EXTREMAL PROBLEMS AND BERGMAN PROJECTIONS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

EXTREMAL PROBLEMS AND BERGMAN PROJECTIONS

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Studying extremal problems on Bergman spaces is rather new and techniques used are usually specific to the problem to be solved. However, a 2014 paper by T. Ferguson developed a systematic method using Bergman projections for solving extremal problems on Bergman spaces A^p on the unit disc with $1 < p < \infty$.

We extended this method to weighted Bergman spaces A^p_α and in some special cases, to extremal problems defined by linear functionals of evaluations at points in the disc other than the origin. We computed the kernels of several such functionals. We also computed the Bergman projections of some functions related to these kernels. Using these projections, we solved a few extremal problems explicitly. Our results have the potential to be extended to the case $p = 1$ and to more general spaces.

We also gave a proof of the existence of the solutions to the extremal problems on the Bergman spaces A^1_α defined by functionals with kernels that extend to the closed disc continuously.

Keywords: Extremal problems, weighted Bergman spaces, Bergman projections.

ÖZET

EKSTREMAL PROBLEMLER VE BERGMAN İZDÜŞÜMLERİ

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Bergman uzayları üzerinde ekstremal problemleri çalışmak oldukça yenidir ve kullanılan teknikler genellikle çözülecek probleme özeldir. Bununla birlikte, T. Ferguson'un 2014 makalesi, Bergman izdüşümleri kullanarak, $1 < p < \infty$ için birim dairedeki A^p Bergman uzaylarında ekstremal problemleri çözmek için sistematik bir yöntem geliştirmiştir.

Biz bu yöntemi A^p_α ağırlıklı Bergman uzaylarına ve bazı özel durumlarda, dairenin 0 dışında noktalarındaki değerleme fonksiyoneli ile tanımlanan ekstremal problemlere genişlettik. Bu türden bir çok fonksiyonelin çekirdeğini hesapladık. Bu çekirdeklerle bağlantılı bazı fonksiyonların Bergman izdüşümlerini de hesapladık. Bu izdüşümleri kullanarak birkaç ekstremal problemi açık olarak çözdük. Sonuçlarımızın $p = 1$ durumuna ve daha genel uzaylara genişletilebilme potansiyeli vardır.

Ayrıca, A^1_α Bergman uzayları üzerinde, çekirdeği kapalı daireye sürekli olarak genişleyen fonksiyonellerle tanımlanan ekstremal problemlerin çözümlerinin varlığının bir ispatını verdik.

Anahtar sözcükler: Ekstremal problemler, ağırlıklı Bergman uzayları, Bergman izdüşümleri.

Contents

Chapter 1

Introduction

Extremal problems in complex analysis have been studied for a long time, and the first result in this field can be considered as the Schwarz lemma. After some initial work on best approximation problems on general Banach spaces, the research shifted to linear problems on specific spaces such as the Hardy spaces. Problems on these spaces are usually better understood, because functions in these spaces have nice properties such as they have boundary values and they can be factored using Blaschke products.

More recently, extremal problems have been considered also on Bergman spaces. Problems on these spaces are usually harder to solve, but far more important from one point of view. The best methods to investigate invariant subspaces and zero sets of Bergman spaces pass through solving extremal problems; see many chapters in [5].

In this thesis, we deal with linear extremal problems on weighted Bergman spaces. So bounded linear functionals, their representations, and thus duality will play important roles. Hence reflexive spaces are easier to deal with, and many sources consider only them as we will also do. The problems we consider will be one of two equivalent types: Either find a function with minimum norm among those functions on which a bounded linear functional is constant, or find a function on which a bounded linear functional is a maximum among those functions with constant norm.

There are no general methods for solving extremal problems on Bergman spaces, and few problems have explicit solutions. However, on Bergman spaces there are the Bergman projections that are very explicit with very useful properties. The paper [4] uses this idea to treat extremal problems on Bergman spaces systematically. In this method, representing given bounded linear functionals with explicit kernels, and then computing Bergman projections of functions become important tools.

We let $\mathbb{C}$ be the complex plane, and for $z = x + iy = re^{i\theta} \in \mathbb{C}$, let $|z| = r = \sqrt{x^2 + y^2}$. We define the open unit disc in $\mathbb{C}$ by

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

Its boundary $\partial\mathbb{D}$ is the unit circle given by $|z| = 1$. Let $\mathbb{D}(z, r)$ be the open disc of radius r centered at z. We denote the normalized area measure on $\mathbb{D}$ by

$$d\mu(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta$$

in rectangular and polar coordinates, respectively. For $\alpha > -1$, we define

$$d\mu_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha\, d\mu(z);$$

These measures are finite and all have mass 1. For $0 < p < \infty$, we let L^p_α be the Lebesgue space corresponding to μ_α. For $p = \infty$, the space of essentially bounded measurable functions with respect to any μ_α is the same and we denote it by L^∞.

Definition 1.0.1. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of all analytic functions f on $\mathbb{D}$ for which

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p\, d\mu_\alpha \right)^{1/p}$$

is finite.

Note that $\|f\|_{A^p_\alpha}$ is a true norm if $1 \le p < \infty$ and has the properties
(i) $\|f\|_{A^p_\alpha} = 0$ if and only if $f = 0$,
(ii) $\|cf\|_{A^p_\alpha} = |c|\|f\|_{A^p_\alpha}$ for $c \in \mathbb{C}$,
(iii) $\|f + g\|_{A^p_\alpha} \le \|f\|_{A^p_\alpha} + \|g\|_{A^p_\alpha}$ (triangle inequality).
For $0 < p < 1$, the triangle inequality is replaced by the inequality
$(iii)'$ $\|f + g\|^p_{A^p_\alpha} \le \|f\|^p_{A^p_\alpha} + \|g\|^p_{A^p_\alpha}$.

For $1 < p < \infty$, the dual space of A^p_α can be identified with A^q_α, where $1/p + 1/q = 1$. Each bounded linear functional $\Phi \in (A^p_\alpha)^*$ has a unique representation

$$\Phi(f) = \int_{\mathbb{D}} f\overline{g}\, d\mu_\alpha, \qquad f \in A^p_\alpha,$$

for some $g \in A^q_\alpha$, where the overbar denotes complex conjugation.

For $1 \leq p < \infty$, $\alpha > -1$, and a given bounded linear functional $\Phi \in (A^p_\alpha)^*$, we investigate the following extremal problems.
(1) Find $F \in A^p_\alpha$ with $\|F\|_{A^p_\alpha} = 1$ such that

$$\Phi(F) = \sup\{\, \Phi(G) \,:\, \|G\|_{A^p_\alpha} = 1,\ \Phi(G) > 0 \,\} = \|\Phi\|.$$

(2) Find $f \in A^p_\alpha$ with $\Phi(f) = 1$ such that

$$\|f\|_{A^p_\alpha} = \inf\{\, \|g\|_{A^p_\alpha} \,:\, g \in A^p_\alpha,\ \Phi(g) = 1 \,\}.$$

We call F an extremal function if F solves the problem (1) for the functional Φ. Also, these two problems are related to each other. If F solves problem (1), then $F/\Phi(F)$ solves problem (2); and if f solves problem (2), then $f/\|f\|_{A^p_\alpha}$ solves problem (1).

In this thesis, we extend the method of [4] to weighted Bergman spaces, and also to extremal problems defined by linear functionals of evaluations at points different from the origin in a few instances. In Chapter 2, we review Bergman spaces and Bergman projections, and also the Bloch space that is the dual of A^1_α. In Chapter 3, we study the basics and general properties of extremal problems. In Chapter 4, we compute the kernels of various bounded linear functionals and Bergman projections of certain functions related to them. In Chapter 5, we solve a few extremal problems. In Chapter 6, we summarize our work and indicate some new problems for future work.

Chapter 2

Bergman Spaces and Projections

2.1 Weighted Bergman Spaces

In this chapter we collect the necessary basic information on weighted Bergman spaces. Firstly, note that in a weighted Bergman space, functions can not grow too fast near the boundary.

Theorem 2.1.1. *Suppose $0 < p < \infty$, and K is a compact subset of $\mathbb{D}$. Then for any function $f \in A^p_\alpha$, we have*

$$|f(z)| \leq C\|f\|_{A^p_\alpha} \qquad z \in K,$$

for some positive constant $C = C(K, p, \alpha)$.

Proof. There is an $R \in (0,1)$ such that $K \subset \mathbb{D}(0,R)$. By the subharmonicity of $|f|^p$,

$$|f(z)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z+re^{i\theta})|^p \, d\theta, \qquad z \in K,\ 0 < r < 1-R.$$

For $w \in \mathbb{D}(z, \frac{1-R}{2})$ we have $|w-z| < \frac{1-R}{2}$, so $|w| - |z| < |w-z| < \frac{1-R}{2}$, then $|w| < \frac{1+R}{2}$ since $|z| < R$, and hence $1 - |w| > \frac{1-R}{2} > 0$. Also $|z| - |w| < |w-z| < \frac{1-R}{2}$, so $-|w| < \frac{1-R}{2}$, and hence $1 - |w| < \frac{3-R}{2} < \infty$. Thus $0 < C_1 < 1 - |w| < (1-|w|)(1+|w|) = 1-|w|^2 < 2(1-|w|) < C_2 < \infty$, that is, $C_3 < (1-|w|^2)^\alpha < C_4$

since $-1 < \alpha < \infty$. Hence

$$\int_0^{\frac{1-R}{2}} |f(z)|^p r\, dr \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\frac{1-R}{2}} |f(z + re^{i\theta})|^p r\, dr\, d\theta.$$

Writing $w = z + re^{i\theta}$,

$$\begin{aligned} |f(z)|^p &\leq \frac{4}{\pi(1-R)^2} \int_{\mathbb{D}(z, \frac{1-R}{2})} |f(w)|^p\, d\mu(w) \\ &= \frac{4C}{\pi(1-R)^2} \int_{\mathbb{D}(z, \frac{1-R}{2})} |f(w)|^p (1 - |w|^2)^\alpha\, d\mu(w) \\ &\leq C \int_{\mathbb{D}(z, \frac{1-R}{2})} |f(w)|^p\, d\mu_\alpha(w) \leq C \int_{\mathbb{D}} |f(w)|^p\, d\mu_\alpha(w). \end{aligned}$$

□

Corollary 2.1.2. *For each $z \in \mathbb{D}$, the point evaluation $\Phi(f) = f(z)$ is a bounded linear functional on the weighted Bergman space $A^p_\alpha(\mathbb{D})$.*

Corollary 2.1.3. *If K is a compact subset of $\mathbb{D}$, f_n and f are in A^p_α and $\|f_n - f\|_{A^p_\alpha} \to 0$ as $n \to \infty$, then $f_n \to f$ uniformly on K; that is, norm convergence implies local uniform convergence.*

Theorem 2.1.4. *A^p_α is complete, that is, every Cauchy sequence in A^p_α converges in norm to some function in A^p_α.*

