ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

THE EXACT SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS WITH LIE SYMMETRY ANALYSIS

DEPARTMENT OF MATHEMATICS

DOCTOR OF PHILOSOPHY THESIS

MİRAÇ GÜNEYSU

AUGUST 2022

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ADVISOR: Assist. Prof. Dr. Melih GÖCEN

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by academic rules and ethical principles all works that are not result of this study are cited properly."

Miraç GÜNEYSU

ABSTRACT

Doctor of Philosophy Thesis

THE EXACT SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS WITH LIE SYMMETRY ANALYSIS

Miraç GÜNEYSU

Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assist. Prof. Dr. Melih GÖCEN

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This thesis involves the study of symmetries to find the exact solutions of some rational difference equations. We obtain the solutions for special cases of the difference equations by using Lie symmetry analysis. This thesis consist of six chapter.

In the first chapter, we introduce the difference equations and Lie symmetry analysis. Also we give a brief literature review of studies about the thesis topic.

In the second chapter, we give some basic definitions and theorems needed in this thesis.

In the third chapter, we consider an example of difference equation which the exact solutions obtained by Lie symmetry analysis.

ABSTRACT (continued)

In the fourth chapter, we investigate the Lie point symmetries of some fifth order difference equations and obtain their exact solutions with Lie symmetry analysis.

In the fifth chapter, we give the exact solutions of some higher order rational difference equation by using Lie symmetry analysis.

In the final chapter, apart from the form of the difference equations which is mentioned in the other chapters, we consider a different type of difference equation and obtain the exact solutions.

Keywords: Difference equation, Lie symmetry analysis, exact solutions, group invariant solutions.

Science Code: 403.06.01

ÖZET

Doktora Tezi

BAZI RASYONEL FARK DENKLEMLERİNİN LİE SİMETRİ ANALİZİ İLE KESİN ÇÖZÜMLERİ

Miraç GÜNEYSU

Zonguldak Bülent Ecevit Üniversitesi
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Bu tez, fark denklemlerinin kesin çözümlerinin simetri ile bulunmasını içerir. Fark denklemlerinin özel durumlarının çözümlerini Lie simetri analizi ile elde ederiz. Bu tez altı bölümden oluşmaktadır.

Birinci bölümde, fark denklemlerini ve Lie simetri analizini tanıtıyoruz. Ayrıca, tez konusu ile ilgili çalışmaların kısa bir literatür taramasını veriyoruz.

İkinci bölümde, bu tezde gerekli olan bazı tanım ve teoremleri veriyoruz.

Üçüncü bölümde, kesin çözümleri Lie simetri analizi ile elde edilen bir fark denklemi örneği ele alıyoruz.

Dördüncü bölümde, bazı beşinci mertebeden fark denklemlerinin Lie nokta simetrilerini araştırıyoruz ve Lie simetri analizi ile kesin çözümlerini elde ediyoruz.

ÖZET (devam ediyor)

Beşinci bölümde, bazı yüksek mertebeden rasyonel fark denklemlerinin kesin çözümlerini Lie simetri analizi kullanarak veriyoruz.

Son bölümde, diğer bölümlerde bahsedilen fark denklemleri formu dışında, farklı bir fark denklem tipini ele alıyoruz ve kesin çözümlerini elde ediyoruz.

AnahtarKelimeler: Fark denklemi, Lie simetri analizi, kesin çözümler, değişmez grup çözümleri.

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LIST OF SYMBOLS

$\mathbb{N}_0$	: {0,1,2,3,...}
$\mathbb{R}$	: Set of real numbers
$\sum$	: Summation symbol
$\prod$	: Multiplication symbol
O(f)	: $\{ g(x): \exists\, A, a \in \mathbb{R}^+$ such that $\lvert g(x)\rvert \leq A\lvert f(x)\rvert$ for all $x > a\}$
$\lfloor x \rfloor$	: $n \leq x < n + 1, n \in \mathbb{Z}$

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEWS

1.1 INTRODUCTION

Difference equations; sometimes it arises from generator functions, sometimes from numerical solutions of differential equations and sometimes it emerges as a mathematical model of a physical event. Difference equations appear in the study of discretization methods for differential equations. Many of the results obtained in the theory of difference equations are almost discrete analogs of the corresponding differential equations. However, the theory of difference equations is richer than the corresponding theory of differential equations. In difference equations, the independent variable is defined on integers. Therefore, instead of derivative terms in difference equations, the difference of the unknown function is found. In this respect, difference equations rather characterize discontinuous problems. Difference equations have attracted the attention of scientists working in various branches of science for a long time. During this time, scientists achieved very effective results on difference equations and ensured the continuous development of this field.

Recently there has been great importance in studying the difference equations in the discipline of mathematics in addition other sciences. One of the reasons for this is importance for techniques applied in difference equations are used in mathematical models of real-life situations such as ecology, genetics, sequence and statistical problems, probabilistic time series, combinatorial analysis, number teory, geometry, electrical circuits, radiation, psychology, sociology, stock market movements in economics, the research of the number of live populations in medicine and biology, ecology, genetics, more importantly the study of cell movements like the ratio of rise in canser cells and behavior HIV virus and so much. For example some environmental scientists used the simple nonlinear difference equation which included a logistic equation, to examine year-to-year changes in the behavior of populations, in1950s. However, in the early 1970s, Robert May studied the types of complex behaviors exhibited by the logistic equation. He investigated on the relationship between these behaviors and fluctations in real populations.

Lately, researchers have used the 'change of variables' approach in which a new variable is introduced and the original equation is transformed into a simpler and solvable equation. However, the change of variables is often not obvious to determine. For this reason, we need an approach which is systematically formal and leads to such changes of variable.

Furthermore, it is important to find the solution forms of difference equations and this is a topic that has been studied recently. One of these studies is the Lie symmetry analysis method, which is one of the new methods. Lie symmetry method provides a theory for finding a proper change of variables. The Norwegian mathematician Sophus Lie studied the group of mappings which leaves the differential equations invariant [1]. With this approach, one can solve difference equations using the group of transformations that leaves the equations invariant, similarly. After the inspiring works of Lie, the invariance properties of these equations under groups of point transformations attracted great interest. The symmetry method has been used to obtain the form of the solutions of difference equations, see for example [2 − 5]. Moreover, Maeda [6] showed how to use symmetry methods and get the solutions of the system of first-order ordinary difference equations.Lie analysis was initiated by the Norwegian mathematician, Sophus Lie [7, 8]. For more applications of this method, see [9, 10 − 13].

The purpose in this thesis is to investigate solutions of some specific nonlinear difference equations with Lie symmetry analysis.This thesis consist of five chapters.

The first chapter is a concise overview of what this thesis is about and also is a literature summary of difference equation theory with Lie symmetry analysis.

The second chapter consists of some basic important definitions and some significant theorems used throughout the thesis.

The third chapter includes solutions of the difference equations

$$x_{n+1} = \frac{x_{n-2}x_n}{x_{n-1}(a_n + b_n x_{n-2}x_n)}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are non-zero real sequences, via symmetries with less restriction on the initial conditions.

The fourth chapter presents solutions of the difference equations

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(a_n + b_n x_n x_{n-2} x_{n-4})}$$

in closed form, where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are non-zero real sequences, via the technique of Lie group analysis.

The fifth chapter contains solutions of the difference equations

$$x_{n+1} = \frac{x_{n-9}}{a_n + b_n x_{n-4} x_{n-9}}$$

for some arbitrary sequences of real numbers (a_n) and (b_n).

The sixth chapter introduces solutions of the difference equations

$$x_{n+1} = \frac{x_n x_{n-2}}{a_n x_{n-2} + b_n x_{n-3}}$$

for some arbitrary sequences of real numbers (a_n) and (b_n).

1.2 REVIEW ON DIFFERENCE EQUATION WITH LIE SYMMETRY ANALYSIS

In this section we give some difference equation that have an important role in the literature for Lie Symmetry Analysis.

- In [14], Ibrahim and Touafek investigated the solutions and properties of the difference equation

  $$x_{n+1} = \frac{x_{n-1} x_{n-2}}{x_n \left(a_n + b_n x_{n-1} x_{n-2}\right)} \tag{1.1}$$

  where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are real two-periodic sequences and the initial values x_{-2}, x_{-1}, x_0 are non-zero real numbers. In [15], Mnguni and Folly-Gbetoula used a Lie symmetry-based method to solved Eq.(1.1). Equivalently, they studied

  $$u_{n+3} = \frac{u_n u_{n+1}}{u_{n+2} \left(A_n + B_n u_n u_{n+1}\right)}$$

  instead, where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are arbitrary sequences.

- In [16], Ibrahim studied the solutions and properties of the difference equation

  $$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} \left(a + b x_n x_{n-2}\right)}$$

where initial values are nonnegative real numbers. In [17], Folly-Gbetoula and Nyirenda extended the work in [16] by obtaining the solutions of difference equations of the form

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}\left(a_n + b_n x_n x_{n-2}\right)}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are non-zero real sequences, via symmetries with less restriction on the initial conditions.Without loss of generality authors instead studied the equation

$$u_{n+3} = \frac{u_n u_{n+2}}{u_{n+1}\left(A_n + B_n u_n u_{n+2}\right)}$$

where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are non-zero real sequences.

- In [18], Elsayed investigated the dynamics and solutions of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{a_n + b_n x_{n-1} x_{n-3} x_{n-5}} \tag{1.2}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are arbitrary sequences and the initial conditions x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0 are arbitrary non zero real numbers. In [19], Mnguni et al. used a Lie symmetry-based method to solve Eq.(1.2). Equivalently, they study

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4}}$$

instead, where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are arbitrary sequences. This means they can only compare x_i with u_{i+5}.

- In [20], Nyirenda and Folly-Gbetoula are inspired by the work of Elsayed in [21] who studied the following recursive sequences:

$$x_{n+1} = \frac{x_{n-3} x_{n-4}}{x_n\left(\pm 1 \pm x_{n-1} x_{n-2} x_{n-3} x_{n-4}\right)}, \quad n = 0, 1, ..., \tag{1.3}$$

where the initial conditions are arbitrary real numbers. Clearly, [21] are special cases of a more general form

$$x_{n+1} = \frac{x_{n-3} x_{n-4}}{x_n\left(a_n + b_n x_{n-1} x_{n-2} x_{n-3} x_{n-4}\right)}, \quad n = 0, 1, ..., \tag{1.4}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are real sequences. Authors utilized symmetry methods to solve this more general difference equation Eq.(1.4). Equivalently, they studied the forward difference equation

$$u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}\left(A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3}\right)}$$

where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are arbitrary sequences.

- In [22], Nyirenda and Folly-Gbetoula used the Lie group analysis method to investigate the invariance properties and the solutions of

$$x_{n+1} = \frac{x_{n-5}x_{n-3}}{x_{n-1}\left(a_n + b_n x_{n-5}x_{n-3}\right)}.$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are real sequences. Without loss of generality, authors equivalently studied the forward difference equation

$$u_{n+6} = \frac{u_n u_{n+2}}{u_{n+4}\left(A_n + B_n u_n u_{n+2}\right)}$$

where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are arbitrary sequences.

- In [23], Folly-Gbetoula and Nyirenda performed a symmetry analysis and derived closed form formulas for solutions of difference equations of the form

$$x_{n+1} = \frac{x_{n-3}x_n}{x_{n-2}\left(a_n + b_n x_{n-3}x_n\right)}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are sequences of real numbers. Without loss of generality, authors equivalently studied the forward difference equation

$$u_{n+4} = \frac{u_n u_{n+3}}{u_{n+1}\left(A_n + B_n u_n u_{n+3}\right)}$$

where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are arbitrary sequences.

- Elsayed, in [24] obtained the solution of

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}.$$

In [25], Folly-Gbetoula and Nyirenda derived solutions of the following difference equations via the invariant of their group of transformations:

$$x_{n+1} = \frac{x_{n-5}}{a_n + b_n x_{n-2}x_{n-5}} \tag{1.5}$$

for some arbitrary sequences of real numbers (a_n) and (b_n). The solution to Eq.(1.5) difference equation provides a generalization of the solution of Elsayed [24].Without loss of generality, authors instead studied the difference equation

$$u_{n+6} = \frac{u_n}{A_n + B_n u_{n+3}u_n}.$$

- In [26], Folly-Gbetoula and Nyirenda studied solutions to the difference equations of the form

$$x_{n+1} = \frac{x_{n-7}x_{n-2}}{x_{n-4}\left(a_n + b_n x_{n-7}x_{n-2}\right)} \tag{1.6}$$

where (a_n) and (b_n) are arbitrary sequences of real numbers, and the initial values are x_{-7}, x_{-6}, ..., x_{-1} and x_0. Authors performed a symmetry analysis and derive formulas for solutions of higher-order difference equations Eq.(1.6) . They considered the forward ordinary difference equations of the Kovalevskaya form

$$u_{n+8} = \frac{u_n u_{n+5}}{u_{n+3}\left(A_n + B_n u_n u_{n+5}\right)}$$

where A_n and B_n are random real sequences, equivalent to Eq.(1.6).

- In [27], Elsayed et al. obtained the formulas of the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-5}x_n}{x_{n-4}\left(\pm 1 \pm x_{n-5}x_n\right)},\ n = 0, 1, ..., \tag{1.7}$$

in which the initial conditions x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0 are arbitrary non zero real numbers. In [28], Folly-Gbetoula and Nyirenda obtained symmetry generators admitted by the difference equations of the form

$$x_{n+1} = \frac{x_{n-5}x_n}{x_{n-4}\left(a_n + b_n x_{n-5}x_n\right)} \tag{1.8}$$

where a_n and b_n are random sequences, and then proceed to find the solutions in closed form via the invariance of the group of transformations admitted by Eq.(1.7). Authors first present the solutions in a unified manner and then split them into different categories based on some properties of the ‘ final constraint’. They considered the forward ordinary difference equations of the Kovalevskaya form

$$u_{n+6} = \frac{u_n u_{n+5}}{u_{n+1}\left(A_n + B_n u_n u_{n+5}\right)}$$

where A_n and B_n are random real sequences, equivalent to Eq.(1.8).

- In [29], Elsayed obtained the formulas of the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-4}x_{n-2}}{x_{n-1}\left(\pm 1 \pm x_{n-4}x_{n-2}\right)},\ n = 0, 1, ...,$$

in which the initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0 are arbitrary non-zero real numbers. In [30], Folly-Gbetoula and Nyirenda obtain the vector fields of

$$x_{n+1} = \frac{x_{n-4}x_{n-2}}{x_{n-1}\left(a_n + b_n x_{n-4}x_{n-2}\right)}$$

where a_n and b_n are random real sequences, and then proceed to find the solutions in closed form via the invariance of the group of transformations and using the method of Lie group analysis. Authors considered the forward ordinary difference equations of the Kovalevskaya form

$$u_{n+5} = \frac{u_n u_{n+2}}{u_{n+3}\left(A_n + B_n u_n u_{n+2}\right)}$$

where A_n and B_n are random real sequences.

