

PRICING INFLATION INDEXED BONDS AND EMBEDDED DEFLATION  
FLOOR OPTIONS: AN ANALYSIS ON TURKISH BOND MARKET

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FLOOR OPTIONS: AN ANALYSIS ON TURKISH BOND MARKET**

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## ABSTRACT

### PRICING INFLATION INDEXED BONDS AND EMBEDDED DEFLATION FLOOR OPTIONS: AN ANALYSIS ON TURKISH BOND MARKET

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Fixed income securities are financial contracts that provide a stream of cash flows to the investors. Such cash flows are exposed to inflation risk. From the perspective of the lender, the purchasing power of provided cash flows might be subject to erosion in inflationary pressures. Inflation indexed bonds issued in a way that to hedge this risk and provide real return to the bond holder. UK issued first inflation indexed bonds as sovereign in 1981. Then, developed countries such as US, Canada, France, Germany, and emerging market countries such as Turkey, Brazil, Mexico had high interest in issuing these bonds.

The principal and coupon payments of these bonds are linked to the changes in the reference price index. Inflation indexed bonds might be issued as plain, which excludes any protection against deflation. On the other hand, bonds might be issued with deflation protection (put) option embedded since bond cash flows will be less than the nominal value in a deflationary economic environment. Deflation protection might cover only principal value as in TIPS issued by US or both principal and coupon payments as inflation indexed bonds issued by Turkish Treasury.

This thesis aims to price deflation protection option premium in Turkish bond market by decomposing bond structure into plain and option components. First, we review the Jarrow-Yildirim model under the HJM framework and the analytical formulas

for derivative prices available in that model. Then, we use historical bond market data to estimate model parameters and the price of the embedded deflation protection option. Finally, we examine historical course of this premium for bonds with different characteristics and inflation expectations.

**Keywords:** Bond Pricing, Debt Securities, Deflation Premium, Fixed Income Market, HJM Framework, Jarrow-Yildirim Model, Inflation Indexed Bonds



## ÖZ

### ENFLASYONA ENDEKSİLİ TAHVİLLERİN VE GÖMÜLÜ DEFLASYON KORUMA OPSİYONLARININ FİYATLANMASI: TÜRK TAHVİL PİYASASI ÜZERİNE ANALİZ

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Sabit getirili menkul kıymetler, yatırımcılara düzenli nakit akımı sağlayan finansal sözleşmelerdir. Bu kıymetlerin sağladığı nakit akımları enflasyon riskine maruzdur. Enflasyonist ortamda, sağlanan nakit akımlarının satın alma gücü borç veren açısından azalmaktadır. Enflasyona endeksli tahviller yatırımcıları bu riskten korumak ve reel getiri sağlamak üzere ihraç edilmektedir. Enflasyona endeksli devlet tahvilleri ilk olarak İngiltere tarafından 1981 yılında ihraç edilmiştir. Daha sonra, ABD, Kanada, Fransa, Almanya gibi gelişmiş ülkelerin yanı sıra Türkiye, Brezilya, Meksika gibi gelişmekte olan ülkeler de bu tahvillere yoğun ilgi göstermiştir.

Tahvillerin anapara ve kupon demeleri referans fiyat endeksindeki değişime bağlıdır. Enflasyona endeksli tahviller deflasyona karşı herhangi bir koruma sağlamayan, yapılandırılmamış, şekilde ihraç edilebilir. Diğer yandan deflasyonist ekonomik ortamda nakit akımının nominal değerinin altına düşmesini engellemek amacıyla deflasyon koruma (satım) opsiyonu gömülü şekilde de ihraç edilebilir. Deflasyon koruması ABD tarafından ihraç edilen tahvillerde olduğu şekilde yalnızca anaparayı kapsayabileceği gibi Türkiye’de Hazine tarafından ihraç edilen tahvillerde olduğu şekilde hem anapara hem de kupon ödemesini kapsayabilmektedir.

Bu çalışma, tahvilleri yapılandırmamış ve opsiyon bileşenlerine ayırarak Türk tah-

vil piyasasında deflasyon koruma primini hesaplamaktadır. İlk olarak, HJM çerçevesinde Jarrow-Yildirim modeli gözden geçirilmekte ve analitik formüller türetilmektedir. Daha sonra, tahvil piyasası verileri kullanılarak parametre tahmini gerçekleştirilmekte ve deflasyon koruma opsiyonu fiyatlanmaktadır. Son olarak, deflasyon priminin tarihsel seyri incelenmekte ve enflasyon beklentileri ile olan ilişkisi incelenmektedir.

**Anahtar Kelimeler:** Tahvil Fiyatlaması, Borçlanma Araçları, Sabit Getiri Menkul Kıymetler Piyasası, HJM Çerçevesi, Jarrow-Yildirim Modeli, Enflasyona Endeksli Tahviller





*To my beloved family*



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## **LIST OF ABBREVIATIONS**

|       |                                                        |
|-------|--------------------------------------------------------|
| BGM   | Brace-Gaterak-Musiela                                  |
| BIS   | Bank for International Settlements                     |
| CPI   | Consumer Price Index                                   |
| DPO   | Deflation Protection Option                            |
| GDDS  | Government Domestic Debt Securities                    |
| GDP   | Gross Domestic Product                                 |
| GFC   | Global Financial Crisis                                |
| HJM   | Heath-Jarrow-Morton                                    |
| IIB   | Inflation Indexed Bond                                 |
| IIC   | Inflation Indexed Caplet                               |
| IIF   | Inflation Indexed Floorlet                             |
| LPI   | Limited Price Indexation                               |
| m-o-m | Month-over-Month                                       |
| NSA   | Non-Seasonally Adjusted                                |
| OECD  | Organization and Economic Co-operation and Development |
| SA    | Seasonally Adjusted                                    |
| TIPS  | Treasury Inflation Protected Securities                |
| y-o-y | Year-over-Year                                         |
| YYIIS | Year on Year Inflation Indexed Swap                    |
| ZCB   | Zero Coupon Bond                                       |
| ZCIIS | Zero Coupon Inflation Indexed Swap                     |

# CHAPTER 1

## INTRODUCTION

Fixed-income securities are financial contracts that pay a stream of pre-determined interest payments and par value over their terms. Bonds are issued by economic agents to borrow money, whereas the buyers effectively agree to lend at the pre-specified yield jointly determined by coupon payments and face value. Such instruments are engineered and issued by several parties, including sovereign and sub-sovereign entities, banks and non-financial corporates to obtain financing. According to BIS, the notional amount outstanding has reached to 25 trillion USD as of December 2019 [2]. One of the most prominent risks associated with bonds' cash flows is the inflation risk. From the perspective of the bond buyer (lender), the purchasing power of the funds composed of interim payments and par value might be subject to erosion in inflationary pressures. To hedge for this risk, market participants can buy inflation indexed bonds. Pioneering examples of inflation-indexed bonds were first issued by the UK in 1981. Then, developed countries such as US, Canada, France, Germany, and emerging market countries such as Turkey, Brazil, Mexico had high interest in issuing these bonds. Although the earliest versions of such instruments were introduced in the Turkish sovereign bond market in 1998, the considerable market share earned by indexed-bonds corresponds to the period after 2008. As of 2019, the nominal outstanding amount of inflation-indexed bonds has elevated to 200 billion TL [28]. Consequently, the proportion of these securities in the Turkish Treasury's total domestic debt stock has risen to almost 20% in 2020.

There exist plain inflation-indexed bonds, which exclude any protection against deflation. In the structure of such bonds, payments are anchored to the realized inflation

rate. In other words, the payoff of a plain inflation-indexed bond is simply the accumulated inflation. However, in the case of deflation, there is the possibility that bond payments might turn out to be lower than what is specified as the nominal amount, and the payoff can become subdued. For this reason, inflation-indexed bonds are also mostly issued with embedded options designed to protect against deflation. This means that such bonds have an embedded put option written on a reference price index and/or inflation rate. This type of bonds pay the par amount even when there is deflation. Pricing deflation-protection is crucial for both lenders and borrowers. Moreover, premium value attached to embedded option might provide information about the future course of the inflation. A higher price for this option would suggest a deflationary expectation. Recent asset pricing literature involves several works analyzing the bond structure into its plain and option components. Seminal works of Chuang et al. [7] and Grishchenko et al. [13] decomposes coupon payments into plain indexed bond and deflation option. This thesis aims to implement the above-mentioned decomposition for inflation indexed bonds issued by the Turkish Treasury and to price deflation premium as of inflation indexed bond price to analyze the cost of deflation protection to the investors. To the best of our knowledge, this has not been done in the currently available literature. Moreover, this work's outcome can be used to create an index to be used as an explanatory variable for the future inflation rate as in Grishchenko et al. [13]. An alternative approach to get information on inflation expectation from bond yields is the computation of an inflation compensation. This idea is applied to the Turkish bond market by Duran and Gülşen [10]; we give a summary of the approach and results of this paper in our literature review, Chapter 3.

The rest of the thesis is structured as follows: Chapter 2 reviews the fixed income market and introduces preliminary conceptual explanations of fixed-income securities, inflation, and indexation mechanism. Chapter 3 reviews the literature regarding the methods used to price inflation indexed bonds and derivatives. Chapter 4 pricing framework for inflation indexed financial securities based on Heath-Jarrow-Morton (HJM) foreign currency analogy introduced by Jarrow and Yildirim [23]. Chapter 5 describes the data used in the analysis and parameter estimation procedure. Chapter 6 presents the empirical findings. Finally, Chapter 7 makes conclusive remarks and mentions about possible future research agenda.

## **CHAPTER 2**

### **PRELIMINARIES**

This chapter gives an overview of the fixed income market and introduces the basic properties of fixed income securities. The basic characteristics and payoff structure of these securities are described. Then we discuss inflation and the problems it may lead to when price indices are used as references in linked bonds.

#### **2.1 Overview of Fixed Income Markets**

Fixed income securities traded in financial markets are designed to pay back predetermined cash flows to the investors at prespecified time intervals. Over time these securities engineered in a way to meet financial market participants' needs such as inter-time transfer of wealth and the alignment of financing expenses with a future stream of potential income. By design, fixed income securities can provide periodical cash flows as well as lump sum cash flow at the maturity date. Securities with periodical coupon payments are called coupon bonds, while those without any interim cash flow are called discount bonds or zero-coupon bonds (ZCB). Besides assuming a certain amount of cash flows, they also offer variable cash flows via indexing coupon payments to a variable of interest. That variable could be an interest rate or macroeconomic variables such as GDP growth rate and inflation rate. Fixed income securities with coupon payments linked to variable interest rates termed as floating rate bonds. On the other hand, others termed as GDP-linked bonds and inflation-linked (indexed) bonds. Historically speaking, the interest of financial market participants on fixed income instruments has been increasing over time. As a general case, the largest issuers

Table 2.1: Marketable Financing Needs (%)

|               | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2013 | 2014 | 2015 | 2016 | 2017 | 2018 | 2019 | 2020 |
|---------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| Short Term    | 50.0 | 55.5 | 45.3 | 44.3 | 44.6 | 45.4 | 43.8 | 40.4 | 39.9 | 40.7 | 40.6 | 41.1 | 41.6 | 42.6 |
| Long Term     | 50.0 | 44.5 | 54.7 | 55.7 | 55.4 | 54.6 | 56.2 | 59.6 | 60.1 | 59.3 | 59.4 | 58.9 | 58.4 | 57.4 |
| Fixed Rate    | 43.9 | 39.9 | 50.6 | 51.4 | 50.9 | 50.3 | 50.9 | 52.3 | 53.5 | 52.8 | 52.8 | 52.5 | 51.5 | 50.7 |
| Index Linked  | 3.3  | 2.5  | 1.9  | 2.4  | 2.9  | 3.3  | 3.7  | 4.0  | 3.7  | 3.5  | 3.5  | 3.3  | 3.5  | 3.3  |
| Variable Rate | 1.6  | 1.1  | 1.0  | 0.9  | 0.7  | 0.3  | 0.9  | 2.6  | 2.4  | 2.4  | 2.6  | 2.7  | 2.9  | 3.0  |
| Other         | 1.3  | 1.0  | 1.1  | 1.0  | 0.9  | 0.7  | 0.7  | 0.6  | 0.5  | 0.5  | 0.6  | 0.5  | 0.5  | 0.4  |

Source: OECD

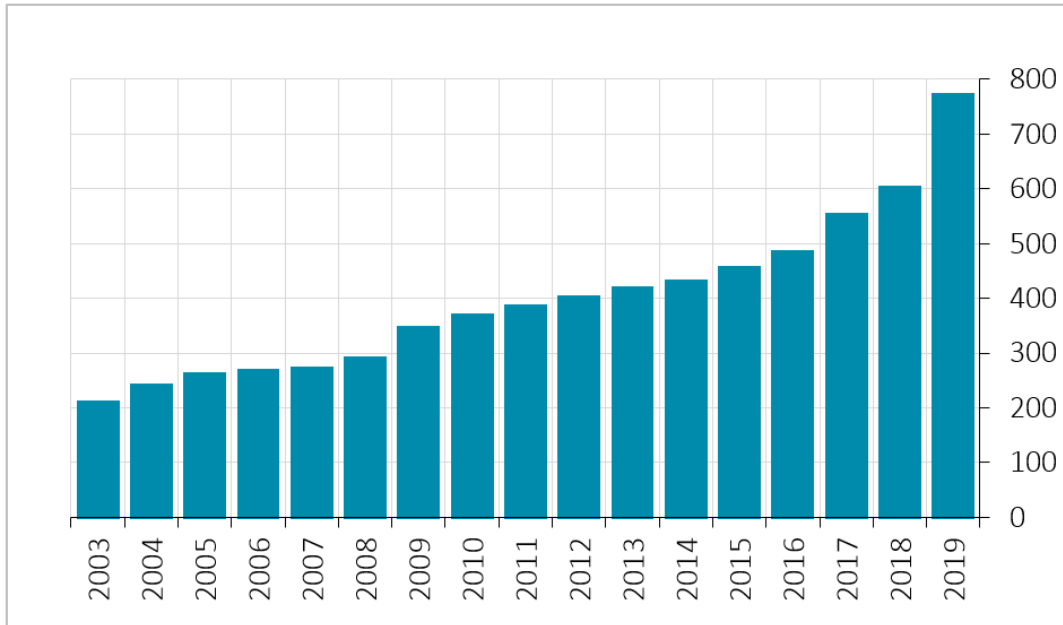


Figure 2.1: Government Domestic Debt Securities

Source: Ministry of Treasury and Finance, nominal outstanding, billions of TL

of these securities are sovereign entities. According to OECD's Sovereign Borrowing Outlook report, marketable debt securities issued by governments hovers around 45 trillion USD in 2019 in OECD countries [29]. It should be noted that an abundance of global liquidity, the risk-seeking behavior of global investors, and improved access to international bond markets have all contributed to the supply of government bonds, especially in the post-Global Financial Crisis (GFC) period and in emerging countries.