Proof. Since L^p_α is complete where $A^p_\alpha \subset L^p_\alpha$ it is enough to show that A^p_α is a closed subspace. Let $f_n \in A^p_\alpha$ and $\{f_n\}$ converge in norm to a function $f \in L^p_\alpha$. Then some subsequence $\{f_{n_k}(z)\}$ converges to $f(z)$ almost everywhere in $\mathbb{D}$. Note that $\{f_n\}$ is a Cauchy sequence in norm, hence a locally uniform Cauchy sequence by Corollary 2.1.3, so it converges locally uniformly to a function g that is analytic in $\mathbb{D}$. Hence, $f(z) = g(z)$ almost everywhere in $\mathbb{D}$, and the space A^p_α is closed and complete. □

So A^p_α is a Banach space for $p \geq 1$, and for $p = 2$, A^2_α is a Hilbert space with the inner product

$$\langle f, g \rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}\, d\mu_\alpha(z).$$

Now let $p = 2$. Since each point evaluation functional $\Phi(f) = f(z)$ is bounded, the Riesz representation theorem for Hilbert spaces guarantees the existence of a unique function $k_{\alpha,z}$ in A^2_α such that $f(z) = \langle f, k_{\alpha,z} \rangle_\alpha$ for every $f \in A^2_\alpha$. The function $K_\alpha(z,\zeta) = \overline{k_{\alpha,z}(\zeta)}$ is called the reproducing kernel of A^2_α. It has the explicit formula

$$K_\alpha(z,\zeta) = \frac{1}{(1-z\overline{\zeta})^{2+\alpha}}, \qquad z,\zeta \in \mathbb{D},$$

and the reproducing property

$$f(z) = \int_{\mathbb{D}} f(\zeta) K_\alpha(z,\zeta)\, d\mu_\alpha(\zeta), \qquad z \in \mathbb{D},\ f \in A^2_\alpha. \tag{2.1.1}$$

Taking $f(\zeta) = K_\alpha(w,\zeta)$ for some $w \in \mathbb{D}$, we obtain

$$\overline{K_\alpha(w,z)} = \int_{\mathbb{D}} \overline{K_\alpha(w,\zeta)} K(z,\zeta)\, d\mu_\alpha(\zeta) = K_\alpha(z,w).$$

Thus the kernel function has the symmetry property $K_\alpha(z,\zeta) = \overline{K_\alpha(\zeta,z)}$. So, $K_\alpha(z,\zeta)$ is analytic in z and conjugate analytic in ζ. Also,

$$K_\alpha(z,z) = \int_{\mathbb{D}} |K_\alpha(z,\zeta)|^2\, d\mu_\alpha(\zeta) = \|K_\alpha(z,\cdot)\|^2_{A^2_\alpha} > 0. \tag{2.1.2}$$

Next let P_α denote the orthogonal projection of L^2_α onto A^2_α since A^2_α is a closed subspace of the Hilbert space L^2_α. We have $P_\alpha f \in A^2_\alpha$ for the function $f \in L^2_\alpha$. So

$$P_\alpha f(z) = \langle P_\alpha f, k_{\alpha,z}\rangle_\alpha = \langle f, k_{\alpha,z}\rangle_\alpha = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-\overline{\zeta}z)^{2+\alpha}}\, d\mu_\alpha(\zeta),$$

where the second equality holds because $f - P_\alpha f$ is orthogonal to $k_{z,\alpha} \in A^2_\alpha$. Thus, the weighted Bergman projection P_α has the formula

$$P_\alpha f(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-\overline{\zeta}z)^{2+\alpha}}\, d\mu_\alpha(\zeta), \qquad f \in L^2_\alpha.$$

The next result is well-known and can be found in [5, Proposition 1.3].

Theorem 2.1.5. *Polynomials are dense in every A^p_α.*

Remark 2.1.6. Since $d\mu_\alpha$ is a finite measure, $A^p_\alpha \subset A^1_\alpha$ for $1 < p < \infty$; in particular $A^2_\alpha \subset A^1_\alpha$. But also A^2_α is dense in A^1_α by Theorem 2.1.5. Then for each $z \in \mathbb{D}$, (2.1.1) holds for all $f \in A^1_\alpha$, and hence for all $f \in A^p_\alpha$ with $1 \le p < \infty$.

Next we will give necessary and sufficient conditions for the boundedness of weighted Bergman projections. The next two technical results are taken from [5, Theorems 1.7 and 1.9].

Lemma 2.1.7. *Let s and t be real numbers satisfying $1 < t < s$. Then there is a constant C, depending only on s and t, such that*

$$\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{t-2}}{|1-\overline{z}\zeta|^s}\, d\mu(\zeta) \leq C(1-|z|^2)^{t-s}, \qquad z \in \mathbb{D}.$$

Theorem 2.1.8. *Let the integral operators T and S be defined by*

$$Tf(z) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{(1-z\bar{w})^{2+a+b}} f(w)\, d\mu(w)$$

and

$$Sf(z) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-z\bar{w}|^{2+a+b}} f(w)\, d\mu(w),$$

where a, b, and c are real numbers. Then for $1 \leq p < \infty$, the following conditions are equivalent:
(i) T is bounded on L^p_c.
(ii) S is bounded on L^p_c.
(iii) $-pa < c+1 < p(b+1)$.

Theorem 2.1.9. *Let $\alpha, \beta > -1$ and $1 \leq p < \infty$. Then P_β is a bounded operator from L^p_α onto A^p_α if and only if $\alpha + 1 < (\beta+1)p$. Further, if $f \in A^p_\alpha$, then $P_\beta f = f$.*

Note that if $1 < p < \infty$, then we can take $\beta = \alpha$ and $P_\alpha : L^p_\alpha \to A^p_\alpha$ is bounded. However, when $p = 1$, we need to take $\beta > \alpha$ to obtain a bounded map $P_\beta : L^1_\alpha \to A^1_\alpha$, because $P_\alpha : L^1_\alpha \to A^1_\alpha$ is not bounded.

The following result is [5, Proposition 1.11].

Corollary 2.1.10. *Let $\alpha > -1$, $1 \leq p < \infty$, and n be a positive integer. An analytic function f on $\mathbb{D}$ belongs to A^p_α if and only if the function $(1-|z|^2)^n f^{(n)}(z)$ is in L^p_α.*

Definition 2.1.11. The dual space of a Banach space X is the space X^* of all bounded linear functionals $\Phi : X \to \mathbb{C}$. Each functional Φ in the dual space has the norm

$$\|\Phi\| = \sup_{\|x\|=1} |\Phi(x)|,$$

under which X^* is itself a Banach space.

By the Riesz representation theorem, the dual space of L^p_α for $1 < p < \infty$ and $-1 < \alpha < \infty$ is isometrically isomorphic to L^p_α, where $1/p + 1/q = 1$. So for every $\Phi \in (L^p_\alpha)^*$, there is a unique $g \in L^q_\alpha$ such that

$$\Phi(f) = \int_{\mathbb{D}} fg\, d\mu_\alpha, \qquad f \in L^p_\alpha.$$

The same representation holds also for the functionals on weighted Bergman spaces. The following result is [5, Theorem 1.16].

Theorem 2.1.12. *For $1 < p < \infty$, the dual space of A^p_α can be identified with A^q_α, where $1/p + 1/q = 1$. To each functional $\Phi \in (A^p_\alpha)^*$, there corresponds a unique $g \in A^q_\alpha$ such that*

$$\Phi(f) = \int_{\mathbb{D}} f\,\overline{g}\, d\mu_\alpha, \qquad f \in A^p_\alpha.$$

Definition 2.1.13. If X and Y are Banach spaces and $T : X \to Y$ is a bounded linear transformation, the adjoint $T^* : Y^* \to X^*$ of T is given by $T^*(\Psi) = \Psi \circ T$ for $\Psi \in Y^*$.

In practice, if X and Y are A^p_α or L^p_α, in view of Theorem 2.1.12, the adjoint of T is found from the identity $\langle Tx, y\rangle_\alpha = \langle x, T^*y\rangle_\alpha$.

The next result is contained in [6, Theorem 5.4].

Theorem 2.1.14. *For $1 < p < \infty$ and $1/p+1/q = 1$, the adjoint of the projection $P_\alpha : L^q_\alpha \to A^q_\alpha$ is the inclusion map $i : A^p_\alpha \to L^p_\alpha$.*

Proof. We need to show that $\langle f, P_\alpha g\rangle_\alpha = \langle i(f), g\rangle_\alpha$ for all $f \in A^p_\alpha$ and $g \in A^q_\alpha$. For such f, g, we have

$$\begin{aligned}
\int_{\mathbb{D}} f\overline{P_\alpha g}\, d\mu_\alpha &= \int_{\mathbb{D}} f(z)\overline{\int_{\mathbb{D}} \frac{g(w)}{(1 - z\bar{w})^{2+\alpha}}\, d\mu_\alpha(w)}\, d\mu_\alpha(z) \\
&= \int_{\mathbb{D}} \overline{g(w)} \int_{\mathbb{D}} \frac{f(z)}{(1 - w\bar{z})^{2+\alpha}}\, d\mu_\alpha(z)\, d\mu_\alpha(w) \\
&= \int_{\mathbb{D}} \overline{g(w)} f(w)\, d\mu_\alpha(w) = \int_{\mathbb{D}} i(f)\,\overline{g}\, d\mu_\alpha
\end{aligned}$$

by the Fubini theorem and Theorem 2.1.9. To justify the use of the Fubini theorem, we write

$$\int_{\mathbb{D}} |f(z)| \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{|1 - z\bar{w}|^{2+\alpha}} |g(w)|\, d\mu(w) d\mu_\alpha(z) = \int_{\mathbb{D}} |f(z)| H(z) d\mu_\alpha(z).$$

By taking $a = 0$ and $b = c = \alpha$ in Theorem 2.1.8, we see that for $g \in L^q_\alpha$, we have $H \in L^q_\alpha$. Then

$$\int_{\mathbb{D}} |f(z)|H(z)d\mu_\alpha(z) \leq \left(\int_{\mathbb{D}} |f(z)|^p \, d\mu_\alpha(z)\right)^{1/p} \left(\int_{\mathbb{D}} H(z)^q \, d\mu_\alpha(z)\right)^{1/q}$$
$$= \|f\|_{A^p_\alpha} \|H\|_{L^q_\alpha} < \infty. \qquad \square$$

To extend the last two results to A^1_α, we need to introduce the Bloch space.

2.2 The Bloch Space

Definition 2.2.1. The Bloch space $\mathcal{B}$ consists of all analytic functions f on $\mathbb{D}$ for which the Bloch "norm"

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)|$$

is finite.

The Bloch space with the true norm $\|f\| = \|f\|_{\mathcal{B}} + |f(0)|$ is a Banach space. The result below shows that every bounded analytic function is in the Bloch space, but although the function $f(z) = \log(1 - z)$ is not bounded, it is in the Bloch space. Bloch functions belong to every weighted Bergman space A^p_α for $0 < p < \infty$ and $-1 < \alpha < \infty$ by Corollary 2.1.10. However, the converse is not true; for example, the function $g(z) = (\log(1 - z))^2$ is not a member of the Bloch space, while it is in A^p_α for every $p < \infty$

We let H^∞ denote the space of bounded holomorphic functions on $\mathbb{D}$ normed with $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$.

