematical convenience it offers. These philosophers consider the potential theoretical inequivalence between the PLA and the EOMs, exploring differences between the two in explanatory,

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[1] See Goldstein (1980), Becker et al (2006), and Scheck (2012).

[2] The action is defined as the integral of a quantity called the Langrangian, which depends on the position and velocity of the dynamical system. Strictly speaking, when the object moves from one point to another, the action is extremised, and so the principle of *least* action is the principle of *stationary* action, but because of the ubiquity of the term, I will continue using PLA.

[3] See P.E.B Jourdain (1912) and Yourgrau and Mandelstam (1979) for a historical overview. See McDonough (2020) for teleology and the scientific revolution.

epistemic, or ontological aspects. Consequently, this alternative perspective will be categorized as the *non-equivalentist* view. It is important to note that both equivalentists and non-equivalentists agree on the empirical equivalence between the PLA and the EOMs; their divergence lies solely in terms of theoretical equivalence.[4]

The purpose of this thesis is to deepen our comprehension of the PLA and its significance within contemporary physics. The PLA is intriguing because of the mix of perspectives it elicits in various modern physics discussions. On one hand, these discussions portray the PLA as merely a mathematical reformulation of the EOMs, reiterating that there is nothing significant to the PLA except that it offers a concise way to express the EOMs (i.e. they hold an equivalentist view). Within the same discussions, however, the PLA receives a much grander treatment than the EOMs. It is heralded as an elegant principle, with the fact that all our EOMs adhere to the action formulation being considered remarkable.[5]

This mix of perspectives, combined with the limited emphasis on the derivation of physics theories from fundamental principles, leaves physics students grappling with the true nature of the PLA. If indeed it is remarkable that all our EOMs are amenable

[4] A few clarificatory remarks are in order. By *empirical equivalence*, I mean that the PLA and the EOMs make the same experimental predictions in all possible circumstances because the action itself cannot be measured, and all predictions proceed through the Euler-Lagrange equations (which are equivalent to the EOMs). Both equivalentists and non-equivalentists agree on the *empirical equivalence* between the PLA and the EOMs, so it is not a point of debate or disagreement. The *equivalentist* view holds that there is nothing significant about the PLA above and beyond its role as a mathematical reformulation of the EOMs. This view is also sometimes referred to as the deflationary view in the literature. According to this view, the EOMs and the PLA are *theoretically equivalent* as well. The notion of *theoretical equivalence* is meant to subsume all forms of equivalence (explanatory, ontological, epistemic), other than empirical. On the other hand, the *non-equivalentist* view holds that there is some theoretical inequivalence between the PLA and the EOMs. This theoretical inequivalence can take different forms based on the particular view. Some of the theoretical inequivalences I will discuss in this thesis include modal inequivalence, which posits that the PLA is a meta-law governing all physical laws, and explanatory inequivalence, which argues that the PLA should take precedence over the EOMs in explanations. Of course, the non-equivalentist, just like the equivalentist, holds that the PLA and the EOMs are empirically equivalent. It is only in the notion of theoretical equivalence described above, do the equivalentist and non-equivalentist views differ.

[5] I will offer further examples in the next section, but to offer one example of such a mix, let's consider Richard Feynmann (1964)'s discussion on the PLA in his lectures on Physics. Following a discussion of how we could consider the action of several different paths from one point to another, Feynmann says "The miracle is that the true path is the one for which that integral is least." (Ch. 19). When discussing the PLA and the EOMs, however, he reiterates that "these are different ways of expressing the same thing" echoing the equivalentist perspective of the PLA as a mere mathematical reformulation.

to an action, what exactly does this remarkability consist in?[6] Reiterating that the PLA merely is a mere mathematical tool while simultaneously marveling at the adaptability of our EOMs to the action principle makes the PLA elusive and challenging to grasp. Unraveling this mystery and advancing towards a deeper understanding of one of the most pivotal principles in physics constitutes the primary objective of this thesis.

The thesis is organized as follows: In Section 1, I will provide a mathematical treatment of the PLA and its relationship with the EOMs. Following this, I will offer a historical analysis by tracing the development of the PLA, starting with its rudimentary origins in the minimum principles of Aristotle, Hero, and Leibniz, and progressing towards the more mathematically formalized concepts introduced by Maupertuis and Lagrange. Through this historical exploration, I aim to shed light on the evolving perspectives surrounding the PLA, potentially explaining why modern physics treatments predominantly favor the equivalentist viewpoint. Additionally, I will explore how modern treatments often incorporate alternative perspectives without explicitly acknowledging the influence of earlier metaphysical views.

In Section 2, I will discuss the two modern perspectives on the PLA: the prevailing equivalentist view in modern physics and the non-equivalentist views expressed by certain modern philosophers interested in the PLA. I will argue that the current state of the non-equivalentist perspective is untenable, as it extends beyond what a thorough examination of the PLA's role in physics can justify. To precisely articulate the non-equivalent treatments found in the literature, I will introduce two criteria: (i) assigning physical significance directly to the action and (ii) proposing the PLA as a meta-law governing dynamics.

In Section 3, I will critique Criterion (i), particularly focusing on the intriguing case of 'alternative Lagrangians' and their implications for non-equivalentist views. The existence of alternative Lagrangians suggests that assigning physical significance to Lagrangians lacks a solid foundation, as experimental data cannot definitively determine their selection. To illustrate the ambiguities that arise from alternative

[6] By being 'amenable' or 'adaptable' to an action formulation, I mean that the EOMs can be derived from an action, or that they can be expressed in terms of an action formulation.

Lagrangians, I will examine the symmetry groups and the quantization of the simple harmonic oscillator system as an example.

In Section 4, I will critique Criterion (ii), which focuses on the modality of the PLA as a meta-law, governing law, or constraint on the EOMs. After providing a brief overview of the literature on laws of nature and meta-laws, I will examine Glick (2023)'s argument proposing the PLA as a constraint analogous to the conservation of energy and argue against its validity. Instead, I will propose that the universality of the PLA is rooted in the prevalence of lower-order differential equations in fundamental laws, particularly second-order equations.

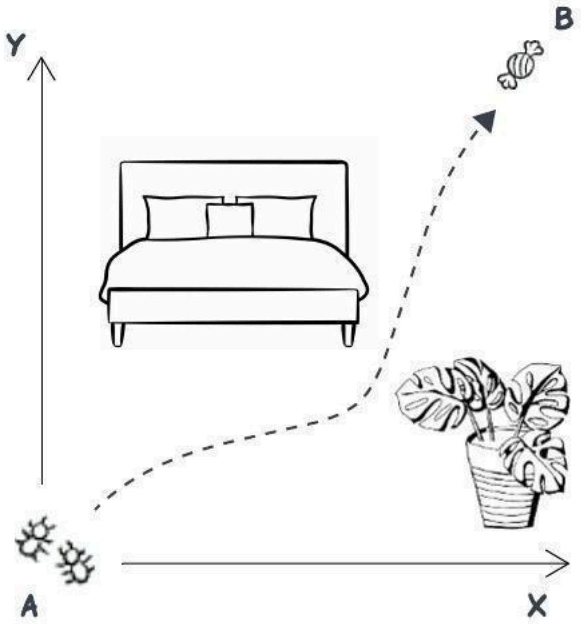
Finally, Section 5 will conclude the thesis, summarizing key findings and insights gained from the analysis conducted in the preceding sections.

CHAPTER 1

PRINCIPLE OF LEAST ACTION: OVERVIEW

1.1 The ant colony (A metaphor for variational calculus)

The particular field of mathematics that provides the technical machinery required to understand the PLA is called the calculus of variations. Before diving into the particular connections between the PLA and the EOMs, let's consider the following metaphor to understand some of the methods involved in the calculus of variations:
Two people have tossed out a plate of toffees. A colony of black ants has spotted the candy, but going straight from their nest (point A) to the candies (point B) is not possible because there are a number of obstacles like ant chalk along one of the walls, a big plant, and a bed in their way. Among the several paths the ants can tread, only one minimizes the distance between the two points while ensuring that precious energy and time are not lost bumping into one of the obstacles. So, which path will the ants choose to get to the toffees?



Point A corresponds to x = 0 and y = 0 in our setup. The various paths we can imagine between points A and B are represented by the function y(x) - for instance, the straight line connecting A and B is y(x) = x. As discussed earlier, we are aiming to optimize a combination of distance and energy. This combination is given the label q(x, y).

Our goal is to find the path y(x) in such a way that when we add up all the q(x, y) values along this path and compare that sum to the sums along other potential paths, we end up with the smallest possible value. In mathematics, summing things up continuously is represented by an integral. So, we have,

$$Q = \int_{a}^{b} q(x, y)\, dx$$

We are trying to find the best path for the ants, the one that has the smallest Q value. Since there are countless possible paths, it's not feasible to try them all one by one and compare Q values. Instead, using the calculus of variations, we seek a path by setting variations in Q, δQ, to zero.[7]

When we set $\delta Q = 0$, we arrive at the Euler-Lagrange equations.

$$\frac{d}{dx}\left(\frac{\partial q}{\partial y'}\right) - \frac{\partial q}{\partial y} = 0$$

Solving these equations provides us with the optimal path, $y(x)$.