As shown in Table 2.1, long term marketable government financing needs have been increasing over time. Due to expansionary policies taken by major central banks during GFC in 2008 and 2009, funding conditions for sovereign entities have been improved. On the other hand, the share of index-linked bonds in long term bonds is around 3%, but it has peaked around 2008 as well. The main reason behind that is both lenders and borrowers had a higher interest in inflation-linked securities to

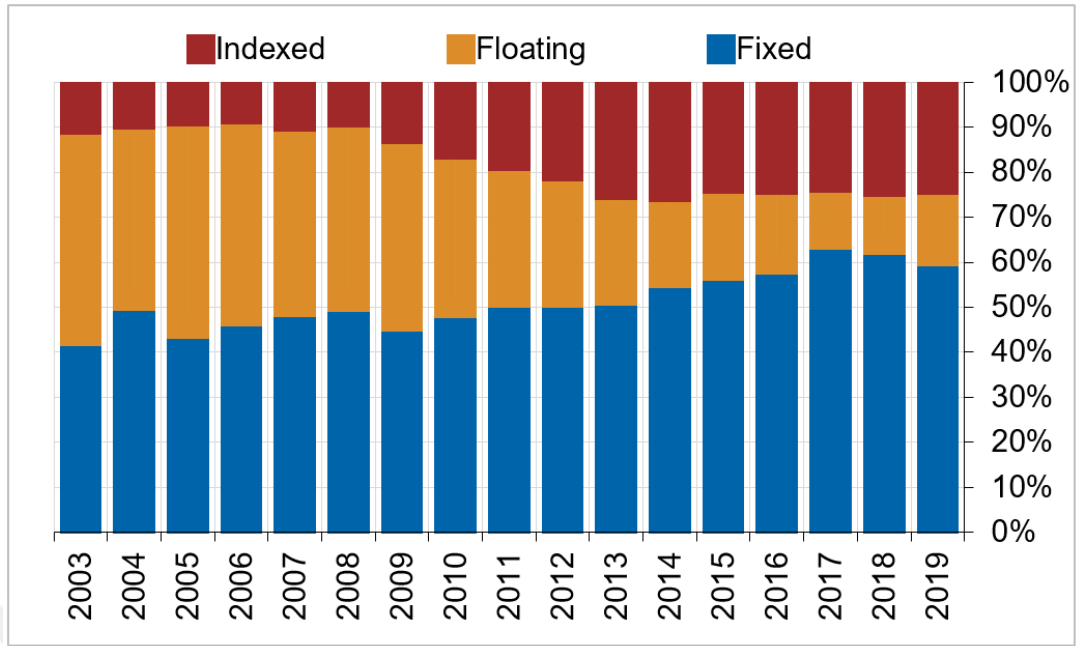


Figure 2.2: Government Debt Stock by Interest Rate Type

Source: Ministry of Treasury and Finance

eliminate risks related to their assets and liabilities.

Turkey has experienced similar phenomena after GFC in terms of the bond market. Nominal outstanding amount of Government Domestic Debt Securities (GDDS) has increased significantly as shown in Figure 2.1. Share of index linked bonds in GDDS has rapidly increased in the same time period. While it was about 8-9% percent of GDDS outstanding before GFC, it has reached to double digits and exceeded 20% in 2012 for the first time in history. Figure 2.2 presents the interest rate type composition of GDDS for the period between 2003 and 2019. It can be concluded that share of inflation indexed bonds is increasing in a low interest rate low inflationary economic environment.

## 2.2 Bond Pricing Basics

In this section we review the basic concepts related to interest rates. All of these definitions and related more concepts can be found in Hull [22] and Brigo, Mercurio [3].

### 2.2.1 Money Market Account

Let  $r$  be risk-free rate, money market account or bank account is defined as

$$M(t) = e^{\int_0^t r(s)ds} \quad (2.1)$$

with

$$dM(t) = r(t)M(t)dt \quad M(0) = 1 \quad (2.2)$$

$$M(0) = 1$$

Money market account represents the future value of the amount of currency that has set aside at time  $t = 0$ .

### 2.2.2 Discount Factor

Discount factor is the current price of a zero coupon bond with one unit of face value. In general terms it can be expressed as

$$D(t, T) = P(t, T) = \frac{M(t)}{M(T)} \quad (2.3)$$

### 2.2.3 Forward Rate

Forward rate is the rate on forward contract at time  $t$  which starts on time  $S$  and expires at time  $T$ . Once spot rate curve is available it is possible to move from spot rate curve to forward rates or vice versa. Forward are at time  $t$  can be expressed as

$$F(t, S, T) = -\frac{\ln P(t, T) - \ln P(t, S)}{T - S} \quad (2.4)$$

### 2.2.4 Instantaneous Forward Rate

Instantaneous forward rate at time  $t$  for the maturity  $T > t$  can be expressed as

$$f(t, T) = \lim_{S \rightarrow T^+} F(t, T, S) = -\frac{\partial \ln P(t, T)}{\partial T} \quad (2.5)$$

### 2.2.5 Short Rate

Short rate is the constant rate that grows amount of  $P(t, T)$  at time  $t$  to 1 unit of currency at time  $T$ . It can be expressed as follows

$$R(t, T) = \frac{\ln P(t, T)}{(T - t)} \quad (2.6)$$

### 2.2.6 Instantaneous Short Rate

The instantaneous short rate at time  $t$  defined as

$$r(t, T) = \lim_{T \rightarrow t^+} R(t, T) \quad (2.7)$$

Then the relation between instantaneous forward rate and instantaneous short rate is as follows

$$r(t, t) = f(t, t)$$

## 2.3 Fundamentals of Fixed Income

In this section we briefly review the pricing framework and cash flow of various fixed income securities. The basic definitions can be found in Hull [22].

### 2.3.1 Zero Coupon Bond

Zero coupon bond (ZCB) is a financial instrument that guarantees to pay notional amount at maturity date. Let  $P(t, T)$  be the price of zero coupon bond at time  $t$  with maturity  $T$  and notional amount one unit of currency. Then the price of a ZCB under continuous time setting and stochastic interest rate process,  $P(t, T)$ ,

$$P(t, T) = e^{-\int_t^T r(s) ds} \quad (2.8)$$

or equivalently under discrete time setting and deterministic interest rate process, price of a bond can be expressed as follows:

$$P(t, T) = \frac{1}{1 + r^{T-t}} \quad (2.9)$$

and  $P(T, T) = 1$ . Payoff of a zero coupon bond is deterministic and it does not depend on interest rate trajectory over the life of bond.

### 2.3.2 Fixed Coupon Bond

Fixed coupon bonds provides regular cash flows, coupon payments, to bond holder. Interim cash flows determined by coupon rate. Coupon payments can be in any frequency but in practice they are mostly semi-annually or quarterly. Price of a fixed coupon bond is equal to sum of present value of future cash flows. Price of a bond with notional amount  $F$ , coupon rate  $c$  and maturity  $T$  at time  $t = 0$ , denoted by  $P_n(t, T)$  is equal to:

$$P_n(t, T) = \sum_{t_i=0}^T \frac{cF}{(1+r)^{t_i}} + \frac{F}{(1+r)^T} \quad (2.10)$$

### 2.3.3 Inflation Indexed Bond

Inflation indexed bonds are designed to provide real return to the bond holders. Both coupon payments and notional amount could be indexed to inflation. Reference rate is generally Consumer Price Index (CPI) but other price indices such as Retail Price Index in U.K. and Harmonised Index of Consumer Prices (HICP) in Euro Area are used as well. Cashflows are adjusted by the change in reference index to provide real return. Let  $I_t$  be the reference index value at time  $t$ , price of an inflation indexed bond at time  $t = 0$ , denoted by  $P_r(0, T)$ ,

$$P_r(t, T) = \sum_{t_i=0}^T \frac{cF}{(1+r)^{t_i}} \frac{I(t_i)}{I(0)} + \frac{F}{(1+r)^T} \frac{I(T)}{I(0)} \quad (2.11)$$

where  $\frac{I(T)}{I(t)}$  is called as Reference Index Multiplier. For example, let inflation index be 100 at  $t_0$  and be 110 at  $t_1$ . Then coupon payment for a bond with nominal 100, coupon rate 2% will be

$$\begin{aligned} c_1 &= cF \frac{I(t)}{I(0)} \\ &= 0.02 * 100 * \frac{110}{100} \\ &= 2.2 \end{aligned}$$

In the equation above both coupon and notional amount payments are purely indexed to the inflation index ratio. But as mentioned before these securities often offer protection against deflation. The deflation protection is offered for only notional value of inflation indexed bonds as in U.S. Treasury issued TIPS, or for both coupon payments and notional value as in inflation indexed bonds issued by Turkish Treasury. The price of a bond which both coupon payments and notional value is expressed as follows:

$$P_r(0, T) = \sum_{i=1}^n \frac{1}{(1+r)^i} \max \left\{ cF \frac{I(i)}{I(0)}, cF \right\} + \frac{1}{(1+r)^n} \max \left\{ F \frac{I(n)}{I(0)}, F \right\} \quad (2.12)$$

### 2.3.4 Cash Flows

The main characteristic difference between fixed coupon nominal bonds and indexed bonds is fixed coupon bonds provide certain cash flow but uncertain real yield while inflation indexed bonds provide uncertain cash flows and certain real yield. Therefore, it is essential to compare cash flow structures of two bonds. To illustrate this, we assume there are two bonds. First, 2-year bond with 5% percent coupon rate and annual coupon frequency and second 2-year bond with 2% percent coupon rate and annual coupon frequency. Under assumption of  $I_{t_0} = 100$ , cash flow of two bonds will be as in Figure 2.3.

First coupon payment for the plain inflation indexed bond mentioned in the example above will be equal to 2.20 if the reference price index goes up to 110 from 100. However, if the reference price index goes down to 95 at time  $t_1$  coupon payment will be 1.9. If the inflation indexed bond has embedded deflation protection option (DPO) coupon payment will be 2.0 as shown in last line of Figure 2.3.

## 2.4 Reference Index and Indexation Lag

### 2.4.1 Inflation

In this study, the data series utilized to represent inflation dynamics in the Turkish economy is retrieved from TurkStat. The most contemporary series produced has 2003 as the base year and it anticipates the overall course of domestic prices of goods

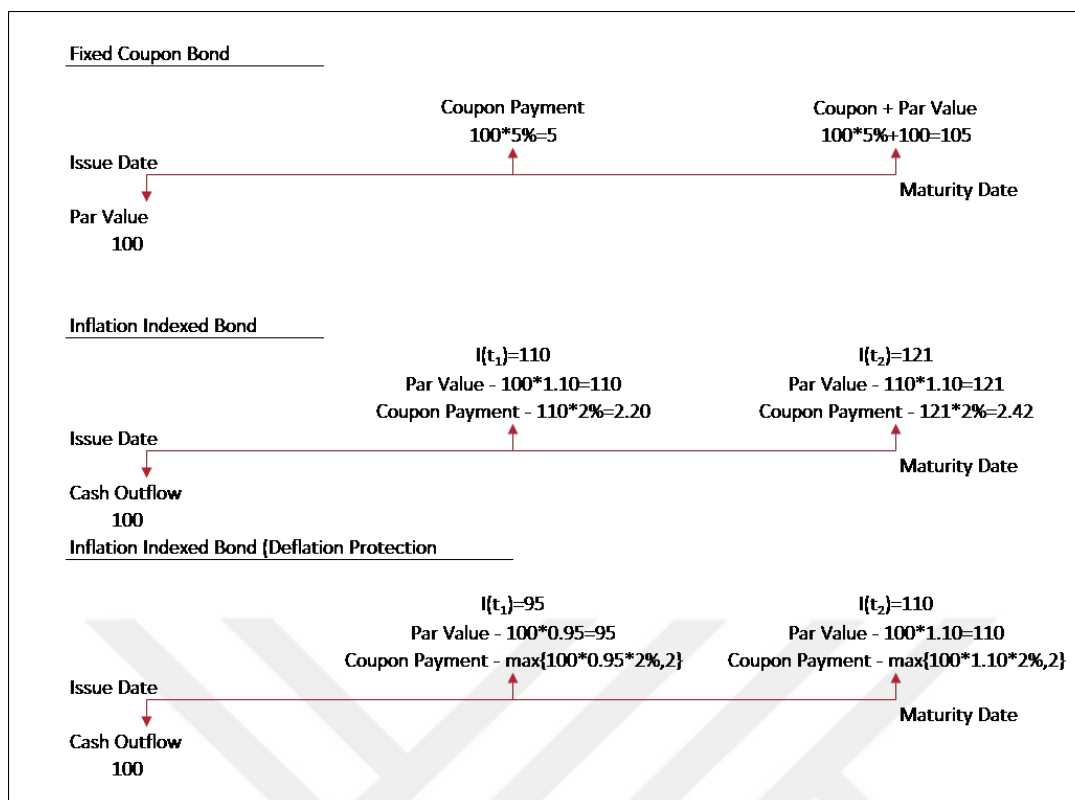


Figure 2.3: Cash Flows of Nominal and Inflation Indexed Bonds

and services consumed by Turkish households. The representative consumer basket, from which indices are obtained, is dynamically created through the Household Budget Survey, which is applied to more than 15000 households from varied socioeconomic status. The weights of individual items are revised on an annual basis depending on the survey results. The main expenditure groups in CPI basket is presented in Table 2.2.

