T.C.

BARTIN ÜNİVERSİTESİ

LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

MATEMATİK ANABİLİM DALI

YÜKSEK LİSANS TEZİ

FARKLI TÜRDEN NÖTROSOFİK DİZİ UZAYLARININ İNCELENMESİ

RÜMEYSA AKBIYIK

DANIŞMAN

PROF. DR. ÖMER KİŞİ

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Danışman	:	Prof. Dr. Ömer KİŞİ
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Üye	:	Doç. Dr. Melih GÖCEN

BARTIN-2025

KABUL VE ONAY

Rümeysa AKBIYIK tarafından hazırlanan "FARKLI TÜRDEN NÖTROSOFİK DİZİ UZAYLARININ İNCELENMESİ" başlıklı bu çalışma, 16.06.2025 tarihinde yapılan savunma sınavı sonucunda oy birliği ile başarılı bulunarak jürimiz tarafından Yüksek Lisans Tezi olarak kabul edilmiştir.

Başkan	:	Prof. Dr. Ömer KİŞİ	……………
Üye	:	Prof. Dr. Emrah ALTUN	……………
Üye	:	Doç. Dr. Melih GÖCEN	……………

Bu tezin kabulü Lisansüstü Eğitim Enstitüsü Yönetim Kurulu'nun ..…/..…/20… tarih ve 20…../…..-….. sayılı kararıyla onaylanmıştır.

Doç. Dr. Zafer CEYLAN
Enstitü Müdürü

BEYANNAME

Bartın Üniversitesi Lisansüstü Eğitim Enstitüsü tez yazım kılavuzuna göre Prof. Dr. Ömer KİŞİ danışmanlığında hazırlamış olduğum "FARKLI TÜRDEN NÖTROSOFİK DİZİ UZAYLARININ İNCELENMESİ" başlıklı yüksek lisans tezimin bilimsel etik değerlere ve kurallara uygun, özgün bir çalışma olduğunu, aksinin tespit edilmesi halinde her türlü yasal yaptırımı kabul edeceğimi beyan ederim.

16.06.2025

Rümeysa AKBIYIK

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Rümeysa AKBIYIK

ÖZET

Yüksek Lisans Tezi

FARKLI TÜRDEN NÖTROSOFİK DİZİ UZAYLARININ İNCELENMESİ

Rümeysa AKBIYIK

Bartın Üniversitesi
Lisansüstü Eğitim Enstitüsü
Matematik Anabilim Dalı

Tez Danışmanı: Prof. Dr. Ömer KİŞİ

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Bu tez çalışmasında, nötrosofik fuzzy G-metrik uzaylar (NFGMS) içinde çift dizilerin istatistiksel yakınsaması, istatistiksel limit noktaları ve istatistiksel yığılma noktaları incelenerek, nötrosofik fuzzy metrik uzay kavramı genişletilmektedir. İlgili teoremler ve örnekler kullanılarak elde edilen sonuçlar desteklenmiş, istatistiksel yakınsamanın temel özellikleri analiz edilmiştir. Ayrıca, çift dizilerin lacunary istatistiksel yakınsaması ve güçlü lacunary yakınsaması ele alınarak, bu kavramlar arasındaki ilişkiler değerlendirilmiştir. Çalışmanın bir diğer amacı, NFGMS'lerde ideal yakınsama kavramını tanıtarak, $\mathcal{I}_2$-Cauchy çift dizileri ve $\mathcal{I}_2$-tamlık kavramlarını incelemektir. İdeal yakınsamanın temel özellikleri analiz edilerek, $\mathcal{I}_2$-limit ve $\mathcal{I}_2$-yığılma noktaları detaylı olarak değerlendirilmiştir. Ayrıca, $\mathcal{I}_2$-istatistiksel yakınsama, $\mathcal{I}_2$-lacunary istatistiksel yakınsama, güçlü $\mathcal{I}_2$-Cesàro toplanabilirlik ve güçlü $\mathcal{I}_2$-lacunary toplanabilirlik kavramları tanımlanmış ve bunlara ait özellikler incelenmiştir. Elde edilen bulgular, cebirsel ve topolojik perspektiften ele alınarak, NFGMS'lerde çift dizilerin yakınsama teorisine katkı sağlamaktadır.

Anahtar Kelimeler: $\mathcal{I}_2$-yakınsaklık, lacunary istatistiksel yakınsaklık, neutrosofik fuzzy G-metrik uzaylar

ABSTRACT

M. Sc. Thesis

INVESTIGATION OF DIFFERENT TYPES OF NEUTROSOPHIC SEQUENCE SPACES

Rümeysa AKBIYIK

Bartın University
Graduate School
Department of Mathematics

Thesis Advisor: Prof. Dr. Ömer KİŞİ
Bartın-2025, pp: 98

This thesis investigates the statistical convergence, statistical limit points, and statistical cluster points of double sequences in neutrosophic fuzzy G-metric spaces (NFGMS), thereby extending the concept of neutrosophic fuzzy metric spaces. Theoretical results are supported by relevant theorems and illustrative examples, and the fundamental properties of statistical convergence are analyzed. Additionally, the study explores lacunary statistical convergence and strong lacunary convergence of double sequences, examining the relationships between these concepts. Another objective of this research is to introduce the notion of ideal convergence in NFGMS and to analyze $\mathcal{I}_2$-Cauchy double sequences and $\mathcal{I}_2$-completeness within these spaces. The fundamental properties of ideal convergence are examined, and the relationships between $\mathcal{I}_2$-limits and $\mathcal{I}_2$-cluster points are discussed in detail. Furthermore, the notions of $\mathcal{I}_2$-statistical convergence, $\mathcal{I}_2$-lacunary statistical convergence, strong $\mathcal{I}_2$-Cesàro summability, and strong $\mathcal{I}_2$-lacunary summability are introduced, and their properties are investigated. The findings of this study contribute to the theory of convergence of double sequences in NFGMS from both algebraic and topological perspectives.

Keywords: $\mathcal{I}_2$-convergence, lacunary sequence, neutrosophic fuzzy G-metric spaces

İÇİNDEKİLER

SİMGELER VE KISALTMALAR DİZİNİ

$(Q,\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$: Nötrosofik fuzzy G-metrik uzay

$S-(\mathfrak{F},\mathfrak{G},\mathfrak{H})$: Nötrosofik fuzzy G-metrik uzayında istatistiksek yakınsak diziler uzayı

$S_{\theta_2}-(\mathfrak{F},\mathfrak{G},\mathfrak{H})$: Nötrosofik fuzzy G-metrik uzayında lacunary istatistiksek yakınsak diziler uzayı $\mathfrak{I}_2$-istatistiksel yakınsak diziler uzayı

$\left[N_{\theta_2}\right]-(\mathfrak{F},\mathfrak{G},\mathfrak{H})$: Nötrosofik fuzzy G-metrik uzayında kuvvetli lacunary yakınsak diziler uzayı

$[\mathrm{C},1,1]-(\mathfrak{F},\mathfrak{G},\mathfrak{H})$: Nötrosofik fuzzy G-metrik uzayında kuvvetli Cesàro yakınsak diziler uzayı

$\Gamma^{\mathcal{J}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$: Nötrosofik fuzzy G-metrik uzayında $\mathcal{J}_2$-yığılma noktaları kümesi

$\Lambda^{\mathcal{J}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$: Nötrosofik fuzzy G-metrik uzayında $\mathcal{J}_2$-limit noktaları kümesi

KISALTMALAR

NFGMU : Nötrosofik fuzzy G-metrik uzay

1. GİRİŞ

Dizilerin yakınsaklığı, matematiğin temel yapı taşlarından biri olup yalnızca bu özelliği taşıyan dizilerin incelenmesiyle sınırlı değildir. Aynı zamanda, yakınsaklık kriterini doğrudan karşılamayan dizilerin de belirli matematiksel çerçeveler içinde analiz edilmesi önemli bir araştırma alanı oluşturmuştur. Bu tür diziler üzerinde yapılan çalışmalar, çeşitli yakınsaklık kavramlarının ortaya çıkmasına ve gelişmesine zemin hazırlamıştır.

Yakınsaklık kavramına önemli bir katkı sağlayan bir diğer temel yaklaşım ise istatistiksel yakınsaklıktır. Pozitif doğal sayıların doğal yoğunluğuna dayalı olarak tanımlanan istatistiksel yakınsaklık, ilk kez 1951 yılında Fast tarafından literatüre kazandırılmıştır (Fast, 1951). Bu kavram çerçevesinde, reel ve kompleks sayı dizileri üzerinde çeşitli özellikler detaylı şekilde araştırılmıştır (Schoenberg, 1959; Fridy, 1985). Çift dizilerde istatistiksel yakınsaklığın uygulanmasına yönelik ilk çalışma ise Mursaleen ve Edely (2003) tarafından yapılmıştır.

Fridy ve Orhan (1993), lacunary diziler kavramını temel alarak, istatistiksel yakınsaklıkla güçlü bağlantılar kuran ve yakınsaklık teorisinde önemli bir yere sahip olan lacunary istatistiksel yakınsaklık kavramını ortaya koymuşlardır. Söz konusu çalışmalarında, özellikle istatistiksel yakınsaklık ve diğer toplanabilirlik yöntemleri ile lacunary istatistiksel yakınsaklık arasındaki ilişkiler ayrıntılı bir şekilde ele alınmıştır. Öte yandan, Çakan vd. (2009) çift lacunary yoğunluğu üzerinde çalışmış ve çift dizilerde istatistiksel yakınsaklık ile lacunary istatistiksel yakınsaklık arasındaki bağlantıları araştırmışlardır.

Kostyrko vd. (2000-2001), istatistiksel yakınsamanın daha genel bir formu olan $\mathcal{I}$-yakınsaklık kavramını geliştirmişlerdir. Das vd. (2008), bu fikri metrik uzayda çift diziler için modelllemiş ve bu yakınsama türünün temel özelliklerini detaylı bir şekilde incelemişlerdir. Aynı temel üzerinde, Das vd. (2011) kavramı daha da derinleştirerek $\mathcal{I}$-istatistiksel yakınsama olarak yeni bir biçimde ele almışlardır. Bu alandaki gelişmeler, özellikle Dündar ve Altay'ın (2014) katkılarının yanı sıra, birçok araştırmacının bu kavramın teorik ve uygulamalı yönlerini detaylı bir şekilde incelemesiyle önemli bir ilerleme kaydetmiştir.

Gähler (1963), 2-metrik uzaylar kavramını incelemiş ve bu tür uzayların, klasik metrik uzayların doğal bir genişlemesi olduğunu savunmuştur. Ancak, bazı araştırmacılar, 2-metrik uzaylar ile geleneksel metrik uzaylar arasında belirgin bir ilişki bulunmadığını iddia etmişlerdir (Ha vd., 1988). Takip eden yıllarda, Dhage (1992), D-metrik uzaylar olarak bilinen yeni bir metrik sınıfını tanımlamıştır. Bununla birlikte, Mustafa ve Sims (2004) tarafından yapılan çalışmalarda, D-metriklerin bazı sınırlı yönleri olduğu vurgulanmıştır. Son olarak, Mustafa ve Sims (2006), G-metrik uzaylar olarak adlandırılan farklı bir genelleştirilmiş metrik uzay sınıfını sunmuşlar ve bu yeni sınıfın, geleneksel metrik uzaylardan daha geniş bir çerçeve sunduğunu kanıtlamışlardır.

Bulanık mantık, matematik, tıp ve mühendislik gibi pek çok disiplinde önemli katkılar sağlamıştır. Bu kavram, ilk kez Zadeh tarafından 1965 yılında literatüre kazandırılmıştır. Zadeh'in bu çalışması, belirsizliğin ölçülmesi açısından bir dönüm noktası olarak kabul edilmiştir. Zadeh, bu çalışmasında, kesin olmayan sınırlarla tanımlanan nesneleri içeren bulanık kümeler teorisini geliştirmiş ve bu kümelerin matematiksel yapısını detaylı bir şekilde incelemiştir. Günlük yaşamda karşılaştığımız belirsizliklerin daha etkili bir şekilde ele alınabilmesi için bulanık kümeler önemli bir yöntem sunmaktadır. Kramosil ve Michálek (1975) ise, geleneksel metrik uzayların daha genel bir versiyonu olarak bulanık metrik uzaylar kavramını önermişlerdir. Kramosil ve Michálek (1975), George ve Veeramani (1994) tarafından geliştirilen bulanık metrik uzaylar üzerinde çeşitli düzenlemeler yapmıştır. Aynı dönemde, Atanassov (1986) sezgisel bulanık kümeler kavramını tanıtmış ve Park (2004), George ve Veeramani (1994) tarafından atılan temelleri geliştirerek bu kavramı sezgisel bulanık metrik uzaylara genişletmiştir.

Smarandache (2005), günlük yaşantımızda karşılaşılan birçok uygulamalı problemin çözümünde ortaya çıkan belirsizliklerin daha hassas bir şekilde ele alınabilmesi amacıyla, bulanık kümeler, sezgisel bulanık kümeler ve aralık değerli sezgisel bulanık kümeler gibi yapılara bir genelleme getiren nötrosofik küme kavramını incelemiştir. Karar verme süreçlerinde, doğrudan yaklaşımların (örneğin evet ya da hayır) ötesinde, karar vericilerin çoğu zaman tereddüt yaşadığı durumlar ortaya çıkabilmektedir. Ayrıca, spor karşılaşmaları veya oylama prosedürleri gibi bazı gerçek yaşam olaylarında üç bileşenli sonuçlar gözlemlenebilmektedir. Tüm bu durumları dikkate alarak, Smarandache, sezgisel bulanık küme teorisini genişletmiş ve nötrosofi adını verdiği, düşüncenin tarafsız bilgisine dayalı yeni bir yaklaşım tanımlamıştır. Bu çerçevede, "nötr" kavramı ile nötr, bulanık ve sezgisel bulanık kümeler ile mantık arasındaki temel farklılık vurgulanmıştır. Tüm bu durumlar göz önünde bulundurularak, Smarandache, sezgisel bulanık küme teorisini, belirsizlik üyelik fonksiyonu adı verilen yeni bir bileşen tanımlayarak genişletmiştir. Buna göre, bir nötrosofik kümeye ait her bir eleman, doğruluk üyelik fonksiyonu (D), belirsizlik üyelik fonksiyonu (B) ve yanlışlık üyelik fonksiyonu (Y) olmak üzere üçlü bir bileşenle tanımlanmaktadır. Bir nötrosofik küme, evrendeki her bir elemanın D, B ve Y değerlerine sahip olduğu bir küme yapısıdır. Bu üç fonksiyon nötrosofik küme yapısında birbirinden bağımsızdır.