- Elsayed, in [31] obtained the exact solutions of

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1}x_{n-3}}. \tag{1.9}$$

In [32], Mnguni et al. obtained solutions of the following difference equations via the invariant of their group of transformations:

$$x_{n+1} = \frac{x_{n-3}}{a_n + b_n x_{n-1}x_{n-3}}$$

for some arbitrary sequences of real numbers (a_n) and (b_n). Thus the solution to the difference equation above extends the solution of Elsayed [31] to equation (1.9) to a more general setting. Without loss of generality, they instead studied the difference equation

$$u_{n+4} = \frac{u_n}{A_n + B_n u_n u_{n+2}}.$$

- In [4], Folly-Gbetoula et al. used symmetry transformations and invariant methods, they derived solutions of the following difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{a_n x_n + b_n x_{n-4}} \tag{1.10}$$

for some arbitrary sequences of real numbers (a_n) and (b_n). They inspired by some finding in [33], Khaliq and Elsayed obtained exact solutions of

$$x_{n+1} = \frac{x_n x_{n-3}}{x_n - x_{n-4}}. \tag{1.11}$$

For the sake of notation and definitions used in this work, authors shall considered the Kovalevskaya form of Eq.(1.10), that is

$$u_{n+5} = \frac{u_{n+1}u_{n+4}}{A_n u_{n+4} + B_n u_n} \tag{1.12}$$

where A_n and B_n are random real sequences. Authors explained how solutions of Eq.(1.10) and Eq.(1.12) are related and showed that solutions in [33], for Eq.(1.11) are indeed special cases of the solutions that they present in [4].

- In [34], Folly-Gbetoula studied the system of difference equations

$$x_{n+4} = \frac{x_n x_{n+1}}{x_{n+3}\left(a_n + b_n x_n x_{n+1}\right)} \tag{1.13}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers. For equation Eq.(1.13), author derived all Lie point symmetries and gived formulas for solutions in closed form and also discussed periodicity and asymptotic behavior of solutions in some cases.

CHAPTER 2

BASIC DEFINITIONS AND THEOREMS

In this chapter, we give some basic definitions and theorems used in this thesis. For details, see $[9, 12, 35, 46-54]$

2.1 DIFFERENCE EQUATIONS

In this section some significant definitions and theorems are presented about difference equations.

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ... \tag{2.1}$$

A solution of Eq.(2.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(2.1) for all $n \geq -k$.

For every set of initial conditions x_{-k}, $x_{-(k+1)}, ...,$ $x_0 \in I$, the difference equation Eq.(2.1) has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

As a special case of above lemma, for every set of initial conditions $x_0, x_{-1} \in I$, the second order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, ... \tag{2.2}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$ and for every set of initial conditions $x_0, x_{-1}, x_{-2} \in I$, the third order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \qquad n = 0, 1, ... \tag{2.3}$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$. (see [48])

Definition 2.1 *A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2.1) If*

$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$

That is

$x_n = \overline{x}$ *for* $n \geq 0$

is a solution of Eq.(2.1) *or equivalently,* $\overline{x}$ *is a fixed point of* f .

Definition 2.2 (Stability) **(a)** *The equilibrium point* $\overline{x}$ *of Eq.*(2.1) *is locally stable if for every* $\varepsilon > 0$*; there exists* $\delta > 0$ *such that for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ *with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \varepsilon, \quad \text{for all } n \geq -k.$$

(b) *The equilibrium point* $\overline{x}$ *of Eq.*(2.1) *is locally asymptotically stable if* $\overline{x}$ *is locally stable solution of*

Eq.(2.1) *and there exists* $\gamma > 0$*, such that for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ *with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(c) *The equilibrium point* $\overline{x}$ *of Eq.*(2.1) *is global attractor if for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ *we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(d) *The equilibrium point* $\overline{x}$ *of Eq.*(2.1) *is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Eq.*(2.1).

(e) *The equilibrium point* $\overline{x}$ *of Eq.*(2.1) *is unstable if* $\overline{x}$ *is not locally stable.*

2.2 LIE SYMMETRY ANALYSIS

We start by giving some important definitions and theorems for Lie symmetry analysis. For more information, see [9, 12].

Definition 2.3 *Let G be a local group of transformations acting on a manifold M. A subset $S \subset M$ is called G-invariant, and G is called symmetry group of S, if whenever $x \in S$ and $g \in G$ is such that $g.x$ is defined, then $g.x \in S$.*

Definition 2.4 *A parameterized set of point transformations,*

$$\Gamma_{\varepsilon} : x \rightarrow \hat{x}\left(x;\varepsilon\right),$$

where $x = x_i$, $i = 1, ..., p$ are continuous variables, is a one-parameter local Lie group of transformations as long as the following conditions are met:

a. Γ_0 *is the identity map if* $\hat{x} = x$ *when* $\varepsilon = 0$.

b. $\Gamma_a \Gamma_b = \Gamma_{a+b}$ *for every* a *and* b *sufficiently close to* 0.

c. *Each* $\hat{x}_i$ *can be represented as a Taylor series (in a neighborhood of* $\varepsilon = 0$ *that is determined by* x*) and therefore*

$$\hat{x}_i\left(x;\varepsilon\right) = x_i + \varepsilon \xi_i\left(x\right) + O(\varepsilon^2), \ i : 1, ..., p.$$

Assume that the forward pth-order difference equation takes the form

$$u_{n+p} = \Omega\left(n, u_n, ..., u_{n+p-1}\right), \ n \in D \tag{2.4}$$

for some smooth function Ω and a regular domain $D \subset \mathbb{Z}$. So as to compute a symmetry group of (2.4), we take into consideration the group of point transformations given as

$$\hat{n} = n,$$

$$\hat{u}_n = u_n + \varepsilon Q\left(n, u_n\right) + O\left(\varepsilon^2\right),$$

$$\vdots$$

$$\hat{u}_{n+j} = u_{n+j} + \varepsilon S^j Q\left(n, u_n\right) + O\left(\varepsilon^2\right), \tag{2.5}$$

where ε (ε is sufficiently small) is the parameter, $Q = Q(n, u_n)$ is a continuous function, referred to as characteristic and S is the shift operator defined as

$$S : n \rightarrow n + 1, \;\; S^i u_n = u_{n+i}.$$

The criterion of invariance is then

$$\hat{u}_{n+p} = \Omega\left(\hat{n}, \hat{u}_n, \hat{u}_{n+1}, ..., \hat{u}_{n+p-1}\right), \tag{2.6}$$

which yields the linearized symmetry condition

$$S^p Q - X\Omega = 0|_{u_{n+p}=\Omega(n,u_n,\ldots,u_{n+p-1})}. \tag{2.7}$$

by substituting (2.5) in (2.6). Observe that

$$X = Q\left(n, u_n\right) \frac{\partial}{\partial u_n} + S^1 Q\left(n, u_n\right) \frac{\partial}{\partial u_{n+1}} +,,, + S^{p-1} Q\left(n, u_n\right) \frac{\partial}{\partial u_{n+p-1}}, \tag{2.8}$$

is the corresponding 'prolonged' infinitesimal of the group of transformations (2.5). Upon knowledge of the function(s) Q, one is able to obtain the invariant V by using the canonical coordinate [35] .

$$S_n = \int \frac{du_n}{Q\left(n, u_n\right)}.$$

Generally, the steps involves are lengthy even though very exact and do not give room to guess work on the perfect choice of invariants.

Definition 2.5 V_n *is invariant under the Lie group of transformations Eq.*(2.5) *if and only if*

$$X\left(V_n\right) = 0$$

where V_n *can be found by solving the characteristic equation*

$$\frac{du_n}{Q} = \frac{du_{n+1}}{SQ} = ... = \frac{du_{n+p-1}}{S^{n+p-1}Q}\left(= \frac{dV_n}{0}\right).$$

CHAPTER 3

AN EXAMPLE OF A THIRD ORDER RATIONAL DIFFERENCE EQUATION

In this chapter we handle an example of difference equation which Folly-Gbetoula and Nyirenda investigated the solutions with Lie symmetry analysis in [17] .

In [16] , Ibrahim investigated the solutions and properties of the difference equation

$$x_{n+1} = \frac{x_{n-2}x_n}{x_{n-1}(a + bx_{n-2}x_n)}$$

where initial values are nonnegative real numbers. In this chapter Folly-Gbetoula and Nyirenda extend the work in [16] by obtaining the solutions of difference equations of the form

$$x_{n+1} = \frac{x_{n-2}x_n}{x_{n-1}(a_n + b_n x_{n-2}x_n)} \tag{3.1}$$

where $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}$ are non-zero real sequences, via symmetries with less restriction on the initial conditions. Because of the terminology they adopt, they instead study the equation

$$u_{n+3} = \frac{u_n u_{n+2}}{u_{n+1}(A_n + B_n u_n u_{n+2})}, \tag{3.2}$$

where $(A_n)_{n\in\mathbb{N}_0}, (B_n)_{n\in\mathbb{N}_0}$ are non-zero real sequences.

Present the solutions of Eq.(3.2) using a symmetry based method, they first find the Lie algebra of Eq.(3.2). Then, they lower the order via the invariant and then solutions are constructed.

It was seen that the solutions obtained in [16] and the solutions obtained in [17] were same.

3.1 REDUCTION AND SOLUTIONS

Le us investigate the difference equation of the form Eq.(3.2), i.e.,

$$u_{n+3} = \Omega = \frac{u_n u_{n+2}}{u_{n+1}(A_n + B_n u_n u_{n+2})}. \tag{3.3}$$

When they apply the invariance criterion Eq.(2.7) to Eq.(3.3), they get

$$Q(n+3,\ \Omega)-Q(n+1,\ u_{n+1})\frac{\partial\Omega}{\partial u_{n+1}}-Q(n+2,\ u_{n+2})\frac{\partial\Omega}{\partial u_{n+2}}-Q(n,\ u_n)\frac{\partial\Omega}{\partial u_n}=0$$

and

$$Q(n+3,\ \Omega)+\frac{u_n u_{n+2}}{u_{n+1}^2(A_n+B_n u_n u_{n+2})}Q(n+1,\ u_{n+1})$$

$$-\frac{A_n u_n}{u_{n+1}(A_n+B_n u_n u_{n+2})^2}Q(n+2,\ u_{n+2})$$

$$-\frac{A_n u_{n+2}}{u_{n+1}(A_n+B_n u_n u_{n+2})^2}Q(n,\ u_n)=0. \tag{3.4}$$

The latter is a functional equation and in order to get rid of u_{n+3}, they apply implicit differentiation with respect to u_n (regarding u_{n+1} as a function of $u_n,\ u_{n+2}$ and Ω) by applying the differential operator

$$L=\frac{\partial}{\partial u_n}-\frac{\partial u_{n+1}}{\partial u_n}\frac{\partial}{\partial u_{n+1}}=\frac{\partial}{\partial u_n}-\frac{(\partial\Omega/\partial u_n)}{(\partial\Omega/\partial u_{n+1})}\frac{\partial}{\partial u_{n+1}}$$

$$\begin{aligned}\frac{\partial}{\partial u_n} \quad &= \quad \frac{2A_nB_nu_{n+2}^2}{u_{n+1}(A_n+B_nu_nu_{n+2})^3}Q\left(n,u_n\right)-\frac{A_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^2}Q'\left(n,u_n\right)\\ &\quad +\frac{A_nu_{n+2}}{u_{n+1}^2\left(A_n+B_nu_nu_{n+2}\right)^2}Q\left(n+1,u_{n+1}\right)+\frac{-A_n^2+A_nB_nu_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^3}Q\left(n+2,u_{n+2}\right)\end{aligned}$$

$$\begin{aligned}\frac{(\partial\Omega/\partial u_n)}{(\partial\Omega/\partial u_{n+1})} \quad &= \quad \frac{A_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^2}\frac{-u_{n+1}^2\left(A_n+B_nu_nu_{n+2}\right)}{u_nu_{n+2}}\\ &= \quad \frac{-A_nu_{n+1}}{u_n\left(A_n+B_nu_nu_{n+2}\right)}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial u_{n+1}} \quad &= \quad \frac{A_nu_{n+2}}{u_{n+1}^2\left(A_n+B_nu_nu_{n+2}\right)^2}Q\left(n,u_n\right)+\frac{A_nu_n}{u_{n+1}^2\left(A_n+B_nu_nu_{n+2}\right)^2}Q\left(n+2,u_{n+2}\right)\\ &\quad +\frac{-2u_nu_{n+2}}{u_{n+1}^3\left(A_n+B_nu_nu_{n+2}\right)}Q\left(n+1,u_{n+1}\right)\\ &\quad +\frac{u_nu_{n+2}}{u_{n+1}^2\left(A_n+B_nu_nu_{n+2}\right)}Q'\left(n+1,u_{n+1}\right)\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial u_n}-\frac{(\partial\Omega/\partial u_n)}{(\partial\Omega/\partial u_{n+1})}\frac{\partial}{\partial u_{n+1}} \quad = \quad & \frac{2A_nB_nu_{n+2}^2}{u_{n+1}(A_n+B_nu_nu_{n+2})^3}Q\left(n,u_n\right) \\
& -\frac{A_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^2}Q^{'}\left(n,u_n\right) \\
& +\frac{A_nu_{n+2}}{u_{n+1}^2\left(A_n+B_nu_nu_{n+2}\right)^2}Q\left(n+1,u_{n+1}\right) \\
& +\frac{-A_n^2+A_nB_nu_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^3}Q\left(n+2,u_{n+2}\right) \\
& +\frac{A_n^2}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^3}Q\left(n+2,u_{n+2}\right) \\
& +\frac{A_n^2u_{n+2}}{u_nu_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^3}Q\left(n,u_n\right) \\
& +\frac{-2A_nu_{n+2}}{u_{n+1}^2\left(A_n+B_nu_nu_{n+2}\right)^2}Q\left(n+1,u_{n+1}\right) \\
& +\frac{A_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^2}Q^{'}n+1,u_{n+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{2A_nB_nu_nu_{n+2}^2+A_n^2u_{n+2}}{u_nu_{n+1}(A_n+B_nu_nu_{n+2})^3}Q\left(n,u_n\right)-\frac{A_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^2}Q^{'}\left(n,u_n\right) \\
& -\frac{A_nu_{n+2}}{u_{n+1}^2\left(A_n+B_nu_nu_{n+2}\right)^2}Q\left(n+1,u_{n+1}\right)+\frac{A_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^2}Q^{'}n+1,u_{n+1}\right) \\
& +\frac{A_nB_nu_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^3}Q\left(n+2,u_{n+2}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{A_nu_{n+2}}{u_{n+1}\left(A_n+B_nu_nu_{n+2}\right)^3}\left[\begin{array}{c}\frac{(A_n+2B_nu_nu_{n+2})}{u_n}Q\left(n,u_n\right)-\left(A_n+B_nu_nu_{n+2}\right)Q'\left(n,u_n\right) \\ -\frac{(A_n+B_nu_nu_{n+2})}{u_{n+1}}Q\left(n+1,u_{n+1}\right)+B_nu_nQ\left(n+2,u_{n+2}\right) \\ +\left(A_n+B_nu_nu_{n+2}\right)Q'\left(n+1,u_{n+1}\right)\end{array}\right] \\
= \quad & 0
\end{aligned}
$$