1.2 The mathematics of the PLA

Consider a generic action in physics,

$$S = \int_{t_a}^{t_b} L(q, q', t)\, dt$$

where S is the action, and L is the Lagrangian, which quantifies a system's dynamics based on its position, q, and velocity, q' . Let us now consider a simple problem in physics, that of a ball rolling down a hill. The Lagrangian L, for classical mechanics, is taken to represent the difference between kinetic and potential energy, $L = KE - PE$.[8] The path the ball takes is the one for which the action is extremized, which can be found by setting variations in the action, δS, to zero. Setting

[7] Similar to finding stationary points on a curve (where the slope is zero, f'(x) = 0), the condition δQ=0 directs us to paths where Q remains stationary. Just as a stationary point in a function can be a minimum or a maximum, a stationary path can also be a maximum or a minimum. Second order derivatives (or variations) would allow us to differentiate between the minima and the maxima.

[8] It is important to note that the Lagrangian is not strictly unique. The same EOMs will be obtained even if we add a total derivative of time to the Lagrangian. This will be discussed further in the next section.

variations in the action to zero yields the Euler-Lagrange equations, which, in physics, are the EOMs for dynamic systems.[9]

The Principle of Least action states: "Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the Lagrangian". (Goldstein 1980).

When variations in the action are set to zero, we obtain the EOMs. The EOMs describe how the position of an object changes over time. For a ball rolling down a hill, Newton's second law provides the EOMs, represented as second-order differential equations:

$$F = ma = m\frac{d^2x}{dt^2}$$

where F is the force, m is the mass, and a is the acceleration. These equations detail how the ball's position and velocity evolve from one moment in time to the next.[10]

Therefore, while the EOMs offer a local, differential depiction of a system's evolution, the PLA offers a global perspective by identifying the path that minimizes the action. Since the action itself (and the Lagrangian) cannot be empirically measured, both the EOMs and the Euler-Lagrange equations derived from the PLA predict the same trajectories for dynamical systems, i.e. the two formulations are empirically equivalent. Despite this empirical equivalence, the PLA has an intriguing historical trajectory that influences its interpretation today. This will be discussed in the next section.

[9] The Euler-Lagrange equations are considered strictly equivalent to EOMs under a few conditions. See Brechtken-Manderscheid (1991), for example, for further details. These subtleties won't impact the discussions here, so I'll use the terms interchangeably.

[10] As mentioned earlier, the EOMs can be derived by setting variations in the action to zero. Conversely, we can perform the inverse procedure, going from the Euler-Lagrange equations (i.e., the EOMs) to the action. This process is known as the inverse problem in the calculus of variations and will be discussed in detail in the upcoming section.

1.3 The history of the PLA: From Aristotle to Lagrange

Throughout history, people have sought a simple set of laws to explain natural phenomena. Various thinkers proposed different fundamental entities as the origins of the universe. For instance, Anaximander, a pre-Socratic Greek philosopher, suggested the concept of the 'Unbounded' or 'apeiron' as the indefinite entity from which everything originates. He believed the universe arose from this boundless substance, distinct from known elements like water. Conversely, Pythagoras, another influential thinker from the same era, theorized that numbers were the foundational elements of the cosmos. He and his followers saw the universe as composed of numerical units, emphasizing the harmony of numerical relationships, an idea famously encapsulated in the Pythagorean theorem.[11] Although steeped in mysticism and theology, these early ideas marked some of the initial attempts to offer a unified explanation of nature. The notion of providing a unified explanation for a broad spectrum of phenomena, or subsuming various phenomena under a concise explanatory umbrella, has persisted throughout history and remains central to our research motives today.

In addition to their belief in unitary fundamental explanations for the range of phenomena in the world, ancient philosophers also believed in inherent aesthetic qualities in the natural world. Aristotle's 'On the Heavens' presents an early example of a 'minimum hypothesis.'[12] In Book II, Aristotle offers a series of arguments for why the shape of the heaven must be spherical. He states:

> ... if the spatial movement of the heavens is the measure of movements because only it is continuous, regular, and eternal, and what is least in each thing is its measure, and if the fastest movement is the least, it is clear that the movement of the heaven would be the fastest of all movements. Moreover, the line of the circle is the least line from a point to itself, and the fastest movement is that along the shortest line. So if the heaven spatially moves in a circle and moves fastest, it is necessary for it to be spherical (p. 40).

[11] See Kirk et al (1983) and Barnes (2001) for further details on Anaximander and Pythagoras.
[12] I am using the translation of the De Caelo by C.D.C Reeves (2020)

Here Aristotle uses the fact that the circle is the shape with the smallest perimeter enclosing a given area to argue that the heavens exhibit circular motion.

Similarly, in 40 AD, the Hero of Alexandria observed that light follows the shortest path between two points, usually a straight line. This led to his ‘principle of least distance,’ confirming that the angle of incidence equals the angle of reflection on a plane mirror. However, this principle fell short when it came to explaining the refraction of light accurately, since light bends when it moves from one medium to another, so it does not always follow the path of least distance. Physicists later refined Hero’s idea, proposing that light takes the path requiring the least time, and not the least distance as Hero had initially suggested. This ‘principle of least time’, attributed to Fermat, addressed the limitations of Hero’s principle, providing a comprehensive explanation for optics (Huygens 1690, Ch.3).

A common theme among the minimum principles discussed so far is that they are directed at offering an explanation for why a physical situation is a certain way. Whereas Hero and Fermat’s principles generate mathematical results from their minimum principles, this is not always the case. Take, for example, Leibniz’s intriguing minimum principle in his philosophy of optimism. Leibniz suggested that out of all possible worlds that could exist, God chose to create the one that contains the maximum amount of good while still allowing for some unavoidable evil.[13] This suggests that the universe, as it exists, is the best possible universe that could have been created. Thus, while Hero and Fermat’s principles focus on explaining physical phenomena like light, Leibniz’s principle extends the concept to considering the optimality of the universe as a whole.

Minimum principles remained a focal point in scientific exploration. Just as Fermat sought the ‘principle of least time’ to understand light’s path, a similar endeavor arose in mechanics, aiming to pinpoint a property that could be minimized to map out particle trajectories. Maupertuis (1750) led this charge, pioneering the formulation of a minimum principle in mechanics. Driven by teleological leanings, he often sought divine rationale for the structure of physical theories. For instance, he pondered why

[13] See E.M Huggard’s (2010) translation of G.W. Leibniz’s Theodicy

the Creator might prefer the law of the inverse square over other laws of attraction. Maupertuis, deeply rooted in faith, saw God's hand in physical theories. In his formulation of the minimum principle, he termed the variable to be minimized as 'the quantity of action,' a combination of a particle's mass and velocity. He viewed the minimization of this quantity as a reflection of God's wisdom.[14] However, Maupertuis's conceptualization of action lacked precision, as he tailored its definition to specific contexts for accurate results.

The pursuit of the correct quantity of action culminated when Lagrange (1788, §3.6) established a more precise mathematical foundation for the principle of least action and identified the correct dynamical quantity that is to be minimized. In his seminal work "Analytical Mechanics" (published in 1788), Lagrange introduced the Lagrangian function, which encapsulates the dynamics of a system. By formulating the Lagrangian and deriving the Euler-Lagrange equations, Lagrange provided a powerful mathematical tool for solving problems in mechanics and understanding the behavior of physical systems. Lagrange laid the groundwork for the development of classical mechanics and had a profound influence on subsequent generations of physicists and mathematicians. His insights into the PLA revolutionized the way physicists approached problems in dynamics and led to significant advances in the understanding of fundamental principles governing the behavior of physical systems. As Jacobi expressed it, the PLA became, in the hands of Lagrange, "the mother of our analytical mechanics" (Jacobi 1866).

It is worth noting that prior to the investigations by Maupertuis and subsequent physicists into minimum principles in mechanics, Newton's (1687) laws had already provided a thorough description of the subject. This raises a pivotal question: Why pursue a minimum principle in mechanics when Newton's laws already furnish a comprehensive framework? Perhaps the very fact that mechanics is amenable to reformulation through a minimum principle enhances our understanding of the

[14]A mention must be made of the difference between minimum principles such as those of Fermat, Maupertuis, and Lagrange, and Ockham's razor. Ockham's razor states that when there are competing explanations for a given phenomenon, the simplest one should be preferred. The main point of distinction is that the minimum principles we have been discussing above consider simplicity or minimization in the physical universe itself, whereas Ockham's razor concerns itself merely with different explanations of physical situations and does not purport to draw any conclusions regarding the ontology of the situation being considered.

universe or provides another perspective on physics. Or, perhaps, the minimum principle simplifies problem-solving in physics, and its strength lies therein. However plausible these reasons may be, historically, it appears that the motivation to explore minimum principles stemmed from the assumption that nature *seeks* to minimize certain quantities. The PLA was considered a reflection of the thriftiness of nature or the wisdom of God. These metaphysical commitments—seeking unifying explanations, formulating overarching principles that subsume phenomena, and assuming that the laws of physics reflect the ontology of the physical universe—influenced even the most rudimentary attempts to explain the physical world quantitatively, shaping the search for the laws of nature.