Sezgisel bulanık kümelerde belirsizlik, üyelik ve üyelik dışılık derecelerine bağlı olarak tanımlanırken; nötrosofik kümelerde belirsizlik, doğruluk ve yanlışlık değerlerinden bağımsız şekilde ele alınmaktadır. D, B ve Y değerleri arasında herhangi bir kısıtlama bulunmadığından, nötrosofik kümeler sezgisel bulanık kümelere kıyasla daha genel bir yapıya sahiptir.

Kirişçi ve Şimşek (2020), sürekli üçgensel normlar ve konormlar kullanarak nötrosofik metrik uzayları yapılandırmıştır. Nötrosofik metrik uzayların çeşitli topolojik ve yapısal özellikleri üzerine birçok çalışma gerçekleştirilmiştir.

Diğer yandan, Menger (1942), metrik uzayın bir genellemesi olarak olasılıksal metrik uzay kavramını tanıtmıştır. Öte yandan, Choi vd. (2018), iki nokta arasındaki mesafe kavramını iki yerine $\mathrm{q}+1$ nokta dikkate alarak genişletmiş ve q mertebesinde g-metrik kavramını ortaya koymuşlardır. Bu kavram üzerine inşa ederek, Abazari (2021), Menger olasılıksal G-metrik uzayın bir genellemesi olarak Menger olasılıksal g-metrik uzayını tanımlamış ve güçlü topolojiye göre istatistiksel yakınsamayı, $\mathbb{N}$ kümesinin q-boyutlu asimptotik yoğunluğu kavramı kullanılarak incelemiştir. Bu gelişmeler, g-metrik kavramının nötrosofik bulanık ortamlara genişletilmesi yönünde araştırmaları teşvik etmektedir. Ayrıca, bu genelleştirilmiş çerçevede, q-boyutlu asimptotik yoğunluk dikkate alınarak istatistiksel yakınsamanın geçerliliği üzerine de bir sorgulama ortaya çıkmaktadır.

Bu tez çalışmasında, literatüre yeni kazandırılan nötrosofik fuzzy G-metrik uzaylar (NFGMU) çerçevesinde çift dizilerin istatistiksel yakınsaması, istatistiksel limit ve küme noktaları gibi kavramlar ele alınarak, nötrosofik fuzzy metrik uzay anlayışı genişletilmiştir. Çalışmada elde edilen bulgular, ilgili teoremler ve örnekler ışığında desteklenmiş, istatistiksel yakınsamanın temel nitelikleri ayrıntılı bir şekilde incelenmiştir. Ayrıca, çift dizilerin lacunary istatistiksel yakınsaması ve güçlü lacunary yakınsaması konularına da yer verilerek, bu iki kavram arasındaki bağlantılar analiz edilmiştir. Araştırmanın bir başka hedefi ise, NFGMS içerisinde ideal yakınsama kavramını tanıtmak ve bu bağlamda $\mathcal{I}_2$-Cauchy çift dizileri ile $\mathcal{I}_2$-tamlık kavramlarını detaylı şekilde araştırmaktır. İdeal yakınsamanın karakteristik özellikleri tartışılmış; $\mathcal{I}_2$-limit ve $\mathcal{I}_2$-yığılma noktaları ayrıntılı biçimde incelenmiştir. Bunun yanında, $\mathcal{I}_2$-istatistiksel yakınsama, $\mathcal{I}_2$-lacunary istatistiksel yakınsama, güçlü $\mathcal{I}_2$-Cesàro toplanabilirlik ve güçlü $\mathcal{I}_2$-lacunary toplanabilirlik gibi yeni kavramlar tanımlanmış ve bunların temel özellikleri kapsamlı şekilde analiz edilmiştir.

2. TANIMLAR VE KAVRAMLAR

Bu bölümde temel tanım, teorem, ve kavramlara yer verilmiştir.

Tanım 2.1. (Schweizer vd., 1960) $\therefore: [0,1] \times [0,1] \longrightarrow [0,1]$ fonksiyonu

(1) $\therefore$ işlemi hem değişmeli hem de birleşmelidir,

(2) Her $\varsigma \in [0,1]$ için $\varsigma = \varsigma \therefore 1$ sağlanır,

(3) Her $0 \leq \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \leq 1$ için, eğer $\varsigma_1 \leq \varsigma_3$ ve $\varsigma_2 \leq \varsigma_4$ ise $\varsigma_1 \therefore \varsigma_2 \leq \varsigma_3 \therefore \varsigma_4$ sağlanır,

(4) $\therefore$ işlemi süreklidir,

koşullarını sağlıyorsa sürekli t-norm olarak tanımlanır.

$*: [0,1] \times [0,1] \longrightarrow [0,1]$ fonksiyonu

$(1')$ $*$ işlemi hem değişmeli hem de birleşmelidir,

$(2')$ Her $\varsigma \in [0,1]$ için $\varsigma = \varsigma * 0$ sağlanır,

$(3')$ Her $0 \leq \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \leq 1$ için, eğer $\varsigma_1 \leq \varsigma_3$ ve $\varsigma_2 \leq \varsigma_4$ ise $\varsigma_1 * \varsigma_2 \leq \varsigma_3 * \varsigma_4$ sağlanır,

$(4')$ $*$ işlemi süreklidir,

koşullarını sağlıyorsa sürekli t-conorm olarak tanımlanır.

$\mathfrak{U} \subseteq \mathbb{N}$ olsun. $\mathfrak{U}$ kümesinin doğal yoğunluğu $d(\mathfrak{U})$ ile gösterilir ve

$$d(\mathfrak{U}) = \lim_{z \to \infty} \frac{1}{z} |\{t \leqslant z : t \in \mathfrak{U}\}|$$

ifadesiyle tanımlanır (eğer bu limit mevcutsa).

Bir reel sayı dizisi (w_z), her $ƍ > 0$ için

$$d(\{z \in \mathbb{N} : |w_z - w_0| > ƍ\}) = 0$$

şartı sağlanıyorsa $w_0 \in \mathbb{R}$ sayısına istatistiksel yakınsaktır denir. $st - w_z = w_0$ sembolü ile gösterilir.

$\mathbb{N}$ kümesinin q-katlı Kartezyen çarpımı $\mathbb{N}^q = \prod_{r=1}^{q} \mathbb{N}^r$ olarak tanımlansın. $\mathrm{Y} \subseteq \mathbb{N}^q$ olmak üzere her $z \in \mathbb{N}$ için

$$Y(z) = \left\{\left(ƀ_1, ƀ_2, \dots, ƀ_q\right) \in Y: ƀ_1, ƀ_2, \dots, ƀ_q \leqslant z\right\}$$

kümesi tanımlansın.

Bu durumda, q. dereceden asimptotik yoğunluğu

$$\mathrm{d}_q(\mathrm{Y}) = \lim_{z\to\infty} \frac{q!}{z^q} |\mathrm{Y}(z)|$$

şeklinde verilir (limit mevcutsa). Burada

$$Y(u) = \left\{\left(u_1, u_2, \dots, u_q\right) \in K^q: u_1, u_2, \dots, u_q \leqslant u\right\}$$

Böylece, Y kümesinin q-boyutlu asimptotik yoğunluğu

$$\mathrm{d_q(Y)} = \lim_{\mathrm{u}\to\infty} \frac{\mathrm{q!}}{\mathrm{u^q}} |\mathrm{Y(u)}|$$

şeklinde verilir.

$K \subseteq \mathbb{N}$ olmak üzere K kümesinin q-boyutlu asimptotik yoğunluğu

$$\mathrm{d_q(\ K)} = \lim_{\mathrm{u}\to\infty} \frac{\mathrm{q!}}{\mathrm{u^q}} |\ \mathrm{K(u)}|$$

biçimde tanımlanır. Burada,

$$K(u) = \left\{\left(u_1, u_2, \dots, u_q\right) \in K^q: u_1, u_2, \dots, u_q \leqslant u\right\}$$

olarak alınır.

Tanım 2.2. (Mustafa ve Sims, 2006) Boş olmayan Q kümesi verilsin. Eğer, $\mathcal{G}: Q \times Q \times Q \to \mathbb{R}^+$ fonksiyonu

1. Eğer $m = n = p$ ise $\mathcal{G}(m, n, p) = 0$,
2. $m \neq n$ olmak üzere, tüm $m, n \in Y$ için $0 < \mathcal{G}(m, m, n)$;
3. $p \neq n$ için tüm $m, n, p \in Y$ için $\mathcal{G}(m, m, n) \leq \mathcal{G}(m, n, p)$,
4. $\mathcal{G}(m, n, p) = \mathcal{G}(m, p, n) = \mathcal{G}(n, p, m) = \cdots$, (tüm değişkenlerde simetrik),

5. Tüm $m, n, p, t \in Y$ için $\mathcal{G}(m, n, p) \leq G(m, t, t) + G(t, n, p)$, (dikdörtgen eşitsizliği)

koşullarını sağlıyorsa, genelleştirilmiş metrik ya da G-metrik olarak adlandırılır ve $(Q, \mathcal{G})$ sembolü ile gösterilir. Aşağıdaki tanım, bu yapının $\mathfrak{q} \in \mathbb{N}$ dereceli bir genellemesidir.

Tanım 2.3. (Choi vd. (2018) Boş olmayan Q kümesi verilsin. $\mathfrak{g}: Q^{\mathfrak{q}+1} \longrightarrow \mathbb{R}_+$ fonksiyonu

1. $m_0 = m_1 = \cdots = m_{\mathfrak{q}}$ ancak ve ancak $\mathfrak{g}(m_0, m_1, \dots, m_{\mathfrak{q}}) = 0$ ise,
2. Herhangi bir $\{0,1, \dots, \mathfrak{q}\}$ üzerindeki π permütasyonu için $\mathfrak{g}(m_0, m_1, \dots, m_{\mathfrak{q}}) = \mathfrak{g}(m_{\pi(0)}, m_{\pi(1)}, \dots, \dots, m_{\pi(\mathfrak{q})})$ ise,
3. Tüm $(m_0, m_1, \dots, m_{\mathfrak{q}}), (n_0, n_1, \dots, n_{\mathfrak{q}}) \in Q^{q+1}$ ve $\{m_j : j = 0,1, \dots, \mathfrak{q}\} \subseteq \{n_j : j = 0,1, \dots, \mathfrak{q}\}$, için $\mathfrak{g}(m_0, m_1, \dots, m_{\mathfrak{q}}) \leq \mathfrak{g}(n_0, n_1, \dots, n_{\mathfrak{q}})$ ise,
4.) Her $m_0, m_1, \dots, m_s, n_0, n_1, \dots, n_t, \curlyvee \in Q$ ve $s + t + 1 = \mathfrak{q}$ için

$$\begin{aligned}\mathfrak{g}(m_0, m_1, \dots, m_s, n_0, n_1, \dots, n_t) \\ \leq \mathfrak{g}(m_0, m_1, \dots, m_s, \curlyvee, \curlyvee, \dots, \curlyvee) + \mathfrak{g}(n_0, n_1, \dots, n_t, \curlyvee, \curlyvee, \dots, \curlyvee)\end{aligned}$$

koşullarını sağlıyorsa, $\mathfrak{q} \in \mathbb{N}$ derecesine sahip bir g-metrik olarak adlandırılır.

3. NÖTROSOFİK FUZZY G-METRİK UZAYLARDA İSTATİSTİKSEL YAKINSAKLIK

Bu bölümde, çift dizilerin nötrosofik fuzzy genelleştirilmiş metrik uzaylarda istatistiksel yakınsama, istatistiksel limit noktaları ve istatistiksel yığılma noktaları incelenmektedir. Nötrosofik fuzzy metrik uzayın daha genel bir yapıya genişletilmesi ele alınarak, istatistiksel yakınsama kavramının bu uzaylardaki temel özellikleri araştırılmaktadır. Son yıllarda, belirsizlik içeren matematiksel yapıların analizi, hem teorik hem de uygulamalı bilimlerde giderek daha fazla ilgi görmektedir. Klasik metrik uzayların genişletilmiş versiyonları, özellikle fuzzy ve nötrosofik yapıların entegre edildiği metrik uzaylar, veri belirsizliği ve karmaşıklığın yüksek olduğu problemlerde önemli bir çerçeve sunmaktadır. Elde edilen teorik sonuçlar, uygun teoremlerle desteklenmiş ve açıklayıcı örneklerle detaylandırılmıştır.

Tanım 3.1. Boş olmayan rastgele bir $\mathcal{Q}$ kümesi, sürekli bir t-norm $\therefore$, sürekli bir t-conorm $*$, ve $\mathcal{Q}^{q+1} \times (0, \infty)$ üzerinde tanımlı bulanık kümeler $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ ele alındığında, altılı $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ ifadesi, eğer aşağıdaki koşullar tüm $\mathfrak{u}, \mathfrak{v} \in (0, \infty)$ için sağlanıyorsa, q. dereceden olan bir nötrosofik fuzzy genelleştirilmiş metrik uzay (kısaca NFGMU) olarak adlandırılır:

(NFGMU-1) Her $\mathscr{f}_0, \mathscr{f}_1, \dots, \mathscr{f}_q \in \mathcal{Q}$ için

$$\mathfrak{F}(\mathscr{f}_0, \mathscr{f}_1, \dots, \mathscr{f}_q, \mathfrak{v}) + \mathfrak{G}(\mathscr{f}_0, \mathscr{f}_1, \dots, \mathscr{f}_q, \mathfrak{v}) + \mathfrak{H}(\mathscr{f}_0, \mathscr{f}_1, \dots, \mathscr{f}_q, \mathfrak{v}) \leq 3$$

sağlanır.

(NFGMU-2) $\mathscr{f}_0 \neq \mathscr{f}_1$ ve her $\mathscr{f}_0, \mathscr{f}_1 \in \mathcal{Q}$ için

$$\mathfrak{F}(\mathscr{f}_0, \mathscr{f}_0, \dots, \mathscr{f}_0, \mathscr{f}_1, \mathfrak{v}) > 0$$

şartı sağlanır.

(NFGMU-3) $\forall \left(\mathscr{f}_0, \mathscr{f}_1, \dots, \mathscr{f}_q\right), \left(\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_q\right) \in \mathcal{Q}^{q+1}$ ve $\left\{\mathscr{f}_0, \mathscr{f}_1, \dots, \mathscr{f}_q\right\} \subsetneq \left\{\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_q\right\}$ için

$$\mathfrak{F}(\mathscr{f}_0, \mathscr{f}_1, \dots, \mathscr{f}_q, \mathfrak{v}) \geq \mathfrak{F}\left(\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_q, \mathfrak{v}\right)$$

eşitsizliği geçerlidir.

