EXISTENTIALLY CLOSED GROUPS

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ABSTRACT

EXISTENTIALLY CLOSED GROUPS

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A group G is called an Existentially Closed Group (Algebraically Closed Group) if for every finite system of equations and inequations with coefficients from G which has a solution in an over group $H \geq G$, has a solution in G. Existentially closed groups were introduced by W. R. Scott in 1951. The notion of existentially closed groups is close to the notion of algebraically closed fields but there are substantial differences. Existentially closed groups were studied and advanced by B. H. Neumann. In this survey thesis we have studied the articles of B. H. Neumann and the paper of W. R. Scott.

Keywords: existentially closed groups, algebraically closed groups, infinite groups

ÖZ

EXISTENTIALLY KAPALI GRUPLAR

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Bir G grubuna eğer katsayıları G den olan eşitlik ve eşitsizliklerden oluşan her sonlu sistemin G'yi içeren bir $H \geq G$ grubunda çözümü olduğunda, G grubunda da varsa bir Existentially Kapalı Grup (Cebirsel Kapalı Grup) denir. Existentially kapalı gruplar 1951'de W.R. Scott tarafından tanıtıldı. Existentially kapalı gruplar kavramı ile cebirsel kapalı cisimler kavramı birbirine benzer olmasına rağmen önemli farklılıklar vardır. Existentially kapalı gruplar B. H. Neumann tarafından incelenmiş ve geliştirilmiştir. Bu derleme tezde, B. H. Neumann'ın makalelerine ve W. R. Scott'ın makalesine çalışılmıştır.

Anahtar Kelimeler: existentially kapalı gruplar, cebirsel kapalı gruplar, sonsuz gruplar

To my father, Ş. Birol GÜREL

Even though he could not see this thesis finished, he is in every page

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CHAPTER 1

INTRODUCTION

The notion of existentially closed (algebraically closed) groups first was introduced by W. R. Scott in 1951 [16]. The motivation might be the well known subject of field theory namely the algebraically closed fields. The study of algebraically closed fields developed in the 19th century. The main question one may ask is which properties of algebraically closed fields are satisfied by existentially closed groups.

A group G is called an **Existentially Closed Group** (Algebraically Closed Group) if for every finite system of equations and inequations with coefficients from G which has a solution in an over group $H \geq G$, has a solution in G. In the study of algebraically closed fields the equations are polynomial equations. Unlike the algebraically closed fields, the power of the variables in the equations of existentially closed groups can also be negative.

W.R. Scott published only one paper on existentially closed groups. He proved that every infinite group is contained in an existentially closed group where the existential closure and that infinite group has the same cardinality. This is exactly the same as in the case of algebraically closed fields. In his paper, W. R. Scott also introduced the nation of κ-existentially groups for any infinite cardinal κ. κ-existentially closed groups are the generalization of existentially closed groups allowing the infinite number of equations and inequations. Namely for an infinite cardinal κ, a group G of cardinality greater than or equal to κ is called a κ-existentially closed group if for every system of equations and inequations with coefficients from G, where the number of equations and inequations are fewer than κ which has a solution in an over group $H \geq G$, there is already a solution in G itself. W. R. Scott also introduced the notion weakly existentially closed groups by

allowing only the finite number of equations. Namely a group G is called a weakly existentially closed (algebraically closed) if for every finite system of equations which has a solution in an over group $H \geq G$, has a solution in G.

Existentially closed groups are studied and advanced by B. H. Neumann in the articles [13], [14]. He proved that a nontrivial weakly existentially closed group G is an existentially closed group. The proof is given in Section 3.2.

Chapter 2 contains preliminaries and background on various topics related to the articles of Scott and Neumann that will be discussed in the forthcoming chapters of the dissertation. These topics are mainly free product of groups, amalgamated free product of groups and the HNN extension of groups [4].

In Section 2.2, we discuss some embedding theorems for groups. Embedding of a group in another group with some prescribed properties is always an interesting study in group theory. In Theorem 2.2.4, it is proven that a countable group can be embedded in a 2-generator group with generators of infinite order. The well known theorem of Higman that states there are finitely generated infinite simple groups is studied in Theorem 2.3.2. This theorem is useful to prove that there are elements of infinite order in any existentially closed group. W. R. Scott has shown that every finite group can be embedded into an arbitrary existentially closed group. His motivation is that every finite group up to isomorphism uniquely determined by its multiplication table. Therefore every existentially closed group contains elements of order n for any positive integer n. i.e. Cyclic groups of order n for any positive integer n are contained in every existentially closed group. By using the above construction of Higman, B. H. Neumann proved that every existentially closed group contains an infinite cyclic subgroup. This is discussed in Section 3.3, Theorem 3.3.10 in a detailed way.

Theorem 2.4.34 is the result of B. H. Neumann which shows the cardinality of pairwise non-isomorphic 2-generator groups is $2^{\aleph_0}$-many. Using this result B. H. Neumann also proved that there are $2^{\aleph_0}$-many many pairwise non-isomorphic countable existentially closed groups. The proof of this which has a very interesting counting argument is in Section 3.3.

In Chapter 3 we have studied the well known paper of W. R. Scott and give the detailed proof of this in Section 3.2. Some of the basic properties of existentially closed groups are discussed in Section 3.3. More particularly the proof of B. H. Neumann that every existentially closed group is simple and existentially closed groups are not finitely generated are given.

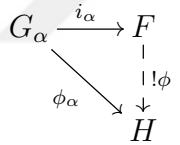
CHAPTER 2

PRELIMINARIES

2.1 Free Products

The results in this section can be found in Robinson's book. [15, Chapter 6]

Definition 2.1.1. *Let $\{G_\alpha \mid \alpha \in \Omega\}$ be arbitrary groups where Ω is any index set. We call a group F as the **free product** of G_α's with a collection of homomorphisms $\{i_\alpha : G_\alpha \to F \mid \alpha \in \Omega\}$ if for any group H with a collection of homomorphism $\{\phi_\alpha : G_\alpha \to H\}$, then there exists a unique homomorphism $\phi : F \to H$ such that $i_\alpha \circ \phi = \phi_\alpha$.*



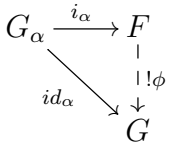
We may denote F as $\underset{\alpha\in\Omega}{Fr}\, G_\alpha$.

Theorem 2.1.2. [15, p. 170] *Let G be a group such that G is generated by its subgroups G_α where $\alpha \in \Omega$ and Ω is any index set. If every element of G has a unique expression of the form $g_1 g_2 \dots g_r$ where $r \geq 0$, $1 \neq g_i \in G_{\alpha_i}$ and $\alpha_i \neq \alpha_{i+1}$, then G is the free product of the G_α's.*

Proof. Let F be the free product of the G_α's with homomorphisms $i_\alpha : G_\alpha \to F$. Then, by definition of free product, for any group H with a collection of homomorphisms $\{\phi_\alpha : G_\alpha \to H \mid \alpha \in \Omega\}$, there exists a unique homomorphism $\phi : F \to H$ such that $i_\alpha \circ \phi = \phi_\alpha$.
Now, take $H = G$ and $\phi_\alpha = id_{G_\alpha}$. By Definition 2.1.1, there exists unique

$$\phi : F \to G$$

such that $i_\alpha \circ \phi = id_{G_\alpha}$.



Since every element of G can be written uniquely as a product of elements in G_α, we have ϕ is a monomorphism.

To show ϕ is onto, we take an arbitrary element of G,

$$g = g_{i_1} g_{i_2} \dots g_{i_k} \qquad g_{i_j} \in G_{\alpha_{i_j}}$$

Then,

$$\begin{aligned}(g^{i_\alpha})^\phi &= ((g_{i_1} g_{i_2} \dots g_{i_k})^{i_\alpha})^\phi \\ &= ((g_{i_1})^{i_{\alpha_{i_1}}})^\phi ((g_{i_2})^{i_{\alpha_{i_2}}})^\phi \ \dots ((g_{i_k})^{i_{\alpha_{i_k}}})^\phi \\ &= g_{i_1} g_{i_2} \dots g_{i_k}\end{aligned}$$

Thus, ϕ is an epimorphism and so $\phi : F \to G$ is an isomorphism. Hence,

$$\langle G_\alpha \mid \alpha \in \Omega \rangle = G \cong F = \underset{\alpha \in \Omega}{Fr}\, G_\alpha$$

□

Definition 2.1.3. *Let $\{G_\alpha \mid \alpha \in \Omega\}$ be arbitrary groups where Ω is any index set and let F be the free product of G_α's,*

$$F = \underset{\alpha \in \Omega}{Fr}\, G_\alpha$$

*If for a fixed group H, there exists a monomorphism $\phi_\alpha : H \to G_\alpha$ for all $\alpha \in \Omega$ then, G is called the **amalgamated free product** of G_λ's with amalgamated subgroup H if*

$$G = F/N$$

where

$$N = \langle (h^{\phi_\alpha})^{-1} h^{\phi_\beta} \mid h \in H \ \alpha, \beta \in \Omega \rangle^F$$

Remark 2.1.4. *If we take the group H as the identity group in Definition 2.1.3, then G is the free product of G_α's in Definition 2.1.1. Thus, we can say that amalgamated free products are the generalized form of the free products.*

2.2 Some Embedding Theorems for Groups

G. Higman, B. H. Neumann and H. Neumann introduced an embedding of a group which is called the HNN extension in [4] as in the following theorem:

Theorem 2.2.1. *Let G be a group with subgroups H, K and let θ be an isomorphism from H to K. Then, there is an embedding $G \hookrightarrow \widehat{G}$ such that θ is induced by an element of $\widehat{G}$*

$$\widehat{G} = \langle G, t \mid t^{-1}ht = h^\theta,\ h \in H,\ h^\theta \in K \rangle$$

and $\widehat{G}$ is called the HNN extension of G.

Proof. Let $\langle a \rangle$ and $\langle b \rangle$ be infinite cyclic groups. Define X, Y as the free products of G and $\langle a \rangle, \langle b \rangle$ respectively.

$$X = G \star \langle a \rangle$$
$$Y = G \star \langle b \rangle$$

Let L be a subgroup of X such that $L = \langle G, H^a \rangle$ and K be a subgroup of Y such that $M = \langle G, K^b \rangle$. Then, by Theorem 2.1.2 we observe,

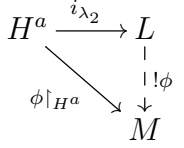
$$L = G \star H^a$$
$$M = G \star K^b$$

Claim. *L and M are isomorphic.*

First, define a homomorphism $\phi : L \to M$ such that $\phi(h^a) = (h^\theta)^b$ for all $h^a \in H^a$ and $\phi_{\restriction G} = id_G$. Then, since L is the free product of G and H^a, by definition of the free product we have

$$\begin{array}{ccc} G & \xrightarrow{i_{\lambda_1}} & L \\ & \searrow^{id_G} & \downarrow !\phi \\ & & M \end{array}$$

and



Observe that, there can be no non-trivial relation of the form $g_1h_1^ag_1h_1^ag_2h_2^a\dots g_nh_n^a = 1$ in the free product and so the kernel of the homomorphism ϕ is trivial which implies ϕ is a monomorphism. It is clear to see ϕ is onto. So, ϕ is an isomorphism. Thus,

$$L \cong M$$

Since, the free product of X and Y, is

$$X \star Y = G \star \langle a \rangle \star G \star \langle b \rangle$$

$L,\ M$ are subgroups of $X \star Y$ with an isomorphism $\phi : L \to M$. Therefore, we can consider the amalgamated free product

$$\widehat{G} = X \star Y/N$$

where

$$N = \langle (g \star h^a)^{-1}(g \star (h^\theta)^b) \mid g \in G, h^a \in H^a,\ (h^\theta)^b \in K^b \rangle^{X \star Y}$$

In $\widehat{G}$,

$$\begin{aligned}(h^a)^{-1}(h^\theta)^b N &= N \\ (h^\theta)^b N &= h^a N \\ h^\theta N &= h^{ab^{-1}} N\end{aligned}$$

So,

$$h^{ab^{-1}} \equiv h^\theta (mod N)$$

where $ab^{-1} \in \widehat{G}$. Notice that, ab^{-1} is an element of infinite order in the free product $X \star Y$ since a and b belong different components. Similarly, in the amalgamated free product $\widehat{G}$, ab^{-1} has infinite order.

Finally, write t as $t = ab^{-1}$ in $\widehat{G}$ that induces θ then, the group $\widehat{G} = \langle G, t \rangle$ is called HNN extension of G. $\square$

The first application of Theorem. 2.2.1. is constructing infinite groups in which all non-trivial elements are conjugate.

Theorem 2.2.2. [4] *Any torsion free group G can be embedded into a group K in which all non-trivial elements are conjugate. Moreover, K is a simple torsion free group.*

Proof. We first embed G to $G^\star$ in which any two non-trivial elements in G are conjugate in $G^\star$ and then construct K using the method of transfinite induction. Since every set is well-ordered, the elements of $G - \{1\}$ can be ordered

$$\{g_\alpha \mid 0 \leq \alpha < \gamma\}$$

for some ordinal γ.
Let $\{G_\alpha \mid \alpha < \gamma\}$ be a collection of groups such that for any $g_\alpha \in G_\alpha$ and $g_\beta \in G$ where $\beta < \alpha$ are conjugate in G_α.

- If α is a limit ordinal, then define G_α as follows,

$$G_\alpha = \bigcup_{\beta<\alpha} G_\beta$$

  Take any two nontrivial element of G_α, say

$$g_n, g_m$$

  Then, g_n and g_m are contained in G_λ for some $\lambda < \alpha$. Thus, by assumption g_n and g_m are conjugate in G_α where $n, m \leq \lambda < \alpha$.

- If α is a successor ordinal, then observe that

$$g_\alpha \in G_\alpha \leq G_{\alpha+1}$$

  By assumption, all g_β are conjugate in G_α for any $\beta < \alpha$.

Since G is a torsion free group,

$$\langle g_\beta \rangle \cong \langle g_\alpha \rangle \cong \mathbb{Z}$$

where $\beta < \alpha$. Then, using HNN extension of G_α we get,

$$G^\star = \langle G_\alpha, t_i \mid g_\beta^{t_i} = g_\alpha,\ \beta < \alpha,\ i \in I \rangle$$

and call this group as $G_{\alpha+1}$. Since every non-trivial element of G_α are conjugate in $G_{\alpha+1}$, we obtain a group tower,

$$G = G_0 \leq G^\star = G_1 \leq G_1^\star = G_2 \leq \cdots$$

So, if we define K as,

$$K = \bigcup_{i=1}^{\infty} G_i$$

then we obtain all nontrivial elements of K are conjugate and torsion free which implies K is a torsion free group. So, every element of K generates an infinite cyclic group. Moreover, normal closure of any nontrivial element is equal to K as any two non-trivial elements in K are conjugate.

$$\langle g^K \mid g \in K \rangle = K$$

Therefore, K is a simple torsion-free group. □

Corollary 2.2.3. [4, Corollary 3] *K is a countable group if G is countable.*

At this point, we know that every countably infinite torsion free group G can be embedded in a torsion free group K such that any two element in K are conjugate and the cardinality of K is countably infinite.

The second application of Theorem 2.2.1 is the following.

Theorem 2.2.4. [4, p. 251] *Any countable group can be embedded into a 2-generator group such that the order of generators are infinite.*

Proof. Let G be a countable group. Then, we may order the elements of G in the following way,

$$G = \{g_{0=1}, g_1, g_2, g_3, \dots\}$$

Let $F = \langle a, b \rangle$ be a group generated by two elements, and $H = G \star F$ the free product of G and F. H has subgroups A and B in the following form

$$A = \langle a, a^b, a^{b^2}, \dots, a^{b^i}, \dots \rangle < F$$
$$B = \langle bg_0, b^a g_1, \dots, b^{a^i} g_i, \dots \rangle$$

One can easily see that there is no relation between the elements of A. So, A is a free group with countably infinite basis $\{a, a^b, a^{b^2}, \ldots, a^{b^i}, \ldots\}$ and similarly B is a free group with countably infinite rank. Thus

$$A \cong B$$

and there exists an isomorphism

$$\begin{aligned} \theta : A &\to B \\ a^{b^i} &\to b^{a^i} g_i \end{aligned}$$

Therefore, we can form HNN extension of H as

$$K = \langle H, t \mid t^{-1} a^{b^i} t = \theta(a^{b^i}) = b^{a^i} g_i,\ a \in A,\ b \in B, \rangle$$

where $i = 0, 1, 2, \ldots$. Now, consider the group generated by a and t, namely $\langle a, t \rangle$.

Claim. $\langle a, t \rangle = K$

Proof of claim. It is obvious to see that the group generated by a and t is contained in K.

Note that,

$$\begin{aligned} a^t = t^{-1} a t &= t^{-1} a^{b^0} t \\ &= \theta(a^{b^0}) \\ &= b^{a^0} g_0 = b g_0 \\ &= b \end{aligned}$$

Thus, $\langle a, b \rangle$ is contained in $\langle a, t \rangle$ and so $F \leq \langle a, t \rangle$. On the other hand, $\langle A, t \rangle$ is a subgroup of $\langle a, b \rangle$ since A is generated by $a, a^b, a^{b^2}, \ldots, a^{b^i}, \ldots$. Observe that

$$t^{-1} a^{b^i} t \in \langle a, t \rangle$$

Since $t^{-1} a^{b^i} t = \theta(a^{b^i}) = b^{a^i} g_i$, we have

$$b^{a^i} g_i \in \langle a, t \rangle$$

Therefore, A, B and G are contained in $\langle a, t\rangle$ which implies

$$K = \langle H, t \mid t^{-1}a^{b^i}t = b^{a^i}g_i,\ a \in A,\ b \in B,\ i = 0, 1, 2, \dots\rangle \leq \langle a, t\rangle$$

Hence, we have the claim $\langle a, t\rangle = K$.

$$G \hookrightarrow H = G \star F$$
$$G \star F \hookrightarrow \underbrace{\langle H, t \mid t^{-1}a^{b^i}t = b^{a^i}g_i,\ a \in A,\ b \in B,\ i = 0, 1, 2, \dots\rangle}_{K=\langle a,t\rangle}$$

So, G can be embedded into two generated group $\langle a, t\rangle$ by HNN extension where t has an infinite order. Moreover, the order of a is infinite as any non-trivial element of a free group has infinite order. □

Corollary 2.2.5. *The group $K = \langle a, t\rangle$ has element of finite order if and only if G has elements of finite order.*

We have proved HNN extension by assuming that we have a group G and two subgroups H and K such that $H \cong K$. So, one can generalize this to the case with a finite or infinite number of isomorphisms.

Theorem 2.2.6. *Let K_i, L_i are isomorphic subgroups of a group G_i for each $i \in I$ and θ_i be an isomorphism between K_i and L_i. Then there exists a group $\hat{G}$ such that*

$$\hat{G} = \langle G, t \mid k_i^t = k_i^{\theta_i} = l_i, \quad k_i \in K_i,\ l_i \in L_i,\ i \in I\rangle$$

By using this generalization, we can prove that there is an embedding in such a way that all elements of the same order in any countable group become conjugate.

