FIRAT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
TÜRKİYE

# AN ENERGY-SAVING MATHEMATICAL MODEL IN THE FRAME OF CONFORMABLE DERIVATIVE

**Hozhin Mohammed BRAIM**

Master's Thesis

DEPARTMENT OF MATHEMATICS

JANUARY 2022

FIRAT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
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Department of Mathematics

Master's Thesis

# AN ENERGY-SAVING MATHEMATICAL MODEL IN THE FRAME OF CONFORMABLE DERIVATIVE

Author
**Hozhin Mohammed BRAIM**

Supervisor
Prof. Dr. Erdal BAŞ

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ELAZIG

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This thesis, which was prepared according to the thesis writing rules of the Graduate School of Natural and Applied Sciences, Fırat University, was evaluated by the committee members who have signed the following signatures and was unanimously approved after the defense exam made open to the academic audience.

| | | Signature |
|---|---|---|
| Supervisor: | Prof. Dr. Erdal BAŞ<br>Firat University, Faculty of Science | Approved |
| Chair: | Prof. Dr. Hasan BULUT<br>Firat University, Faculty of Science | Approved |
| Member: | Dr. Öğr. Üyesi Mine BABAOĞLU<br>Kahramanmaraş Sütçü İmam University, Faculty of Education | Approved |

This thesis was approved by the Administrative Board of the Graduate School on

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Signature

Prof. Dr. Kürşat Esat ALYAMAÇ
Director of the Graduate School

# DECLARATION

I hereby declare that I wrote this Master's Thesis titled "An energy-saving mathematical model in the frame of conformable derivative" in consistent with the thesis writing guide of the Graduate School of Natural and Applied Sciences, Firat University. I also declare that all information in it is correct, that I acted according to scientific ethics in producing and presenting the findings, cited all the references I used, express all institutions or organizations or persons who supported the thesis financially. I have never used the data and information I provide here in order to get a degree in any way.

10 January 2022

**Hozhin Mohammed BRAIM**

# PREFACE

First of all, I am grateful to God Almighty for providing me with the ability to succeed in my efforts to finish this thesis on time.

I want to express my heartfelt gratitude to my loving parents and entire family for their continuous encouragement, inspiration, and support during my years of research. They played an important role in my success and always encouraged me to pursue my goals. I am grateful to them for their devotion, prayers, and sacrifices in educating and training me for my future.

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**Hozhin Mohammed BRAIM**
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# ABSTRACT

## An energy-saving mathematical model in the frame of conformable derivative

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In this study, there are four chapters. A general history of fractional calculations and conformable derivatives is given in the first chapter. In the second chapter, the fundamental definitions, theorems of the Laplace transform are given in detail. In the third chapter, an energy-saving model for building heating and cooling is considered. An energy-saving model for building heating and cooling is explained which has a huge effect on some aspects of everyday life, as well as a much more sustainable energy system all over the world. Furthermore the model is considered in a general format with local derivatives and the results are given. In the fourth chapter, the scientific results of the studies carried out in the thesis are given in detail.

**Keywords:** Conformable derivative, Energy, Heating-Cooling, Model.

# ÖZET

## Uyumlu türev ile enerji tasarrufu sağlayan matematiksel bir model

**Hozhin Mohammed BRAIM**

Yüksek Lisans Tezi

FIRAT ÜNİVERSİTESİ
Fen Bilimleri Enstitüsü

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Bu çalışmada dört bölüm bulunmaktadır. Birinci bölümde kesirli hesaplamaların ve uyumlu türevlerin genel bir tarihçesi verilmiştir. İkinci bölümde temel tanım teoremler ve Laplace dönüşümleri detaylı olarak verilmiştir. Üçüncü bölümde binaların soğutulması ısıtılması için enerji tasarruf modeli göz önüne alınmıştır. Günlük yaşamın bazı yönleri üzerinde büyük etkisi olan ve tüm dünyada çok daha sürdürülebilir bir enerji sistemi bina ısıtma ve soğutma için enerji tasarrufu sağlayan bir model ile ilgili sonuçlar açıklanmıştır. Ayrıca bu modelleme lokal türevlerle daha genel bir formatta ele alınmış ve sonuçları verilmiştir. Dördüncü bölümde ise tezde yapılan çalışmaların bilimsel sonuçları ayrıntılı olarak verilmiştir.

**Anahtar Kelimeler:** Uyumlu türev, Enerji, Isıtma ve Soğutma, Modelleme.

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| Symbols | |
|---|---|
| $\frac{d^n}{dx^n}$ | : A derivative of order $n$ concerning $x$. |
| $\mathcal{D}_\alpha^a$ | : The fractional derivative operator with order $\alpha$. |
| $T_\alpha^a$ | : Left conformable derivative with order $\alpha$. |
| ${}_\alpha^b T$ | : Right conformable derivative with order $\alpha$. |
| $\mathbb{R}$ | : Set of real numbers. |
| $\mathbb{N}$ | : Set of natural numbers. |
| $I_\alpha^a$ | : Left conformable integral with order $\alpha$. |
| ${}_\alpha^b I$ | : Right conformable integral with order $\alpha$. |
| $\Gamma$ | : Gamma function. |
| $\mathcal{L}_\alpha^{t_0}$ | : Conformable Laplace transform of order $\alpha$, where $t_0 \in \mathbb{R}$. |
| $\mathcal{L}$ | : Normal Laplace transform. |
| $\lambda$ | : Lambda. |
| K | : Positive constant. |

# 1. INTRODUCTION

Relating to economic development and environmental benefits, one of the most important principles for a long-term global energy system is energy efficiency. As a result, to create a more sustainable environment, a strong focus on energy conservation should be put. Simply said, energy efficiency implies using less energy to do the same work - in other words, reducing energy waste. Utilization management is also important for keeping the correct balance between energy efficiency and demand. As a result, it will conserve electricity to use mathematical approaches to investigate the heating and cooling of buildings in some different ways. Many of us dedicate a significant amount of time in structures like residences, businesses, hotels, and hospitals, which is why these models are so important. It is possible to pass on significant savings by reducing energy usage in homes, which account for a substantial percentage. Alternatively, there is a close relationship between inside and outside temperature because when the outside temperature changes, the inside temperature changes as well. It is important to raise the inside temperature to compensate for the decreasing temperature outside, and as the difference between the inside and external temperatures increases, so must the need to raise the inside temperature, so does the risk of frostbite, a significant amount of energy is used. Energy can be saved by adopting energy-saving behaviors, installing insulation, and updating appliances, among other things. The energy-saving model listed for building heating and cooling has a strong link to Newton's law of cooling and furthermore economical models [1-4]. Furthermore, compartmental analysis is used to model the temperature of a building, which treats a building as one unit. The name for this method is a one-compartment system, reduces the complexity of the model by identifying the proper parameters for estimating the temperature of buildings at any given moment. This model that defines throughout a 24-hour period, the inside temperature of a structure as a function of outside temperature, as well as some other factors including heat produced inside the building, furnace heating, and air conditioner cooling, is denoted by [5]

$$\frac{dT(t)}{dt} = K\{M(t) - T(t)\} + H(t) + F(t). \tag{1.1}$$

T(t) is the temperature of inside the building, $H(t)$ is the heat created by lights, machines, and people, $F(t)$ is the heating or cooling produced by the furnace or air conditioner. Also the outside temperature is generated by $M(t)$, the physical properties of the building denoted by the positive constant $K$. Many previous methods of communicating the model may be incomplete. As a result, we present this model in terms of heating and cooling from an arbitrary order perspective to obtain significantly more useful results [5]. Some local derivatives, the most common of which is the conformable derivative, have become increasingly popular in recent years, which is defined by

$$T_{\alpha}^{a}(t) = \lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon(t-a)^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{1.2}$$

Where $t > 0$ and $0 < \alpha \leq 1$, This limit-based local derivative without memory is well-behaved since it conforms to the widely used properties in calculus. In several ways, such as derivative of the product and quotient of two functions integration by parts formulas and chain rule, Taylor power series, and Laplace transform, this efficient derivative approach to the classical derivative outperforms the classical derivative. In this effort, the conformable fractional integral sense of the fractional Laplace transform is reflected. There are numerous publications in the literature about fractional Laplace transforms their properties and applications. Abdeljawad [6-7], on the other hand, defined the conformable fractional Laplace transform with a few basic properties. Here, several other significant properties are meticulously explored as part of this study's further investigation.

Another way for changing temperature in the buildings, we install a thermostat to compare the required temperature $T(t)$ to the temperature in the building, and the furnace heating is activated when the real temperature is lower than the desired temperature. When the temperature rises above the intended level, however, the air conditioner is activated to keep the room cool. If we assume that the quantity of heating or cooling is equivalent to the difference in temperature, that is $F(t) = k_F[T_w - T(t)]$ where $K_F$ is any positive constant. In this case, the equation (1.1) becomes the following equation

$$\frac{dT(t)}{dt} = K\{N(t) - T(t)\} + C(t) + K_F[T_w - T(t)]. \tag{1.3}$$

During the last and present centuries, many scientists found appropriate definitions for the notion of integral or derivative with non-integer order, From that time which L'Hospital 1695 asked if we have $n = \frac{1}{2}$, so what does it mean of $\frac{d^n f}{dx^n}$. To indicate differentiation and integration in any order, the fractional name was used. In recent years, the significance of fractional calculations in the fields of pure applied science and engineering has begun to grow significantly. Many definitions of fractional calculus have become popular, such as Riemann-Liouville, Caputo [8-10], Weyl, Fourier, Abel, Leibniz, Grünwald, and Letnikov definitions. Furthermore, the most commonly used definitions are

1) ***Riemann-Liouville definition*** [11] **.** It is straightforward to substitute the iterated integral with a single integral direct Leibniz-Cauchy formula and to replace the fractional function with the Gamma function. The integration of the metric $dt$ also yields the arbitrary order Riemann-Liouville. For $n-1 \leq \alpha < n$, where $n$ belongs to natural numbers, the derivative of the function $f$ with order $\alpha$ on the Riemann-Liouville definition is given by [10]

$$(T_{\alpha}^{a} f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(t)}{(t-x)^{\alpha-n+1}} dx. \tag{1.4}$$

Which $T_{\alpha}^{a}$ is an operator of the left fractional derivative with order is not integer number, also $\frac{d^n}{dt^n}$ is an operator of the derivative with an integer number.

2) ***Caputo definition.*** [11] This definition utilized an integral form of the fractional derivative. Let $\alpha \in [n-1, n), n \in \mathbb{N}$, then the fractional derivative of $f$ is defined as [11]

$$^{c}T_{a}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(f)^n(t)}{(t-x)^{\alpha-n+1}} dx. \tag{1.5}$$

3) ***Grünwald –Letnikov definition*** [11] **.** The limit definition of the derivative should be iterated in this law to generate a quantity with a specific binomial coefficient, which should then be fractionalized by using the Gamma function instead of the fractional in the binomial coefficient. Grunwald-Letnikov defines a fractional derivative as

$$(T_{a}^{\alpha} f)(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j \to 0}^{\frac{x-a}{h}} (-1)^j \binom{\alpha}{j} f(x - jh). \tag{1.6}$$

A wide variety of studies and models related to non-local and local derivatives are available in the literature [12-14, 18-26].