(NFGMU-4)

$$\mathfrak{F}(ꞙ_0, ꞙ_1, \dots, ꞙ_q, \mathfrak{v}) = 1 \Leftrightarrow ꞙ_0 = ꞙ_1 = \cdots = ꞙ_q,$$

geçerlidir.

(NFGMU-5) Herhangi bir $\{0,1, \dots, q\}$ üzerindeki π permütasyonu için

$$\mathfrak{F}(ꞙ_0, ꞙ_1, \dots, ꞙ_q, \mathfrak{v}) = \mathfrak{F}(ꞙ_{\pi(0)}, ꞙ_{\pi(1)}, \dots, ꞙ_{\pi(q)}, \mathfrak{v})$$

şartı sağlanır.

(NFGMU-6) Her $ꞙ_0, ꞙ_1, \dots, ꞙ_{\mathfrak{g}}, \mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_{\mathfrak{f}}, \Upsilon \in Q$ ve $\mathfrak{g} + \mathfrak{f} + 1 = q$ için

$$\begin{aligned} &\mathfrak{F}\left(ꞙ_0, ꞙ_1, \dots, ꞙ_{\mathfrak{g}}, \Upsilon, \dots, \Upsilon, \mathfrak{u}\right) \therefore \mathfrak{F}(\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_{\mathfrak{f}}, \Upsilon, \dots, \Upsilon, \mathfrak{v}) \\ &\leq \mathfrak{F}\left(ꞙ_0, ꞙ_1, \dots, ꞙ_{\mathfrak{g}}, \mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_{\mathfrak{f}}, \mathfrak{u} + \mathfrak{v}\right) \end{aligned}$$

eşitsizliği sağlanır.

(NFGMU-7)

$$\mathfrak{F}(ꞙ_0, ꞙ_1, \dots, ꞙ_q, .) : (0, \infty) \longrightarrow [0,1]$$

sürekli bir fonksiyondur.

(NFGMU-8) Her $ꞙ_0, ꞙ_1, \dots, ꞙ_q \in Q$ için

$$\lim_{\mathfrak{v} \to \infty} \mathfrak{F}\left(ꞙ_0, ꞙ_1, \dots, ꞙ_q, \mathfrak{v}\right) = 1$$

sağlanır.

(NFGMU-9) $ꞙ_0 \neq ꞙ_1$ ve her $ꞙ_0, ꞙ_1 \in Q$ için

$$\mathfrak{G}(ꞙ_0, ꞙ_0, \dots, ꞙ_0, ꞙ_1, \mathfrak{v}) < 1$$

şartı sağlanır.

(NFGMU-10) $\forall \left(\not{f}_0, \not{f}_1, \dots, \not{f}_q\right), (\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_q) \in \mathcal{Q}^{q+1}$ ve $\left\{\not{f}_0, \not{f}_1, \dots, \not{f}_q\right\} \subsetneq \{\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_q\}$ için

$$\mathfrak{G}(\not{f}_0, \not{f}_1, \dots, \not{f}_q, \mathfrak{v}) \leq \mathfrak{G}\left(\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_q, \mathfrak{v}\right)$$

eşitsizliği geçerlidir.

(NFGMU-11)

$$\mathfrak{G}(\not{f}_0, \not{f}_1, \dots, \not{f}_q, \mathfrak{v}) = 1 \Leftrightarrow \not{f}_0 = \not{f}_1 = \cdots = \not{f}_q,$$

geçerlidir.

(NFGMU-12) Herhangi bir $\{0,1, \dots, q\}$ üzerindeki π permütasyonu için

$$\mathfrak{G}(\not{f}_0, \not{f}_1, \dots, \not{f}_q, \mathfrak{v}) = \mathfrak{G}(\not{f}_{\pi(0)}, \not{f}_{\pi(1)}, \dots, \not{f}_{\pi(q)}, \mathfrak{v})$$

şartı sağlanır.

(NFGMU-13) Her $\not{f}_0, \not{f}_1, \dots, \not{f}_{\mathfrak{g}}, \mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_{\mathfrak{f}}, \Upsilon \in \mathcal{Q}$ ve $\mathfrak{g} + \mathfrak{f} + 1 = q$ için

$$\begin{aligned} &\mathfrak{G}\left(\not{f}_0, \not{f}_1, \dots, \not{f}_{\mathfrak{g}}, \Upsilon, \dots, \Upsilon, \mathfrak{u}\right) * \mathfrak{G}(\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_{\mathfrak{f}}, \Upsilon, \dots, \Upsilon, \mathfrak{v}) \\ &\geq \mathfrak{G}\left(\not{f}_0, \not{f}_1, \dots, \not{f}_{\mathfrak{g}}, \mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_{\mathfrak{f}}, \mathfrak{u} + \mathfrak{v}\right) \end{aligned}$$

eşitsizliği sağlanır.

(NFGMU-14)

$$\mathfrak{G}(\not{f}_0, \not{f}_1, \dots, \not{f}_{\mathfrak{g}}, .): (0, \infty) \longrightarrow [0,1]$$

sürekli bir fonksiyondur.

(NFGMU-15) Her $\not{f}_0, \not{f}_1, \dots, \not{f}_q \in \mathcal{Q}$ için

$$\lim_{\mathfrak{v} \to \infty} \mathfrak{G}\left(\not{f}_0, \not{f}_1, \dots, \not{f}_q, \mathfrak{v}\right) = 0$$

sağlanır.

(NFGMU-16) $\not{f}_0 \neq \not{f}_1$ ve her $\not{f}_0, \not{f}_1 \in \mathcal{Q}$ için

$$\mathfrak{H}(\not{f}_0, \not{f}_0, \dots, \not{f}_0, \not{f}_1, \mathfrak{v}) < 1$$

şartı sağlanır.

(NFGMU-17) $\forall \left(\not{f}_0, \not{f}_1, \dots, \not{f}_q\right), \left(\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_q\right) \in \mathcal{Q}^{q+1}$

ve

$$\left\{\not{f}_0, \not{f}_1, \dots, \not{f}_q\right\} \subsetneq \left\{\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_q\right\}$$

için

$$\mathfrak{H}\left(\not{f}_0, \not{f}_1, \dots, \not{f}_q, \mathfrak{v}\right) \leq \mathfrak{H}\left(\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_q, \mathfrak{v}\right)$$

eşitsizliği geçerlidir.

(NFGMU-18)

$$\mathfrak{H}\left(\not{f}_0, \not{f}_1, \dots, \not{f}_q, \mathfrak{v}\right) = 1 \Leftrightarrow \not{f}_0 = \not{f}_1 = \dots = \not{f}_q,$$

geçerlidir.

(NFGMU-19) Herhangi bir $\{0,1, \dots, q\}$ üzerindeki π permütasyonu için

$$\mathfrak{H}\left(\not{f}_0, \not{f}_1, \dots, \not{f}_q, \mathfrak{v}\right) = \mathfrak{H}\left(\not{f}_{\pi(0)}, \not{f}_{\pi(1)}, \dots, \not{f}_{\pi(q)}, \mathfrak{v}\right)$$

şartı sağlanır.

(NFGMU-20) Her $\not{f}_0, \not{f}_1, \dots, \not{f}_{\mathfrak{g}}, \mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_{\mathfrak{f}}, \mathfrak{r} \in \mathcal{Q}$ ve $\mathfrak{g} + \mathfrak{f} + 1 = q$ için

$$\begin{aligned}&\mathfrak{H}\left(\not{f}_0, \not{f}_1, \dots, \not{f}_{\mathfrak{g}}, \mathfrak{r}, \dots, \mathfrak{r}, \mathfrak{u}\right) * \mathfrak{H}\left(\mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_{\mathfrak{f}}, \mathfrak{r}, \dots, \mathfrak{r}, \mathfrak{v}\right) \\ &\geq \mathfrak{H}\left(\not{f}_0, \not{f}_1, \dots, \not{f}_{\mathfrak{g}}, \mathfrak{h}_0, \mathfrak{h}_1, \dots, \mathfrak{h}_{\mathfrak{f}}, \mathfrak{u} + \mathfrak{v}\right)\end{aligned}$$

eşitsizliği sağlanır.

(NFGMU-21)

$$\mathfrak{H}\left(\not{f}_0, \not{f}_1, \dots, \not{f}_{\mathfrak{g}}, .\right) : (0, \infty) \longrightarrow [0,1]$$

sürekli bir fonksiyondur.

(NFGMU-22) Her $ꞙ_0, ꞙ_1, \ldots, ꞙ_q \in Q$ için

$$\lim_{\mathfrak{v}\to\infty} \mathfrak{H}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_q, \mathfrak{v}\right) = 0$$

sağlanır.

Ayrıca, $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ üçlüsüne Q üzerinde nötrosofik fuzzy G-metrik olarak adlandırılır.

Örnek 3.1. (Q, g), q. dereceden bir g-metrik uzay olsun. $\mathfrak{v} > 0$ için $\forall ς_1, ς_2 \in [0,1]$ için $ς_1 \therefore ς_2 = ς_1 \cdot ς_2$ ve $ς_1 * ς_2 = \min\{ς_1 + ς_2, 1\}$ tanımlansın.

$$\mathfrak{F}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_q, \mathfrak{v}\right) = \frac{\mathfrak{v}}{\mathfrak{v} + \mathrm{g}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_q\right)},$$

$$\mathfrak{G}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_q, \mathfrak{v}\right) = \frac{\mathrm{g}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_q\right)}{\mathfrak{v} + \mathrm{g}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_q\right)}$$

ve

$$\mathfrak{H}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_q, \mathfrak{v}\right) = \frac{\mathfrak{v}}{\mathrm{g}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_q\right)}$$

olarak alınırsa $(Q, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU belirtir.

İspat. (NFGMU-6) $\mathfrak{v}, \mathfrak{u} > 0$ ve $\mathfrak{g} + \mathfrak{f} + 1 = q$ için $ꞙ_0, ꞙ_1, \ldots, ꞙ_{\mathfrak{g}}, \mathfrak{h}_0, \mathfrak{h}_1, \ldots, \mathfrak{h}_{\mathfrak{f}}, ɤ \in Q$ olsun. Bu durumda

$$\mathrm{g}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_{\mathfrak{g}}, \mathfrak{h}_0, \mathfrak{h}_1, \ldots, \mathfrak{h}_{\mathfrak{f}}\right) \leq \mathrm{g}\left(ꞙ_0, ꞙ_1, \ldots, ꞙ_{\mathfrak{g}}, ɤ, \ldots, ɤ\right) + \mathrm{g}\left(\mathfrak{h}_0, \mathfrak{h}_1, \ldots, \mathfrak{h}_{\mathfrak{f}}, ɤ, \ldots, ɤ\right) \qquad (3.1)$$

sağlanır.

Bununla birlikte

$$\mathfrak{F}(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r},\mathfrak{v}) \therefore \mathfrak{F}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r},\mathfrak{u}\right) = \frac{\mathfrak{v}}{\mathfrak{v}+\mathrm{g}(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r})}$$

$$\therefore \frac{\mathfrak{u}}{\mathfrak{u}+\mathrm{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r}\right)}$$

$$\leq \frac{\mathfrak{v}\mathfrak{u}}{\mathfrak{v}\mathfrak{u}+\mathfrak{v}.\,\mathrm{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r}\right)+\mathfrak{u}.\,\mathrm{g}(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r})}$$

$$= \frac{1}{1+\dfrac{\mathrm{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r}\right)}{\mathfrak{u}}+\dfrac{\mathrm{g}(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r})}{\mathfrak{v}}} =$$

$$\leq \frac{1}{1+\dfrac{\mathrm{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r}\right)}{\mathfrak{v}+\mathfrak{u}}+\dfrac{\mathrm{g}(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r})}{\mathfrak{v}+\mathfrak{u}}}$$

$$= \frac{\mathfrak{v}+\mathfrak{u}}{\mathfrak{v}+\mathfrak{u}+\mathrm{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r}\right)+\mathrm{g}(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r})}$$

yazılır.

(3.1) eşitsizliği kullanılırsa

$$\mathfrak{F}(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r},\mathfrak{v}) \therefore \mathfrak{F}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r},\mathfrak{u}\right)$$

$$\leq \frac{\mathfrak{v}+\mathfrak{u}}{\mathfrak{v}+\mathfrak{u}+\mathrm{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r}\right)+\mathrm{g}(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r})}$$

$$= \mathfrak{F}\left(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{v}+\mathfrak{u}\right)$$

elde edilir.

(NFGMU-13) Aynı koşulları sağlayan $\mathfrak{v},\mathfrak{u},\mathfrak{g},\mathfrak{f}$ için

$$\mathrm{g}\left(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}}\right) \leq \mathrm{g}\left(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r}\right)+\mathrm{g}(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r})$$

sağlanır. Bu durumda

$$\Rightarrow 1+\frac{\mathfrak{v}+\mathfrak{u}}{\mathrm{g}\left(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}}\right)}$$

$$\geq 1+\frac{\mathfrak{v}+\mathfrak{u}}{\mathrm{g}\left(\mathfrak{x}_0,\mathfrak{x}_1,\dots,\mathfrak{x}_{\mathfrak{g}},\mathfrak{r},\dots,\mathfrak{r}\right)+\mathrm{g}(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{r},\dots,\mathfrak{r})}.$$

$$\Rightarrow 1+\frac{\mathfrak{g}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}}\right)}{\mathfrak{v}+\mathfrak{u}+\mathfrak{g}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}}\right)}$$

$$\leq \frac{\mathfrak{g}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{x},\dots,\mathfrak{x}\right)+\mathfrak{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{x},\dots,\mathfrak{x}\right)}{\mathfrak{v}+\mathfrak{g}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{x},\dots,\mathfrak{x}\right)+\mathfrak{u}+\mathfrak{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{x},\dots,\mathfrak{x}\right)}$$

$$\leq \frac{\mathfrak{g}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{x},\dots,\mathfrak{x}\right)}{\mathfrak{v}+\mathfrak{g}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{x},\dots,\mathfrak{x}\right)}+\frac{\mathfrak{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{x},\dots,\mathfrak{x}\right)}{\mathfrak{u}+\mathfrak{g}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{x},\dots,\mathfrak{x}\right)}$$

elde edilir. Bu durumda

$$\mathfrak{G}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{v}+\mathfrak{u}\right)$$
$$\leq \mathfrak{G}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{x},\dots,\mathfrak{x},\mathfrak{v}\right)+\mathfrak{G}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{x},\dots,\mathfrak{x},\mathfrak{u}\right)$$

bulunur.

$$\mathfrak{G}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{v}+\mathfrak{u}\right)\leq 1$$

olduğundan

$$\mathfrak{G}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{v}+\mathfrak{u}\right)$$
$$\leq \min\left\{\mathfrak{G}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{x},\dots,\mathfrak{x},\mathfrak{v}\right)+\mathfrak{G}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{x},\dots,\mathfrak{x},\mathfrak{u}\right),1\right\}$$
$$=\mathfrak{G}\left(\not{f}_0,\not{f}_1,\dots,\not{f}_{\mathfrak{g}},\mathfrak{x},\dots,\mathfrak{x},\mathfrak{v}\right)*\mathfrak{G}\left(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_{\mathfrak{f}},\mathfrak{x},\dots,\mathfrak{x},\mathfrak{u}\right)$$

sağlanır. Benzer sonuç (NFGMU-20) göz önünde bulundurulduğunda sağlanmış olur.