Proposition 2.2.2. *The space H^∞ is contained in $\mathcal{B}$ and $\|f\|_{\mathcal{B}} \leq \|f\|_\infty$ for every $f \in H^\infty$.*

Proof. Let f be analytic in $\mathbb{D}$ with $\|f\|_\infty \leq 1$. Then by the Schwarz-Pick lemma,

$$(1 - |z|^2)|f'(z)| \leq 1 - |f(z)|^2 \leq 1, \qquad z \in \mathbb{D}.$$

It follows that $H^\infty \subset \mathcal{B}$ with $\|f\|_{\mathcal{B}} \leq \|f\|_\infty$. $\square$

The weighted Bergman projection is a useful tool also on the Bloch space and identifies it as the dual of any A^1_α. For $\alpha > -1$, let's use the notation

$$D_\alpha f(z) = f(z) + \frac{1}{2+\alpha} z f'(z) \qquad \text{and} \qquad I_\alpha f(z) = \frac{2+\alpha}{1+\alpha}(1-|z|^2) D_\alpha f(z).$$

The next result is taken from [2, Theorem 2 and Corollary 13].

Theorem 2.2.3. *For any $\alpha > -1$, the map $P_\alpha : L^\infty \to \mathcal{B}$ is bounded. Moreover, if $f \in \mathcal{B}$, then $P_\alpha I_\alpha f(z) = f(z)$.*

The following result is contained in [6, Remark 7.3].

Theorem 2.2.4. *For any $\alpha > -1$, the dual of A^1_α can be identified with $\mathcal{B}$. For every $\Phi \in (A^1_\alpha)^*$, there is a unique $g \in \mathcal{B}$ such that*

$$\Phi(f) = \int_{\mathbb{D}} f\, \overline{I_\alpha g}\, d\mu_\alpha, \qquad f \in A^1_\alpha.$$

There are other ways to represent the duality $(A^1_\alpha)^* = \mathcal{B}$, but this suits our needs better.

Theorem 2.2.5. *The adjoint of the projection $P_\alpha : L^\infty \to \mathcal{B}$ under the pairing in Theorem 2.2.4 is the inclusion map $i : A^1_\alpha \to L^1_\alpha$.*

Proof. Let $f \in A^1_\alpha$ and $g \in L^\infty$; so $P_\alpha g \in \mathcal{B}$. We show $\langle f, I_\alpha P_\alpha g\rangle_\alpha = \langle i(f), g\rangle_\alpha$. By differentiation under the integral sign, Fubini theorem and Theorem 2.1.9, we obtain

$$\begin{aligned}
\int_{\mathbb{D}} f\, \overline{I_\alpha P_\alpha g}\, d\mu_\alpha &= \frac{2+\alpha}{1+\alpha} \int_{\mathbb{D}} f(z) \overline{(1-|z|^2) D_\alpha \int_{\mathbb{D}} \frac{g(w)}{(1-z\bar{w})^{2+\alpha}}\, d\mu_\alpha(w)}\, d\mu_\alpha(z) \\
&= \int_{\mathbb{D}} f(z) \int_{\mathbb{D}} \frac{\overline{g(w)}}{(1-\bar{z}w)^{3+\alpha}}\, d\mu_\alpha(w)\, d\mu_{\alpha+1}(z) \\
&= \int_{\mathbb{D}} \overline{g(w)} \int_{\mathbb{D}} \frac{f(z)}{(1-w\bar{z})^{3+\alpha}}\, d\mu_{\alpha+1}(z)\, d\mu_\alpha(w) \\
&= \int_{\mathbb{D}} \overline{g(w)} f(w) d\mu_\alpha(w) = \int_{\mathbb{D}} i(f)\, \overline{g}\, d\mu_\alpha.
\end{aligned}$$

To justify the use of the Fubini theorem, we consider

$$J = \int_{\mathbb{D}} |f(z)| \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-w\bar{z}|^{3+\alpha}} |g(w)|\, d\mu(w) d\mu_{\alpha+1}(z).$$

Now $g \in L^\infty$ and we handle the inner integral using Lemma 2.1.7 with $s = 3 + \alpha$ and $t = 2 + \alpha$. Then

$$J \leq C \int_{\mathbb{D}} |f(z)| \frac{1}{1 - |z|^2} \, d\mu_{\alpha+1}(z) \leq C \|f\|_{A^1_\alpha} < \infty.$$

$\square$

Chapter 3

Extremal Problems

3.1 Introduction

In this chapter, we will investigate the existence and uniqueness of the solutions in the weighted Bergman spaces A^p_α of the two problems (1) and (2) of Chapter 1. We will give necessary and sufficient conditions for the existence of solutions separately for $1 < p < \infty$ and $p = 1$, and also prove their uniqueness, which requires an extra slight condition on the kernels when $p = 1$ that is satisfied for all our kernels.

Let's remark that solving extremal problems on Hilbert spaces can be trivial as the following argument adapted from [3, p. 9] shows. Consider the reproducing property (2.1.1) of the Bergman kernel $K_\alpha(z, \zeta)$ on $f \in A^2$. Applying the Cauchy-Schwarz inequality to the reproducing property and using (2.1.2) yield $|f(z)| \leq \sqrt{K_\alpha(z,z)}\, \|f\|_{A^2_\alpha}$ for $f \in A^2_\alpha$. For a given $a \in \mathbb{D}$, if $f(a) = 1$, then $\|f\|_{A^2_\alpha} \geq 1/\sqrt{K_\alpha(z,z)}$. Equality occurs for $f_0(z) = K_\alpha(z,a)/K_\alpha(a,a) \in A^2_\alpha$. Defining the linear functional $\Phi : A^2_\alpha \to \mathbb{C}$ by $\Phi(f) = f(a)$, the extremal problem of finding $f \in A^2_\alpha$ with $\Phi(f) = 1$ and of minimal norm is solved by f_0. As a result, we will ignore the Hilbert spaces A^2_α in this thesis and concentrate on the case $p \neq 2$, although our methods are valid for $p = 2$ too.

Theorem 3.1.1. *The problems* (1) *and* (2) *given in Chapter* 1 *are equivalent.*

Proof. Given (1), let $f = \dfrac{F}{\Phi(F)}$ and $g = \dfrac{G}{\Phi(G)}$. Then, $\Phi(f) = \Phi(g) = 1$. Taking norms of two sides, we get

$$\|g\|_{A^p_\alpha} = \frac{\|G\|_{A^p_\alpha}}{\Phi(G)} = \frac{1}{\Phi(G)}$$

and

$$\begin{aligned}\|f\|_{A^p_\alpha} = \frac{\|F\|_{A^p_\alpha}}{\Phi(F)} &= \frac{1}{\|\Phi(F)\|} = \frac{1}{\|\Phi(F)\|} = \frac{1}{\sup_{\|G\|_{A^p_\alpha}=1} \|\Phi(G)\|} \\ &= \inf_{\|G\|_{A^p_\alpha}=1} \frac{1}{\|\Phi(G)\|} = \inf_{\Phi(g)=1} \|g\|_{A^p_\alpha}.\end{aligned}$$

So if F solves problem (1), then $\dfrac{F}{\Phi(F)}$ solves problem (2).

Given (2), let $F = \dfrac{f}{\|f\|_{A^p_\alpha}}$ and $G = \dfrac{g}{\|g\|_{A^p_\alpha}}$. Then $\|F\|_{A^p_\alpha} = \|G\|_{A^p_\alpha} = 1$. Applying the functional Φ to both sides, we get $\Phi(G) = \dfrac{\Phi(g)}{\|g\|_{A^p_\alpha}} = \dfrac{1}{\|g\|_{A^p_\alpha}} > 0$ and $\Phi(F) = \dfrac{\Phi(f)}{\|f\|_{A^p_\alpha}} = \dfrac{1}{\inf_{\Phi(g)=1} \|g\|_{A^p_\alpha}} = \sup_{\Phi(g)=1} \dfrac{1}{\|g\|_{A^p_\alpha}} = \sup_{\|G\|_{A^p_\alpha}=1} \Phi(G)$. Also if f solves problem (2), then $\dfrac{f}{\|f\|_{A^p_\alpha}}$ solves problem (1). $\square$

If f solves problem (2), then $\|f\|_{A^p_\alpha} = \dfrac{1}{\|\Phi\|}$ by the equivalence of the two problems.

Before dealing with the extremal problems on weighted Bergman spaces, we have to make sure that they have solutions in these spaces. We need some notions of convexity for it.

Definition 3.1.2. A subset E of a vector space X is said to be convex if for every pair of points $x, y \in E$, the line segment

$$\{\, tx + (1-t)y \,:\, 0 \le t \le 1 \,\}$$

joining them lies in E.

Definition 3.1.3. A Banach space X is strictly convex if for each $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$, it is true that $\|\frac{1}{2}(x+y)\| < 1$. That is, if the midpoint of the segment joining two distinct points on the unit sphere lies strictly inside the sphere.

Definition 3.1.4. A Banach space X is uniformly convex if for each $\epsilon > 0$ there is a $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|\frac{1}{2}(x+y)\| > 1-\delta$ imply $\|x-y\| < \epsilon$.

An equivalent requirement for uniform convexity is that for any pair of sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| \to 2$ as $n \to \infty$, it is necessarily true that $\|x_n - y_n\| \to 0$. Every uniformly convex space is strictly convex, but not conversely. The following theorems show the importance of strict and uniform convexity.

Theorem 3.1.5. *In a convex subset of a strictly convex Banach space there is at most one element of minimal norm.*

Theorem 3.1.6. *Each closed convex subset of a uniformly convex Banach space contains a unique element of minimal norm.*

Every Hilbert space is uniformly convex. The space L^p_α is known to be uniformly convex for $1 < p < \infty$, but not for $p = 1$ or $p = \infty$. Every subspace of a uniformly convex space is uniformly convex, so the weighted Bergman space A^p_α with $1 < p < \infty$ is uniformly convex. On the other hand, A^1_α is not uniformly convex, but it is strictly convex.

For a functional Φ on a vector space X, we use the term null space, denoted $\operatorname{null}\Phi$ to mean the subspace of all $x \in X$ such that $\Phi(x) = 0$.

Lemma 3.1.7. *Let V be a vector space over $\mathbb{C}$ and $\Phi, \Phi_1, \ldots, \Phi_N$ be linear functionals on V. If $\bigcap_{j=1}^N \operatorname{null}\Phi_j \subseteq \operatorname{null}\Phi$, then there are constants $c_j \in \mathbb{C}$ such that $\Phi(v) = \sum_{j=1}^N c_j\Phi_j(v)$ for every v in V.*

Proof. Without loss of generality, $\bigcap_{k\neq j} \operatorname{null}\Phi_k \neq \bigcap_{k=1}^N \operatorname{null}\Phi_k$ for $1 \le j \le N$. So, for $1 \le j \le N$, there is a y_j in $\bigcap_{k\neq j} \operatorname{null}\Phi_k$, but not in $\bigcap_{k=1}^N \operatorname{null}\Phi_k$. So $\Phi_k(y_j) = 0$ for $k \neq j$, but $\Phi_j(y_j) \neq 0$. Let $v_j = \dfrac{y_j}{\Phi_j(y_j)}$. Hence $\Phi_j(v_j) = 1$ and $\Phi_k(v_j) = 0$ for $k \neq j$. Put $c_j = \Phi(v_j)$. If $v \in V$, let $y = v - \sum_{j=1}^N \Phi_j(v)v_j$. Then $\Phi_k(y) = \Phi_k(v) - \sum_{j=1}^N \Phi_j(v)\Phi_k(v_j) = 0$ for $1 \le k \le N$. By hypothesis, $\Phi(y) = 0$ too. Thus $0 = \Phi(v) - \sum_{j=1}^N \Phi_j(v)\Phi(v_j) = \Phi(v) - \sum_{j=1}^N c_j\Phi_j(v)$ for $v \in V$. That is, $\Phi(v) = \sum_{j=1}^N c_j\Phi_j(v)$ for every $v \in V$. $\square$

Notation 3.1.8. If X is a closed subspace of a normed space Y and $f \in Y$, we write $f \perp X$ if $\|f\| \leq \|f + h\|$ for all $h \in X$.