This thesis is structured as follows: The basic definitions, theorems, and properties of the truncated conformable derivative and integral are discussed in chapter 2. In chapter 3, the Laplace transform of truncated conformable derivative and integral is used to solve the aforementioned model (An energy-saving mathematical model in the frame of conformable derivative) and numerous details regarding this useful method are provided. Some figures are included to help understand the energy-saving model at the end of this thesis.

# 2. BASIC DEFINITIONS AND THEOREMS

## 2.1. Conformable derivative

**Definition 2.1.1.** [17] Assume that $\boldsymbol{f}: (\mathbf{0}, \infty) \to \mathbb{R}$ be a given function, then a left conformable derivative of the function $f$ with order $\mathbf{0} < \boldsymbol{\alpha} \leq \mathbf{1}$ is defined as follows

$$T_\alpha^a(t) = \lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon(t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{2.1}$$

For every $t > a$. If $a = 0$, then we can write by $T_\alpha$. From this definition if a left conformable derivative $T_\alpha^a$ exists on the interval$(a, b)$, then we can infer that

$$(T_\alpha^a f)(t) = \lim_{t \to a^+} (T_\alpha^a f)(t). \tag{2.2}$$

**Note:** Let $f$is a differentiable function and $0 < \alpha \leq 1$, after that, we may say

$$(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(t). \tag{2.3}$$

**Definition 2.1.2**. [15-17] The right conformable derivative of the function $\boldsymbol{f}: (\mathbf{0}, \infty) \to \mathbb{R}$ with order $\boldsymbol{\alpha}$, where $\mathbf{0} < \boldsymbol{\alpha} \leq \mathbf{1}$, is defined as follows

$$({}_\alpha^b T)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon(b-t)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{2.4}$$

For every $t < b$. From this definition if $\left({}_\alpha^b T f\right)(t)$ exists on the interval $(a, b)$, then we can assume that

$$({}_\alpha^b T f)(t) = \lim_{t \to b^-} ({}_\alpha^b T f)(t). \tag{2.5}$$

**Note**: [5] We can also define the right conformable derivative as

$$({}_\alpha^b T f)(t) = -(b-t)^{1-\alpha} f'(t). \tag{2.6}$$

Where $\alpha \in (0,1]$ and $f$ is differentiable. Assume that $f$ be a given function and conformable derivative of the function $f$ is exists, so that $f$ is said to be $\alpha$-differentiable function. Let $\alpha \in (0,1]$ and consider two $\alpha$-differentiable functions, f and g at $a < t$, then the definition of conformable derivative satisfies all of the following properties [5,17]

1/ $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for every $a, b$ that the numbers belong to $\mathbb{R}$.

2/ $T_\alpha(t^p) = pt^{p-t}$, for every $p \in \mathbb{R}$.

3/ $T_\alpha(fg) = gT_\alpha(f) + fT_\alpha(g)$.

4/ $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.

5/ $T_\alpha(\lambda)=0$, for all $f(t)=\lambda$ where $\lambda$ is a constant.

6/ If $f$ is differentiable in addition, then $(T_\alpha f)(t)=t^{1-\alpha}\frac{df}{dt}(t)$.

**The conformable derivative of some functions** [17,18]

1/ $T_\alpha(t^p)=pt^{p-\alpha}$, for every $p\in\mathbb{R}$.

2/ $T_\alpha(1)=0$.

3/ $T_\alpha(e^{cx})=cx^{1-\alpha}e^{cx}$, where $c\in\mathbb{R}$.

4/ $T_\alpha(sinbx\,)=bx^{1-\alpha}(cosbx), b\in\mathbb{R}$.

5/ $T_\alpha(cosbx)=-bx^{1-\alpha}sinbx, b\in\ \mathbb{R}$.

**Through this, we found the conformable derivative of basic functions as follows**

$$T_\alpha\left(sin\frac{t^\alpha}{\alpha}\right)=cos\frac{t^\alpha}{\alpha}.$$

$$T_\alpha\left(cos\frac{t^\alpha}{\alpha}\right)=-sin\frac{t^\alpha}{\alpha}.$$

$$T_\alpha(e^{\frac{t^\alpha}{\alpha}})=e^{\frac{t^\alpha}{\alpha}}.$$

**Remark 2.1.3.** [18] We must see that the function could not be a differentiable function, but it could be α-differentiable at a point, for example

Take $f(t)=3\sqrt[3]{t}$. Then $(T_{\frac{1}{3}}f)(0)=\lim_{t\to0^+}(T_{\frac{1}{3}}f)(t)=1$, where $(T_{\frac{1}{3}}f)(t)=1$, for $t>0$. But $\frac{df}{dt}(0)$ does not exist.

**Definition 2.1.4.** [9,17] Let $f$ be a n-differentiable at $\boldsymbol{t}$, where $\boldsymbol{t}>\boldsymbol{0}$. Then the conformable derivative of $f$ with order $\boldsymbol{\alpha}\in(\boldsymbol{n},\boldsymbol{n}+\boldsymbol{1})$, $n\in\mathbb{N}$ is given by

$$T_\alpha f(t)=\lim_{\varepsilon\to0}\frac{f^{(\lceil\alpha\rceil-1)}(t+\varepsilon t^{\lceil\alpha\rceil-\alpha})-f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}. \tag{2.7}$$

Where the number $\lceil\alpha\rceil$ is the smallest integer number which $\lceil\alpha\rceil\geq a$. Also, we can assume that

$$T_\alpha f(t)=t^{(\lceil\alpha\rceil-\alpha)}f^{\lceil\alpha\rceil}(t). \tag{2.8}$$

Where $\alpha \in (n, n+1]$ and $n$ is a positive integer, we say that $f$ is (n+1)-differentiable of a non-negative number $t$.

**Definition 2.1.5.** [17] suppose that $f: (0, \infty) \to \mathbb{R}$ be a given function and $f^{(n)}(t)$ exists, then the left conformable derivative starting from the number $a$ to the function $f$ with order $\alpha$ where $n < \alpha \leq n+1, n \in \mathbb{N}$, and $\beta = \alpha - 1$ is defined as

$$T_\alpha^a f(t) = T_\beta^a f^{(n)}(t). \tag{2.9}$$

We write $T_\alpha$ and $T_\beta$ where $a = 0$.

**Definition 2.1.6.** [17-19] The right conformable derivative $f: (0, \infty) \to \mathbb{R}$ ended at $b$, with order $\alpha$ where $n < \alpha \leq n+1$, $n \in \mathbb{N}$ and $\beta = \alpha - 1$ is given by

$$({}_\alpha^b T f)(t) = (-1)^{n+1} ({}_\beta^b T f^{(n)})(t). \tag{2.10}$$

Notice that, if we have the number $\alpha$ such that $\alpha = n+1$ then $\beta = \alpha - n = n+1-n \Longrightarrow \beta = 1$ and the conformable derivative will become $f^{(n+1)}(t)$. Also, Definition 2.1.4 and Definition 2.1.5 are respectively will become the same as Definition 2.1.1 and Definition 2.1.2, where $n = 0$ or $\alpha \epsilon (0,1)$.

**Theorem 2.1.7. (Rolle's theorem to conformable differentiable functions)** [6,20]

Assume that $a > 0$, if $f: (0, \infty) \to \mathbb{R}$ is a given function such that satisfy the following conditions

i. $f$ is continuous function on interval $(a, b)$.
ii. $f$ is $\alpha$-differentiable function for some $\alpha \in (0,1)$.
iii. $f(a) = f(b)$.

Then there exist $c \,\epsilon\, (a, b)$, such that $(T_\alpha f\ )(c) = 0$ .

***Proof.*** By conditions $(i)$ and $(iii)$ we have $c \,\epsilon\, (a, b)$, which is the extrema point. Assume that c is a local minimum point with no loss of generality. So,

$$(T_\alpha f)(c) = \lim_{\varepsilon \to 0^+} \frac{f\left(c + \varepsilon c^{1-\alpha}\right) - f(c)}{\varepsilon} = \lim_{\varepsilon \to 0^-} \frac{f\left(c + \varepsilon c^{1-\alpha}\right) - f(c)}{\varepsilon} \tag{2.11}$$

hence $(T_\alpha f)(c) = 0$ because the limit is non-positive as $\varepsilon \to 0^-$ and is non-negative as $\varepsilon \to 0^+$.

**Theorem 2.1.8. (Mean value theorem to conformable differentiable functions)** [19,21]

Let $a > 0$ and $f: (0, \infty) \to \mathbb{R}$ be a given function that satisfy the following conditions

i. $f$ is continuous on interval $(a, b)$.

ii. $f$ is $\alpha$-differentiable function for some $\alpha \in (0,1)$.

Then there exist the number $c \epsilon (a, b)$, such that

$$(T_\alpha f)(c) = \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}.$$

***Proof.*** Take the function

$$g(c) = f(x) - f(\alpha) - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right) \quad (2.12)$$

$$(T_\alpha g)(x) = (T_\alpha f)(x) - 0 - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}, \quad (2.13)$$

because the function g fulfills Rolle's theorem assumptions, then there is the number $c \epsilon (a, b)$ such that $(T_\alpha g)(c) = 0$, so

$$(T_\alpha f)(c) = \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}. \quad (2.14)$$

**Note:** If $f$ is a differentiable function with $0 < \alpha < 1$, so that $(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(x)$ and $\left({}_\alpha^b T f\right)(t) = -(b-t)^{1-\alpha} f'(t)$. The conformable derivative of constant function is zero. Likewise, if $(T_\alpha^a f)(t) = 0$ on interval $(a, b)$ so, by the mean value theorem of conformable differentiable function for all $t \epsilon (a, b)$ we can easily show that $f(t) = 0$, the conformable mean value theorem can also be used where we can prove that if the conformable derivative of the function $f$ on interval $(a, b)$ is positive (negative) then the graph of $f$ is increasing (decreasing).

**Theorem 2.1.9.** [19] Assume that $f: [a, b] \to \mathbb{R}$ be a given function, $a > 0$ and $0 < \alpha \leq 1$, if the function $f$ satisfy the following conditions

i. $f$ is continuous on interval $[a, b]$.
ii. $f$ is α-differentiable on interval $(a, b)$.

Then we can say that

a) If $(T_\alpha f)(t) < 0$ for all $t \in (a, b)$, then $f$ is decreasing on $[a, b]$.
b) If $(T_\alpha f)(t) > 0$ for all $t \in (a, b)$, then $f$ are increasing on $[a, b]$.