CHAPTER 2

EQUIVALENTISTS VS NON-EQUIVALENTISTS

Despite the successful formulation of mechanics in terms of minimum principles, by the late 18th century, the earlier metaphysical connotations associated with the principle itself, whether as a description of the swiftness of nature or the wisdom of God, had diminished. Veldman (2024) argues that much of the metaphysics linked to the PLA was refuted because it could not be adequately modeled in terms of mathematics. That is, the mathematization of the PLA led to several dissociations of metaphysical ideas from it. As mentioned earlier, the PLA states that all physical processes aim to minimize the 'action of nature,' striving for efficiency, economy, or simplicity.

According to Veldman, two crucial factors stemming from the mathematization process contributed to the abandonment of metaphysical interpretations surrounding the PLA. Firstly, most metaphysical notions revolve around the idea that nature selects the path minimizing the action from an infinite number of available paths for the particle. However, this notion is not entirely accurate. The only true statement is that trajectories result from setting the variations in action to zero. Strictly speaking, the trajectory could be a minimum, a maximum, or simply a saddle point, rendering a metaphysical interpretation centered on minimization incorrect. Secondly, metaphysical ideas suggest that action is a fundamental property of nature, and its minimization represents the economy of nature, as the quantity of action is optimized in a particle's motion. This optimization supposedly reflects nature's thriftiness. However, the action is insensitive to the sign of the integral, meaning both negative and positive versions of an action quantity would yield the same trajectory. Consequently, the action does not readily lend itself to a realist interpretation.

This implies that assertions such as "the minimization reflects nature's thriftiness or economy, or God's wisdom" lack substantial support.[15] Lagrange appears to be the only early physicist who recognized this, as he remarked after discussing that the action could be a minimum or a maximum, "This is the principle to which I improperly gave the name of Least Action and which I view not as a metaphysical principle but as a simple and general result of the laws of mechanics" (p.183).[16]

Although the mathematization led to a lot of the previous theological and teleological views on the PLA being abandoned, appreciation for the PLA transformed on different grounds in the late 19th century. Before Lagrange's time, only optics and mechanics had found themselves an interpretation in terms of the PLA. However, the surge in the development of physical theories during the late 19th century saw an increased activity regarding the PLA too. During this period, several physicists succeeded in representing various theories in terms of the PLA, including thermodynamics and electromagnetic theory, leading to the view that the PLA has a universality and generality that renders it superior to each of the individual theories themselves.

Furthermore, because of the elegance and simplicity of representation that the PLA affords, previously unsolvable or difficult to solve problems could now be solved, and it was considered analogously to symmetry principles that also make the mathematics of the problem more apparent. A crucial aspect of this elegance and convenience stems from the fact that the action formulation deals with coordinate-independent quantities. These two factors together, i.e the generality of the PLA and its usefulness

[15] It is important to highlight that while metaphysical perspectives like those of Maupertuis may appear outdated to modern readers, the idea that action represents some aspect of the universe is quite intuitive. Consider our ant metaphor: the quantity to be minimized is a combination of distance and energy, which naturally lends itself to a simple interpretation, i.e., a combination of energy and distance. However, it is only through a thorough mathematical understanding of the connections between experimental data, EOMs, and actions that we realize the action is not mathematically unique and is not sensitive to sign. Consequently, a realist interpretation of the action is difficult to defend. This concept will be further explored in the next section. It is also worth noting that the language often used in variational calculus implies that the actual path is special in some sense. For example, in our ant example, the path for the ants that minimizes the combination of distance and energy is called the 'optimal' path. This invokes connotations of economy and thriftiness, etc. So it must be realized that although, in a modern sense, the teleological or theological notions held by earlier thinkers may seem far off, they are not entirely far off.

[16] It supports Veldman's argument that the mathematization of the PLA removed early metaphysical notions. Lagrange, arguably one of the few early physicists who refrained from offering grandiose interpretations of the PLA, was indeed the first to thoroughly mathematize it.

in problem-solving, led to a transformation on how the PLA was received. In the 1860s, Lord Kelvin and Tait showed that the PLA offered an excellent method for solving problems in hydro-dynamics and elasticity. Furthermore, Helmholtz later explored whether the kinetic potentials in different theories could be used to write Lagrangians for them. He showed, for the first time, that the PLA could indeed be applied to the three great branches of physics - mechanics, electrodynamics and thermodynamics.[17]

Helmholtz (1886) regarded the PLA as a leitmotif, praising it for its heuristic role in the development of new theories. He states:

> In every case, it seems to me that the general validity of the principle of least action has been ensured to the extent that it can occupy a high position as a heuristic principle and a guide for any attempts to formulate the laws of new classes of phenomena (p. 210).

Max Planck (1915) admired the PLA's comprehensive nature, noting:

> As long as physical science exists, the highest goal to which it aspires is the solution of the problem of embracing all natural phenomena, observed and still to be observed, in one simple principle which will allow all past and, especially, future occurrences to be calculated. Among the more or less general laws, the discovery of which characterize the development of physical science during the
> last century, the principle of Least Action is at present certainly one which, by its form and comprehensiveness, may be said to have approached most closely to the ideal aim of theoretical inquiry. (p. 69).

[17] It's important to note that these physicists were not exclusively working with Lagrange's Principle of Least Action but rather with transformed versions derived later by Hamilton and Jacobi. In all the works considered for this thesis, the Principle of Least Action encompasses these transformed versions as well.

Corry (1997) also discusses the central role the PLA played in Hilbert's axiomatic method in physics. Hilbert offered a derivation of Einstein's field equations in general relativity that took the PLA as one of its foundations[18]:

> Furthermore, the universal validity of variational principles as the key to deriving the main equations of physics was a central underlying assumption of all of Hilbert's work on physics, and that kind of reasoning appears throughout [Hilbert's 1905] lectures as well (p. 185)

However, alongside the perspective that the PLA is significant in and of itself, a deflationary view also emerged. Drawing on Mach's philosophy, the logical empiricists regarded the PLA merely as a mathematical redescription of the EOMs.[19] This view arose from the fact that both the PLA and the EOMs render empirically equivalent results. This *equivalentist* stance, later notably advocated by Yourgrau and Mandelstam (1979), argues that since the precise form of the action for a specific theory cannot be predefined and must instead be derived from experimental evidence and theoretical considerations, attributing significance to the principle itself — suggesting that it allows a deeper understanding of nature — lacks a solid foundation. From the equivalentist standpoint, the PLA is seen as lacking intrinsic significance, chosen for brevity rather than inherent superiority.

Most contemporary physicists are equivalentists and expect to articulate the laws of motion both differentially, through the EOMs, and integrally, through the action. It is to be noted, however, that although the prevailing view in physics is indeed equivalentist, several treatments of the PLA seem to retain elements from the

[18]When quoting these historical figures, my aim is not to categorize their perspectives into the same philosophical framework. Each of these figures held distinct philosophical views. Planck embraced a convergent realist perspective, where the increased unification facilitated by the PLA justified stronger ontological conclusions. In contrast, Hilbert rejected ontological reductionism. To him, the PLA's role in unification underscored the significance of mathematics in physics (Stöltzner 2003). However, the common thread among these thinkers was their agreement that the PLA was something separate from the EOMs in some sense. This common ground forms the basis for my analysis.

[19] Mach's 'economy of thought' is a concept that emphasizes the preference for simple, concise explanations and theories that are parsimonious and avoid unnecessary complexity. Since the action is not directly observable, the PLA would be considered superfluous within this framework. Note that the concept of "economy of thought" shares similarities with Ockham's razor in that both emphasize simplicity in our descriptions of nature. However, neither makes claims about the simplicity of nature itself. See Stöltzner (2003) for details on the logical empiricists' influence on the interpretation of the PLA.

historical metaphysical views discussed above. Note, for example, Anthony Zee's (1999) discussion about the PLA:

> The entire physical world is described by one single action. When physicists dream of writing down the entire theory of the physical universe on a cocktail napkin, they mean to write down the action of the universe. It would take a lot more room to write down all the equations of motion (p. 109).