Theorem 2.2.7. [4, Corollary 1] *Any group G can be embedded in a countable group L in which all elements of the same order are conjugate.*

Proof. Let G be a countable group. First, we embed G into a group $G^\star$ such that any two elements in G of the same order are conjugate in $G^\star$.

To construct $G^\star$, we consider the set of all ordered pairs (x_i, y_i) for all $i \in I$ such that x_i and y_i have the same order. Observe that, we have countably many pairs of

elements of the same order since G is a countable group. Moreover, $\langle x_i \rangle \cong \langle y_i \rangle$ for each $i \in I$. So, we can define HNN extension of G with isomorphisms $\varphi_i : \langle x_i \rangle \to \langle y_i \rangle$ and obtain

$$G^\star = \langle G, t_i \mid t_i^{-1} x_i t_i = x_i^{\varphi_i} = y_i \quad i \in I \rangle$$

Note that the elements in G of the same order are conjugate in $G^\star$

Now, we obtain that

$$G_0 = G \hookrightarrow G_1 = G^\star \hookrightarrow G_2 = G_1^\star \hookrightarrow \cdots \hookrightarrow G_i = G_{i-1}^\star \hookrightarrow \cdots$$

and define

$$L = \bigcup_{i=1}^{\infty} G_i$$

Finally, take two elements of the same order from L. Then, they are contained in one of the G_i's and so are conjugate in $G_{i+1} = G_i^\star$. Thus, they are conjugate in L.

Since L is countable union of countable groups, it is countable. □

Definition 2.2.8. *A group G is called **divisible** if for any $g \in G$ and $n \in \mathbb{N}$, there exists an element $h \in G$ such that $h^n = g$.*

Theorem 2.2.9. *Let G be a countable group. Then G can be embedded in a countable simple divisible group.*

Proof. Given a countable group G, we may find an embedding from G to a group K which contains elements of any order. K is constructed as

$$K = G \times \underset{n=1}{\overset{\infty}{Dr}} C_n \times \mathbb{Z}$$

where C_n is the cyclic group of order n. Observe that the order of K is countably infinite. Next, we consider the free product of K and an infinite cyclic group, say $K_1 = K \star \langle x \rangle$. This product K_1 has countably infinite order. By Theorem 2.2.4 we can embed K_1 into a 2-generator group K_2 in which generators have infinite order and K_2 is also countable. Using Theorem 2.2.7, K_2 can be embedded into a group L

such that any two elements of the same order are conjugate.

$$G \hookrightarrow K = G \times \underset{n=1}{\overset{\infty}{Dr}} C_n \times \mathbb{Z}$$
$$K \hookrightarrow K_1 = K \times \langle x \rangle \hookrightarrow K_2$$
$$K_2 \hookrightarrow L$$

- L is countable

  The proof comes naturally by construction of L.

- L is simple

  Let N be a nontrivial normal subgroup of L and g be a nontrivial element of N of order n (n may be infinite). Then, g is an element of L. Since the subgroup K of L contains elements of any order, there exists $k \in K$ such that k and g has the same order. By the fact that, any two element of the same order are conjugate in L, we can find $t \in L$ such that $g^t = k$. So, $k \in N$

  First, consider the element

  $$x^{-1}kxk^{-1} \in K \star \langle x \rangle < K_2 < L$$

  where x is the special element of $K \star \langle x \rangle$. On the otherhand,

  $$\underbrace{x^{-1}kx}_{\in N}\underbrace{k^{-1}}_{\in N} \in N$$

  The order of $x^{-1}kxk^{-1}$ is infinite in $K \star \langle x \rangle$ and so in K_2. The group K_2 is generated by two elements of infinite order, say a and b. Remember, any two elements of the same order are conjugate in L and so there exists s_1 and s_2 in L such that

  $$(x^{-1}kxk^{-1})^{s_1} = a$$
  $$(x^{-1}kxk^{-1})^{s_2} = b$$

  Since $x^{-1}kxk^{-1} \in N$ and N is normal in L, we obtain a and b are contained in N. This means that

  $$N \geq K_2 \geq K_1 \geq K$$

  i.e. N contains elements of every order.

Let g_1, g_2 be elements of order n in L for $n \in \mathbb{N}$. Then there exists $t \in L$ such that $g_1^t = g_2$. By above observation, g_1 is contained N and also normality of N implies g_2 is contained in N. Thus, every element of L is contained in N. Hence L is a simple group.

- L is divisible

  Let $g \in L$ such that the order of g is m (m may be infinite) and let $n \in \mathbb{N}$. We need to find an element $h \in L$ such that $h^n = g$. Observe that there is an element of order mn in $K \leq L$, say

  $$t^{mn} = 1$$

  Then, t^n and g have the same order in L and so are conjugate in L. In other words, there exists an element $z \in L$ such that

  $$(t^n)^z = z^{-1}t^n z = (z^{-1}tz)^n = g$$

  Thus, L is a divisible group.

□

2.3 Some Remarks on Finitely Generated Groups

We start with a very famous result from G. Higman's paper, namely, the existence of a finitely generated infinite simple group [3]. To construct such a group, we need the following lemma,

Lemma 2.3.1. [3] *If n is an integer greater than 1, then the least prime factor of n is less than the least prime factor of $2^n - 1$.*

Proof. Let p be a prime factor of $2^n - 1$ and s be the least positive integer such that $p \mid 2^s - 1$. Then, $2^s \equiv 1(mod\ p)$, $s \neq 1$ and $s \mid n$. Let q be a prime factor of s. We need to show that $q < p$. Since $gcd(2, p) = 1$, by Fermat's little theorem

$$2^{p-1} \equiv 1(mod\ p)$$

which implies $s \mid p-1$ since s is the least positive integer such that $2^s \equiv 1(mod\ p)$ by assumption.
Moreover, $q \mid s$ and $s \mid p-1$ implies $q \mid n$ and $q \mid p-1$. Thus,

$$q < p$$

Thus, the least prime divisors of n is less than the least prime divisor of $2^n - 1$. This completes the proof of lemma. □

Theorem 2.3.2. [3] *There exists an infinite simple group G which is finitely generated.*

Proof. Our construction of G starts with taking groups G_i generated by x_i and y_i such that $x_i^{-1} y_i x_i = y_i^2$ is satisfied for $i = 1, 2, 3, 4$. Observe that the subgroups of G_i generated by x_i 's and y_i's are freely generated infinite cyclic groups. So, they are isomorphic as infinite cyclic groups and we can consider the free product M_1 of $G_1 = \langle x_1, y_1 \rangle$ and $G_2 = \langle x_2, y_2 \rangle$ with amalgamated subgroup $\langle x_2 \rangle \cong \langle y_1 \rangle$. Notice that, the intersection of the subgroups $\langle x_1 \rangle$ of G_1 and $\langle y_2 \rangle$ of G_2 is equal to identity. So, the subgroup K_1 of M_1 generated by x_1 and y_2 is actually the free product of $\langle x_1 \rangle$ and $\langle y_2 \rangle$ by Theorem 2.1.2.

Similarly, we consider the free product M_2 of G_3 and G_4 with amalgamated subgroups $\langle y_3 \rangle \cong \langle x_4 \rangle$. Then, the subgroup K_2 of M_2 generated by x_3 and y_4 is the free product of $\langle x_3 \rangle$ and $\langle y_4 \rangle$.

Next, we form the free product of M_1 and M_2 with amalgamating $K_1 \cong K_2$, call this product as G. So simply to say, in the group G, we identify $x_1 = y_4$ and $y_2 = x_3$. The identifications we made are $x_1 = y_4$, $x_2 = y_1$, $x_3 = y_2$, $x_4 = y_3$. Thus, we may say that G is generated by

$$x_1, x_2, x_3, x_4$$

with the defining relations

$$\begin{aligned} x_1^{-1} x_2 x_1 &= x_2^2 \\ x_2^{-1} x_3 x_2 &= x_3^2 \\ x_3^{-1} x_4 x_3 &= x_4^2 \\ x_4^{-1} x_1 x_4 &= x_1^2 \end{aligned} \tag{2.1}$$

G is finitely generated and infinite since it is a free product of infinite groups. So, the only thing we need to prove is that G has no proper normal subgroup of finite index. This is equivalent to say the above relations cannot satisfied in a finite group unless $x_i = 1$ for all $i = 1, 2, 3, 4$. So, suppose that the relations (1.1) are satisfied in a finite group generated by x_i such that x_i's are not all equal to 1 and the order of x_i is equal to n_i where $i = 1, 2, 3, 4$. Then, using the relations (1.1) for any m we have;

$$\begin{aligned}
x_1^{-m} x_2 x_1^m &= x_1^{-m+1} \underbrace{x_1^{-1} x_2 x_1}_{x_2^2} x_1^{m-1} \\
&= x_1^{-m+2} \underbrace{x_1^{-1} x_2^2 x_1}_{x_2^4} x_1^{m-2} = x_1^{-m+2} x_2^{2^2} x_1^{m-2} \\
&= x_1^{-m+3} \underbrace{x_1^{-1} x_2^4 x_1}_{x_2^8} x_1^{m-3} = x_1^{-m+3} x_2^{2^3} x_1^{m-3} \\
&\vdots \\
&= x_1^{-m+m} x_2^{2^m} x_1^{m-m} \\
&= x_2^{2^m}
\end{aligned}$$

Next, choose $m = n_1$ as $\mid x_1 \mid = n_1$, then

$$\begin{aligned}
x_1^{-n_1} x_2 x_1^{n_1} &= x_2^{2^{n_1}} \\
x_2 &= x_2^{2^{n_1}} \\
1 &= x_2^{2^{n_1}-1}
\end{aligned}$$

The order of x_2 is n_2 by assumption. So,

$$n_2 \mid 2^{n_1} - 1$$

Similiarly, we get

$$\begin{aligned}
n_3 &\mid 2^{n_2} - 1 \\
n_4 &\mid 2^{n_3} - 1 \\
n_1 &\mid 2^{n_4} - 1
\end{aligned}$$

Thus, if one of n_i equals to 1, say $n_1 = 1$, then $n_2 = n_3 = n_4 = 1$. This leads us a contradiction. Therefore, we may suppose that x_i's are different from identity and $n_i > 1$ for all $i = 1, 2, 3, 4$.

Furthermore, let p' be the least prime factor of $n_1n_2n_3n_4$, then without loss of generality, assume that $p' \mid n_2$. Above, we have shown that $n_2 \mid 2^{n_1} - 1$. Then by Lemma 2.3.1, the least prime factor of n_1, say q', is less than p'. i.e. $q' < p'$. Then, $q' \mid n_1n_2n_3n_4$. This contradicts the assumption of p' being the least prime factor dividing $n_1n_2n_3n_4$. Therefore in any finite group generated by x_i for all $i = 1, 2, 3, 4$ the relations (1.1) cannot be satisfied unless the group is trivial. Hence, G has no normal subgroup of finite index.

In 1950, G. Higman proved that the above argument is true if G has more than four generators however, if G has

- two generators, say, x_1, x_2 then we have

$$x_1^{-1}x_2x_1 = x_2^2 \quad \Longrightarrow \quad x_2^{-1}x_1^{-1}x_2x_1 = x_2$$
$$x_2^{-1}x_1x_2 = x_1^2 \quad \Longrightarrow \quad x_1^{-1}x_2^{-1}x_1x_2 = x_1$$

So, if we multiply them together

$$x_1x_2 = x_1^{-1}\, x_2^{-1}\, x_1 \underbrace{x_2x_2^{-1}}_{1}\, x_1^{-1}\, x_2\, x_1$$
$$= 1$$

Then, we get

$$x_1^{-1}x_2^{-1}\underbrace{x_1x_2}_{1} = x_1$$
$$x_1^{-1} = x_1x_2$$
$$= 1$$

So,

$$x_1 = x_2 = 1$$

Hence, the group

$$G = \langle x_1, x_2 \mid x_1^{-1}x_2x_1 = x_2^2,\ x_2^{-1}x_1x_2 = x_1^2 \rangle$$

is trivial.

Now, we show that the group generated by x_1, x_2, x_3 with the given relations is trivial.

- three generators, x_1, x_2, x_3, then we have

$$x_1^{-1}x_2x_1 = x_2^2 \implies x_2x_1 = x_1x_2^2 \implies \underbrace{x_2x_1x_2^{-1} = x_1x_2}_{(i)}$$

$$x_2^{-1}x_3x_2 = x_3^2 \implies x_3x_2 = x_2x_3^2 \implies x_3x_2x_3^{-1} = x_2x_3 \implies \underbrace{x_2x_3^{-1} = x_3^{-1}x_2x_3}_{(ii)}$$

$$\underbrace{x_3^{-1}x_1x_3 = x_1^2}_{(iii)}$$

Now, we take the conjugate of (i) by x_3 and using the relations (ii) and (iii), we get;

$$x_3^{-1}\underbrace{x_2x_1x_2^{-1}}_{x_1x_2}x_3 = x_3^{-1}x_1x_2x_3$$

$$= \underbrace{x_3^{-1}x_1x_3}_{x_1^2}\underbrace{x_3^{-1}x_2x_3}_{x_2x_3^{-1}}$$

$$\underbrace{x_3^{-1}x_2x_3}_{x_2x_3^{-1}}\underbrace{x_3^{-1}x_1x_3}_{x_1^2}\underbrace{x_3^{-1}x_2^{-1}x_3}_{(x_2x_3^{-1})^{-1}} = x_1^2x_2x_3^{-1}$$

$$x_2\underbrace{x_3^{-1}x_1^2x_3}_{x_1^4}x_2^{-1} = x_1^2x_2x_3^{-1}$$

$$x_2x_1^4x_2^{-1} = x_1^2x_2x_3^{-1}$$

Then,

$$x_2x_1^4x_2^{-1}x_3 = x_1^2x_2$$

Hence,

$$x_3 = x_2x_1^{-4}x_2^{-1}x_1^2x_2$$

Thus,

$$\langle x_1, x_2, x_3\rangle = \langle x_1, x_2\rangle$$

Moreover, we have

$$x_1^{-1}x_2x_1 = x_2^2 \implies x_2^{-1}x_1^{-1}x_2x_1 = x_2$$

$$x_2^{-1}x_3x_2 = x_3^2 \implies x_3^{-1}x_2^{-1}x_3x_2 = x_3$$

$$x_3^{-1}x_1x_3 = x_1^2 \implies x_1^{-1}x_3^{-1}x_1x_3 = x_1$$

which tells us that G is a perfect group. i.e. $G = G' = G''$ but $\langle x_1, x_2 \rangle$ has a derived subgroup generated by x_2 which is cyclic group and has second derived group which equals to identity. Therefore $G \cong \langle x_1, x_2 \rangle$ is solvable of derived length 2 this leads us a contradiction since G is perfect.

Hence, the group G is trivial.

Here, we would like to point out that there is another way which includes some basic group theoretical axioms to prove

$$G = \langle x_1, x_2, x_3 \mid R_1, R_2, R_3 \rangle$$

where

$$\begin{aligned} R_1 &:= x_1^{-1} x_2 x_1 = x_2^2 \\ R_2 &:= x_2^{-1} x_3 x_2 = x_3^2 \\ R_3 &:= x_3^{-1} x_1 x_3 = x_1^2 \end{aligned}$$

is the identity group.

We start with taking the inverse of R_1 and get

$$x_1^{-1} x_2^{-1} x_1 = x_2^{-2}$$

Then

$$\begin{aligned} x_1^{-1} x_2^{-1} x_1 = x_2^{-2} \quad &\Longrightarrow \quad x_1^{-1} x_2^{-1} x_1 x_2 = x_2^{-1} \\ &\Longrightarrow \quad x_2^{-1} x_1 x_2 = x_1 x_2^{-1} \qquad (R_4) \end{aligned}$$

Claim. *For any $i \in \mathbb{N}$, relations $x_2^{-i} x_1 x_2^i = x_1 x_2^{-i}$ holds.*

Proof of claim. We use induction on i.

- If $i = 1$, then the equation becomes R_4.
- Assume that the equation $x_2^{-(i-1)} x_1 x_2^{i-1} = x_1 x_2^{-(i-1)}$ holds.