Lemma 3.1.9. *Let $1 \leq p < \infty$. Suppose $\Phi \in (A^p_\alpha)^*$, and let $f \in A^p_\alpha$ such that $\Phi(f) = 1$. Then f solves problem (2) if and only if $f \perp \operatorname{null}\Phi$.*

Proof. ($\Rightarrow$): If $h \in \operatorname{null}\Phi$, then $\Phi(f + h) = \Phi(f) + \Phi(h) = \Phi(f) = 1$. Put $g = f + h$; then $\|f\|_{A^p_\alpha} \leq \|g\|_{A^p_\alpha} = \|f + h\|_{A^p_\alpha}$.

($\Leftarrow$): If $g \in A^p_\alpha$ satisfies $\Phi(g) = 1$, let $h = g - f$; so $h \in \operatorname{null}\Phi$. Then $\|f\|_{A^p_\alpha} \leq \|f + h\|_{A^p_\alpha} = \|g\|_{A^p_\alpha}$. Thus f solves problem (2). □

Definition 3.1.10. Let $1 \leq p < \infty$ and a $\Phi \in (A^p_\alpha)^*$ be given. There is a $g \in A^q_\alpha$ ($p > 1$) or a $g \in \mathcal{B}$ ($p = 1$) uniquely such that the integral representation in Theorem 2.1.12 or in Theorem 2.2.4 hold. We denote this g by k_Φ and call it the kernel of Φ.

The expression kernel of Φ comes from the case $p = 2$ and $\Phi(f) = f(a)$ for some $a \in \mathbb{D}$. In such a case, $k_\Phi(z) = K_\alpha(a, z)$, the reproducing kernel of A^2_α, as discussed in Chapter 2.

Notation 3.1.11. For a function f on $\mathbb{D}$ and $z \in \mathbb{D}$, we put $\operatorname{sgn} f(z) = \dfrac{f(z)}{|f(z)|}$ if $f(z) \neq 0$, and otherwise we set $\operatorname{sgn} f(z) = 0$.

For $f(z) \neq 0$, we have two other forms for $\operatorname{sgn} f(z)$: $\operatorname{sgn} f(z) = \dfrac{|f(z)|}{\overline{f(z)}}$, and $\operatorname{sgn} f(z) = e^{i\theta(z)}$, where $f(z) = |f(z)|e^{i\theta(z)}$ with $\theta(z) \in [-\pi, \pi)$. Clearly $|\operatorname{sgn} f(z)| = 1$ for $f(z) \neq 0$.

3.2 The Case $1 < p < \infty$

The proof of the following result is adapted from [10, Proposition 1].

Proposition 3.2.1. *For $1 < p < \infty$, each of the problems (1), (2) has a unique solution, and the two solutions are constant multiples of each other.*

Proof. Let

$$K = \{ f \in A^p_\alpha \,:\, \Phi(f) = 1 \}.$$

Note that Φ is linear, hence K is a convex set, and K is closed by the continuity of Φ. We mentioned in Section 3.1 that A^p_α is uniformly convex for $1 < p < \infty$. It follows from Theorem 3.1.6 that K has a unique element of minimum norm. Thus, there is a unique extremal function for problem (2). Let f denote that extremal function. Then, by the proof of Theorem 3.1.1, $F = \dfrac{f}{\|f\|_{A^p_\alpha}}$ is the unique extremal function for problem (1). □

Our next result is based on [7, Theorem 4.2.1].

Theorem 3.2.2. *Suppose that X is a closed subspace of L^p_α for $1 < p < \infty$, and $f \in L^p_\alpha$. Then $f \perp X$ if and only if $\int_{\mathbb{D}} |f|^{p-1}(\operatorname{sgn} \bar{f}) h \, d\mu_\alpha = 0$ for every $h \in X$.*

Proof. ($\Leftarrow$): Without loss of generality, assume $\|f\|_{L^p_\alpha} = 1$. Let $g = |f|^{p-1} \operatorname{sgn} \bar{f}$. Then $g \in L^q_\alpha$ and $\|g\|_{L^q_\alpha} = \|f\|_{L^p_\alpha} = 1$. Then for each $h \in X$, we have

$$\|f\|_{L^p_\alpha} = 1 = \int_{\mathbb{D}} fg \, d\mu_\alpha = \int_{\mathbb{D}} (f+h) g \, d\mu_\alpha \le \|f+h\|_{L^p_\alpha} \|g\|_{L^q_\alpha} = \|f+h\|_{L^p_\alpha}.$$

($\Rightarrow$): Let $f \perp X$. By the Hahn-Banach theorem, there is an $M : L^p_\alpha \to \mathbb{C}$ with $\|M\| = \|f\|^{-1}_{L^p_\alpha}$ such that $Mf = 1$ and $Mh = 0$ for all $h \in X$. By the Riesz representation theorem, there is a $g \in L^q_\alpha$ such that $Mf = \int_{\mathbb{D}} fg \, d\mu_\alpha = 1$, where $\|M\| = \|g\|_{L^q_\alpha} = \|f\|^{-1}_{L^p_\alpha}$ and $Mh = \int_{\mathbb{D}} hg \, d\mu_\alpha = 0$. Then $\int_{\mathbb{D}} fg \, d\mu_\alpha = \|f\|_{L^p_\alpha} \|g\|_{L^q_\alpha}$. Now the conditions for equality in Hölder inequality yield that $\operatorname{sgn} \bar{f} = \operatorname{sgn} g$ and $c|f|^p = |g|^q$ a.e. for some constant c. Hence $g = c|f|^{p-1} \operatorname{sgn} \bar{f}$ for some constant c. Since $Mh = 0$, we obtain the desired result. □

The next two results also depend on their unweighted versions in [10, Theorem 2 and Corollary 3].

Theorem 3.2.3. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. For $\Phi \in (A^p_\alpha)^*$ and $f \in A^p_\alpha$ with $\Phi(f) = 1$, the following statements are equivalent:*

(*i*) f *is the extremal function for* Φ *and problem* (2).
(*ii*)

$$\int_{\mathbb{D}} |f|^{p-1}(\operatorname{sgn}\overline{f})h\,d\mu_\alpha = 0,$$

for all $h \in A^p_\alpha$ *with* $\Phi(h) = 0$.
(*iii*)

$$\int_{\mathbb{D}} |f|^{p-1}(\operatorname{sgn}\overline{f})h\,d\mu_\alpha = \|f\|^p_{A^p_\alpha}\Phi(h),$$

for all $h \in A^p_\alpha$.
(*iv*)

$$k_\Phi = P_\alpha\left(\frac{|f|^{p-1}\operatorname{sgn} f}{\|f\|^p_{A^p_\alpha}}\right).$$

Proof. (*i*) $\Leftrightarrow$ (*ii*): By Lemma 3.1.9, (*i*) is equivalent to that $f \perp \operatorname{null}\Phi$. This condition is equivalent to (*ii*) by Theorem 3.2.2 with $X = \operatorname{null}\Phi$.

(*ii*) $\Leftrightarrow$ (*iii*): One direction is obvious. For the other, let $f \in A^p_\alpha$ satisfy (*ii*). For arbitrary $h \in A^p_\alpha$, the function $g = h - \Phi(h)f \in A^p_\alpha$ satisfies $\Phi(g) = 0$. Substituting g in (*ii*) in place of h, we obtain (*iii*).

(*iii*) $\Leftrightarrow$ (*iv*): By part (*iii*) and Theorem 2.1.14,

$$\Phi(h) = \int_{\mathbb{D}} i(h)\frac{|f|^{p-1}(\operatorname{sgn}\overline{f})}{\|f\|^p_{A^p_\alpha}}\,d\mu_\alpha = \int_{\mathbb{D}} hP_\alpha\left(\frac{|f|^{p-1}(\operatorname{sgn}\overline{f})}{\|f\|^p_{A^p_\alpha}}\right)d\mu_\alpha = \int_{\mathbb{D}} h\overline{k_\Phi}\,d\mu_\alpha.$$

Note that $k_\Phi \in A^q_\alpha$ as noted in the proof of Theorem 3.2.2. The converse is just the above in reverse order. □

Corollary 3.2.4. *Let* $1 < p < \infty$ *and* $1/p + 1/q = 1$. *For* $\Phi \in (A^p_\alpha)^*$ *and* $F \in A^p_\alpha$ *with* $\|F\|_{A^p_\alpha} = 1$, *the following statements are equivalent:*
(*i*) F *is the extremal function for* Φ *and problem* (1).
(*ii*)

$$\int_{\mathbb{D}} |F|^{p-1}(\operatorname{sgn}\overline{F})h\,d\mu_\alpha = 0,$$

for all $h \in A^p_\alpha$ *such that* $\Phi(h) = 0$.
(*iii*)

$$\int_{\mathbb{D}} |F|^{p-1}(\operatorname{sgn}\overline{F})h\,d\mu_\alpha = \frac{\Phi(h)}{\Phi(F)},$$

for all $h \in A^p_\alpha$.
(*iv*)

$$k_\Phi = \Phi(F)P_\alpha(|F|^{p-1}\operatorname{sgn} F).$$

Proof. This is obtained from the above theorem using the equivalence of the problems (1) and (2). □

Then last two results in this section are the weighted versions of [4, Theorems 2.2 and 2.4].

Theorem 3.2.5. *Let $1 < p < \infty$ and $1/p+1/q = 1$. Let $F \in A^p_\alpha$ with $\|F\|_{A^p_\alpha} = 1$. Then F is the extremal function for problem (1) for the functional Ψ with kernel*

$$k_\Psi = P_\alpha(|F|^{p-1}\operatorname{sgn} F) \in A^q_\alpha.$$

If F is the extremal function for problem (1) for some functional $\Phi \in (A^p_\alpha)^*$ with kernel $k_\Phi \in A^q_\alpha$, then

$$P_\alpha(|F|^{p-1}\operatorname{sgn} F) = \frac{k_\Phi}{\|\Phi\|}.$$

Proof. The functional $\Psi \in (A^p_\alpha)^*$ has the form

$$\Psi(f) = \int_{\mathbb{D}} f|F|^{p-1}(\operatorname{sgn}\bar{F})\, d\mu_\alpha, \qquad f \in A^p_\alpha.$$

Then by Hölder inequality

$$|\Psi(f)| \le \|f\|_{L^p_\alpha}\||F|^{p-1}\|_{L^q_\alpha} = \|f\|_{L^p_\alpha}\|F\|^{p/q}_{L^p_\alpha} = \|f\|_{L^p_\alpha}.$$

So $\|\Psi\| \le 1$. We also have $\Psi(F) = \|F\|_{L^p_\alpha} = 1$, which implies that Ψ has norm exactly 1, and since $\Psi(F) = \|\Psi\| = 1$, F is the extremal function for Ψ. The rest follows from the equivalence of (i) and (iv) in Corollary 3.2.4 since $\Phi(F) = \|\Phi\|$ for an extremal F in problem (1). □

Theorem 3.2.6. *Let $1 < p < \infty$ and $\Phi_1, \Phi_2, \ldots, \Phi_N \in (A^p_\alpha)^*$ be linearly independent. For $F \in A^p_\alpha$, we have*

$$\|F\|_{A^p_\alpha} = \inf\{\, \|f\|_{A^p_\alpha} : \Phi_1(f) = \Phi_1(F), \ldots, \Phi_N(f) = \Phi_N(F)\,\}$$

if and only if $P_\alpha(|F|^{p-1}\operatorname{sgn} F)$ is a linear combination of the kernels of $\Phi_1, \ldots, \Phi_N$.