Here we see an amalgam of both perspectives. On the one hand, the action is thought to constitute a singular description of the universe. On the other hand, there is the equivalentist implication that we use the action formulation only because it is more concise than the EOMs. Consider also, Taylor's (2003) vivid remark, "At the stone moving with non relativistic speed in a region of small space-time curvature, nature shouts: *Follow the path of least action*!" (p. 423) – a teleological remark that implies nature has a need or a tendency to minimize the quantity of action. Taylor's discussion, otherwise, ascribes to an equivalentist treatment of the PLA.[20] Perhaps these treatments are meant to induce awe in physics students, but they make the PLA appear rather elusive and mysterious. The analysis in this thesis is hoped to unshroud some of this mystery.

Returning again to the main stem of our discussion, despite the general prevalence of the equivalentist view in physics literature, in recent decades, philosophical discourse has explored the potential theoretical inequivalence between the PLA and the EOMs, i.e inequivalences in explanatory, epistemic or ontological respects.[21] I will now provide a summary of these non-equivalentist views, aiming to distill common themes and perspectives on the PLA. Towards the end, I will put forward two criteria to categorize these views, laying the groundwork for a more thorough analysis in subsequent sections.

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[20] Both Zee and Taylor do not acknowledge the non-uniqueness of the quantity of action and how the mathematical quantity itself is not of significance. I will discuss this in further detail in the next section. For further examples of discussions like that of Zee and Taylor, see the sections on the PLA in Marion & Thornton (1995, Ch.7), and Feyman's (1964, Ch.19) Vol.II lectures.

[21] As I clarified earlier, by theoretical equivalence I mean any form of equivalence except for empirical equivalence.

An influential philosopher to challenge the equivalentist view was Stöltzner (1994) who states "For [Yourgrau and Madelstam], variational principles are a mere reformulation of the equations of motion, which is physically equivalent to them" (p. 33). In contrast, Stöltzner emphasizes that "a well-defined action principle is *stronger* than the Euler-Lagrange equations derived from it" (p. 46). To show this, he provides examples to support the case that distinct actions have different physical meanings, highlighting the possibility of ascribing physical meaning to the action itself.

The discourse on the PLA was deepened by debates on its compatibility with dispositional essentialism, a view that considers the dispositions of objects as the fundamental natural properties that determine their dynamics.[22] Katzav (2004) argued that dispositional essentialism is incompatible with the PLA. According to him, the PLA ensures that the quantity of action is extremized in all dynamics. This implies that the motion of a physical system is not solely determined by the intrinsic properties and dispositions of objects. This contradicts the idea that dispositions alone determine the EOMs, leading to the conclusion that the PLA and dispositional essentialism are incompatible.

Katzav, in arguing for his main thesis, proposed that the PLA should have *explanatory* precedence over the EOMs, as deductions from the PLA have a unifying force and because it is non-accidental that all dynamics obey the PLA (p.212).[23] He also asserts, albeit incorrectly, that the fact physicists typically use the PLA to deduce the corresponding EOMs, not vice versa, illustrates that the explanation proceeds from the PLA to the EOMs (p. 213).[24]

Ellis (2005) challenged Katzav's argument, characterizing it as 'naive' because Katzav assumed that dispositions are the only real explanation for physical change. Ellis proposes a more sophisticated view based on complex hierarchies of natural kinds that can accommodate the PLA. In this framework, laws of nature describe essential properties, and Ellis suggests that the property 'Lagrangianism' is a "truly

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[22] For the purposes of this paper, I will not engage with the broader debate on dispositional essentialism but only extract views on the PLA.
[23] It is non-accidental because the PLA ensures that all dynamics have extremized quantities of action.
[24] Katzav's claim is incorrect because one can just as easily go from the EOMs to the PLA. See Thebault and Smart (2015) for a critical analysis of Katzav's arguments.

universal property — one possessed by every object in the universe". Furthermore, "every continuing object must be Lagrangian, disposed to evolve in accordance with the principle of least action" (p. 91).

Thebault and Smart (2013) agree with Katzav in acknowledging that the adaptability of all our EOMs to an action formulation is not a mere accident (p. 17). Terekhovich (2017) argues that the PLA possesses greater physical necessity than the EOMs, proposing a two-level modality in which the EOMs are contingent on the PLA. Lastly, Glick (2023) also emphasizes that the EOM's adaptability to an action formulation is non-accidental and argues that the PLA serves as a constraint on possible dynamical laws, akin to the laws of conservation of energy (p. 12).

To navigate this rich tapestry of views, I will group them broadly into two criteria.

1) **Physical significance** - Distinct actions have different physical meanings. The PLA has a physical significance of its own, beyond its role in deriving the EOMs.
2) **Necessary amenability** - Physically meaningful EOMs should inherently allow for an action formulation, as the extremization of action in the dynamics of objects is not arbitrary but essential. The PLA operates as a governing law or meta-law, ensuring that action quantities are extremized across all dynamics.

These criteria group the non-equivalentist perspectives discussed above based on the primary claims they make in maintaining their non-equivalentist stance. 'Physical significance' primarily aligns with Stöltzner's views, while 'Necessary amenability' encompasses perspectives from Katzav, Thebault and Smart, Ellis, and Glick. In the upcoming sections, I will elaborate and analyze the non-equivalentist views based on these criteria. While the primary focus will be on critiquing non-equivalentist perspectives, the analysis will simultaneously contribute to a broader understanding of the PLA and address common questions that arise among physics students regarding its application and significance.

CHAPTER 3

CRITERION 1: PHYSICAL SIGNIFICANCE

In this section, I examine the non-equivalentist claim that the action can be afforded physical significance in and of itself. I argue that such significance cannot be ascribed to the action. To support this argument, I will discuss ambiguities arising in the inverse problem within the calculus of variations — the inverse problem concerns constructing an action when provided with the EOM.

There is a common assumption that a unique Lagrangian exists for a given EOM.[25] In our previous example of the ball, the Lagrangian, $L = KE - PE$, corresponds to the Newtonian EOM. Several non-equivalentist treatments presume that a particular dynamical system is associated with a singular, distinctive Lagrangian. Katzav (2004) reinforces this viewpoint:

> … deductions [from the PLA] appeal to a *single scalar quantity* to deduce how the object within a physical system will evolve, no matter how complex the system and its evolution (p. 212).

Similarly, Thebault and Smart (2015) state:

> For each physical system, one can *uniquely* specify a function called the 'Lagrangian'. This function is determined by the nature of the physical system in question; that is, the properties and forces inherent in it. (p. 388).

These views imply that given a particular system, a Lagrangian can be uniquely specified for it, allowing the non-equivalentist to ascribe ontological significance to this unique entity.[26] However, a detailed consideration of how we progress from

[25] Recall Zee's statement : "The entire physical world is described by one single action."

[26] By ontological significance, I mean that the quantity of action itself is considered significant. For example, the historical views that thought the quantity of action represented the economy of nature consider the action to be ontologically significant because they hold that there is such a thing as the 'action of the universe'. I am drawn to think that Zee also holds that there is an 'action of the universe'

experimental data gathered from a particular dynamical system to the lagrangians shows that the Lagrangian is not necessarily unique.

So, how do we translate direct observations into mathematical structures like EOMs and actions? Initially, one might imagine a straightforward link between observations and equations, where observations uniquely dictate the mathematical structures, which are then subject to further empirical scrutiny. However, the connection between observations, EOMs and actions involves several ambiguities and substantial overdetermination.

We observe various trajectories, such as an arrow in flight, a falling apple, or traces in a cloud chamber. Measurements of specific observables from these trajectories, such as position, lack inherent structure. We must make topological and differential assumptions to construct a differentiable manifold Q , representing the configuration space of all possible values of the observables., representing the configuration space of all possible values of the observables. Transforming this information into a set of differential equations involves considerations like isotropy, symmetry, and a preference for second-order equations. This requires a lot of theoretical judgment, and there is no deductive procedure for constructing the dynamical theory which encompasses the experimental data.[27] From the differential equations, we can construct the Lagrangian - this step is our focal point of interest because it involves the mathematical progression from EOMs to actions.

The question of finding a Lagrangian when given an arbitrary EOM falls within the inverse problem of the calculus of variations.[28] If we can find a solution to the inverse problem, i.e. if we can find a Lagrangian for a given EOM, we always obtain a set of Lagrangians. This is expected since we are deriving a global variational principle (the action) from local stationary paths. However, these Lagrangians are ‘gauge-equivalent,’ meaning they share the same global properties and are related by

although what exactly this would mean seems insubstantial to the rest of his discussion. Later in this section, I will discuss Stoltzner’s view on the significance of the action, and this will make the idea of ‘ontological significance’ clearer.

[27] See Marmo et al. (1985, Ch.6), Marmo & Saletan (1977) and Cariñena (2015) for details about further ambiguities and examples.

[28] The wider mathematical problem considers whether an arbitrary differential equation can be formulated in terms of a variational principle - hence the term ‘calculus of variations’.

a time derivative.[29] In these situations, we have a single class of gauge-equivalent Lagrangians associated with a specific differential equation. For simplicity, we'll refer to these cases as having a 'unique' Lagrangian.