Then,

$$\begin{aligned} x_2^{-i}x_1x_2^i &= x_2^{-1}\underbrace{x_2^{-(i-1)}x_1x_2^{i-1}}_{x_1x_2^{-(i-1)}}x_2 \\ &= x_2^{-1}x_1x_2^{-(i-1)}x_2 \\ &= \underbrace{x_2^{-1}x_1x_2}_{x_1x_2^{-1}}x_2^{-(i-1)} \\ &= x_1x_2^{-1}x_2^{-(i-1)} \\ &= x_1x_2^{-i} \end{aligned}$$

Thus, the proof of the claim is done. So, for each i we have

$$x_2^{-i}x_1x_2^{-i} = x_1x_2^{-i} \qquad (R5)$$

Next, we take the inverse of R_2 and obtain,

$$x_2^{-1}x_3^{-1}x_2 = x_3^{-2}$$

Then,

$$\begin{aligned} x_2^{-1}x_3^{-1}x_2 = x_3^{-2} \quad &\implies \quad x_2^{-1}x_3^{-1}x_2x_3 = x_3^{-1} \\ &\implies \quad x_3^{-1}x_2x_3 = x_2x_3^{-1} \qquad (R_6) \end{aligned}$$

Similarly, using induction on i, one can show that for each $i \in \mathbb{N}$ the following relation holds.

$$x_3^{-i}x_2x_3^i = x_2x_3^{-i} \qquad (R_7)$$

Now, take the conjugate of (R_3) by x_2

$$
\begin{aligned}
x_2^{-1}x_3^{-1}x_1x_3x_2 &= x_3^{-2}\underbrace{x_2^{-1}x_1x_2}_{x_1x_2^{-1}}x_3^2 \quad by\ (R_7)\\
&= x_3^{-2}x_1x_2^{-1}x_3^2\\
&= x_3^{-2}x_1x_3^2\underbrace{x_3^{-2}x_2^{-1}x_3^2}_{x_3^2x_2^{-1}} \quad by\ (R_7)\\
&= x_3^{-2}x_1x_3^2x_3^2x_2^{-1} = x_3^{-2}x_1x_3^4x_2^{-1}\\
&= x_3^{-1}\underbrace{x_3^{-1}x_1x_3^1}_{x_1^2}x_3^3x_2^{-1} = x_3^{-1}x_1^2x_3^3x_2^{-1} \quad by\ (R_3)\\
&= x_3^{-1}x_1^2x_3^3x_2^{-1}\\
&= \underbrace{x_3^{-1}x_1^2x_3}_{(x_3^{-1}x_1x_3)^2}x_3^2x_2^{-1} = \underbrace{(x_3^{-1}x_1x_3)}_{x_1^2}{}^2x_3^2x_2^{-1} \quad by\ (R_3)\\
&= x_1^4x_3^2x_2^{-1}
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
x_2^{-1}\underbrace{x_3^{-1}x_1x_3}_{x_1^2}x_2 &= x_1^4x_3^2x_2^{-1}\\
x_2^{-1}x_1^2x_2 &= x_1^4x_3^2x_2^{-1}\\
\underbrace{(x_2^{-1}x_1x_2)}_{x_1x_2^{-1}}{}^2 &= x_1^4x_3^2x_2^{-1} \quad by\ (R_4)\\
(x_1x_2^{-1})^2 &= x_1^4x_3^2x_2^{-1}\\
x_1x_2^{-1}x_1x_2^{-1} &= x_1^4x_3^2x_2^{-1}\\
x_1x_2^{-1}x_1 &= x_1^4x_3^2\\
x_1^{-3}x_2^{-1}x_1 &= x_3^2\\
x_1^{-2}\underbrace{x_1^{-1}x_2^{-1}x_1}_{(x_1^{-1}x_2x_1)^{-1}} &= x_3^2\\
x_1^{-2}x_2^{-2} &= x_3^2 \quad by\ (R_1)
\end{aligned}
$$

So, we have a new relation

$$x_1^{-2}x_2^{-2} = x_3^2 \qquad (R_8)$$

Then, we consider

$$\begin{aligned} x_3^2 x_2 x_3^{-2} &= x_1^{-2} x_2^{-2} x_2 x_2^2 x_1^2 = x_1^{-2} x_2 x_1^2 \quad by\ (R_8) \\ &= x_1^{-1} \underbrace{x_1^{-1} x_2 x_1}_{x_2^2} x_1 \quad by\ (R_1) \\ &= x_1^{-1} x_2^2 x_1 = (x_1^{-1} x_2 x_1)^2 \\ &= (x_2^2)^2 = x_2^4 \end{aligned}$$

Using the relation (R_7) when $i = 2$ and above calculation, we obtain

$$\begin{aligned} x_3^{-2} x_2 x_3^2 &= x_2 x_3^{-2} \\ \underbrace{x_3^2 x_3^{-2}}_{1} x_2 x_3^2 &= \underbrace{x_3^2 x_2 x_3^{-2}}_{x_2^4} \\ x_2 x_3^2 &= x_2^4 \\ x_3^2 &= x_2^3 \qquad (R_9) \end{aligned}$$

Clearly, the relation (R_9) implies that x_3^2 commutes with x_2. By the relation (R_2),

$$\begin{aligned} x_2^{-1} x_3 x_2 &= x_3^2 \\ x_3 &= x_2^{-1} x_3^2 x_2 \\ &= x_3^2 \end{aligned}$$

implies $x_3 = 1$. Then, put $x_3 = 1$ in the relation (R_3) and observe

$$\begin{aligned} x_3 x_1 x_3^{-1} &= x_1^2 \\ x_1 &= x_1^2 \\ x_1 &= 1 \end{aligned}$$

Similarly using (R_1), we get $x_2 = 1$. Hence, $G = \langle x_1, x_2, x_3 \mid R_1, R_2, R_3 \rangle$ is the trivial group. □

The following theorem is one of the important results of Marshall Hall, on [2].

Theorem 2.3.3. *Let G be a finitely generated group. Then, the number of subgroups of index n in G is finite. If H is a subgroup of finite index n, then there exists a characteristic subgroup K in H such that $[G : K]$ is finite.*

Proof. Let n be a positive integer. For each subgroup H of G with finite index n, we form the set of right cosets of H in G, say

$$\Omega = \{Hc_1, Hc_2, \ldots, Hc_n\}$$

where $c_1 = 1$ and $c_i \in G$. Then, G acts from right cosets of H in G and there exists a homomorphism

$$\theta_H : G \hookrightarrow S_n \cong Sym(\Omega)$$
$$g \to \theta_H(g)$$

where

$$\theta_H(g) : \Omega \to \Omega$$
$$Hc_i \to Hc_ig$$

It is clear that, if $h \in H$ then $Hc_1h = Hh = H$. i.e. the group H stabilizes $H \in \Omega$. Moreover if $x \in G$ stabilizes $H \in \Omega$, then $Hx = H$ and this happens if and only if $x \in H$.

Now, consider another subgroup of G of finite index n, say T. So, we have another homomorphism, say

$$\theta_T : G \hookrightarrow S_n \cong Sym(\Omega)$$
$$g \to \theta_T(g)$$

If $H \neq T$, then there exists $t \in T$ such that $t \notin H$. Consider

$$\theta_T(t) = Hc_1t = Ht \neq H$$

Thus, $\theta_H \neq \theta_T$. So, θ_H is a homomorphism from finitely generated group G which is uniquely determined by the generators of G. So, we may define at most $\mid S_n \mid^m$ homomorphisms from G into S_n. This means that we may have at most finitely many subgroups of finite index n.

Now, let $H_1, H_2, \ldots H_s$ be the set of all subgroups of index n in G. i.e.

$$[G : H_i] = n \qquad \forall i \in \{1, 2, \ldots, s\}$$

and consider the subgroup K of G defined as

$$K = \bigcap_{i=1}^{s} H_i$$

Claim. *K is a characteristic subgroup of finite index in G.*

Proof of the claim. We have $[G : K]$ is finite since intersection of finitely many subgroups of finite index in G is of finite index in G. Moreover, for each i, K is a subgroup of H_i. So, all we have to prove is that $K = charG$. i.e. $\alpha(K) = K$ for all $\alpha \in Aut(G)$.

But, this follows from the fact that if H is a subgroup of finite index in G, then

$$\{Hc_1 = H,\ Hc_2,\ Hc_3,\ \ldots,\ Hc_n\}$$

is the set of right cosets of H in G.

Then, take $\alpha \in Aut(G)$ and consider $\alpha(H)$. It is a subgroup of G of index n because

$$G = \bigcup_{i=1}^{n} Hc_i$$
$$\alpha(G) = \bigcup_{i=1}^{n} \alpha(H)\alpha(c_i)$$

So, $\alpha(H)$ has index n in G and if we consider the subgroup K under α, then

$$\alpha(K) = \bigcap_{i=1}^{s} \alpha(H_i) = K$$

since α permutes the subgroups of index n.

Thus, K is characteristic subgroup of G of finite index. □

Before we go further, let's recall some basic definitions and observations about residually finite groups.

Definition 2.3.4. *A group G is called **residually finite** if for any non-trivial $g \in G$, there exists a normal subgroup N_g of G of finite index such that $g \notin N_g$. Equivalently one can tell, if $\Sigma = \{H \leq G : [G : H] < \infty\}$ and $\bigcap_{H \in \Sigma} H = 1$, then G is residually finite group.*

Example 2.3.5. *Some examples of residually finite groups are stated.*

(i) $\mathbb{Z}$ *is residually finite.*

For any $n \in \mathbb{Z}$*, consider the subgroup* $\langle p\mathbb{Z}\rangle$ *satisfying* $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p$ *where* p *is any prime number is finite and* $n \notin \langle p\mathbb{Z}\rangle$*. Moreover,* $\bigcap_{p\in\mathcal{P}} \mathbb{Z} = \{0\}$ *where* $\mathcal{P}$ *is the set of all primes.*

(ii) Every finite group is residually finite.

Let G be a finite group and $H = \{1\}$*. Then for any* $g \in G$ *such that* $g \neq 1$*, we have* $H \trianglelefteq G$ *such that* $g \notin H$ *and* $[G : H] =\mid G \mid < \infty$*.*

(iii) Let $G = \underset{i\in I}{Dr}\, G_i$ *such that* $\{G_i \mid i \in I\}$ *be a set of finite groups and* I *is any index. Then G is a residually finite group.*

Take a non-trivial element g *from* G*. Then there exists* k *such that* $g = g_1 g_2 \dots g_k$ *and* $g_{k+j} = 1$ *for all* $j \geq 1$*. Now take* $H = \underset{j=1}{\overset{\infty}{Dr}}\, G_{k+j}$*. Observe that* $g \notin H$*. But,* G/H *is finite since each* $G_1, G_2 \dots G_k$ *is finite and so* $\underset{i=1}{\overset{k}{Dr}}\, G_i$ *is finite.*

(iv) Free group of finite rank is residually finite.

Remark 2.3.6. *Subgroup of a residually finite group is residually finite.*

Proof. Let G be a residually finite group and H be a subgroup of G. If we take a nontrivial element from H, say $h \neq 1$, then there exists a normal subgroup N_h of G such that $[G : N_h] < \infty$ and $h \notin N_h$ as $H \leq G$ and G is residually finite.

Now, $H \cap N_h \triangleleft H$. Since $h \notin N_h$, we have $h \notin H \cap N_h$. Moreover,

$$H \cong HN_h/N_h \cong H/(H \cap N_h) \leq G/N_h$$

Since $\mid G/N_h \mid$ is finite, we have $\mid H/(H \cap N_h) \mid$ is finite. Thus, the required normal subgroup in H is $H \cap N_h$. Hence H is residually finite. □

Remark 2.3.7. *Homomorphic image of a residually finite group may not be residually finite.*

Proof. This follows from the fact that, every group is a homomorphic image of a free group. So, every infinite simple group is a homomorphic image of a free group. Free groups of finite rank are residually finite by (iv) in Example 2.3.5 but there are finitely generated infinite simple groups. So, they cannot be residually finite. See, the examples of G. Higman (1973) and R. Thompson (1969)
[10, p. 191]. □

We are ready to prove the following theorem of G. Baumslag [1].

Theorem 2.3.8. *Let G be a finitely generated residually finite group. Then, the automorphism group $Aut(G)$ of G is also residually finite.*

Proof. We want to show that for any non-trivial automorphism α of $Aut(G)$, there exists a normal subgroup N of $Aut(G)$ of finite index in $Aut(G)$ and $\alpha \notin N$. So, let $\alpha \in Aut(G)$ then there exists $g \in G$ such $g \neq 1$ such that $\alpha(g)g^{-1} \neq 1$. Since G is finitely generated and residually finite, there exists a characteristic subgroup K of G such that G/K is finite by Theorem 2.3.3. As K is a characteristic subgroup of G, we have $\alpha(K) = K$ for any $\alpha \in Aut(G)$. Then, every automorphism of G induces an automorphism of G/K, namely

$$\begin{aligned} \theta : Aut(G) &\to Aut(G/K) \\ \beta &\to \theta(\beta) = \theta_\beta \end{aligned}$$

where $(gk)\theta_\beta = \beta(g)K$. Notice that β is an automorphism of G,

$$\begin{aligned} (g_1g_2)K\theta_\beta &= \beta(g_1g_2)K \\ &= \beta(g_1)\beta(g_2)K \\ &= \beta(g_1)K\beta(g_2)K \\ &= (g_1K)\theta_\beta(g_2K)\theta_\beta(g_2K) \end{aligned}$$

and so,

$$\theta_\beta \in Aut(G/K)$$

Moreover, $\theta : Aut(G) \to Aut(G/K)$ is a homomorphism because if $\beta, \gamma \in Aut(G)$, then $\theta_{\beta\gamma} = \theta_\beta\theta_\gamma$. Therefore we can define the quotient group $Aut(G)/Ker\theta$ which is isomorphic to a subgroup of $Aut(G/K)$ as $Ker\theta = K$. Since the index of K in G

is finite, we get $\mid Aut(G/K) \mid$ is finite. So automorphism group of G has a subgroup, $Ker\theta$ of finite index. On the other hand, by assumption we know that $\alpha(g)g^{-1} \neq 1$ and so the following is true

$$(gK)\theta_\alpha = \alpha(g)K \neq gK$$

if and only if $\alpha(g)g^{-1} \notin K = Ker\theta$.

Hence, θ_α is a non-trivial automorphism of G/K. □

Remark 2.3.9. *If G is not finitely generated, then Theorem 2.3.8 may not be true.*

Proof. Consider the free group $F = \langle x_i \mid i \in \mathbb{N} \rangle$ with countably infinite basis. Observe that $Aut(F)$ contains $Sym(\mathbb{N})$ as permutations of the basis elements gives automorphisms of F. We know that $Alt(\mathbb{N})$ is an infinite symmetric subgroup of $Sym(\mathbb{N})$ and by Remark 2.3.6 subgroups of a residually finite group is residually finite. But, infinite simple group cannot have a non-trivial subgroup of finite index. If $[G : H] = n < \infty$, then consider the following

$$\theta_H : G \hookrightarrow Sym(n)$$

The quotient group $G/Ker\theta_H$ is isomorphic to a subgroup of the symmetric group of n letters. If G is simple, $Ker(\theta_H) = \{1\}$ and this yields us to a contradiction. □

2.4 Some Remarks on Infinite Groups

In this section, we will investigate some famous results from *Some Remarks on Infinite Groups* that is written by B. H. Neumann in 1937 [12]. First, our aim is to prove that every finitely generated group has a maximal subgroup. Then we will prove that there exists uncountable many non-isomorphic pairwise 2-generator group. Before moving on, some basic group theoretical information will be given.

Recall that the subgroup M of a group G is a maximal subgroup if $M \neq G$ and there exists no subgroup K of G such that $M \underset{\neq}{<} K \underset{\neq}{<} G$. If G is a finite group, then every proper subgroup of G is contained in a maximal subgroup. On the otherhand, there are infinite groups which has no maximal subgroup. For example, $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ or more generally any divisible group abelian group does not have a maximal group.

Definition 2.4.1. *An element $t \in G$ is called a non-generator for G if for any subset X of G $\langle t, X\rangle = G$ implies $X = G$.*

Note that the subgroup generated by all non-generators is a subgroup of G and one can easily prove the followings

- The product of non-generators is a non-generator
- Identity is a non-generator.
- Inverse element of a non-generator is also a non-generator.

Theorem 2.4.2. *(Frattini) Let G be a finite group, then the group of non-generators is equal to the intersection of maximal subgroups of G.*

Proof. Assume that t is a non-generator of G. Since G is a finite group, we know that G has maximal subgroups. Let M be maximal subgroup which does not contain t. But, t is a non-generator and by definition of a non-generator we have if $\langle t, M\rangle = G$, then $M = G$ which is impossible. So, t must be in M. Similarly, one can show that this is true for any maximal subgroup of G. Therefore, every maximal subgroup of G contains t. i.e. $t \in \bigcap_{M\, max\, in\, G} M$.

Conversly, take an element y from the intersection of maximal subgroups of G and show that y is a non-generator of G. Suppose the contrary, i.e. there exists a set X such that $\langle y, X\rangle = G$ but $\langle X\rangle \neq G$. Since G is finite, there is a maximal subgroup M of G containing $\langle X\rangle$. Then, $\langle y, M\rangle = G$ but this is a contradiction as $y \in M$. □

The above proof can be modified to prove the following theorem.

Theorem 2.4.3. *If every proper subgroup of a group G is contained in a maximal subgroup, then the subgroup consisting of non-generators is equal to the intersection of maximal subgroups.*

Definition 2.4.4. *The subgroup consisting of non-generators is called Frattini subgroup of G and denoted by*

$$\Phi(G) = \bigcap_{M\, max\, in\, G} M$$

Moreover, if G has no maximal subgroup, we write $\Phi(G) = G$.

Example 2.4.5. *$\mathbb{Z}$ has maximal subgroups of the form $p\mathbb{Z}$ where p is a prime number. Then,*

$$\Phi(\mathbb{Z}) = \bigcap_{p \in \mathcal{P}} p\mathbb{Z} = 0$$

where $\mathcal{P}$ is the set of prime numbers.

Remark 2.4.6. *Frattini subgroup of G is a characteristic subgroup of G.*

Proof. If $\alpha \in Aut(G)$ and M is a maximal subgroup of G, then $\alpha(M)$ is again a maximal subgroup of G. So,

$$\Phi(G) = \bigcap_{M\, max\, in\, G} M$$

$$\alpha(\Phi(G)) = \bigcap_{M\, max\, in\, G} \alpha(M) = \bigcap_{M\, max\, in\, G} M = \Phi(G)$$

Thus, Frattini subgroup of G is a characteristic subgroup of G. In particular

$$\Phi(G) \trianglelefteq G$$

□

We are ready to look into some results from [12],

Lemma 2.4.7. [12, p. 121] *Let g be an element of a group G and let H be a subgroup of G not containing g. Then, there exists a subgroup K of G containing H but not g and K is called relatively maximal subgroup of G.*

Proof. We will prove this by transfinite induction. Since every set can be well -ordered, we may order the elements of G in the following way;

$$\{g_\alpha \mid \alpha < \gamma\} \quad \text{where } \gamma \text{ is the order of } G$$

We start with fixing $H_0 = H$ and $g \notin H$. Then, assume that we have constructed H_i such that $H < H_i$ and $g \notin H_i$. Now, consider H_{i+1}.

- If $g \in \langle g_i, H_i \rangle$, then $H_{i+1} = H_i$
- If $g \notin \langle g_i, H_i \rangle$, then $H_{i+1} = \langle g_i, H_i \rangle$

So, we have a chain of subgroups. If β is a limit ordinal, then one can define,

$$H_\beta = \bigcup_{i<\beta} H_i$$

Observe that, H_β does not contain the element g since none of the H_i contains g. Hence, H_β is a subgroup of G which does not contain the given element g. Then we define a group K in the following way,

$$K = \bigcup_{j<\gamma} H_j$$

Note that K is a subgroup of G and K does not contain g. Moreover, for any element g' in G, $\langle K, g' \rangle$ contains the element g.

Thus, K is relatively maximal subgroup of G. □

Remark 2.4.8. *If K is a relatively maximal subgroup of a group G not containing the element $g \in G$, then any group containing K contains the element g.*

Lemma 2.4.9. [12, p. 121] *A relatively maximal subgroup K of G not containing the element g is a maximal subgroup of G if and only if $\langle K, g \rangle = G$.*

Proof. Let K be a relatively maximal subgroup of G not containing the element $g \in G$. First, assume that K is a maximal subgroup of G. Then $\langle K, g \rangle = G$ as K is maximal and $g \notin K$. Conversely, suppose that $\langle K, g \rangle = G$ for the relatively maximal subgroup K not containing g. Then, any group containing K must contain the element g as K is relatively maximal subgroup. Thus, K is a maximal subgroup of G. □

Theorem 2.4.10. *The intersection of maximal subgroups of a group G coincide with the subgroup C of G which is generated by non-generators of G.*
i.e.

$$C = \bigcap_{M\, max\, in\, G} M \quad \textit{where C is the subgroup of non-generators of G}$$

Proof. First, we will show that $\bigcap_{M\, max\, in\, G} M$ contains C. Assume that t is an element of C but $t \notin \bigcap_{M\, max\, in\, G} M$. Then, there exists a maximal subgroup M' of G such that $t \notin M'$. By definition of maximal subgroup, we have $\langle M', t \rangle = G$ but this is

impossible since t is a non-generator. So, we have a contradiction and so t is contained in $\bigcap_{M\, max\, in\, G} M$. i.e. $C \subseteq \bigcap_{M\, max\, in\, G} M$

For the other direction, we assume that x is an element of $\bigcap_{M\, max\, in\, G} M$ but $x \notin C$. Then by Lemma 2.4.7, there exists a relatively maximal subgroup K of G such that $C \subseteq K$ and $x \notin K$. So, every subgroup of G containing K properly contains the element x. i.e. K is a maximal subgroup and $\langle K, x\rangle = G$. Since $x \in \bigcap_{M\, max\, in\, G} M$ and K is maximal, we have $\langle K, x\rangle = K$ which implies that x is contained in C. Therefore, $\bigcap_{M\, max\, in\, G} M \subseteq C$.