After simplification, the resulting equation yields

$$(A_n+B_nu_nu_{n+2})Q'(n+1,\ u_{n+1})-\frac{(A_n+B_nu_nu_{n+2})}{u_{n+1}}Q(n+1,\ u_{n+1})$$

$$+B_nu_nQ(n+2,\ u_{n+2})-(A_n+B_nu_nu_{n+2})Q'(n,\ u_n)$$

$$+(2B_nu_{n+2}+\frac{A_n}{u_n})Q(n,\ u_n)=0. \tag{3.5}$$

The symbol $'$ denotes the derivative with regarding to the continuous variable. Differentiation of Eq.(3.5) with respect to u_n twice, keeping u_{n+2} and u_{n+1} fixed, yields the equation

$$
\begin{aligned}
& \left(B_n u_{n+2}\right) Q'\left(n+1, u_{n+1}\right) - \frac{B_n u_{n+2}}{u_{n+1}} Q\left(n+1, u_{n+1}\right) + B_n Q\left(n+2, u_{n+2}\right) \\
& -\left(B_n u_{n+2}\right) Q'\left(n, u_n\right) - \left(A_n + B_n u_n u_{n+2}\right) Q''\left(n, u_n\right) - \frac{A_n}{u_n^2} Q\left(n, u_n\right) \\
& +\left(2 B_n u_{n+2} + \frac{A_n}{u_n}\right) Q'\left(n, u_n\right) \\
= \; & 0
\end{aligned}
$$

$$
\begin{aligned}
& -\left(B_n u_{n+2}\right) Q''\left(n, u_n\right) - \left(B_n u_{n+2}\right) Q''\left(n, u_n\right) - \left(A_n + B_n u_n u_{n+2}\right) Q'''\left(n, u_n\right) \\
& +\frac{2 A_n}{u_n^3} Q\left(n, u_n\right) - \frac{A_n}{u_n^2} Q'\left(n, u_n\right) - \frac{A_n}{u_n^2} Q\left(n, u_n\right) + \left(2 B_n u_{n+2} + \frac{A_n}{u_n}\right) Q''\left(n, u_n\right) \\
= \; & 0
\end{aligned}
$$

$$
\begin{aligned}
& -B_n u_n u_{n+2} Q'''(n,\ u_n) - A_n Q'''(n,\ u_n) + \frac{A_n}{u_n} Q''(n,\ u_n) - \frac{2A_n}{u_{n^2}} Q'(n,\ u_n) \\
& + \frac{2A_n}{u_{n^3}} Q(n,\ u_n) = 0.
\end{aligned} \tag{3.6}
$$

The characteristic in Eq.(3.6) does not depend on u_{n+2} and so Eq.(3.6) splits into the following system by comparing exponents of u_{n+2} :

$$1 \ : \ Q'''(n,\ u_n) - \frac{1}{u_n} Q''(n,\ u_n) + \frac{2}{u_{n^2}} Q'(n,\ u_n) - \frac{2}{u_{n^3}} Q(n,\ u_n) = 0 \tag{3.7}$$

$$u_{n+2} : Q'''(n,\ u_n) = 0 \tag{3.8}$$

They find that the solution to Eq.(3.8) is:

$$Q(n,\ u_n) = \alpha_n u_n^2 + \beta_n u_n \tag{3.9}$$

for some arbitrary functions α_n and β_n of n. Substituting Eq.(3.9) and its first, second and third shifts in Eq.(3.4) and then replacing the expression of u_{n+3} given in Eq.(3.2) in the resulting equation yields

$$
\begin{aligned}
& \alpha_{n+3} u_{n+3}^2 + \beta_{n+3} u_{n+3} + \frac{u_n u_{n+2}}{u_{n+1}^2 \left(A_n + B_n u_n u_{n+2}\right)} \left(\alpha_{n+1} u_{n+1}^2 + \beta_{n+1} u_{n+1}\right) \\
& - \frac{A_n u_n}{u_{n+1} \left(A_n + B_n u_n u_{n+2}\right)^2} \left(\alpha_{n+2} u_{n+2}^2 + \beta_{n+2} u_{n+2}\right) \\
& - \frac{A_n u_{n+2}}{u_{n+1} \left(A_n + B_n u_n u_{n+2}\right)^2} \left(\alpha_n u_n^2 + \beta_n u_n\right) \\
= \; & 0
\end{aligned}
$$

$$
\begin{aligned}
& \alpha_{n+3}\left(\frac{u_n u_{n+2}}{u_{n+1}\left(A_n+B_n u_n u_{n+2}\right)}\right)^2+\beta_{n+3}\frac{u_n u_{n+2}}{u_{n+1}\left(A_n+B_n u_n u_{n+2}\right)} \\
& +\frac{\alpha_{n+1} u_n u_{n+2} u_{n+1}^2}{u_{n+1}^2\left(A_n+B_n u_n u_{n+2}\right)}+\frac{\beta_{n+1} u_n u_{n+2} u_{n+1}}{u_{n+1}^2\left(A_n+B_n u_n u_{n+2}\right)} \\
& -\frac{\alpha_{n+2} A_n u_n u_{n+2}^2}{u_{n+1}\left(A_n+B_n u_n u_{n+2}\right)^2}-\frac{\beta_{n+2} A_n u_n u_{n+2}}{u_{n+1}\left(A_n+B_n u_n u_{n+2}\right)^2} \\
& -\frac{\alpha_n A_n u_n^2 u_{n+2}}{u_{n+1}\left(A_n+B_n u_n u_{n+2}\right)^2}-\frac{\beta_n A_n u_n u_{n+2}}{u_{n+1}\left(A_n+B_n u_n u_{n+2}\right)^2} \\
= \; & 0
\end{aligned}
$$

$$
\begin{aligned}
& \alpha_{n+3} u_n^2 u_{n+2}^2+\beta_{n+3} u_n u_{n+1} u_{n+2}\left(A_n+B_n u_n u_{n+2}\right)+\alpha_{n+1} u_n u_{n+1}^2 u_{n+2}\left(A_n+B_n u_n u_{n+2}\right) \\
& +\beta_{n+1} u_n u_{n+1} u_{n+2}\left(A_n+B_n u_n u_{n+2}\right)-\alpha_{n+2} A_n u_n u_{n+1} u_{n+2}^2-\beta_{n+2} A_n u_n u_{n+1} u_{n+2} \\
& -\alpha_n A_n u_n^2 u_{n+1} u_{n+2}-\beta_n A_n u_n u_{n+1} u_{n+2} \\
= \; & 0
\end{aligned}
$$

$$
\begin{aligned}
& B_n u_n^2 u_{n+2}^2 u_{n+1}^2 \alpha_{n+1}+B_n u_n^2 u_{n+2}^2 u_{n+1}(\beta_{n+1}+\beta_{n+3}) \\
& -A_n u_n^2 u_{n+2} u_{n+1} \alpha_n-A_n u_n u_{n+2}^2 u_{n+1} \alpha_{n+2}+A_n u_n u_{n+2} u_{n+1}^2 \alpha_{n+1} \\
& +u_n^2 u_{n+2}^2 \alpha_{n+3}-A_n u_n u_{n+1} u_{n+2}(\beta_n+\beta_{n+2}-\beta_{n+1}-\beta_{n+3})=0. \qquad (3.10)
\end{aligned}
$$

Equating coefficients of all powers of shifts of u_n to zero and simplifying the resulting system, they get its reduced form

$$\alpha_n=0, \qquad (3.11)$$

$$\beta_n+\beta_{n+2}=0. \qquad (3.12)$$

The two solutions of the linear second-order difference equation Eq.(3.12) are now taken by

$$\beta_n=\beta^n \text{ and } \beta_n=\overline{\beta}^n,$$

where $\beta=\exp(i\pi/2)$ and $\overline{\beta}=-\exp(i\pi/2)$ is the complex conjugate of β. The characteristics are

$$Q_1(n,\ u_n)=\beta^n u_n \text{ and } Q_2(n,\ u_n)=\overline{\beta}^n u_n,$$

and hence, the symmetry operators are given by

$$X_1=\beta^n u_n\frac{\partial}{\partial_{u_n}}+\beta^{n+1}u_{n+1}\frac{\partial}{\partial_{u_{n+1}}}+\beta^{n+2}u_{n+2}\frac{\partial}{\partial_{u_{n+2}}},$$

$$X_2 = \overline{\beta}^n u_n \frac{\partial}{\partial_{u_n}} + \overline{\beta}^{n+1} u_{n+1} \frac{\partial}{\partial_{u_{n+1}}} + \overline{\beta}^{n+2} u_{n+2} \frac{\partial}{\partial_{u_{n+2}}}.$$

Utilizing the canonical coordinate

$$S_n = \int \frac{du_n}{Q_1(n, u_n)} = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n| \tag{3.13}$$

and the relation Eq.(3.11), they derive the invariant function $\tilde{\mathcal{V}}_n$ as:

$$\tilde{\mathcal{V}}_n = S_n \beta^n + S_{n+2} \beta^{n+2}. \tag{3.14}$$

$$\begin{aligned} &= \frac{1}{\beta^n} \ln |u_n| \, \beta^n + \frac{1}{\beta^{n+2}} \ln |u_{n+2}| \, \beta^{n+2} \\ &= \ln |u_n| + \ln |u_{n+2}| \\ &= \ln |u_n u_{n+2}| \end{aligned}$$

Note that

$$X_1(\tilde{\mathcal{V}}_n) = \beta^n + \beta^{n+2} = 0$$

and

$$X_2(\tilde{\mathcal{V}}_n) = \overline{\beta}^n + \overline{\beta}^{n+2} = 0.$$

For the sake of simplicity, they use

$$|\mathcal{V}_n| = \exp\{-\tilde{\mathcal{V}}_n\}, \tag{3.15}$$

instead.

$$\begin{aligned} |\mathcal{V}_n| &= \exp\{-\ln |u_n u_{n+2}|\} \\ &\Rightarrow V_n = \pm \frac{1}{(u_n u_{n+2})} \end{aligned}$$

One can show by using Eq.(3.3) and Eq.(3.15) that

$$\begin{aligned} V_{n+1} &= \frac{\pm 1}{u_{n+1} u_{n+3}} = \frac{\pm 1}{u_{n+1} \frac{u_n u_{n+2}}{u_{n+1}(A_n + B_n u_n u_{n+2})}} = \pm \frac{u_{n+1}\,(A_n + B_n u_n u_{n+2})}{u_{n+1} u_n u_{n+2}} \\ &= \pm \frac{A_n}{u_n u_{n+2}} \pm \frac{B_n u_n u_{n+2}}{u_n u_{n+2}} = A_n V_n \pm B_n \end{aligned}$$

$$\Rightarrow \mathcal{V}_{n+1} = A_n \mathcal{V}_n \pm B_n. \tag{3.16}$$

At this stage, obtaining the solution of Eq.(3.3) is straightforward. They first use Eq.(3.13) to get

$$|u_n| = \exp(\beta_n S_n).$$

They then invoke Eq.(3.14) to obtain

$$|u_n| = \exp(\beta^n c_1 + \overline{\beta}^n c_2 - \frac{1}{2}\sum_{k_1=0}^{n-1} \beta^n \overline{\beta}^{k_1} \tilde{\mathcal{V}}_{k_1} - \frac{1}{2}\sum_{k_2=0}^{n-1} \overline{\beta}^n \beta^{k_2} \tilde{\mathcal{V}}_{k_2}). \tag{3.17}$$

Finally, invoking Eq.(3.15) leads to

$$|u_n| = \exp(\beta^n c_1 + \overline{\beta}^n c_2 + \frac{1}{2}\sum_{k_1=0}^{n-1} \beta^n \overline{\beta}^{k_1} \ln|\mathcal{V}_{k_1}| + \frac{1}{2}\sum_{k_2=0}^{n-1} \overline{\beta}^n \beta^{k_2} \ln|\mathcal{V}_{k_2}|)$$

$$= \exp(\beta^n c_1 + \overline{\beta}^n c_2 + \sum_{k=0}^{n-1} \mathrm{Re}(\gamma(n,\ k)) \ln|\mathcal{V}_k|), \tag{3.18}$$

where V_k satisfies Eq.(3.16) with $\gamma(n,\ k) = \beta^n \overline{\beta}^k$. Observe that c_1 and c_2 satisfy

$$c_1 + c_2 = \ln|u_0| \ \textit{and}\ \beta(c_1 - c_2) = \ln|u_1|. \tag{3.19}$$

Note that the equation in Eq.(3.18) give rise to solutions of Eq.(3.3) in a unified manner. Moreover, for the function $\gamma(n,\ k) = \beta^n \overline{\beta}^k$, we have

$$\begin{aligned} \gamma(0,1) &= \overline{\beta},\ \gamma(1,0) = \beta,\ \gamma(n,\ n) = 1,\ \gamma(n+2,\ k) = -\gamma(n,\ k), \\ \gamma(n,\ k+2) &= -\gamma(n,\ k),\ \gamma(4n,\ k) = \gamma(0,\ k),\ \gamma(n,\ 4k) = \gamma(n,\ 0) \end{aligned} \tag{3.20}$$

Utilizing the expression of u_n in Eq.(3.18) and Eq.(3.20), note that

$$|u_{4n+j}| = \exp(H_j + \sum_{k_1=0}^{4n+j-1} \mathrm{Re}(\gamma(j,\ k_1)) \ln|\mathcal{V}_{k_1}|)$$

where

$$H_j = \beta^j c_1 + \overline{\beta}^j c_2.$$

For $j = 0$, they have,

$$|u_{4n}| = \exp(H_0 + \ln|\mathcal{V}_0| - \ln|\mathcal{V}_2| + \ldots + \ln|\mathcal{V}_{4n-4}| - \ln|\mathcal{V}_{4n-2}|)$$

$$= \exp(H_0) \prod_{s=0}^{n-1} |\frac{\mathcal{V}_{4s}}{\mathcal{V}_{4s+2}}|. \tag{3.21}$$

It can be shown that the absolute value function is not necessary by using the fact that

$$\mathcal{V}_i = \frac{1}{u_i u_{i+2}}$$

To find $\exp(H_0)$, set $n=0$ in Eq.(3.21) and note that $|u_0| = \exp(H_0)$. Thus

$$u_{4n} = u_0 \prod_{s=0}^{n-1} \frac{\mathcal{V}_{4s}}{\mathcal{V}_{4s+2}}. \tag{3.22}$$