In several cases, however, multiple *non-trivially* related Lagrangians exist, leading to identical solutions for the resulting Euler-Lagrange equations and, therefore, describing the same dynamical system. These are termed 'alternative Lagrangians' and hold profound implications for the role of the Lagrangian and, therefore, the PLA in physics (Rosen 1969).[30] Let's consider an example to illustrate their significance.

Example: Symmetry group of the Simple Harmonic Oscillator (SHO)
The one-dimensional SHO is defined by the Newtonian equation,

$$q'' + q = 0$$

and the standard Lagrangian,

$$L = KE - PE = (q')^2 - q^2$$

We now consider the symmetry group of the SHO system. Using Lie theory to study the invariance of the EOM, Anderson and Davison (1974) have shown that the complete symmetry group of the SHO is the *eight-parameter* Lie group $SL(3, R)$. One would expect that studying the invariance of the standard Lagrangian would also yield the expected eight-parameter symmetry group. However, Lutzky (1978) shows that it produces only a *five-parameter* subgroup of $SL(3, R)$. Why? The standard Lagrangian is not unique; there are alternative Lagrangians that describe the same SHO system. Lutzky identifies three alternatives, each associated with a one-parameter symmetry group. One such alternative is the following:

$$L_1 = 3q(q')^2 cos(t) - 3q^3 cos(t) - (q')^3 sin(t) - 3q'q^2 sin(t)$$

Only when the standard Lagrangian is considered *together* with the alternative Lagrangians can we recover the complete eight-parameter symmetry group. This example highlights that the symmetries of the standard Lagrangian may not furnish

[29] Consider integrating a function; an arbitrary constant always arises, but these functions are related to each other in a trivial manner. Similarly, gauge-equivalent Lagrangians are related to each other by a time derivative, so they are only trivially different from each other.
[30] I must emphasize that 'alternative Lagrangians' are not limited to a small class of problems. See Leubner (1981), Marmo & Saletan (1977a), Ghosh et al. (2004), Henneaux (1982) and Santilli (1983) for detailed discussions.

the full symmetry group of a physical system. Exploring alternative Lagrangians becomes crucial to capturing the complete symmetry group. Furthermore, we must note that the *entire* symmetry group could be recovered from the EOM, whereas only a subgroup could be recovered from the standard Lagrangian. This poses a problem for the non-equivalentist because a single Lagrangian does not offer a comprehensive description of the system. Moreover, alternative Lagrangians, differing significantly from the standard one, can describe the same system. While we usually interpret the standard Lagrangian as the difference between kinetic and potential energies, seeking to interpret the alternative Lagrangian, L_1, in a physical sense seems rather ad hoc. This challenges the idea of ascribing physical significance to the Lagrangian because if we interpret one of them physically, then we must interpret the rest of them physically too because there is no clear way of privileging one over the other. This cautions against interpreting the action in a physical manner or as an inherent property of the universe.

We have established that multiple Lagrangians can describe the same classical system. Stöltzner (1994) was one of the first to discuss the importance of alternative Lagrangians in philosophical literature. Despite acknowledging their existence, his views on the PLA are still non-equivalentist. He states that "a well-defined action principle is *stronger* than the Euler-Lagrange equations derived from it." (p. 46) He illustrates that while alternative Lagrangians may yield identical EOMs, their quantization results in distinct, yet classically indistinguishable, quantum systems. He adds:

> Hence different Lagrangians, and in consequence different actions, really have a different *physical meaning*, even though their classical limit leads to the same equations of motion. (p. 51).

According to him, "[the alternative Lagrangians] give different quantum theories" (p. 50), challenging the notion that the action formulation provides no additional information beyond the EOMs. If Stöltzner's views prove accurate, it undermines the concerns I raised earlier about attributing physical significance to the Lagrangian because, in his framework, each alternative Lagrangian holds a unique physical meaning as they yield different quantum theories.

To evaluate Stöltzner's claims, it is essential first to discuss quantization, the process which facilitates the transition from a classical to a quantum system. In classical physics, we often think of properties like energy as continuous, meaning they can have any value within a given range. However, in the quantum world, physical quantities, such as energy and position, exist only in discrete levels.[31] Quantization is performed through canonical quantization or Feynman's path integral formulation, both grounded in the Lagrangian.[32] Stöltzner asserts that, although alternative Lagrangians may be indistinguishable classically, they lead to genuinely different quantum systems once quantized.

However, Stöltzner overlooks a key point: although alternative Lagrangians can indeed be quantized, the resulting quantum systems exhibit incompatible energy spectra and uncertainty relations. Experimental evidence supports only one of these quantized Lagrangians (Edwards 1979, Senitzky 1960). For example, if we were to quantise the alternative Lagrangians describing the harmonic oscillator system, we would indeed get a different quantum theory from each Lagrangian. However, only one of the quantum systems is 'correct' when subject to experimental scrutiny. This means the action formulation does not inherently provide *more* information than the EOMs. It only means that the alternative Lagrangians can be plugged into the quantization procedure. Just because we can plug Lagrangians into the quantization procedure, it does not mean that we are actually obtaining *bona fide* quantum systems from them.

Various methods have been proposed to determine the 'correct' Lagrangian. Some insist that the Hamiltonian must be equal to the energy for proper quantization, although the authors admit there is no clear justification for this choice except that it yields correct empirical results in most cases (Kerner 1958). This condition also

[31] Bound states have discrete energy levels but the free particle has a continuum of energies and momenta even in quantum mechanics.

[32] The Lagrangian is a technical requirement, not a fundamental one. Alternative quantization methods, such as Yang-Feldman and stochastic quantization, proceed directly through the EOMs.

proves inadequate for dissipative systems (Lemos 1981). Others suggest introducing a small perturbation, gradually removing it, and observing the Lagrangian to which the system converges (Morandi et al. 1990). These ambiguities highlight that alternative Lagrangians do not generate genuine quantum systems. Only one of these quantized systems is deemed the 'correct' one, and there is no clear-cut way of establishing which one.[33] This challenges Stöltzner's view that alternative Lagrangians have different physical meanings and strengthens the initial position I endorse that Lagrangians should not be ascribed physical significance.

A non-equivalentist might raise the concern that we do indeed recover one Lagrangian as the 'correct' one upon quantisation. The non-equivalentist would argue that this Lagrangian represents reality and has physical meaning, while the others are merely mathematical degeneracies. To understand why this argument is insufficient, let's now examine the two cases I've discussed—the harmonic oscillator system and quantization—together. When identifying the symmetries of the harmonic oscillator, the alternative Lagrangians were necessary for recovering the complete symmetry group. However, when quantising a system, only one of the Lagrangians achieved empirically correct results. In some cases then, like when ascertaining symmetries, we *need* alternative Lagrangians. In other cases, like when quantising a system, if we have alternative Lagrangians, we need to 'choose' one over the others. This significantly complicates the task for a non-equivalentist who wants to advocate for the physical significance of the Lagrangians. Unless we establish a clear rationale beforehand for favoring one Lagrangian over the others, which works in all cases, we cannot argue that the privileged Lagrangian is indeed 'physical'. Even if such a rationale could be established, the privileged Lagrangian may not furnish a complete description of the system, as shown in our symmetries example.

The discussion in this section cautions against interpreting the Lagrangian as a unique quantity encompassing all information about a system or as a physically meaningful entity. It also weakens some of the arguments for elevating the PLA over the EOMs.

[33] Determining the correct Lagrangian solely from the mathematical structure is not straightforward. For example, one could suggest that the epistemic value of simplicity would be sufficient to choose the 'correct' Lagrangian. This does not always hold, as shown in Redmount et al. (1999, Ex.1), and associating the correct Lagrangian with the simplest EOM doesn't always work either. There are cases where EOMs with identical solutions differ only by a constant, while the alternative Lagrangians differ significantly (DJ Saunders 2010, p. 51).

CHAPTER 4

CRITERION 2: NECESSARY AMENABILITY

In this section, I critically examine the non-equivalentist assertion that the PLA functions as a meta-law, governing law, or constraint on admissible EOMs. This entails considering the modality of the PLA in relation to the EOMs and analyzing the non-equivalentist claims of its modal distinction from the EOMs. I will begin by presenting these non-equivalentist views. Following that, I will offer a brief overview of the literature on the laws of nature, meta-laws, and constraints, with a particular emphasis on Marc Lange's accounts on these matters. Subsequently, I will analyze Glick (2023)'s non-equivalentist account in detail. This account situates the PLA within Lange's broader framework on constraints in physics. I will critique Glick's arguments by examining the relationship between the amenability of the EOMs to an action formulation and our preference for second-order EOMs.