Proof. First suppose $P_\alpha(|F|^{p-1}\operatorname{sgn} F)$ is a linear combination of $k_{\Phi_1}, \ldots, k_{\Phi_N}$. Let $h \in A^p_\alpha$ be such that $\Phi_{\Phi_j}(h) = 0$ for $1 \le j \le N$. Then by Theorem 2.1.14 twice,

$$\begin{aligned}\|F\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} F|F|^{p-1}\operatorname{sgn}\bar{F}\,d\mu_\alpha = \int_{\mathbb{D}} FP_\alpha(|F|^{p-1}\operatorname{sgn}\bar{F})\,d\mu_\alpha \\ &= \int_{\mathbb{D}} (F+h)P_\alpha(|F|^{p-1}\operatorname{sgn}\bar{F})\,d\mu_\alpha = \int_{\mathbb{D}} (F+h)|F|^{p-1}(\operatorname{sgn}\bar{F})\,d\mu_\alpha \\ &\le \|F+h\|_{A^p_\alpha}\||F|^{p-1}\|_{A^q_\alpha} = \|F+h\|_{A^p_\alpha}\|F\|^{p-1}_{A^p_\alpha}.\end{aligned}$$

Hence $\|F\|_{A^p_\alpha} \le \|F+h\|_{A^p_\alpha}$. Let $f = F + h$ so that $\Phi_j(f) = \Phi_j(F)$. Thus F is extremal.

Next suppose $\|F\|_{A^p_\alpha} = \inf\{\, \|f\|_{A^p_\alpha} : \Phi_1(f) = \Phi_1(F), \ldots, \Phi_N(f) = \Phi_N(F)\,\}$. Let $0 \ne h \in A^p_\alpha$ be such that $\Phi_1(h) = \cdots = \Phi_N(h) = 0$. Since there are only finitely many functionals Φ_j, such a function h exists. Then $\|F\|_{A^p_\alpha} \le \|F+h\|_{A^p_\alpha}$ by the extremality of F. By Theorem 3.2.2 using $X = \bigcap_{j=1}^N \operatorname{null}\Phi_j$, we have $\int_{\mathbb{D}} |F|^{p-1}(\operatorname{sgn}\bar{F})h\,d\mu_\alpha = 0$, which gives $\int_{\mathbb{D}} P_\alpha(|F|^{p-1}\operatorname{sgn}\bar{F})h\,d\mu_\alpha = 0$ by Theorem 2.1.14. Now define $\Psi(f) = \int_{\mathbb{D}} fP_\alpha(|F|^{p-1}\operatorname{sgn}\bar{F})\,d\mu_\alpha$ for $f \in A^p_\alpha$. By Lemma 3.1.7, we obtain $\Psi = \sum_{j=1}^N c_j\Phi_j$ for some constants c_j. Thus, $P_\alpha(|F|^{p-1}\operatorname{sgn} F)$, which is the kernel of Ψ, is a linear combination of $k_{\Phi_1}, \ldots, k_{\Phi_N}$. $\square$

3.3 The Case $p = 1$

Let's restate the extremal problems for $p = 1$:
(1^*) Find $F \in A^1_\alpha$ with $\|F\|_{A^1_\alpha} = 1$ such that

$$\Phi(F) = \sup\{\, \Phi(G) : \|G\|_{A^1_\alpha} = 1,\ \Phi(G) > 0\,\} = \|\Phi\|.$$

(2^*) Find $f \in A^1_\alpha$ with $\Phi(f) = 1$ such that

$$\|f\|_{A^1_\alpha} = \inf\{\, \|g\|_{A^1_\alpha} : g \in A^1_\alpha,\ \Phi(g) = 1\,\}.$$

We also write $\Phi(g) = \int_{\mathbb{D}} g(z)\overline{k_\Phi(z)}\,d\mu_\alpha(z)$.

Notation 3.3.1. We denote by $C(\overline{\mathbb{D}})$ the Banach space of all complex-valued continuous functions on $\overline{\mathbb{D}}$ normed with $\|\cdot\|_\infty$, and by $M(\overline{\mathbb{D}})$ the Banach space of all regular complex Borel measures on $\overline{\mathbb{D}}$ normed by $\|\nu\| = |\nu|(\overline{\mathbb{D}})$, where $|\nu|$ is total variation of ν. By the Riesz representation theorem, we have $M(\overline{\mathbb{D}}) = C(\overline{\mathbb{D}})^*$

isometrically, that is, for every bounded linear functional Ψ on $C(\overline{\mathbb{D}})$, there is a unique $\nu \in M(\overline{\mathbb{D}})$ such that $\Psi(f) = \displaystyle\int_{\overline{\mathbb{D}}} f\, d\nu$ for every $f \in C(\overline{\mathbb{D}})$, and $\|\nu\| = \|f\|_\infty$.

Remark 3.3.2. We consider $A^1_\alpha \subset L^1_\alpha \subset M(\overline{\mathbb{D}}) \subset C(\overline{\mathbb{D}})^*$ by way of the isometry $g \mapsto g\, d\mu_\alpha$ and extending $g(z)(1-|z|^2)^\alpha$ to $\overline{\mathbb{D}}$ by setting it to be 0 on $\partial\mathbb{D}$. So A^1_α and A^1 can be thought of subspaces in a dual space, and we can talk about weak* convergence in them.

We need the following two results which are special cases of [8, Theorem 8.2 and Corollary 2] since $\partial\mathbb{D}$ is smooth.

Lemma 3.3.3. *Let $f_n \in A^1$ and $\mu \in M(\overline{\mathbb{D}})$, and suppose that $f_n\, d\mu \to d\lambda$ weak*. Then there exists an $f \in A^1$ such that $d\lambda = f\, d\mu$.*

Lemma 3.3.4. *Let $f_n, f \in A^1$. Then $f_n \to f$ weak* in A^1 if and only if $\{\|f_n\|\}$ is bounded and $f_n(z) \to f(z)$ for all $z \in \mathbb{D}$.*

Our uniqueness result for $p = 1$ is adapted from [1].

Theorem 3.3.5. *If $k_\Phi \in C(\overline{\mathbb{D}})$, then problem (2^*) has a unique solution.*

Proof. Let $\{g_n\} \subset A^1_\alpha$ be such that $\Phi(g_n) = 1$ and $\lim\limits_{n\to\infty} \|g_n\|_{A^1_\alpha} = \inf\{\, \|g\|_{A^1_\alpha} : g \in A^1_\alpha,\ \Phi(g) = 1\,\}$. By Remark 3.3.2, $\{g_n\, d\mu_\alpha\}$ is a bounded sequence in the dual of $C(\overline{\mathbb{D}})$. Since $C(\overline{\mathbb{D}})$ is separable, by the Banach-Alaoglu theorem, there is a subsequence $\{g_{n_m}\}$ such that $g_{n_m}\, d\mu_\alpha$ converges weak* to $d\lambda \in M(\overline{\mathbb{D}})$. That is, $\lim\limits_{m\to\infty} \displaystyle\int_{\mathbb{D}} h g_{n_m}\, d\mu_\alpha = \int_{\mathbb{D}} h\, d\lambda$ for all $h \in \mathbb{C}(\overline{\mathbb{D}})$. By Lemma 3.3.3, there is an $\widetilde{f} \in A^1$ such that $d\lambda = \widetilde{f}\, d\mu$. Write $d\lambda(z) = \dfrac{1}{\alpha+1}\widetilde{f}(z)(1-|z|^2)^{-\alpha}\, d\mu_\alpha(z) = f(z)\, d\mu_\alpha(z)$ with $f \in A^1_\alpha$. In particular,

$$\lim_{m\to\infty} \int_{\mathbb{D}} k_\Phi g_{n_m}\, d\mu_\alpha = \lim_{m\to\infty} \Phi(g_{n_m}) = 1 = \Phi(f) = \int_{\mathbb{D}} k_\Phi f\, d\mu_\alpha.$$

By Lemma 3.3.4, $(1+\alpha)g_{n_m}(z)(1-|z|^2)^\alpha \to \widetilde{f}(z)$ and hence $g_{n_m}(z) \to f(z)$ for all $z \in \mathbb{D}$. By the Fatou lemma, $\|f\|_{A^1_\alpha} \le \lim\limits_{m\to\infty} \|g_{n_m}\|_{A^1_\alpha}$, and this implies $\|f\|_{A^1_\alpha} = \inf\{\, \|g\|_{A^1_\alpha} : g \in A^1_\alpha,\ \Phi(g) = 1\,\}$. Thus (2^*) has a solution. By the strict convexity of A^1_α and Theorem 3.1.5, this solution is unique. Equivalently, (1^*) has a unique solution on A^1_α. □

Our next result is based on [7, Theorem 4.2.2].

Theorem 3.3.6. *Suppose that X is a closed subspace of L^1_α, and $f \in L^1_\alpha$. Then $f \perp X$ if and only if $\int_{\mathbb{D}} (\operatorname{sgn} \bar{f}) h \, d\mu_\alpha = 0$ for every $h \in X$.*

Proof. ($\Leftarrow$): Without loss of generality, assume $\|f\|_{L^1_\alpha} = 1$. Let $g = \operatorname{sgn} \bar{f}$. Then $g \in L^\infty$ and $\|f\|_{L^\infty} = \|f\|_{L^1_\alpha} = 1$. Then for each $h \in X$ we have

$$\|f\|_{L^1_\alpha} = 1 = \int_{\mathbb{D}} f\, g \, d\mu_\alpha = \int_{\mathbb{D}} (f+h)\, g \, d\mu_\alpha \leq \|f+h\|_{L^1_\alpha} \|g\|_{L^\infty} = \|f+h\|_{L^1_\alpha}.$$

($\Rightarrow$): Let $f \perp X$, by the Hahn-Banach theorem, there is an $M : L^1_\alpha \to \mathbb{C}$ with $\|M\| = 1$ such that $Mf = \|f\|_{L^1_\alpha}$ and $Mh = 0$ for $h \in X$. By the Riesz representation theorem, there is a $g \in L^\infty$ such that $Mf = \displaystyle\int_{\mathbb{D}} fg \, d\mu_\alpha = \|f\|_{L^1_\alpha}$, where $\|M\| = \|g\|_{L^\infty} = 1$ and $Mh = \displaystyle\int_{\mathbb{D}} hg \, d\mu_\alpha = 0$. Then $\|f\|_{L^1_\alpha} = \displaystyle\int_{\mathbb{D}} |f| \, d\mu_\alpha = \displaystyle\int_{\mathbb{D}} fg \, d\mu_\alpha$. Thus $fg \geq 0$ and $|f| = fg$ a.e., which gives $g = \operatorname{sgn} \bar{f}$. Finally $Mh = 0$ yields $\int_{\mathbb{D}} (\operatorname{sgn} \bar{f}) h \, d\mu_\alpha = 0$. ☐

Theorem 3.3.7. *For $\Phi \in (A^1_\alpha)^*$ and $f \in A^1_\alpha$ with $\Phi(f) = 1$, the following are equivalent:*

(i) f solves problem (2).*

(ii)

$$\int_{\mathbb{D}} (\operatorname{sgn} \bar{f}) h \, d\mu_\alpha = 0$$

for all $h \in A^1_\alpha$ with $\Phi(h) = 0$.