□

Theorem 2.4.11. [12, p. 122] *Let G be a finitely generated group. Then, every proper subgroup of G is contained in a maximal subgroup.*

Proof. Let $\{g_1, g_2, \ldots g_n\}$ be the set of generators of G and let H be a proper subgroup of G. Then, H cannot contain all the generators of G. Say $g_1 \notin H$. By Lemma 2.4.7, there exists a relatively maximal subgroup K of G such that $g_1 \notin K$ but any group containing K contains g_1.

- If $\langle K, g_1 \rangle = G$, then K is maximal subgroup of G by Lemma 2.4.9.
- If $\langle K, g_1 \rangle = K_1 \neq G$, then K_1 is the proper subgroup of G and so K_1 cannot contain one of the generators of G, say $g_2 \notin K_1$. Similarly, there exists a relatively maximal subgroup K_2 such that $g_2 \notin K_2$ but any group containing K_2 contains g_2.
  - If $\langle K_2, g_2 \rangle = G$, then K_2 is maximal subgroup of G containing g_1, g_2.
  - If not, we continue as before. i.e. Construct a relatively maximal subgroup K_3 and so on.

Note that, we can continue at most finitely many step since G is finitely generated. Thus in the last step, say $g_n \notin K_n$ and $\langle K_n, g_n\rangle = G$ implying K_n is a maximal subgroup of G.

In particular, every finitely generated group has maximal subgroups. □

Remark 2.4.12. *Converse of the above theorem is not true in general. As an example, one can give infinite dimensional vector spaces over $\mathbb{Z}_p$ whihc is a finite field with p elements where p is a prime number. If we remove one vector from the basis it generates a subspace of codimension one. So, this subspace is maximal subgroup of the vector space. But, the vector space is infinitely generated.*

Lemma 2.4.13. *Let G be a group and L be a normal subgroup of G not containing the element $g \in G$. Then, L is a relatively maximal normal subgroup of G and any normal subgroup containing L contains the element g.*

Proof. Let N be a proper normal subgroup of a group G. The technique of this proof is similar to the proof of Lemma 2.4.7. So, we start with ordering the elements of G since every set can be well-ordered.

$$\{g_\alpha \mid \alpha < \gamma\} \quad \text{where } \gamma \text{ is an ordinal and } \mid G \mid = \gamma$$

Then, fix $N_0 = N$ and $g \notin N$. Assume that we have constructed N_i such that $N < N_i$ and $g \notin N_i$. Next is to consider N_{i+1},

- If $g \in \langle N_i, g_{i+1} \rangle^G$, then $N_{i+1} = N_i$.
- If $g \notin \langle N_i, g_{i+1} \rangle^G$, then $N_{i+1} = \langle N_i, g_{i+1} \rangle^G$.

So, we have a chain of normal subgroups. If β is a limit ordinal, then we define

$$N_\beta = \bigcup_{i<\beta} N_i$$

Since each N_i is a normal subgroup of G and none of the N_i's contain g, the union N_β is normal in G and does not contain g. Now, we define

$$L = \bigcup_{j<\gamma} N_j$$

Then, L is a normal subgroup of G and $g \notin L$. So, L is a relatively maximal normal subgroup of G and any normal subgroup M of G containing L contains g. □

Lemma 2.4.14. *A relatively maximal normal subgroup L of a group G not containing the element $g \in G$ is a maximal normal subgroup of G if and only if $\langle L, g \rangle^G = G$.*

Proof. Let L be a relatively maximal normal subgroup of G not containing the element $g \in G$. Suppose that L is a maximal normal subgroup of G. Then, $\langle L, g\rangle^G = G$ as L is maximal normal and $g \notin L$. Conversely, assume that $\langle L, g\rangle^G = G$ for the relatively maximal normal subgroup L not containing g. Then, any normal subgroup containing L must contain the element g since L is relatively maximal normal. Therefore, L is a maximal normal subgroup of G. □

Theorem 2.4.15. *Let G be a finitely generated group. If N is a proper normal subgroup of G, then N is contained in a maximal normal subgroup of G.*

Proof. Let $g_1, g_2 \dots g_n$ be the generators of G and let N be a proper normal subgroup of G. So, all the generators of G cannot be contained in N, say $g_1 \notin N$. Then by Lemma 2.4.13, there exist a relative maximal normal subgroup L such that $g_1 \notin L$.

- If $\langle L, g_1\rangle^G = G$, then L is a maximal normal subgroup containing N by Lemma 2.4.14.
- If $\langle L, g_1\rangle^G \underset{\neq}{\lhd} G$, then there exists another generatorog G, say g_2 such that $g_2 \notin \langle L, g_1\rangle^G$. So, there is a relatively maximal normal subgroup L_1 such that $\langle L, g_1\rangle^G \trianglelefteq L_1$ and any normal subgroup of G containing L_1 contains the element g_2.

We continue like this at most n steps since G is finitely generated groups. The last step we reach a relatively maximal normal subgroup L_k such that $\langle g_1, g_2, \dots, g_{n-1}\rangle \leq L_k$ and any normal subgroup of G containing L_k contains the element g_n.

Therefore, L_k is a maximal normal subgroup of G containing N. In other words, every proper normal subgroup of a finitely generated group G is contained in a maximal normal subgroup of G. □

Remark 2.4.16. *Let $H_1 \leq H_2 \leq \dots$ be a sequence of subgroups of a group G. We say that $H_1 \leq H_2 \leq \dots$ is a proper ascending chain of subgroups if for all $i \in I$, $H_i \underset{\neq}{\leq} H_{i+1}$ where I is any index set.*

Lemma 2.4.17. [12, p. 123] *If G is a finitely generated group, then G cannot be written as a union of proper ascending sequence of subgroups.*

Proof. Let G be a group generated by $g_1, g_2, \ldots, g_n$. If $G = \bigcup_{i \in I} H_i$ where $H_1 \leq H_2 \leq \ldots$ is a proper ascending chain of subgroups of G, then there exists $i_1, i_2, \ldots, i_n$ such that $g_j \in H_{i_j}$ and $j \in \{1, 2, \ldots, n\}$.
Observe that the chain is strictly ascending, all g_k's where $k \in \{1, 2, \ldots, n\}$ are contained in H_{i_m} for some m. Thus,

$$G = \langle g_1, g_2, \ldots, g_n \rangle \leq H_{i_m}$$

So, after H_{i_m} we cannot have strict inequality. Hence, the result follows. □

Now, we will discuss generators of a group and corresponding defining relations. The following theorem shows that if G is a finitely generated group, then for any finite generating set for G, there exists finitely many relations such that G is again a finitely presented group but the number of generators and the number of relations could be different.

Theorem 2.4.18. [12, p. 124] *Let G be a finitely presented group with m generators and s relations. If there is another set of generators for G of cardinality n, then G can be defined by at most $r \leq n + s$ relations.*

Proof. Let $g_1, g_2, \ldots, g_m$ be the generators and $r_1, r_2, \ldots, r_s$ be the relations of G. i.e $G = \langle g_1, g_2, \ldots, g_m \mid r_i(g_1, g_2, \ldots, g_m) = 1 \rangle$ where $i \in \{1, 2, \ldots, s\}$. Now, assume that $\{h_1, h_2, \ldots, h_n\}$ is another set of generators of G. So, we may write h_j's in terms of g_k's and g_k's in terms of h_j's where $j \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, m\}$.

$$\begin{aligned} h_j &= H_j(g) = H_j(g_1, g_2, \ldots, g_m) \\ g_k &= G_k(h) = G_k(h_1, h_2, \ldots, h_n) \end{aligned}$$

Infact, G can be generated by the g_k and h_j with relations

$$\begin{aligned} r_i(g_1, g_2, \ldots, g_m) &= 1, \quad i \in \{1, 2, \ldots s\} \\ h_j = H_j(g_1, g_2, \ldots, g_m), &\quad j \in \{1, 2, \ldots n\} \end{aligned} \tag{2.2}$$

and if we add the following relations to (2.2)

$$g_k = G_k(h_1, h_2, \ldots, h_n), \quad k \in \{1, 2, \ldots m\} \tag{2.3}$$

then we can express (2.2) in terms of h_j's in the following way,

$$\begin{aligned}
&r_1(G_1(h_1,h_2,\ldots,h_n),G_2(h_1,h_2,\ldots,h_n),\ldots,G_m(h_1,h_2,\ldots,h_n))=1\\
&r_2(G_1(h_1,h_2,\ldots,h_n),G_2(h_1,h_2,\ldots,h_n),\ldots,G_m(h_1,h_2,\ldots,h_n))=1\\
&\quad\vdots\\
&r_s(G_1(h_1,h_2,\ldots,h_n),G_2(h_1,h_2,\ldots,h_n),\ldots,G_m(h_1,h_2,\ldots,h_n))=1\\
&h_1=H_1(G_1(h_1,h_2,\ldots,h_n),G_2(h_1,h_2,\ldots,h_n),\ldots,G_m(h_1,h_2,\ldots,h_n))\\
&h_2=H_1(G_1(h_1,h_2,\ldots,h_n),G_2(h_1,h_2,\ldots,h_n),\ldots,G_m(h_1,h_2,\ldots,h_n))\\
&\quad\vdots\\
&h_l=H_1(G_1(h_1,h_2,\ldots,h_n),G_2(h_1,h_2,\ldots,h_n),\ldots,G_m(h_1,h_2,\ldots,h_n))
\end{aligned}\tag{2.4}$$

In particular, by using (2.3) and (2.4) we obtain,

$$G=\langle g_k,h_j \mid r_i(G_k(h))=1,\ h_j=H_j(G_k(h)),\ g_k=G_k(h_j)\rangle \tag{2.5}$$

where $i\in\{1,2,\ldots,s\}$, $j\in\{1,2,\ldots,n\}$ and $k\in\{1,2,\ldots,m\}$.

Note that, in (2.5) we may delete the generators g_k's with the relations (2.2) because in (2.2) g_k's are defined in terms of the h_j's by the relations (2.3). So, in (2.5) g_k's are not used. Therefore,

$$G=\langle h_j \mid r_i(G_k(h))=1,\ h_j=H_j(G_k(h))\rangle$$

with $i\in\{1,2,\ldots,s\}$, $j\in\{1,2,\ldots,n\}$ and $k\in\{1,2,\ldots,m\}$.

Hence, we have at most $s+n$ relations with respect to the generators $h_1,h_2,\ldots h_n$.

□

Remark 2.4.19. *The numbers n,m,r and s need not to be finite.*

The above proof shows that the numbers n,m,r and s finite. In fact we have the following corollary

Corollary 2.4.20. *Let G be a finitely generated group. If G cannot be defined in finitely many relations in these generators, then it cannot be defined in finitely many relations in any set of generators of G.*

Theorem 2.4.21. [12, p. 126] *Let G be a group generated by two elements a_1 and a_2 with the defining relations*

$$r_n = [a_1^{-(u_n-2)} a_2 a_1^{u_n-2}, a_2] = 1$$

where $u_n = 2^{2^n} + 1$ and $n = 1, 2, \ldots$. Then G cannot be defined by finitely many relations.

To prove this theorem, we need to recall some important definitions and theorems of alternating groups.

Lemma 2.4.22. *Alternating group of finite order $n \geq 3$, A_n is generated by cycles of length 3.*

Proof. Note that

- if $n = 1$ and $n = 2$, then A_n is the identity group.
- if $n = 3$, then $A_n = \{(1), (123), (132)\} = \langle(123)\rangle$

So, we will consider A_n for $n \geq 4$. By definition, every element of an alternating group is a product of even number of 2-cycles. Moreover, (ab)(cd) can have at most one element in the intersection of $\{a, b\}$ and $\{c, d\}$ otherwise the product will be identity. Infact,

- if $\{a, b\} \cap \{c, d\} \neq \emptyset$, say $a = d$, then (ab)(ac)=(abc). So, (ab)(ac) is contained in the group generated by 3-cycles.
- if $\{a, b\} \cap \{c, d\} = \emptyset$, then (ab)(cd)=(abc)(cad). Thus, (ab)(cd) is a product of two cycles of length 3 and so is contained in the group generated by 3-cycles.

Therefore, every element of A_n is a product of 3-cycles where $n \geq 2$. Hence, A_n is generated by 3-cycles. □

Theorem 2.4.23. [6, Theorem 12.1.1] *A_n is simple group for $n \neq 4$.*

Proof. We know that $A_1 = A_2 = 1$ and $A_3 = \langle(123)\rangle$ which is isomorphic to a simple group $\mathbb{Z}_3$. For $n = 4$, A_4 has a normal subgroup, namely Klein 4-group, $V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$ which implies that it is not a simple group. So, we may take $n \geq 5$

1. Take $n = 5$ and consider the alternating group on $\{a, b, c, d, e\}$.

   Assume that N is a normal subgroup of A_5 and let α be a nontrivial element of N. Clearly, α is one of the following cycle form;

   (a) $\alpha = (abc) \in N$

   Suppose that α' is arbitrary 3-cycle in A_5. Then α and α' must have at least one number in common. So without loss of generality, we may take $\alpha' = (ade)$. Moreover we know that the element $\gamma = (bd)(ce)$ is in A_5 and if we take the conjugate of α by γ, we get

   $$\alpha^\gamma = (abc)^{(bd)(ce)} = \underbrace{(ade)}_{\alpha'}$$

   By the normality of N, we see that α' is contained in N. Similarly, if the cycles α and α' has more than one number in common, one can find the same results. Thus, we can obtain all 3-cycles in A_5 are contained in N. By Lemma 2.4.22, we know that A_5 is generated by cycles of length 3. Thus $N = A_5$.

   (b) $\alpha = (abcde) \in N$

   The element $\sigma = (ab)(cd)$ is contained in A_5. Consider the conjugate of α by σ and call this product as π.

   $$\pi = \alpha^\sigma = (abcde)^{(ab)(cd)} = (adceb)$$

   Since N is normal, we have π is in N. Then the product,

   $$\pi\alpha = (adceb)(abcde) = (aec)$$

   must be in N which means N contains a 3-cycle. Using (a), one can obtain $N = A_5$.

(c) $\alpha = (ab)(cd) \in N$

The element $\sigma = (abe)$ is in A_5. Similiar to above case, we take the conjugate of α by σ and call this elemet as π

$$\pi = \alpha^{\sigma} = (ab)(cd)^{(abe)} = (be)(cd)$$

Since N is normal, π is in N. Next is to consider the product of π and α,

$$\pi\,\alpha = \underbrace{(be)(cd)}_{\in N}\,\underbrace{(ab)(cd)}_{\in N} = (abe) \in N$$

So, N contains a 3-cycle and by (a) we have $N = A_5$

Hence A_5 is simple.

2. Take $n = 6$ and consider the alternating group on $\{a, b, c, d, e, f\}$.

Assume that N is a normal subgroup of A_6. We will consider two cases,

(a) Suppose that the stabilizer of each symbol in N is trivial.

Let α be a nontrivial element of N. Then the cycle structure of α is $(ab)(cdef)$ or $(abc)(def)$. If $\alpha = (ab)(cdef)$, then

$$\alpha^2 = (ab)(cdef)(ab)(cdef) = (a)(b)(ce)(df)$$

but by assumption it cannot be an element of N. So, $\alpha = (abc)(def)$. Since N is a normal subgroup of A_6, if we take $(cde) \in A_6$, then the conjugate of α by (cde) must be contained in N.

$$\alpha^{(cde)} = (abc)(def)^{(cde)} = (abd)(cfe) \in N$$

Then call this element as $\pi = \alpha^{(cde)}$ and consider the product

$$\begin{aligned}\alpha\pi &= (abc)(def)(abd)(cfe)\\ &= (adcbf)(e)\end{aligned}$$

It must be contained in N but this contradicts to our assumption. Thus, this case is impossible.

(b) Suppose that the stabilizer of some symbol in N is nontrivial.

Let x be an element of N fixes f and H be the stabilizer of f in A_6. It is obvious that H is isomorphic to A_5 which is simple by (1). So H is

simple. Consider the intersection $H \cap N$, it is not trivial since $x \in H \cap N$ and normal in H. So, one can observe that $H \cap N = H$. Hence $H \cong A_5$ is a subgroup of N. By Lemma 2.4.22, N contains all 3-cycles in A_5. Take one of 3-cycles, say $\alpha = (abc) \in N$. Our aim is to show $N = A_6$. So, we take an arbitrary 3-cycle α' in A_6. If α and α' have at least one element in common, then using similar calculations we did in the case (1a), we can find an element γ of A_6 such that $\alpha^\gamma = \alpha' \in N$ and thus $N = A_6$. If α and α' are disjoint 3-cycles, then there is a permutation $\beta \in S_6$ such that β sends $a \to d$, $b \to e$ and $c \to f$. i.e.

$$\begin{aligned} \alpha^\beta &= (abc)^\beta \\ &= (abc)^{(ad)(be)(cf)} = (def) \end{aligned}$$

Note that if β is not even permutation, then we may multiply β with a cycle (ij) where i and j is not in $\{a, b, c\}$ so that it does not effect conjugacy relation. So, (def) is in N. Therefore $N = A_6$.

Hence, A_6 is simple

3. If $n \geq 7$ and N is a normal subgroup of A_n, then

   (a) If N contains a 3-cycle, then one can easily prove that $N = A_n$ by using the same calculations we did in (2a) and (2b).

   (b) If N contains a cycle of length greater or equal to 4, say $\alpha = (abcd\dots) \in N$ then consider α^σ where $\sigma = (abc) \in A_n$. Since N is normal, $\alpha^\sigma = (abcd\dots)^{(abc)} = (bcad\dots)$ must be in N. Then,

$$\begin{aligned} \alpha\alpha^\sigma &= \underbrace{(abcd\dots)}_{\in N}\underbrace{(bcad\dots)}_{\in N} \\ &= (abd) \in N \end{aligned}$$

   So, a 3-cycle is contained in N and by (3a) we obtain $N = A_n$

   Thus, we may assume that all cycles contained in N must be a product of even number of disjoint 2-cycles or 3-cycles.

   (c) N contains a product of two 3-cycles.

Let $\alpha = (abc)(def) \in N$ where (abc) and (def) are disjoint cycles. Similiarly to (2a), take the conjugate of (cde) by α and multiply from left by α,

$$\alpha\alpha^{(cde)} = (abc)(def)(abd)(cfe) = (adcbf)(e)$$

But in (3b) we proved that if N contains a cycle of lenght greater or equal to 4, then $N = A_n$

Hence, each element of N is a product of even number of disjoint 2-cycles.