Table 2.2: Weights by main expenditure groups of consumer price index (CPI)

| Main expenditure group           | 2010  | 2011  | 2012  | 2013  | 2014  | 2015  | 2016  | 2017  | 2018  | 2019  | 2020  |
|----------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Food and non-alcoholic beverages | 27.60 | 26.78 | 26.22 | 24.09 | 24.45 | 24.25 | 23.68 | 21.77 | 23.03 | 23.29 | 22.77 |
| Alcoholic beverages and tobacco  | 5.31  | 5.90  | 5.21  | 5.07  | 5.29  | 4.82  | 4.98  | 5.87  | 5.14  | 4.23  | 6.06  |
| Clothing and footwear            | 7.30  | 7.22  | 6.87  | 6.83  | 7.17  | 7.38  | 7.43  | 7.33  | 7.21  | 7.24  | 6.96  |
| Housing                          | 16.83 | 16.46 | 16.44 | 16.68 | 16.41 | 15.79 | 15.93 | 14.85 | 14.85 | 15.16 | 14.34 |
| Furnishings, household equipment | 6.78  | 6.93  | 7.45  | 7.28  | 7.52  | 7.78  | 8.02  | 7.72  | 7.66  | 8.33  | 7.77  |
| Health                           | 2.55  | 2.40  | 2.29  | 2.22  | 2.44  | 2.57  | 2.66  | 2.63  | 2.64  | 2.58  | 2.80  |
| Transportation                   | 13.90 | 15.15 | 16.73 | 17.99 | 15.54 | 15.38 | 14.31 | 16.31 | 17.47 | 16.78 | 15.62 |
| Communication                    | 4.94  | 4.64  | 4.60  | 4.64  | 4.70  | 4.38  | 4.42  | 4.12  | 3.91  | 3.69  | 3.80  |
| Recreation and culture           | 2.83  | 2.70  | 2.98  | 2.95  | 3.36  | 3.54  | 3.81  | 3.62  | 3.39  | 3.29  | 3.26  |
| Education                        | 2.48  | 2.32  | 2.18  | 1.91  | 2.26  | 2.53  | 2.56  | 2.69  | 2.67  | 2.40  | 2.58  |
| Hotels, cafes and restaurants    | 5.51  | 5.89  | 5.63  | 6.18  | 6.58  | 6.98  | 7.47  | 8.05  | 7.27  | 7.86  | 8.67  |
| Miscellaneous goods and services | 3.97  | 3.61  | 3.40  | 4.16  | 4.28  | 4.60  | 4.73  | 5.04  | 4.76  | 5.15  | 5.37  |
| Source: TurkStat                 |       |       |       |       |       |       |       |       |       |       |       |

As of 2020, food expenditures constitute almost 23% of all basket, whereas housing and transportation expenditures amounted to approximately 14% and 16%, respec-

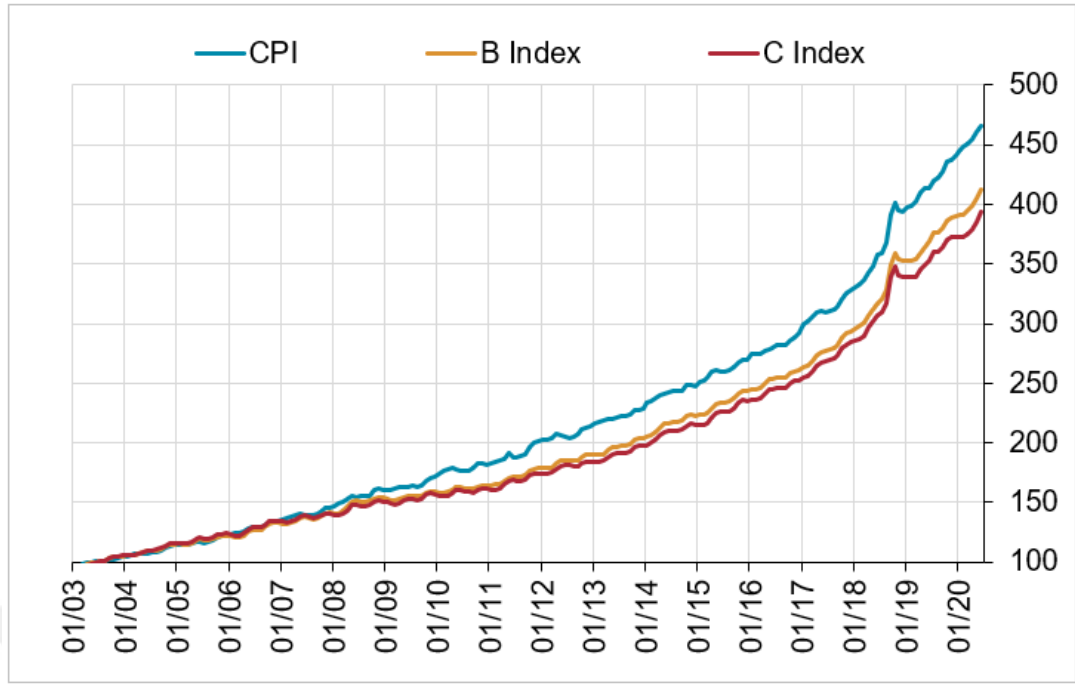


Figure 2.4: Indicators for the CPIs having specified coverages

Source: Turkstat, 2003=100

tively. As price movements in some items can display the impact of exogenous factors and/or specific pricing patterns, TurkStat also constructs CPI indicators with specified coverages [32]. To name a few, B index is generated by excluding unprocessed food, energy, alcoholic beverages, tobacco, and gold from the general basket, while C index does not take all food prices into consideration in addition to the abovementioned items which shown in Figure 2.4.

In this study, we prefer to focus on headline CPI index, as it is used for reference index in inflation indexed bonds issued by Ministry of Finance and Treasury, hereafter Treasury, in Turkey to proxy the inflation as it has the broadest specification and representation. Basically, we define inflation as the positive rate of change in headline CPI index, whereas disinflation refers to the decline in rate of increase in CPI index. More relevant to our work, deflation is categorized as the negative rate of change in representative consumer prices in any particular examination horizon.

Although, Turkish economy did not experience any realization of deflation in the sample period, inflationary pressures tend to prevail as a general case since the inflation rate has mostly hovered above both the inflation expectations and the medium-term target within explicit inflation-targeting framework adopted in Turkey after 2006. On

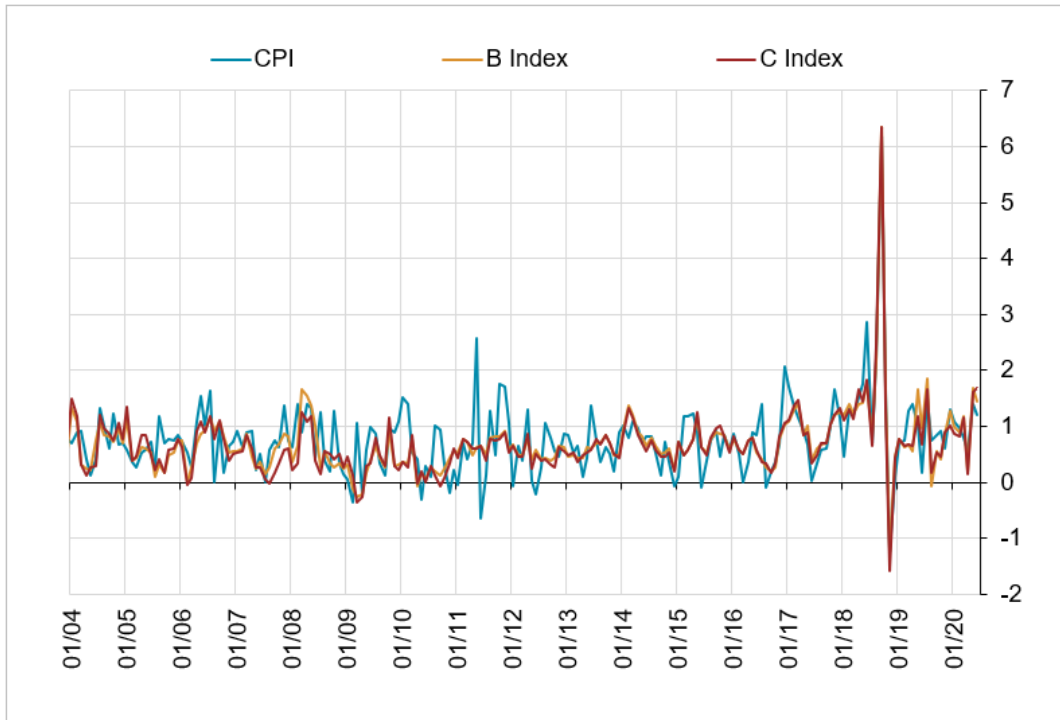


Figure 2.5: Indicators for the CPIs having specified coverages, m-o-m change  
Source: Turkstat, (%), SA

occasional cases, as seen in 2018 following the financial market volatilities involving sizeable currency depreciation, annual and monthly change in consumer prices can reach excessive levels as can be seen in Figure 2.5 and Figure 2.6.

#### 2.4.2 Inflation Market Participants

There are three types of market participants in inflation market. First, inflation receivers who have liabilities strongly linked to changes in inflation index. Pension funds are natural member of this group since they use inflation linked financial products to eliminate inflation risk in their liabilities. This group of market participants are structurally long inflation. Second, inflation payers who are willing to pay inflation rate as floating liability. Inflation payers mostly consists of governments. Main motivation behind this action is to believe inflation rate will fall. This group of participants is structurally short inflation. Last, speculators are interested not only in paying inflation but also receiving it as their market belief points so. This group aim to make money out of movements in inflation rate.

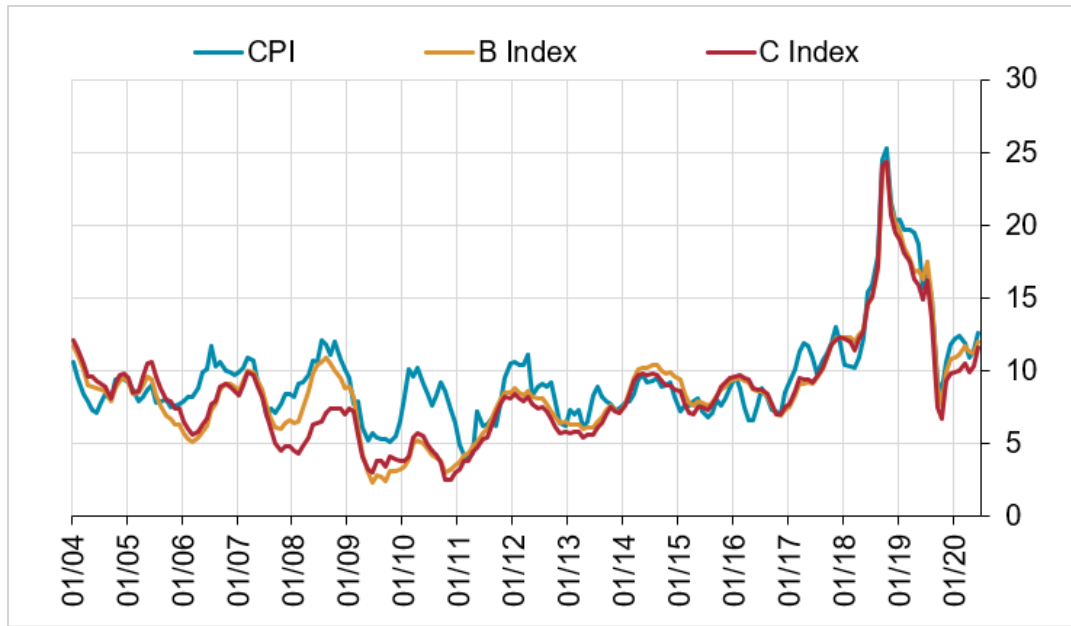


Figure 2.6: Indicators for the CPIs having specified coverages, y-o-y change  
Source: Turkstat, (%), NSA

### 2.4.3 Indexation Lag

As mentioned before, main feature of inflation indexed bonds is to provide real return to the bond holders. Fair real return can be provided if cash flows are strictly indexed to the inflation trajectory. This phenomena is called as perfect indexation. However, in practice perfect indexation does not exist yet. Because inflation index is not certain when cash flows are intended to pay. Inflation index for a specific time is announced with 1 to 3 month lag for most of the countries [8]. Then, cash flows have 1 to 3 month indexation lag. Figure 2.7 shows the timeline for indexation lag problem. Turkish bond market works under similar conditions. The guideline for inflation in-

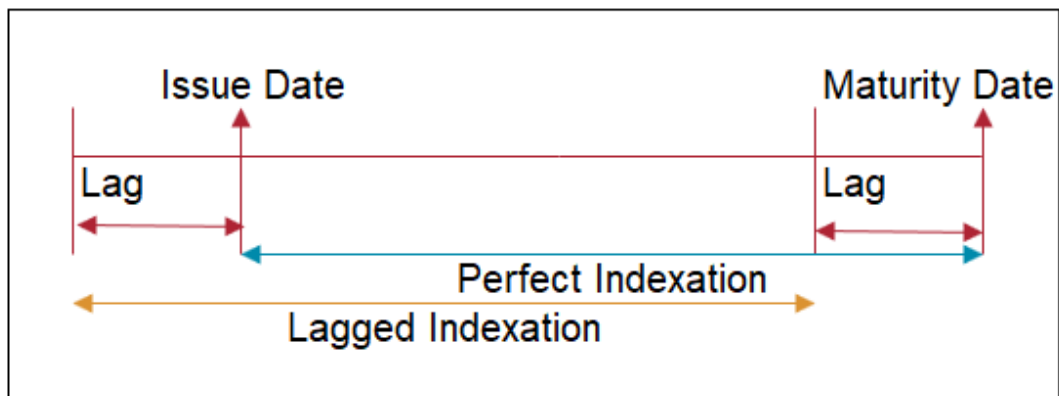


Figure 2.7: Indexation Lag

dexed bonds and daily reference index is announced by Treasury. According to the guideline, daily reference index is calculated as follows:

$$\text{Daily Reference Index} = CPI_{a-3} + \frac{g-1}{AG} * (CPI_{a-2} - CPI_{a-3}) \quad (2.13)$$

where

- $CPI_{a-t}$  is CPI of (a-t) month
- $g$  is number of past days in (a) month and
- $AG$  is the number of total days in (a) month.