Nevertheless, from Eq.(3.16), using the plus sign they have

$$\begin{aligned}
\mathcal{V}_1 &= A_0\mathcal{V}_0 + B_0 \\
\mathcal{V}_2 &= A_1\mathcal{V}_1 + B_1 = A_1\left(A_0\mathcal{V}_0 + B_0\right) + B_1 \\
\mathcal{V}_3 &= A_2\mathcal{V}_2 + B_2 = A_2\left(A_1\left(A_0\mathcal{V}_0 + B_0\right) + B_1\right) + B_2 \\
\mathcal{V}_4 &= A_3\mathcal{V}_3 + B_3 = A_3\left(A_2\left(A_1\left(A_0\mathcal{V}_0 + B_0\right) + B_1\right) + B_2\right) + B_3 \\
\mathcal{V}_5 &= A_4\mathcal{V}_4 + B_4 = A_4\left(A_3\left(A_2\left(A_1\left(A_0\mathcal{V}_0 + B_0\right) + B_1\right) + B_2\right) + B_3\right) + B_4 \\
\mathcal{V}_6 &= A_5\mathcal{V}_5 + B_5 = A_5\left(A_4\left(A_3\left(A_2\left(A_1\left(A_0\mathcal{V}_0 + B_0\right) + B_1\right) + B_2\right) + B_3\right) + B_4\right) + B_5 \\
&\vdots \\
\mathcal{V}_n &= \mathcal{V}_0(\prod_{k_1=0}^{n-1} A_{k_1}) + \sum_{l=0}^{n-1}(B_l \prod_{k_2=l+1}^{n-1} A_{k_2}),
\end{aligned}$$

where $\mathcal{V}_0 = 1/(u_0u_2)$. Hence, using Eq.(3.22), they obtain

$$\begin{aligned}
u_{4n} &= u_0 \prod_{s=0}^{n-1} \frac{\mathcal{V}_{4s}}{\mathcal{V}_{4s+2}} \\
&= u_0 \prod_{s=0}^{n-1} \frac{\mathcal{V}_0\left(\prod_{k_1=0}^{4s-1} A_{k_1}\right) + \sum_{l=0}^{4s-1}\left(B_l \prod_{k_2=l+1}^{4s-1} A_{k_2}\right)}{\mathcal{V}_0\left(\prod_{k_1=0}^{4s+1} A_{k_1}\right) + \sum_{l=0}^{4s+1}\left(B_l \prod_{k_2=l+1}^{4s+1} A_{k_2}\right)} \\
&= u_0 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{4s-1} A_{k_1}\right) + u_0u_2\sum_{l=0}^{4s-1}\left(B_l \prod_{k_2=l+1}^{4s-1} A_{k_2}\right)}{\left(\prod_{k_1=0}^{4s+1} A_{k_1}\right) + u_0u_2\sum_{l=0}^{4s+1}\left(B_l \prod_{k_2=l+1}^{4s+1} A_{k_2}\right)}.
\end{aligned}$$

For $j=1$, they get

$$\begin{aligned}
u_{4n+1} &= u_1 \prod_{s=0}^{n-1} \frac{\mathcal{V}_{4s+1}}{\mathcal{V}_{4s+3}} \\
&= u_1 \prod_{s=0}^{n-1} \frac{\mathcal{V}_0\left(\prod_{k_1=0}^{4s} A_{k_1}\right) + \sum_{l=0}^{4s}\left(B_l \prod_{k_2=l+1}^{4s} A_{k_2}\right)}{\mathcal{V}_0\left(\prod_{k_1=0}^{4s+2} A_{k_1}\right) + \sum_{l=0}^{4s+2}\left(B_l \prod_{k_2=l+1}^{4s+2} A_{k_2}\right)}
\end{aligned}$$

$$= u_1 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{4s} A_{k_1}\right) + u_0 u_2 \sum_{l=0}^{4s} \left(B_l \prod_{k_2=l+1}^{4s} A_{k_2}\right)}{\left(\prod_{k_1=0}^{4s+2} A_{k_1}\right) + u_0 u_2 \sum_{l=0}^{4s+2} \left(B_l \prod_{k_2=l+1}^{4s+2} A_{k_2}\right)}.$$

For $j = 2$, they have,

$$u_{4n+2} = u_2 \prod_{s=0}^{n-1} \frac{\mathcal{V}_{4s+2}}{\mathcal{V}_{4s+4}}$$

$$= u_2 \prod_{s=0}^{n-1} \frac{\mathcal{V}_0 \left(\prod_{k_1=0}^{4s+1} A_{k_1}\right) + \sum_{l=0}^{4s+1} \left(B_l \prod_{k_2=l+1}^{4s+1} A_{k_2}\right)}{\mathcal{V}_0 \left(\prod_{k_1=0}^{4s+3} A_{k_1}\right) + \sum_{l=0}^{4s+3} \left(B_l \prod_{k_2=l+1}^{4s+3} A_{k_2}\right)}$$

$$= u_2 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{4s+1} A_{k_1}\right) + u_0 u_2 \sum_{l=0}^{4s+1} \left(B_l \prod_{k_2=l+1}^{4s+1} A_{k_2}\right)}{\left(\prod_{k_1=0}^{4s+3} A_{k_1}\right) + u_0 u_2 \sum_{l=0}^{4s+3} \left(B_l \prod_{k_2=l+1}^{4s+3} A_{k_2}\right)}.$$

Finally, for $j = 3$, they have

$$u_{4n+3} = u_3 \prod_{s=0}^{n-1} \frac{\mathcal{V}_{4s+3}}{\mathcal{V}_{4s+5}}$$

$$= u_3 \prod_{s=0}^{n-1} \frac{\mathcal{V}_0 \left(\prod_{k_1=0}^{4s+2} A_{k_1}\right) + \sum_{l=0}^{4s+2} \left(B_l \prod_{k_2=l+1}^{4s+2} A_{k_2}\right)}{\mathcal{V}_0 \left(\prod_{k_1=0}^{4s+4} A_{k_1}\right) + \sum_{l=0}^{4s+4} \left(B_l \prod_{k_2=l+1}^{4s+4} A_{k_2}\right)}$$

$$= u_3 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{4s+2} A_{k_1}\right) + u_0 u_2 \sum_{l=0}^{4s+2} \left(B_l \prod_{k_2=l+1}^{4s+2} A_{k_2}\right)}{\left(\prod_{k_1=0}^{4s+4} A_{k_1}\right) + u_0 u_2 \sum_{l=0}^{4s+4} \left(B_l \prod_{k_2=l+1}^{4s+4} A_{k_2}\right)}.$$

More compactly, they have that

$$u_{4n+j} = u_j \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{4s+j-1} A_{k_1}\right) + u_0 u_2 \sum_{l=0}^{4s+j-1} \left(B_l \prod_{k_2=l+1}^{4s+j-1} A_{k_2}\right)}{\left(\prod_{k_1=0}^{4s+j+1} A_{k_1}\right) + u_0 u_2 \sum_{l=0}^{4s+j+1} \left(B_l \prod_{k_2=l+1}^{4s+j+1} A_{k_2}\right)},$$

$j = 0, 1, 2, 3$. Hence, the solution to their equation Eq.(3.1) becomes

$$x_{4n+j-2} = x_{j-2} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{4s+j-1} a_{k_1} \right) + x_{-2}x_0 \sum_{l=0}^{4s+j-1} \left(b_l \prod_{k_2=l+1}^{4s+j-1} a_{k_2} \right)}{\left(\prod_{k_1=0}^{4s+j+1} a_{k_1} \right) + x_{-2}x_0 \sum_{l=0}^{4s+j+1} \left(b_l \prod_{k_2=l+1}^{4s+j+1} a_{k_2} \right)},$$

where $j = 0, 1, 2, 3$, provided that the denominators do not vanish. In the other sections, they give special cases of solutions.

3.2 THE CASE a_n AND b_n ARE 1-PERIODIC SEQUENCES

In this case, $a_n = a$ and $b_n = b$ where $a,\ b \in \mathbb{R}$. They split this depending on whether or not $a = 1$.

3.2.1 The Case $a \neq 1$

The solution becomes

$$x_{4n+j-2} = x_{j-2} \prod_{s=0}^{n-1} \frac{a^{4s+j} + bx_{-2}x_0(\frac{1-a^{4s+j}}{1-a})}{a^{4s+j+2} + bx_{-2}x_0(\frac{1-a^{4s+j+2}}{1-a})} \tag{3.23}$$

where $j = 0, 1, 2, 3$ and

$a^{4s+j+2}(1-a) + bx_{-2}x_0(1-a^{4s+j+2}) \neq 0 \forall (s,\ j) \in \{0, 1, 2,\ \ldots,\ n-1\} \times \{0, 1, 2, 3\}$.

3.2.2 The Case $a = -1$

In this case, Eq.(3.23) reduces to

$$x_{4n+j-2} = x_{j-2} \prod_{s=0}^{n-1} \frac{(-1)^j + bx_{-2}x_0(\frac{1-(-1)^j}{2})}{(-1)^j + bx_{-2}x_0(\frac{1-(-1)^j}{2})}$$

$= x_{j-2}$.

Note that remark in this case $\{x_n\}_{n=-2}^{\infty}$ is periodic with period 4.

3.2.3 The Case $a = 1$

They obtain the following solution which, for $b = \pm 1$, appears in [16] (see Theorems 1 and 3).

$$x_{4n+j-2} = x_{j-2} \prod_{s=0}^{n-1} \frac{1 + bx_{-2}x_0(4s+j)}{1 + bx_{-2}x_0(4s+j+2)}.$$

More explicitly, in terms of the initial values $x_{-2},\ x_{-1},\ x_0$ they have

$$x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1+4sbx_{-2}x_0}{1+(4s+2)x_{-2}x_0b},$$

$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1+bx_{-2}x_0(4s+1)}{1+bx_{-2}x_0(4s+3)},$$

$$x_{4n} = x_0\prod_{s=0}^{n-1}\frac{1+bx_{-2}x_0(4s+2)}{1+bx_{-2}x_0(4s+4)}$$

and

$$x_{4n+1} = \frac{x_{-2}x_0}{x_{-1}(a+bx_{-2}x_0)}\prod_{s=0}^{n-1}\frac{1+bx_{-2}x_0(4s+3)}{1+bx_{-2}x_0(4s+5)}.$$

3.3 THE CASE a_n AND b_n ARE 2-PERIODIC SEQUENCES

In this case, for $j = 1, 3$ the solution becomes

$$x_{4n+j-2} = x_{j-2}\prod_{s=0}^{n-1}\frac{(a_0a_1)^{\frac{\lfloor 4\mathrm{s}+j-1\rfloor}{2}}a_0 + x_{-2}x_0f(j,a_0,b_0)}{(a_0a_1)^{\frac{\lfloor 4\mathrm{s}+j+1\rfloor}{2}}a_0 + x_{-2}x_0f(j+2,a_0,b_0)}$$

where

$$f\ (\ j,\ a_0,\ b_0)\ = b_0\sum_{l=0,l\ even}^{4s+j-1}(a_0a_1)\frac{\lfloor 4\mathrm{s}+j-1-l\rfloor}{2} + a_0b_1\sum_{l=0,l\ odd}^{4s+j-1}(a_0a_1)\frac{\lfloor 4\mathrm{s}+j-1-l\rfloor}{2}$$

For $j = 0, 2$ they have

$$x_{4n+j-2} = x_{j-2}\prod_{s=0}^{n-1}\frac{(a_0a_1)^{\frac{\lfloor 4\mathrm{s}+j-1\rfloor}{2}}a_0 + x_{-2}x_0g(j,a_0,b_0)}{(a_0a_1)^{\frac{\lfloor 4\mathrm{s}+j+1\rfloor}{2}}a_0 + x_{-2}x_0g(j+2,a_0,b_0)}$$

where

$$g(j,\ a_0,\ b_0) = a_1b_0\sum_{l=0,l\text{ even}}^{4s+j-1}(a_0a_1)\frac{\lfloor 4\mathrm{s}+j-1-l\rfloor}{2} + B_1\sum_{l=1,l\text{ odd}}^{4s+j-1}(a_0a_1)\frac{\lfloor 4\mathrm{s}+j-1-l\rfloor}{2}$$

As a result, in chapter three, we first gave an example to step by step details for explain Lie symmetry analysis which we used along this thesis.

CHAPTER 4

SOLUTIONS OF THE EQUATION $x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(a_n + b_n x_n x_{n-2} x_{n-4})}$ WITH LIE SYMMETRY ANALYSIS

First of all, we say that the results of this chapter are cited from [45] which has been accepted for publication.

In this chapter, we obtain exact solutions of the following rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(a_n + b_n x_n x_{n-2} x_{n-4})},$$

where a_n and b_n are random real sequences, by using the technique of Lie symmetry analysis. Moreover, we discuss the periodic nature and behavior of solutions for some special cases.

In [16], Ibrahim investigated the solutions and properties of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}$$

where initial values are nonnegative real numbers.

Elsayed and Ibrahim [38] obtained the solutions of the following difference equations of order five

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(\pm 1 \pm x_n x_{n-2} x_{n-4})}, \ n = 0, 1, 2, ...$$

where the initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0 are arbitrary real numbers.

Our aim in this chapter is to give the form of the solutions of the following difference equation

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(a_n + b_n x_n x_{n-2} x_{n-4})} \tag{4.1}$$

in closed form, where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are non-zero real sequences, via the technique of Lie group analysis. In this work, we are motivated by the results of Elsayed and Ibrahim [38] .

For the sake of convenience, we instead investigate the Kovalevskaya form of Eq.(4.1):

$$u_{n+5} = \frac{u_n u_{n+2} u_{n+4}}{u_{n+1} u_{n+3}(A_n + B_n u_n u_{n+2} u_{n+4})}. \tag{4.2}$$

4.1 REDUCTION AND SOLUTIONS

Consider difference equations of the form Eq.(4.2), i.e.,

$$u_{n+5} = \Omega = \frac{u_n u_{n+2} u_{n+4}}{u_{n+1} u_{n+3}(A_n + B_n u_n u_{n+2} u_{n+4})} \tag{4.3}$$

To get the symmetry algebra of equation Eq.(4.3), we apply the invariance criterion Eq.(2.7) to Eq.(4.3) to get

$$\begin{aligned}
&Q(n+5,\Omega) - \frac{u_{n+2}u_{n+4}A_n}{u_{n+1}u_{n+3}(A_n + B_n u_n u_{n+2} u_{n+4})^2} Q(n,u_n) \\
&+ \frac{u_{n+2}u_{n+4}u_n}{u_{n+3}u_{n+1}^2(A_n + B_n u_n u_{n+2} u_{n+4})} Q(n+1,u_{n+1}) \\
&- \frac{u_{n+4}u_n A_n}{u_{n+1}u_{n+3}(A_n + B_n u_n u_{n+2} u_{n+4})^2} Q(n+2,u_{n+2}) \\
&+ \frac{u_{n+4}u_{n+2}u_n}{u_{n+1}u_{n+3}^2(A_n + B_n u_n u_{n+2} u_{n+4})} Q(n+3,u_{n+3}) \\
&- \frac{u_{n+2}u_n A_n}{u_{n+1}u_{n+3}(A_n + B_n u_n u_{n+2} u_{n+4})^2} Q(n+4,u_{n+4}) = 0.
\end{aligned} \tag{4.4}$$

By acting the partial differential operator

$$L = \frac{\partial}{\partial u_n} - \frac{\partial u_{n+3}}{\partial u_n}\frac{\partial}{\partial u_{n+3}} = \frac{\partial}{\partial u_n} - \frac{(\partial\Omega/\partial u_n)}{(\partial\Omega/\partial u_{n+3})}\frac{\partial}{\partial u_{n+3}},$$

on Eq.(4.4), we have

$$\begin{aligned}
&\left(2u_{n+2}u_{n+4}B_n + \frac{A_n}{u_n}\right) Q - (A_n + B_n u_n u_{n+2} u_{n+4})Q' + (B_n u_n u_{n+4})S^{(2)}Q \\
&- \frac{(A_n + B_n u_n u_{n+2} u_{n+4})}{u_{n+3}} S^{(3)}Q + (A_n + B_n u_n u_{n+2} u_{n+4})\, S^{(3)}Q' + (B_n u_n u_{n+2})\, S^{(4)}Q = 0.
\end{aligned} \tag{4.5}$$