**Lemma 2.1.10.** Suppose that $f, h: [a, \infty) \to R$ be given functions such that $T_\alpha^a$ is exists for every $a < t$. If $f$ is a differentiable function on the interval $(a, \infty)$ and $T_\alpha^a f(t) = (t-a)^{1-\alpha} h(t)$, so that $h(t) = \frac{df}{dt}$ for all $t > 0$.

The foundation for the proof is the definition and putting in place $h = \varepsilon(t-a)^{1-\alpha}$ then $h \to 0$ as $\varepsilon \to 0$. We can state this lemma as a result.

**Theorem 2.1.11. (Chain Rule)** [17] Suppose that $\alpha \in (0,1]$ and $g, f: [a,\infty) \to \mathbb{R}$ left $\alpha$-differentiable functions. If we assume $h(t) = f\big(g(t)\big)$. So $h(t)$ become left $\alpha$-differentiable function and for all values of $t$ when $t \neq 0$ and the function $g(t) \neq 0$ we obtain

$$(T_\alpha^a h)(t) = (T_\alpha^a f)\big(g(t)\big) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1} \tag{2.15}$$

and if $t = a$, then

$$(T_\alpha^a h)(a) = \lim_{t \to a^+} (T_\alpha^a f)\big(g(t)\big) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1}. \tag{2.16}$$

***Proof***. Since the function g is a continuous and by putting $u = t + \varepsilon(t-a)^{1-\alpha}$ in definition then we get

$$(T_\alpha^a h)(t) = (t-a)^{1-\alpha} \frac{d}{dt}[\mathrm{h(t)}]$$

$$= \lim_{u \to t} \frac{f\big(g(u)\big) - f(g(t))}{(u-t)} (t-a)^{1-\alpha}$$

$$= \lim_{u \to t} \frac{f\big(g(u)\big) - f(g(t))}{g(u) - g(t)} \cdot \lim_{u \to t} \frac{g(u) - g(t)}{(u-t)} (t-a)^{1-\alpha}$$

$$= \lim_{g(u) \to g(t)} \frac{f\big(g(u)\big) - f(g(t))}{g(u) - g(t)} \cdot g(t)^{1-\alpha} . T_\alpha^a g(t) \cdot g(t)^{\alpha-1}$$

$$= (T_\alpha^a f)\big(g(t)\big) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1}.$$

**Theorem 2.1.12.** [19] Assume $\alpha, \beta$ are two positive constant numbers, such that $0 < \alpha, \beta < 1$, and $f$ be a given function (non-constant) two times differentiable on an open interval $(a, b)$, $a, b \in \mathbb{R}$, The conformable derivatives must follow the rules below.

$$T_\alpha[T_\beta f(x)] \neq T_\beta[T_\alpha(f(x)]. \tag{2.17}$$

***Proof.*** We may get the following relationship from the definition:

$$T_\alpha\big[T_\beta f(x)\big] = T_\alpha\big[x^{1-\beta}\big] f'(x) + x^{1-\beta} T_\alpha[(f'(x)]$$

$$= x^{1-\beta-\alpha}[(1-\beta)f'(x) + x f''(x)] \tag{2.18}$$

$$T_\beta[T_\alpha f(x)] = T_\beta[x^{1-\alpha}] f'(x) + x^{1-\alpha} T_\beta[f'(x)] = x^{1-\beta-\alpha}[(1-\alpha)f'(x) + x f''(x)].$$

As a result, the proof is finished.

**Proposition 2.1.13.** Suppose that $0 < \alpha, \beta \leq 1$ and the function $f: [a, \infty) \to \mathbb{R}$ such that $1 < \alpha + \beta \leq 2$. If $f$ is twice differentiable on $(a, \infty)$ so that

$$(T_\alpha^a T_\beta^a f)(t) = T_{\alpha+\beta}^a f(t) + (1-\beta)(t-\alpha)^{-\beta} T_\alpha^a f(t). \tag{2.19}$$

***Proof.*** The function $f$ is twice differentiable and by conformable product rule we see that

$$\begin{aligned}(T_\alpha^a T_\beta^a f)(t) &= (t-a)^{1-\alpha} \frac{d}{dt}\left[T_\beta^a f(t)\right] \\ &= (t-a)^{1-\alpha} \frac{d}{dt}\left[(t-a)^{1-\beta} f'(t)\right] \\ &= (t-a)^{1-\alpha}[(t-a)^{1-\beta} f''(t) + (1-\beta)(t-a)^{-\beta} f'(t)] \\ &= T_{\alpha+\beta}^a f(t) + (1-\beta)(t-\alpha)^{-\beta} T_\alpha^a f(t).\end{aligned}$$

We also have $(T_\alpha^a T_\beta^a f)(t) = T_2 f(t) = f''(t)$ when $\alpha, \beta \to 1$.

In this section, for a few smooth functions, we calculate the conformable derivative at a number $a$ in the left case and a number $b$ in the right case. Suppose $\alpha \in (n-1, n)$, $n$ is a natural number and $f: [a, \infty) \to \mathbb{R}$ be given such that $f^{(n)}(t)$ exists and continuous. Then, by definition 2.1.4 we can see this,

$$\begin{aligned}T_\alpha^a f(t) &= T_{\alpha+1-n}^a f^{(n-1)}(t) \\ &= (t-a)^{n-\alpha} f^{(n)}(t) \end{aligned} \tag{2.20}$$

and

$$T_\alpha^a f(a) = \lim_{t\to a^+} (t-a)^{n-\alpha} f^{(n)}(t) = 0.$$

Also, from the right case when $f: (-\infty, b) \to \mathbb{R}$ is a function such that $f^{(n)}(t)$ exists and continuous, then we have

$$\left({}_\alpha^b T f\right)(b) = \lim_{t\to b^+} (b-t)^{n-\alpha} f^{(n)}(t) = 0.$$

Now, assume $\alpha \in (0,1]$ and $n$ is positive integer number, then the sequential conformable derivative with order n on the left side is defined as follows

$${}^{(n)}T_\alpha^a f(t) = T_\alpha^a T_\alpha^a \dots T_\alpha^a f(f), \quad n - times \tag{2.21}$$

also, in the right case defined by

$${}_\alpha^b T^{(n)} f(t) = {}_\alpha^b T\, {}_\alpha^b T \dots {}_\alpha^b T f(t), \quad n - times. \tag{2.22}$$

Suppose that $f:[a,\infty)\to\mathbb{R}$ is a second continuously differentiable and $\alpha\in(0,\frac{1}{2}]$, the collections on the other hand, show that

$$T_{\alpha}^{a}T_{\alpha}^{a}f(t)=\begin{cases}(1-\alpha)(t-a)^{1-2\alpha}f'(t)+(t-a)^{2-2\alpha}f''(t), & for \quad t>a,\\ 0, & for \quad t=a.\end{cases} \tag{2.23}$$

Also, when the function $f:(-\infty,b)\to\mathbb{R}$ is second continuously differentiable and $\alpha\in(0,\frac{1}{2}]$, then the direct collections in the right case show that

$${}_{\alpha}^{b}T\,{}_{\alpha}^{b}Tf(t)=\begin{cases}(1-\alpha)(b-t)^{1-2\alpha}f'(t)+(b-t)^{2-2\alpha}f''(t), & if \quad b>t,\\ 0, & if \quad b=t.\end{cases} \tag{2.24}$$

As a result of this procedure, we are notified that the second-order sequential conformable derivative is not possible to be continuous function when $f$ is second continuously differentiable function for $\alpha\in(\frac{1}{2},1)$. If we consider an inductive step, then we can obtain $n^{th}$-order sequential conformable derivative function which is continuous and disappears at the endpoints $b$ in the right case and $a$ in the left case when $f$ is $n$-continuously differentiable function for $\alpha\in(0,\frac{1}{n}]$.

**Theorem 2.1.14.** [19] Suppose that $f$, $g$ are two differentiable functions, as a result $f$ is differentiable function at any $g(t)$ and $g$ is differentiable at any $t$, the conformable derivative follow chain rule, it means

$$T_{\alpha}(fog(x))=x^{1-\alpha}g(x)^{1-\alpha}g'(x)T_{\alpha}(f(t))|_{t=g(x)}.$$

***Proof.*** Assume that $g$ is differentiable function at any $t$, and $f$ is differentiable function at any $g(t)$, so that by definition, we get

$$T_{\alpha}[(fog)(x)]=\lim_{\varepsilon\to 0}\frac{(fog)(x+\varepsilon x^{1-\alpha})-(fog)(x)}{\varepsilon} \tag{2.25}$$

putting $k=\varepsilon x^{1-\alpha}$ , we get

$$T_{\alpha}[(fog)(x)]=x^{1-\alpha}\lim_{k\to 0}\frac{(fog)(x+k)-(fog)(x)}{k} \tag{2.26}$$

with

$$=\lim_{k\to 0}\frac{(fog)(x+k)-(fog)(x)}{k}.$$

The derivative of a composite function, if we use it the chain rule for the ordinary derivative, we obtain

$$\lim_{k\to 0}\frac{(fog)(x+k)-(fog)(x)}{k}=\lim_{k\to 0}\frac{f(g(x)+k)-f(g(x))}{k} \ . \tag{2.27}$$

We also have $k = \varepsilon g(x)^{1-\alpha}$.

The equation above can be rewritten as follows

$$\lim_{k\to 0}\frac{(fog)(x+k)-(fog)(x)}{k} = g(x)^{1-\alpha}\lim_{\varepsilon\to 0}\frac{f\big(g(x)+\varepsilon g(x)^{1-\alpha}\big)-f(g(x))}{\varepsilon} \tag{2.28}$$

replacing (2.28) in (2.26) we get the following

$$T_\alpha\big((fog)(x)\big) = x^{1-\alpha}g(x)^{1-\alpha}\lim_{\varepsilon\to 0}\frac{f\big(g(x)+\varepsilon g(x)^{1-\alpha}\big)-f(g(x))}{\varepsilon} \tag{2.29}$$

with of course

$$T_\alpha\big(f(t)\big)|_{t=g(x)} = \lim_{\varepsilon\to 0}\frac{f\big(g(x)+\varepsilon g(x)^{1-\alpha}\big)-f(g(x))}{\varepsilon}.$$

This completes the proof.