For example, CPI index in December 2019 is 440.50 and in January 2020 is 446.45. Then the reference CPI Index value on March 15, 2020 is calculated as follows:

$$\begin{aligned} CPI_{15\text{March}2020} &= CPI_{\text{December}2019} + \frac{15-1}{31} * (CPI_{\text{January}2020} - CPI_{\text{December}2019}) \\ &= 443.1871 \end{aligned}$$

which is in line with Daily Reference Indexed announced by Treasury.<sup>1</sup> Please note that due to indexation lag, reference CPI Index used on first day of month  $t$  is equal to the CPI Index of month  $t - 3$ .

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<sup>1</sup> Daily reference CPI Index is announced at <https://en.hmb.gov.tr/information-for-investors>

## CHAPTER 3

### LITERATURE REVIEW

In this chapter, studies relevant to pricing of inflation-indexed securities are reviewed. There are two main types of pricing inflation indexed securities in the finance literature. First group is known to propose foreign currency analogy. In this strand of the literature, real and nominal economies considered as domestic and foreign economy. Inflation rate, on the other hand, considered as an exchange rate between domestic and foreign economies. Second group of studies proposes to model nominal, real and inflation processes separately. These processes are connected by correlated Brownian motions and pricing is mostly done by using joint distribution of processes.

The foreign currency analogy introduced in Macroeconomic Theory by Fisher [11]. The well-known Fisher equation, defines the inflation rate as the difference between nominal and real rates with some correction terms.

The fundamental background of foreign currency analogy in finance literature is mainly developed by Hughston [19]. That study proposes a pricing model in which the economy has nominal interest rates and real interest rates derived from inflation-linked instruments then the price level plays an important role as foreign exchange parity. Then pricing dynamic of inflation-linked instrument is simply reduced to pricing cross-currency interest. Closed-form solution for vanilla options on inflation is also provided.

Dodgson and Kainth [9] propose to price inflation-linked derivatives using correlated Hull-White model. In this approach short rate and inflation rate are Ornstein-Uhlenbeck, mean reverting diffusive processes. Therefore closed form solutions are

available under constant volatility assumption. By loosening constant volatility assumption, they introduce local stochastic volatility into the model to capture dependence of market prices. Finally, they price more complex derivatives, namely inflation indexed caps and floors, using Monte Carlo sampling.

Jarrow, Yildirim [23] and Brigo, Mercurio [3] propose an approach based on foreign currency analogy. There are nominal and real rate processes in the model which are correlated Hull-White short rate processes. In this model nominal and real rates are one factor Gaussian processes while inflation index is lognormal process with a drift equal to difference between nominal and real rates under risk-neutral measure. Jarrow-Yildirim model has attracted attention and become the benchmark approach on the grounds of its mathematical tractability.

Stewart [30] employs foreign currency analogy, extended HJM framework, introduced by Jarrow-Yildirim [23] to price inflation-indexed derivatives. This study particularly addresses how nominal derivatives can be expressed in terms of zero-coupon bonds and priced using same dynamics. Initial term structures are derived from nominal bonds and inflation swaps. Then nominal volatility parameters are fitted by using at-the-money caps traded in the market.

Brody, Crosby and Li [4] propose of multi-factor version of Jarrow-Yildirim model. The study introduce closed form solution for convexity adjustment for zero-coupon inflation swaps and period-on-period inflation swaps with delayed payments. Then the results are applied to price Limited Price Indexation (LPI) swaps which are exotic type of inflation derivatives.

Leung and Wu [25] establish a market model for term structure of forward inflation rate. The study further extends market model to take volatility smile into account and shows that inflation forward rate is consistent with real forward rate which is natural outcome of HJM model.

Mercurio [26] uses Jarrow-Yildirim approach to model inflation derivatives with extension of two market models. Market models extend Jarrow-Yildirim model by assuming nominal and real forward rates are lognormal martingales under respective T-forward measures. This assumption comes from Brace Gatarek Musiela (BGM)

Model. Aforementioned study also compares pricing performance of Jarrow-Yildirim model and two market models.

Kazziha [24] and Belgrade et al. [1] assume that forward inflation index is lognormal and follows driftless Brownian motion under relevant T-forward measures. Mercurio and Moreni [27] extend this approach by introducing stochastic volatility to have flexibility to capture fat tails and skewness of the market implied distribution.

Hughston and Macrina [20] propose discrete-time stochastic models setting for the pricing of inflation-indexed financial securities. The model builds nominal and real pricing kernels, in terms of which the price index can be expressed, then uses Sidrauski-type utility function depending on microeconomic variables related to money supply to determine continuous-time dynamics of inflation index.

Hinnerich [18] extends HJM model by proposing that forward rates and inflation index are driven by multidimensional Brownian motions and general marked point process. The study also provides closed form solution for options on inflation indexed bonds.

Guney [14] incorporates jumps by proposing instantaneous forward rates, inflation index and bond prices driven by standard Brownian motions and finite number of Poisson processes.

Christensen et al. [6] determine the value of deflation floor in inflation indexed bonds by employing an arbitrage-free term structure with spanned stochastic volatility. However, this study lacks of time dependence of option value. Therefore explanatory power of embedded option to future inflation remains weak.

Grishchenko et al. [13] propose a model that two state variables govern bond price under a continuous-time. State variables are interest rate and inflation index and they are jointly Gaussian. Therefore Grishchenko et al. [13] obtain a closed-form solution for both nominal and inflation indexed derivatives. Moreover, the study aims to value the embedded option in inflation indexed bond, which also known as deflation floor, and constructs an option pricing index. Then it shows option pricing index is statistically significant in explaining future inflation both in-sample and out-of-sample.

Tekmen [31] is the first study reviewing inflation bond market and probable applications to Turkish bond market. This study examines the benefits of inflation indexed bond issuance for Turkey. Also, it covers extensive examples from inflation bond markets around the world.

Duran and Gülşen [10] use inflation compensation derived from nominal and real yield curve to measure inflation expectation in the market. Inflation compensation is defined as the rate makes investor indifferent between investing in inflation linked bond and nominal bond. Calculation of inflation compensation is simply the difference between nominal and real zero coupon rates for different maturities. On the basis of higher frequency and reliable data available with bonds traded in the market, they conclude information content of inflation compensation is competent alternative of inflation surveys.

## CHAPTER 4

### MODEL FOR PRICING

This section firstly presents the details of Jarrow-Yildirim model based on foreign currency analogy. General framework is defined and processes are given under risk neutral measure and correlation between these processes is taken into account. Second, inflation indexed derivatives are introduced and pricing dynamics is described.

#### 4.1 Jarrow-Yildirim Framework

Jarrow and Yildirim [23] employed the Heath Jarrow Morton [17] three factor model to price Treasury Inflation Protected Securities (TIPS) and options on inflation index. Jarrow-Yildirim (JY) method is based on foreign currency analogy which considers nominal rates as domestic economy rates and real rates as foreign economy rates. Therefore, inflation index is nothing but the exchange rate between those economies. In JY model, nominal instantaneous forward rate,  $f_n$  and real instantaneous forward rate,  $f_r$  dynamics under real world probability measure  $(\Omega, \mathcal{F}, P)$  with filtration  $\mathcal{F}_t$ , is defined as

$$df_n(t, T) = \alpha_n(t, T)dt + \sigma_n(t, T)dW_n^P(t) \quad (4.1)$$

$$df_r(t, T) = \alpha_r(t, T)dt + \sigma_r(t, T)dW_r^P(t) \quad (4.2)$$

and dynamics of inflation index is defined as

$$dI(t) = I(t)\mu(t)dt + \sigma_I I dW_I^P(t) \quad (4.3)$$

where  $W_n^P$ ,  $W_r^P$  and  $W_I^P$  are Brownian motions with correlations  $\rho_{n,r}$ ,  $\rho_{n,I}$ ,  $\rho_{r,I}$  and  $\alpha_n$ ,  $\alpha_r$ ,  $\mu$  are adapted processes with respect to filtration  $\mathcal{F}_t$  while  $\sigma_n$ ,  $\sigma_r$  are deter-

ministic functions and  $\sigma_I$  is positive constant. Moreover, initial conditions are characterized as

$$f_n(0, T) = f_n^M(0, T) \quad (4.4)$$

$$f_r(0, T) = f_r^M(0, T) \quad (4.5)$$

$$I(0) = I_0 > 0 \quad (4.6)$$

where

- $f_n^M$  is market observed nominal instantaneous forward rate at time 0
- $f_r^M$  is market observed real instantaneous forward rate at time 0 for every maturity  $T$ .

Following Mercurio [27] let us denote the nominal and forward instantaneous short rates as

$$n(t) = f_n(t, t) \quad (4.7)$$

$$r(t) = f_r(t, t) \quad (4.8)$$

and choose the volatility functions in forward rate dynamics as

$$\sigma_n(t, T) = \sigma_n e^{-\alpha_n(T-t)} \quad (4.9)$$

$$\sigma_r(t, T) = \sigma_r e^{-\alpha_r(T-t)} \quad (4.10)$$

where  $\sigma_n, \sigma_r, \alpha_n$  and  $\alpha_r$  are positive constants. Under nominal risk neutral measure,  $Q$ , nominal, real rate and inflation index processes are

$$dn(t) = [\theta_n(t) - \alpha_n n(t)]dt + \sigma_n dW_n^Q(t) \quad (4.11)$$

$$dr(t) = [\theta_r(t) - \rho_{r,I}\sigma_I\sigma_r - \alpha_r r(t)]dt + \sigma_r dW_r^Q(t) \quad (4.12)$$

$$dI(t) = I(t)[n(t) - r(t)]dt + \sigma_I I(t)dW_I^Q(t) \quad (4.13)$$

where  $W_n, W_r$  and  $W_I$  are Brownian motions with correlations  $\rho_{n,r}, \rho_{n,I}, \rho_{r,I}$ .

Nominal and real rates are modelled as one factor Hull-White [21] hence  $\theta_n(t), \theta_r(t)$  are deterministic functions to fit initial term structure such that

$$\theta_x(t) = \frac{\partial f_x(0, t)}{\partial t} + \alpha_x f_x(0, t) + \frac{\sigma_x^2}{2\alpha_x}(1 - e^{-2\alpha_x t}), \quad x \in \{n, r\} \quad (4.14)$$

and Brownian motions  $W_n, W_r$  and  $W_I$  are correlated through the correlation matrix

$$\rho = \begin{pmatrix} 1 & \rho_{n,r} & \rho_{n,I} \\ \rho_{n,r} & 1 & \rho_{r,I} \\ \rho_{n,I} & \rho_{r,I} & 1 \end{pmatrix} \quad (4.15)$$

Instantaneous forward rates are Normally distributed under risk neutral measure as assumed in Jarrow Yildirim [23]. Therefore inflation index  $I(t)$  is lognormally distributed under risk neutral measure as proven in respective work. Then inflation index at time  $T$  can be expressed as

$$I(T) = I(t)e^{\int_t^T [n(u)-r(u)]du + \frac{1}{2}\sigma_I^2(T-t) + \sigma_I(W_I(T)-W_I(t))} \quad (4.16)$$

## 4.2 Inflation Indexed Derivatives

The framework of [23] reviewed in the previous section leads to explicit formulas for the basic options written on inflation indexed bonds; these formulas were also derived in [23] (see also [3]). In this section we present these formulas and review their derivation. The most common derivatives are inflation indexed swaps and options on inflation index. First, zero coupon inflation indexed swaps will be examined and model-independent analytical formula will be reviewed. Then, year on year inflation indexed swap will be considered. Finally, analytical formulas for inflation indexed cap and floor will be reviewed.

### 4.2.1 Zero Coupon Inflation Indexed Swaps

Zero coupon inflation indexed swap (ZCIIS) is an agreement between two parties where party A pays inflation rate for a specific period and party B pays a fixed rate. Assuming maturity of swap is  $M$  years, notional amount is  $N$  and fixed rate is  $K$  then payoff of a fixed rate leg will be

$$N [(1 + K)^M - 1] \quad (4.17)$$

and payoff of a inflation leg will be

$$N \left[ \frac{I(T_M)}{I(0)} - 1 \right] \quad (4.18)$$

By no arbitrage, value of inflation leg of ZCIIS at time  $t$  is

$$ZCIIS(t, T_M, I_0, N) = N \mathbb{E}_n \left[ e^{-\int_t^{T_M} n(u) du} \left( \frac{I(T_M)}{I_0} - 1 \right) | \mathcal{F}_t \right] \quad (4.19)$$

By foreign currency analogy, price of a real zero coupon bond is equal to price of a nominal zero coupon bond which pays one unit of inflation rate over the life of bond.