The notation $'$ stands for the derivative with respect to the continuous variable. Differentiation of Eq.(4.5) with respect to u_n twice, while u_{n+2}, u_{n+3} and u_{n+4} are fixed, yields the equation

$$\frac{2A_n}{u_n^3}Q - \frac{2A_n}{u_n^2}Q' + \frac{A_n}{u_n}Q'' - A_n Q''' - (B_n u_n u_{n+2} u_{n+4})Q''' = 0. \tag{4.6}$$

The equation above is solved by separation of variables in powers of shifts of u_n. Thus, we have the system

$$\begin{aligned} 1 &: \; Q'''(n,u_n) - \frac{1}{u_n}Q''(n,u_n) + \frac{2}{u_n^2}Q'(n,u_n) - \frac{2}{u_n^3}Q(n,u_n) = 0 \\ u_{n+2}u_{n+4} &: \; Q'''(n,u_n) = 0 \end{aligned}$$

whose solution is as follows:

$$Q(n,u_n) = \alpha_n u_n^2 + \beta_n u_n, \tag{4.7}$$

or some arbitrary functions α_n and β_n of n to be found. Substituting Eq.(4.7) and its first, second and third shifts in Eq.(4.4), and thereafter, replacing the expression of u_{n+5} given in Eq.(4.3) in the obtained equation, it follows that

$$\begin{aligned} & u_n^2 u_{n+2}^2 u_{n+4}^2 \alpha_{n+5} - A_n\alpha_n u_n^2 u_{n+1}u_{n+2}u_{n+3}u_{n+4} + A_n\alpha_{n+1}u_n u_{n+1}^2 u_{n+2}u_{n+3}u_{n+4} \\ & -A_n\alpha_{n+2}u_n u_{n+1}u_{n+2}^2 u_{n+3}u_{n+4} + A_n\alpha_{n+3}u_n u_{n+1}u_{n+2}u_{n+3}^2 u_{n+4} \\ & -A_n\alpha_{n+4}u_n u_{n+1}u_{n+2}u_{n+3}u_{n+4}^2 + \alpha_{n+1}B_n u_n^2 u_{n+2}^2 u_{n+4}^2 u_{n+3}u_{n+1}^2 \\ & +\alpha_{n+3}B_n u_n^2 u_{n+2}^2 u_{n+4}^2 u_{n+3}^2 u_{n+1} + A_n u_n u_{n+1}u_{n+2}u_{n+3}u_{n+4}(\beta_{n+5} - \beta_n + \beta_{n+1} \\ & -\beta_{n+2} + \beta_{n+3} - \beta_{n+4}) + B_n u_n^2 u_{n+2}^2 u_{n+4}^2 u_{n+1}u_{n+3}(\beta_{n+5} + \beta_{n+1} + \beta_{n+3}) \\ = \; & 0. \end{aligned} \tag{4.8}$$

Equating coefficients of all powers of shifts of u_n to zero and simplifying the resulting system, we get its reduced form

$$\alpha_n = 0, \qquad \beta_n + \beta_{n+2} + \beta_{n+4} = 0 \tag{4.9}$$

and its solutions are

$$\alpha_n = 0 \quad \text{and} \quad \beta_n = (-1)^n\beta^n c_1 + (-1)^n\overline{\beta}^n c_2 + \overline{\beta}^n c_3 + \beta^n c_4$$

for some arbitrary constants c_i, $i = 1, ..., 4$ and where $\beta = \exp(\pi i/3)$. So, we get four characteristics given by

$$\begin{aligned} & Q_1(n,u_n) = (-1)^n\beta^n u_n, Q_2(n,u_n) = (-1)^n\overline{\beta}^n u_n, Q_3(n,u_n) = \overline{\beta}^n u_n, \\ & Q_4(n,u_n) = \beta^n u_n. \end{aligned}$$

The four corresponding symmetry generators X_1, X_2, X_3 and X_4 are as follows:

$$\begin{aligned}
X_1 &= \sum_{j=0}^{4}(-\beta)^{n+j}u_{n+j}\partial u_{n+j},\\
X_2 &= \sum_{j=0}^{4}(-\bar{\beta})^{n+j}u_{n+j}\partial u_{n+j},\\
X_3 &= \sum_{j=0}^{4}(\bar{\beta})^{n+j}u_{n+j}\partial u_{n+j},\\
X_4 &= \sum_{j=0}^{4}(\beta)^{n+j}u_{n+j}\partial u_{n+j}.
\end{aligned}$$

Here, using Q_4, we introduce the canonical coordinate [35]

$$S_n = \int \frac{du_n}{Q_4(n,u_n)} = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n}\ln|u_n|\,. \tag{4.10}$$

Using relation Eq.(4.9), we derive the function

$$\widetilde{V}_n = S_n\beta^n + S_{n+2}\beta^{n+2} + S_{n+4}\beta^{n+4}$$

and let

$$|V_n| = \exp\left\{-\widetilde{V}_n\right\}. \tag{4.11}$$

In other words, $V_n = \pm 1/(u_n u_{n+2} u_{n+4})$. One can show by using Eq.(4.3) and Eq.(4.11) that

$$V_{n+1} = A_n V_n + B_n,$$

that is,

$$V_n = V_0\prod_{k_1=0}^{n-1}A_{k_1} + \sum_{l=0}^{n-1}B_l\prod_{l+1=0}^{n-1}A_{k_2}. \tag{4.12}$$

Here, we first use Eq.(4.10) to have

$$|u_n| = \exp(S_n\beta^n).$$

Then, invoking Eq.(4.11) yields

$$|u_n| = |H_n|\exp\left[\frac{2\sqrt{3}}{3}\sum_{k=0}^{n-1}\left(\cos\frac{(n-k)\pi}{2}\cos\frac{(n-k+1)\pi}{6}\right)\ln|V_k|\right] \tag{4.13}$$

where V_k is given in Eq.(4.12) and note that the H_n's satisfy

$$H_0 = x_0, \quad H_1 = x_1, \quad H_2 = x_2, \quad H_3 = x_3, \quad H_4 = \frac{1}{x_0 x_2}, \quad H_5 = \frac{1}{x_1 x_3}, \quad H_{6n+j} = H_j.$$

We can simplify the solution Eq.(4.13) further by splitting it into six categories. We have

$$|u_{6n}| = H_{6n} \exp\left[\frac{2\sqrt{3}}{3}\sum_{k=0}^{6n-1}\left(\cos\frac{(6n-k)\pi}{2}\cos\frac{(6n-k+1)\pi}{6}\right)\ln|V_k|\right]$$

$$= H_0 \exp\left(\ln V_0 - \ln V_2 + \ln V_6 - \ln V_8 + ... + \ln V_{6n-6} - \ln V_{6n-4}\right)$$

$$= x_0 \prod_{k=0}^{n-1}\left|\frac{V_{6k}}{V_{6k+2}}\right|.$$

Thus

$$u_{6n} = u_0 \prod_{k=0}^{n-1}\frac{V_{6k}}{V_{6k+2}}. \tag{4.14}$$

It can be verified, using $V_n = 1/(u_n u_{n+2} u_{n+4})$, that there is no need for absolute value function in Eq.(4.14). Similarly, for any $j = 0, \ldots, 5$ we obtain the following:

$$u_{6n+j} = u_j \prod_{k=0}^{n-1}\frac{V_{6k+j}}{V_{6k+j+2}}$$

For $j = 0$, we get

$$u_{6n} = u_0 \prod_{k=0}^{n-1}\frac{V_{6k}}{V_{6k+2}}$$

$$= u_0 \prod_{k=0}^{n-1}\frac{\left(\prod_{k_1=0}^{6k-1} A_{k_1}\right) + u_0u_2u_4\sum_{l=0}^{6k-1}\left(B_l \prod_{k_2=l+1}^{6k-1} A_{k_2}\right)}{\left(\prod_{k_1=0}^{6k+1} A_{k_1}\right) + u_0u_2u_4\sum_{l=0}^{6k+1}\left(B_l \prod_{k_2=l+1}^{6k+1} A_{k_2}\right)}$$

For $j = 1$, we have

$$u_{6n+1} = u_1 \prod_{k=0}^{n-1}\frac{V_{6k+1}}{V_{6k+3}}$$

$$= u_1 \prod_{k=0}^{n-1}\frac{\left(\prod_{k_1=0}^{6k} A_{k_1}\right) + u_0u_2u_4\sum_{l=0}^{6k}\left(B_l \prod_{k_2=l+1}^{6k} A_{k_2}\right)}{\left(\prod_{k_1=0}^{6k+2} A_{k_1}\right) + u_0u_2u_4\sum_{l=0}^{6k+2}\left(B_l \prod_{k_2=l+1}^{6k+2} A_{k_2}\right)}$$

For $j = 2$, we obtain

$$u_{6n+2} = u_2 \prod_{k=0}^{n-1}\frac{V_{6k+2}}{V_{6k+4}}$$

$$= u_2 \prod_{k=0}^{n-1} \frac{\left(\prod_{k_1=0}^{6k+1} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+1} \left(B_l \prod_{k_2=l+1}^{6k+1} A_{k_2}\right)}{\left(\prod_{k_1=0}^{6k+3} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+3} \left(B_l \prod_{k_2=l+1}^{6k+3} A_{k_2}\right)}$$

For $j = 3$, we find

$$u_{6n+3} = u_3 \prod_{k=0}^{n-1} \frac{V_{6k+3}}{V_{6k+5}}$$

$$= u_3 \prod_{k=0}^{n-1} \frac{\left(\prod_{k_1=0}^{6k+2} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+2} \left(B_l \prod_{k_2=l+1}^{6k+2} A_{k_2}\right)}{\left(\prod_{k_1=0}^{6k+4} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+4} \left(B_l \prod_{k_2=l+1}^{6k+4} A_{k_2}\right)}$$

For $j = 4$, we have

$$u_{6n+4} = u_4 \prod_{k=0}^{n-1} \frac{V_{6k+4}}{V_{6k+6}}$$

$$= u_4 \prod_{k=0}^{n-1} \frac{\left(\prod_{k_1=0}^{6k+3} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+3} \left(B_l \prod_{k_2=l+1}^{6k+3} A_{k_2}\right)}{\left(\prod_{k_1=0}^{6k+5} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+5} \left(B_l \prod_{k_2=l+1}^{6k+5} A_{k_2}\right)}$$

Finally, for $j = 5$, we get

$$u_{6n+5} = u_5 \prod_{k=0}^{n-1} \frac{V_{6k+5}}{V_{6k+7}}$$

$$= u_5 \prod_{k=0}^{n-1} \frac{\left(\prod_{k_1=0}^{6k+4} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+4} \left(B_l \prod_{k_2=l+1}^{6k+4} A_{k_2}\right)}{\left(\prod_{k_1=0}^{6k+6} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+6} \left(B_l \prod_{k_2=l+1}^{6k+6} A_{k_2}\right)}$$

More compactly, we have the solutions of Eq.(4.3) as follows:

$$u_{6n+j} = u_j \prod_{k=0}^{n-1} \frac{\left(\prod_{k_1=0}^{6k+j-1} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+j-1} \left(B_l \prod_{k_2=l+1}^{6k+j-1} A_{k_2}\right)}{\left(\prod_{k_1=0}^{6k+j+1} A_{k_1}\right) + u_0u_2u_4 \sum_{l=0}^{6k+j+1} \left(B_l \prod_{k_2=l+1}^{6k+j+1} A_{k_2}\right)}$$

$j = 0, 1, 2, 3, 4, 5$. Therefore the solutions of Eq.(4.1) become

$$x_{6n+j-4} = x_{j-4} \prod_{k=0}^{n-1} \frac{\left(\prod_{k_1=0}^{6k+j-1} a_{k_1}\right) + x_{-4}x_{-2}x_0 \sum_{l=0}^{6k+j-1} \left(b_l \prod_{k_2=l+1}^{6k+j-1} a_{k_2}\right)}{\left(\prod_{k_1=0}^{6k+j+1} a_{k_1}\right) + x_{-4}x_{-2}x_0 \sum_{l=0}^{6k+j+1} \left(b_l \prod_{k_2=l+1}^{6k+j+1} a_{k_2}\right)}, 0 \le j \le 5. \tag{4.15}$$

4.2 THE CASE a_n AND b_n ARE 1-PERIODIC SEQUENCES

In this case, $a_n = a$ and $b_n = b$ where $a,\ b \in \mathbb{R}$. Then, equations in Eq.(4.15) simplify to

$$x_{6n+j-4} = x_{j-4}\prod_{k=0}^{n-1}\frac{a^{6k+j}+x_{-4}x_{-2}x_0 b\left(\sum\limits_{l=0}^{6k+j-1}a^l\right)}{a^{6k+j+2}+x_{-4}x_{-2}x_0 b\left(\sum\limits_{l=0}^{6k+j+1}a^l\right)}, 0 \le j \le 5. \tag{4.16}$$

Theorem 4.1 *The difference equation*

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(a+bx_n x_{n-2}x_{n-4})} \tag{4.17}$$

has a periodic solution of period six if and only if $x_{-4}x_{-2}x_0 = \frac{1-a}{b}$, $a \neq 1$.

Proof. Here, $x_{-4}x_{-2}x_0 = \frac{1-a}{b}$ and then Eq.(4.16) simplifies to

$$\begin{aligned} x_{6n+j-4} =& x_{j-4}\prod_{k=0}^{n-1}\frac{a^{6k+j}+(1-a)\left(\sum\limits_{l=0}^{6k+j-1}a^l\right)}{a^{6k+j+2}+(1-a)\left(\sum\limits_{l=0}^{6k+j+1}a^l\right)}, 0 \le j \le 5 \\ =& x_{j-4}\prod_{k=0}^{n-1}\frac{a^{6k+j}+(1-a)\left(\frac{1-a^{6k+j}}{1-a}\right)}{a^{6k+j+2}+(1-a)\left(\frac{1-a^{6k+j+2}}{1-a}\right)}, 0 \le j \le 5 \\ =& x_{j-4}, \end{aligned}$$

for $i = 0, \ldots 5$. ■

Theorem 4.2 *Let* $\{x_n\}$ *be a well-definied solution to the sequence*

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(1+bx_n x_{n-2}x_{n-4})}$$

with $b \neq 0$. *Then*

$$\lim_{n\to\infty} x_n = 0.$$

Proof. Using Eq.(4.16), with $a = 1$, we have

$$\begin{aligned} x_{6n+j-4} =& x_{j-4}\prod_{k=0}^{n-1}\frac{1+(6k+j)bx_{-4}x_{-2}x_0}{1+(6k+j+2)bx_{-4}x_{-2}x_0} \\ =& x_{j-4}\prod_{k=0}^{n-1}\left[1-\frac{2bx_{-4}x_{-2}x_0}{1+(6k+j+2)bx_{-4}x_{-2}x_0}\right]. \end{aligned}$$

Clearly, $1 + (6k + j + 2)bx_{-4}x_{-2}x_0 \to \infty$ as $k \to \infty$. Therefore, we can always find a sufficiently large $n_0 \in \mathbb{N}$ such that for all $k > n_0$, we have

$$1 + (6k + j + 2)bx_{-4}x_{-2}x_0 \simeq (6k + j + 2)bx_{-4}x_{-2}x_0.$$

So

$$x_{6n+j-4} = x_{j-4}\Gamma(n_0) \prod_{k=n_0+1}^{n-1} \left(1 - \frac{2}{6k + j + 2}\right)$$

$$= x_{j-4}\Gamma(n_0) \prod_{k=n_0+1}^{n-1} \exp\left[\ln\left(1 - \frac{2}{6k + j + 2}\right)\right]$$

Note that

$$\Gamma(n_0) = \prod_{k=0}^{n_0} \left(1 - \frac{2}{6k + j + 2}\right).$$

Now, using the fact that $\ln(1 + x) = x + O(x^2)$ for small x, we obtain

$$x_{6n+j-4} = x_{j-4}\Gamma(n_0) \prod_{k=n_0+1}^{n-1} \exp\left[-\frac{2}{6k + j + 2} + O\left(\frac{1}{(6k + j + 2)^2}\right)\right]$$

$$= x_{j-4}\Gamma(n_0) \exp\left[-\sum_{k=n_0+1}^{n-1} \left(\frac{2}{6k + j + 2}\right)\right] \prod_{k=n_0+1}^{n-1} \exp\left[O\left(\frac{1}{(6k + j + 2)^2}\right)\right],$$

$j = 0, \ldots, 5$, and hence $x_n \to \infty$ as $n \to \infty$. ■

Next, we deal with the cases $a = \pm 1$ and $b = \pm 1$.