(d) N contains a product of even number of disjoint 2-cycles.

Let $\alpha = (ab)(cd) \in N$ where (ab) and (cd) are disjoint cycles in S_n. If we take a 3-cycle $(aef) \in A_n$, then $\alpha^{(aef)}$ is in N since N is normal.

$$\begin{aligned}\sigma = \alpha^{(aef)} &= (ab)(cd)^{(aef)} \\ &= (be)(cd)\end{aligned}$$

Now, consider the product of α and σ in the folllowing way,

$$\begin{aligned}\alpha\sigma &= (ab)(cd)(be)(cd) \\ &= (aeb)(c)(d)\end{aligned}$$

So, $\alpha\sigma$ is a 3-cycle and so by (3a) we get $N = A_n$. It follows that if $\alpha = (ab)(cd)(ef)(gh)$ is an element of N, then consider the conjugate of α by the element $(bc)(de)$ of A_n

$$\begin{aligned}\alpha^{(bc)(de)} &= (ab)(cd)(ef)(gh)^{(bc)(de)} \\ &= (ac)(be)(df)(gh)\end{aligned}$$

must be contained in N. Call this element $\sigma = \alpha^{(bc)(de)}$ and multiply from left by α

$$\begin{aligned}\underbrace{\alpha}_{\in N}\underbrace{\sigma}_{\in N} &= (ab)(cd)(ef)(gh)(ac)(be)(df)(gh) \\ &= (aed)(bcf)(g)(h) \in N\end{aligned}$$

By (3c), if N contains a product of two 3-cycles, $N = A_n$

Therefore, for $n \neq 4$, the alternating group A_n does not have any non-trivial normal subgroup. i.e. A_n is simple. □

Theorem 2.4.24. *A_n is generated by 3-cycles of the form $(1ij)$ where $n \geq 3$.*

Proof. Since we are interested in 3-cycles of the form (abc) where none of a, b and c is equal to 1. By using Lemma 2.4.22 , it is enough to write (abc) as a product of 3-cycles of the form $(1ij)$.

$$\begin{aligned}(abc) &= (1bc)(1ab) \\ &= (1)(bca) = (abc)\end{aligned}$$

Thus, 3-cycles of type $(1ij)$ generates A_n for $n \geq 3$. □

Remark 2.4.25. *There are $(n-1)(n-2)$ many 3-cycles of type $(1ij)$ in A_n.*

Theorem 2.4.26. *3-cycles of type $(12j)$ generate A_n for $n \geq 3$.*

Proof. By the previous Theorem 2.4.24, we know that for $n \geq 4$, A_n is generated by 3-cycles of the form $(1ij)$. So, it is enough to show that any 3-cycle of the form $(1ij)$ can be written as a product of 3-cycles of the form $(12j)$.

$$\begin{aligned}(1ij) &= (12j)(12i)\underbrace{(12j)(12j)}_{(1j2)} \\ &= (12j)(12i)(1j2) \\ &= (1ij)\end{aligned}$$

So, 3-cycles of type $(12j)$ generates A_n where $n \geq 3$.

□

Remark 2.4.27. *A_n contains $(n-2)$ many 3-cycles of type $(12j)$ in A_n*

Theorem 2.4.28. *For $n \geq 3$, the consecutive 3-cycles of the form $(i\, i+1\, i+2)$ generate A_n.*

Proof.

- If $n = 3$, then $A_3 = \langle (123) \rangle$

- If $n = 4$, then $A_4 = \langle (123), (124) \rangle$ by Theorem 2.4.26. On the other hand, we may write (124) as a product of consecutive 3-cycles.

$$\begin{aligned}(124) &= (123)(234)(123)^2 \\ &= (123)(234)(132)\end{aligned}$$

So, we can write A_4 as a product of consecutive 3-cycles. i.e. $A_4 = \langle (123), (234) \rangle$

- If $n \geq 5$, we can use induction on n. Assume that all 3-cycles of the form $(12\,j)$ can be written as a product of consecutive 3-cycles where $3 \leq j \leq n-1$. Then consider

$$(12\,n) = (12\,n-1)(12\,n-2)(n-2\,n-1\,n)(12\,n-1)(12\,n-2)$$

Therefore, 3-cycles of consecutive form generate A_n for all $n \geq 3$.

□

Theorem 2.4.29. *For $n \geq 4$, A_n is generated by*

(i) (123) *and* $(12\ldots n)$ *if n is odd*

(ii) (123) *and* $(23\ldots n)$ *if n is even*

Proof. First we prove the case (i). Let n be an odd number. Then, $(12\ldots n)$ is contained in A_n. If we show that all consecutive 3-cycles is contained in the $\langle (123), (12\ldots n) \rangle$, then by Theorem 2.4.28 one can prove this theorem.

If we take $\sigma = (12\ldots n) \in A_n$

$$\begin{aligned}(123)^{\sigma} &= (234) \\ (123)^{\sigma^2} &= (345) \\ (123)^{\sigma^3} &= (456) \\ &\vdots \\ (123)^{\sigma^i} &= (i+1\,i+2\,i+3)\end{aligned}$$

Thus, $A_n = \langle (123), (12\ldots n) \rangle$.

For the second part of the theorem, we let n be an even number. Choose $\sigma = (23\dots n) \in A_n$,

$$\begin{aligned}(123)^{\sigma} &= (134)\\ (123)^{\sigma^2} &= (145)\\ &\vdots\\ (123)^{\sigma^i} &= (1\,i+2\,i+3)\end{aligned}$$

Observe that

$$\begin{aligned}(123)^{\sigma^{k-1}}(123)^{\sigma^{k-2}} &= (1\,k+1\,k+2)(1\,k\,k+1)\\ &= (k\,k+1\,k+2)\end{aligned}$$

Thus $A_n = \langle (123), (23\dots n)\rangle$

In particular, A_n is 2-generator group for $n \geq 4$. □

We would like to point out the following example,

Example 2.4.30. *Let G be a group generated by $\sigma = (\cdots -3\ -2\ -1\,0\,1\,2\,3\dots)$ and $\theta = (123)$. Then,*

$$G = \langle \sigma, \theta\rangle \leq Sym(\mathbb{Z})$$

Now, consider

$$\begin{aligned}\sigma^2 &= (\dots -4\ -2\ 0\,2\,4\,6\dots)(\dots -3\ -1\ 0\,1\,3\,5\dots)\\ \sigma^3 &= (\dots -6\ -3\ 0\,3\,6\,9\dots)(\dots -5\ -2\ 1\,4\,7\dots)(\dots -4\ -1\ 2\,5\,8\dots)\\ &\vdots\\ \sigma^n &= (\dots -2n\ -n\ 0\ n\ 2n\ 3n\dots)(\dots -2n+1\ -n+1\ 1\ n+1\ 2n+1\dots)\dots\\ &\quad\dots(\dots -n-1\ -1\ n-1\ 2n-1\dots)\end{aligned}$$

and

$$\sigma^{-1} = (\dots 3\,2\ 1\,0\ -1\ -2\ -3\dots)$$

So, each cycle is a congruence class of modulo n. Note that $G = \langle \sigma, \theta\rangle$ is 2-generator

group and the conjugate of θ by σ

$$\begin{aligned}\theta^{\sigma} = (123)^{\sigma} &= (234)\\ (123)^{\sigma^2} = (234)^{\sigma} &= (345)\\ (123)^{\sigma^n} &= (n+1\, n+2\, n+3)\end{aligned}$$

is an element of G which implies that all consecutive 3-cycles of the form $(i\ i{+}1\, i{+}2)$ where $i \in \mathbb{Z}$ is contained in G. By Theorem 2.4.28, we know A_n is generated by 3-cycles of consecutive form. So A_n is a subgroup of G for any n. In particular,

$$\underbrace{\langle (i\ i+1\, i+2) \ \mid \quad i \in \mathbb{Z} \rangle}_{Alt(\mathbb{Z})} \leq G$$

Recall the Baer-Schreier-Ulam theorem,

- Let Ω be a infinite set of cardinality κ. Then, the only normal subgroups of $Sym(\Omega)$ are

$$Sym^{\kappa}(\Omega) = \{\ \alpha \in Sym(\Omega) \mid \quad \mid supp(\alpha) \mid < \kappa\}$$

$$1 \trianglelefteq Alt(\Omega) \trianglelefteq FSym(\omega) \cdots \trianglelefteq Sym(\Omega)$$

In our case, the chain is in the following way,

$$1 \ \trianglelefteq Alt(\mathbb{Z}) \trianglelefteq FSym(\mathbb{Z}) \trianglelefteq Sym(\mathbb{Z})$$

Observe that $Alt(\mathbb{Z}) \trianglelefteq G = \langle \sigma, \theta \rangle$ and consider the quotient group, namely

$$G/Alt(\mathbb{Z}) = \langle \sigma, \theta \rangle / Alt(\mathbb{Z})$$

But $\theta = (123) \in Alt(\mathbb{Z})$ implies that

$$\begin{aligned}G/Alt(\mathbb{Z}) &= \langle \sigma \rangle Alt(\mathbb{Z})/Alt(\mathbb{Z})\\ &\cong \langle \sigma \rangle / (Alt(\mathbb{Z}) \cap \langle \sigma \rangle)\end{aligned}$$

As $Alt(\mathbb{Z}) \cap \langle \sigma \rangle = 1$, we get

$$G/Alt(\mathbb{Z}) \cong \langle \sigma \rangle$$

which implies $G/Alt(\mathbb{Z})$ is an infinite cyclic group.

Observe that $Alt(\mathbb{Z})$ is a locally finite group. i.e. Every finitely generated subgroup of $Alt(\mathbb{Z})$ is a finite group but $Alt(\mathbb{Z})$ is not finitely generated even if it is a subgroup of 2-generator group $G = \langle \sigma, \theta \rangle$.

Remark 2.4.31. *The alternating group A_t is generated by $a = (a_1\, a_2\, \ldots a_t)$ and $b = (a_1\, a_2\, a_3)$ if t is odd. Moreover,*

$$[a^{-(t-2)}ba^{t-2}, b] \neq 1$$

and

$$[a^{-v}ba^{v}, b] = 1$$

for $3 \leq v < t-3$.

Let's consider an easy example to understand the above remark.

Example 2.4.32. *If we take $t = 5$,*

Let $a = (a_1a_2\ a_3\ a_4a_5)$ and $b = (a_1a_2a_3)$. Then, if we take the conjugate of b by a, we obtain

$$b^a = (a_1a_2a_3)^a = (a_1a_2a_3)^{(a_1a_2\ a_3a_4a_5)} = (a_2a_3a_4)$$

Consider

$$\begin{aligned} bb^a &= (a_1a_2a_3)(a_2a_3a_4) \\ &= (a_1a_3)(a_2a_4) \end{aligned}$$

and

$$\begin{aligned} b^ab &= (a_2a_3a_4)(a_1a_2a_3) \\ &= (a_1a_2)(a_3a_4) \end{aligned}$$

Thus,

$$bb^a \neq b^ab$$

which implies that

$$[a^{-1}ba, b] \neq 1$$

Similarly, if we take the conjugate of b by a^2

$$\begin{aligned} b^{a^2} &= (b^a)^a = (a_2a_3a_4)^{(a_1a_2\ a_3\ a_4a_5)} \\ &= (a_3a_4a_5) \end{aligned}$$

Then,

$$\begin{aligned} bb^{a^2} &= (a_1a_2a_3)(a_3a_4a_5) \\ &= (a_1a_2a_4a_5a_3) \end{aligned}$$

and

$$\begin{aligned} b^{a^2}b &= (a_3a_4a_5)(a_1a_2a_3) \\ &= (a_1a_2a_3a_4a_5) \end{aligned}$$

So,

$$bb^{a^2} \neq b^{a^2}b$$

which also implies that

$$[a^{-2}ba^2, b] \neq 1$$

Indeed, if we consider

$$\begin{aligned} b^{a^3} &= (a_3\, a_4\, a_5)^{(a_1a_2\ a_3\ a_4a_5)} \\ &= (a_1\, a_4\, a_5) \end{aligned}$$

Then,

$$\begin{aligned} bb^{a^3} &= (a_1a_2a_3)(a_1a_4a_5) = (a_1a_2\ a_3\ a_4a_5) \\ b^{a^3}b &= (a_1a_4a_5)(a_1a_2a_3) = (a_1a_4\ a_5\ a_2a_3) \end{aligned}$$

and so we have

$$[a^{-3}ba^3, b] \neq 1$$

But if we take $t = 7$*,*

Let $a = (a_1\, a_2\ \ a_3\ \ a_4\, a_5\, a_6\, a_7)$ *and* $b = (a_1\, a_2\, a_3)$*.*

Consider the followings,

$$\begin{aligned} b^a &= (a_2\, a_3\, a_4) \\ b^{a^2} &= (a_3\, a_4\, a_5) \\ b^{a^3} &= (a_4\, a_5\, a_6) \\ b^{a^4} &= (a_4\, a_5\, a_6)^a = (a_5\, a_6\, a_7) \\ b^{a^5} &= (a_5\, a_6\, a_7)^a = (a_6\, a_7\, a_1) \\ b^{a^6} &= (a_6\, a_7\, a_1)^a = (a_7\, a_1\, a_2) \end{aligned}$$

So,

$$\begin{aligned} [a^{-4}ba^4, b] &= 1 \\ [a^{-3}ba^3, b] &= 1 \end{aligned}$$

since b^{a^4}, b^{a^3} and b represents disjoint cycles. On the other hand,

$$[a^{-5}ba^5, b] \neq 1$$

as a_1 is common with b^{a^5} and b.

If $t \geq 7$ is odd, take $a = (a_1\, a_2\, \ldots,\, a_t)$ and $b = (a_1\, a_2\, a_3)$. Consider $b^{a^3} = (a_4\, a_5\, a_6)$ and it commutes with b. Observe that followings

$$\begin{aligned} b^{a^{t-3}} &= (a_{t-2}\, a_{t-1}\, a_t) \\ b &= (a_1\, a_2\ \ a_3\ ) \end{aligned}$$

imply

$[a^{-(t-3)}ba^{t-3}, b] = 1$ since $(a_{t-2}a_{t-1}a_t)$ and b have no common index if $t-3 > 3$

$[a^{-(t-2)}ba^{t-2}, b] \neq 1$ since $(b^{a^{t-3}})^a = b^{a^{t-2}} = (a_{t-1}\, a_t\, a_1)$

Thus, if we consider A_{u_m} where $u_m = 2^{2^m} + 1$, then $[a^{-(u_n-2)}ba^{u_n-2}, b] = 1$ where $n < m$ and $3 \leq u_n - 2 \leq u_N - 2$ with $n < N$.

But in A_{u_n}, we obtain $[a^{-(u_n-2)}ba^{u_n-2}, b] \neq 1$.

Each relation r_n is depending on n and the values are changed. So the relation $r_n = [a^{-(u_n-2)}ba^{u_n-2}, b]$ is independent of the rest relations.

Now, we are ready to prove Theorem 2.4.21, namely

- Let G be a group generated by two elements a and b with the defining relations

$$r_n = [a^{-(u_n-2)}ba^{u_n-2}, b] = 1$$

where $u_n = 2^{2^n} + 1$ and $n = 1, 2, \ldots$. Then G cannot be defined by finitely many relations.

Proof. Let F be a free group with generators a and b. Inside F, we have the following subgroups;

$$H_n = \langle r_1, r_2, \ldots, r_n \rangle$$

where $r_n = [a^{-(u_n-2)}ba^{u_n-2}, b]$. These subgroups form a properly ascending sequence of subgroups. In fact these subgroups H_i are verbal subgroups of F and so they are fully invariant, in particular normal subgroups of F. Moreover if we consider the union of these groups, say H, then H is a normal subgroup of F.

$$H_1 \leq H_2 \leq \cdots \leq H_n \leq H_{n+1} \leq \cdots$$

Note that each H_n is finitely generated but the union

$$H = \bigcup_{i=1}^{\infty} H_i$$

is not finitely generated.

If we choose G as F/H, then

$$G = \langle a, b \mid \ r_i = 1 \rangle$$

where $i = 1, 2, \ldots$

Hence, we are done. □

We have constructed finitely generated groups infact 2-generator groups such that they cannot be defined by finitely many relations.

B. H. Neumann proved his very famous theorem that is the number of different 2-generator groups is uncountable [12, p. 126]. To understand the proof, we give some examples,

Example 2.4.33. *Let $G = \langle a, b\rangle$ where*

$$a = (a_1\, a_2\, a_3)(a_4\, a_5\, a_6\, a_7\, a_8)(a_9\, a_{10}\, a_{11}\, a_{12}\, a_{13}\, a_{14}\, a_{15}\,)$$
$$b = (a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_9\, a_{10}\, a_{11})$$

- *Consider*

$$a^2 = (a_1\, a_3\, a_2)(a_4\, a_6\, a_8\, a_5\, a_7)(a_9\, a_{11}\, a_{13}\, a_{15}\, a_{10}\, a_{12}\, a_{14}\,)$$
$$a^3 = (a_1)(a_2)(a_3)(a_4\, a_7\, a_5\, a_8\, a_6)(a_9\, a_{12}\, a_{15}\, a_{11}\, a_{14}\, a_{10}\, a_{13}\,)$$
$$a^4 = (a_1\, a_2\, a_3)(a_4\, a_8\, a_7\, a_6\, a_5)(a_9\, a_{13}\, a_{10}\, a_{14}\, a_{11}\, a_{15}\, a_{12}\,)$$
$$a^5 = (a_1\, a_3\, a_2)(a_4)(a_5)(a_6)(a_7)(a_8)(a_9\, a_{14}\, a_{12}\, a_{10}\, a_{15}\, a_{13}\, a_{11})$$
$$a^6 = (a_1)(a_2)(a_3)(a_4\, a_5\, a_6\, a_7\, a_8)(a_9\, a_{15}\, a_{14}\, a_{13}\, a_{12}\, a_{11}\, a_{10})$$
$$a^7 = (a_1a_2a_3)(a_4\, a_6\, a_8\, a_5\, a_7)(a_9)(a_{10})\ldots(a_{15})$$
$$(a^7)^2 = (a_1a_3a_2)(a_4\, a_8\, a_7\, a_6\, a_5)(a_9)(a_{10})\ldots(a_{15})$$
$$(a^7)^3 = (a_1)(a_2)(a_3)(a_4\, a_5\, a_6\, a_7\, a_8)(a_9)(a_{10})\ldots(a_{15})$$
$$(a^7)^4 = (a_1a_2a_3)(a_4\, a_7\, a_5\, a_8\, a_6)(a_9)(a_{10})\ldots(a_{15})$$
$$(a^7)^5 = (a_1a_3a_2)(a_4)(a_5)(a_6)(a_7)(a_8)(a_9)(a_{10})\ldots(a_{15})$$

Let's call the element a^{35} as t. Observe that $t \in \langle a\rangle < G = \langle a, b\rangle$. Moreover,

$$\begin{aligned} t^a &= ((a_1a_3a_2)(a_4)(a_5)(a_6)(a_7)(a_8)(a_9)(a_{10})\ldots(a_{15}))^a \\ &= (a_1a_3a_2)^{(a_1\, a_2\, a_3)(a_4\, a_5\, a_6\, a_7\, a_8)(a_9\, a_{10}\, a_{11}\, a_{12}\, a_{13}\, a_{14}\, a_{15}\,)} = (a_1a_3a_2) \\ t^b &= ((a_1a_3a_2)(a_4)(a_5)(a_6)(a_7)(a_8)(a_9)(a_{10})\ldots(a_{15}))^b \\ &= (a_1a_3a_2)^{(a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_9\, a_{10}\, a_{11})} = (a_1a_3a_2) \\ t^at &= (a_1a_3a_2)(a_1a_3a_2) \\ &= (a_1a_2a_3) \end{aligned}$$

Consider the alternating group generated by $\{a_1, a_2, a_3\}$ which is isomorphic to A_3 and then by Theorem 2.4.24 one can get $A_3 = \langle(a_1a_3a_2)\rangle$. Thus $\langle t\rangle^G \cong A_3$ and $\langle t\rangle^G \trianglelefteq G$.