Aynı örnek, işlemler $\forall\varsigma_1,\varsigma_2\in[0,1]$ için $\varsigma_1 \therefore \varsigma_2=\min\{\varsigma_1,\varsigma_2\}$ ve $\varsigma_1 * \varsigma_2=\max\{\varsigma_1,\varsigma_2\}$ şeklinde tanımlandığında da geçerliliğini korur. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ uzayı g-metrikten üretilmiş olup standart NFGMU belirtir.

Önerme 3.1. Her $\varsigma\in[0,1]$ için $\varsigma\therefore\varsigma=\varsigma$ ve $\varsigma*\varsigma=\varsigma$ için $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ NFGMU yapısı altında

(a)

$$\mathfrak{F}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{z\text{-kez}},\mathscr{p},\dots,\mathscr{p},\mathfrak{v}) \geqslant \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^{z-1}}\right),$$
$$\mathfrak{G}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{z\text{-kez}},\mathscr{p},\dots,\mathscr{p},\mathfrak{v}) \leqslant \mathfrak{G}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^{z-1}}\right),$$
$$\mathfrak{H}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{z\text{-kez}},\mathscr{p},\dots,\mathscr{p},\mathfrak{v}) \leqslant \mathfrak{H}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^{z-1}}\right);$$

(b)

$$\mathfrak{F}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{z\text{-kez}},\mathscr{p},\dots,\mathscr{p},\mathfrak{v}) \geqslant \mathfrak{F}\left(\mathscr{p},\mathscr{f},\dots,\mathscr{f},\frac{\mathfrak{v}}{2^{q-1}}\right),$$
$$\mathfrak{G}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{z\text{-kez}},\mathscr{p},\dots,\mathscr{p},\mathfrak{v}) \leqslant \mathfrak{G}\left(\mathscr{p},\mathscr{f},\dots,\mathscr{f},\frac{\mathfrak{v}}{2^{q-1}}\right),$$
$$\mathfrak{H}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{z\text{-kez}},\mathscr{p},\dots,\mathscr{p},\mathfrak{v}) \leqslant \mathfrak{H}\left(\mathscr{p},\mathscr{f},\dots,\mathscr{f},\frac{\mathfrak{v}}{2^{q-1}}\right)$$

eşitsizlikleri geçerlidir.

İspat. a) (NFGMU-6) koşulu yardımıyla

$$\begin{aligned}
\mathfrak{F}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{z\text{-kez}},\mathscr{p},\dots,\mathscr{p},\mathfrak{v}) &\geq \mathfrak{F}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{(z-1)\text{-kez}},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2}) \therefore \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2}\right) \\
&\geq \mathfrak{F}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{(z-2)\text{-kez}}\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^2}) \therefore \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^2}\right) \therefore \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2}\right)
\end{aligned}$$

Her $\varsigma \in [0,1]$ için $\varsigma \therefore \varsigma = \varsigma$ olduğundan

$$\begin{aligned}
\mathfrak{F}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{z\text{-kez}},\mathscr{p},\dots,\mathscr{p},\mathfrak{v}) &\geq \mathfrak{F}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{(z-2)\text{-kez}},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^2}) \therefore \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^2}\right) \\
&\geq \mathfrak{F}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{(z-3)\text{-kez}},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^3}) \therefore \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^3}\right) \therefore \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^2}\right) \\
&\geq \mathfrak{F}(\underbrace{\mathscr{f},\mathscr{f},\dots,\mathscr{f}}_{(z-3)\text{-kez}},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^3}) \therefore \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^3}\right) \\
&\vdots \\
&\geq \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^{z-1}}\right) \therefore \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^{z-1}}\right) \\
&= \mathfrak{F}\left(\mathscr{f},\mathscr{p},\dots,\mathscr{p},\frac{\mathfrak{v}}{2^{z-1}}\right)
\end{aligned}$$

elde edilir. (NFGMU-13), koşulu yardımıyla

$$\begin{aligned}\mathfrak{G}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{z\text{-kez}},y,\dots,y,\mathfrak{v}) &\leq \mathfrak{G}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{(z-1)\text{-kez}},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2}) * \mathfrak{G}\left(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2}\right)\\ &\leq \mathfrak{G}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{(z-2)\text{-kez}},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^2}) * \mathfrak{G}\left(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^2}\right) * \mathfrak{G}\left(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2}\right)\end{aligned}$$

elde edilir. Her $\varsigma \in [0,1]$ için $\varsigma * \varsigma = \varsigma$ olduğundan

$$\begin{aligned}\mathfrak{G}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{z\text{-kez}},y,\dots,y,\mathfrak{v}) &\leq \mathfrak{G}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{(z-2)\text{-kez}},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^2}) * \mathfrak{G}\left(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^2}\right)\\ &\leq \mathfrak{G}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{(z-3)\text{-kez}},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^3}) * \mathfrak{G}\left(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^3}\right) * \mathfrak{G}\left(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^2}\right)\\ &\geq \mathfrak{G}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{(z-3)\text{-kez}},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^3}) * \mathfrak{G}\left(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^3}\right)\\ &\;\vdots\\ &\leq \mathfrak{G}(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^{z-1}}) * \mathfrak{G}(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^{z-1}})\\ &= \mathfrak{G}(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^{z-1}})\end{aligned}$$

sağlanır.

Benzer şekilde

$$\mathfrak{H}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{z\text{-kez}},y,\dots,y,\mathfrak{v}) \leqslant \mathfrak{H}\left(\mathfrak{x},\mathfrak{y},\dots,\mathfrak{y},\frac{\mathfrak{v}}{2^{z-1}}\right)$$

eşitsizliği de elde edilir.

(b): (a) şıkkından yararlanılarak

$$\begin{aligned}\mathfrak{F}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{z\text{-kez}},\mathfrak{y},\dots,\mathfrak{y},\mathfrak{v}) &= \mathfrak{F}(\underbrace{\mathfrak{y},\mathfrak{y},\dots\mathfrak{y},}_{(q+1-z)\text{-kez}}\mathfrak{x},\dots,\mathfrak{x},\mathfrak{v})\\ &\geq \mathfrak{F}\left(\mathfrak{y},\mathfrak{x},\dots,\mathfrak{x},\frac{\mathfrak{v}}{2^{q-l}}\right),\end{aligned}$$

$$\mathfrak{G}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{z\text{-kez}},\mathfrak{y},\dots,\mathfrak{y},\mathfrak{v}) \leq \mathfrak{G}\left(\mathfrak{y},\mathfrak{x},\dots,\mathfrak{x},\frac{\mathfrak{v}}{2^{q-l}}\right)$$

ve

$$\mathfrak{H}(\underbrace{\mathfrak{x},\mathfrak{x},\dots,\mathfrak{x}}_{z\text{-kez}},\mathfrak{y},\dots,\mathfrak{y},\mathfrak{v}) \leq \mathfrak{H}\left(\mathfrak{y},\mathfrak{x},\dots,\mathfrak{x},\frac{\mathfrak{v}}{2^{q-l}}\right)$$

eşitsizlikleri geçerlidir.

Tanım 3.2. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $\not f_0 \in \mathcal{Q}$ olsun. $\mathfrak{v}>0$ için $\not f_0$ merkezli $r \in (0,1)$ yarıçaplı açık yuvar

$$\mathcal{B}_{\not f_0}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},r)=\{\mu\in\mathcal{Q}:\mathfrak{F}(\not f_0,\mu,\mu,\dots,\mu,\mathfrak{v})>1-r,\mathfrak{G}(\not f_0,\mu,\mu,\dots,\mu,\mathfrak{v})<r \text{ ve } \mathfrak{H}(\not f_0,\mu,\mu,\dots,\mu,\mathfrak{v})<r\}$$

şeklinde tanımlanır.

Uyarı 3.1. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun.

$$\mathcal{T}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}=\{\mathfrak{J}\subset\mathcal{Q}:\ \forall\not f_0\in\mathfrak{J},\exists r\in(0,1) \text{ ve } \mathfrak{v}>0 \text{ için öyle ki } \mathcal{B}_{\not f_0}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},r)\subset\mathfrak{J}\}$$

kümesi $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ tarafından üretilen topoloji tanımlar.

$$\left\{\mathcal{B}_{\not f_0}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\frac{1}{\mathfrak{m}},\frac{1}{\mathfrak{m}}\right)\right\}$$

kümesinin $\not f_0\in\mathcal{Q}$ için yerel bir baz oluşturduğu açıktır. Bu durum $\mathcal{T}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$ topolojisinin birinci sayılabilir olduğunu gösterir. Ek olarak, her açık yuvar $\mathcal{T}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$ topoloji içinde açık küme belirtir.

Tanım 3.3. Bir $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ NFGMU üzerinde tanımlı $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ üçlüsü eğer her $\mathfrak{v}>0$ ve $(\not f_0,\not f_1,\dots,\not f_\mathfrak{q}),(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_\mathfrak{q})\in\mathcal{Q}^{\mathfrak{q}+1}$ için

$$\{\not f_0,\not f_1,\dots,\not f_\mathfrak{q}\}=\{\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_\mathfrak{q}\}$$

eşitliği sağlandığında

$$\mathfrak{F}(\not f_0,\not f_1,\dots,\not f_\mathfrak{q},\mathfrak{v})=\mathfrak{F}(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_\mathfrak{q},\mathfrak{v}),$$
$$\mathfrak{G}(\not f_0,\not f_1,\dots,\not f_\mathfrak{q},\mathfrak{v})=\mathfrak{G}(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_\mathfrak{q},\mathfrak{v})$$

ve

$$\mathfrak{H}(\not f_0,\not f_1,\dots,\not f_\mathfrak{q},\mathfrak{v})=\mathfrak{H}(\mathfrak{h}_0,\mathfrak{h}_1,\dots,\mathfrak{h}_\mathfrak{q},\mathfrak{v})$$

eşitlikleri sağlanıyorsa, lineer bağımsız olarak adlandırılır.

Önerme 3.2. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. Bazı $\mathfrak{d}\in[0,1]$ ve $\mathfrak{v}>0$ için $\mathfrak{F}(\not f_0,\not f_1,\dots,\not f_\mathfrak{q},\mathfrak{v})>1-\mathfrak{d}$, $\mathfrak{G}(\not f_0,\not f_1,\dots,\not f_\mathfrak{q},\mathfrak{v})<\mathfrak{d}$ ve $\mathfrak{H}(\not f_0,\not f_1,\dots,\not f_\mathfrak{q},\mathfrak{v})<\mathfrak{d}$ koşulları sağlanıyorsa aşasıdaki koşullar sağlanır:

a) Eğer $\mathfrak{q}(\{\not f_0,\not f_1,\dots,\not f_\mathfrak{q}\})\geqslant 3$ ise her $\mathrm{j}\in\{0,1,\dots,\mathfrak{q}\}$ için $\not f_\mathrm{j}\in\mathcal{B}_{\not f_0}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},r)$ sağlanır.

b) Eğer $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ lineer bağımsız ve $\mathfrak{q}(\{\not f_0,\not f_1,\dots,\not f_\mathfrak{q}\})\geqslant 2$ ise her $\mathrm{j}\in\{0,1,\dots,\mathfrak{q}\}$ için $\not f_\mathrm{j}\in$

$\mathcal{B}_{\mathscr{f}_0}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathscr{r})$ sağlanır.

NFGMU'ya ilişkin tanım ve temel özellikler sunulduktan sonra, konunun derinlemesine anlaşılmasını destekleyecek temel tanım ve teoremler ele alınacaktır.

Tanım 3.4. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}'$da bir dizi olsun. Eğer her $\mathfrak{d}\in(0,1)$ ve her $\mathfrak{v}>0$ için $\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q\geqslant\hbar_0$ ve $\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q\geqslant\hbar_0$ olduğunda

$$\mathfrak{F}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)>1-\mathfrak{d},\mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)<\mathfrak{d}$$

ve

$$\mathfrak{H}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)<\mathfrak{d}$$

koşullarını sağlayan bir $\hbar_0\in\mathbb{N}$ sayısı bulunuyorsa $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisine $\mathscr{f}\in\mathcal{Q}$ elemanına $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre yakınsaktır denir.

Teorem 3.1. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $\mathcal{T}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$, $\mathcal{Q}$ uzayı üzerinde bir topoloji olsun. $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\mathscr{f}\in\mathcal{Q}$ elemanına yakınsak olması için gerek ve yeter koşul her $\mathfrak{v}>0$ için $\mathfrak{y},\mathfrak{z}\to\infty$ iken $\mathfrak{F}(\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\dots,\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f},\mathfrak{v})\to1$, $\mathfrak{G}(\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\dots,\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f},\mathfrak{v})\to0$ ve $\mathfrak{H}(\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\dots,\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f},\mathfrak{v})\to0$ sağlanmasıdır.

İspat. $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\mathscr{f}\in\mathcal{Q}$ elemanına yakınsak olsun. Bu durumda, her $\mathfrak{d}\in(0,1)$ ve her $\mathfrak{v}>0$ için $\mathfrak{y},\mathfrak{z}\geqslant\hbar_0$ iken $\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\in\mathcal{B}_{\mathscr{f}}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})$ koşulunu sağlayan $\hbar_0\in\mathbb{N}$ sayısı mevcuttur. Bu durumda $\mathfrak{y},\mathfrak{z}\geqslant\hbar_0$ için

$$\mathfrak{F}(\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\dots,\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f},\mathfrak{v})>1-\mathfrak{d},\mathfrak{G}(\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\dots,\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f},\mathfrak{v})<\mathfrak{d},\text{ and }\mathfrak{H}(\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\dots,\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f},\mathfrak{v})<\mathfrak{d}$$

elde edilir. Buradan

$1-\mathfrak{F}(\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\dots,\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f},\mathfrak{v})<\mathfrak{d}$, $\mathfrak{G}(\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\dots,\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f},\mathfrak{v})<\mathfrak{d}$ ve $\mathfrak{H}(\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\dots,\mathscr{f}_{\mathfrak{y}\mathfrak{z}},\mathscr{f},\mathfrak{v})<\mathfrak{d}$ yazılır.