(iii)

$$\int_{\mathbb{D}} (\operatorname{sgn} \bar{f}) h \, d\mu_\alpha = \|f\|_{A^1_\alpha} \Phi(h)$$

for all $h \in A^1_\alpha$.

(iv)

$$k_\Phi = P_\alpha \left(\frac{\operatorname{sgn} f}{\|f\|_{A^1_\alpha}} \right).$$

Proof. This is just a repetition of the proof of Theorem 3.2.3, except that in the last part we use Theorem 2.2.5 instead. ☐

Corollary 3.3.8. *For* $\Phi \in (A^1_\alpha)^*$ *and* $F \in A^1_\alpha$ *with* $\|F\|_{A^1_\alpha} = 1$*, the following are equivalent:*
(i) F *solves problem* (1^*)*.*
(ii)

$$\int_{\mathbb{D}} (\operatorname{sgn} \bar{F}) h \, d\mu_\alpha = 0$$

for all $h \in A^1_\alpha$ *with* $\Phi(h) = 0$*.*
(iii)

$$\int_{\mathbb{D}} (\operatorname{sgn} \bar{F}) h \, d\mu_\alpha = \frac{\Phi(h)}{\|\Phi\|}$$

for all $h \in A^1_\alpha$*.*
(iv)

$$k_\Phi = \|\Phi\| P_\alpha(\operatorname{sgn} F).$$

Proof. This is obtained from the above theorem using the equivalenec of the problems (1^*) and (2^*). □

Chapter 4

Kernels of Functionals on A^p_α

We will now give some important results on the connection between the weighted Bergman projections and extremal problems. Here, we also do some calculations of weighted Bergman projections which will be helpful for the solutions of extremal problems. We follow the techniques developed in [4] where the case $\alpha = 0$ is taken care of. Throughout this chapter, when we use A^p_α, we will always have $1 < p < \infty$ and $\alpha > -1$.

4.1 Some General Results

Notation 4.1.1. The Pochhammer symbol is $(a)_b = \dfrac{\Gamma(a+b)}{\Gamma(a)}$, where Γ is the gamma function, and $a > 0$, $b \geq 0$. In particular, if $b = k$ is a positive integer, then $(a)_k = a(a+1)\cdots(a+k-1)$, which is a shifted factorial, and $(1)_k = k!$. We also take $(a)_0 = 1$. In this notation, $\dfrac{1}{(1-z)^a} = \displaystyle\sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k$ for $a > 0$ and $|z| < 1$.

Proposition 4.1.2. *Let $m, n \geq 0$, and $\alpha > -1$. Then*

$$P_\alpha(z^m \bar{z}^n) = \frac{(1+m+n)_n z^{m-n}}{(2+\alpha+m-n)_n}$$

for $m \geq n$, and it is otherwise 0.

Proof. For $m, n \geq 0$ and $\alpha > -1$,

$$\begin{aligned}
P_\alpha(z^m \bar{z}^n) &= \int_{\mathbb{D}} \frac{w^m \bar{w}^n (1-|w|^2)^\alpha}{(1-z\bar{w})^{2+\alpha}} (1+\alpha)\, d\mu(w) \\
&= \int_{\mathbb{D}} w^m \bar{w}^n (1-|w|^2)^\alpha \sum_{k=0}^{\infty} \frac{(1+\alpha)_{k+1}}{k!} z^k \bar{w}^k \, d\mu(w) \\
&= \frac{(1+\alpha)_{m-n+1}}{(m-n)!} z^{m-n} \int_{\mathbb{D}} |w|^{2m} (1-|w|^2)^\alpha \, d\mu(w) \\
&= \frac{(1+\alpha)_{m-n+1}}{(m-n)!} z^{m-n} \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^{2m} (1-r^2)^\alpha \, d\theta r \, dr \\
&= \frac{(1+\alpha)_{m-n+1}}{(m-n)!} z^{m-n} \int_0^1 u^m (1-u)^\alpha \, du \\
&= \frac{\Gamma(2+\alpha+m-n)}{\Gamma(1+\alpha)\Gamma(1+m-n)} z^{m-n} \frac{\Gamma(1+m)\Gamma(1+\alpha)}{\Gamma(2+\alpha+m)} \\
&= \frac{(1+m-n)_n z^{m-n}}{(2+\alpha+m-n)_n}
\end{aligned}$$

for $m \geq n$ since the integral is nonzero only for $m = n+k$. So, also $P_\alpha(z^m \bar{z}^n) = 0$ for $n > m$. $\square$

The following result is [4, Theorem 3.2].

Theorem 4.1.3. *Let $1 < q_1, q_2 \leq \infty$. Let p_1 and p_2 be the conjugate exponents of q_1 and q_2. Let*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$$

and suppose that $1 < q < \infty$. Let p be the conjugate exponent of q. Suppose that $k_\Psi \in A^{q_1}_\alpha$ and that $g \in A^{q_2}_\alpha$. Define the functional Ψ by $\Psi(f) = \int_{\mathbb{D}} f \bar{k}_\Psi \, d\mu_\alpha$ for $f \in A^{p_1}_\alpha$. Then $P_\alpha(k\bar{g})$ is the kernel of the functional $\Phi \in (A^p_\alpha)^*$ defined by

$$\Phi(f) = \Psi(fg), \qquad f \in A^p_\alpha.$$

Proof. We have $\Phi(f) = \Psi(fg) = \int_{\mathbb{D}} f(z)\, g(z)\, \bar{k}_\Psi(z)\, d\mu_\alpha(z)$. Note that $f \in A^p_\alpha$, $g \in A^{q_2}$ and $k_\Psi \in A^{q_1}$. Also, from the conjugate exponents $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$, we get $\frac{1}{p} + \frac{1}{q_1} + \frac{1}{q_2} = 1$, $\frac{1}{q_1} + (\frac{1}{p} + \frac{1}{q_2}) = 1$ and $\frac{1}{p_1} = \frac{1}{p} + \frac{1}{q_2}$. So, $fg \in A^{p_1}_\alpha$ and $1 < p, q, p_1, p_2 < \infty$. Then by Theorem 2.1.14,

$$\Phi(f) = \int_{\mathbb{D}} f\, g\, \bar{k}_\Psi \, d\mu_\alpha = \int_{\mathbb{D}} i(f)\, g\, \bar{k}_\Psi \, d\mu_\alpha = \int_{\mathbb{D}} f\, \overline{P_\alpha(\bar{g} k_\Psi)} \, d\mu_\alpha.$$

Hence, the kernel of Φ is $k_\Phi = P_\alpha(\bar{g} k_\Psi)$. $\square$

4.2 Evaluations at the Origin

Proposition 4.2.1. *The kernel for the functional* $\Phi : f \mapsto f^{(n)}(0)$ *in* $(A^p_\alpha)^*$ *is* $k_\Phi(z) = (\alpha+2)_n z^n$.

Proof. Since $f \in A^p_n$, let $f = \sum_{k=1}^{\infty} c_k z^k$. Then

$$\begin{aligned}
\int_{\mathbb{D}} f(z)\overline{z}^n \, d\mu_\alpha(z) &= \int_{\mathbb{D}} (\sum_k c_k z^k)(\alpha+1)(1-|z|^2)^\alpha \overline{z}^n \, d\mu(z) \\
&= \sum_k c_k(\alpha+1) \int_{\mathbb{D}} z^k (1-|z|^2)^\alpha \overline{z}^n \, d\mu(z) \\
&= \frac{\alpha+1}{\pi} \sum_k c_k \int_0^1 r^{k+n+1}(1-r^2)^\alpha \, dr \int_0^{2\pi} e^{i\theta(k-n)} \, d\theta \\
&= 2c_n(\alpha+1) \int_0^1 r^{2n+1}(1-r^2)^\alpha \, dr \\
&= c_n(\alpha+1) \int_0^1 x^n(1-x)^\alpha \, dx = c_n(\alpha+1)\frac{n!}{(\alpha+1)_{n+1}} \\
&= c_n \frac{n!}{(\alpha+2)_n} = \frac{f^{(n)}(0)}{n!}\frac{n!}{(\alpha+2)_n} = \frac{f^{(n)}(0)}{(\alpha+2)_n}.
\end{aligned}$$

Thus $f^{(n)}(0) = \int_{\mathbb{D}} f(z)[(\alpha+2)_n \overline{z}^n] \, d\mu_\alpha(z)$, so $k_\Phi(z) = (\alpha+2)_n z^n$. Note that for the unweighted situation, we have the kernel $k_\Phi = (n+1)! z^n$. □

Corollary 4.2.2. *If* $g \in A^p_\alpha$, *then*

$$P_\alpha(z^n \overline{g(z)}) = \frac{1}{(2+\alpha)_n} \sum_{j=0}^{n} \binom{n}{j} (2+\alpha)_j \, \overline{g^{n-j}(0)} z^j.$$

Proof. By Theorems 4.1.3 and 4.2.1, $P_\alpha(z^n \overline{g(z)})$ is the kernel for the functional taking $f \in A^p_\alpha$ to

$$\frac{(fg)^{(n)}(0)}{(\alpha+2)_n} = \frac{1}{(\alpha+2)_n} \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(0) g^{(n-j)}(0).$$

But by the linearity of the problem and Proposition 4.2.1 again, this kernel is also equal to $\frac{1}{(\alpha+2)_n} \sum_{j=0}^{n} \binom{n}{j} (\alpha+2)_j \, \overline{g^{(n-j)}(0)} z^j$. Equating the two expressions, we obtain the desired result. □

Corollary 4.2.3. *Let f be a polynomial of degree at most N and $g \in A^p_\alpha$. Then, $P_\alpha(f\bar{g})$ is a polynomial of degree at most N.*