## 2.2. Conformable integral

**Definition 2.2.1.** [17,22] Suppose that $f:(0,\infty)\to\mathbb{R}$ is a function, so that the left conformable integral of a function $f$ of order $\alpha$ where $0<\alpha\leq 1$ is defined as follows

$$I_\alpha^a f(t) = \int_a^t f(x)d_\alpha(x,a).$$

If $a = 0$ then in $I_\alpha^a$ and $d_\alpha(x,a)$ we can write $I_\alpha$ and $d_\alpha x$ .

$$= \int_a^t (x-a)^{\alpha-1}f(x)\,dx. \tag{2.30}$$

**Definition 2.2.2.** [15] Assume that $f:(0,\infty)\to\mathbb{R}$ be a given function then the right conformable integral of a function $f$ of order $\alpha$ where $0<\alpha\leq 1$ is defined by

$${}_\alpha^b I f(t) = \int_t^b f(x)d_\alpha(b,x)$$

$$= \int_t^b (b-x)^{\alpha-1}f(x)\,dx. \tag{2.31}$$

**Theorem 2.2.3**. [22] Assume that $f:(0,\infty)\to\mathbb{R}$ be a continuous function and $0<\alpha\leq 1$, then for every $a<t$

$$T_\alpha^a I_\alpha^a f(t) = f(t). \tag{2.32}$$

***Proof.*** since $f$ is continuous so $I_\alpha^a f(t)$ is a differentiable function, hence

$$T_\alpha^a(I_\alpha^a f)(t) = (t-a)^{1-\alpha}\frac{d}{dt}(I_\alpha^a f)(t)$$

$$= (t-a)^{1-\alpha} \frac{d}{dt} \int_a^t (x-a)^{\alpha-1} f(x)\, dx$$

$$= (t-a)^{1-\alpha} (x-a)^{\alpha-1} f(t)$$

$$= f(t).$$

**Theorem 2.2.4.** Suppose that $f: (0, \infty) \to \mathbb{R}$ be a continuous function and $0 < \alpha \leq 1$, then for every $t > a$ we have

$$ {}_{\alpha}^{b}T\, {}_{\alpha}^{b}I f(t) = f(t). \tag{2.33}$$

***Proof.*** Since $f$ is a continuous function, then ${}_{\alpha}^{b}I f(t)$ is differentiable. Hence,

$${}_{\alpha}^{b}T\big({}_{\alpha}^{b}I(f)\big)(t) = t^{1-\alpha} \frac{d}{dt}\, {}_{\alpha}^{b}I(f(t))$$

$$= t^{1-\alpha} \frac{d}{dt} \int_b^t \frac{f(x)}{x^{1-\alpha}} dx$$

$$= t^{1-\alpha} \frac{f(t)}{t^{1-\alpha}}$$

$$= f(t).$$

**Definition 2.2.5.** [17] Assume $\beta = \alpha - n$ and $f$ be a given function, so that the left fractional integral starting from $a > 0$ of a function $f$ of order $n < \alpha \leq n+1$ is defined by

$$I_{\alpha}^{a} f(t) = I_{n+1}^{a}((t-a)^{\beta-1} f) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x) dx \tag{2.34}$$

be careful of if $\alpha = n+1$ then we get $\beta = 1$, As a result, the iterative integral of $f$ could be determined by using Cauchy formula with $n+1$ times over $(a, t]$ is given by

$$I_{\alpha}^{a} f(t) = I_{n+1}^{a} f(t) = \frac{1}{n!} \int_a^t (t-x)^n f(x) dx. \tag{2.35}$$

Remember that the left Riemann-Liouville conformable integral of order any positive number $\alpha$ which start from $a$ is defined as follows

$$I_{\alpha}^{a} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{2.36}$$

Notice that $\Gamma(n+1) = n!$ for $n$ is a natural number then from eq. (2.32) and (2.33), we can see this

$$I_{\alpha}^{a} f(t) = I_{\alpha}^{a} f(t)$$

where $\alpha = n+1$ for $n = 0, 1, 2, \ldots$

**Application 2.2.6.** We have

$$(I_{\alpha}^{a}(t-a)^{\mu-1})(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)}(x-a)^{\alpha+\mu-1} \tag{2.37}$$

where $\alpha,\mu>0$, so that we can easily calculate the left conformable fractional integral of $(t-a)^{\mu}$ of order $\alpha\in(n,n+1]$ and $\mu\in\mathbb{R}$ such that $\alpha+\mu-n>0$ as

$$I_{\alpha}^{a}(t-a)^{\mu}(x)=I_{n+1}^{a}(t-a)^{\mu+\alpha-n-1}(x)=\frac{\Gamma(\alpha+\mu-n)}{\Gamma(\alpha+\mu+1)}(x-a)^{\alpha+\mu}$$

likewise, the right conformable fractional integral of $(t-a)^{\mu}$ of order $\alpha\in(n,n+1]$ and $\mu\in\mathbb{R}$ such that $\alpha+\mu-n>0$ where $\alpha,\mu>0$ is given by

$${}_{\alpha}^{b}I(b-t)^{\mu}(x)={}_{n+1}^{b}I(b-t)^{\mu+\alpha-n-1}(x)=\frac{\Gamma(\alpha+\mu-n)}{\Gamma(\alpha+\mu+1)}(b-x)^{\alpha+\mu}.$$

From the explanation above, the conformable fractional integrals of polynomial functions and the Riemann fractional integrals and are different from a constant multiple and correspond with natural orders.

**Proposition 2.2.7.** [17] If $0<\alpha,\mu\leq 1$ and $f:[a,\infty)\to\mathbb{R}$ be a given function, such that $1<\alpha+\mu\leq 2$. Then

$$I_{\alpha}I_{\mu}f(t)=\frac{t^{\mu}}{\mu}(I_{\alpha}f)(t)+\frac{1}{\mu}\left(I_{\alpha+\mu}f\right)(t)-\frac{t}{\mu}\int_{0}^{t}s^{\alpha+\mu-2}f(s)ds. \tag{2.38}$$

***Proof.*** Interchange the integral order and note that

$$I_{\alpha+\mu}f(t)=\left(I_{2}s^{\alpha+\mu-2}f(s)\right)(t) \tag{2.39}$$

$$=\int_{0}^{t}(t-s)s^{\alpha+\mu-2}f(s)ds$$

then, we see that

$$I_{\alpha}I_{\mu}f(t)=\int_{0}^{t}\left[\int_{0}^{t_1}f(s)s^{\alpha-1}ds\right]t_1{}^{\mu-1}dt_1$$

$$=\int_{0}^{t}f(s)s^{\alpha-1}\left[\int_{s}^{t}t_1{}^{\mu-1}dt_1\right]ds$$

$$=\int_{0}^{t}f(s)s^{\alpha-1}\left[\frac{t^{\mu}}{\mu}-\frac{s^{\mu}}{\mu}\right]ds$$

$$=\int_{0}^{t}\frac{t^{\mu}}{\mu}s^{\alpha-1}f(s)\,ds-\int_{0}^{t}\frac{s^{\mu}}{\mu}s^{\alpha-1}f(s)\,ds$$

$$=\int_{0}^{t}\frac{t^{\mu}}{\mu}s^{\alpha-1}f(s)\,ds+\int_{0}^{t}\frac{t}{\mu}s^{\alpha+\mu-2}f(s)\,ds-\int_{0}^{t}\frac{t}{\mu}s^{\alpha+\mu-2}f(s)\,ds-\int_{0}^{t}\frac{s^{\mu}}{\mu}s^{\alpha-1}f(s)\,ds$$

$$= \int_0^t \frac{t^\mu}{\mu} s^{\alpha-1} f(s)\, ds + \frac{1}{\mu}\left[\int_0^t t s^{\alpha+\mu-2} f(s)\, ds - \int_0^t s^{\alpha+\mu-1} f(s)\, ds - t\int_0^t s^{\alpha+\mu-2} f(s)\, ds\right]$$

$$= \frac{t^\mu}{\mu}(I_\alpha f)(t) + \frac{1}{\mu}\left[\left(I_{\alpha+\mu} f\right)(t) - t\int_0^t s^{\alpha+\mu-2}\, f(s) ds\right].$$

**Lemma 2.2.8.** [17,22] suppose that $f\colon [a,\infty) \to \mathbb{R}$ be a function such that $f^n(t)$ is continuous and $n < \alpha \leq n+1$ where $\beta = \alpha - n$, then for all $t > a$ we have

$$T_\alpha^a I_\alpha^a f(t) = f(t). \tag{2.40}$$

***Proof.*** By definition of the conformable derivative and integral we get

$$T_\alpha^a I_\alpha^a f(t) = T_\beta^a [\frac{d^n}{dt^n} I_\alpha^a f(t)]$$

$$= T_\beta^a [\frac{d^n}{dt^n} I_{n+1}^a ((t-a)^{\beta-1} f(t))].$$

There are derivatives of order $n$ and integral with order $n+1$ where $n$ is positive integer, so

$$T_\alpha^a I_\alpha^a f(t) = T_\beta^a [I_1^a ((t-a)^{\beta-1} f(t))].$$

Since $\beta = \alpha - n$ where $n < \alpha \leq \mathrm{n} + 1$, $n$ belongs to natural numbers and for $0 < \beta \leq 1$ so that

$$I_1^a [(t-a)^{\beta-1} f(t)] = I_\beta^a f(t)$$

$$T_\alpha^a I_\alpha^a f(t) = T_\beta^a I_\beta^a f(t).$$

Hence by Theorem 2.2.3, we get $T_\alpha^a I_\alpha^a f(t) = f(t)$.

**Lemma 2.2.9.** Suppose that $f\colon (-\infty, b] \to \mathbb{R}$ be a given function and $n < \alpha \leq n+1$ such that $f^n(t)$ is a continuous function, where $\beta = \alpha - n$ then for every $b > t$ we have

$${}_\alpha^b T\, {}_\alpha^b I f(t) = f(t).$$

We can prove the previous lemma.

**Corollary 2.2.10.** Assume that $f\colon [a,b) \to \mathbb{R}$ be a given function. If $I_\alpha^a T_\alpha^a f(t)$ exists for $a < t < b$, then $f(t)$ is differentiable on the open interval $(a,b)$.

**Lemma 2.2.11.** [8] Suppose that $\boldsymbol{f}\colon [a,b) \to \mathbb{R}$ be a differentiable function, and $0 < \alpha \leq 1$ , then for every $t > 0$

$$I_\alpha^a T_\alpha^a f(t) = f(t) - f(a). \tag{2.41}$$

***Proof.*** From the Theorem 2.1.7 and Definition 2.2.1, since $f$ is differentiable then we have

$$I_\alpha^a T_\alpha^a f(t) = \int_a^t (x-a)^{a-1} T_\alpha(f)(x)\, dx$$

$$= \int_a^t (x-a)^{a-1}(x-a)^{1-a} f'(x)\, dx$$

$$= f(t) - f(a).$$

**Proposition 2.2.12.** Let $f: [a,b) \to \mathbb{R}$ is $(n+1)$ times differentiable function and $\alpha \in (n, n+1]$, then for every $t > a$ we have

$$I_\alpha^a T_\alpha^a f(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}. \tag{2.42}$$

***Proof.*** By definition of conformable derivative and conformable integral we get

$$I_\alpha^a T_\alpha^a f(t) = I_{n+1}^a[(t-a)^{\beta-1} T_\beta^a f^{(n)}(t)]$$

$$= I_{n+1}^a[(t-a)^{\beta-1}(t-a)^{1-\beta} f^{(n+1)}(t)]$$

$$= I_{n+1}^a f^{(n+1)}(t)$$

$$= f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}.$$

Similarly, we have the right case of the form, which is as follows.