4.1 Non-equivalentists on the modality of the PLA

In the summary of non-equivalentist perspectives presented in Chapter 2, it was noted that proponents of these positions often assert that physically meaningful EOMs naturally lend themselves to an action formulation. This viewpoint manifests in various forms. For example, Ellis (2005) proposes that 'Lagrangianism' is a universal property of all objects, suggesting that every persisting object evolves in accordance with the PLA. Ellis's argument serves as a response to Katzav's (2004) thesis that the PLA is incompatible with dispositional essentialism. Katzav argued that the PLA ensures that the action is extremized in all dynamics, suggesting that the motion of objects cannot be solely grounded in their dispositional properties. This perceived contradiction leads Katzav to conclude that the PLA and dispositional essentialism are incompatible with each other.

Ellis critiques Katzav's dispositional essentialism as naive and presents a hierarchical, more sophisticated view of essentialism that, according to Ellis, better accommodates the PLA. Ellis outlines three hierarchies of natural kinds: objects or substances, events or processes, and properties or relationships. He asserts that understanding the ontological dependencies within this hierarchy is essential for explaining the necessity of natural laws and the widespread applicability of global principles such as the PLA. Ellis states:

> Lagrange's principle of least action applies to all physical systems, and I would suppose it to be of the essence of the global kind in the category of objects or substances. If this is so, then, of course, every continuing object must be Lagrangian, i.e. be disposed to evolve in accordance with the principle of least action. A sophisticated dispositionalist therefore has no trouble in accommodating Lagrange's principle, or explaining its metaphysical necessity. (p.91)

Several other non-equivalentists find it remarkable that all our EOMs can be formulated using an action principle. Tonti (1969) argues that the amenability of EOMs to an action formulation is a consequence of their inherent structure. He emphasizes this point by stating, "What is interesting about the variational principles is that the equations describing many physical laws have just such a structure to possess a variational formulation" (p. 139).

Another perspective holds that the PLA functions as a governing law or meta-law, ensuring that all dynamics have extremized quantities of action. This view is advocated by Katzav (2004), Thebault and Smart (2013), Glick (2023), and Terekhovich (2017). Katzav (2004) suggests that the non-accidental nature of the PLA and its unificatory role argue for its explanatory precedence over the EOMs (p. 212). Stoltzner (2003) expresses a similar sentiment when discussing the subtleties of the PLA: "There were general mathematical arguments—such as coordinate invariance or *the non-trivial fact* that a variational principle could be set up." (p. 289). Thebault and Smart describe the PLA as a "fundamental meta-law" (p. 387), while Glick perceives it as a constraint analogous to the law of conservation of energy. A common theme among these views is their recognition of the significance of the PLA's universality or wide applicability. Given this wide applicability, these non-equivalentists suggest that it would be rather puzzling if all of our EOMs would coincidentally obey the PLA without any underlying reason.

4.2 Laws, meta-laws and constraints

In order to analyze the non-equivalentist views on the modality of the PLA, it is essential to understand the metaphysical distinctions among laws, meta-laws, constraints, and coincidences. A concise exploration of these concepts in the literature is warranted.

Firstly, let's highlight some differences between laws of nature and accidental regularities with examples.[34] Consider the phenomena of metals expanding when heated and the height of US presidents. It is true that all metals expand when they are heated. It is also true that the height of all US presidents is under 2m tall. While both exhibit regularities, the crucial distinction lies in their stability under varying conditions. For instance, it is a *law of nature* that metals expand when heated, as this phenomenon remains stable irrespective of factors such as the size of the metal sample or the mode of heating used. Conversely, the observation that all US presidents so far have been under 2 meters tall is merely an *accidental regularity*, as there is nothing preventing the presidential candidate from meeting this height requirement, so there is no stability as the conditions are varied.

This distinction between an accidental regularity and a law of nature also implies different levels of necessity. For example, if any unheated piece of metal were to be heated, it would expand—it is physically impossible for a metal to fail to expand when heated. However, while it is possible for the next US president to be under 2 meters tall, it is not a necessity for someone to meet this height requirement to become president. This illustrates the difference between something being necessarily the case and something merely happening to be the case.[35]

[34] I am using Beebee (2015)'s examples here.

[35] This discussion can become complex rapidly, with a vast amount of literature exploring the precise definition of a law of nature. Questions arise regarding whether we should conceptualize the world in terms of laws of nature and which criteria are pertinent in determining something as a law of nature. For the purpose of this discussion, I operate under the assumption that laws of nature exist and that

The literature on the laws of nature is extensive, but the specific framework I will focus on is the 'Anti-reductionism' account, advocated in various forms by Carroll (1994) and Lange (2000, 2009). A central tenet of this account is the differentiation between laws of nature and non-laws by their non-coincidental character. In order for something to qualify as a law of nature, there must be an explanation, or an underlying reason, for its non-coincidental character.[36] In Lange's (2000) framework, a crucial aspect is the resilience of laws under counterfactual conditionals. This means that even in scenarios where the world appears markedly different, the laws of nature would remain resilient and stable.

Now, we elevate the distinction between laws and accidental regularities to a higher level by applying it to the laws themselves. When considering the laws of physics, we observe certain regularities among them. For instance, all our laws of physics are expressed in terms of mathematical equations. Additionally, they all adhere to specific symmetry principles. Similarly, all laws of physics conform to the PLA and can be derived from an action.

The question significant for our purposes is whether all laws of physics obey the PLA because it serves as a meta-law or a law governing other laws, or if they happen to obey the PLA due to coincidence. The non-equivalentist accounts I presented earlier suggest that all our laws are amenable to an action formulation because the PLA functions as a meta-law or governing law of sorts. It seems implausible that all our EOMs are amenable to an action formulation purely by coincidence. Investigating this claim is the primary focus of my inquiry.[37]

distinctions exist between laws of nature and accidental regularities. However, I will refrain from delving into the intricacies of this debate because the basics are sufficient for our purposes.

[36] This seems to be the sense in which the non-equivalentist accounts I presented above hold that the PLA is 'non-accidental' or 'non-coincidental', i.e that there must be an underlying reason why the PLA holds.

[37]Another significant point, which I won't delve into in great detail, is the distinction between the PLA and the EOMs. While the PLA is often regarded as a principle, the EOMs are typically introduced as a quantification of dynamics. This raises the question: Why is the PLA considered a principle, and where does it fit into the distinctions between laws and accidental regularities? Does its modality differ from that of the laws of nature? The distinction between a law of nature and a mere accidental regularity has been extensively discussed in the literature, with scholars emphasizing various aspects of this difference. However, there appears to be less discussion specifically focused on principles. In some cases, such as with the principle of relativity, the principle is more general and serves as a foundational assumption upon which the laws themselves are built. It applies across a range of laws or theories, similar to how the principle of relativity underpins both Newtonian Mechanics and Einstein's relativity,

I will focus on a specific non-equivalentist account, namely Glick's (2023) interpretation of the PLA. Glick's argument stands out among other non-equivalentist accounts because he grounds his views on the PLA within Lange's (2007, 2016) framework of meta-laws, constraints, and coincidences regarding the metaphysics of laws of nature. Lange distinguishes between meta-laws and byproducts in his earlier (2007) work, and later (2016), between constraints and coincidences. According to Lange, meta-laws, situated at the highest level of natural necessity, can explain the constraints, which in turn explain the first-order laws, establishing a hierarchical account.

Lange explores the difference between constraints and coincidences in his book "Because without Cause." His investigation focuses on the principles that all laws of physics adhere to. Regarding such principles, Feynman (1967) says "across the variety of these detailed laws there sweep great general principles which all the laws seem to follow." (p. 59) Lange proposes a distinction between these overarching principles as either constraints on fundamental forces or mere coincidences of the laws themselves.[38] It's unsurprising, then, that Glick employs Lange's framework to argue that the PLA, akin to these overarching principles, functions as a constraint.

If a principle is a coincidence, it can be explained by the force laws themselves whereas if a principle is a constraint, it helps to explain why the laws have certain features. Moreover, if a principle functions as a constraint, it possesses greater modal strength than the force laws, as it applies universally across the laws of physics. Lange articulates this distinction, stating that:

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albeit in slightly different forms. This seems to be the sense in which the PLA functions as a principle—it provides a foundation for the development of laws and theories. However, there are other cases in which the principle itself is considered a law of nature, as is the case with conservation laws. Recently, such principles have been considered potential candidates for meta-laws (or laws of laws) that govern how the laws behave in a non-causal manner. It appears, therefore, that principles sometimes serve as foundations for other theories, are sometimes considered laws themselves, and have also been argued to be meta-laws. There doesn't seem to be a consensus on what exactly constitutes a principle or what its modality is, so I won't discuss this aspect of the PLA any further. Instead, I will briefly explore the role of the PLA as a heuristic principle at the end of the thesis.

[38] As I mentioned, this includes principles such as the conservation of energy and symmetry principles which all the laws of physics seem to obey. The PLA is also a principle that all the laws of physics obey. Lange argues that these principles can either be coincidences or constraints.