Proof. Although this result is obvious from Corollary 4.2.2, we give a direct computational proof that also yields the exact coefficients.Since f is analytic and polynomial of degree at most N, we can write it as $f(z) = \sum_{m=0}^{N} a_m z^m$. Then

$$\begin{aligned}
P_\alpha(f\bar{g}) &= \int_{\mathbb{D}} \sum_{m=0}^{N} a_m \frac{\zeta^m \zeta^n}{(1-\bar{\zeta}z)^{2+\alpha}} (\alpha+1)(1-|\zeta|^2)^\alpha \, d\sigma(\zeta) \\
&= \sum_{m=0}^{N} a_m \int_{\mathbb{D}} \frac{\zeta^m \zeta^n}{(1-\bar{\zeta}z)^{2+\alpha}} (\alpha+1)(1-|\zeta|^2)^\alpha \, d\sigma(\zeta) \\
&= \sum_{m=0}^{N} a_m \int_{\mathbb{D}} \zeta^m \bar{\zeta^n} (\alpha+1)(1-|\zeta|^2)^\alpha (1-\bar{\zeta}z)^{-2-\alpha} \, d\sigma(\zeta) \\
&= \sum_{m=0}^{N} a_m \sum_{k=0}^{\infty} \frac{(\alpha+2)_k}{k!} z^k \int_{\mathbb{D}} \zeta^m \bar{\zeta^n} (\alpha+1)(1-|\zeta|^2)^\alpha \bar{\zeta}^k \, d\sigma(\zeta) \\
&= \sum_{m=0}^{N} a_m \sum_{k=0}^{\infty} \frac{(\alpha+1)_{k+1}}{k!} z^k \int_0^1 r^{m+n+k+1}(1-r^2)^\alpha \, dr \frac{1}{\pi} \int_0^{2\pi} e^{i\theta(m-n-k)} \, d\theta \\
&= \sum_{m=0}^{N} a_m \frac{(\alpha+1)_{m-n+1}}{(m-n)!} z^{m-n} \int_0^1 2r^{2m+1}(1-r^2)^\alpha \, dr \\
&= \sum_{m=0}^{N} a_m \frac{(\alpha+1)_{m-n+1}}{(m-n)!} z^{m-n} \int_0^1 x^m (1-x)^\alpha \, dx \\
&= \sum_{m=0}^{N} a_m \frac{(\alpha+1)_{m-n+1}}{(m-n)!} z^{m-n} \frac{m!}{(\alpha+1)_{m+1}} \\
&= \sum_{m=0}^{N} a_m \frac{(m-n+1)_n}{(m-n+2+\alpha)_\alpha} z^{m-n}. \qquad \square
\end{aligned}$$

Theorem 4.2.4. *The function $P_\alpha(|F|^{p-1} \operatorname{sgn} F)$ is a polynomial of degree at most N if and only if*

$$\|F\|_{A^p_\alpha} = \inf\{ \|f\|_{A^p_\alpha} : f(0) = F(0), \ldots, f^{(N)}(0) = F^{(N)}(0) \}.$$

It is a polynomial of degree exaxtly N if and only if N is the smallest integer such that the above conditions hold.

Proof. This result is immediate from Theorem 3.2.6 once we notice that the kernels of evaluations at 0 are monomials by Proposition 4.2.1. □

Now let's note the identity

$$|F|^{p-1}\operatorname{sgn} F = \frac{|F|^p}{\overline{F}} = F^{p/2}\,\overline{F}^{p/2-1} \tag{4.2.1}$$

for an analytic F on $\mathbb{D}$. We will apply P_α to this function when $F^{p/2}$ is a polynomial.

Using Theorem 4.2.4, identity (4.2.1), and Corollary 4.2.3, we obtain the following result.

Theorem 4.2.5. *Suppose that $F \in A^p_\alpha$ and $F^{p/2}$ is a polynomial of degree N. If $p < 2$, suppose also $F^{p/2-1} \in A^{p_1}_\alpha$ for some $p_1 > 1$. Then*

$$\|F\|_{A^p_\alpha} = \inf\{\, \|f\|_{A^p_\alpha} \,:\, f(0) = F(0), \ldots, f^{(N)}(0) = F^{(N)}(0)\,\}.$$

4.3 Evaluations at $a \in \mathbb{D}$

Proposition 4.3.1. *For $a \in \mathbb{D}$ and $n = 0, 1, 2, \ldots$, the kernel for the functional $f \mapsto f^{(n)}(a)$ is*

$$\frac{(\alpha+2)_n z^n}{(1-\bar{a}z)^{2+\alpha+n}}.$$

Proof. We know $f(a) = \displaystyle\int_{\mathbb{D}} K_\alpha(a,z) f(z)\, d\mu_\alpha(z)$ Differentiating n times with respect to a, we obtain

$$\frac{\partial^n K_\alpha(a,z)}{\partial a^n} = \frac{(2+\alpha)(2+\alpha+1)\cdots(2+\alpha+n-1)}{(1-a\bar{z})^{2+\alpha+n}}\bar{z}^n = \frac{(\alpha+2)_n}{(1-\bar{a}z)^{2+\alpha+n}}\bar{z}^n$$

So, $f^{(n)}(a) = \displaystyle\int_{\mathbb{D}} f(z)\frac{(\alpha+2)_n \bar{z}^n}{(1-a\bar{z})^{2+\alpha+n}}\, d\mu_\alpha(z)$ and hence the kernel for the functional $\Phi : f \mapsto f^{(n)}(a)$ is $k_\Phi(z) = \dfrac{(\alpha+2)_n z^n}{(1-\bar{a}z)^{2+\alpha+n)}}$, $n = 0, 1, 2, \ldots$. For the unweighted situation, $\alpha = 0$, and the kernel is $\dfrac{(n+1)! z^n}{(1-\bar{a}z)^{n+2}}$, $n = 0, 1, 2, \ldots$. □

Corollary 4.3.2. *If $g \in A^p_\alpha$ and $a \in \mathbb{D}$, then*

$$P_\alpha\left(\frac{z^n}{(1-\bar{a}z)^{2+\alpha+n}}\overline{g(z)}\right) = \frac{1}{(2+\alpha)_n}\sum_{j=0}^{n}\binom{n}{j}(2+\alpha)_j\,\overline{g^{(n-j)}(a)}\frac{z^j}{(1-\bar{a}z)^{2+\alpha+j}}.$$

Proof. By Theorems 4.1.3 and 4.3.1, $P_\alpha\left(\frac{z^n}{(1-\bar{a}z)^{2+\alpha+n}}\overline{g(z)}\right)$ is the kernel for the functional taking $f \in A^p_\alpha$ to

$$\frac{(fg)^{(n)}(a)}{(\alpha+2)_n} = \frac{1}{(\alpha+2)_n}\sum_{j=0}^{n}\binom{n}{j}f^{(j)}(a)g^{(n-j)}(a).$$

But by the linearity of the problem and Proposition 4.3.1 again, this kernel is also equal to $\frac{1}{(\alpha+2)_n}\sum_{j=0}^{n}\binom{n}{j}(\alpha+2)_j\,\overline{g^{(n-j)}(a)}\frac{z^j}{(1-\bar{a}z)^{2+\alpha+n}}$. Equating the two expressions, we obtain the desired result. □

Proposition 4.3.3. *For each $a \in \mathbb{D}$ with $a \neq 0$, there are numbers $c_0, \ldots, c_n$ with $c_n \neq 0$ such that the function*

$$\frac{1}{(1-\bar{a}z)^{2+\alpha+n}}$$

is the kernel for the functional $f \mapsto c_0 f(a) + c_1 f'(a) + \cdots + c_n f^{(n)}(a)$.

Proof. By [4, Theorem 3.5], we have $\frac{1}{(1-\bar{a}z)^{2+n}} = \sum_{j=0}^{n} c_j \frac{z^j}{(1-\bar{a}z)^{2+j}}$. We multiply both sides by $\frac{1}{(1-\bar{a}z)^{\alpha}}$. □

Corollary 4.3.4. *Let $a \in \mathbb{D}$, f be a product of $\frac{1}{(1-\bar{a}z)^{2+\alpha}}$ and a polynomial of degree at most N in the variable $\frac{z}{1-\bar{a}z}$, and $g \in A^p_\alpha$. Then $P_\alpha(f\bar{g})$ has the same form as f.*

Proof. The proof is immediate from Corollary 4.3.2. □

The following special case will be sufficient for solving some practical extremal problems. The difference of this from Corollary 4.3.4 is that we switch variables from $\frac{z}{1-\bar{a}z}$ to $\frac{z-a}{1-\bar{a}z}$, but only for $N = 0$ or $N = 1$.

Corollary 4.3.5. *Let $a \in \mathbb{D}$, f be a product of $\dfrac{1}{(1-\bar{a}z)^{2+\alpha}}$ and a polynomial of degree at most 1 in the variable $\dfrac{z-a}{1-\bar{a}z}$, and $g \in A^p_\alpha$. Then $P_\alpha(f\bar{g})$ has the same form as f.*

Proof. By Proposition 4.3.3, the extra term $\dfrac{-a}{(1-\bar{a}z)^{3+\alpha}}$ is a linear combination of $\dfrac{1}{(1-\bar{a}z)^{2+\alpha}}$ and $\dfrac{z}{(1-\bar{a}z)^{3+\alpha}}$. Hence f is a product of $\dfrac{1}{(1-\bar{a}z)^{2+\alpha}}$ and a polynomial of degree at most 1 in the variable $\dfrac{z}{1-\bar{a}z}$. By Corollary 4.3.4, $P_\alpha(f\bar{g})$ has this last form too. Finally, $P_\alpha(f\bar{g})$ can also be written as a product of $\dfrac{1}{(1-\bar{a}z)^{2+\alpha}}$ and a polynomial of degree at most 1 in the variable $\dfrac{z-a}{1-\bar{a}z}$ by comparing the coefficients. $\square$

Theorem 4.3.6. *Suppose $a \in \mathbb{D}$, $F \in A^p_\alpha$, and $F^{p/2}$ is a product of $\dfrac{1}{(1-\bar{a}z)^{2+\alpha}}$ and a polynomial of degree at most N in the variable $\dfrac{z}{1-\bar{a}z}$. If $p<2$, suppose also $F^{p/2-1} \in A^{p_1}_\alpha$ for some $p_1>1$. Then*

$$\|F\|_{A^p_\alpha} = \inf\big\{\, \|f\|_{A^p_\alpha} \,:\, f(a)=F(a),\ldots,f^{(N)}(a)=F^{(N)}(a)\,\big\}.$$

Proof. This result is obtained exactly the same way as Theorem 4.2.5, and it reduces to it when $a=0$. $\square$

Corollary 4.3.7. *Suppose $a \in \mathbb{D}$, $F \in A^p_\alpha$, and $F^{p/2}$ is a product of $\dfrac{1}{(1-\bar{a}z)^{2+\alpha}}$ and a polynomial of degree at most 1 in the variable $\dfrac{z-a}{1-\bar{a}z}$. If $p<2$, suppose also that $F^{p/2-1} \in A^{p_1}_\alpha$ for some $p_1>1$. Then*

$$\|F\|_{A^p_\alpha} = \inf\big\{\, \|f\|_{A^p_\alpha} \,:\, f(a)=F(a),\ldots,f^{(N)}(a)=F^{(N)}(a)\,\big\}.$$

Proof. We use Corollary 4.3.5 instead. $\square$

Chapter 5

Solutions of Extremal Problems

In this chapter, we solve specific extremal problems whose conditions are stated at some arbitrary $0 \neq a \in \mathbb{D}$ as opposed to $a = 0$ which is usually the case in the literature. We also solve the problems in weighted Bergman spaces A^p_α with arbitrary $\alpha > -1$. Yet we have to impose the condition that $p > 1$. It turns out that our standard weights have little effect on the form of the solution, but an $a \neq 0$ have a large effect.