- *Consider*

$$\begin{aligned}
a^2 &= (a_1\, a_3\, a_2)(a_4\, a_6\, a_8\, a_5\, a_7)(a_9\, a_{11}\, a_{13}\, a_{15}\, a_{10}\, a_{12}\, a_{14}\,)\\
a^3 &= (a_1)(a_2)(a_3)(a_4\, a_7\, a_5\, a_8\, a_6)(a_9\, a_{12}\, a_{15}\, a_{11}\, a_{14}\, a_{10}\, a_{13}\,)\\
a^4 &= (a_1\, a_2\, a_3)(a_4\, a_8\, a_7\, a_6\, a_5)(a_9\, a_{13}\, a_{10}\, a_{14}\, a_{11}\, a_{15}\, a_{12}\,)\\
a^5 &= (a_1\, a_3\, a_2)(a_4)(a_5)(a_6)(a_7)(a_8)(a_9\, a_{14}\, a_{12}\, a_{10}\, a_{15}\, a_{13}\, a_{11})\\
a^6 &= (a_1)(a_2)(a_3)(a_4\, a_5\, a_6\, a_7\, a_8)(a_9\, a_{15}\, a_{14}\, a_{13}\, a_{12}\, a_{11}\, a_{10})\\
a^7 &= (a_1a_2a_3)(a_4\, a_6\, a_8\, a_5\, a_7)(a_9)(a_{10})\ldots(a_{15})\\
(a^7)^2 &= (a_1a_3a_2)(a_4\, a_8\, a_7\, a_6\, a_5)(a_9)(a_{10})\ldots(a_{15})\\
(a^7)^3 &= (a_1)(a_2)(a_3)(a_4\, a_5\, a_6\, a_7\, a_8)(a_9)(a_{10})\ldots(a_{15})
\end{aligned}$$

Then, calculate $b^{a^{21}}$ and call this element as t_1.

$$\begin{aligned}
t_1 &= b^{a^{21}}\\
&= ((a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_9\, a_{10}\, a_{11}))^{(a_1)(a_2)(a_3)(a_4\, a_5\, a_6\, a_7\, a_8)(a_{10})\ldots(a_{15})}\\
&= (a_1\, a_2\, a_3)(a_5\, a_6\, a_7)(a_9\, a_{10}\, a_{11})
\end{aligned}$$

Take the conjugate of t_1 by a^{21},

$$\begin{aligned}
t_2 &= t_1^{a^{21}}\\
&= ((a_1\, a_2\, a_3)(a_5\, a_6\, a_7)(a_9\, a_{10}\, a_{11}))^{(a_4\, a_5\, a_6\, a_7\, a_8)}\\
&= (a_1\, a_2\, a_3)(a_6\, a_7\, a_8)(a_9\, a_{10}\, a_{11})
\end{aligned}$$

Consider the conjugate of t_2 by a^{21},

$$\begin{aligned}
t_3 &= t_2^{a^{21}}\\
&= (a_1\, a_2\, a_3)(a_6\, a_7\, a_8)(a_9\, a_{10}\, a_{11})^{(a_4\, a_5\, a_6\, a_7\, a_8)}\\
&= (a_1\, a_2\, a_3)(a_7\, a_8\, a_4)(a_9\, a_{10}\, a_{11})
\end{aligned}$$

Multiply t_3 from left by t_2^2,

$$\begin{aligned}
t_2^2t_3 &= ((a_1\, a_2\, a_3)(a_6\, a_7\, a_8)(a_9\, a_{10}\, a_{11}))^2(a_1\, a_2\, a_3)(a_7\, a_8\, a_4)(a_9\, a_{10}\, a_{11})\\
&= (a_1\, a_3\, a_2)(a_6\, a_8\, a_7)(a_9\, a_{11}\, a_{10})(a_1\, a_2\, a_3)(a_7\, a_8\, a_4)(a_9\, a_{10}\, a_{11})\\
&= (a_1)(a_2)(a_3)(a_6\, a_4\, a_7)(a_9)(a_{10})(a_{11})\\
&= (a_4a_7a_6)
\end{aligned}$$

So, $t_2^2t_3$ is contained in $G = \langle a, b\rangle$. Moreover,

$$\begin{aligned}(t_2^2t_3)^a &= (a_4a_7a_6)^{(a_1\,a_2\,a_3)(a_4\,a_5\,a_6\,a_7\,a_8)(a_9\,a_{10}\,a_{11}\,a_{12}\,a_{13}\,a_{14}\,a_{15}\,)}\\ &= (a_5a_8a_7)\\ (t_2^2t_3)^b &= (a_4a_7a_6)^{(a_1\,a_2\,a_3)(a_4\,a_5\,a_6)(a_9\,a_{10}\,a_{11})}\\ &= (a_4a_5a_7)\end{aligned}$$

and

$$\begin{aligned}(t_2^2t_3)^b(t_2^2t_3) &= (a_4a_5a_7)(a_4a_7a_6) = (a_4a_5a_6)\\ (t_2^2t_3)^a(t_2^2t_3)^b &= (a_5a_8a_7)(a_4a_5a_7) = (a_4a_5a_8)\end{aligned}$$

Consider the alternating group on $\{a_4, a_5, a_6, a_7, a_8\}$ which is isomorphic to A_5. By Theorem 2.4.26, we have $A_5 \cong \langle (a_4a_5a_6), (a_4a_5a_7), (a_4a_5a_8)\rangle$. So, above observations leads us to $\langle t_2^2t_3\rangle^G \cong A_5 \lhd G$.

- *Consider*

$$\begin{aligned}a^2 &= (a_1\,a_3\,a_2)(a_4\,a_6\,a_8\,a_5\,a_7)(a_9\,a_{11}\,a_{13}\,a_{15}\,a_{10}\,a_{12}\,a_{14}\,)\\ a^3 &= (a_1)(a_2)(a_3)(a_4\,a_7\,a_5\,a_8\,a_6)(a_9\,a_{12}\,a_{15}\,a_{11}\,a_{14}\,a_{10}\,a_{13}\,)\\ a^4 &= (a_1\,a_2\,a_3)(a_4\,a_8\,a_7\,a_6\,a_5)(a_9\,a_{13}\,a_{10}\,a_{14}\,a_{11}\,a_{15}\,a_{12}\,)\\ a^5 &= (a_1\,a_3\,a_2)(a_4)(a_5)(a_6)(a_7)(a_8)(a_9\,a_{14}\,a_{12}\,a_{10}\,a_{15}\,a_{13}\,a_{11})\\ (a^5)^2 &= (a_1\,a_2\,a_3)(a_4)(a_5)(a_6)(a_7)(a_8)(a_9\,a_{12}\,a_{15}\,a_{11}\,a_{14}\,a_{10}\,a_{13})\\ (a^5)^3 &= (a_1)(a_2)(a_3)(a_4)(a_5)(a_6)(a_7)(a_8)(a_9\,a_{10}\,a_{11}\,a_{12}\,a_{13}\,a_{14}\,a_{15})\end{aligned}$$

Take the conjugate of b by a^{15} and call this element as t_1.

$$\begin{aligned}t_1 &= b^{a^{15}}\\ &= ((a_1\,a_2\,a_3)(a_4\,a_5\,a_6)(a_9\,a_{10}\,a_{11}))^{(a_9\,a_{10}\,a_{11}\,a_{12}\,a_{13}\,a_{14}\,a_{15})}\\ &= (a_1\,a_2\,a_3)(a_4\,a_5\,a_6)(a_{10}\,a_{11}\,a_{12})\end{aligned}$$

Then, consider the conjugate of t_1 by a^{15} and call this element as t_2

$$\begin{aligned}t_2 &= t_1^{a^{15}}\\ &= ((a_1\,a_2\,a_3)(a_4\,a_5\,a_6)(a_{10}\,a_{11}\,a_{12}))^{(a_9\,a_{10}\,a_{11}\,a_{12}\,a_{13}\,a_{14}\,a_{15})}\\ &= (a_1\,a_2\,a_3)(a_4\,a_5\,a_6)(a_{11}\,a_{12}\,a_{13})\end{aligned}$$

Now, we take the conjugate of t_2 *by* a^{15}

$$\begin{aligned} t_3 &= t_2^{a^{15}} \\ &= ((a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_{11}\, a_{12}\, a_{13}))^{(a_9\, a_{10}\, a_{11}\, a_{12}\, a_{13}\, a_{14}\, a_{15})} \\ &= (a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_{12}\, a_{13}\, a_{14}) \end{aligned}$$

and consider the conjugate of t_3 *by* a^{15}

$$\begin{aligned} t_4 &= t_3^{a^{15}} \\ &= ((a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_{12}\, a_{13}\, a_{14}))^{(a_9\, a_{10}\, a_{11}\, a_{12}\, a_{13}\, a_{14}\, a_{15})} \\ &= (a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_{13}\, a_{14}\, a_{15}) \end{aligned}$$

Similarly, take the conjugate of t^4 *by* a^{15}

$$\begin{aligned} t_5 &= t_4^{a^{15}} \\ &= ((a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_{13}\, a_{14}\, a_{15}))^{(a_9\, a_{10}\, a_{11}\, a_{12}\, a_{13}\, a_{14}\, a_{15})} \\ &= (a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_9\, a_{14}\, a_{15}) \end{aligned}$$

Calculate $t_5 t_4^2$ *and call this element as* s

$$\begin{aligned} s = t_4^2 t_5 &= (a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_9\, a_{14}\, a_{15})((a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_{13}\, a_{14}\, a_{15}))^2 \\ &= (a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_9\, a_{14}\, a_{15})(a_1\, a_3\, a_2)(a_4\, a_6\, a_5)(a_{13}\, a_{15}\, a_{14}) \\ &= (a_9\, a_{13}\, a_{15}) \end{aligned}$$

and consider

$$\begin{aligned} s^a &= (a_9\, a_{13}\, a_{15})^{(a_1\, a_2\, a_3)(a_4\, a_5\, a_6\, a_7\, a_8)(a_9\, a_{10}\, a_{11}\, a_{12}\, a_{13}\, a_{14}\, a_{15}\,)} \\ &= (a_9\, a_{10}\, a_{14}) \\ s^b &= (a_9\, a_{13}\, a_{15})^{(a_1\, a_2\, a_3)(a_4\, a_5\, a_6)(a_9\, a_{10}\, a_{11})} \\ &= (a_{10}\, a_{13}\, a_{15}) \end{aligned}$$

Moreover,

$$\begin{aligned} s^2 s^b &= (a_9\, a_{13}\, a_{15})^2 (a_{10}\, a_{13}\, a_{15}) \\ &= (a_9\, a_{15}\, a_{13})(a_{10}\, a_{13}\, a_{15}) \\ &= (a_9\, a_{10}\, a_{13}) \\ s(s^2)^b &= (a_9\, a_{13}\, a_{15})(a_9\, a_{15}\, a_{13})^b \\ &= (a_9\, a_{13}\, a_{15})(a_{10}\, a_{15}\, a_{13}) \\ &= (a_9\, a_{10}\, a_{15}) \\ (s(s^2)^b)^a &= (a_9\, a_{10}\, a_{11}) \end{aligned}$$

Observe that $\langle s \rangle^G \triangleleft G$. Similarly above cases, if we consider the alterneting group on $\{a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}\}$, then by Theorem 2.4.26, we obtain

$$\langle (a_9\, a_{10}\, a_{11}), (a_9\, a_{10}\, a_{12}), (a_9\, a_{10}\, a_{13}), (a_9\, a_{10}\, a_{14}), (a_9\, a_{10}\, a_{15}) \rangle \cong A_7$$

So, there is a normal subgroup of G which is isomorhic to A_7.

Thus, G has a normal subgroup isomorphic to $A_3 \times A_5 \times A_7$. Observe that G has no normal subgroups isomorphic to A_4 or A_6.

Theorem 2.4.34. [12, p. 126] *The number of pairwise non-isomorphic groups generated by two elements is $2^{\aleph_0}$.*

Proof. Let $U = (u_1, u_2, u_3, \dots)$ be a strictly increasing infinite sequence of odd numbers where $u_1 \geq 5$ and let $\{\sigma_{\mu\nu} \mid \mu = 1, 2, 3, \dots,\ \nu = 1, 2, 3, \dots,\ u_\mu\}$ be a set of symbols. Define a group G_U as follows,

$$G_U = \langle a, b \rangle$$

where

$$\begin{aligned} a &= (\sigma_{11}\sigma_{12} \dots \sigma_{1u_1})(\sigma_{21}\sigma_{22} \dots \sigma_{2u_2}) \dots \\ b &= (\sigma_{11}\sigma_{12}\sigma_{13})(\sigma_{21}\sigma_{22}\sigma_{23}) \dots \end{aligned}$$

Then, G_U is countably infinite since it is a 2-generator group. Observe that the cyclic group generated by a, $\langle a \rangle$, is an infinite cyclic group and the cyclic group generated by b, $\langle b \rangle$, is of order 3.

G_U permutes the $\sigma_{\mu\nu}$ for a fixed μ, in the same way as the alternating group A_μ on this symbols. Every normal subgroup of G_U permutes the $\sigma_{\mu\nu}$ for a fixed μ according to A_μ or else leaves them invariant. Note that a finite normal subgroup of G leaves all but a finite number of symbols fixed otherwise it contains alternating groups of arbitrary high order. So, the alternating group A_β is a normal subgroup of G_U if β is one of u_α for some α belonging U.

By Remark 2.4.31, in the alternating group A_{u_μ},

$$r_{u_m} = [b^{a^{u_m-2}}, b] = 1 \quad \text{if} \quad \mu > m$$

and

$$r_{u_m} = [b^{a^{u_m-2}}, b] \neq 1 \quad \text{in} \quad A_{u_m}$$

Therefore, r_{u_m} and its conjugates generates a normal subgroup of G_U which leaves the symbols $\sigma_{\mu\nu}$ with $\mu > \nu$ invariant but not the $\sigma_{\mu\nu}$ itself. Consider the finitely generated subgroup $\langle r_{u_1}, r_{u_2}, \dots, r_{u_m} \rangle$ of G_U, then $\langle r_{u_1}, r_{u_2}, \dots, r_{u_m} \rangle^{G_U}$ is a normal subgroup of G_U which is isomorphic to A_{u_m} almost always because $r_{u_1} = r_{u_2} = \cdots = r_{u_{m-1}} = 1$ in A_{u_m}.

So, the group G_U contains normal subgroups isomorphic to A_{u_m}. The alternating group A_ν is a normal subgroup of G_U if and only if ν is one of u_i belong to the given sequence U. If U and U' are different strictly increasing infinite sequence of odd numbers, then G_U and $G_{U'}$ are not isomorphic since they have different finite normal subgroup to the positions where given sequences are different.

Thus, each sequence U uniquely defines a 2-generator group G_U. As we can find $2^{\aleph_0}$-many sequences, there are $2^{\aleph_0}$-many 2-generator group. □

These groups G_U are quite interesting. By above observations, given the sequence

$$U = (u_1, u_2, \dots)$$

consider $G_U = \langle a, b \rangle$ where $a = (\sigma_{11}\sigma_{12}\dots\sigma_{1u_1})(\sigma_{21}\sigma_{22}\dots\sigma_{2u_2})\dots$ and $b = (\sigma_{11}\sigma_{12}\sigma_{13})(\sigma_{21}\sigma_{22}\sigma_{23})\dots$. Then as in the Example 2.4.33, G_U contains finite normal subgroups isomorphic to A_{u_i} for all $i = 1, 2, \dots$. If we let N to be the direct product of A_{u_i} 's where $i = 1, 2, \dots$, then N becomes a normal subgroup of G_U.

Thus, one can consider the quotient group, namely

$$G_U/N \cong \langle a, b\rangle/N$$

Observe that the elements in N are of the form $(\sigma_{11}\sigma_{12}\dots\sigma_{1u_i})$ with $i = 1, 2, \dots$. Since $a = (\sigma_{11}\sigma_{12}\dots\sigma_{1u_1})(\sigma_{21}\sigma_{22}\dots\sigma_{2u_2})\dots$, we may remove finitely many u_i-cycles from a. The element aN in G_U/N has infinite order as the order of a is infinite. Similarly, we may remove finitely many 3-cycles from b as $(\sigma_{11}\sigma_{12}\sigma_{13}) \in N$ and the element bN in G_U/N has order 3.

Hence, independent of the choice of the sequence U, the quotient group G_U/N is isomorphic to the group generated by $(\cdots, -2, -1, 0, 1, 2, \cdots)$ and (123), Moreover, we have shown in Example 2.4.30 that $Alt(\mathbb{Z})$ is a normal subgroup of $\langle(\cdots, -2, -1, 0, 1, 2, \cdots), (123)\rangle$ which implies that G_U/N has a normal subgroup which is isomorphic to $Alt(\mathbb{Z})$.

Theorem 2.4.35. *There are uncountably many pairwise non-isomorphic simple countable groups.*

Proof. We have seen in Theorem 2.2.9 that every countable group can be embedded into a countable simple group. So, every 2-generator group can be embedded into a countable simple group. Suppose that there exists only countably many non-isomorphic simple countable groups, then by pigeonhole principle $2^{\aleph_0}$-many 2-generator groups should be a subgroup of a countable simple group. But, a countable simple group can have at most countably many countable 2-generator subgroups.