So for time  $t < T$  following equation holds:

$$I(t)P_r(t, T_M) = I(t)\mathbb{E}_r \left[ e^{-\int_t^{T_M} r(u) du} | \mathcal{F}_t \right] = \mathbb{E}_n \left[ e^{-\int_t^{T_M} n(u) du} I(T_M) | \mathcal{F}_t \right] \quad (4.20)$$

Therefore (4.19) becomes

$$ZCIIS(t, T_M, I_0, N) = N \left[ \frac{I(t)}{I_0} P_r(t, T_M) - P_n(t, T_M) \right] \quad (4.21)$$

and when  $t = 0$

$$ZCIIS(0, T_M, N) = N [P_r(0, T_M) - P_n(0, T_M)] \quad (4.22)$$

Hence the price of a zero coupon inflation indexed swap is model independent. Moreover, fair strike of a zero coupon inflation indexed swap is

$$K = \left( \frac{P_r(0, T_M)}{P_n(0, T_M)} \right)^{\frac{1}{M}} - 1 \quad (4.23)$$

After constructing both nominal and real yield curve in following section it will be possible to form term structure of ZCIIS fixed rates.

#### 4.2.2 Year on Year Inflation Indexed Swap

Year on Year Inflation Indexed Swap (YYIIS) is an agreement between two parties where Party A pays the periodic inflation rate and Party B pays fixed rate over the life of the contract. As the name of the contract reveals payment frequency is mostly annual but there might be some other contracts where payment frequency is other than annual. Value of a inflation leg at time  $t < T_{i-1}$

$$YYIIS(t, T_{i-1}, T_i, \phi_i, N) = N \phi_i \mathbb{E}_n \left[ e^{-\int_t^{T_i} n(u) du} \left( \frac{I(T_i)}{I(T_{i-1})} - 1 \right) | \mathcal{F}_t \right] \quad (4.24)$$

By taking  $\mathcal{F}_t$ -measurable part of the integral to outside of expectation, the equation above can be expressed as

$$N\phi_i \mathbb{E}_n \left[ e^{-\int_t^{T_{i-1}} n(u)du} \mathbb{E}_n \left[ e^{-\int_{T_{i-1}}^{T_i} n(u)du} \left( \frac{I(T_i)}{I(T_{i-1})} - 1 \right) | \mathcal{F}_{T_{i-1}} \right] | \mathcal{F}_t \right] \quad (4.25)$$

Inner expectation is equal to zero coupon inflation indexed swap. Therefore Eq.(4.25) becomes

$$N\phi_i \mathbb{E}_n \left[ e^{-\int_t^{T_{i-1}} n(u)du} P_r(T_{i-1}, T_i) | \mathcal{F}_t \right] - N\phi_i P_n(t, T_i) \quad (4.26)$$

When change of numerier is applied using Corollary 2 of Theorem 1 of Geman et al. [12] real instantenous rate process under  $Q_n^{T_i-1}$  becomes

$$dr(t) = [-\rho_{n,r}\sigma_n\sigma_r B_n(t, T_{i-1}) + \theta_r(t) - \rho_{r,I}\sigma_I\sigma_r - \alpha_r r(t)]dt + \sigma_r dW_r^{T_{i-1}}(t) \quad (4.27)$$

and  $W_r^{T_{i-1}}$  is a  $Q_n^{T_i-1}$  Brownian motion and therefore real bond price  $P_r(T_{i-1}, T_i)$  is lognormally distributed under  $Q_n^{T_i-1}$ . In the Section 1 - One Factor Interest Rate Models of Hull-White [21], zero-coupon bond price process is defined as follows

$$P_r(t, T) = A_r(t, T) e^{-B_r(t, T)r(t)} \quad (4.28)$$

where

$$B_r(t, T) = \frac{1}{a_r} [1 - e^{-a_r(T-t)}] \quad (4.29)$$

and

$$A_r(t, T) = \frac{P_r^M(0, T)}{P_r^M(0, t)} e^{Br(t, T)f_r^M(0, t) - \frac{\sigma_r^2}{4a_r}(1 - e^{-2a_r t})B_r(t, T)^2} \quad (4.30)$$

Then using real instantaneous rate process in Eq.(4.27) we obtain YYIIS as

$$YYIIS(t, T_{i-1}, T_i, \phi_i, N) = N\phi_i P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} - N\phi_i P_n(t, T_i) \quad (4.31)$$

where correction term, C, is

$$C(t, T_{i-1}, T_i) = \sigma_r B_r(T_{i-1}, T) \left[ B_r(t, T_{i-1}) \left( \rho_{r,I}\sigma_I - \frac{1}{2}\sigma_r B_r(t, T_{i-1}) \right) + \frac{\rho_{n,r}\sigma_n}{a_n + a_r} (1 + a_r B_n(t, T_{i-1})) - \frac{\rho_{n,r}\sigma_n}{a_n + a_r} B_n(t, T_{i-1}) \right] \quad (4.32)$$

### 4.2.3 Inflation Indexed Caplet and Floorlet

Inflation Indexed Caplet (IIC) is a call option on inflation rate where Inflation Indexed Floorlet (IIF) is a put option on inflation rate. Payoff of an option on inflation rate at

maturity  $T_i$  is equal to

$$N\psi_i \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa \right) \right]^+ \quad (4.33)$$

where

- $N$  is notion,
- $\kappa$  is strike and
- $\psi$  is the year fraction of the period  $[T_i, T_{i-1}]$ .

When  $\omega = 1$  payoff above denotes the payoff of a caplet and when  $\omega = -1$  it stands for payoff of a floorlet. Then price of the option at  $t < T_{i-1}$  under  $T_i$ -forward measure

$$\begin{aligned} IICF(t, T_{i-1}, T_i, \psi_i, K, N, \omega) &= N\psi_i \mathbb{E}_n \left[ e^{\int_t^{T_i} n(u) du} \left[ \omega \frac{I(T_i)}{I(T_{i-1})} - K \right]^+ | \mathcal{F}_t \right] \\ &= N\psi_i P_n(t, T_i) \mathbb{E}_n \left[ \left[ \omega \frac{I(T_i)}{I(T_{i-1})} - K \right]^+ | \mathcal{F}_t \right] \end{aligned} \quad (4.34)$$

Jarrow-Yildirim assuming both nominal and real rates are Gaussian and this implies inflation rate  $\frac{I(T_i)}{I(T_{i-1})}$  is lognormally distributed under  $Q_n$ . Moreover, inflation rate conditional on  $\mathcal{F}_t$  is still lognormally distributed under nominal forward measure,  $Q_n^{T_i}$ . Since inflation rate is lognormally distributed, the option prices can be derived using following property of lognormal distribution. If  $X$  is a lognormal distributed random variable with  $\mathbb{E}[X] = m$  and the variance of  $\ln(X) = v^2$  then

$$\mathbb{E} [\omega(X - K)^+] = \omega m \Phi \left( \omega \frac{\ln \frac{m}{K} + \frac{1}{2}v^2}{v} \right) - \omega K \Phi \left( \omega \frac{\ln \frac{m}{K} - \frac{1}{2}v^2}{v} \right) \quad (4.35)$$

where  $\Phi$  is standard normal cumulative distribution function. Expected value of  $\frac{I(T_i)}{I(T_{i-1})}$  can be derived using YYIS pricing formula. Recall that YYIS price under nominal forward measure  $Q_n^{T_i}$

$$YYIS(t, T_{i-1}, T_i) = P_n(t, T_i) \mathbb{E}_n^{T_i} \left[ \frac{I(T_i)}{I(T_{i-1})} - 1 | \mathcal{F}_t \right] \quad (4.36)$$

or equivalently under nominal forward measure  $Q_n^{T_{i-1}}$  it can be expressed as

$$YYIS(t, T_{i-1}, T_i) = P_n(t, T_{i-1}) \mathbb{E}_n^{T_{i-1}} [P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i) | \mathcal{F}_t] \quad (4.37)$$

when equate two expressions we get the following

$$\begin{aligned}\mathbb{E}_n^{T_i} \left[ \frac{I(T_i)}{I(T_{i-1})} - 1 | \mathcal{F}_t \right] &= \frac{P_n(t, T_{i-1})}{P_n(t, T_i)} \mathbb{E}_n^{T_{i-1}} [P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i) | \mathcal{F}_t] \quad (4.38) \\ &= \frac{P_n(t, T_{i-1})}{P_n(t, T_i)} \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, (T_{i-1}, T_i))}\end{aligned}$$

where  $C(t, T_{i-1}, T_i)$  is same convexity term derived in YYHS pricing formula. Note that it vanishes when  $t = T_{i-1}$  and expectation becomes

$$\mathbb{E}_n^{T_i} \left[ \frac{I(T_i)}{I(t)} - 1 | \mathcal{F}_t \right] = \frac{P_r(t, T_i)}{P_n(t, T_i)} \quad (4.39)$$

same as in ZCHS.

### 4.3 Options on Inflation Index

Option on inflation indexed can be priced with complete HJM model under no-arbitrage assumption. Consider a European option on inflation index with strike  $K$  and maturity  $T$ . Payoff of a call option will be

$$C_T = \max[I(T) - K, 0] \quad (4.40)$$

and for put option

$$P_T = \max[K - I(T), 0] \quad (4.41)$$

Under risk neutral measure value of a call option at time  $t$  is

$$C_t = \mathbb{E}_n^Q \left[ \max[I(T) - K, 0] e^{-\int_t^T n(u) du} \right] \quad (4.42)$$

Using Eq.(4.13) and extended Vasicek model framework, call option value is

$$\begin{aligned}C_t &= I(t)P_r(t, T)\Phi \left( \frac{\ln \left( \frac{I(t)P_r(t, T)}{KP_n(t, T)} \right) + \frac{1}{2}v^2}{v} \right) \\ &\quad - KP_n(t, T)\Phi \left( \frac{\ln \left( \frac{I(t)P_r(t, T)}{KP_n(t, T)} \right) - \frac{1}{2}v^2}{v} \right)\end{aligned} \quad (4.43)$$

and value of a put option is

$$\begin{aligned}P_t &= KP_n(t, T)\Phi \left( -\frac{\ln \left( \frac{I(t)P_r(t, T)}{KP_n(t, T)} \right) - \frac{1}{2}v^2}{v} \right) \\ &\quad - I(t)P_r(t, T)\Phi \left( -\frac{\ln \left( \frac{I(t)P_r(t, T)}{KP_n(t, T)} \right) + \frac{1}{2}v^2}{v} \right)\end{aligned} \quad (4.44)$$

where  $\Phi$  is standard Normal cumulative distribution function and  $v^2$  is

$$v^2 = \int_t^T \sigma_n(u, T)^2 du + \int_t^T \sigma_r(u, T)^2 du + 2 \int_t^T \rho_{n,r} \sigma_n(u, T) \sigma_r(u, T) du \quad (4.45)$$

$$+ 2\rho_{n,I} \sigma_I \int_t^T \sigma_n(u, T) du - 2\rho_{r,I} \sigma_I \int_t^T \sigma_r(u, T) du + \sigma_I^2(T - t)$$

#### 4.4 Deflation Protection Option Embedded in Inflation Indexed Bonds

Consider a zero coupon inflation indexed bond with deflation protection at par value,  $P$ . Payoff of this bond at maturity  $T$  is

$$\max [F(T), P] = F(T) + \frac{P}{I(0)} DPO(T) \quad (4.46)$$

where

- $F(T)$  is inflation adjusted par value such that  $F(T) = P \frac{I(T)}{I(0)}$
- $I(0)$  is reference inflation index at initial
- $I(T)$  is reference inflation index at maturity
- $DPO(T)$  is deflation protection (put) option with maturity  $T$  and payoff  $\max[I(0) - I(T), 0]$

If inflation index at maturity is greater than inflation index at initial time then deflation protection option is worthless. Hence, option will not be exercised and payoff of a bond will be equal to the inflation adjusted par value. If inflation index at maturity is less than inflation index at initial then deflation protection option will be exercised and payoff will be equal to par value. Therefore, taking a long position in abovementioned zero coupon bond is equal to taking a long position on inflation indexed bond without protection and long position on  $\frac{P}{I(0)}$  units of deflation protection option [7].

## CHAPTER 5

### DATA AND PARAMETER ESTIMATION

This section first describes the data available for deriving nominal and real yield curves. Then, historical development of yield curves is addressed and ZCIIS yield curve is constructed. Interest rate, volatility and correlation parameters are estimated using time varying volatility function proposed by Jarrow-Yildirim [23], and the statistical estimators of these parameters are presented.

#### 5.1 Data

Treasury has issued first inflation indexed government bond in 1998. Maturity of the bonds was mostly a year and they were offering high real yield reflecting the interest rate developments and inflationary environment. In 2007, Treasury started to issue CPI linked bonds to diversify the instruments issued, to broaden investor base and to extend funding maturity. The new term bonds have 5 years or more maturity with semi-annual coupon payments. There are 20 inflation indexed outstanding bonds in the market around 200 billion TL nominal value as of June 2020. Figure 5.1 shows total outstanding nominal value of IIB's since 2012.

##### 5.1.1 Nominal Yield Curve

Nominal yield curve presents the relation between nominal interest rates and maturity. Analyzing the evolution of yield curve over time is important for two reasons. First, inflation indexed bonds are exposed to interest rate risk as nominal bonds are.