Sonuç olarak, $\mathfrak{y},\mathfrak{z}\to\infty$ iken

$$\mathfrak{F}(\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f}_{\mathfrak{y}\mathfrak{z}},\dots,\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f},\mathfrak{v}) \to 1, \mathfrak{G}(\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f}_{\mathfrak{y}\mathfrak{z}},\dots,\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f},\mathfrak{v}) \to 0$$

ve $\mathfrak{H}(\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f}_{\mathfrak{y}\mathfrak{z}},\dots,\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f},\mathfrak{v}) \to 0$ bulunur.

Tersine, her $\mathfrak{v} > 0$ için $\mathfrak{y},\mathfrak{z} \to \infty$ iken

$$\mathfrak{F}(\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f}_{\mathfrak{y}\mathfrak{z}},\dots,\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f},\mathfrak{v}) \to 1, \mathfrak{G}(\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f}_{\mathfrak{y}\mathfrak{z}},\dots,\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f},\mathfrak{v}) \to 0$$

ve $\mathfrak{H}(\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f}_{\mathfrak{y}\mathfrak{z}},\dots,\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f},\mathfrak{v}) \to 0$ olsun. Buradan, her $\mathfrak{d} \in (0,1)$ ve her $\mathfrak{v} > 0$ için $\mathfrak{y},\mathfrak{z} \geqslant \hbar_0$ iken

$$1 - \mathfrak{F}(\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f}_{\mathfrak{y}\mathfrak{z}},\dots,\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f},\mathfrak{v}) < \mathfrak{d}, \mathfrak{G}(\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f}_{\mathfrak{y}\mathfrak{z}},\dots,\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f},\mathfrak{v}) < \mathfrak{d}$$

ve $\mathfrak{H}(\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f}_{\mathfrak{y}\mathfrak{z}},\dots,\not{f}_{\mathfrak{y}\mathfrak{z}},\not{f},\mathfrak{v}) < \mathfrak{d}$ koşulunu sağlayan $\hbar_0 \in \mathbb{N}$ sayısı mevcuttur. Sonuç olarak, $\mathfrak{y},\mathfrak{z} \geqslant \hbar_0$ iken $\not{f}_{\mathfrak{y}\mathfrak{z}} \in \mathcal{B}_{\not{f}}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})$ olur. Böylece, $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\not{f} \in Q$ elemanına yakınsaklığı elde edilir.

Tanım 3.5. $(Q,\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\not{f}_{\mathfrak{y}\mathfrak{z}})$, Q'da bir dizi olsun. Eğer her $\mathfrak{d} \in (0,1)$ ve her $\mathfrak{v} > 0$ için $\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q} \geqslant \hbar_0$ ve $\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q} \geqslant \hbar_0$ olduğunda

$$\mathfrak{F}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\right) > 1 - \mathfrak{d}, \mathfrak{G}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\right) < \mathfrak{d}$$

ve

$$\mathfrak{H}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\right) < \mathfrak{d}$$

koşullarını sağlayan bir $\hbar_0 \in \mathbb{N}$ sayısı bulunuyorsa $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisine $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre Cauchy dizisi denir.

$(Q,\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ NFGMU içinde her $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ Cauchy dizisi yakınsak ise uzaya tam NFGMU denir.

Teorem 3.2. $(Q,\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. Bu durumda, Q'da her yakınsak dizi bir Cauchy dizisidir.

Tanım 3.6. $(Q,\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\not{f}_{\mathfrak{y}\mathfrak{z}})$, Q'da bir dizi olsun. Her $\mathfrak{d} \in (0,1)$ ve her $\mathfrak{v} > 0$ için $\not{f} \in Q$'nun $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre $(\mathfrak{v},\mathfrak{d})$-komşuluğu

$$\mathcal{V}_{\not{f}}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d}) =\{(\not{f}_1,\not{f}_2,\dots,\not{f}_{\mathfrak{q}}) \in \mathcal{Q}^{\mathfrak{q}}: \mathfrak{F}(\not{f},\not{f}_1,\not{f}_2,\dots,\not{f}_{\mathfrak{q}},\mathfrak{v}) > 1-\mathfrak{d} \text{ ve } \mathfrak{G}(\not{f},\not{f}_1,\not{f}_2,\dots,\not{f}_{\mathfrak{q}},\mathfrak{v}) < \mathfrak{d}, \mathfrak{H}(\not{f},\not{f}_1,\not{f}_2,\dots,\not{f}_{\mathfrak{q}},\mathfrak{v}) < \mathfrak{d}\},$$

şeklinde tanımlanır.

Tanım 3.7. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\not{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}'$da bir dizi olsun. Eğer her $\mathfrak{d} \in (0,1)$ ve her $\mathfrak{v} > 0$ için

$$d_{\mathfrak{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}: \mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) \leqslant 1-\mathfrak{d} \text{ yada } \mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) \geqslant \mathfrak{d}\right\}\right) = 0,$$

veya denk olarak

$$\lim_{\mathfrak{y},\mathfrak{z}\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\left| \left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}: \mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}} \leqslant \mathfrak{y}, \mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}} \leqslant \mathfrak{z}, \mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) \leqslant 1-\mathfrak{d} \text{ yada } \mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) \geqslant \mathfrak{d}\right\}\right| = 0,$$

koşulları sağlanıyorsa $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisine $\not{f} \in \mathcal{Q}$ elemanına $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre istatistiksel yakınsaktır denir ve $\not{f}_{\mathfrak{y}\mathfrak{z}} \overset{st_2-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\rightarrow} \not{f}$ ya da $st_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olarak gösterilir. NFGMU'da tüm istatistiksel yakınsak diziler uzayı $st_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$ ile ifade edilir.

Lemma 3.1. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. Her $\mathfrak{d} \in (0,1)$ ve her $\mathfrak{v} > 0$ için aşağıdaki ifadeler denktir:

(a) $st_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$.

(b)

$$d_{\mathfrak{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}: \mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) > 1-\mathfrak{d} \text{ ve } \mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) < \mathfrak{d}\right\}\right) = 1,$$

(c)

$$d_{\mathfrak{q}}\left(\left\{\left(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)\leqslant 1-\mathfrak{d}\right\}\right)=0,$$

ve

$$d_{\mathfrak{q}}\left(\left\{\left(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)\geqslant\mathfrak{d}\right\}\right)=0,$$

$$d_{\mathfrak{q}}\left(\left\{\left(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)\geqslant\mathfrak{d}\right\}\right)=0.$$

(d)

$$d_{\mathfrak{q}}\left(\left\{\left(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)> 1-\mathfrak{d}\right\}\right)=1,$$

veya

$$d_{\mathfrak{q}}\left(\left\{\left(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\mathfrak{d}\right\}\right)=1,$$

$$d_{\mathfrak{q}}\left(\left\{\left(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\mathfrak{d}\right\}\right)=1.$$

Teorem 3.3. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$, $\mathcal{Q}'$da bir dizi olsun. Her $\mathfrak{d}\in(0,1)$ ve her $\mathfrak{v}>0$ için aşağıdaki ifadeler denktir:

(a) $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$;

(b) $d_{\mathfrak{q}}\left(\left\{\left(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}\right)\notin\mathcal{V}_{\not{f}}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})\right\}\right)=0.$

İspat. İspat, istatistiksel yakınsaklık tanımından doğrudan elde edilir.

Teorem 3.4. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. $\left(\fint_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\mathcal{Q}$'da $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\fint_{\mathfrak{y}\mathfrak{z}}=\fint(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olsun. Bu durumda, her $\mathfrak{d}\in(0,1)$, $\mathfrak{v}>0$ ve keyfi $m\in\{1,2,\dots,\mathfrak{q}\}$ için

$$d_{\mathfrak{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\fint_{\mathfrak{s}_m\mathfrak{w}_m}\notin\mathcal{B}_{\fint}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})\right\}\right)=0$$

sağlanır.

İspat. Her $\mathfrak{d}\in(0,1)$, $\mathfrak{v}>0$ ve her $m\in\{1,2,\dots,\mathfrak{q}\}$ için

$$\begin{aligned}\mathbb{K}(\mathfrak{v},\mathfrak{d})=\Big\{&\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{F}\left(\fint,\fint_{\mathfrak{s}_1\mathfrak{w}_1},\fint_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\fint_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)>1-\mathfrak{d}\\ &\text{and } \mathfrak{G}\left(\fint,\fint_{\mathfrak{s}_1\mathfrak{w}_1},\fint_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\fint_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\mathfrak{d},\mathfrak{H}\left(\fint,\fint_{\mathfrak{s}_1\mathfrak{w}_1},\fint_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\fint_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\mathfrak{d}\Big\},\end{aligned}$$

ve

$$\mathbb{L}(\mathfrak{v},\mathfrak{d})=\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\fint_{\mathfrak{s}_m\mathfrak{w}_m}\in\mathcal{B}_{\fint}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})\right\}$$

tanımlansın.

$$st_2-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\fint_{\mathfrak{y}\mathfrak{z}}=\fint(\mathfrak{F},\mathfrak{G},\mathfrak{H})$$

olduğundan $d_{\mathfrak{q}}(\mathbb{K}(\mathfrak{v},\mathfrak{d}))=1$ yazılır. $(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\in\mathbb{K}(\mathfrak{v},\mathfrak{d})$ alınsın.

Bu durumda

$$\mathfrak{F}\left(\fint,\fint_{\mathfrak{s}_1\mathfrak{w}_1},\fint_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\fint_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)>1-\mathfrak{d},\ \mathfrak{G}\left(\fint,\fint_{\mathfrak{s}_1\mathfrak{w}_1},\fint_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\fint_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\mathfrak{d}\text{ ve}$$

$$\mathfrak{H}\left(\fint,\fint_{\mathfrak{s}_1\mathfrak{w}_1},\fint_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\fint_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\mathfrak{d}$$

olur. Önerme 3.2'e göre her $m\in\{1,2,\dots,\mathfrak{q}\}$ için $\fint_{\mathfrak{s}_m\mathfrak{w}_m}\in\mathcal{B}_{\fint}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})$ elde edilir. Dolayısıyla, $(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\in\mathbb{L}(\mathfrak{v},\mathfrak{d})$, ve bu nedenle $\mathbb{K}(\mathfrak{v},\mathfrak{d})\subseteq\mathbb{L}(\mathfrak{v},\mathfrak{d})$ yazılır. Bu durum, $d_{\mathfrak{q}}(\mathbb{L}(\mathfrak{v},\mathfrak{d}))=1$ eşitliğini sağlar ve istenen sonucu verir.

Teorem 3.5. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. $\left(\fint_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\fint\in\mathcal{Q}$ elemanına $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre yakınsak ise $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\fint_{\mathfrak{y}\mathfrak{z}}=\fint(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlanır.

İspat. $\left(\fint_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\fint\in\mathcal{Q}$ elemanına yakınsak olsun. Bu durumda, her $\mathfrak{d}\in(0,1)$ ve her $\mathfrak{v}>$

0 için $\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_\mathfrak{q} \geqslant \hbar_0$ ve $\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_\mathfrak{q} \geqslant \hbar_0$ olduğunda

$$\mathfrak{F}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{v}\right) > 1 - \mathfrak{d}, \mathfrak{G}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{v}\right) < \mathfrak{d}$$

ve

$$\mathfrak{H}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{v}\right) < \mathfrak{d}$$

koşullarını sağlayan bir $\hbar_0 \in \mathbb{N}$ sayısı bulunur.

$$\begin{aligned}\mathbb{K}(\mathfrak{y}, \mathfrak{z}) = \{&\left((\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_\mathfrak{q}), (\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_\mathfrak{q})\right) \in \mathbb{N}^\mathfrak{q} \times \mathbb{N}^\mathfrak{q} : \mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_\mathfrak{q} \leqslant \mathfrak{y}, \\ &\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_\mathfrak{q} \leqslant \mathfrak{z}, \mathfrak{F}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{v}\right) > 1 - \mathfrak{d}, \\ &\text{ve } \mathfrak{G}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{v}\right) < \mathfrak{d}\Big\}\end{aligned}$$

kümesi tanımlansın. Bu durumda

$$|\mathbb{K}(\mathfrak{y}, \mathfrak{z})| \geqslant \binom{\mathfrak{y}\mathfrak{z} - \hbar_0^2}{\mathfrak{q}}. \Longrightarrow \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^\mathfrak{q}} |\mathbb{K}(\mathfrak{y}, \mathfrak{z})| \geqslant \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^\mathfrak{q}} \binom{\mathfrak{y}\mathfrak{z} - \hbar_0^2}{\mathfrak{q}} = 1,$$

elde edilir. Böylece

$$\lim_{\mathfrak{y},\mathfrak{z}\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^\mathfrak{q}} |(\mathbb{K}(\mathfrak{y}, \mathfrak{z}))^c| = 0,$$

bulunur. Bu ise $st_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ sonucunu verir.

Teorem 3.5'in tersinin geçerli olmadığını göstermek için aşağıdaki örneği sunuyoruz.

Örnek 3.2. Örnek 3.1'de tanımlandığı gibi $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ NFGMU alınsın. Burada, $\mathcal{Q} = \mathbb{R}$ olsun ve $\mathrm{g}: \mathbb{R}^{\mathfrak{q}+1} \longrightarrow \mathbb{R}^+$ fonksiyonu $\mathrm{g}\left(\not{f}_0, \not{f}_1, \ldots, \not{f}_\mathfrak{q}\right) = \max_{0 \leqslant u,v \leqslant \mathfrak{q}} \{|\not{f}_u - \not{f}_v|\}$ şeklinde tanımlansın. $\mathbb{R}$'de $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi, $u, v \in \mathbb{N}$ için

$$\not{f}_{\mathfrak{y}\mathfrak{z}} = \begin{cases} \mathfrak{y}\mathfrak{z}, & \text{eğer } \mathfrak{y} = u^3, \mathfrak{z} = v^3, \\ 1, & \text{diğer durumlarda,} \end{cases}$$

alınsın. Bu durumda, $st_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = 1(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ olduğu halde yakınsak değildir.

Teorem 3.6. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\mathscr{f}\in\mathcal{Q}$ elemanına $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre istatistiksel yakınsak ise $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}$ istatistiksel limiti tektir.