4.2.1 The Case $a = 1$ and $b = \pm 1$.

We can verify our results from [38] (see Theorems 1 and 6) that

$$x_{6n+j-4} = x_{j-4}\prod_{k=0}^{n-1}\frac{1 + b(6k + j)x_{-4}x_{-2}x_0}{1 + b(6k + j + 2)x_{-4}x_{-2}x_0}$$

Clearly, in terms of the initial values $x_{-4}, x_{-3}, x_{-2},\ x_{-1},\ x_0$, we have

$$x_{6n-4} = x_{-4} \prod_{k=0}^{n-1} \frac{1 + 6bkx_{-4}x_{-2}x_0}{1 + (6k + 2)bx_{-4}x_{-2}x_0},$$

$$x_{6n-3} = x_{-3} \prod_{k=0}^{n-1} \frac{1 + (6k + 1)bx_{-4}x_{-2}x_0}{1 + (6k + 3)bx_{-4}x_{-2}x_0},$$

$$x_{6n-2} = x_{-2} \prod_{k=0}^{n-1} \frac{1 + (6k + 2)bx_{-4}x_{-2}x_0}{1 + (6k + 4)bx_{-4}x_{-2}x_0},$$

$$x_{6n-1} = x_{-2}\prod_{k=0}^{n-1}\frac{1+(6k+3)bx_{-4}x_{-2}x_0}{1+(6k+5)bx_{-4}x_{-2}x_0},$$

$$x_{6n} = x_0\prod_{k=0}^{n-1}\frac{1+(6k+4)bx_{-4}x_{-2}x_0}{1+(6k+6)bx_{-4}x_{-2}x_0}$$

and

$$x_{6n+1} = \frac{x_{-4}x_{-2}x_0}{x_{-1}x_{-3}(1+bx_{-4}x_{-2}x_0)}\prod_{k=0}^{n-1}\frac{1+(6k+5)bx_{-4}x_{-2}x_0}{1+(6k+7)bx_{-4}x_{-2}x_0}$$

4.2.2 The Case $a=-1$ and $b=\pm 1$

In this case, the solution becomes

$$\begin{aligned} x_{6n+j-4} &= x_{j-4}\prod_{k=0}^{n-1}\frac{(-1)^j + bx_{-4}x_{-2}x_0\left(\dfrac{1-(-1)^j}{2}\right)}{(-1)^j + bx_{-4}x_{-2}x_0\left(\dfrac{1-(-1)^j}{2}\right)} \\ &= x_{j-4} \end{aligned}$$

where $j = 0,1,2,3,4,5$.

Therefore, in this case it is easy to see that the solutions are periodic with period six. It is clear that, here, the extra condition $x_{-4}x_{-2}x_0 = \frac{1-a}{b}$ in the above theorem is not needed. We can check our results with the ones which is obtained in [38] (see Theorems 3 and 8).

Finally, we obtained four symmetry generators of shifted equation and used one of these generators to obtain exact solutions with Lie symmetry analysis to difference equations of the form

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(a_n + b_n x_n x_{n-2} x_{n-4})}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are non-zero real sequences and this was the extension to the work done by Elsayed E M and Ibrahim T F [38]. However, the method we used is completely different from the above mentioned study.

CHAPTER 5

THE EXACT SOLUTIONS OF SOME HIGHER ORDER DIFFERENCE EQUATIONS VIA LIE SYMMETRY ANALYSIS

In [26], Folly-Gbetoula and Nyirenda studied the solutions and properties of the difference equation

$$x_{n+1} = \frac{x_{n-7}x_{n-2}}{x_{n-4}(a_n + b_n x_{n-7}x_{n-2})}$$

where a_n and b_n are arbitrary sequences of real numbers and initial values are x_{-7}, x_{-6}, .., x_{-1} and x_0. Mnguni et al. [43] investigated

$$x_{n+1} = \frac{x_{n-4}}{a_n + b_n x_n x_{n-1}x_{n-2}x_{n-3}x_{n-4}}$$

where (a_n) and (b_n) are arbitrary sequences of real numbers.

Elsayed [24] , obtained the solutions of the following difference equations of order six

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-2}x_{n-5}}, \tag{5.1}$$

and Folly-Gbetoula and Nyirenda [25] perform a Lie symmetry analysis, derive formulas for solutions of Eq.(5.1) .

Elsayed [42] , found the solution of the following difference equations of order ten

$$x_{n+1} = \frac{x_{n-9}}{\pm 1 \pm x_{n-4}x_{n-9}}. \tag{5.2}$$

In this chapter we give solutions of the following difference equations with the invariant of their group of transformations:

$$x_{n+1} = \frac{x_{n-9}}{a_n + b_n x_{n-4}x_{n-9}} \tag{5.3}$$

for some arbitrary sequences of real numbers (a_n) and (b_n) .

The solution to our difference equation provides a generalization of the solution of Elsayed [42] to equation Eq.(5.2) in a more general setting.

Without loss of generality, we instead study the difference equation

$$u_{n+10} = \frac{u_n}{A_n + B_n u_{n+5} u_n} \tag{5.4}$$

Imposing the symmetry condition Eq.(2.7) to the equation above, we obtain

$$Q\left(n+10, u_{n+10}\right) + \frac{B_n u_n^2}{\left(A_n + B_n u_n u_{n+5}\right)^2} Q(n+5, u_{n+5}) - \frac{A_n}{\left(A_n + B_n u_n u_{n+5}\right)^2} Q(n, u_n) = 0 \tag{5.5}$$

The characteristic Q takes different arguments, and solving for it becomes diffucult. So we first eliminate u_{n+10} by implicit differentiability with respect to u_n. This yields

$$\begin{aligned} & \frac{A_n}{\left(A_n + B_n u_n u_{n+5}\right)^2} Q^{'}(n+5, u_{n+5}) - \frac{A_n}{\left(A_n + B_n u_n u_{n+5}\right)^2} Q^{'}(n, u_n) \\ & + \frac{2A_n}{u_n \left(A_n + B_n u_n u_{n+5}\right)^2} Q(n, u_n) \\ = \; & 0 \end{aligned}$$

and after simplification, we get

$$Q^{'}(n+5, u_{n+5}) - Q^{'}(n, u_n) + \frac{2}{u_n} Q(n, u_n) = 0,$$

where $'$ denotes the derivative with respect to the continuous variable. We now eliminate u_{n+5} by differentiating the above equation with respect to u_n and get the following equation:

$$Q^{''}(n, u_n) - \frac{2}{u_n} Q^{'}(n, u_n) + \frac{2}{u_n^2} Q(n, u_n) = 0, \tag{5.6}$$

Thus, the solution of Eq.(5.6) is

$$Q = \alpha_n u_n + \beta_n u_n^2 \tag{5.7}$$

where α_n and β_n are functions of n. We substitute Eq.(5.7) in Eq.(5.5) and with simplification, we are led to

$$\alpha_{n+10} A_n u_n + \alpha_{n+10} B_n u_n^2 u_{n+5} + \beta_{n+10} u_n^2 + \alpha_{n+5} B_n u_n^2 u_{n+5} + \beta_{n+5} B_n u_n^2 u_{n+5}^2 - \alpha_n A_n u_n - \beta_n A_n u_n^2 = 0$$

To solve for α_n and β_n, we use the method of seperation by powers of shifs of u_n and we obtain the following reduced system:

$$\begin{aligned} \alpha_n + \alpha_{n+5} &= 0, \\ \beta_n &= 0. \end{aligned} \tag{5.8}$$

The solution of Eq.(5.8) are $(-1)^n, \exp(\pm n\pi/5), \exp(\pm 3n\pi/5)$. Let $\theta_1 = \exp(n\pi/5)$ and $\theta_2 = \exp(3n\pi/5)$. The implication is that we have five non trivial generators

$$\begin{aligned}
X_1 &= \sum_{s=0}^{9} \theta_1^{n+s} u_{n+s} \frac{\partial}{\partial u_{n+s}}, \\
X_2 &= \sum_{s=0}^{9} \overline{\theta}_1^{n+s} u_{n+s} \frac{\partial}{\partial u_{n+s}}, \\
X_3 &= \sum_{s=0}^{9} \theta_2^{n+s} u_{n+s} \frac{\partial}{\partial u_{n+s}}, \\
X_4 &= \sum_{s=0}^{9} \overline{\theta}_2^{n+s} u_{n+s} \frac{\partial}{\partial u_{n+s}}, \\
X_5 &= \sum_{s=0}^{9} (-1)^{n+s} u_{n+s} \frac{\partial}{\partial u_{n+s}}.
\end{aligned}$$

5.1 REDUCTION

We use the known choice of canonical coordinate [9],

$$s_n = \int \frac{du_n}{Q(n, u_n)}.$$

Using X_1 (we could have chosen X_2, X_3, X_4 or X_5), we have

$$s_n = \int \frac{du_n}{Q_1(n, u_n)} = \frac{1}{\theta_1^n} \ln |u_n|.$$

Using Eq.(5.8), it can be shown that

$$X_1(\theta_1^{n+5} s_{n+5} + \theta_1^n s_n) = 0$$

and so

$$\widetilde{V}_n = \theta_1^n s_n + \theta_1^{n+5} s_{n+5}$$

is an invariant of X_1. We introduce

$$|V_n| = \exp\left\{-\widetilde{V}_n\right\},$$

i.e., $V_n = \pm 1/u_n u_{n+5}$. For the rest of the work, we use $V_n = 1/u_n u_{n+5}$. One can show that

$$V_{n+5} = A_n V_n + B_n \tag{5.9}$$

and it can be verified that the closed form solution of Eq.(5.9) is

$$V_{5n+j} = V_j \left(\prod_{k_1=0}^{n-1} A_{5k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{5l+j} \prod_{k_2=l+1}^{n-1} A_{5k_2+j} \right), \ j = 0, 1, 2, 3, 4 \tag{5.10}$$

5.2 EXACT SOLUTIONS OF EQUATION (5.3)

Working backward to express the result in terms of the original variable u_n, we get

$$|u_n| = \exp\left(\begin{array}{c} \theta_1^n c_1 + \overline{\theta}_1^n c_2 + \theta_2^n c_3 + \overline{\theta}_2^n c_4 + (-1)^n c_5 + \frac{1}{5}\sum_{k_0=0}^{n-1}(-1)^{n-k_0}\ln|V_{k_0}| \\ +\frac{1}{5}\sum_{k_1=0}^{n-1}\theta_1^n\theta_1^{-k_1}\ln|V_{k_1}| + \frac{1}{5}\sum_{k_2=0}^{n-1}\overline{\theta}_1^n\theta_1^{k_2}\ln|V_{k_2}| \\ +\frac{1}{5}\sum_{k_3=0}^{n-1}\theta_2^n\theta_2^{-k_3}\ln|V_{k_3}| + \frac{1}{5}\sum_{k_4=0}^{n-1}\overline{\theta}_2^n\theta_2^{k_4}\ln|V_{k_4}| \end{array}\right)$$

$$= \exp\left(H_n + \frac{1}{5}\sum_{k_0=0}^{n-1}(-1)^{n-k_0}\ln|V_{k_0}| + \frac{2}{5}\sum_{k_1=0}^{n-1}\mathrm{Re}\left[\gamma_1(n,k_1)\right]\ln|V_{k_1}| + \frac{2}{5}\sum_{k_2=0}^{n-1}\mathrm{Re}\left[\gamma_2(n,k_2)\right]\ln|V_{k_2}|\right) \tag{5.11}$$

where $H_n = \theta_1^n c_1 + \overline{\theta}_1^n c_2 + \theta_2^n c_3 + \overline{\theta}_2^n c_4 + (-1)^n c_5$ and $\gamma_s(n,k) = \theta_s^n\theta_s^{-k}$, $s = 1, 2$.
Note that the functions $\gamma_s(n,k) = \theta_s^n\theta_s^{-k}$, $s = 1, 2$ satisfy the following properties:

$$\begin{aligned} \gamma_s(n+5,k) &= \gamma_s(n,k+5) = -\gamma_s(n,k), \gamma_s(n+10,k) = \gamma_s(n,k+10) = \gamma_s(n,k), \quad (5.12) \\ \gamma_s(10n,k) &= \gamma_s(0,k), \gamma_s(n,10k) = \gamma_s(n,0), s = 0, 1. \end{aligned}$$

We recall that $\theta_1 = \exp(n\pi/5)$ and $\theta_2 = \exp(3n\pi/5)$.
From the expression of u_n given in Eq.(5.11) and from the above properties Eq.(5.12), it is clear that

$$|u_{10n+j}| = \exp\left(H_j + \frac{1}{5}\sum_{k=0}^{10n+j-1}((-1)^{j-k} + 2\,\mathrm{Re}\left[\gamma_1(j,k) + \gamma_2(j,k)\right])\ln V_k\right), \ \textit{where}\ 0 \le j \le 9. \tag{5.13}$$

For $j = 0$, we have that

$$\begin{aligned} |u_{10n}| &= \exp(H_0 + \ln|V_0| - \ln|V_5| + ... + \ln|V_{10n-10}| + \ln|V_{10n-5}|) \quad (5.14) \\ &= \exp(H_0)\prod_{s=0}^{n-1}\left|\frac{V_{10s}}{V_{10s+5}}\right|. \end{aligned}$$