**Proposition 2.2.13.** Suppose that $f: [a,b) \to \mathbb{R}$ be $(n+1)$ times differentiable function and $n < \alpha \leq n+1$, then for every $a < t$ we have

$$^{b}_{\alpha}I\, ^{b}_{\alpha}T(f)(t) = f(t) - \sum_{k=0}^{n} \frac{(-1)^k f^{(k)}(b)(b-t)^k}{k!} \tag{2.43}$$

but when $n = 0$ or $0 < \alpha \leq 1$, then

$$^{b}_{\alpha}I\, ^{b}_{\alpha}T(f)(t) = f(t) - f(b). \tag{2.44}$$

**Theorem 2.2.14. (Conformable integration by parts)** [17,19] Suppose that $f$, $g$ are two differentiable functions and $f, g: [a,b] \to \mathbb{R}$, then

$$\int_a^b f(x)\, T_\alpha^a(g)(x) d_\alpha(x,a) = fg|_a^b - \int_a^b g(x)\, T_\alpha^a(f)(x) d_\alpha(x,a). \tag{2.45}$$

**Proof.** From the Definition 2.1.1 and Definition 2.2.1 we can say that

$$\int_a^b f(x)\, T_\alpha^a(g)(x) d_\alpha(x,a) = \int_a^b (x-a)^{\alpha-1} f(x)\, (\mathrm{x}-a)^{1-\alpha} g'(x) dx \tag{2.46}$$

by integration by parts we have

$$= \int_a^b f(x)\, g'(x) dx \tag{2.47}$$

$$\int_a^b f(x)\, T_\alpha^a(g)(x) d_\alpha(x,a) = fg|_a^b - \int_a^b g(x)\, f'(x) dx$$

$$= fg|_a^b - \int_a^b (\mathrm{x}-a)^{1-\alpha}(x-a)^{\alpha-1} g(x)\, f'(x) dx$$

$$= fg|_a^b - \int_a^b g(x)\, T_\alpha^a(f)(x) d_\alpha(x,a).$$

## 2.3. Fractional power series expansion

We construct fractional power series expansions for functions which the Taylor power series expansion is not accessible because they are not continuously differentiable at some positions.

**Theorem 2.3.1.** [17,23] Let $0 < \alpha \leq 1$ and $f$ is an infinitely $\alpha$-differentiable function at a neighborhood of a point $t_0$, so that the power series of the function $f$ is defined by

$$f(t) = \sum_{k=0}^{\infty} \frac{(T_\alpha^{t_0} f)^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!}, \tag{2.48}$$

where $t_0 < t < t_0 + R^{\frac{1}{\alpha}}, R > 0$ and $(T_\alpha^{t_0})^{(k)}(t_0)$ this is k times of the fractional derivatives have been applied.

***Proof.*** Let for $t_0 < t < t_0 + R^{\frac{1}{\alpha}}, R > 0$ the function $f$ is defined by

$$f(t) = c_0 + c_1(t-t_0)^\alpha + c_2(t-t_0)^{2\alpha} + c_3(t-t_0)^{3\alpha} + \cdots$$

then $f(t_0) = c_0$. Taking $T_\alpha^{t_0}$ of $f$ and evaluate at $t_0$ we get

$$\left(T_\alpha^{t_0} f\right)(t_0) = c_1 \alpha \Longrightarrow c_1 = \frac{\left(T_\alpha^{t_0} f\right)(t_0)}{\alpha}$$

$$\left(T_\alpha^{t_0} T_\alpha^{t_0} f\right)(t_0) = 2\alpha^2 c_2 \Longrightarrow c_2 = \frac{\left(T_\alpha^{t_0} T_\alpha^{t_0} f\right)(t_0)}{2\alpha^2}$$

$$\left(T_\alpha^{t_0} T_\alpha^{t_0} T_\alpha^{t_0} f\right)(t_0) = 6\alpha^3 c_3 \Longrightarrow c_3 = \frac{\left(T_\alpha^{t_0} T_\alpha^{t_0} T_\alpha^{t_0} f\right)(t_0)}{6\alpha^3} \dots .$$

Inductively proceed and applying $T_\alpha^{t_0}$ of function $f$, $n$-times and evaluating at $t_0$ we get

$$c_n = \frac{\left(T_\alpha^{t_0} f\right)^{(n)}(t_0)}{\alpha^n n!}.$$

Therefore (2.48) is obtained and the proof is finished.

**Proposition 2.3.2. (Fractional Taylor inequality**) [17] Suppose that $f$ is $\alpha$-differentiable function at a neighborhood of a point $t_0$ and has a Taylor power series representation (2.48), and $\alpha \in (0,1]$ such that $|(T_\alpha^a f)^{n+1}| \leq M,\ \ M > 0$ for some $n$ belongs to natural numbers. If

$$R_n^\alpha(t) = \sum_{k=n+1}^{\infty} \frac{(T_\alpha^{t_0})^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!}$$

$$= f(x) - \sum_{k=0}^{n} \frac{(T_\alpha^{t_0})^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!}$$

then we have

$$|R_n^\alpha(t)| \leq \frac{M}{\alpha^{n+1}(n+1)!}(t-t_0)^{\alpha(n+1)}.$$

**Application 2.3.3.** [17-23] Let the fractional exponential function $f(t) = e^{\frac{(t-t_0)^\alpha}{\alpha}}\ and$ for $\alpha \in (0,1)$ the function $f(t)$ does not have a Taylor power series representation about $t_0$ because it is not differentiable at a point $t_0$. We know that $\left(T_\alpha^{t_0} f\right)^{(n)}(t_0) = 1$ for every $n$, therefore

$$e^{\frac{(t-t_0)^\alpha}{\alpha}} = \sum_{k=0}^{\infty} \frac{(t-t_0)^{k\alpha}}{\alpha^k k!}. \tag{2.49}$$

This series converges to $f$ on intervals, as shown by the ratio test $[t_0, \infty)$.

**Application 2.3.4.** Suppose that $f(t) = sin\frac{(t-t_0)^\alpha}{\alpha}$ and $g(t) = cos\frac{(t-t_0)^\alpha}{\alpha}$, know that $f$, $g$ do not have Taylor power series about $t = t_0$ for $\alpha \in (0,1)$ because they are not differentiable functions there, by the help of (2.48),

$$T_\alpha^{t_0} sin\frac{(t-t_0)^\alpha}{\alpha} = cos\frac{(t-t_0)^\alpha}{\alpha}$$

$$T_\alpha^{t_0} cos\frac{(t-t_0)^\alpha}{\alpha} = -sin\frac{(t-t_0)^\alpha}{\alpha}$$

we have

$$sin\frac{(t-t_0)^\alpha}{\alpha} = \sum_{k=0}^{\infty}(-1)^k \frac{(t-t_0)^{(2k+1)\alpha}}{\alpha^{(2k+1)}(2k+1)!}, \quad t_0 \leq t < \infty \tag{2.50}$$

and

$$cos\frac{(t-t_0)^\alpha}{\alpha} = \sum_{k=0}^{\infty}(-1)^k \frac{(t-t_0)^{(2k)\alpha}}{\alpha^{(2k)}(2k)!}, \qquad t_0 \leq t < \infty\,. \tag{2.51}$$

**Application 2.3.5.** Assume that $0 < \alpha < 1$ then the function $f(t) = \frac{1}{1-\frac{1}{\alpha}t^{\alpha}}$ is not differentiable at $t = 0$ , hence it does not have Taylor power series about $t = 0$. Now with the help of (2.48), we can see that

$$\frac{1}{1-\frac{t^{\alpha}}{\alpha}} = \sum_{k=0}^{\infty} t^{\alpha k}, \qquad 0 \le t < 1. \tag{2.52}$$

For more generally we get

$$\frac{1}{1-\frac{(t-t_0)^{\alpha}}{\alpha}} = \sum_{k=0}^{\infty} (t-t_0)^{\alpha k}, \qquad t_0 \le t < t_0 + 1. \tag{2.53}$$

**Remark 2.3.6.** In case $f$ is not differentiable function at $a$ where $f$ is a function such that defined on $(-\infty, a)$, then for some $\alpha \in (0,1]$ We want to find its conformable right fractional-order derivatives ${}_{\alpha}^{a}T$ at $a$ and apply it for fractional Taylor series on some open interval $(a - R, a)$, where $R > 0$. For example, the functions $\frac{(t-t_0)^{\alpha}}{\alpha}$ and $\sin\left(\frac{(t-t_0)^{\alpha}}{\alpha}\right)$ and so on.

# 3. MATHEMATICAL MODELING OF HEATING AND COOLING OF BUILDINGS USING LAPLACE TRANSFORM

In this chapter, we consider some conformable applications of Laplace transform

$$\mathcal{L}_{\alpha}^{t_0}\{f(t)\}(s) = \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)}{\alpha}} f(t)(t-t_0)^{\alpha-1} \tag{3.1}$$

where $t_0 \in \mathbb{R}$ $and$ $0 < \alpha < 1$, we apply this definition to identify a solution of Laplace conformable differential equation of order $\alpha$.

## 3.1. Laplace Transform

**Definition 3.1.1.** [24] Suppose that $f(t)$ be a function on the interval $[0, \infty)$, then the Laplace transform of $f$ is the function $F$ defined by

$$F(s) = \int_0^{\infty} e^{-st} f(t)dt. \tag{3.2}$$

The domain of the function $F(s)$ is all the values of $s$ for when the integral in (3.2) exists. The Laplace transform of $f$ is denoted by both $\mathcal{L}\{f\}$ and $F$.