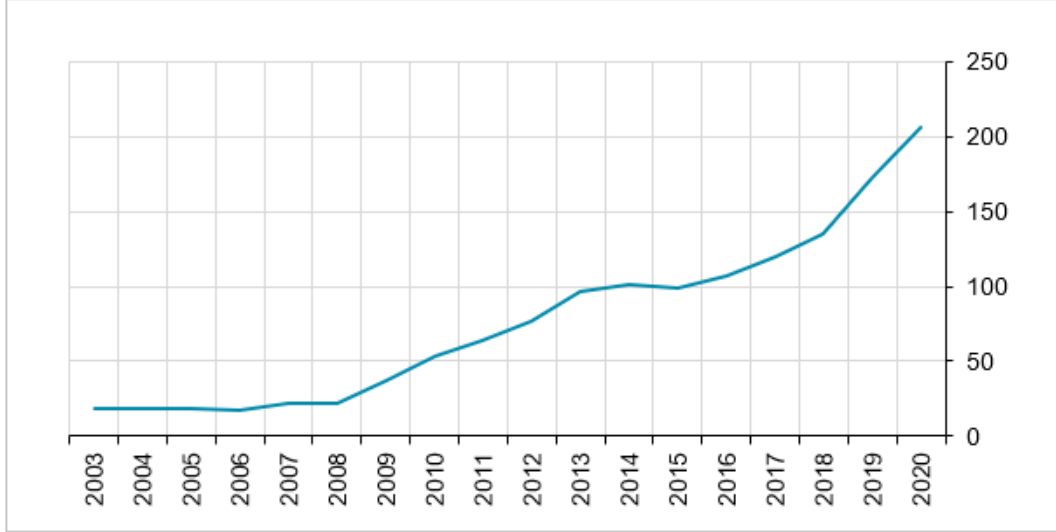


Figure 5.1: Nominal Value of Outstanding Inflation Indexed Bonds

Source: Ministry of Treasury and Finance, nominal outstanding, billions of TL

Therefore, its price changes as shape of the yield curve changes. Second, time series of yield curve data is used to estimate parameters.

We will follow Nelson-Siegel method as proposed by Gurkaynak et al [15]. This is because parametric models are better at capturing changes along the yield curve than non-parametric models. Daily price of zero coupon and fixed coupon bonds is obtained from Bloomberg for the period of January 2014 and March 2020. Nelson-Siegel method proposes a functional form for zero rate as following [15]

$$r(m, \beta, \tau) = \beta_0 + \beta_1 \left( \frac{1 - e^{-\frac{m}{\tau}}}{-\frac{m}{\tau}} \right) + \beta_2 \left( \frac{1 - e^{-\frac{m}{\tau}}}{-\frac{m}{\tau}} - e^{-\frac{m}{\tau}} \right) \quad (5.1)$$

where

- $m$  is time to maturity and
- $\beta = (\beta_0, \beta_1, \beta_2)$  and  $\tau$  are parameters to be estimated.

Estimation process aims to minimize squared difference between market price and fitted price of bonds. In this context each coupon bond considered as series of zero coupon bonds. And the price difference weighted by inverse of duration to minimize estimation error for short term rates in the yield curve. Therefore the objective

function is

$$\min_{\beta, \tau} \sum_{i=1}^k \left( \frac{P_t^{i,m} - P_t^{i,fitted}}{D_i} \right)^2 \quad (5.2)$$

where

- $P_t^{i,m}$  is market price of i-th bond at time  $t$
- $P_t^{i,fitted}$  is fitted price of i-th bond at time  $t$
- $D_i$  is duration of i-th bond.

Average values of nominal rates for  $\beta_0$ ,  $\beta_1$ ,  $\beta_2$ , and  $\tau$  are shown in Table 5.1.  $\beta$  parameters interpreted as level, slope and curvature [5].

Table 5.1: Average Parameters for Nominal Rates (2014-2020)

| $\beta_0$ | $\beta_1$ | $\beta_2$ | $\tau$ |
|-----------|-----------|-----------|--------|
| 10.54     | 1.71      | 2.07      | 2.24   |

Figure 5.2 shows evolution of nominal yield curve. See Appendix A for time series of 2, 5 and 10 year nominal rates.

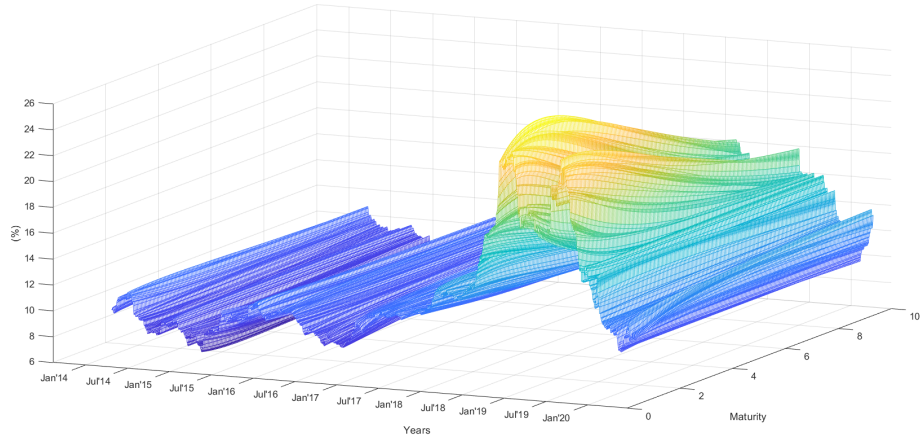


Figure 5.2: Term Structure of Nominal Interest Rate

Source: Author's calculations

### 5.1.2 Real Yield Curve

Real yield curve presents the relation between real interest rates and maturity. Eq (5.1) and Eq (5.2) will be used to form real yield curve as well. Daily prices of

inflation indexed bonds also obtained from Bloomberg for the same period. However, inflation indexed bond prices are quoted as gross price such that

$$Gross\ Price = (Clean\ Price + Accrued\ Interest) \frac{I(t)}{I(0)} \quad (5.3)$$

- $I(t)$  is reference index at time  $t$  and
- $I(0)$  is reference index at time issuance of bond.

To derive real yield curve, we need to multiply each bond price with inverse of respective index ratio. Consider a bond with clean price 100, accrued interest 1.2 and reference index be 110 at time  $t$  while be 100 at initiation. Then the gross price quoted for this bond will be

$$Gross\ Price = (100 + 1.2) \frac{110}{100} = 111.32$$

Average values of real rates for  $\beta_0$ ,  $\beta_1$ ,  $\beta_2$ , and  $\tau$  are shown in Table 5.2.

Table 5.2: Average Parameters for Real Rates (2014-2020)

| $\beta_0$ | $\beta_1$ | $\beta_2$ | $\tau$ |
|-----------|-----------|-----------|--------|
| 3.10      | 9.91      | -15.26    | 0.86   |

Figure 5.3 shows evolution of real yield curve. It can be concluded from both Figure 5.2 and Figure 5.3 that real rates are more volatile than nominal rates for short end of the yield curve. See Appendix B for time series of 2, 5 and 10 year real rates.

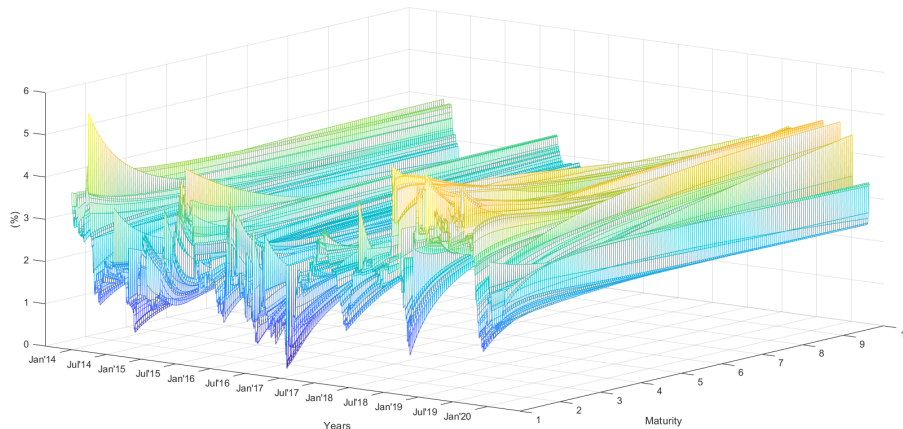


Figure 5.3: Term Structure of Real Interest Rate

Source: Author's calculations

There is no ZCIIS on Turkey's inflation traded in the market. But given yield curves of nominal and inflation indexed bonds term structure of ZCIIS rates can be calculated by (4.23). Figure 5.4 shows the time series of ZCIIS rates.

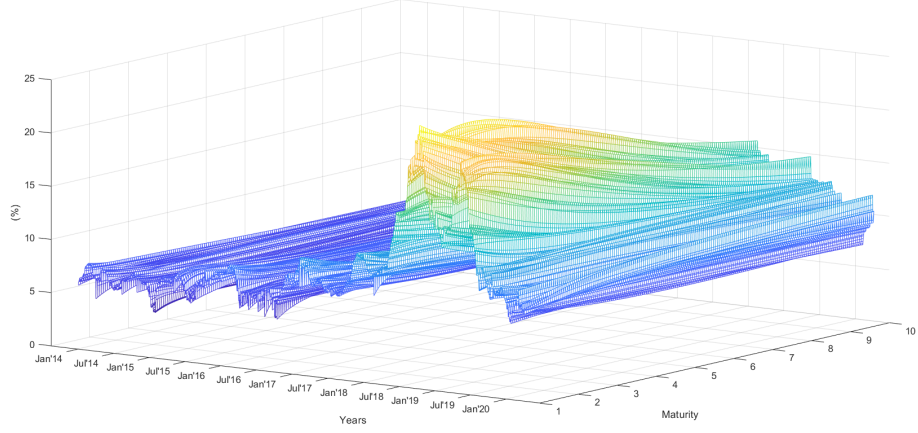


Figure 5.4: Term Structure of Zero Coupon Inflation Indexed Swap

Source: Author's calculations

## 5.2 Parameter Estimation

### 5.2.1 Real Rate Parameters

Using Proposition 1 of Jarrow-Yildirim [29], we state that real bond price dynamics in HJM model under martingale measure follows

$$\frac{dP_r(t, T)}{P_r(t, T)} = \left[ r(t) - \rho_{rI}\sigma_I(t) + \int_t^T \sigma_r(t, s)ds \right] dt - \int_t^T \sigma_r(t, s)ds dW_r^Q(t) \quad (5.4)$$

Consider a one factor time varying volatility function in the form of

$$\sigma_r(t, T) = \sigma_r e^{-a_r(T-t)} \quad (5.5)$$

where  $\sigma_r > 0$ ,  $a_r > 0$  are constants. Therefore bond return evolves with Normal distribution

$$\frac{dP_r(t, T)}{P_r(t, T)} - \left[ r(t) - \rho_{rI}\sigma_I(t) + \int_t^T \sigma_r(t, s)ds \right] \Delta t \sim N \left[ 0, \left( \int_t^T \sigma_r(t, s)ds \right)^2 \Delta t \right] \quad (5.6)$$

When we use daily observation of bond prices, ( $\Delta t = 1/360$ ), expected return on the bond,  $\left[ r(t) - \rho_{rI}\sigma_I(t) + \int_t^T \sigma_r(t, s)ds \right] \Delta t$ , is smaller than its standard deviation,

$\int_t^T \sigma_r(t, s) ds$ . Then, we can disregard it for parameter estimation. Hence, variance of a zero coupon bond is expressed as following

$$var \left( \frac{\Delta P_r(t + \Delta t, T)}{P_r(t, T)} \right) = \sigma_r^2 \left( \frac{e^{-a_r(T-t)} - 1}{a_r} \right)^2 \Delta t \quad (5.7)$$

Using the time series of zero coupon real bond prices, derived from real yield curve presented in the section above, we compute the sample variance for different maturities. Later, we run a cross section non-linear regression to estimate  $\sigma_r$  and  $a_r$ . We get the parameters as in Table 5.3.

Table 5.3: Nominal and Real Rate Parameters

|            |         |
|------------|---------|
| $\sigma_r$ | $a_r$   |
| 0.00153    | 0.4251  |
| $\sigma_n$ | $a_n$   |
| 0.00529    | 0.14010 |

### 5.2.2 Nominal Rate Parameters

Again, using Proposition 1 of Jarrow-Yildirim [29], we state that nominal bond price dynamics in HJM model under martingale measure follows

$$\frac{dP_n(t, T)}{P_n(t, T)} = n(t)dt - \int_t^T \sigma_n(t, s) dW_n^Q(t) \quad (5.8)$$

Considering same volatility function with constant parameters,  $\sigma_n$  and  $a_n$  we get

$$var \left( \frac{\Delta P_n(t + \Delta t, T)}{P_n(t, T)} \right) = \sigma_n^2 \left( \frac{e^{-a_n(T-t)} - 1}{a_n} \right)^2 \Delta t \quad (5.9)$$

Using the time series of zero coupon nominal bonds we run same procedure for nominal rate parameters. We get the parameters as in Table 5.3.

### 5.2.3 Inflation Parameters

In this subsection volatility of inflation will be estimated. we assume volatility,  $\sigma_I$  is constant and estimated by the following

$$\sigma_I = \left\{ \frac{1}{\Delta t} var \left( \frac{\Delta I(t)}{I(t)} \right) \right\}^{\frac{1}{2}} \quad (5.10)$$

Inflation index data is obtained from Turkstat [?]cpi2020. Even though we have reference index data on daily basis we prefer to use monthly release because linear interpolation distorts estimation procedure.

#### 5.2.4 Correlation Parameters

Correlation between processes is estimated by following equations.

$$\rho_{n,I} = \text{corr} \left( \Delta r_n(t), \frac{\Delta I(t)}{I(t)} \right) \quad (5.11)$$

$$\rho_{r,I} = \text{corr} \left( \Delta r_r(t), \frac{\Delta I(t)}{I(t)} \right) \quad (5.12)$$

$$\rho_{n,r} = \text{corr} (\Delta r_n(t), \Delta r_r(t)) \quad (5.13)$$

Estimated correlations are shown in Table 5.4. It is remarkable that relation between nominal rates and inflation index in the long term is different than the short term. This is because inflation expectations have more weight on nominal interest rates in longer terms.