İspat. Kabul edelim ki $\mathscr{f},\mathscr{g}\in\mathcal{Q}$ için $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{g}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olsun. $\mathscr{f}=\mathscr{g}$ olduğu gösterilecektir. $\mathfrak{d}\in(0,1)$ için $(1-\mathfrak{d}_1)\therefore(1-\mathfrak{d}_1)>1-\mathfrak{d}$ ve $\mathfrak{d}_1*\mathfrak{d}_1<\mathfrak{d}$ koşullarını sağlayan $\mathfrak{d}_1\in(0,1)$ seçilsin.

Şimdi, $\mathfrak{v}>0$ için aşağıdaki kümeler tanımlansın:

$$\begin{aligned}\mathbb{A}(\mathfrak{v},\mathfrak{d}_1)\coloneqq\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}:\ &\mathfrak{F}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\\ &\leqslant 1-\mathfrak{d}_1\text{ , yada }\mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\\ &\geqslant\mathfrak{d}_1,\mathfrak{H}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\geqslant\mathfrak{d}_1\Big\},\end{aligned}$$

$$\begin{aligned}\mathbb{B}(\mathfrak{v},\mathfrak{d}_1):=\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}:\ &\mathfrak{F}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\\ &\leqslant 1-\mathfrak{d}_1\text{ , yada }\mathfrak{G}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\\ &\geqslant\mathfrak{d}_1,\mathfrak{H}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\geqslant\mathfrak{d}_1\Big\},\end{aligned}$$

$$\begin{aligned}\mathbb{A}^c(\mathfrak{v},\mathfrak{d}_1):=\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}:\ &\mathfrak{F}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\\ &>1-\mathfrak{d}_1\text{ ve }\mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\\ &<\mathfrak{d}_1,\mathfrak{H}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)<\mathfrak{d}_1\Big\},\end{aligned}$$

$$\begin{aligned}\mathbb{B}^c(\mathfrak{v},\mathfrak{d}_1):=\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}:\ &\mathfrak{F}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\\ &>1-\mathfrak{d}_1\text{ ve }\mathfrak{G}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\\ &<\mathfrak{d}_1,\mathfrak{H}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)<\mathfrak{d}_1\Big\}.\end{aligned}$$

$st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{g}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğundan $d_{\mathrm{q}}\big(\mathbb{A}(\mathfrak{v},\mathfrak{d}_1)\big)=0$ ve $\mathrm{d}_{\mathrm{q}}\big(\mathbb{B}(\mathfrak{v},\mathfrak{d}_1)\big)=0$ elde edilir. Lemma 3.1'den $d_{\mathrm{q}}\big(\mathbb{A}^c(\mathfrak{v},\mathfrak{d}_1)\big)=d_{\mathrm{q}}\big(\mathbb{B}^c(\mathfrak{v},\mathfrak{d}_1)\big)=1$ yazılır. Böylece

$$d_{\mathrm{q}}\big(\mathbb{A}(\mathfrak{v},\mathfrak{d}_1)\cup\mathbb{B}(\mathfrak{v},\mathfrak{d}_1)\big)=0,$$

dolayısıyla

$$d_{\mathrm{q}}\Big(\big(\mathbb{A}(\mathfrak{v},\mathfrak{d}_1)\cup\mathbb{B}(\mathfrak{v},\mathfrak{d}_1)\big)^{\mathrm{c}}\Big)=d_{\mathrm{q}}\big(\mathbb{A}(\mathfrak{v},\mathfrak{d}_1)^{\mathrm{c}}\cap\mathbb{B}(\mathfrak{v},\mathfrak{d}_1)^{\mathrm{c}}\big)=1$$

olur.

$(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\in\mathbb{A}(\mathfrak{v},\mathfrak{d}_1)^{\mathrm{c}}\cap\mathbb{B}(\mathfrak{v},\mathfrak{d}_1)^{\mathrm{c}}$ alınsın.

(NFGMU-3), (NFGMU-6), ve Tanım 2.1’in 3’üncü maddesinden

$$\begin{aligned}\mathfrak{F}(\mathscr{f},\mathscr{g},\dots,\mathscr{g},\mathfrak{v}) &\geqslant \mathfrak{F}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\therefore\mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{g},\dots,\mathscr{g},\frac{\mathfrak{v}}{2}\right)\\ &\geqslant \mathfrak{F}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\therefore\mathfrak{F}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\\ &\geqslant (1-\mathfrak{d}_1)\therefore(1-\mathfrak{d}_1)>1-\mathfrak{d}\end{aligned}$$

elde edilir.

(NFGMU-10), (NFGMU-13), ve Tanım 2.1’in (3′)’üncü maddesinden

$$\begin{aligned}\mathfrak{G}((\mathscr{f},\mathscr{g},\dots,\mathscr{g},\mathfrak{v}) &\leqslant \mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)*\mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{g},\dots,\mathscr{g},\frac{\mathfrak{v}}{2}\right)\\ &\leqslant \mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)*\mathfrak{G}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\frac{\mathfrak{v}}{2}\right)\leqslant\mathfrak{d}_1*\mathfrak{d}_1\\ &<\mathfrak{d}\end{aligned}$$

bulunur.

Aynı zamanda (NFGMU-17), (NFGMU-20), ve Tanım 2.1’in (3′)’üncü maddesinden

$$\mathfrak{H}((\mathscr{f},\mathscr{g},\dots,\mathscr{g},\mathfrak{v})<\mathfrak{d}$$

yazılır. $\mathfrak{d}\in(0,1)$ keyfi sabit değeri ve $\forall\mathfrak{v}>0$ için

$$\mathfrak{F}(\mathscr{f},\mathscr{g},\dots,\mathscr{g},\mathfrak{v})=1\text{ ve }\mathfrak{G}((\mathscr{f},\mathscr{g},\dots,\mathscr{g},\mathfrak{v})=0,\mathfrak{H}((\mathscr{f},\mathscr{g},\dots,\mathscr{g},\mathfrak{v})=0,$$

elde edilir ki bu da $\mathscr{f}=\mathscr{g}$ olması demektir. Böylece $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}$ istatistiksel limitin tekliği gösterilmiş olur.

Tanım 3.8. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $\left(\mathscr{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$, $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin $\mathcal{Q}'$da bir alt dizisi olsun. $\{(\mathfrak{y}_m,\mathfrak{z}_n): m,n\in\mathbb{N}\}$ indeks kümesi $\mathbb{N}^2$'de istatistiksel yoğun ise yani $d_{\mathrm{q}}(\{(\mathfrak{y}_m,\mathfrak{z}_n): m,n\in\mathbb{N}\})=1$ sağlanıyorsa $\left(\mathscr{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ dizisine $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin $\mathcal{Q}'$da istatistiksel yoğun alt dizisi denir.

Teorem 3.7. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)$, $\mathcal{Q}'$da bir dizi olsun. Aşağıdaki ifadeler denktir:

(1) $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$.

(2) Hemen hemen $\mathfrak{y},\mathfrak{z}$ için $\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}$ olacak şekilde $\mathcal{Q}$'da yakınsak $\left(\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi mevcuttur.

(3) $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin, istatistiksel yoğun ve yakınsak olan $\left(\mathscr{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ biçiminde bir alt dizisi vardır.

İspat. $(1)\Longrightarrow(2)$ $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $\mathscr{f}\in\mathcal{Q}$ olsun. Lemma 3.1'e gore bazı $\mathfrak{d}\in(0,1)$ ve $\mathfrak{v}>0$ için

$$d_{\mathrm{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}:\mathfrak{F}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)>1-\mathfrak{d}\right.\right.$$
$$\left.\left.\text{ve } \mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)<\mathfrak{d},\mathfrak{H}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)<\mathfrak{d}\right\}\right)=1,$$

elde edilir.

Her $\mathfrak{r}\in\mathbb{N}$ için

$$\mathcal{S}(\mathfrak{v},\mathfrak{r})=\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}:\mathfrak{F}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)>1-\frac{1}{\mathfrak{r}}\right.$$
$$\left.\text{ve } \mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)<\frac{1}{\mathfrak{r}},\mathfrak{H}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)<\frac{1}{\mathfrak{r}}\right\}$$

kümesi tanımlansın.

Her $\mathfrak{r}\in\mathbb{N}$ için $\mathcal{S}(\mathfrak{v},\mathfrak{r}+1)$ kümesi $\mathcal{S}(\mathfrak{v},\mathfrak{r})$ kümesinin bir alt kümesi olduğu açıktır. $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğundan,

$$d_{\mathrm{q}}(\mathcal{S}(\mathfrak{v},\mathfrak{r})=1 \tag{3.2}$$

eşitliği sağlanır.

Şimdi, rastgele seçilmiş $\left(\mathfrak{s}_1^1,\mathfrak{s}_2^1,\dots,\mathfrak{s}_{\mathfrak{q}}^1\right),\left(\mathfrak{w}_1^1,\mathfrak{w}_2^1,\dots,\mathfrak{w}_{\mathfrak{q}}^1\right)\in\mathcal{S}(\mathfrak{v},1)$ alınsın. Bu durumda

$$\mathfrak{m}_1=\max\left\{\mathfrak{s}_1^1,\mathfrak{s}_2^1,\dots,\mathfrak{s}_{\mathfrak{q}}^1\right\},\mathfrak{n}_1=\max\left\{\mathfrak{w}_1^1,\mathfrak{w}_2^1,\dots,\mathfrak{w}_{\mathfrak{q}}^1\right\}$$

olsun.

(3.2) sağlandığından $\left(\mathfrak{s}_1^2,\mathfrak{s}_2^2,\dots,\mathfrak{s}_{\mathfrak{q}}^2\right),\left(\mathfrak{w}_1^2,\mathfrak{w}_2^2,\dots,\mathfrak{w}_{\mathfrak{q}}^2\right)\in\mathcal{S}(\mathfrak{v},2)$ olacak şekilde $\mathfrak{m}_2>\mathfrak{m}_1$, and $\mathfrak{n}_2>\mathfrak{n}_1$ sağlayan

$$\mathfrak{m}_2=\max\left\{\mathfrak{s}_1^2,\mathfrak{s}_2^2,\dots,\mathfrak{s}_{\mathfrak{q}}^2\right\},\mathfrak{n}_2=\max\left\{\mathfrak{w}_1^2,\mathfrak{w}_2^2,\dots,\mathfrak{w}_{\mathfrak{q}}^2\right\}$$

mevcuttur ve $\mathfrak{y}\geqslant\mathfrak{m}_2,\mathfrak{z}\geqslant\mathfrak{n}_2$ değerleri için

$$\begin{aligned}\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\ &\Big\{\Big(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\Big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\\ &\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{F}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)>1-\frac{1}{2}\\ &\text{ve }\ \mathfrak{G}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\frac{1}{2},\mathfrak{H}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\frac{1}{2}\Big\}\Big|>\frac{1}{2}\end{aligned}$$

yazılır.

Bu süreç, (3.2) kullanarak devam ettirilirse, $\left(\mathfrak{s}_1^3,\mathfrak{s}_2^3,\dots,\mathfrak{s}_{\mathfrak{q}}^3\right),\left(\mathfrak{w}_1^3,\mathfrak{w}_2^3,\dots,\mathfrak{w}_{\mathfrak{q}}^3\right)\in\mathcal{S}(\mathfrak{v},3)$ olacak şekilde,

$$\mathfrak{m}_3=\max\left\{\mathfrak{s}_1^3,\mathfrak{s}_2^3,\dots,\mathfrak{s}_{\mathfrak{q}}^3\right\}>\mathfrak{m}_2\ \text{ ve }\ \mathfrak{n}_3=\max\left\{\mathfrak{w}_1^3,\mathfrak{w}_2^3,\dots,\mathfrak{w}_{\mathfrak{q}}^3\right\}>\mathfrak{n}_2$$

şartlarını sağlayan elemanlar bulunur.

Tüm $\mathfrak{y}\geqslant\mathfrak{m}_3,\mathfrak{z}\geqslant\mathfrak{n}_3$ için,

$$\begin{aligned}\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\ &\Big\{\Big(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\Big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\\ &\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{F}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)>1-\frac{1}{3}\\ &\text{and }\ \mathfrak{G}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\frac{1}{3},\mathfrak{H}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)<\frac{1}{3}\Big\}\Big|>\frac{2}{3}\end{aligned}$$

elde edilir.

Bu süreci sürdürerek, $\left(\mathfrak{s}_1^{\mathfrak{r}},\mathfrak{s}_2^{\mathfrak{r}},\dots,\mathfrak{s}_{\mathfrak{q}}^{\mathfrak{r}}\right),\left(\mathfrak{w}_1^{\mathfrak{r}},\mathfrak{w}_2^{\mathfrak{r}},\dots,\mathfrak{w}_{\mathfrak{q}}^{\mathfrak{r}}\right)\in\mathcal{S}(\mathfrak{v},\mathfrak{r})$ olacak şekilde $\mathfrak{m}_{\mathfrak{r}}=$

$\max\{\mathfrak{s}_1^{\mathfrak{k}}, \mathfrak{s}_2^{\mathfrak{k}}, \dots, \mathfrak{s}_q^{\mathfrak{k}}\}, \mathfrak{n}_{\mathfrak{k}} = \max\{\mathfrak{w}_1^{\mathfrak{k}}, \mathfrak{w}_2^{\mathfrak{k}}, \dots, \mathfrak{w}_q^{\mathfrak{k}}\}$ biçiminde tanımlanan doğal sayılar kümesinde artan $(\mathfrak{m}_{\mathfrak{k}})$ ve $(\mathfrak{n}_{\mathfrak{k}})$ dizileri elde edilir.

Tüm $\mathfrak{y} \geqslant \mathfrak{m}_{\mathfrak{k}}, \mathfrak{z} \geqslant \mathfrak{n}_{\mathfrak{k}}$ için

$$\lim_{\mathfrak{y},\mathfrak{z}\to\infty} \frac{q!}{(\mathfrak{y}\mathfrak{z})^q} \Big| \{\big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\big) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q \leqslant \mathfrak{y},$$
$$\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q \leqslant \mathfrak{z}, \mathfrak{F}\big(\mathfrak{f}, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\big) > 1 - \frac{1}{\mathfrak{k}} \text{ ve}$$
$$\mathfrak{G}\big(\mathfrak{f}, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\big) < \frac{1}{\mathfrak{k}}, \mathfrak{H}\big(\mathfrak{f}, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\big) < \frac{1}{\mathfrak{k}}\Big\} \Big| > \frac{\mathfrak{k}-1}{\mathfrak{k}}$$

bulunur.