**Theorem 3.1.2.** [7] Let $f$, $f_1$ $and$ $f_2$ are functions which Laplace transform exists for $\alpha < s$ and assume that $c$ be a constant. Then for $s > \alpha$ ,

1) $\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$. (3.3)

2) $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$, c is constant. (3.4)

***Proof.*** We can use the linearity properties of integration, we get for $s > \alpha$

$$\mathcal{L}\{f_1 + f_2\}(s) = \int_0^{\infty} e^{-st} [f_1(t) + f_2(t)]dt$$

$$= \int_0^{\infty} e^{-st} f_1(t)dt + \int_0^{\infty} e^{-st} f_2(t)dt$$

$$= \mathcal{L}\{f_1\}(s) + \mathcal{L}\{f_2\}(s).$$

Hence, equation (3.3) is satisfied. Similarly, we can see this.

$$\mathcal{L}\{cf\}(\mathrm{s}) = \int_0^{\infty} e^{-st} (cf(t))dt = c\int_0^{\infty} e^{-st} f(t)dt.$$

$$= c\,\mathcal{L}\{f\}(s).$$

**Theorem 3.1.3.** If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, so that

$$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)\,. \tag{3.5}$$

***Proof.*** We can calculate

$$\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^{\infty} e^{-st}e^{at}f(t)dt$$

$$= \int_0^{\infty} e^{-(s-a)t}f(t)dt$$

$$= F(s-a).$$

**Theorem 3.1.4.** [24] Assume that $f'(t)$ be piecewise continuous function on the interval $[0,\infty)$ and $f(t)$ is a continuous function on the interval $[0,\infty)$, with both of exponential order $\alpha$ . Then, for $s > \alpha$

$$\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0). \tag{3.6}$$

***Proof.*** Since $\mathcal{L}\{f'\}\ exists$, we can utilize integration by parts with ($u = e^{-st}$ and $dv = f'(t)dt$) to have

$$\mathcal{L}\{f'\}(s) = \int_0^{\infty} e^{-st}f'dt = \lim_{N\to\infty}\int_0^{N} e^{-st}f'(t)dt$$

$$= \lim_{N\to\infty}\left[e^{-st}f(t)|_0^N + s\int_0^{\infty} e^{-st}f(t)dt\right]$$

$$= \lim_{N\to\infty} e^{-sN}\ f(N) - f(0) + s\lim_{N\to\infty}\int_0^{N} e^{-st}f(t)dt$$

$$= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\,\mathcal{L}\{f\}(s)$$

to investigate $\lim_{N\to\infty} e^{-sN}\ f(N)$, we have noticed because the function $f(t)$ is of exponential order $\alpha$, there exists a constant $M$ such that for $N$ large,

$$\left|e^{-sN}f(N)\right| \le e^{-sN}Me^{\alpha N} = Me^{-(s-\alpha)N}$$

consequently, for $s > \alpha$,

$$0 \le \lim_{N\to\infty}\left|e^{-sN}f(N)\right| \le \lim_{N\to\infty} Me^{-(s-\alpha)N} = 0.$$

So, $\lim_{N\to\infty} e^{-sN}f(N) = 0.$

For $s > \alpha$ equation (3.6) reduced to

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

**Table 3.1.** Basic functions of Laplace transform [17,25]

| $f(t)$ | $\mathcal{L}\{f\}(s) = \mathrm{F(s)}$ | |
|---|---|---|
| $1$ | $\frac{1}{s}$, | $s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}$ | $s > a$ |
| $t^n$, $\quad n = 1,2,3, \ldots$ | $\frac{n!}{s^{n+1}}$ | $s > 0$ |
| $\sin\ bt$ | $\frac{b}{s^2+b^2}$ | $s > 0$ |
| $\cos bt$ | $\frac{s}{s^2+b^2}$ | $s > 0$ |
| $e^{at}t^n$, $\quad n = 1,2,3 \ldots$ | $\frac{n!}{(s-a)^{n+1}}$ | $s > a$ |
| $e^{at} \sin bt$ | $\frac{b}{(s-a)^2+b^2}$ | $s > a$ |
| $e^{at} \cos bt$ | $\frac{s-a}{(s-a)^2+b^2}$ | $s > a$ |

**Definition 3.1.5.** Assume that $F(s)$ be a given function, if there is a function $f(t)$ which is continuous on $[0, \infty)$ and satisfies $\mathcal{L}\{f\} = F$. Then we can say that $f(t)$ is the inverse Laplace transform of $F(s)$ and employ the notation $f = \mathcal{L}^{-1}\{F\}$.

## 3.2. Energy-saving model by using differential Laplace transform

Our objective in this section is to produce a mathematical model that explains the 24-hour temperature profile in the buildings as a function of outside temperature, the heat generated inside the structure, furnace and air conditioner heating or cooling. The model that was identified is defined as follows

$$T'(t) = K\{M(t) - T(t)\} + H(t) + F(t) \qquad\qquad T(0) = T_0 \qquad (3.7)$$

where the inside temperature of a building at time $t$ is denoted by $T(t)$, $H(t)$is the additional heating rate, $F(t)$ is heating (or cooling) generated by the furnace (or air conditioner), the physical properties of the building is a positive constant $K$, and $M(t)$ is the outside temperature of the building.

We consider the Laplace transform of equation (3.7)

$$\mathcal{L}\{T'(t)\} = \mathcal{L}(K\{M(t) + T(t)\} + H(t) + F(t))$$

$$s\mathcal{L}\{T(t)\} - T(0) = -K\mathcal{L}\{T(t)\} + \mathcal{L}\{KM(t) + H(t) + F(t)\}$$

$$s\mathcal{L}\{T(t)\} + K\mathcal{L}\{T(t)\} = \mathcal{L}\big(KM(t) + H(t) + F(t)\big) + T_0.$$

Let $G(t) = K[M(t) + H(t) + F(t)]$

$$\mathcal{L}\{T(t)\} = \frac{\mathcal{L}\{G(t)\}}{s+k} + \frac{T_0}{s+k}$$

by taking inverse Laplace transform

$$T(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+k}\mathcal{L}\{G(t)\}\right\} + \mathcal{L}^{-1}\{\frac{T_0}{s+k}\}$$

$$T(t) = \int_0^t e^{-k(t-x)}G(t)dx + T_0e^{-kt}.$$

## 3.3. Laplace transform of the modified conformable derivative

We apply the Laplace transform in the case of the modified conformable derivative to solve arbitrary-order differential equations, and we introduce the Laplace transform of some important functions.

**Definition 3.3.1.** Suppose that $f: [t_0 \to \infty)$ be a real-valued function and $t_0 \in \mathbb{R}$, $\alpha \in (0,1]$, then the fractional Laplace transform of order $\alpha$ starting from $a$ of the function $f$ is defined as follows

$$\mathcal{L}_\alpha^{t_0}\{f(t)\}(s) = F_\alpha^{t_0}(s) = \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t)d\alpha(t, t_0)$$

$$= \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t)(t-t_0)^{\alpha-1}\, dt. \tag{3.8}$$

**Theorem 3.3.2.** Suppose that $f: [t_0, \infty) \to \mathbb{R}$ be a real-valued function for $\alpha \in (0,1]$ and $\alpha \in \mathbb{R}$ then

$$\mathcal{L}_\alpha^{t_0}\{T_\alpha^a(f)(t)\}(s) = sF_\alpha^a(s) - f(a). \tag{3.9}$$

***Proof.*** From the definition of conformable Laplace transform

$$\mathcal{L}_\alpha^{t_0}\{T_\alpha^a(f)(t)\}(s) = \int_a^{\infty} e^{-s\frac{(t-a)^\alpha}{\alpha}}(t-a)^{1-\alpha} f'(t)(t-a)^{\alpha-1}dt$$

$$= \int_a^{\infty} e^{-s\frac{(t-a)^\alpha}{\alpha}} f'(t)dt$$

$$= f(t)e^{-s\frac{(t-a)^\alpha}{\alpha}}|_a^\infty + s\int_a^{\infty}(t-a)^{\alpha-1} e^{-s\frac{(t-a)^\alpha}{\alpha}} f(t)dt$$

$$= -f(a) + sF_\alpha^a(\mathrm{s}).$$

**Lemma 3.3.3.** This lemma connects the conformable and normal Laplace transforms, assume that $f: [t_0, \infty) \to \mathbb{R}$ is a function and $\mathcal{L}_\alpha^{t_0}\{f(t)\}(s) = F_\alpha^{t_0}(s)$ exists then we get

$$F_\alpha^{t_0}(s) = \mathcal{L}\{(f(t_0 + (\alpha t)^{\frac{1}{\alpha}}))\}(s)$$

which for $g: [0, \infty) \to \mathbb{R}$

$$\mathcal{L}\{g(t)\}(s) = \int_0^{\infty} e^{-st} g(t)dt. \tag{3.10}$$

***Proof.*** Let $u = \frac{(t-t_0)^\alpha}{\alpha} \Rightarrow t - t_0 = (\alpha u)^{\frac{1}{\alpha}}$ $\;$ *then* $dt = (\alpha u)^{\frac{1}{\alpha}-1}du$

$$F_\alpha^{t_0}(s) = \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t)(t-t_0)^{\alpha-1}dt$$

$$= \int_{t_0}^{\infty} e^{-su} f(t_0 + (\alpha u)^{\frac{1}{\alpha}}) \left((\alpha u)^{\frac{1}{\alpha}}\right)^{\alpha-1} (\alpha u)^{\frac{1}{\alpha}-1}du$$

$$= \mathcal{L}\{f(t_0 + (\alpha u)^{\frac{1}{\alpha}})\}(s).$$

**Lemma 3.3.4.** Let $f: [0, \infty) \to \mathbb{R}$, $\mathcal{L}_\alpha^{t_0}\{f(t)\}(s) = \mathrm{F}_\alpha^{t_0}(s)$ exists and $\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} e^{-st} f(t)dt$ is classical Laplace transform, so that

$$\mathcal{L}_\alpha^{t_0}\{f(t)\}(s) = \mathrm{F}_\alpha^{t_0}(s) = \mathcal{L}\{f(\frac{t^\alpha}{\alpha})\}. \tag{3.11}$$

**Application 3.3.5.**[17,26] We provide conformable Laplace transforms for particular functions in this condition.

1) $\mathcal{L}_\alpha^{t_0}\{1\}(s) = \frac{1}{s}$, for $s > 0$.

2) $\mathcal{L}_\alpha^{t_0}\{t\}(s) = \mathcal{L}\left\{f\left(t_0 + (\alpha u)^{\frac{1}{\alpha}}\right)\right\}(s) = \frac{t_0}{s} + \alpha^{\frac{1}{\alpha}} \frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{s^{1+\frac{1}{\alpha}}}$, $s > 0$.

3) $\mathcal{L}_\alpha^{0}\{t^p\}(s) = \frac{\alpha^{\frac{p}{\alpha}}}{s^{1+\frac{p}{\alpha}}}\, \Gamma\left(1 + \frac{1}{\alpha}\right)$, $s > 0$.

4) $\mathcal{L}_\alpha^{0}\left\{e^{\frac{t^\alpha}{\alpha}}\right\}(s) = \frac{1}{s-1}$, $s > 1$.

5) $\mathcal{L}_\alpha^{0}\left\{sinw\frac{t^\alpha}{\alpha}\right\}(s) = \mathcal{L}\{sinwt\}(s) = \frac{1}{w^2+s^2}$ .

6) $\mathcal{L}_\alpha^{0}\left\{cosw\frac{t^\alpha}{\alpha}\right\}(s) = \mathcal{L}\{coswt\}(s) = \frac{s}{w^2+s^2}$.