Table 5.4: Correlation Parameters

| Maturity | $\rho_{r,I}$ | $\rho_{n,I}$ | $\rho_{n,r}$ |
|----------|--------------|--------------|--------------|
| 1        | -0.4506      | -0.0510      | 0.1848       |
| 2        | -0.4520      | -0.0756      | 0.3016       |
| 3        | -0.4057      | -0.0909      | 0.3802       |
| 4        | -0.3555      | -0.0901      | 0.4077       |
| 5        | -0.3093      | -0.0768      | 0.4074       |
| 6        | -0.2696      | -0.0531      | 0.3942       |
| 7        | -0.2354      | -0.01961     | 0.3735       |
| 8        | -0.2048      | 0.0229       | 0.3459       |
| 9        | -0.1767      | 0.0728       | 0.3117       |
| 10       | -0.1510      | 0.1270       | 0.2725       |



## CHAPTER 6

### EMPIRICAL RESULTS

The derivation procedure outlined in previous chapters has been implemented on 9 specific inflation-indexed bonds issued by Turkish Treasury. In specific, the issue dates of examined instruments cover the years between 2012 and 2018. Some descriptive information belonging to these bonds such as issue date, maturity date, coupon rate and par values are presented in Table 6.1 (See Appendix C for detailed information about all the bonds covered in data set).

Table 6.1: Inflation Indexed Bonds

| No. | ISIN         | Issue Date | Maturity Date | Coupon Rate (Semi-Annual) | Reference Index - I(0) |
|-----|--------------|------------|---------------|---------------------------|------------------------|
| 1   | TRT230222T13 | 7/3/2012   | 23/2/2022     | 1.50%                     | 201.07                 |
| 2   | TRT261022T10 | 7/11/2012  | 26/10/2022    | 1.00%                     | 205.85                 |
| 3   | TRT081123T10 | 20/11/2013 | 8/11/2023     | 1.40%                     | 223.29                 |
| 4   | TRT080524T17 | 21/5/2014  | 8/5/2024      | 1.20%                     | 236.24                 |
| 5   | TRT180924T11 | 1/10/2014  | 18/9/2024     | 1.00%                     | 243.17                 |
| 6   | TRT160425T17 | 29/4/2015  | 16/4/2025     | 1.00%                     | 252.12                 |
| 7   | TRT140126T11 | 27/1/2016  | 14/1/2026     | 1.35%                     | 268.69                 |
| 8   | TRT070727T13 | 19/7/2017  | 7/7/2027      | 1.45%                     | 310.03                 |
| 9   | TRT280628T18 | 11/7/2018  | 28/6/2028     | 1.65%                     | 344.57                 |

The inclusion of multiple indexed bonds allows one to analyze time-varying reaction of the deflation premium to different periods characterized by varied interest rate and inflation outlook. As shown in Table 6.2, it is found that the deflation premium ranges from 0 to 0.3419%, with average value corresponding to 0.0107%.

Table 6.2: Deflation Protection Premium Statistics

| Maximum | Minimum | Number of Observations |
|---------|---------|------------------------|
| 0.00342 | 0       | 480                    |

Historical course of deflation premium derived from examined instruments are depicted in Figure 6.1. It is asserted that deflation premium turned out to be relatively substantial during stable inflationary dynamics throughout 2015. This finding is related to the fundamental principle of option pricing: the feature of moneyness amplifies the option value. On the other hand, it is known that it takes more time for an option to go deep out-of-money in the case of low inflation environment. This relationship can be observed for deflation premium extracted from IIB-4 and IIB-5. In parallel to the acceleration of inflation rate from the levels of 6-7% to over 9% throughout the second half of 2015, a concordant sharp decline is experienced in the deflation premium estimates. Furthermore, similar reaction is detected when downward movements in inflation rate prevails. In the beginning of second half of 2019 the economy has experienced low inflation rate due to base effects. It can clearly be seen for IIB-9 for that period. Deflation premium has increased to about 0.02% from almost zero.

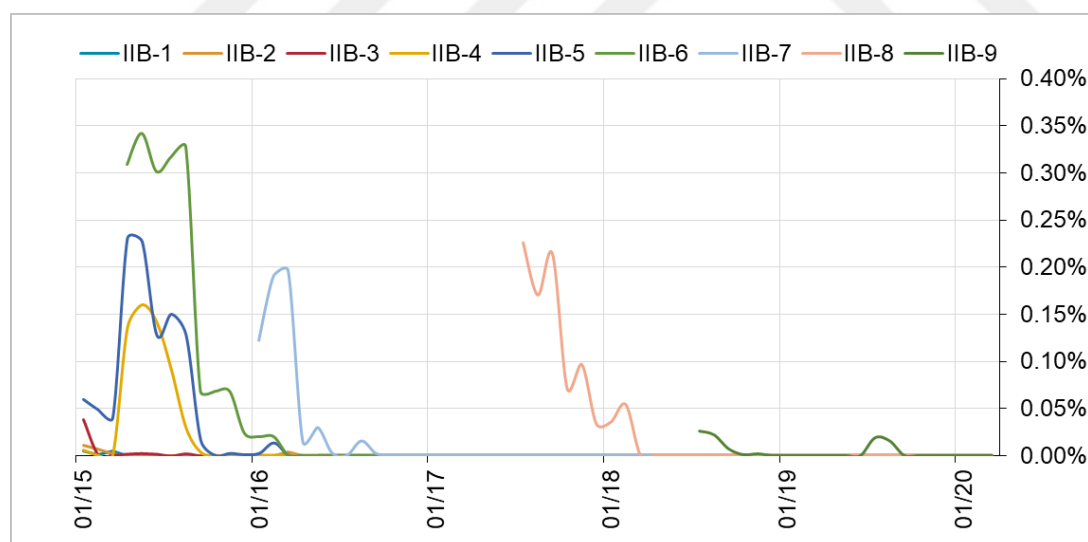


Figure 6.1: Deflation Premium

Source: Author's calculations

Apart from inflation realizations, interest rate stands as an important parameter in determining the option value. Deflation premium hovers around 0.10%-0.20% levels for bond issues undertaken from 2016 to 2018. However, it emerges as less than

0.05%, especially after the second half of 2018. It is not surprising to see such pricing tendencies because interest rate went up to more than 20% in that period, initiated by the monetary policy tightening of CBRT, financial market volatilities and non-accommodative financial conditions. In addition to this, another structural component influencing the deflation premium as option value seems to be the maturity. According to the results in Figure 6.1, option value elevates as time to maturity increases. Combined with the feature of being deep out-of-the-money, we can speculate that options are mostly worthless for the rest of their life unless dramatic shifts occur in interest rate and inflation. In the following step, we investigate whether or not obtained

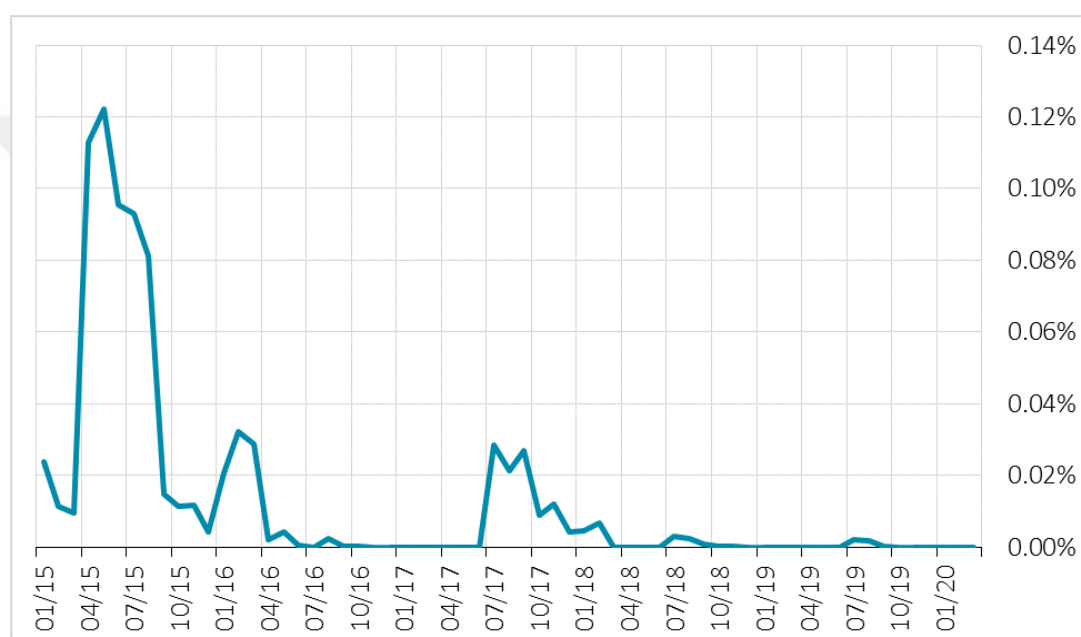


Figure 6.2: Average Deflation Premium

Source: Author's calculations

deflation premium is associated with the inflation expectations. To this end, point inflation expectations regarding 12-months and 24-months horizons are obtained from CBRT Inflation Tendency Survey (See the Appendix D for detailed series). On top of point forecasts, the standard deviations of individual survey respondents' inflation expectations are retrieved to proxy for inflation uncertainty, in other words, to what extent inflation expectations are dispersed. These series are both taken in 12 and 24-months ahead forecast to cover shorter and relatively medium-term expectation formation. Table 6.3 documents the sizeable correlations between inflation expectations and average deflation premium which is observed both for 12 and 24-months ahead value.

Table 6.3: Correlation between Deflation Premium and Inflation Expectations (CBRT Inflation Tendency Survey)

| 12-Months Ahead Inflation Expectations | Inflation Uncertainty (12-Months Ahead) | 24-Months Ahead Inflation Expectations | Inflation Uncertainty (24-Months Ahead) |
|----------------------------------------|-----------------------------------------|----------------------------------------|-----------------------------------------|
| -0.3047                                | -0.3751                                 | -0.3325                                | -0.3976                                 |

As the final phase, we calculate a single unique deflation premium series shown in Figure 6.2 by taking the simple average of unique deflation premium attached to examined 9 bonds. On top of level expectations, deflation premium is also found to be negatively correlated with inflation uncertainty as shown in Figure 6.3. This finding is sensible considering the fact that stable course of current inflation would be translated into subdued inflation expectations and decline inflation uncertainty. This might eventually be priced in bond markets, ultimately leading to higher deflation premium.

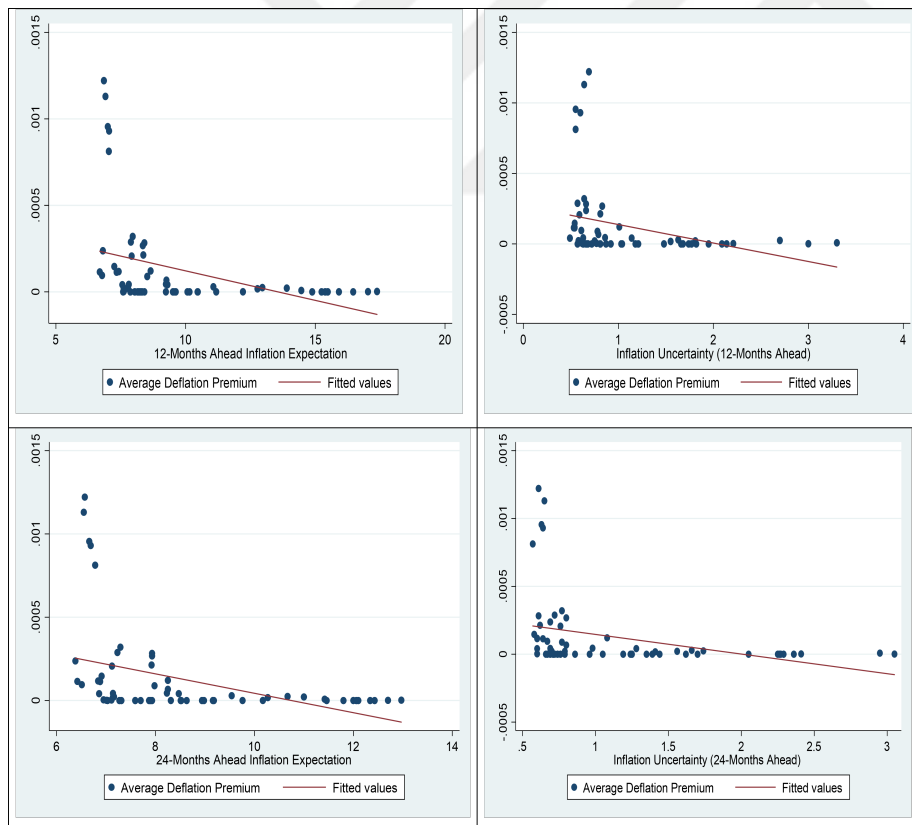


Figure 6.3: Scatterplots of Deflation Premium and Inflation Expectations

Source: Author's calculations

## **CHAPTER 7**

### **CONCLUSION**

Fixed income securities are financial contracts that pay a stream of pre-determined interest payments and par value over their terms. One of the main risks related to bond cash flows is inflation risk. From the perspective of the bond buyer (lender), the purchasing power of the funds composed of interim payments and par value might be subject to erosion in inflationary pressures. Inflation indexed bonds are issued to allow market participants to hedge this risk. Hence, it has considerable interest from market participants and according to BIS nominal amount of these bonds has reached 25 trillion USD by end of 2019.

Inflation indexed bonds might be issued as plain bonds which exclude any protection against deflation. In the structure of such bonds, payments are anchored to the realized inflation rate. However, in the case of deflation, there is the possibility that bond payments might turn out to be lower than what is specified as the nominal amount, and the payoff can become subdued. For this reason, inflation-indexed bonds are also mostly issued with embedded options designed to protect against deflation.

Deflation premium is calculated for 9 inflation indexed bonds, selected with different characteristics for the period of 2015 to 2020. It is asserted that deflation premium turned out to be relatively substantial during stable inflationary dynamics throughout 2015. This finding is related to the fundamental principle of option pricing: the feature of moneyness amplifies the option value. On the other hand, it is known that it takes more time for an option to go deep out-of-money in the case of low inflation environment. Same phenomena can be seen for second half of 2019 where the economy experienced sluggish inflation rate due to base effects. It can clearly be

seen that deflation premium has increased. Apart from inflation realizations, interest rate stands as an important parameter in determining the option value. It is not surprising to see deflation premium diminishes in times of high interest environment initiated by the monetary policy tightening of CBRT, financial market volatilities and non-accommodative financial conditions. Finally, another structural component influencing the deflation premium as option value seems to be the maturity. Option value elevates as time to maturity increases. Combined with the feature of being deep out-of-money, we can speculate that options are mostly worthless for the rest of their life unless dramatic shifts occurs in interest rate and inflation.