Şimdi ise

$$\mathcal{T}_1 = \{(\mathfrak{c}, \mathfrak{e}) \in \mathbb{N}^2 : 1 < \mathfrak{c} < \mathfrak{m}_1, 1 < \mathfrak{e} < \mathfrak{n}_1\},$$

ve

$$\mathcal{T}_2 = \bigcup_{\mathfrak{k}\in\mathbb{N}} \{\mathfrak{c} = \max\{\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_q\}, \mathfrak{e}$$
$$= \max\{\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_q\} : (\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_q),\ (\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_q) \in \mathcal{S}(\mathfrak{v}, \mathfrak{k}), \mathfrak{m}_{\mathfrak{k}} \leqslant \mathfrak{c}$$
$$< \mathfrak{m}_{\mathfrak{k}+1}, \mathfrak{n}_{\mathfrak{k}} \leqslant \mathfrak{e} < \mathfrak{n}_{\mathfrak{k}+1}\}$$

kümeleri tanımlansın.

Son olarak, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ kümesi ve

$$\mathfrak{p}_{\mathfrak{ce}} = \begin{cases} \mathfrak{f}_{\mathfrak{ce}}, & \text{eğer } (\mathfrak{c}, \mathfrak{e}) \in \mathcal{T} \\ \mathfrak{f}, & \text{diğer durumlarda,} \end{cases}$$

dizisi tanımlansın.

$\mathfrak{d} \in (0,1)$ değeri için $\frac{1}{\mathfrak{k}} < \mathfrak{d}$ koşulunu sağlayan $\mathfrak{k} \in \mathbb{N}$ seçilsin. Bu durumda $1 - \frac{1}{\mathfrak{k}} > 1 - \mathfrak{d}$ olur. Bu ise $(\mathfrak{p}_{\mathfrak{ce}})$ dizisinin $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ normuna göre $\mathfrak{f}$'e yakınsadığını gösterir.

Sabit $\mathfrak{y}, \mathfrak{z} \in \mathbb{N}$ değerleri için ve $\mathfrak{m}_{\mathfrak{k}} \leqslant \mathfrak{y} < \mathfrak{m}_{\mathfrak{k}+1}, \mathfrak{n}_{\mathfrak{k}} \leqslant \mathfrak{z} < \mathfrak{n}_{\mathfrak{k}+1}$ olduğundan

$$
\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:
$$
$$
\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{f}_{\mathfrak{s}_m\mathfrak{w}_n}\neq\mathfrak{p}_{\mathfrak{s}_m\mathfrak{w}_n},m,n\in\{1,2,\dots,\mathfrak{q}\}\Big\}
$$
$$
\subseteq\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z}\Big\}
$$
$$
\setminus\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},
$$
$$
\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{F}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)>1-\frac{1}{\mu}\text{ ve}
$$
$$
\mathfrak{G}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\frac{1}{\mu},\mathfrak{H}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\frac{1}{\mu}\Big\}
$$

elde edilir.

Buradan

$$
\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:
$$
$$
\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{f}_{\mathfrak{s}_m\mathfrak{w}_n}\neq\mathfrak{p}_{\mathfrak{s}_m\mathfrak{w}_n},m,n\in\{1,2,\dots,\mathfrak{q}\}\Big\}\Big|
$$
$$
\leqslant\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:
$$
$$
\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z}\Big\}\Big|
$$
$$
-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},
$$
$$
\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{F}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)>1-\frac{1}{\mu}\text{ ve}
$$
$$
\mathfrak{G}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\frac{1}{\mu},\mathfrak{H}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\frac{1}{\mu}\Big\}\Big|
$$
$$
\leqslant 1-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:
$$

$$
\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)>1-\frac{1}{\mu}\text{ ve}
$$
$$
\mathfrak{G}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\frac{1}{\mu},\mathfrak{H}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\frac{1}{\mu}\Big\}\Big|\leqslant\frac{1}{\mu}<\mathfrak{d}
$$

yazılır.

$\mathfrak{d}>0$ keyfi değer olduğundan

$$
\mathrm{d}_{\mathfrak{q}}\Big(\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:
$$
$$
\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{f}_{\mathfrak{s}_m\mathfrak{w}_n}\neq\mathfrak{p}_{\mathfrak{s}_m\mathfrak{w}_n},m,n\in\{1,2,\dots,\mathfrak{q}\}\Big\}\Big)=0
$$

bulunur. Böylece hemen hemen $\mathfrak{y},\mathfrak{z}$ değerleri için $\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}$ elde edilir.

(2) $\Longrightarrow$ (3) Hemen hemen $\mathfrak{y},\mathfrak{z}$ için $\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}$ olacak şekilde $\mathcal{Q}$'da yakınsak $(\mathfrak{p}_{\mathfrak{y}\mathfrak{z}})$ dizisi alınsın. Bu durumda, $\mathbb{A}=\{(\mathfrak{y},\mathfrak{z})\in\mathbb{N}^2:\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}\}$ kümesi için $\mathrm{d}_{\mathfrak{q}}(\mathbb{A})=1$ olur. Sonuç olarak, $(\mathfrak{p}_{\mathfrak{y}\mathfrak{z}})_{(\mathfrak{y},\mathfrak{z})\in\mathbb{A}}$ dizisi $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin yakınsak istatistiksel yoğun alt dizisidir.

(3) $\Longrightarrow$ (1) $(\mathfrak{f}_{\mathfrak{y}_m\mathfrak{z}_n})$ dizisi, $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin istatistiksel olarak yoğun bir alt dizisi olup $\mathfrak{f}\in\mathcal{Q}$ elamanına istatistiksel yakınsak olsun. $\mathbb{K}=\{(\mathfrak{y}_m,\mathfrak{z}_n):m,n\in\mathbb{N}\}$ indeks kümesi için $\mathrm{d}_{\mathfrak{q}}(\mathbb{K})=1$ sağlanır. Bazı $\mathfrak{d}\in(0,1)$ ve $\mathfrak{v}>0$ değerleri için

$$\begin{aligned}&\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\\&\quad\leqslant\mathfrak{z},\mathfrak{F}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)>1-\mathfrak{d}\text{ ve }\mathfrak{G}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)\\&\quad<\mathfrak{d},\mathfrak{H}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\mathfrak{d}\Big\}\\&\quad\supseteq\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{K}^{\mathfrak{q}}\times\mathbb{K}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\\&\quad\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{F}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)\\&\quad>1-\mathfrak{d}\text{ ve }\mathfrak{G}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)\\&\quad<\mathfrak{d},\mathfrak{H}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\mathfrak{d}\Big\}\end{aligned}$$

elde edilir. Böylece

$$\begin{aligned}&\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\\&\quad\leqslant\mathfrak{z},\mathfrak{F}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)>1-\mathfrak{d}\text{ ve }\mathfrak{G}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)\\&\quad<\mathfrak{d},\mathfrak{H}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\mathfrak{d}\Big\}\Big|\\&\quad\geqslant\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{K}^{\mathfrak{q}}\times\mathbb{K}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\\&\quad\leqslant\mathfrak{y},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{F}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)\\&\quad>1-\mathfrak{d}\text{ ve }\mathfrak{G}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)\\&\quad<\mathfrak{d},\mathfrak{H}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big)<\mathfrak{d}\Big\}\Big|\end{aligned}$$

yazılır. Böylece, $st_2-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ bulunur.

Teorem 3.7 doğrudan aşağıdaki sonucu vermektedir.

Sonuç 3.1. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU ve $\left(\nmid_{\mathfrak{y}\mathfrak{z}}\right)$, $\mathcal{Q}'$da istatistiksel yakınsak bir dizi olsun. Bu durumda, $\left(\nmid_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin yakınsak bir alt dizisi vardır.

Yukarıdaki ifadenin tersi genel olarak geçerli değildir; başka bir deyişle, yakınsak bir alt dizi içeren ancak istatistiksel yakınsak olmayan diziler mevcuttur. Aşağıdaki örnek bu durumu açıklamaktadır.

Örnek 3.3. $(\mathbb{R}, \mathrm{g})$ uzayı

$$g\left(\nmid_0, \nmid_1, \dots, \nmid_{\mathfrak{q}}\right) = \max_{0 \leqslant \mathfrak{m}, \mathfrak{n} \leqslant \mathfrak{q}} \{|\nmid_{\mathfrak{m}} - \nmid_{\mathfrak{n}}|\}, \forall \nmid_0, \nmid_1, \dots, \nmid_{\mathfrak{q}} \in \mathbb{R}$$

$\mathfrak{q}$. dereceden bir g-metrik uzay olsun.

Örnek 3.1'de tanımlanan $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ üçlüsünü ele alınsın. $\forall \mathfrak{d}_1, \mathfrak{d}_2 \in [0,1]$ içi $\mathfrak{d}_1 \therefore \mathfrak{d}_2 = \min\{\mathfrak{d}_1, \mathfrak{d}_2\}$ ve $\mathfrak{d}_1 * \mathfrak{d}_2 = \max\{\mathfrak{d}_1, \mathfrak{d}_2\}$ işlemleri tanımlansın. Bu durumda, $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ NFGMU oluşturmaktadır. Şimdi $\mathfrak{m}, \mathfrak{n} \in \mathbb{N}$ için $\left(\nmid_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi

$$\nmid_{\mathfrak{y}\mathfrak{z}} = \begin{cases} \dfrac{1}{\mathfrak{y}\mathfrak{z}}, & \text{eğer } \mathfrak{y} = \mathfrak{m}^2, \mathfrak{z} = \mathfrak{n}^2, \\ (\mathfrak{y}\mathfrak{z})^2, & \text{diğer durumlarda,} \end{cases}$$

şeklinde tanımlansın. Böylece $\left(\nmid_{\mathfrak{m}^2\mathfrak{n}^2}\right)$ dizisi $\left(\nmid_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin bir alt dizisini oluşturmakta olup, 0'a yakınsaktır. Ancak $\left(\nmid_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin istatistiksel yakınsak olmadığı açıktır.

Şimdi, $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ NFGMU'larda çift diziler için istatistiksel Cauchy dizi kavramı tanımlanacak ve bu kavramın temel özellikleri ayrıntılı olarak incelenecektir.

Tanım 3.9. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU ve $\left(\nmid_{\mathfrak{y}\mathfrak{z}}\right)$, $\mathcal{Q}'$da bir dizi olsun. Eğer her $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için

$$\begin{aligned} \mathrm{d}_{\mathfrak{q}} \Big(\big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}) \big) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} &: \mathfrak{F}\left(\nmid_{\mathfrak{s}_1\mathfrak{w}_1}, \nmid_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \nmid_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \nmid_{\mathbb{U}\mathbb{V}}, \mathfrak{v}\right) \\ &\leqslant 1 - \mathfrak{d} \text{ yada } \mathfrak{G}\left(\nmid_{\mathfrak{s}_1\mathfrak{w}_1}, \nmid_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \nmid_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \nmid_{\mathbb{U}\mathbb{V}}, \mathfrak{v}\right) \\ &\geqslant \mathfrak{d}, \mathfrak{H}\left(\nmid_{\mathfrak{s}_1\mathfrak{w}_1}, \nmid_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \nmid_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \nmid_{\mathbb{U}\mathbb{V}}, \mathfrak{v}\right) \geqslant \mathfrak{d} \Big) = 0 \end{aligned}$$

koşulunu sağlayan $\mathbb{U} = \mathbb{U}(\mathfrak{d}) \in \mathbb{N}, \mathbb{V} = \mathbb{V}(\mathfrak{d}) \in \mathbb{N}$ varsa $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisine $\mathcal{Q}'$da $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ normuna göre istatistiksel Cauchy dizisi denir.

Şimdi ise NFGMU'larda istatistiksel yakınsak diziler ve istatistiksel Cauchy dizisi arasındaki ilişki verilecektir.

Teorem 3.8. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU ve $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)$, $\mathcal{Q}'$da istatistiksel yakınsak bir dizi olsun. Bu durumda, $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)$ $\mathcal{Q}'$da istatistiksel Cauchy dizisidir.

İspat. $st_2 - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} \mathscr{f}_{\mathfrak{y}\mathfrak{z}} = \mathscr{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ olsun. Verilen $\mathfrak{d} \in (0,1)$ sayısı için $(1 - \mathfrak{d}_1) \therefore (1 - \mathfrak{d}_1) > 1 - \mathfrak{d}$ ve $\mathfrak{d}_1 * \mathfrak{d}_1 < \mathfrak{d}$ koşulunu sağlayan bir $\mathfrak{d}_1 \in (0,1)$ seçilsin. Keyfi bir $\mathfrak{v} > 0$ için

$$P(\mathfrak{d}_1) = \Big\{\Big(\left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_\mathrm{q}\right), \left(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_\mathrm{q}\right)\Big) \in \mathbb{N}^\mathrm{q} \times \mathbb{N}^\mathrm{q} \colon \mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) \leqslant 1 - \mathfrak{d}_1 \text{ yada } \mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) \geqslant \mathfrak{d}_1, \mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) \geqslant \mathfrak{d}_1\Big\},$$

ve

$$P^c(\mathfrak{d}_1) = \Big\{\Big(\left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_\mathrm{q}\right), \left(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_\mathrm{q}\right)\Big) \in \mathbb{N}^\mathrm{q} \times \mathbb{N}^\mathrm{q} \colon \mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) > 1 - \mathfrak{d}_1 \text{ ve } \mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1, \mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1\Big\}$$

kümeleri tanımlansın.

$st_2 - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} \mathscr{f}_{\mathfrak{y}\mathfrak{z}} = \mathscr{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ olmasından $\mathrm{d}_\mathrm{q}\big(P(\mathfrak{d}_1)\big) = 0$ ve $\mathrm{d}_\mathrm{q}\big(P^c(\mathfrak{d}_1)\big) = 1$ olur.

$\left(\mathscr{k}_1, \mathscr{k}_2, \dots, \mathscr{k}_\mathrm{q}\right), \left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_\mathrm{q}\right) \in P^c(\mathfrak{d}_1)$ alınsın. Bu durumda

$$\mathfrak{F}\left(\mathscr{f}_{\mathscr{k}_1\mathfrak{h}_1}, \mathscr{f}_{\mathscr{k}_2\mathfrak{h}_2}, \dots, \mathscr{f}_{\mathscr{k}_\mathrm{q}\mathfrak{h}_\mathrm{q}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) > 1 - \mathfrak{d}_1,$$

ve

$$\mathfrak{G}\left(\mathscr{f}_{\mathscr{k}_1\mathfrak{h}_1}, \mathscr{f}_{\mathscr{k}_2\mathfrak{h}_2}, \dots, \mathscr{f}_{\mathscr{k}_\mathrm{q}\mathfrak{h}_\mathrm{q}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1, \mathfrak{H}\left(\mathscr{f}_{\mathscr{k}_1\mathfrak{h}_1}, \mathscr{f}_{\mathscr{k}_2\mathfrak{h}_2}, \dots, \mathscr{f}_{\mathscr{k}_\mathrm{q}\mathfrak{h}_\mathrm{q}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1$$

yazılır.