> [a principle] is a constraint exactly when it would still have held, even if there had been additional types of fundamental interactions. Thus, a constraint possesses a certain distinctive kind of invariance under counterfactual perturbations." (p. 48) [39]

Glick (2023) employs this criterion of 'resilience under counterfactuals' to contend that the PLA functions as a constraint. He articulates his argument as follows:

> A principle like the PLA may be regarded as a coincidentally true generalization, or alternatively, as a limitation on what dynamical laws are possible... when we look at the considerations Lange appeals to when discussing conservation laws, many of those same points apply to the PLA as well. In particular, Lange claims that energy conservation is resilient, in that it applies to a wide range of different dynamical laws. (p. 24)

Glick maintains that viewing a principle as possessing resilience indicates its function as a constraint, as a mere coincidence wouldn't be expected to persist if the underlying dynamical laws were altered. Glick acknowledges that the PLA could be perceived either as a coincidence or a constraint. However, he contends that since several dynamical laws follow from the PLA, regarding it a coincidence would be puzzling. He concludes, "If scientists regard the generalization as having the kind of resilience discussed above, then it should be regarded as a constraint."

4.3 Resilience due to second-order EOMs

I agree with Glick that within the account put forward by Lange, a reasonable case can be made for why the PLA is a constraint. After all, nearly all branches of physics are amenable to an action formulation, so the PLA is indeed resilient. However, in this section, I will attempt to show that the resilience exhibited by the PLA is not because

[39] Resilience under counterfactual conditionals was also a key element that distinguished a law from a mere accidental regularity, highlighting the fact that Lange is applying his account of the laws of nature on a higher level. Lange presents a detailed account of how exactly this resilience can be realized in terms of nomic stability. Furthermore, he discusses how constraints can derive their status from being associated with meta-laws. The meta-laws non-causally explain the constraints. These details, however, do not contribute to the main stem of the argument in my paper so I shall not discuss them any further.

of some feature that allows it to be a constraint in terms of increased modal necessity, but rather because of the mathematical structure of the EOMs.[40] In this respect, an important question to consider is the following: how remarkable is it that all EOMs are amenable to an action formulation?[41] Posed more mathematically, given an arbitrary EOM, what mathematical conditions must it meet for it to be expressed in terms of an action?[42] This will allow us to understand whether amenability to an action formulation is something special or remarkable.

In order to be expressed in terms of an action, a differential equation must meet the Helmholtz conditions (Santilli 1978).[43] However, the fact that EOMs can be derived from a Lagrangian might not be as surprising as the non-equivalentist presumes. For a condition to be stringent, there should be cases where certain differential equations are amenable to an action formulation while others are not. However, a broad spectrum of equations can meet the Helmholtz conditions (Tartaglia 1983). Furthermore, studies such as those by Talukdar et al. (2020) demonstrate that the theory of self-adjoint differential equations provides a framework for constructing a Lagrangian even when the original equation fails to satisfy the Helmholtz criteria, which implies a lack of stringency in finding an action for a given EOM. Furthermore, EOMs that represent systems that are not considered fundamental such as systems with friction, drag or damped oscillators can also be expressed in terms of an action formulation (Santilli 1978). This means that if the Helmholtz conditions are met, the EOM obeys the PLA. But since practically any differential equation that can be solved can also be expressed in terms of an action, there doesn't seem to be anything remarkable or special about the fact that all our fundamental EOMs can be expressed in terms of an action too. It is noteworthy that this lack of stringency applies to low-order differential equations, which constitute our fundamental EOMs—typically second or first-order.[44]

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[40] My argument is meant to show that the amenability to the action formulation is indeed 'non-accidental'. However, the non-coincidental nature is not because the PLA is a meta-law, but rather because the EOMs are low order.
[41] This will help us gauge whether the resilience of the PLA is something substantial.
[42] In other words, what conditions should an arbitrary EOM satisfy to be the Euler-Lagrange equations of some action?
[43] Specific details of the conditions are not necessary for our purposes.
[44] The mathematical works cited above work with second-order differential equations. Whether higher order equations would be amenable to an action formulation does not seem to be explored in the literature given that EOMs are predominantly second order with a few first-order equations.

This also means that the resilience condition is met because of the mathematical structure of the EOMs, not because of the modality of the PLA.[45] The PLA, then, is neither a coincidence nor a constraint in the sense of Lange. It is not a constraint, because its resilience is a byproduct of the mathematical structure of the laws themselves, so the amenability to an action formulation can be explained by appealing to the structure of the laws themselves. The PLA is not a coincidence either because there is indeed an underlying reason why the EOMs are amenable to an action formulation, i.e they are low order.

Why do we have only low order differential equations in physical laws? Although exploring this question in detail goes beyond the scope of this thesis, a brief discussion is worthwhile. According to Swanson (2019), the answer lies in the Ostrogradski instability.[46] If EOMs exceed second-order, the energy function becomes unbounded from below, leading to instability. The Ostogradski instability is one of the most powerful theoretical constraints on the construction of new field theories.[47] However, Swanson warns that an unbounded energy doesn't necessarily render a theory unphysical. It only means that the EOMs corresponding to the higher order do not have well-behaved solutions. Interpreting considerations about the Ostrogradski instability—whether from a minimalist mathematical perspective or as a metaphysical restriction—is a further task. Swanson himself maintains a 'metaphysically agnostic' stance, leaving room for other philosophical interpretations.

These considerations mean that labeling the PLA a 'meta-law' or a fundamental law requires justification, not acceptance at face value, and currently, no non-equivalentist account has provided such justification. Although Glick situates his views on the PLA within Lange's account of meta-laws and the PLA is indeed resilient, its resilience is not because of the modality of the PLA, but rather because of the mathematical structure of the EOMs. The focal point for non-equivalentists is addressing the puzzle

[45] Perhaps this poses a challenge to Lange's account that if a principle is a constraint, it is modally stronger than the first order laws and if it is a coincidence, then there is no common reason why all the laws obey the particular principle.
[46] Easwaran (2014) also offers an account of why physics has second-order equations. Swanson's account, however, overcomes limitations that Easwaran's account faces.
[47] See Motohashi & Suyama (2015) for details.

of second-order EOMs and in doing so, perhaps we could gain a deeper understanding of the PLA as well.

CHAPTER 5

CLOSING DISCUSSIONS

This thesis aimed to deepen our understanding of the PLA and its significance in physics. To contextualize modern perspectives on the PLA, I initially provided a historical overview of its development, illustrating its parallel evolution with EOMs and its unique origins rooted in the theological and teleological views of early physicists. Subsequently, I categorized modern perspectives into two main camps: equivalentist, which posits theoretical equivalence between the PLA and EOMs, and non-equivalentist, which explores the potential theoretical disparities between them. While the former perspective predominates among contemporary physicists, the latter has garnered attention from philosophers in recent decades.

In scrutinizing non-equivalentist viewpoints, I outlined two criteria and critically evaluated each. Firstly, the contention that the action (or Lagrangians) can be interpreted physically or ascribed physical significance faces significant challenges posed by alternative Lagrangians, for which clear rationale for resolution remains elusive. Secondly, the assertion that the PLA's wide applicability across theoretical physics suggests its role as a meta-law or constraint governing dynamics is undermined by the fact that all EOMs, being second-order, are amenable to action formulations irrespective of the PLA's modality.

This analysis leads us to contemplate the significance of the PLA in physics. I believe the PLA serves as a valuable heuristic principle, guiding the search for new theories in conjunction with other theoretical assumptions. It is crucial to recognize that the formulation of the Lagrangian depends on specific theoretical frameworks; thus, the PLA is not a fundamental law in isolation but rather contingent on theoretical underpinnings. While some may speculate that a 'theory of everything' could be encapsulated by an equation like $\delta S = 0$, such an equation only holds meaning within the context of accompanying theoretical assumptions. With that said, the PLA plays a pivotal role in the development of new theories, particularly when working with complex theories. Working from experimental evidence towards EOMs can be a rather intricate process. However, starting from the action allows for a much easier derivation and makes certain features, such as the gauge property, much more evident.

Furthermore, the PLA's conciseness and utility in simplifying the presentation of complex theories cannot be overstated. However, claims suggesting its explanatory precedence over EOMs due to its 'unificatory' role are unfounded. The PLA's purported unificatory function is subsumed under its role as a heuristic tool in theory development. In conclusion, while the PLA offers conciseness and utility in the development and derivation of the EOMs, it does not possess inherent unificatory power or a higher modality than the EOMs. Recognizing the limitations of the PLA is essential for a nuanced understanding of its role in physics.

REFERENCES

Anderson, I. M., & Davison, S. M. (1974). A generalization of Lie's "counting" theorem for second-order ordinary differential equations. *Journal of Mathematical Analysis and Applications, 48,* 301-315.