Deflation premium is not only associated with the inflation realizations but also with inflation expectations. We calculate single deflation premium value for the selected bonds by taking average of deflation premiums. We see that deflation premium is negatively correlated with inflation expectations and inflation uncertainty for 12 and 24 months ahead.

Further studies might focus on Grishchenko et al. [13] methodology by creating an index that based on deflation premium or option return to used as explanatory variable for future inflation rate to examine statistical and economic significance. Also, it might be promising alternative indicator to inflation compensation to explain dynamics of inflation expectation.

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## APPENDIX A

### TIME SERIES OF NOMINAL RATES

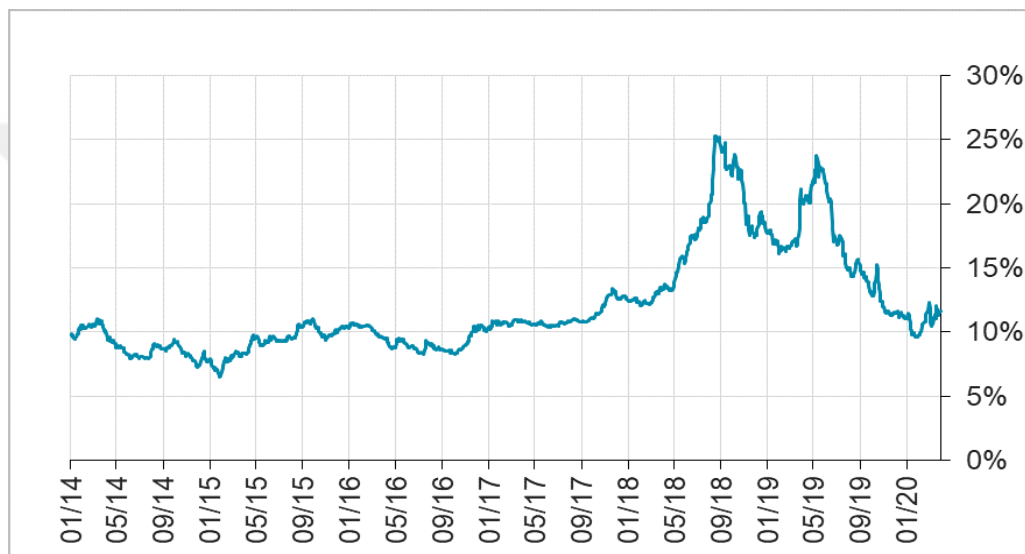


Figure A.1: 2-Year Nominal Rate

Source: Author's calculations

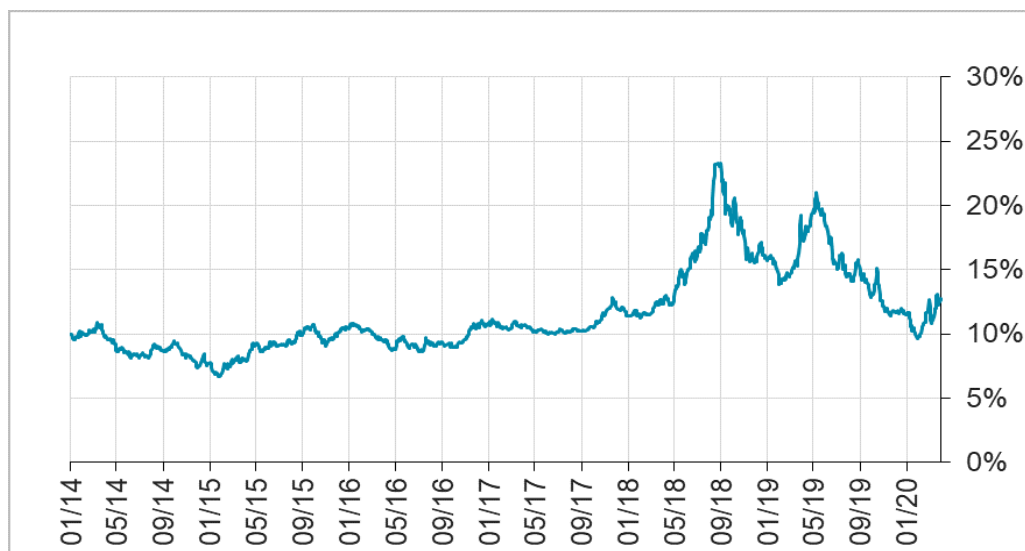


Figure A.2: 5-Year Nominal Rate

Source: Author's calculations



Figure A.3: 10-Year Nominal Rate

Source: Author's calculations

## APPENDIX B

### TIME SERIES OF REAL RATES

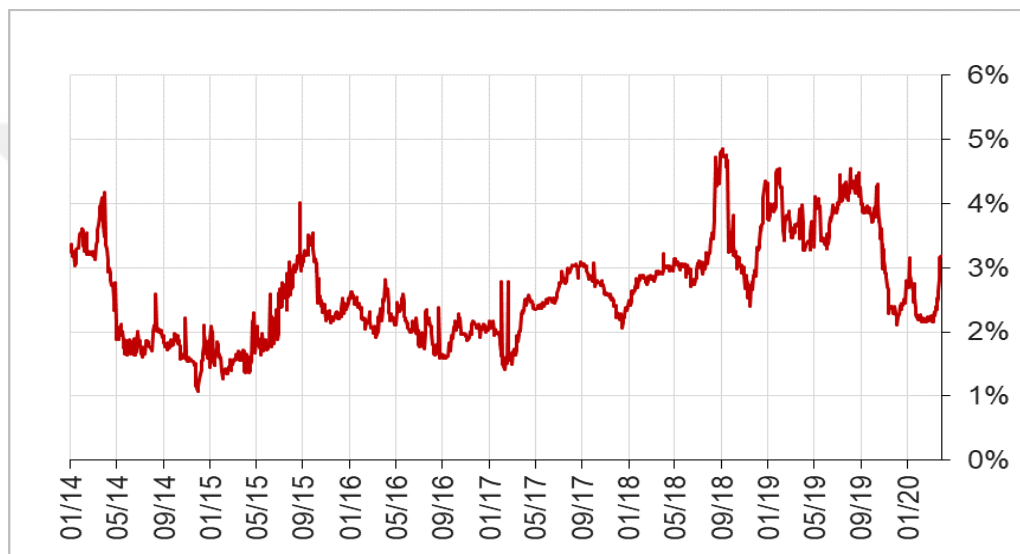


Figure B.1: 2-Year Real Rate

Source: Author's calculations



Figure B.2: 5-Year Real Rate

Source: Author's calculations



Figure B.3: 10-Year Real Rate

Source: Author's calculations

## APPENDIX C

### INFLATION INDEXED BOND DATA SET

Table C.1: Inflation Indexed Bonds in Data Set

| No. | ISIN         | Issue Date | Maturity Date | Coupon Rate<br>(Semi-Annual) | Reference Index |
|-----|--------------|------------|---------------|------------------------------|-----------------|
| 1   | TRT110215T16 | 17/02/2010 | 11/02/2015    | 2.25%                        | 170.52          |
| 2   | TRT010420T19 | 14/04/2010 | 01/04/2020    | 2.00%                        | 175.16          |
| 3   | TRT290415T14 | 05/05/2010 | 29/04/2015    | 2.00%                        | 176.72          |
| 4   | TRT060121T16 | 19/01/2011 | 06/01/2021    | 1.50%                        | 182.38          |
| 5   | TRT040516T11 | 11/05/2011 | 04/05/2016    | 1.25%                        | 184.18          |
| 6   | TRT210721T11 | 03/08/2011 | 21/07/2021    | 1.50%                        | 190.63          |
| 7   | TRT230222T13 | 07/03/2012 | 23/02/2022    | 1.50%                        | 201.07          |
| 8   | TRT261022T10 | 07/11/2012 | 26/10/2022    | 1.00%                        | 205.85          |
| 9   | TRT030523T13 | 15/05/2013 | 03/05/2023    | 0.50%                        | 218.04          |
| 10  | TRT020823T11 | 14/08/2013 | 02/08/2023    | 1.50%                        | 220.77          |
| 11  | TRT081123T10 | 20/11/2013 | 08/11/2023    | 1.40%                        | 223.29          |
| 12  | TRT200219T11 | 26/02/2014 | 20/02/2019    | 1.75%                        | 228.90          |
| 13  | TRT080524T17 | 21/05/2014 | 08/05/2024    | 1.20%                        | 236.24          |
| 14  | TRT180924T11 | 01/10/2014 | 18/09/2024    | 1.00%                        | 243.17          |
| 15  | TRT160425T17 | 29/04/2015 | 16/04/2025    | 1.00%                        | 252.12          |
| 16  | TRT140126T11 | 27/01/2016 | 14/01/2026    | 1.35%                        | 268.69          |
| 17  | TRT070727T13 | 19/07/2017 | 07/07/2027    | 1.45%                        | 310.03          |
| 18  | TRT120128T11 | 24/01/2018 | 12/01/2028    | 1.45%                        | 323.95          |

**Table C.1 continued from previous page**

| No. | ISIN         | Issue Date | Maturity Date | Coupon Rate<br>(Semi-Annual) | Reference Index |
|-----|--------------|------------|---------------|------------------------------|-----------------|
| 19  | TRT280628T18 | 11/07/2018 | 28/06/2028    | 1.65%                        | 344.57          |
| 20  | TRT100124T17 | 16/01/2019 | 10/01/2024    | 2.10%                        | 398.47          |
| 21  | TRT050624T35 | 12/06/2019 | 05/06/2024    | 2.05%                        | 405.31          |
| 22  | TRT061124T11 | 13/11/2019 | 06/11/2024    | 1.60%                        | 424.52          |
| 23  | TRT290125T15 | 05/02/2020 | 29/01/2025    | 1.18%                        | 437.70          |

## APPENDIX D

### INFLATION EXPECTATIONS

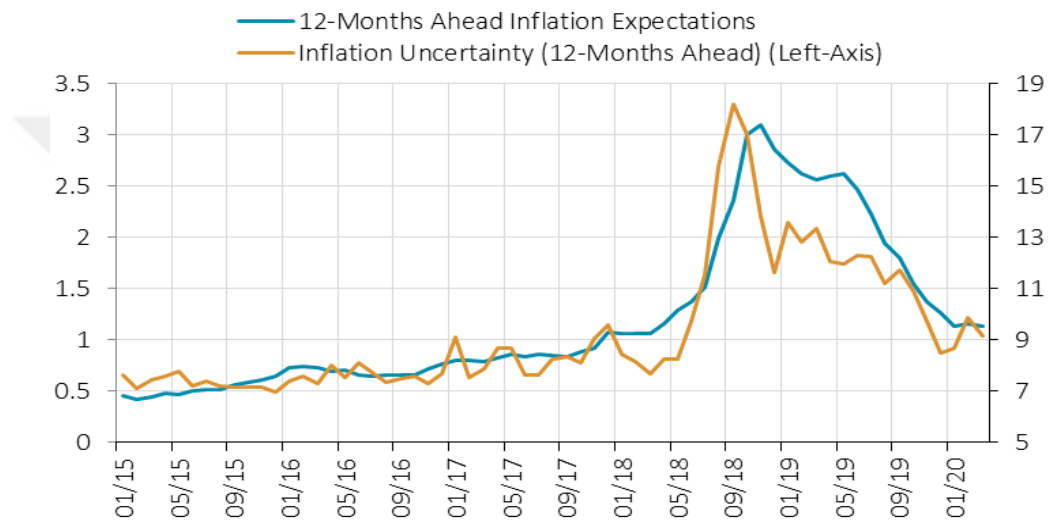


Figure D.1: 12 Months Ahead Inflation Expectation

Source: CBRT Inflation Tendency Survey

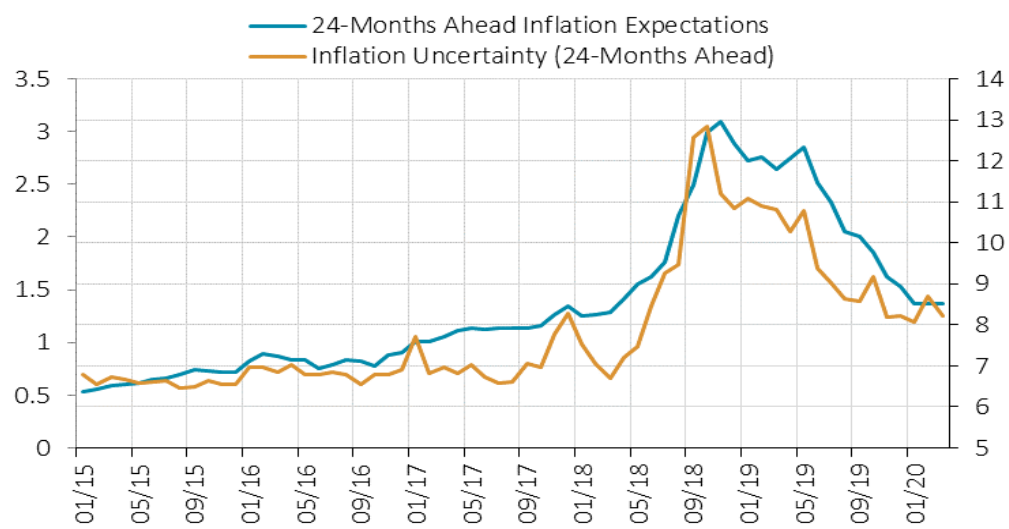


Figure D.2: 24 Months Ahead Inflation Expectation

Source: CBRT Inflation Tendency Survey