Hence, there are $2^{\aleph_0}$-many pairwise non-isomorphic simple countable groups. □

Remark 2.4.36. *There are only countably many finitely presented groups.*

Proof. Let G be a finitely presented group. First assume that G is generated by one element, namely $G = \langle x\rangle$. The relations in G are of the form $x^n = 1$ for some n. For any given integer $n \geq 1$, there exists unique cyclic group of order n. Since there are only countably many cyclic groups, we have countably many 1-generator finitely presented groups. Then, suppose that G is generated by two elements, namely

$G = \langle x, y \rangle$. Each relation in G is an element of a free group generated by x and y. Since the free group of rank 2 has countably many elements, we have countably many relations for G. Thus, we have countably many 2-generator finitely presented groups. Observe that the same calculations can be made for n-generator group because it has countably many elements.

Therefore, there are only countably many finitely presented groups. □

CHAPTER 3

ALGEBRAICALLY CLOSED GROUPS

3.1 Motivation

In this section, we give some basic results from *Field Theory* and the proofs can be found in any graduate level algebra book, see [9, Chapter 5].

A field K is algebraically closed if every polynomial $f(x) \in K[x]$ such that $deg(f(x)) \geq 1$ has a root in K. So, every polynomial in one variable can be written as a product of polynomials of degree 1 over algebraically closed field K. In particular every polynomial in $K[x]$ of degree greater than or equal to 2 is reducible over K.

Theorem 3.1.1. [9, Theorem 2.5] *Every field is contained in an algebraically closed field.*

Definition 3.1.2. [9, p. 239] *An algebraic closure of a field F is an algebraic extension K of F that is algebraically closed. Algebraic closure of a field is unique up to isomorphism.*

Example 3.1.3. *$\mathbb{Q}$ is a field of rational numbers. The algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ is algebraically closed field. Moreover, $\overline{\mathbb{Q}}$ is countably infinite. i.e. $\mid \overline{\mathbb{Q}} \mid = \mid \mathbb{Q} \mid$*

Proof. To show why $\overline{\mathbb{Q}}$ is algebraically closed, first note that finite extensions are algebraic extensions. If F is any field and $f(x) \in F[x]$, then there is a finite extension K_f of F such that $f(x)$ has a root in K_f. Moreover, K_f is an algebraic extension of F.

In $\mathbb{Q}[x]$, there are countably infinite many polynomials with coefficients in $\mathbb{Q}$. (i.e.

$\mathbb{Q}[x]$ is a countably infinite integral domain).

Each polynomial has only finitely many roots. Next, we adjoin to $\mathbb{Q}$ the roots of all non-constant polynomials in $\mathbb{Q}[x]$. Then we obtain an infinite degree extension $\mathbb{Q}_{(1)}$ of $\mathbb{Q}$.

$[\mathbb{Q}_{(1)} : \mathbb{Q}] = \infty$.

Similarly, we define $\mathbb{Q}_{(i+1)}$ by adjoining the roots of all non-constant polynomials in $\mathbb{Q}_{(i)}[x]$ to $\mathbb{Q}_{(i)}$.

Now, let

$$\overline{\mathbb{Q}} = \bigsqcup_{i=1}^{\infty} \mathbb{Q}_{(i)}$$

Take

$$h(x) \in \overline{\mathbb{Q}}[x]$$

Then,$h(x)$ is a polynomial with coefficients in $\mathbb{Q}_{(j)}[x]$ for some j. Therefore, roots of $h(x)$ are in $\mathbb{Q}_{(j+1)} \subseteq \overline{\mathbb{Q}}$.

Hence, $\overline{\mathbb{Q}}$ is algebraically closed. □

Remark 3.1.4. *Given any infinite cardinal number κ there exist a field K of cardinality κ and an algebraically closed field $\overline{K}$ of cardinality κ such that $\overline{K} \supseteq K$ where $\mid K \mid = \mid \overline{K} \mid = \kappa$.*

3.2 Algebraically Closed Groups

Algebraically closed groups are analogues to algebraically closed fields.

Let $X = \{x_i \mid i \in \mathcal{I}\}$ where $\mathcal{I}$ is any index set.

Let F_X be the free group on the set X. Then, consider the free product of a given group G and F_X,

$$G \star F_X$$

Observe that

- The elements of $G \star F_X$ are the **words** with coefficients from group G.
- The elements in $G \star F_X$ of the form $g \in G$ are called **constants**.
- For a word $w \in G \star F_X$, the statement $w = 1$ is called an **equation** (equality) over G.
- For a word $w \in G \star F_X$, the statement $w \neq 1$ is called an **inequation** (inequality) over G.

We know that the empty word is xx^{-1} which corresponds to identity in F_X. So, the equation

$$xx^{-1} = g$$

for a nontrivial element $g \in G$ has no solution in any group $H \supseteq G$. This shows us that some of the equations defined over G may not have any solution in any group H containing G.

Example 3.2.1. *Let G be a nontrivial group and let $a, b \in G$ have different orders. So, we have;*

$$a^n = 1,\ b^m = 1 \text{ and } n \neq m$$

Then, the equation

$$x^{-1}ax = b$$

has no solution in H for any $H \supseteq G$.

Remark 3.2.2. *This point is **different** in field case because for any field F and for any polynomial $f(x) \in F[x]$ with $degf \geq 1$, there exist an extension K_f of F containing a root of $f(x)$.*

Definition 3.2.3. *Let G be a group and $w(x_i, g_k)$ is a word in the indeterminates x_i where $i \in \mathcal{I}$ and coefficients $g_k \in G$. A nonempty finite set of equations and inequations over the group G*

$$w_i(x_i, g_k) = 1 \quad i \in \mathcal{I}$$
$$v_j(x_j, g_k) \neq 1 \quad j \in \mathcal{J}$$

*is **consistent** over G if there exist a group*

$$H \supseteq G$$

and an embedding

$$\varphi : G \star F_X \hookrightarrow H$$

such that

- $\varphi(g) = g \quad \forall g \in G$
- $w_i \in Ker(\varphi) \quad \forall i \in \mathcal{I}$
- $v_j \notin Ker(\varphi) \quad \forall j \in \mathcal{J}$

Example 3.2.4. *Let G be a torsion free group. Then, the equation*

$$x^n = 1$$

has no non-trivial solution in G but has a non-trivial solution in

$$G \times C_n$$

where C_n is cyclic group of order n. Thus, $x^n = 1$ is consistent over G.

W. R. Scott [16] first introduced algebraically closed groups and weakly algebraically closed groups.

Definition 3.2.5. *A group G is called **algebraically closed** if for every nonempty finite set of equations and inequations*

$$w_i(x_i, g_k) = 1 \quad i \in \mathcal{I} \tag{3.1}$$

$$v_j(x_j, g_k) \neq 1 \quad j \in \mathcal{J} \tag{3.2}$$

that is consistent over G, has a solution in G.

Definition 3.2.6. *A group G is called **weakly algebraically closed** if for every finite set of equations which is consistent over G has a solution in G.*

It is obvious by definition that every algebraically closed group is weakly algebraically closed. Conversly, B. H. Neumann [13] proved that every weakly algebraically closed group is algebraically closed if it is different from the identity group. Here one can observe that if G is an identity group, then the equation $x \neq 1$ has a solution in a nontrivial group $H \supseteq G$. So, $x \neq 1$ has a solution over G but it has no solution in G. Thus, G is not algebraically closed. To prove Neumann's theorem, we first need two lemmas.

Lemma 3.2.7. [13, Lemma A] *Let G be an arbitrary group and let the set of equations (3.1) and inequations (3.2) has a solution in $H \supseteq G$. Then, there exists a group*

$$\widehat{H} \supseteq H$$

such that the system of (3.1), (3.2) and the further equations

$$c^2 = 1,\ d^2 = g_0,\ \ (cd)^2 = 1 \tag{3.3}$$

$$s_j^2 = 1,\ (s_j v_j(x_j, g_k))^3 = 1 \tag{3.4}$$

$$t_j^{-1} s_j t_j = c \quad \forall j \in \mathcal{J} \tag{3.5}$$

where g_0 is an arbitrary element, has a solution in G.

Proof. Let the order of g_0 in G be γ and D be the dihedral group.

- If γ is infinite, D is an infinite dihedral group.

$$D = \langle a, b \mid\ a^2 = 1,\ (ab)^2 = 1 \rangle$$

- If $\gamma \in \mathbb{N}$, then D is a dihedral group of order 4γ.

$$D = \langle a, b \mid\ a^2 = 1,\ b^{2\gamma} = 1\ (ab)^2 = 1 \rangle$$

Notice that, the order of b^2 and g_0 are equal to γ. Thus we can take the free product of H and D amalgamating g_0 and b^2. Call this product as H_1.

Now, consider the equations (3.3) in H_1 and take

$$\begin{aligned} a = c \quad &\implies a^2 = 1 \\ (ab) = (cd) \quad &\implies (ab)^2 = 1 \\ b = d \implies \quad b^2 = d^2 = g_0 \quad &\implies \quad b^2 = g_0 \end{aligned}$$

Then, the equations (3.3) is satisfied in H_1. Infact, the system of (3.1) and (3.2) has a solution in H_1 since $H_1 \supset H$.

Observe that H satisfies (3.2) , so we can consider the solutions of (3.2) in H, namely $h_1, h_2, \ldots, h_n$. Then,

$$v_j(h_1, h_2, \ldots, h_n, g_k) = z_j \quad \text{where } z_j \neq 1$$

Let the order of z_j be $\alpha(i)$. Then, define A_j as follows;

$$A_j = \langle a_j, b_j \mid a_j^2 = 1,\ b_j^{\alpha(j)} = 1,\ (a_j b_j)^3 = 1 \rangle$$

So, the orders of b_j and z_j are equal to $\alpha(j)$. Therefore, we can form H_2 as the free product of H_1 and all groups A_j amalgamating b_j and z_j;

$$H_2 = H_1 \star A_1 \star A_2 \star \cdots \star A_j$$

amalgamating $\langle b_j \rangle (\cong \langle z_j \rangle)$.

In the group H_2, if we take

$$\begin{aligned} h_i &= x_i \\ a_j &= s_j \quad \Longrightarrow\ a_j^2 = 1 \end{aligned}$$

then, equation (3.4) is satisfied in H_2

Finally, in H_2,

$$s_j^2 = 1 \quad \& \quad a^2 = c^2 = 1$$

So, we can use HNN extention. Then, there exists a group $\widehat{H}$ containing H_2 such that

$$\widehat{H} = \langle H_2, t_j \mid t_j^{-1} s_j t_j = a, \rangle$$

Hence, equation (3.5) is satisfied in $\widehat{H}$. □

Lemma 3.2.8. [13, Lemma B] *Let G be a group such that $1 \neq g_0 \in G$. If any such of the system of equations (3.1), (3.3), (3.4), (3.5) has a solution in G or in any group H containing G, then the inequalities (3.2) is also satisfied in H.*

Proof. For some solutions of (3.1), (3.3), (3.4), (3.5) assume that one of the inequations (3.2) is not satisfied, say,

$$v_2(h_1, h_2, \ldots, h_n, g_k) = 1$$

By equation (3.4), we have;

$$(s_2)^2 = 1 \quad \Longrightarrow \quad (s_2 v_2(h_j, g_k))^3 = 1$$

Then,

$$(s_2)^3 = 1$$

It follows that

$$s_2 = 1$$

By equation (3.5), we get;

$$t_2^{-1} s_2 t_2 = c \quad \Longrightarrow \quad c = 1$$

Finally, using equation (3.3) we obtain,

$$(cd)^2 = 1 \quad \Longrightarrow \quad d^2 = 1$$

But, $d^2 = 1$ implies $d^2 = g_0 = 1$

This is a contradiction since g_0 is nontrivial. □

Theorem 3.2.9. [13] *If a weakly algebraically closed group G is nontrivial, then it is an algebraically closed group.*

Proof. Let G be a non-trivial weakly algebraically closed group. Let H be a group containing G such that the system of equations (3.1) and inequations (3.2) has a solution in H.

By Lemma 3.2.7 implies that there exists a group H_2 containing H such that the system of equations (3.1), (3.3), (3.4) and (3.5) has a solution in H_2 but G is a weakly algebraically closed group so (3.1), (3.3), (3.4) and (3.5) are satisfied in G.

Finally, by Lemma 3.2.8, any such solution satisfying (3.1), (3.3), (3.4) and (3.5) in G also satisfies the inequalities (3.2).

Hence, G is algebraically closed group. □

Here, we would like to point out that Scott's definition of algebraically closed group [16] is also called existentially closed group. So, another terminology is commonly used to define algebraically closed groups and weakly algebraically closed groups

that is

existentially closed groups instead of algebraically closed groups

algebraically closed groups instead of non-trivial weakly algebraically closed groups

From now on, we will be using existentially closed groups instead of algebraically closed groups in this thesis. For example, one can read Theorem 3.2.9 in the following way,

- Every non-trivial weakly algebraically closed group is algebraically closed and hence existentially closed.

We will discuss the results of the first paper [16] in existentially closed groups in this part.

Theorem 3.2.10. [16, Theorem 3] *Every group can be embedded in an existentially closed group.*

Proof. We will consider three cases to prove this theorem

Case 1. Let H be a finite group of order $n \neq 1$.

Then,

$$H = \{h_{0=1}, h_1, \ldots, h_{n-1}\}$$

such that

$$h_i h_j = h_k \quad \text{where } i, j, k \in \{0, 1, \ldots, n-1\}$$

Consider the multiplication table and form the equations and inequations with respect to all the products in multiplication table as

$$\begin{aligned} x_\alpha &\neq 1 \quad \text{where } \alpha \in \{1, 2, \ldots, n-1\} \\ x_i x_j &= x_k \quad \text{where } i, j, k \in \{1, 2, \ldots, n-1\} \end{aligned} \tag{3.6}$$

Observe that there are finitely many equations and inequations.

Next, we let G be an existentially closed group. Then, the finite set of equations and inequations (3.6) have a solution in

$$G \times H$$

Since G is an existentially closed group, it has a solution in G, namely,

$$g_0, g_1, \ldots, g_{n-1}$$

We know that multiplication table of groups is unique up to isomorphism. Therefore, H is isomorphic to the subgroup of G containing the elements $g_0, g_1, \ldots, g_{n-1}$.

$$H \cong \langle g_0, g_1, \ldots, g_{n-1} \rangle < G$$

This implies that an existentially closed group contains all finite groups. Hence the cardinality of an existentially closed group is infinite.

Case 2. Let H be a countably infinite group.

Let Σ be the set of all finite set of equations and inequations with coefficients in H.

$$\Sigma = \{ S_i \mid i \in \mathcal{I} \}$$

where S_i is a finite set of equations and inequations with coefficients in H. Notice that, the cardinality of Σ is the same as the cardinality of H. So, Σ is a countable set. Then, since every set can be well-ordered, we may order the elements S_i in Σ as;

$$S_1, S_2, S_3, \ldots$$

First, we will try to construct a countable group

$$H^\star$$

such that every consistent system with coefficients in H has a solution in $H^\star \supseteq H$.

Let $H_1 = H$. Assume that we have constructed countable group H_i such that all the consistent system of finitely many equations and inequations with coefficients in H, say

$$S_1, S_2, \ldots, S_i$$

have a solution in H_i. Then, consider the finite set of equations and inequations, S_{i+1},

- If S_{i+1} is not consistent over H_i, then $H_i = H_{i+1}$.
- If S_{i+1} is consistent over H_i, then there exists a group K and an embedding $\varphi : H_i \hookrightarrow K$ such that the system S_{i+1} has a solution in K, namely,

  $$k_1, k_2, \ldots, k_t$$

  So, define H_{i+1} as follows;

  $$H_{i+1} = \langle \varphi(H_i), k_1, k_2, \ldots, k_t \rangle$$

  Now, H_{i+1} is a countable group since H_i is countable. Moreover, H_{i+1} contains an isomorphic copy of H and all the solutions of the consistent systems $S_1, S_2, \ldots, S_i, S_{i+1} \in \Sigma$.

We continue like this and obtain;

$$H^\star = \bigcup_{i=1}^{\infty} H_i$$

The union of countably many countable sets is countable. Thus, $H^\star$ is countably infinite and every consistent system in Σ has a solution in $H^\star$.

Next step is to construct an existentially closed group G.

Let $G_1 = H$ and $G_2 = H^\star$. Consider the set of all finitely many equations and inequations with coefficients from $H^\star$, call this set as Σ_1. We know that, the cardinality of $H^\star$ and Σ_1 are the same and so Σ_1 is a countable set.

Now, as above we obtain a new group

$$G_2^\star = G_3$$

such that all the consistent systems in Σ_1 have a solution in G_3.

We continue like this and get;

$$G_1 \leq G_2 \leq G_3 \leq \cdots \leq G_n \leq \ldots$$

Define G as follows;

$$G = \bigcup_{i=1}^{\infty} G_i$$

Observe that G is countable since it is countable union of countable groups.

Claim. *G is an existentially closed group.*

If we take a finite system of equations and inequations with coefficients in G, then there exists $n \in \mathbb{N}$ such that all the coefficients in the equations and inequations are coming from G_n and the system is consistent over G_n. This consistent system has a solution in G_{n+1} and so in G. Therefore, for every finite set of equations and inequation which is consistent over G, has a solution in G.

Hence, G is a countable existentially closed group and it contains an isomorphic copy of H which is countable.

Case 3. Let H be an uncountably infinite group.

This case is similar to the previous case. Instead of starting with a countable group, we start with an arbitrary group.

Let Σ be the set of all finite equations and inequations with coefficients in H as we did in the previous theorem. So, the cardinality of Σ is uncountable. We can order the elements of Σ as;

$$\Sigma = \{S_\alpha \mid 1 \leq \alpha < \beta\}$$

where β is a cardinal.

Now, we will construct a group $H^\star \supseteq H$ such that for every finite system of equations and inequations in Σ is consistent over H has a solution in $H^\star$.

We start with $H_1 = H$. Let $\gamma \geq 1$ is an ordinal. Assume that we have constructed H_γ such that all the consistent system of finitely many equations and inequations with coefficients in H, say $\{S_\alpha \mid 1 \leq \alpha < \gamma + 1\}$ have a solution in H_γ. Then, similar to Case 2 we define;

$$H_{\gamma+1} = \langle \varphi(H_\gamma), k_1, k_2, \ldots, k_t \rangle$$

Observe, $H_{\gamma+1}$ contain an isomorphic copy of H and also H_γ.

If γ is a limit ordinal, then define $H_\gamma = \bigcup_{\alpha<\gamma} H_\alpha$

Let

$$H^\star = \bigcup_{\alpha<\beta} H_\alpha$$

Then, every consistent system in Σ with coefficients in H has a solution in $H^\star$.

Next, we try to construct an existentially closed group G as we did in Case 2.

Let $G_1 = H$ and $G_2 = H^\star$. Consider the set of all finitely many equations and inequations with coefficients from $H^\star$ and obtain a new group

$$G_2^\star = G_3$$

such that all the consistent systems have a solution in it. If i is a successor ordinal then

$$G_{i+1} = G_i^\star$$

but if i is a limit ordinal, then

$$G_i^\star = \bigcup_{j<i} G_j$$

So, we continue like this and define G as follows;

$$G = \bigcup_{i<\beta} G_i$$

where β is some ordinal.