Anderson, R. L., & Duchamp, T. (1980). On the existence of global variational principles. *American Journal of Mathematics, 102*(5), 781-868.

Barnes, J. (2001). *Early Greek philosophy*. Penguin Books.

Becker, K., Becker, M., & Schwarz, J. H. (2006). *String theory and M-theory: A modern introduction*. Cambridge University Press.

Beebee, H. (2015). Causes and laws: Philosophical aspects.

Brechteken-Mandersch, U. (1991). *Introduction to the calculus of variations*. CRC Press.

Cariñena, J. F., Ibort, A., Marmo, G., & Morandi, G. (2015). *Geometry from dynamics, classical and quantum*. Springer Netherlands.

Carroll, J. W. (1994). *Laws of nature*. Cambridge University Press.

Corry, L. (1997). David Hilbert and the axiomatization of physics (1894–1905). *Archive for History of Exact Sciences, 51,* 83-198.

Easwaran, K. (2014). Why physics uses second derivatives. *The British Journal for the Philosophy of Science*.

Edwards, I. K. (1979). Quantization of inequivalent classical Hamiltonians. *American Journal of Physics, 47*(2), 153-155.

Ellis, B. (2005). Katzav on the limitations of dispositionalism. *Analysis, 65*(1), 90-92.

Feynman, R. P., Leighton, R. B., & Sands, M. (1964). *The Feynman lectures on physics* (Vol. 2). Addison-Wesley.

Feynman, R. P. (1967). *The character of physical law*. MIT Press.

Ghosh, S., Shamanna, J., & Talukdar, B. (2004). Inequivalent Lagrangians for the damped harmonic oscillator. *Canadian Journal of Physics, 82*(7), 561-567.

Glick, D. (2023). The principle of least action and teleological explanation in physics. *Synthese, 202*(1), 25.

Goldstein, H. (1980). *Classical mechanics* (2nd ed.). Addison-Wesley.

Helmholtz, H. v. (1886). Über die physikalische Bedeutung des Prinzip der kleinsten Wirkung [On the physical meaning of the principle of least action]. In H. v. Helmholtz (Ed.), *Wissenschaftliche Abhandlungen* (Vol. III, pp. 203–248). Leipzig: J.A. Barth.

Henneaux, M. (1982). Equations of motion, commutation relations and ambiguities in the Lagrangian formalism. *Annals of Physics, 140*(1), 45-64.

Hilbert, D. (1905). Logische Principien des mathematischen Denkens [Logical principles of mathematical thinking]. Universität Göttingen.

Huygens, C. (1690). *Traité de la lumière* [Treatise on light]. Pieter van der Aa.

Jacobi, C. G. J. (1866). *Vorlesungen über Dynamik* [Lectures on dynamics]. G. Reimer.

Jourdain, P. E. (1912). Maupertuis and the principle of least action. *The Monist, 22*(4), 414-459.

Katzav, J. (2004). Dispositions and the principle of least action. *Analysis, 64*(3), 206-214.

Kelvin, W. T. B., & Tait, P. G. (1867). *Treatise on natural philosophy* (Vol. 1). Claredon Press.

Kerner, E. H. (1958). Note on the forced and damped oscillator in quantum mechanics. *Canadian Journal of Physics, 36*(3), 371-377.

Kirk, G. S., Raven, J. E., & Schofield, M. (1983). *The presocratic philosophers: A critical history with a selection of texts*. Cambridge University Press.

Lagrange, J. L. (1788). *Analytical mechanics* (Mécanique analytique).

Lange, M. (2000). *Natural laws in scientific practice*. Oxford University Press.

Lange, M. (2007). Laws and meta-laws of nature: Conservation laws and symmetries. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics, 38*(3), 457-481.

Lange, M. (2009). *Laws and lawmakers: Science, metaphysics, and the laws of nature*. Oxford University Press.

Lange, M. (2016). *Because without cause: Non-causal explanations in science and mathematics*. Oxford University Press.

Leibniz, G. W. (2010). *Theodicy*. Cosimo, Inc.

Leubner, C. (1981). Inequivalent Lagrangians from constants of the motion. *Physics Letters A, 86*(2), 68-70.

Lemos, N. A. (1981). Physical consequences of the choice of the Lagrangian. *Physical Review D, 24*(4), 1036.

Lie, S. (1922). *Gesammelte Abhandlungen* (F. Engel & P. Heegaard, Eds.).

Lutzky, M. (1978). Symmetry groups and conserved quantities for the harmonic oscillator. *Journal of Physics A: Mathematical and General, 11*(2), 249-258.

Marion, J. B., & Thornton, S. T. (1995). *Classical dynamics of systems and particles*. Thomson.

Marmo, G., Saletan, E. J., Simoni, A., & Vitale, B. (1985). *Dynamical systems: A differential geometric approach to symmetry and reduction*. John Wiley & Sons.

Marmo, G., & Saletan, E. J. (1977a). Ambiguities in the Lagrangian and Hamiltonian formalism: Transformation properties. *Nuovo Cimento B Serie, 40*(1), 67-89.

Marmo, G., & Saletan, E. J. (1977b). Other symmetries and constants of the motion. In *Group theoretical methods in physics* (pp. 379-384). Academic Press.

Maupertuis, P. L. M. de. (1750). *Essai de cosmologie* [Essay on cosmology].

McDonough, J. K. (2020). Not dead yet: Teleology and the scientific revolution. In *Teleology: A history* (pp. 150-179). Oxford University Press.

Morandi, G., Ferrario, C., Vecchio, G. L., Marmo, G., & Rubano, C. (1990). The inverse problem in the calculus of variations and the geometry of the tangent bundle. *Physics Reports, 188*(3-4), 147-284.

Motohashi, H., & Suyama, T. (2015). Third order equations of motion and the Ostrogradsky instability. *Physical Review D, 91*(8), 085009.

Newton, I. (1687). *Philosophiæ Naturalis Principia Mathematica* [Mathematical principles of natural philosophy].

Planck, M. (1915). Das Prinzip der kleinsten Wirkung [The principle of least action]. In M. Planck (Ed.), *Wege zur physikalischen Erkenntnis. Reden und Vorträge* (pp. 68–78). S. Hirzel.

Reeve, C. D. C. (2020). *De Caelo*. Hackett Publishing.

Redmount, I., Suen, W. M., & Young, K. (1999). Ambiguities in quantizing a classical system. *arXiv preprint gr-qc/9904042*.

Rosen, G. (1969). *Formulations of classical and quantum dynamical theory*. Academic Press.

Santilli, R. M. (1978). *Foundations of theoretical mechanics* (Vol. 1). Springer.

Santilli, R. M. (1983). *Foundations of theoretical mechanics II*. Springer-Verlag.

Saunders, D. J. (2010). Thirty years of the inverse problem in the calculus of variations. *Reports on Mathematical Physics, 66*(1), 43-53.

Scheck, F. (2012). *Classical field theory: On electrodynamics, non-Abelian gauge theories and gravitation*. Springer-Verlag.

Senitzky, I. R. (1960). Dissipation in quantum mechanics: The harmonic oscillator. *Physical Review, 119*(2), 670.

Stöltzner, M. (1994). Action principles and teleology. In *Inside Versus Outside: Endo-and Exo-Concepts of Observation and Knowledge in Physics, Philosophy and Cognitive Science* (pp. 33-62). Springer.

Stöltzner, M. (2003). The principle of least action as the logical empiricist's shibboleth. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics, 34*(2), 285-318.

Swanson, N. (2022). On the Ostrogradski instability; or, why physics really uses second derivatives. *The British Journal for the Philosophy of Science*.

Talukdar, B., Chatterjee, S., & Ali, S. G. (2020). On the analytic representation of Newtonian systems. *Pramana, 94*, 1-11.

Tartaglia, A. (1983). Non-conservative forces, Lagrangians and quantisation. *European Journal of Physics, 4*(4), 231.

Taylor, E. F. (2003). A call to action. *American Journal of Physics, 71*(5), 423-425.

Terekhovich, V. (2017). Metaphysics of the principle of least action. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics, 62,* 189-201.

Thebault, K. P., & Smart, B. T. (2013). On the metaphysics of least action.

Thébault, K. P., & Smart, B. T. (2015). Dispositions and the principle of least action revisited. *Analysis, 75*(3), 386-395.

Tonti, E. (1969). Variational formulation of nonlinear differential equations (I). *Bulletins de l'Académie Royale de Belgique, 55*(1), 137-165.

Veldman, M. (2024). Mathematizing metaphysics: The case of the principle of least action. *Philosophy of Science, 91*(2), 351-369.

Yourgrau, W., & Mandelstam, S. (1979). *Variational principles in dynamics and quantum theory*. Courier Corporation.

Zee, A. (1999). *Fearful symmetry: The search for beauty in modern physics*. Princeton University Press.