7) $\mathcal{L}_\alpha^{t_0}\left\{e^{-k\frac{(t-t_0)^\alpha}{\alpha}} f(t)\right\}(s) = \mathcal{L}\{e^{-kt} f(t_0 + (\alpha u)^{\frac{1}{\alpha}})\}$.

**Theorem 3.3.6. (Convolution theorem**) [26] Suppose that $f, g: [0,\infty) \to \mathbb{R}, \alpha \in (0,1]$, $s > 0$ and there exists $f(t), g(t)$ are two functions such that $\mathcal{F}_\alpha^{t_0}(s) = \mathcal{L}_\alpha^{t_0}\{f(t)\}\ and\ \mathcal{G}_\alpha^{t_0}(s) = \mathcal{L}_\alpha^{t_0}\{g(t)\}$, then we get the following relation [25]

$$\mathcal{L}_\alpha^{t_0}\{f * g\}(t) = F_\alpha^{t_0}(s)\ \ G_\alpha^{t_0}(s) \tag{3.12}$$

where $(f * g)$ is the convolution of the functions $f\ and\ g$.

***Proof.*** With the help of relation (3.11) and because the classical Laplace transform has the convolution property, we have

$$\mathcal{L}_\alpha^{t_0}\{f(t) * g(t)\}(t) = \mathcal{L}\left\{\int_0^{t_0} f(u)g(t-u)du\right\} = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}(\int_0^{t_0} f(u)g(t-u)du)dt$$

$$= \int_0^\infty \int_0^{t_0} e^{-s\frac{t^\alpha}{\alpha}} f(u)g(t-u)dudt$$

now let $t - u = z$ , $dt = dz$,

$$= \int_{u=0}^\infty \int_{t=u}^\infty e^{-s\frac{t^\alpha}{\alpha}} f(u)g(t-u)dtdu$$

$$= \int_{u=0}^\infty \int_{z=0}^\infty e^{-s\frac{(u+z)^\alpha}{\alpha}} f(u)g(z)dzdu$$

$$= \int_{u=0}^{\infty} e^{-st} \int_{z=0}^{\infty} f(u)g(z)dzdu$$

$$= \mathcal{L}\left\{\int_0^t f(t-u)g(u)du\right\} = \mathcal{L}\{(f*g)\}.$$

Therefore,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{(f*g)(t)\}\} = (f*g)(t) \quad .$$

### 3.4. Energy-saving model by using conformable Laplace transform

In this part, we'll show how to use the Laplace transform of the truncated conformable derivative to present solutions for the energy-saving model to heating and cooling buildings of any order. Consequently, we propose using the additional parameter $\alpha$ in the truncated conformable Laplace transform function to calculate how the internal temperature of buildings affects the outside temperature. The following is the underlying model in the truncated conformable derivative form

$$\mathcal{L}_{\alpha}^{t_o}\{f(t)\}(s) = \int_{t_o}^{\infty} e^{-s\frac{(t-t_o)}{\alpha}} f(t)(t-t_o)^{\alpha-1}. \tag{3.13}$$

Where $t_0 \epsilon\, \mathbb{R}$, and $0 < \alpha < 1$. This definition is used to investigate the solution of the Laplace conformable differential equation of order α. such as the observed model is given by

$$\frac{dT}{dt} = K\{M(t) - T(t)\} + H(t) + F(t). \tag{3.14}$$

Which the additional heating rate that is always non-negative is denoted by $H(t)$ and $F(t)$ is negative for air conditioner cooling and positive for furnace. the temperature $T(t)$ is proportional to the difference between $M(t)$ which is the outside temperature and the inside temperature $T(t)$ and $K$ be any positive constant, which is depends on the physical properties of the building.
The model can be given in Laplace form as follows

$$T_{\alpha}T(t) = K\{M(t) - T(t)\} + H(t) + F(t) \qquad \text{where, } T(0) = T_0 .$$

If we use Laplace transform for the above equation we get

$$\mathcal{L}_{\alpha}\{T_{\alpha}\mathrm{T(t)}\} = \mathcal{L}_{\alpha}\{K\{M(t) - T(t)\} + H(t) + F(t)\}.$$

By using this relation

$$\mathcal{L}_{\alpha}\{T_{\alpha}T(t)\} = s\mathcal{L}_{\alpha}(T(t) - T(0))$$

$$s\mathcal{L}_{\alpha}\{T(t) - T(0)\} = -K\mathcal{L}_{\alpha}\{T(t)\} + \mathcal{L}_{\alpha}\{KM(t) + H(t) + F(t)\}$$

$$\mathcal{L}_\alpha\{T(t)\} = \frac{T_o}{s+K} + \frac{\mathcal{L}_\alpha\{G(t)\}}{s+K}$$

where $G(t) = KM(t) + H(t) + F(t)$.

By taking inverse Laplace transform, we obtain

$$T(t) = \mathcal{L}_\alpha^{-1}\left\{\frac{T_o}{s+K} + \frac{\mathcal{L}_\alpha\, G(t)}{s+K}\right\}$$

$$T(t) = T_o\, \mathcal{L}_\alpha^{-1}\left\{\frac{1}{s+K}\right\} + \mathcal{L}_\alpha^{-1}\left\{\frac{1}{s+K}\mathcal{L}_\alpha\{G(t)\}\right\}$$

$$T(t) = T_0\, e^{-k\frac{t^\alpha}{\alpha}} + e^{-k\frac{t^\alpha}{\alpha}}\, .\, \mathrm{G(t)}$$

by using the convolution theorem, we get

$$T(t) = T_0\, e^{-k\frac{t^\alpha}{\alpha}} + e^{-k\frac{t^\alpha}{\alpha}} * \mathrm{G(t)}$$

$$T(t) = T_0\, e^{-k\frac{t^\alpha}{\alpha}} + \int_0^t e^{-k\frac{(t-\tau)^\alpha}{\alpha}}\, .\, \mathrm{G}(\tau)\, t^{\alpha-1}\ \mathrm{d}\tau\ .$$

## 3.5. Description of building heating and cooling model by classical derivative.

The main goal of this part is to explain the creation of a fundamental mathematical model for representing building temperature profiles over a 24-hour period by using the classical Laplace transform. The observed model is given by [5]

$$\frac{d}{dt}\big(T(t)\big) = K\{N(t) - T(t)\} + C(t) + kf(T_w - T(t))\ . \tag{3.15}$$

Where the 24-hours temperature inside the building is T(t) which is given in the form of a function of the outside temperature N(t), the heat generated in the building is denoted by C(t), and the heating of the furnace or cooling of the air conditioner F(t) generally, We assume that the amount of heating or cooling is proportionate to the temperature difference, which $F(t) = K_F[T_w - T(t)]$, where $K_F$ is positive constant, $T_w$ is desire temperature. $F(t)$ and $C(t)$ are defined as energy per unit time and $K$ is a positive constant depending on the physical properties of the building.

The observed model is given by

$$\frac{d\,T(t)}{dt} = K\{N(t) - T(t)\} + C(t) + K_F[T_w - T(t)] \tag{3.16}$$

by taking Laplace transform

$$s\mathcal{L}\{T(t)\} - T(0) = \mathcal{L}\{K\{N(t) - T(t)\} + C(t) + K_F[T_w - T(t)\}$$

$$s\mathcal{L}\{T(t)\} - T(0) = -K\mathcal{L}\{T(t)\} + K\mathcal{L}\{N(t)\} + \mathcal{L}\{C(t)\} - K_F\mathcal{L}\{T(t)\} + K_F\mathcal{L}\{T_w\}$$

$$\mathcal{L}\{T(t)\}(s + K + K_F) = T_0 + K\mathcal{L}\{N(t)\} + \mathcal{L}\{C(t)\} + K_F\mathcal{L}\{T_w\}$$

$$\mathcal{L}\{T(t)\} = \frac{T_0}{s+K+K_F} + \frac{\mathcal{L}\{KN(t)\}+C(t)}{s+K+K_F} + \frac{K_F\mathcal{L}\{T_w\}}{s+K+K_F} \tag{3.17}$$

by taking inverse Laplace to transform for (3.17),

$$T(t) = T_0\, e^{-(K+K_F)t} + e^{-(K+K_F)t}.G(t) + K_F\, e^{-(K+K_F)t}T_w$$

by convolution theorem,

$$T(t) = T_0\, e^{-(K+K_F)t} + \int_0^t e^{-(K+K_F)(t-\tau)} * G(\tau)d\tau + K_F \int_0^t e^{-(K+K_F)(t-\tau)}\, T_\tau\, d\tau.$$

## 3.6. Description of building heating and cooling model by conformable derivative.

In this section, the fractional building heating and cooling model was introduced by utilizing Laplace transform to the model. This fractional model can be expressed by

$$\frac{dT(t)}{dt} = K[N(t) - T(t)] + C(t) + K_F(T_w - T(t))\ . \tag{3.18}$$

By taking Laplace transform for equation (3.18)

$$\mathcal{L}_\alpha\{T_\alpha T(t)\} = \mathcal{L}_\alpha\{K[N(t) - T(t)] + C(t) + K_F(T_w - T(t)\}.$$

By using this relation

$$\mathcal{L}_\alpha\{T_\alpha T(t)\} = s\mathcal{L}_\alpha\{T(t)\} - T(0)$$

$$s\mathcal{L}_\alpha\{T(t)\} - T_0 = -K\mathcal{L}_\alpha\{T(t)\} + K\mathcal{L}_\alpha\{N(t)\} + \mathcal{L}_\alpha\{C(t)\} + K_F\mathcal{L}_\alpha\{T(t)\} + K_F\mathcal{L}_\alpha\{T(t)\}$$

$$s\mathcal{L}_\alpha\{T(t)\} + K\mathcal{L}_\alpha\{T(t)\} + K_F\mathcal{L}_\alpha\{T(t)\} = \mathcal{L}_\alpha\{KN(t) + C(t)\} + K_F\mathcal{L}_\alpha\{T_w\} + T_0$$

$$\mathcal{L}_\alpha\{T(t)\}(s + K + K_F) = T_0 + \mathcal{L}_\alpha\{KN(t) + C(t)\} + K_F\mathcal{L}_\alpha\{T_w\}.$$

By taking inverse Laplace transform

$$T(t) = \mathcal{L}^{-1}\{\frac{T_0}{s+K+K_F} + \frac{\mathcal{L}_\alpha\{G(t)\}}{s+K+K_F} + \frac{K_F\mathcal{L}_\alpha\{T_w\}}{s+K+K_F}\}.$$

Where $G(t) = KN(t) + C(t)$

$$T(t) = T_0\, e^{-(K+K_F)\frac{t^\alpha}{\alpha}} + e^{-(K+K_F)\frac{t^\alpha}{\alpha}} * G(t) + K_F\, e^{-(K+K_F)\frac{t^\alpha}{\alpha}}\, T_w$$

$$= T_0\, e^{-(K+K_F)\frac{t^\alpha}{\alpha}} + \int_0^t e^{-(K+K_F)\frac{(t-\tau)^\alpha}{\alpha}}.\, G(\tau)\, t^{\alpha-1} d\tau + K_F \int_0^t e^{-(K+K_F)\frac{(t-\tau)^\alpha}{\alpha}}.\, G(\tau)\, t^{\alpha-1} d\tau.$$