By setting $n = 0$ in Eq.(5.14), we get $\exp(H_0) = u_0$ and so

$$u_{10n} = u_0\prod_{s=0}^{n-1}\left|\frac{V_{10s}}{V_{10s+5}}\right|.$$

Using $1/\left(u_n u_{n+5}\right)$, it can be shown that there is no need of absolute values. More generally, for any $j=0,1,...,9$ we obtain the following:

$$u_{10n+j}=u_j\prod_{s=0}^{n-1}\left|\frac{V_{10s+j}}{V_{10s+j+5}}\right|.$$

Using Eq.(5.10), where $n:=2s$ and $j=0$ for V_{10s}, and $n:=2s+1$ and $j=0$ for $V_{10s+5,}$ we have

$$\begin{aligned}
V_{10s} &= V_0\left(\prod_{k_1=0}^{2s-1}A_{5k_1}\right)+\sum_{l=0}^{2s-1}\left(B_{5l}\prod_{k_2=l+1}^{2s-1}A_{5k_2}\right)\\
&= V_0\left(\prod_{k_1=0}^{2s-1}A_{5k_1}+\frac{1}{V_0}\sum_{l=0}^{2s-1}B_{5l}\prod_{k_2=l+1}^{2s-1}A_{5k_2}\right)\\
&= \frac{1}{u_0u_5}\left(\prod_{k_1=0}^{2s-1}A_{5k_1}+u_0u_5\sum_{l=0}^{2s-1}B_{5l}\prod_{k_2=l+1}^{2s-1}A_{5k_2}\right)
\end{aligned}$$

and

$$\begin{aligned}
V_{10s+5} &= V_0\left(\prod_{k_1=0}^{2s}A_{5k_1}\right)+\sum_{l=0}^{2s}\left(B_{5l}\prod_{k_2=l+1}^{2s}A_{5k_2}\right)\\
&= V_0\left(\prod_{k_1=0}^{2s}A_{5k_1}+\frac{1}{V_0}\sum_{l=0}^{2s}B_{5l}\prod_{k_2=l+1}^{2s}A_{5k_2}\right)\\
&= \frac{1}{u_0u_5}\left(\prod_{k_1=0}^{2s}A_{5k_1}+u_0u_5\sum_{l=0}^{2s}B_{5l}\prod_{k_2=l+1}^{2s}A_{5k_2}\right).
\end{aligned}$$

Thus

$$u_{10n}=u_0\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s-1}A_{5k_1}+u_0u_5\sum\limits_{l=0}^{2s-1}B_{5l}\prod\limits_{k_2=l+1}^{2s-1}A_{5k_2}}{\prod\limits_{k_1=0}^{2s}A_{5k_1}+u_0u_5\sum\limits_{l=0}^{2s}B_{5l}\prod\limits_{k_2=l+1}^{2s}A_{5k_2}},$$

which implies that

$$x_{10n-10}=x_{-10}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s-1}a_{5k_1}+x_{-10}x_{-5}\sum\limits_{l=0}^{2s-1}b_{5l}\prod\limits_{k_2=l+1}^{2s-1}a_{5k_2}}{\prod\limits_{k_1=0}^{2s}a_{5k_1}+x_{-10}x_{-5}\sum\limits_{l=0}^{2s}b_{5l}\prod\limits_{k_2=l+1}^{2s}a_{5k_2}}.$$

For $j=1$, we have that

$$x_{10n-9}=x_{-9}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s-1}a_{5k_1+1}+x_{-9}x_{-4}\sum\limits_{l=0}^{2s-1}b_{5l+1}\prod\limits_{k_2=l+1}^{2s-1}a_{5k_2+1}}{\prod\limits_{k_1=0}^{2s}a_{5k_1+1}+x_{-9}x_{-4}\sum\limits_{l=0}^{2s}b_{5l+1}\prod\limits_{k_2=l+1}^{2s}a_{5k_2+1}}.$$

For $j = 2$, we get that

$$x_{10n-8} = x_{-8} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} a_{5k_1+2} + x_{-8}x_{-3} \sum_{l=0}^{2s-1} b_{5l+2} \prod_{k_2=l+1}^{2s-1} a_{5k_2+2}}{\prod_{k_1=0}^{2s} a_{5k_1+2} + x_{-8}x_{-3} \sum_{l=0}^{2s} b_{5l+2} \prod_{k_2=l+1}^{2s} a_{5k_2+2}}.$$

For $j = 3$, we obtain that

$$x_{10n-7} = x_{-7} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} a_{5k_1+3} + x_{-7}x_{-2} \sum_{l=0}^{2s-1} b_{5l+3} \prod_{k_2=l+1}^{2s-1} a_{5k_2+3}}{\prod_{k_1=0}^{2s} a_{5k_1+3} + x_{-7}x_{-2} \sum_{l=0}^{2s} b_{5l+3} \prod_{k_2=l+1}^{2s} a_{5k_2+3}}.$$

For $j = 4$, we have that

$$x_{10n-6} = x_{-6} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} a_{5k_1+4} + x_{-6}x_{-1} \sum_{l=0}^{2s-1} b_{5l+4} \prod_{k_2=l+1}^{2s-1} a_{5k_2+4}}{\prod_{k_1=0}^{2s} a_{5k_1+4} + x_{-6}x_{-1} \sum_{l=0}^{2s} b_{5l+4} \prod_{k_2=l+1}^{2s} a_{5k_2+4}}.$$

For $j = 5$, we find that

$$x_{10n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} a_{5k_1} + x_{-5}x_{-10} \sum_{l=0}^{2s} b_{5l} \prod_{k_2=l+1}^{2s-1} a_{5k_2}}{\prod_{k_1=0}^{2s+1} a_{5k_1} + x_{-5}x_{-10} \sum_{l=0}^{2s+1} b_{5l} \prod_{k_2=l+1}^{2s+1} a_{5k_2}}.$$

For $j = 6$, we get that

$$x_{10n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} a_{5k_1+1} + x_{-9}x_{-4} \sum_{l=0}^{2s} b_{5l+1} \prod_{k_2=l+1}^{2s} a_{5k_2+1}}{\prod_{k_1=0}^{2s+1} a_{5k_1+1} + x_{-9}x_{-4} \sum_{l=0}^{2s+1} b_{5l+1} \prod_{k_2=l+1}^{2s+1} a_{5k_2+1}}.$$

For $j = 7$, we obtain that

$$x_{10n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} a_{5k_1+2} + x_{-8}x_{-3} \sum_{l=0}^{2s} b_{5l+2} \prod_{k_2=l+1}^{2s} a_{5k_2+2}}{\prod_{k_1=0}^{2s+1} a_{5k_1+2} + x_{-8}x_{-3} \sum_{l=0}^{2s+1} b_{5l+2} \prod_{k_2=l+1}^{2s+1} a_{5k_2+2}}.$$

For $j=8$, we have that

$$x_{10n-2}=x_{-2}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s}a_{5k_1+3}+x_{-7}x_{-2}\sum\limits_{l=0}^{2s}b_{5l+3}\prod\limits_{k_2=l+1}^{2s}a_{5k_2+3}}{\prod\limits_{k_1=0}^{2s+1}a_{5k_1+3}+x_{-7}x_{-2}\sum\limits_{l=0}^{2s+1}b_{5l+3}\prod\limits_{k_2=l+1}^{2s+1}a_{5k_2+3}}.$$

For $j=9$, we find that

$$x_{10n-1}=x_{-1}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s}a_{5k_1+4}+x_{-6}x_{-1}\sum\limits_{l=0}^{2s}b_{5l+4}\prod\limits_{k_2=l+1}^{2s}a_{5k_2+4}}{\prod\limits_{k_1=0}^{2s+1}a_{5k_1+4}+x_{-6}x_{-1}\sum\limits_{l=0}^{2s+1}b_{5l+4}\prod\limits_{k_2=l+1}^{2s+1}a_{5k_2+4}}.$$

5.3 THE CASE a_j AND b_j ARE 1-PERIODIC

5.3.1 The Case $a_0 \neq 1$

In this case, we have the solution defined by the following equations

$$x_{10n-10}=x_{-10}\prod_{s=0}^{n-1}\frac{a_0^{2s}+x_{-10}x_{-5}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1}+x_{-10}x_{-5}b_0\frac{1-a_0^{2s+1}}{1-a_0}},\quad x_{10n-9}=x_{-9}\prod_{s=0}^{n-1}\frac{a_0^{2s}+x_{-9}x_{-4}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1}+x_{-9}x_{-4}b_0\frac{1-a_0^{2s+1}}{1-a_0}}$$

$$x_{10n-8}=x_{-8}\prod_{s=0}^{n-1}\frac{a_0^{2s}+x_{-8}x_{-3}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1}+x_{-8}x_{-3}b_0\frac{1-a_0^{2s+1}}{1-a_0}},\quad x_{10n-7}=x_{-7}\prod_{s=0}^{n-1}\frac{a_0^{2s}+x_{-7}x_{-2}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1}+x_{-7}x_{-2}b_0\frac{1-a_0^{2s+1}}{1-a_0}}$$

$$x_{10n-6}=x_{-6}\prod_{s=0}^{n-1}\frac{a_0^{2s}+x_{-6}x_{-1}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1}+x_{-6}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}},\quad x_{10n-5}=x_{-5}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+x_{-5}x_{-10}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2}+x_{-5}x_{-10}b_0\frac{1-a_0^{2s+2}}{1-a_0}}$$

$$x_{10n-4}=x_{-4}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+x_{-4}x_{-9}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2}+x_{-4}x_{-9}b_0\frac{1-a_0^{2s+2}}{1-a_0}},\quad x_{10n-3}=x_{-3}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+x_{-3}x_{-8}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2}+x_{-3}x_{-8}b_0\frac{1-a_0^{2s+2}}{1-a_0}}$$

$$x_{10n-2}=x_{-2}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+x_{-2}x_{-7}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2}+x_{-2}x_{-7}b_0\frac{1-a_0^{2s+2}}{1-a_0}},\quad x_{10n-1}=x_{-1}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+x_{-1}x_{-6}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2}+x_{-1}x_{-6}b_0\frac{1-a_0^{2s+2}}{1-a_0}}$$

5.3.2 The Case $a_0=-1$ and $b_0=1$

In this case, we obtain the following solution which appears in [42] (see Theorem 4)

$$x_{10n-1}=x_{-1}(-1+x_{-1}x_{-6})^n,\quad x_{10n-2}=x_{-2}(-1+x_{-2}x_{-7})^n$$

$$x_{10n-3}=x_{-3}(-1+x_{-3}x_{-8})^n,\quad x_{10n-4}=x_{-4}(-1+x_{-4}x_{-9})^n$$

$$x_{10n-5} = x_{-5}(-1 + x_{-5}x_{-10})^{-n}, \quad x_{10n-6} = x_{-6}(-1 + x_{-6}x_{-1})^{-n}$$

$$x_{10n-7} = x_{-7}(-1 + x_{-7}x_{-2})^{-n}, \quad x_{10n-8} = x_{-8}(-1 + x_{-8}x_{-3})^{-n}$$

$$x_{10n-9} = x_{-9}(-1 + x_{-9}x_{-4})^{-n}, \quad x_{10n-10} = x_{-10}(-1 + x_{-10}x_{-5})^{n}$$

where $x_{-1}x_{-6} \neq 1, x_{-2}x_{-7} \neq 1, x_{-3}x_{-8} \neq 1, x_{-4}x_{-9} \neq 1, x_{-5}x_{-10} \neq 1.$

5.3.3 The Case $a_0 = -1$ and $b_0 = -1$

In this case, one obtains the solution defined by the following equations and appears in [42] (see Theorem 9)

$$x_{10n-1} = x_{-1}(-1 - x_{-1}x_{-6})^{n}, \quad x_{10n-2} = x_{-2}(-1 - x_{-2}x_{-7})^{n},$$

$$x_{10n-3} = x_{-3}(-1 - x_{-3}x_{-8})^{n}, \quad x_{10n-4} = x_{-4}(-1 - x_{-4}x_{-9})^{n},$$

$$x_{10n-5} = x_{-5}(-1 - x_{-5}x_{-10})^{-n}, \quad x_{10n-6} = x_{-6}(-1 - x_{-6}x_{-1})^{-n},$$

$$x_{10n-7} = x_{-7}(-1 - x_{-7}x_{-2})^{-n}, \quad x_{10n-8} = x_{-8}(-1 - x_{-8}x_{-3})^{-n},$$

$$x_{10n-9} = x_{-9}(-1 - x_{-9}x_{-4})^{-n}, \quad x_{10n-10} = x_{-10}(-1 - x_{-10}x_{-5})^{n}$$

where $x_{-1}x_{-6} \neq -1, x_{-2}x_{-7} \neq -1, x_{-3}x_{-8} \neq -1, x_{-4}x_{-9} \neq -1, x_{-5}x_{-10} \neq -1.$

5.3.4 The Case $a_0 = 1$ and $b_0 = 1$

In this case, we have the solution defined by the following equations and appears in [42] (see Theorem 1)

$$x_{10n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1 + (2s+1)\,x_{-1}x_{-6}}{1 + (2s+2)\,x_{-1}x_{-6}}, \quad x_{10n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1 + (2s+1)\,x_{-2}x_{-7}}{1 + (2s+2)\,x_{-2}x_{-7}},$$

$$x_{10n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1 + (2s+1)\,x_{-3}x_{-8}}{1 + (2s+2)\,x_{-3}x_{-8}}, \quad x_{10n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{1 + (2s+1)\,x_{-4}x_{-9}}{1 + (2s+2)\,x_{-4}x_{-9}},$$

$$x_{10n-5} = x_{-5}\prod_{s=0}^{n-1}\frac{1 + (2s+1)\,x_{-5}x_{-10}}{1 + (2s+2)\,x_{-5}x_{-10}}, \quad x_{10n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{1 + 2sx_{-6}x_{-1}}{1 + (2s+1)\,x_{-6}x_{-1}},$$

$$x_{10n-7} = x_{-7}\prod_{s=0}^{n-1}\frac{1 + 2sx_{-7}x_{-2}}{1 + (2s+1)\,x_{-7}x_{-2}}, \quad x_{10n-8} = x_{-8}\prod_{s=0}^{n-1}\frac{1 + 2sx_{-8}x_{-3}}{1 + (2s+1)\,x_{-8}x_{-3}},$$

$$x_{10n-9} = x_{-9}\prod_{s=0}^{n-1}\frac{1 + 2sx_{-9}x_{-4}}{1 + (2s+1)\,x_{-9}x_{-4}}, \quad x_{10n-10} = x_{-10}\prod_{s=0}^{n-1}\frac{1 + 2sx_{-10}x_{-5}}{1 + (2s+1)\,x_{-10}x_{-5}}$$

where $jx_{-1}x_{-6} \neq 1, jx_{-2}x_{-7} \neq 1, jx_{-3}x_{-8} \neq 1, jx_{-4}x_{-9} \neq 1, jx_{-5}x_{-10} \neq 1$ for all $j = 1, 2, ..., 2n$.

Finally, we get five symmetries of the difference equations Eq.(5.3) . As a result, solutions to the equations were obtained with Lie symmetry analysis and the solutions found were compared with [42] .

CHAPTER 6

THE EXACT SOLUTIONS OF $x_{n+1} = \frac{x_n x_{n-2}}{a_n x_{n-2} + b_n x_{n-3}}$ USING LIE SYMMETRY ANALYSIS

In [4], Folly-Gbetoula et al. investigated the solutions and properties of the difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{a_n x_n + b_n x_{n-4}}$$

for some arbitrary sequences of real numbers $\{a_n\}$ and $\{b_n\}$.