5.1 Series Expansions at $a \in \mathbb{D}$

We are interested in the expansion of an analytic function f on the unit disc $\mathbb{D}$ in the form

$$f(z) = \frac{1}{(1-\bar{a}z)^{2+\alpha}} \left[c_0 + c_1 w + \frac{c_2}{2!} w^2 + \cdots + \frac{c_N}{N!} w^N + \cdots \right],$$

for a fixed $a \in \mathbb{D}$, where

$$w = \frac{z-a}{1-\bar{a}z}$$

in this chapter. Note that $w = 0$ if and only if $z = a$. The c_k are uniquely determined by the derivatives of f at a as

$$c_k = \frac{d^k}{dw^k}\big((1-\bar{a}z)^{2+\alpha} f(z)\big)\Big|_{w=0} = \frac{d^k}{dw^k}\big((1-\bar{a}z)^{2+\alpha} f(z)\big)\Big|_{z=a}.$$

In particular, $c_0 = f(a)(1-|a|^2)^{2+\alpha}$. If $a = 0$, then this expansion is the same as the Taylor expansion of f. Let's call this expansion a non-Taylor expansion.

Lemma 5.1.1. *Suppose $a \in \mathbb{D}$, $c_0, \ldots, c_N \in \mathbb{C}$, $c_0 \neq 0$,*

$$f(z) = \frac{1}{(1-\bar{a}z)^{2+\alpha}} \left[c_0 + c_1 w + \cdots + \frac{c_N}{N!} w^N \right],$$

and $p \in \mathbb{R}$. Let $g(z) = f^p(z)$ for some fixed branch of the pth power function in some neighborhood of a, and let the coefficients in the non-Taylor expansion of g be b_k, $k = 0, 1, \ldots$. Let

$$G(z) = \frac{1}{(1-\bar{a}z)^{2+\alpha}} \left[b_0 + b_1 w + \cdots + \frac{b_N}{N!} w^N \right]$$

and $F = G^{1/p}$. Then the first $N+1$ coefficients in the non-Taylor expansion of F are $c_0, \ldots, c_N$.

Proof. We follow the proof of [4, Lemma 4.1]. Let U be a neighborhood of a such that there are $r_0 > 0$ and $\theta_0 \in \mathbb{R}$ such that

$$f(U) \subset \left\{ re^{i\theta} \,:\, r > r_0 \text{ and } \theta_0 - \frac{\pi}{p} < \theta < \theta_0 + \frac{\pi}{p} \right\}.$$

Then the pth power function can be defined analytically in $f(U)$, and $g(z)$ makes sense for $z \in U$. Also $V = f(U)^p$ does not contain 0 and is included in a simply connected domain, and hence the $1/p$-th power function can be defined analytically in V such that it is the inverse of the pth power function in $f(U)$, that is, $g^{1/p}(z) = f(z)$ for $z \in U$. Note that the kth coefficient in the non-Taylor expansion of a function is determined by its kth derivative. Then g and G and hence $g^{1/p}$ and $G^{1/p}$ agree up to order N in the non-Taylor expansion. Thus up to order N in the non-Taylor expansion, $F = G^{1/p} = g^{1/p} = f$ in U. $\square$

We will use this lemma with $p/2$ in place of p.

Theorem 5.1.2. *Let $a \in \mathbb{D}$ and $N = 0$ or $N = 1$. Let also $d_0, \ldots, d_N \in \mathbb{C}$ with $d_0 \neq 0$. Suppose $F \in A^p_\alpha$, $F^{(k)}(a) = d_k$, $k = 0, \ldots, N$, and*

$$\|F\|_{A^p_\alpha} = \inf\left\{ \|f\|_{A^p_\alpha} \,:\, f(a) = d_0, \ldots, f^{(N)}(a) = d_N \right\}.$$

Let $c_0, \ldots, c_N$ be the coefficients in the non-Taylor expansion of F, and let $b_0, \ldots, b_N$ be related to $c_0, \ldots, c_N$ as in Lemma 5.1.1 using $p/2$ in place of p. Define

$$G(z) = \frac{1}{(1-\bar{a}z)^{2+\alpha}} \left[b_0 + b_1 w + \cdots + \frac{b_N}{N!} w^N \right].$$

Suppose also that $G^{1-2/p} \in A^{p_1}_\alpha$ for some $p_2 > 1$ and G has no zeros in $\mathbb{D}$. Then $F = G^{2/p}$.

Proof. First, $G^{2/p}$ is defined in $\mathbb{D}$ since G has no zeros there. By Lemma 5.1.1, the first N non-Taylor coefficients of $G^{2/p}$ are $c_0, \ldots, c_N$, and hence its first N derivatives at a are $d_0, \ldots, d_N$. We also have to show that $G^{2/p}$ is the solution to the extremal problem. But $P_\alpha(|G^{2/p}|^p / \bar{G}^{2/p}) = P_\alpha(G\bar{G}^{1-2/p})$ is a product of $\dfrac{1}{(1-\bar{a}z)^{2+\alpha}}$ and a polynomial of degree at most 1 in the variable w since G is too by Corollary 4.3.5. Then by Corollary 4.3.7, we see that $F = G^{2/p}$. $\square$

5.2 Specific Extremal Problems

Example 5.2.1. Given $a \in \mathbb{D}$ and $N = 0$, we find $F \in A^p_\alpha$ with minimal $\|F\|_{A^p_\alpha}$ such that $F(a) = 1$. By Theorem 5.1.2, $d_0 = 1$ and $F(z) = \left(\dfrac{b_0}{(1-\bar{a}z)^{2+\alpha}} \right)^{2/p}$, where b_0 must be selected to have $F(a) = 1$. So $b_0 = (1-|a|^2)^{2+\alpha}$ and the extremal function is

$$F(z) = \left(\frac{1-|a|^2}{1-\bar{a}z} \right)^{(2+\alpha)2/p}.$$

This is exactly the same solution under the equivalence of problems (1) and (2) obtained in [9] by another method.

Example 5.2.2. We solve the following extremal problem: Given $a \in \mathbb{D}$ and $N = 1$, find $F \in A^p_\alpha$ with minimal $\|F\|_{A^p_\alpha}$ such that $F(a) = 1$ and $F'(a) = d_1$. By Theorem 5.1.2, F must have the form

$$F(z) = \left(\frac{1}{(1-\bar{a}z)^{2+\alpha}} \left(b_0 + b_1 \frac{z-a}{1-\bar{a}z} \right) \right)^{2/p}.$$

We determine b_0 and b_1 such that F satisfies the given conditions at a. So $b_0 = (1-|a|^2)^{2+\alpha}$ as in Example 5.2.1. Also

$$F'(z) = \frac{2}{p}\left(\frac{1}{(1-\bar{a}z)^{2+\alpha}}\left(b_0 + b_1\frac{z-a}{1-\bar{a}z}\right)\right)^{2/p-1} \cdot$$
$$\cdot \left[\frac{b_1}{(1-\bar{a}z)^{2+\alpha}}\frac{1-|a|^2}{1-\bar{a}z} + \left(b_0 + b_1\frac{z-a}{1-\bar{a}z}\right)\frac{(2+\alpha)\bar{a}}{(1-\bar{a}z)^{3+\alpha}}\right],$$

and so using the above value of b_0,

$$F'(a) = \frac{2}{p}\frac{1}{(1-|a|^2)^{3+\alpha}}\big(b_1 + (1-|a|^2)^{2+\alpha}(2+\alpha)\bar{a}\big) = d_1.$$

Solving for b_1, we obtain

$$b_1 = (1-|a|^2)^{2+\alpha}\left[\frac{p}{2}d_1(1-|a|^2) - (2+\alpha)\bar{a}\right].$$

Substituting the values of b_0 and b_1, the extremal function is

$$F(z) = \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{(2+\alpha)2/p}\left(1 + \left[\frac{p}{2}d_1(1-|a|^2) - (2+\alpha)\bar{a}\right]\frac{z-a}{1-\bar{a}z}\right)^{2/p}. \quad (5.2.1)$$

We also have to make sure that F is never 0 to apply Theorem 5.1.2. This is satisfied when the modulus of the quantity in square brackets in F is ≤ 1. For example, if $a = 1/2$, $p = 4$, and $\alpha = 1$, then d_1 must satisfy $|d_1 - 1| \leq 2/3$. Picking also $d_1 = 5/3$, we see that

$$F(z) = \left(\frac{3}{2}\right)^{3/2}\frac{\sqrt{1+z}}{(2-z)^2}.$$

When $a = 0$, the extremal function F in (5.2.1) reduces to the function given in [4, Example 4.3] no matter what $\alpha > -1$ is. So the extremal function with data at 0 is the same in all A^p_α.

Chapter 6

Discussion and Directions for Future Work

In [4], Ferguson described a structured approach to solving extremal problems in Bergman spaces on the unit disc. He used the idea that for $1 < p < \infty$ the dual of A^p is A^q for representing bounded linear functionals on A^p by so-called kernels in A^q, because many extremal problems deal with finding a function in A^p with minimal norm among those functions in A^p that fix the values of some linear functionals on A^p. Then the existence of the Bergman projection P on A^p from L^p is used to determine the form of the extremal function. The evaluation of Bergman projections of certain functions then allows one to actually compute the extremal functions.

Bergman projections is a good method that exist in very general situations. So the method of Ferguson has the potential to be extended to more general spaces. In this thesis we reworked the method in weighted Bergman spaces A^p_α on which the weighted Bergman projections P_α act from weighted Lebesgue spaces L^p_α. We computed the kernels of some standard bounded linear functionals on A^p_α and solved a few simple extremal problems that involve evaluation of functions and their derivatives at 0 first. In this case, it so happens that the extremal functions are the same both in the unweighted and weighted Bergman spaces as indicated in Example 5.2.2. This is possibly due to the fact that the standard weights $(1 - |z|^2)^\alpha$ we use are all the same and equal to 1 when $z = 0$.

We also worked in adapting the method to solving extremal problems whose conditions are stated at a point $a \neq 0$ in the disc. This turned out to be a bigger challenge, because the kernels for evaluations at a are considerably more complicated and cannot be handled by simple polynomials, as the results of Chapter 4 show. As a result, we had limited success here and handled only evaluations of functions and their first derivatives. To handle higher derivatives, we need stronger and general results on decomposing certain functions in terms kernels of evaluations of these higher derivatives, and also we need to compute Bergman projections of more complicated functions. Both requirements involve complicated calculations, and we did not have time for them. So we are content with the modest examples in Chapter 5, which already have complicated extremal functions.

All of the above are for $1 < p < \infty$. However, weighted Bergman projections exist also on A^1_α as shown in Chapter 2, and using them we can identify the dual A^1_α as the Bloch space $\mathcal{B}$. However, the pairing of the dual is further complicated. To use these ideas in extremal problems on A^1_α, we need to work out all of Chapters 3 and 4 for the new forms of the kernels of linear functionals. Doing this goes beyond the scope of this thesis.

Moreover, there are the Besov spaces B^p_α that extend the weighted Bergman spaces for $\alpha \leq -1$, and Bergman projections exist on them too. This is even a longer and more complicated project than handling $p = 1$ in Bergman spaces.

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