Thus, if we take any finite system of equations and inequations with coefficients from G, we obtain that the system has a solution in G as in the previous case. In other words, G is existentially closed. □

The following corollary is obvious.

Corollary 3.2.11. *Every countable group can be embedded in a countable existentially closed group.*

3.3 Some Remarks on Existentially Closed Groups

The following lemma is from the book of G. Higman and E. Scott, [5, p. 3].

Lemma 3.3.1. [5, Lemma 1.3] *Let g be an element of existentially closed group $G^\star$ and let $\Sigma = \{x_1^{-1}gx_1 = g^2,\ x_2^{-1}g^2x_2 = g^2,\ x_2^{-1}gx_2 \neq g\}$ be a set of equations and inequations with variables x_1 and x_2. Then g has infinite order if and only if Σ has a solution in $G^\star$.*

Proof. Let $G^\star$ be existentially closed group. Suppose that $g \in G^\star$ has an infinite order. Then, the subgroup generated by g is an infinite cyclic group and it is isomorphic to the subgroup generated by g^2 since the order of g^2 is infinite. i.e. $\langle g\rangle \cong \langle g^2\rangle$. Now, we are able to form the HNN extension of $G^\star$ with isomorphism $\theta : \langle g\rangle \to \langle g^2\rangle$. So, we have the following group,

$$H = \langle G^\star, t \mid t^{-1}gt = \theta(g) = g^2\rangle$$

Then, we try to form the HNN extention of H with the identity isomorphism on $\langle g^2\rangle$ and obtain a new group;

$$\hat{H} = \langle H, s \mid s^{-1}g^2s = g^2 \quad \rangle \geq G^\star$$

Moreover, by Britton's theorem which is reviewed in [5, p. 3] that is

"Let G be a group and A, B be subgroups of G. If $\theta : A \to B$ is an isomorphism, then $H = \langle G, t \mid t^{-1}at = \theta(a)\ a \in A\rangle$ is the HNN extension of G. If $g_0t^{\epsilon_1}g_1 \dots t^{\epsilon_n}g_n$ where $\epsilon_i \in \{-1, 1\}$ and $g_j \in G$ with $1 \leq i \leq n,\ 0 \leq j \leq n$, then for some $q \in \{1, 2, \dots, n\}$ either

- $\epsilon_q = -\epsilon_{q+1} = -1$ and $g_q \in A$
- $\epsilon_q = -\epsilon_{q+1} = 1$ and $g_q \in B$"

By Britton's lemma, $g^{-1}s^{-1}gs = 1$ implies that g must be contained in $\langle g^2\ \rangle$, but since $g \notin \langle g^2\ \rangle$, we get a contradiction that is $g^{-1}s^{-1}gs \neq 1$. Thus, in $\hat{H}$ we have an inequation,

$$s^{-1}gs \neq g.$$

If we consider the system of equations and inequation

$$\Sigma = \{x_1^{-1}gx_1 = g^2,\ x_2^{-1}g^2x_2 = g^2,\ x_2^{-1}gx_2 \neq g\}$$

with variables x_1 and x_2, then one can easily see that the system Σ has a solution in $\hat{H}$, namely $x_1 = t$ and $x_2 = s$. Since $G^\star$ is existentially closed and so by definition, Σ has a solution in $G^\star$.

For the other side, suppose that order of g is finite and Σ has a solution in $G^\star$. Then, we have

$$t^{-1}gt = g^2$$

implying that the order of g and g^2 are the same.

Claim. *If $g \in G$ is an element of order m where $m \in \mathbb{N}$, then m is an odd number.*

Suppose that the order of g is an even number. i.e. $m = 2k$ for some $k \in \mathbb{Z}$, then one can observe that $(g^2)^k = 1$. So we obtain $\mid g^2 \mid = k$ and $m = k$ which is a contradiction. This contradiction leads us that the order of g must be an odd number, namely $g^{2k-1} = 1$ and shows us that for some integer k

$$(g^2)^k g^{-1} = 1 \quad \Longrightarrow \quad (g^2)^k = g$$

On the other hand, if we consider the equation $s^{-1}g^2s = g^2$ in Σ we get

$$s^{-1}\underbrace{(g^2)^k}_{g}s = \underbrace{(g^2)^k}_{g}$$

This is impossible since $s^{-1}gs \neq g$. So, Σ has no solution if g has finite order.

Hence, we proved that g has an infinite order if and only if there is a finite set of equations and inequations that has a solution in $G^\star$. □

Theorem 3.3.2. *An existentially closed group is infinite.*

Proof. Suppose that G is an existentially closed group of order n where $1 < n \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $k > n$ and $(k, n) = 1$. Then, the equation

$$x^k = 1$$

has no solution in G.

Observe that, $x^k = 1$ has a solution over G, namely

$$G \times C_k$$

where C_k is a cyclic group of order k. But since G is an existentially closed group, it has a solution in G which is a contradiction.

Hence, existentially closed groups are infinite. □

Theorem 3.3.3. *Every existentially closed group is simple.*

Proof. Let G be an existentially closed group and let a, g be two nontrivial elements of G. To prove that G is simple, we only need to show $G \leq \langle a \rangle^G$. Consider the free product of G and $\langle x \rangle$,

$$G \star \langle x \rangle$$

Now, in the free product the elements

$$axa^{-1}x^{-1},\ gxa^{-1}x^{-1}$$

have infinite order since these elements starts with an element from G and ends with an element in $\langle x \rangle$. Thus, we have;

$$\langle axa^{-1}x^{-1} \rangle \cong \langle gxa^{-1}x^{-1} \rangle$$

So, we can form a HNN extention

$$H = \langle G \star \langle x \rangle, t \mid taxa^{-1}x^{-1}t^{-1} = gxa^{-1}x^{-1} \rangle$$

If we regard t and x as variables, we get an equation with coefficients from G. Moreover this equation has a solution in $H \supset G$ and as G is existentially closed has a solution in G. Let g_1, g_2 be the solution and substitute $t = g_1$ and $x = g_2$. So, we have;

$$g_1 a g_2 a^{-1} g_2^{-1} g_1^{-1} = g g_2 a^{-1} g_2^{-1}$$

Then, solve this for g

$$g_1 a g_2 a^{-1} g_2^{-1} a^{-1} g_2 a g_2^{-1} = g$$

$$g_1 a \underbrace{g_2 a^{-1} g_2^{-1}}_{\in \langle a \rangle^G} g_1^{-1} g_2 a g_2^{-1} = g$$

$$g_1 \underbrace{a g_2 a^{-1} g_2^{-1}}_{\in \langle a \rangle^G} g_1^{-1} g_2 a g_2^{-1} = g$$

$$\underbrace{g_1 a g_2 a^{-1} g_2^{-1} g_1^{-1}}_{\in \langle a \rangle^G} \underbrace{g_2 a g_2^{-1}}_{\in \langle a \rangle^G} = g$$

implying that g is contained in $\langle a \rangle^G$. Since g is an arbitrary element of G, we obtain $G \leq \langle a \rangle^G$.

Hence $G = \langle a \rangle^G$. i.e. G is a simple group. □

B. H. Neumann proved the following theorem which has a quite interesting argument in the proof. .

Theorem 3.3.4. [14, Theorem 1.3] *The number of pairwise non-isomorphic countable existentially closed groups is* $2^{\aleph_0}$.

Proof. Using Theorem 2.2.4 and Corollary 3.2.11 , we understand that every 2-generator group can be embedded in a countable existentially closed group. Suppose that there are κ-many non-isomorphic countable existentially closed groups for a cardinal κ. Since each of these can only contain countably many 2-generator groups as a subgroup, we see that the number of non-isomorphic 2-generator groups which embed into these existentially closed groups is κ. But by Theorem 2.4.34, there are $2^{\aleph_0}$-many non-isomorphic 2-generator groups. Thus $\kappa = 2^{\aleph_0}$. □

In the same article of Neumann [14], he proved an existentially closed group cannot be finitely generated nor finitely related.

Theorem 3.3.5. [14, Theorem 2.1] *If G is an existentially closed group, then*

(i) G is not finitely generated.

(ii) G is not finitely related.

Proof. Let G be an existentially closed group.

(i) First we show that G is not finitely generated.

Let K be a finitely generated subgroup of G, say

$$K = \langle g_1, g_2, \ldots, g_n \rangle$$

where $g_1, g_2, \ldots, g_n$ in G.

We will prove that every finitely generated subgroup of G has a nontrivial centralizer.

Consider the equations and inequations

$$y^{-1}g_1y = g_1,\ y^{-1}g_2y = g_2,\ \ldots,\ y^{-1}g_ny = g_n\ \&\ y \neq 1$$

This system has a solution over G, namely,

$$G \times \langle y \rangle$$

and has a solution in G since G is an existentially closed group. So, there exists $z \in G$ such that

$$z^{-1}g_iz = g_i \quad \text{where } i = 1, 2, \ldots, n \quad \& \quad z \neq 1$$

So,

$$C_G(K) \neq 1$$

Thus, every finitely generated subgroup of an existentially closed group has a nontrivial centralizer.

If G is finitely generated it has a nontrivial center. Since center is always a normal subgroup and by Theorem 3.3.3 we know that every existentially closed group is simple which leads us a contradiction. Hence G cannot be finitely generated.

(ii) Now, we show that G is not finitely related.

Suppose that G is a finitely related existentially closed group. By (i), G has infinitely many generators. So,

$$G = \langle g_1, g_2, \ldots \mid r_1 = 1, r_2 = 1 \ldots r_m = 1 \rangle$$

Observe, infinitely many of these generators of G cannot be seen in any of the defining relations. Let S be the finite subset of G such that the elements which appear in the above finitely many relations and L be the set such that $L = G \setminus S$. So the elements in L generate freely a free group, say,

$$F_L < G$$

On the other hand, consider the subgroup of G which is generated by the generators which occur in the defining relations and call this subgroup as

$$R_S < G$$

Then, it is easy to see that

$$G = \langle F_L, R_S \rangle$$

and by Theorem 2.1.2 we get

$$G = F_L \star R_S$$

Next, we take any two generators from F_L, say g_1, g_2. Consider the free group they generate

$$K = \langle g_1, g_2 \rangle$$

In F_L every element has a unique expression. So, $kz = zk$ is impossible by unique writing in the free group. Thus, $C_{F_L}(K) = 1$. Since G is the free product of F_L and R_S, we get $C_G(K) = 1$. This is a contradiction because by (i) we know that $C_G(K) \neq 1$

Hence, G is not finitely related. □

Definition 3.3.6. [10, p. 229] *A group H is residually embeddable in G if $1 \neq w \in H$, there is an embedding $\phi : H \hookrightarrow G$ such that $\phi(w) \neq 1$.*

Theorem 3.3.7. [10, p. 229] *Every finitely presented group is residually embeddable into an existentially closed group.*

Proof. Let G be an existentially closed group and H be a finitely presented group such that

$$H = \langle h_1, h_2, \ldots, h_n \mid r_1 = 1, r_2 = 1, \ldots, r_k = 1 \rangle$$

Then, consider the finite system of equations and inequations where $x_1, x_2, \ldots, x_n$ are variables and $r_1, r_2, \ldots, r_k$ are relations in G

$$\begin{aligned} & r_i(x_1, x_2, \ldots, x_n) = 1 \quad i \in \{1, 2, \ldots, k\} \\ & w(x_1, x_2, \ldots, x_n) \neq 1 \\ & w \neq 1, \quad w \in H \end{aligned}$$

This system has a solution in $G \times H$ but G is existentially closed so it has a solution in G, namely

$$g_1, g_2, \ldots, g_n$$

such that

$$r_i(g_1, g_2, \ldots, g_n) = 1 \quad i \in \{1, 2, \ldots, k\}$$
$$w(g_1, g_2, \ldots, g_n) \neq 1$$
$$w \neq 1 \quad w \in H$$

We need to use Von Dyck's theorem,

Theorem. [15, p. 51] *Let G_1 and G_2 be groups with presentations $\pi_1 : F \to G_1$ and $\pi_2 : F \to G_2$ such that each relator of π_1 is also a relator of π_2 ($Ker\pi_1 \leq Ker\pi_2$). Then the function from G_1 to G_2 is a well-defined epimorphism.*

So by Von Dyck's theorem, there exist a homomorphism $\phi : H \hookrightarrow G$ such that

- $\phi(h_i) = g_i$
- $\phi(w) = w(g_1, g_2, \ldots, g_n) \neq 1$
- All the defining relations in H goes to identity by ϕ.

Therefore, H is residually embeddable into an existentially closed group G. □

A notation of an absolute presentation was introduced by B. H. Neumann [14] when he was working on the isomorphism problem for existentially closed groups.

Definition 3.3.8. *Let G be a group with a set V of generators together with a set R of relations and a set S of irrelations among those generators. The group G is said to be **absolutely presented** group if the following holds*

- *G has a presentation of the form $G = \langle V \mid R \rangle$*
- *If $\phi : G \to H$ is a homomorphism such that the set of irrelations S are satisfied in $\phi(G)$, then $G \cong \phi(G)$*

Theorem 3.3.9. [14, Theorem 2.1] *A finitely absolutely presented group can be embedded in an existentially closed group.*

Proof. Let H be a finitely absolutely presented group

$$H = \langle h_1, h_2, \ldots, h_k \mid \ r_i(h_1, h_2, \ldots h_k) = 1, s_j(h_1, h_2, \ldots h_k) \neq 1 \rangle$$

where $1 \leq i \leq m$ and $1 \leq j \leq n$. Let G be an existentially closed group.

Consider the system of equations and inequations, where $x_1, x_2, \ldots, x_k$ are the variables, r_i are the relations and s_j are the irrelations in G

$$r_1(x_1, x_2, \ldots x_k) = 1, r_2(x_1, x_2, \ldots x_k) = 1, \ldots, r_m(x_1, x_2, \ldots x_k) = 1 \tag{3.7}$$

$$s_1(x_1, x_2, \ldots x_k) \neq 1, s_2(x_1, x_2, \ldots x_k) \neq 1, \ldots, s_n(x_1, x_2, \ldots x_k) \neq 1 \tag{3.8}$$

This system has a solution in $G \times H$, but G is existentially closed. So, it has a solution in G which shows the finite set of equations (2.7) and inequations (2.8) is consistent over H. Then, there exist a homomorphism

$$\varphi : H \hookrightarrow G$$

such that

- $r_i \in Ker(\varphi) \quad \forall i = 1, 2, \ldots, m$
- $s_j \notin Ker(\varphi) \quad j = 1, 2, \ldots, n$

H is a finitely absolutely presented group, so the set of irrelations (2.8) in H are satisfied in $\varphi(H) < G$ and

$$H \cong \varphi(H) < G$$

Hence, a finitely absolutely presented group can be embedded in an existentially closed group. □

Lemma 3.3.10. [14, Lemma 2.5] *Every existentially closed group contains cyclic groups of every order.*

Proof. We know that finite groups are contained in existentially closed groups by the first case of the proof of the Theorem 3.2.10. Thus, cyclic groups of finite order are contained in existentially closed groups. For an infinite cyclic group, we first let $G^\star$ be

an existentially closed group. Then, consider the system of equations and inequations where x_1, x_2, x_3, x_4 are the variables

$$x_1^{x_2} = x_1^2, x_2^{x_3} = x_2^2, x_3^{x_4} = x_3^2, x_4^{x_1} = x_4^2, x_1 \neq 1$$

This system has a solution in the group G which is constructed by G. Higman in 1951 [3] that is

$$G = \langle a_1, a_2, a_3, a_4 \mid a_1^{a_2} = a_1^2, a_2^{a_3} = a_2^2, a_3^{a_4} = a_3^2, a_4^{a_1} = a_4^2, a_1 \neq 1 \rangle$$

and G has no normal subgroup of finite index. This system has a solution in

$$G \times G^\star$$

but $G^\star$ is existentially closed so it has a solution in $G^\star$. Next, consider an embedding from G to $G^\star$,

$$\phi : G \hookrightarrow G^\star$$

In the proof of G not having finite index normal subgroups, we showed that if one of the $\phi(x_i)$ is finite, then $\phi(G)$ will be finite. Since $\phi(x_1) \neq 1$ we obtain $\phi(x_i)$ is an element of $G^\star$ and $\phi(x_i)$ has infinite order in $G^\star$. □

Corollary 3.3.11. [14, Corollary 2.4] *Existentially closed groups cannot be periodic.*

Proof. By Theorem 3.3.10, infinite cyclic groups are contained in existentially closed groups. So, there are elements of infinite order. Thus, existentially closed group cannot be periodic. □

Lemma 3.3.12. *Every existentially closed group is divisible.*

Proof. Let G be an existentially closed group. Let g be an element of G and $n \in \mathbb{N}$. By Lemma 3.3.10, we know that cyclic group of any order is contained in G. Consider the subgroup of G generated by the element g.

- If g is infinite, then take $\langle c \rangle$ be an infinite cyclic group and so $\langle c^n \rangle \cong \langle g \rangle$.
- If g is finite of order m, then $\langle c \rangle$ is a cyclic group of order mn, and so $\langle c^n \rangle \cong \langle g \rangle$

Now, consider the amalgamated free product of $G \star \langle c \rangle$ with amalgamated subgroup $\langle c^n \rangle \cong \langle g \rangle$ and call this product as H. Then, the system of equations and inequations

$$\Sigma = \{x^{-n}g = 1,\ x^{-(n-1)}g \neq 1,\ x^{-(n-2)}g \neq 1, \ldots, x^{-1}g \neq 1\}$$

has a solution in H. Since G is existentially closed the system Σ has a solution in G. i.e. There exists an element $c \in G$ such that $c^n = g$.

Hence, G is a divisible group. $\square$

CHAPTER 4

GENERALIZATIONS

4.1 κ-Existentially Closed Groups

In the definition of an existentially closed group we have only finitely many equations and inequations. We may generalize the definition of an existentially closed group to κ-existentially closed group for any infinite cardinal κ.

Definition 4.1.1. *Let κ be an infinite cardinal. A group G is called* ***κ-existentially closed*** *if for every set of equations and inequations which has fewer than κ constants and variables has a solution over G, has a solution in G.*

The generalization of existentially closed groups are mentioned in the paper of Scott and also in Macintyre's paper [11]. Observe that, an existentially closed group which is exactly Scott's definition of algebraically closed group [16] is the same as $\aleph_0$-existentially closed group. In Chapter 3, we showed some properties of existentially closed groups but we didn't mention any existence, uniqueness or structure of them.

For κ- existentially closed groups, in 2017 Otto H. Kegel and Mahmut Kuzucuoğlu published an article [8] in which uniqueness of these groups are shown, provided that they exist. Their proof is quite interesting but needs some set theoretical background. Moreover, in 2018, Burak Kaya, Otto H. Kegel and Mahmut Kuzucuoğlu [7] proved that there exist a κ- existentially closed group of cardinality κ for each regular cardinal κ and no κ- existentially closed group of cardinality κ for singular cardinal κ.

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