Elsayed and Khaliq [55] obtained the solution of the following difference equations of order four

$$x_{n+1} = \frac{a x_n x_{n-2}}{b x_{n-2} + c x_{n-3}}$$

where the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers and a, b, c are positive constants.

In this chapter, we derive solutions of the following difference equations via the invariant of their group of transformations:

$$x_{n+1} = \frac{x_n x_{n-2}}{a_n x_{n-2} + b_n x_{n-3}} \tag{6.1}$$

for some arbitrary sequences of real numbers $\{a_n\}$ and $\{b_n\}$.

For the sake of notation and definitions used in this work, we shall consider the Kovalevskaya form of Eq.(6.1), that is

$$u_{n+4} = \frac{u_{n+3} u_{n+1}}{A_n u_{n+1} + B_n u_n} \tag{6.2}$$

where A_n and B_n are random real sequences.

6.1 SYMMETRIES

Consider the difference equation Eq.(6.2). Assume that the Lie point transformations are of the form Eq.(2.8) and let the corresponding symmetry generator of the characteristic

Q be

$$X = Q(n,u_n)\frac{\partial}{\partial u_n} + S^{(1)}Q(n,u_n)\frac{\partial}{\partial u_{n+1}} + ... + S^{(3)}Q(n,u_n)\frac{\partial}{\partial u_{n+3}}. \quad (6.3)$$

The symmetry generator X satisfies the linearized symmetry condition Eq.(2.7), i.e.,

$$\begin{aligned} & S^{(4)}Q(n,u_n) - \frac{u_{n+1}}{A_n u_{n+1} + B_n u_n} S^{(3)}Q(n,u_n) \\ & -\frac{B_n u_n u_{n+3}}{(A_n u_{n+1} + B_n u_n)^2} S^{(1)}Q(n,u_n) \\ & +\frac{B_n u_{n+1} u_{n+3}}{(A_n u_{n+1} + B_n u_n)^2} Q(n,u_n) \\ = \; & 0. \end{aligned} \quad (6.4)$$

In order to solve for Q, we firstly eliminate the first term in the equation above, by applying the operator

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+3}}{\partial u_n}\frac{\partial}{\partial u_{n+3}}$$

on Eq.(6.4), that is, we differentiate the functional equation Eq.(6.4) implicitly with respect to u_n by assuming u_{n+3} as a function of u_n; u_{n+1} and u_{n+4}. This yields the following (after simplification):

$$\begin{aligned} & \frac{A_n u_{n+1} + B_n u_n}{u_{n+3}} Q(n+3,u_{n+3}) - A_n Q(n+1,u_{n+1}) \\ & -B_n Q(n,u_n) + (A_n u_{n+1} + B_n u_n) Q'(n,u_n) \\ & -(A_n u_{n+1} + B_n u_n) Q'(n+3,u_{n+3}) \\ = \; & 0. \end{aligned}$$

We differentiate the above equation with respect to u_n twice, keeping u_{n+3} constant, and obtain the following equation:

$$B_n Q''(n,u_n) + (A_n u_{n+1} + B_n u_n) Q'''(n,u_n) = 0$$

whose solution is

$$Q(n,u_n) = \alpha_n u_n + \beta_n \quad (6.5)$$

for some functions α and β to be found. We now substitute Eq.(6.5) in Eq.(6.4) and get the overdetermined equation

$$\begin{aligned} & \alpha_{n+4}\left(A_n u_{n+1}^2 u_{n+3} + B_n u_n u_{n+3} u_{n+1}\right) + \beta_{n+4}\left(A_n u_{n+1} + B_n u_n\right)^2 \\ & -\alpha_{n+3}\left(A_n u_{n+1}^2 u_{n+3} + B_n u_n u_{n+1} u_{n+3}\right) \\ & -\beta_{n+3}\left(A_n u_{n+1}^2 + B_n u_n u_{n+1}\right) - \alpha_{n+1}\left(u_{n+1} B_n u_n u_{n+3}\right) \\ & -\beta_{n+1}\left(B_n u_n u_{n+3}\right) + \alpha_n\left(B_n u_n u_{n+1} u_{n+3}\right) + \beta_n\left(B_n u_{n+1} u_{n+3}\right) \\ = & \ 0 \end{aligned}$$

which separation by monomials gives

$$\begin{aligned} \alpha_{n+4} - \alpha_{n+3} - \alpha_{n+1} + \alpha_n &= 0, \\ \alpha_{n+4} - \alpha_{n+3} &= 0, \\ \beta_{n+4} - A_n \beta_{n+3} &= 0, \\ \beta_{n+4} &= 0, \\ \beta_{n+4} - 2A_n \beta_{n+3} &= 0, \\ \beta_{n+1} &= 0, \\ \beta_n &= 0. \end{aligned}$$

We obtain the following reduced system:

$$\alpha_{n+1} - \alpha_n = 0, \qquad \beta_n = 0.$$

We use the known choice of canonical coordinate Eq.(2.6),

$$S_n = \int \frac{du_n}{\alpha_n u_n} = \frac{1}{\alpha_n} \ln |u_n|$$

and let

$$|r_n| = \exp\left[\alpha_n S_n - \alpha_{n+1} S_{n+1}\right].$$

We obtain one (new) independent variable (also an invariant) as follows

$$r_n = \frac{u_n}{u_{n+1}} \tag{6.6}$$

and

$$r_{n+3} = A_n + B_n r_n. \tag{6.7}$$

The general form of Eq.(6.7) after iterations is

$$r_{3n} = r_0 \left(\prod_{k_1=0}^{n-1} B_{3k_1} \right) + \sum_{l=0}^{n-1} \left(A_{3l} \prod_{k_2=l+1}^{n-1} B_{3k_2} \right)$$

or simply

$$r_{3n} = \frac{u_0}{u_1} \left(\prod_{k_1=0}^{n-1} B_{3k_1} \right) + \sum_{l=0}^{n-1} \left(A_{3l} \prod_{k_2=l+1}^{n-1} B_{3k_2} \right).$$

It can be verified that the closed form solutions of Eq.(6.7) is

$$r_{3n+i} = r_i \left(\prod_{k_1=0}^{n-1} B_{3k_1+i} \right) + \sum_{l=0}^{n-1} \left(A_{3l+i} \prod_{k_2=l+1}^{n-1} B_{3k_2+i} \right) \tag{6.8}$$

$i = 0, 1, 2.$

Invoking Eq.(6.6), we have

$$u_n = \left(\prod_{k_1=0}^{n-1} \frac{1}{r_{k_1}} \right) u_0 \tag{6.9}$$

and

$$u_{3n} = \left(\prod_{k_1=0}^{3n-1} \frac{1}{r_{k_1}} \right) u_0.$$

More generally, for any $j = 0, 1, 2$ we obtain the following:

$$u_{3n+j} = \left(\prod_{k_1=0}^{3n-1} \frac{1}{r_{k_1+j}} \right) u_j$$

from Eq.(6.9) for $j = 0$, we have that

$$u_{3n} = u_0 \prod_{s=0}^{n-1} \left(\frac{1}{r_{3s} r_{3s+1} r_{3s+2}} \right), \tag{6.10}$$

for $j = 1$, we obtain that

$$u_{3n+1} = \left(\prod_{k_1=0}^{3n} \frac{1}{r_{k_1}} \right) u_0,$$

$$u_{3n+1} = u_1 \prod_{s=0}^{n-1} \left(\frac{1}{r_{3s+1} r_{3s+2} r_{3s+3}} \right), \tag{6.11}$$

for $j = 2$, we get that

$$u_{3n+2} = \left(\prod_{k_1=0}^{3n+1} \frac{1}{r_{k_1}} \right) u_0,$$

$$u_{3n+2} = u_2 \prod_{s=0}^{n-1} \left(\frac{1}{r_{3s+2} r_{3s+3} r_{3s+4}} \right). \tag{6.12}$$

6.2 SOLUTIONS OF EQUATION (6.1)

Substituting Eq.(6.8) in Eq.(6.10) yields:

$$
\begin{aligned}
u_{3n} &= u_0 \prod_{s=0}^{n-1} \left(\frac{1}{r_{3s}}\right)\left(\frac{1}{r_{3s+1}}\right)\left(\frac{1}{r_{3s+2}}\right) \\
&= u_0 \prod_{s=0}^{n-1} \left[\frac{1}{r_0\left(\prod_{k_1=0}^{s-1} B_{3k_1}\right)+\sum_{l=0}^{s-1}\left(A_{3l}\prod_{k_2=l+1}^{s-1} B_{3k_2}\right)}\right] \\
&\quad \times \left[\frac{1}{r_1\left(\prod_{k_1=0}^{s-1} B_{3k_1+1}\right)+\sum_{l=0}^{s-1}\left(A_{3l+1}\prod_{k_2=l+1}^{s-1} B_{3k_2+1}\right)}\right] \\
&\quad \times \left[\frac{1}{r_2\left(\prod_{k_1=0}^{s-1} B_{3k_1+2}\right)+\sum_{l=0}^{s-1}\left(A_{3l+2}\prod_{k_2=l+1}^{s-1} B_{3k_2+2}\right)}\right].
\end{aligned}
$$

Substituting Eq.(6.8) in Eq.(6.11) gives

$$
\begin{aligned}
u_{3n+1} &= u_1 \prod_{s=0}^{n-1} \left(\frac{1}{r_{3s+1}}\right)\left(\frac{1}{r_{3s+2}}\right)\left(\frac{1}{r_{3(s+1)}}\right) \\
&= u_1 \prod_{s=0}^{n-1} \left[\frac{1}{r_1\left(\prod_{k_1=0}^{s-1} B_{3k_1+1}\right)+\sum_{l=0}^{s-1}\left(A_{3l+1}\prod_{k_2=l+1}^{s-1} B_{3k_2+1}\right)}\right] \\
&\quad \times \left[\frac{1}{r_2\left(\prod_{k_1=0}^{s-1} B_{3k_1+2}\right)+\sum_{l=0}^{s-1}\left(A_{3l+2}\prod_{k_2=l+1}^{s-1} B_{3k_2+2}\right)}\right] \\
&\quad \times \left[\frac{1}{r_0\left(\prod_{k_1=0}^{s} B_{3k_1}\right)+\sum_{l=0}^{s}\left(A_{3l}\prod_{k_2=l+1}^{s} B_{3k_2}\right)}\right].
\end{aligned}
$$

Substituting Eq.(6.8) in Eq.(6.12) gives

$$\begin{aligned}
u_{3n+2} &= u_2\prod_{s=0}^{n-1}\left(\frac{1}{r_{3s+2}}\right)\left(\frac{1}{r_{3(s+1)}}\right)\left(\frac{1}{r_{3(s+1)+1}}\right)\\
&= u_2\prod_{s=0}^{n-1}\left[\frac{1}{r_2\left(\prod_{k_1=0}^{s-1}B_{3k_1+2}\right)+\sum_{l=0}^{s-1}\left(A_{3l+2}\prod_{k_2=l+1}^{s-1}B_{3k_2+2}\right)}\right]\\
&\quad\times\left[\frac{1}{r_0\left(\prod_{k_1=0}^{s}B_{3k_1}\right)+\sum_{l=0}^{s}\left(A_{3l}\prod_{k_2=l+1}^{s}B_{3k_2}\right)}\right]\\
&\quad\times\left[\frac{1}{r_1\left(\prod_{k_1=0}^{s}B_{3k_1+1}\right)+\sum_{l=0}^{s}\left(A_{3l+1}\prod_{k_2=l+1}^{s}B_{3k_2+1}\right)}\right].
\end{aligned}$$

Note that $r_0 = \frac{u_0}{u_1}, r_1 = \frac{u_1}{u_2}, r_3 = \frac{u_2}{u_3}$.

For $A = B = 1$,

$$\begin{aligned}
u_{3n} &= u_0\prod_{s=0}^{n-1}\left(\frac{1}{r_0+s}\right)\left(\frac{1}{r_1+s}\right)\left(\frac{1}{r_2+s}\right)\\
&= u_0\prod_{s=0}^{n-1}\left(\frac{u_1}{u_0+su_1}\right)\left(\frac{u_2}{u_1+su_2}\right)\left(\frac{u_3}{u_2+su_3}\right)\\
&= u_1^n u_2^n u_3^n u_0\prod_{s=0}^{n-1}\frac{1}{[u_0+su_1]\,[u_1+su_2]\,[u_2+su_3]},
\end{aligned}$$

$$\begin{aligned}
u_{3n+1} &= u_1\prod_{s=0}^{n-1}\left(\frac{1}{r_1+s}\right)\left(\frac{1}{r_2+s}\right)\left(\frac{1}{r_0+s+1}\right)\\
&= u_1\prod_{s=0}^{n-1}\left(\frac{u_2}{u_1+su_2}\right)\left(\frac{u_3}{u_2+su_3}\right)\left(\frac{u_1}{u_0+(s+1)\,u_1}\right)\\
&= u_1^{n+1}u_2^n u_3^n\prod_{s=0}^{n-1}\frac{1}{[u_1+su_2]\,[u_2+su_3]\,[u_0+(s+1)\,u_1]},
\end{aligned}$$

$$\begin{aligned} u_{3n+2} &= u_2 \prod_{s=0}^{n-1} \left(\frac{1}{r_2 + s} \right) \left(\frac{1}{r_0 + s + 1} \right) \left(\frac{1}{r_1 + s + 1} \right) \\ &= u_2 \prod_{s=0}^{n-1} \left(\frac{u_3}{u_2 + s u_3} \right) \left(\frac{u_1}{u_0 + (s+1) u_1} \right) \left(\frac{u_2}{u_1 + (s+1) u_2} \right) \\ &= u_1^n u_2^{n+1} u_3^n \prod_{s=0}^{n-1} \frac{1}{[u_2 + s u_3] [u_0 + (s+1) u_1] [u_1 + (s+1) u_2]}. \end{aligned}$$

It follows that (after shifting back three times) the formulas for solutions of Eq.(6.1) are given by

$$x_{3n-2} = \frac{x_{-2}^{n+1} x_{-1}^n x_0^n}{\prod_{i=0}^{n-1} (x_{-3} + (i+1) x_{-2}) (x_{-2} + i x_{-1}) (x_{-1} + i x_0)},$$

$$x_{3n-1} = \frac{x_{-2}^n x_{-1}^{n+1} x_0^n}{\prod_{i=0}^{n-1} (x_{-3} + (i+1) x_{-2}) (x_{-2} + (i+1) x_{-1}) (x_{-1} + i x_0)},$$

$$x_{3n} = \frac{x_{-2}^n x_{-1}^n x_0^{n+1}}{\prod_{i=0}^{n-1} (x_{-3} + (i+1) x_{-2}) (x_{-2} + (i+1) x_{-1}) (x_{-1} + (i+1) x_0)}.$$

Finally, in this chapter, we studied fourth order recurrence equation and obtained their respective solutions of the difference equation and this was the extension to the work done by Elsayed E M and Khaliq A [55]. However, the method we used is completely different from [55] .

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