# 4. CONCLUSIONS

In this part, heating and cooling of buildings for an energy-saving model, we provide several experiments and visual observations, which have a significant impact on economics, the environment, health effects, and a much more sustainable energy system all over the world. As mentioned previously, the observed model is provided by [5]

$$T'(t) = K\{M(t) - T(t)\} + H(t) + F(t), \qquad T(0) = T_0. \tag{4.1}$$

By using the Laplace transform to solve the above equation (4.1) with classical differentiation, we get

$$T = T_0 e^{-kt} + \int_0^t e^{-k(t-\tau)} G(\tau) d\tau. \tag{4.2}$$

When we compare this integer-order solution to the alternative solution produced in the main findings using the truncated conformable derivative with the arbitrary order provided below, we find that the integer-order solution is preferable, we can see that this integer-order solution is preferable.

$$\mathrm{T(t)} = T_0\, e^{-k\frac{t^\alpha}{\alpha}} + \int_0^t e^{-k\frac{(t-\tau)^\alpha}{\alpha}} \,.\, \mathrm{G}\,(\tau - 1)\, t^{\alpha-1}\, \mathrm{d}\tau\,. \tag{4.3}$$

We can easily see that the solution (4.3) consists of two parameters $\alpha$ and $\tau$, where $\alpha \in (0,1)$.
In other words, a solution of arbitrary order can be applied in a finite number of ways, that is a huge benefit to a real-world problem like calculating the temperature of a building at any given time and observing its behavior in greater detail [3]. We do this by analyzing the heating and cooling of buildings including some visuals consisting of various values of $\alpha$. However, as shown in Figure 1, the association between inside and outside temperature, that is an effective indication for inside temperature, is found to support the results. When the weather is warm, the temperature of inside has a strong connection with outside temperature but, when the weather is cold because people try to maintain a consistent temperature by employing heating systems while the outside temperature lowers, measurement error may be considerable. As a result, investigating building heating and cooling from an arbitrary order perspective, can be much more useful than a standard technique, especially in cold winter situations. Figure 4.1 and Figure 4.2 are obtained when $H(t)$ is a constant, $F(t) = 0$ and outdoor temperature $M(t)$ varies as a sinus curve, such that $M(t) = M_0 - Bcos\theta t$, where $B$ is a positive constant and $\theta = \frac{\pi}{12}$. In Figure 4.1 we utilize the distinct value of $\alpha = 1, \alpha = 0.98$ while in Figure 4.2 the values are $\alpha = 1, \alpha = 0.94$. We also look at the solution curves for $\alpha = 1, \alpha = 0.90$, and $\alpha = 1, \alpha = 0.86$ respectively in Figure 4.3 and Figure 4.4. The obtained

results clearly show that conformable derivatives can provide less measurement error for energy-saving models in terms of heating and cooling infinite numbers of orders.

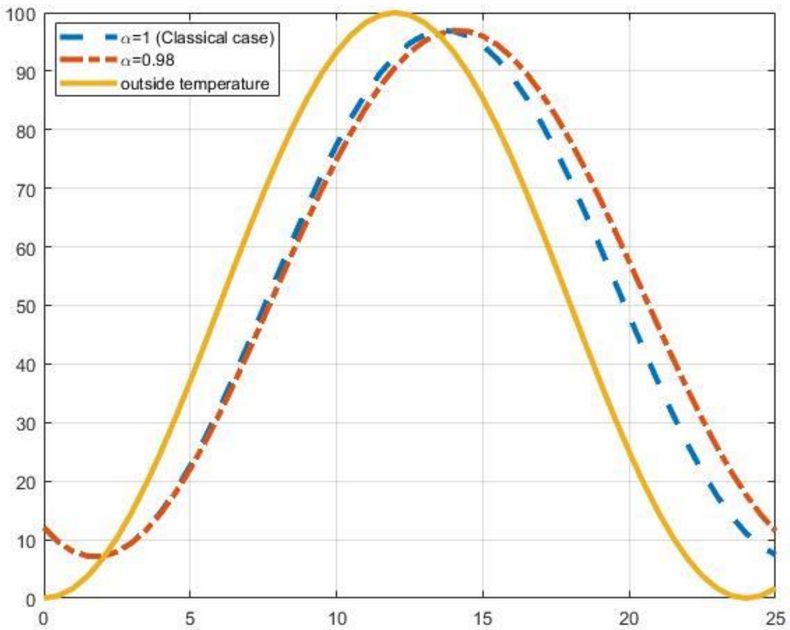


**Figure 4.1.** Comparative analysis with truncated conformable derivative When $\alpha = 1,\ \alpha = 0.98$

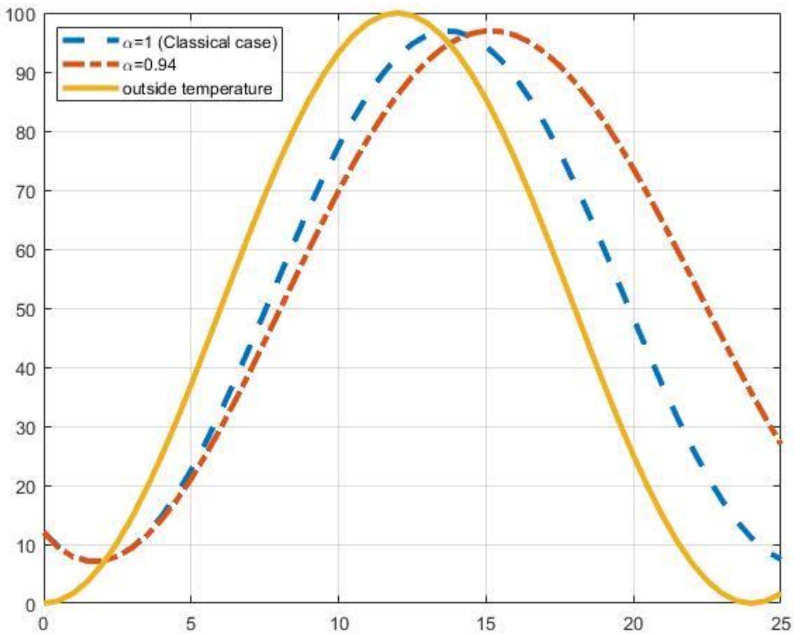


**Figure 4.2**. Comparative analysis with truncated conformable derivative When $\alpha = 1,\ \alpha = 0.94$

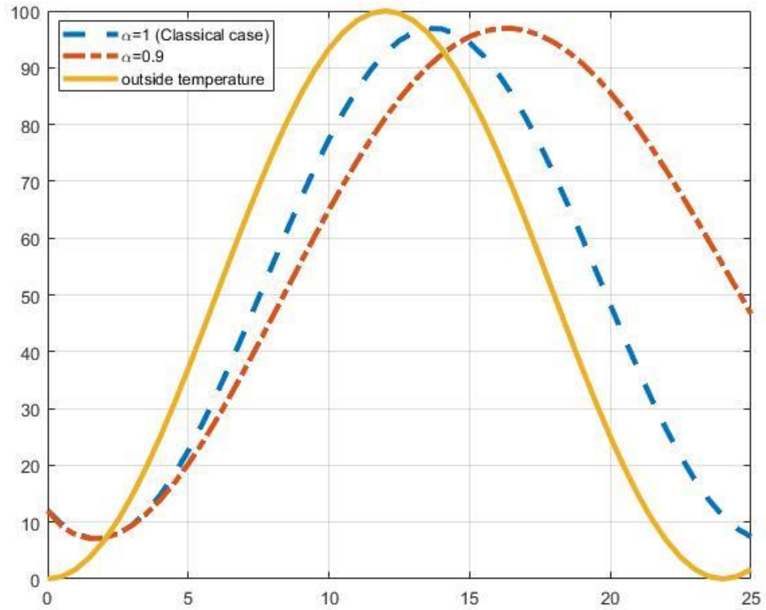


**Figure 4.3.** Comparative analysis with truncated conformable derivative When $\alpha = 1,\ \alpha = 0.90$

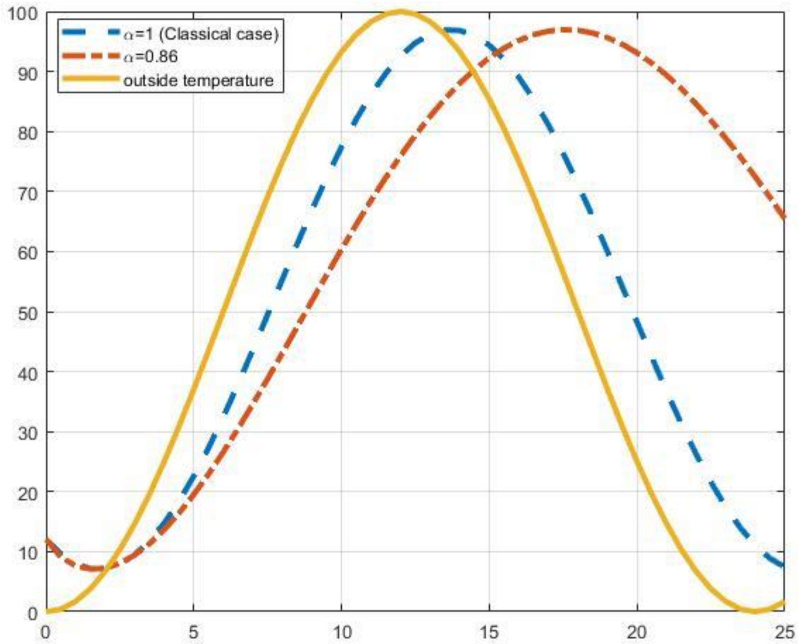


**Figure 4.4.** Comparative analysis with truncated conformable derivative When $\alpha = 1,\ \alpha = 0.86$

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# CURRICULUM VITAE

Hozhin Mohammed BRAIM

**PERSONAL INFORMATION**

[redacted] [redacted]
[redacted] [redacted]
[redacted] [redacted]
[redacted] [redacted]
[redacted] [redacted]
[redacted] [redacted]

**ARAŞTIRMACI BİLGİLERİ**

[redacted] [redacted]
[redacted] [redacted]

**EDUCATION**

[redacted] [redacted]
[redacted] [redacted]

**RESEARCH EXPERIENCES**

- ✓ Computer Programming languages available (PASCAL PROGRAMING, SPSS, MATLAB)

**WORK EXPERIENCE**

**2019 – ... : Awring School**
**2018– 2019: LEZAN private school**