Bazı $\mathfrak{y}, \mathfrak{z} \in \{1,2, \dots, \mathfrak{q}\}$ için sabit $\mathscr{k}_{\mathfrak{y}}, \mathfrak{h}_{\mathfrak{z}} \in \mathbb{N}$ değerleri için,

$$\mathfrak{F}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathscr{f}, \dots, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) \geqslant \mathfrak{F}\left(\mathscr{f}_{\mathscr{k}_1\mathfrak{h}_1}, \mathscr{f}_{\mathscr{k}_2\mathfrak{h}_2}, \dots, \mathscr{f}_{\mathscr{k}_{\mathfrak{q}}\mathfrak{h}_{\mathfrak{q}}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) > 1 - \mathfrak{d}_1$$

ve

$$\mathfrak{G}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathscr{f}, \dots, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) \leqslant \mathfrak{G}\left(\mathscr{f}_{\mathscr{k}_1\mathfrak{h}_1}, \mathscr{f}_{\mathscr{k}_2\mathfrak{h}_2}, \dots, \mathscr{f}_{\mathscr{k}_{\mathfrak{q}}\mathfrak{h}_{\mathfrak{q}}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1,$$
$$\mathfrak{H}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathscr{f}, \dots, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) \leqslant \mathfrak{H}\left(\mathscr{f}_{\mathscr{k}_1\mathfrak{h}_1}, \mathscr{f}_{\mathscr{k}_2\mathfrak{h}_2}, \dots, \mathscr{f}_{\mathscr{k}_{\mathfrak{q}}\mathfrak{h}_{\mathfrak{q}}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1$$

elde edilir.

$(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}) \in P^c(\mathfrak{d}_1)$ için

$$\begin{aligned}\mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathfrak{v}\right) &\geqslant \mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) \therefore \mathfrak{F}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathscr{f}, \dots, \mathscr{f}, \frac{\mathfrak{v}}{2}\right)\\ &> (1 - \mathfrak{d}_1) \therefore (1 - \mathfrak{d}_1) > 1 - \mathfrak{d}\end{aligned}$$

ve

$$\begin{aligned}\mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathfrak{v}\right) &\leqslant \mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) * \mathfrak{G}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathscr{f}, \dots, \mathscr{f}, \frac{\mathfrak{v}}{2}\right)\\ &< \mathfrak{d}_1 * \mathfrak{d}_1 < \mathfrak{d}\end{aligned}$$

$$\begin{aligned}\mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathfrak{v}\right) &\leqslant \mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathscr{f}, \frac{\mathfrak{v}}{2}\right) * \mathfrak{H}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathscr{f}, \dots, \mathscr{f}, \frac{\mathfrak{v}}{2}\right)\\ &< \mathfrak{d}_1 * \mathfrak{d}_1 < \mathfrak{d}\end{aligned}$$

sağlanır. Böylece

$$\begin{aligned}P^c(\mathfrak{d}_1) \subseteq \Big\{&\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathfrak{v}\right)\\ &> 1 - \mathfrak{d} \text{ ve } \mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathfrak{v}\right)\\ &< \mathfrak{d}, \mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}}, \mathfrak{v}\right) < \mathfrak{d}\Big\}\end{aligned}$$

yazılır.

Sonuç olarak

$$\mathrm{d}_{\mathfrak{q}}\big(P^{c}(\mathfrak{d}_1)\big) \leqslant \mathrm{d}_{\mathfrak{q}}\Big(\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\big)\Big)\in \mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:$$
$$\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{k_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\Big) > 1-\mathfrak{d} \text{ ve}$$
$$\mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{k_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\Big) < \mathfrak{d}, \mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{k_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\Big) < \mathfrak{d}\Big)$$

elde edilir.

Dolayısıyla

$$\mathrm{d}_{\mathfrak{q}}\Big(\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\big)\Big)\in \mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:$$
$$\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{k_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\Big) > 1-\mathfrak{d} \text{ ve}$$
$$\mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{k_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\Big) < \mathfrak{d}, \mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{k_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\Big) < \mathfrak{d}\Big) = 1$$

olup

$$\mathrm{d}_{\mathfrak{q}}\Big(\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\big)\Big)\in \mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:$$
$$\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{k_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\Big) \leq 1-\mathfrak{d} \text{ yada}$$
$$\mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{k_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\Big) \geq \mathfrak{d}, \mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{k_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\Big) \geq \mathfrak{d}\Big) = 0$$

sağlanmış olur. Böylece teorem ispatlanmış olur.

Teorem 3.8'nin tersi genellikle geçerli değildir. Bunu göstermek için aşağıdaki örneği ele alınsın.

Örnek 3.4. $\mathcal{Q} = (0,1]$ olsun. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ uzayı Örnek 3.2'de tanımlanan NFGMU olsun. $\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$ dizisi

$$\not{f}_{\mathfrak{y}\mathfrak{z}} = \begin{cases} 1, & \text{eğer } \mathfrak{y} = k^3, \mathfrak{z} = l^3, \\ \dfrac{1}{\mathfrak{y}\mathfrak{z}}, & \text{diğer durumlarda}, \ (\forall k, l \in \mathbb{N}) \end{cases}$$

şeklinde tanımlansın. Bu durumda, $\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$ dizisi $\mathcal{Q}$'da istatistiksel Cauchy dizisidir ancak istatistiksel yakınsak değildir.

Tanım 3.10. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ uzayında her istatistiksel Cauchy dizisi istatistiksel yakınsak ise uzaya istatistiksel tam uzay denir.

Teorem 3.9. Eğer $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ istatistiksel tam uzay ise tam uzaydır.

İspat. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ istatistiksel tam uzay olsun. Eğer $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\mathcal{Q}$'da bir Cauchy dizisi ise, bu durumda $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\mathcal{Q}$'da istatistiksel Cauchy dizisidir. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ uzayı istatistiksel tam uzay olduğundan istatistiksel yakınsaktır. Sonuç 3.1.'e göre $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n})$ yakınsak bir alt dizisi vardır. $(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n})$ dizisi $\not{f}$'e yakınsak olsun. Verilen $\mathfrak{d} \in (0,1)$ sayısı için $(1-\mathfrak{d}_2) \therefore (1-\mathfrak{d}_2) > 1-\mathfrak{d}_1$, $\mathfrak{d}_3 * \mathfrak{d}_3 < \mathfrak{d}_1$, $(1-\mathfrak{d}_1) \therefore (1-\mathfrak{d}_4) > 1-\mathfrak{d}$ ve $\mathfrak{d}_1 * \mathfrak{d}_4 < \mathfrak{d}$ koşullarını sağlayan $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4 \in (0,1)$ değerleri mevcuttur.

$\mathfrak{d}_5 = \min\{\mathfrak{d}_2, \mathfrak{d}_3\}$ olsun. $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ Cauchy dizisi olduğundan verilen $\mathfrak{v} > 0$ için $\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_\mathrm{q} \geqslant \mathfrak{y}_0$ ve $\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_\mathrm{q} \geqslant \mathfrak{z}_0$ olduğunda

$$\mathfrak{F}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \frac{\mathfrak{v}}{4}\right) > 1-\mathfrak{d}_5,$$

ve

$$\mathfrak{G}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \frac{\mathfrak{v}}{4}\right) < \mathfrak{d}_5,$$

$$\mathfrak{H}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \frac{\mathfrak{v}}{4}\right) < \mathfrak{d}_5$$

koşulunu sağlayan $\exists \mathfrak{y}_0, \mathfrak{z}_0 \in \mathbb{N}$ mevcuttur. Aynı zamanda, $(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n})$ dizisi $\not{f}$'e yakınsaktır. Bu durumda $\mathfrak{s}_{\mathfrak{y}_1}, \mathfrak{s}_{\mathfrak{y}_2}, \dots, \mathfrak{s}_{\mathfrak{y}_\mathrm{q}} \geqslant \mathfrak{y}_1$ ve $\mathfrak{w}_{\mathfrak{z}_1}, \mathfrak{w}_{\mathfrak{z}_2}, \dots, \mathfrak{w}_{\mathfrak{z}_\mathrm{q}} \geqslant \mathfrak{z}_1$ olduğunda

$$\mathfrak{F}\left(\not{f}_{\mathfrak{s}_{\mathfrak{y}_1}\mathfrak{w}_{\mathfrak{z}_1}}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_2}\mathfrak{w}_{\mathfrak{z}_2}}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_\mathrm{q}}\mathfrak{w}_{\mathfrak{z}_\mathrm{q}}}, \not{f}, \frac{\mathfrak{v}}{4}\right) > 1-\mathfrak{d}_4,$$

ve

$$\mathfrak{G}\left(\not{f}_{\mathfrak{s}_{\mathfrak{y}_1}\mathfrak{w}_{\mathfrak{z}_1}}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_2}\mathfrak{w}_{\mathfrak{z}_2}}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_\mathrm{q}}\mathfrak{w}_{\mathfrak{z}_\mathrm{q}}}, \not{f}, \frac{\mathfrak{v}}{4}\right) < \mathfrak{d}_4,$$

$$\mathfrak{H}\left(\not{f}_{\mathfrak{s}_{\mathfrak{y}_1}\mathfrak{w}_{\mathfrak{z}_1}}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_2}\mathfrak{w}_{\mathfrak{z}_2}}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_\mathrm{q}}\mathfrak{w}_{\mathfrak{z}_\mathrm{q}}}, \not{f}, \frac{\mathfrak{v}}{4}\right) < \mathfrak{d}_4$$

koşullarını sağlayan $\mathfrak{y}_1, \mathfrak{z}_1 \in \mathbb{N}$ mevcuttur.

$\mathbb{U} = \max\{\mathfrak{y}_0, \mathfrak{y}_1\}$, $\mathbb{V} = \max\{\mathfrak{z}_0, \mathfrak{z}_1\}$ olarak tanımlansın. $\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_\mathrm{q}, \mathfrak{s}_{\mathfrak{y}_1}, \mathfrak{s}_{\mathfrak{y}_2}, \dots, \mathfrak{s}_{\mathfrak{y}_\mathrm{q}} \geqslant \mathbb{U}$,

$\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q, \mathfrak{w}_{\mathfrak{z}_1}, \mathfrak{w}_{\mathfrak{z}_2}, \dots, \mathfrak{w}_{\mathfrak{z}_q} \geqslant \mathbb{V}$ için (NFGMU-3) ve (NFGMU-6) kullanıldığında

$$\begin{aligned}
\mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}, \mathfrak{v}\right) &\geqslant \mathfrak{F}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) \\
&\therefore \mathfrak{F}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \dots, , \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}, \frac{\mathfrak{v}}{2}\right) \geqslant \mathfrak{F}\left(\not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \not{f}, \frac{\mathfrak{v}}{4}\right) \\
&\therefore \mathfrak{F}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \dots, \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \frac{\mathfrak{v}}{4}\right) \therefore \mathfrak{F}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) \\
&\geqslant \mathfrak{F}\left(\not{f}_{\mathfrak{s}_{\mathfrak{y}_1}\mathfrak{w}_{\mathfrak{z}_1}}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_2}\mathfrak{w}_{\mathfrak{z}_2}}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \not{f}, \frac{\mathfrak{v}}{4}\right) \\
&\therefore \mathfrak{F}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_1}\mathfrak{w}_{\mathfrak{z}_1}}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_2}\mathfrak{w}_{\mathfrak{z}_2}}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \frac{\mathfrak{v}}{4}\right) \therefore \mathfrak{F}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{4}\right) \\
&> (1-\mathfrak{d}_4) \therefore (1-\mathfrak{d}_5) \therefore (1-\mathfrak{d}_5) > (1-\mathfrak{d}_4) \therefore (1-\mathfrak{d}_2) \therefore (1-\mathfrak{d}_2) \\
&> (1-\mathfrak{d}_4) \therefore (1-\mathfrak{d}_1) > (1-\mathfrak{d})
\end{aligned}$$

elde edilir.

Benzer şekilde (NFGMU-10) ve (NFGMU-13) kullanıldığında

$$\begin{aligned}
&\mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}, \mathfrak{v}\right) \\
&\qquad \leqslant \mathfrak{G}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) * \mathfrak{G}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \dots, , \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}, \frac{\mathfrak{v}}{2}\right) \\
&\qquad \leqslant \mathfrak{G}\left(\not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \not{f}, \frac{\mathfrak{v}}{4}\right) * \mathfrak{G}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \dots, \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \frac{\mathfrak{v}}{4}\right) \\
&\qquad * \mathfrak{G}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) \\
&\qquad \leqslant \mathfrak{G}\left(\not{f}_{\mathfrak{s}_{\mathfrak{y}_1}\mathfrak{w}_{\mathfrak{z}_1}}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_2}\mathfrak{w}_{\mathfrak{z}_2}}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \not{f}, \frac{\mathfrak{v}}{4}\right) \\
&\qquad * \mathfrak{G}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_1}\mathfrak{w}_{\mathfrak{z}_1}}, \not{f}_{\mathfrak{s}_{\mathfrak{y}_2}\mathfrak{w}_{\mathfrak{z}_2}}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \frac{\mathfrak{v}}{4}\right) * \mathfrak{G}\left(\not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \not{f}_{\mathfrak{s}_0\mathfrak{w}_0}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{y}_q}\mathfrak{w}_{\mathfrak{z}q}}, \frac{\mathfrak{v}}{4}\right) \\
&\qquad < \mathfrak{d}_4 * \mathfrak{d}_5 * \mathfrak{d}_5 < \mathfrak{d}_4 * \mathfrak{d}_3 * \mathfrak{d}_3 < \mathfrak{d}_4 * \mathfrak{d}_1 < \mathfrak{d}.
\end{aligned}$$

bulunur.

Benzer şekilde

$$\mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}, \mathfrak{v}\right) < \mathfrak{d}$$

yazılır.

Böylece $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\not{f}$’e yakınsaktır ve sonuç olarak $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ uzayı tam uzaydır.

Tanım 3.11. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}'$da bir dizi olsun. Eğer bazı $\mathscr{f}_0 \in \mathcal{Q}$ için,

$$\mathrm{d}_{\mathrm{q}}\Big(\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\big)\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}}:$$
$$\mathfrak{F}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f}_0,\mathfrak{v}_0\Big) \le 1-\mathfrak{d}_0 \text{ and}$$
$$\mathfrak{G}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f}_0,\mathfrak{v}_0\Big) \ge \mathfrak{d}_0, \mathfrak{H}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f}_0,\mathfrak{v}_0\Big) \ge \mathfrak{d}_0\Big) = 0$$

koşunu sağlayan $\mathfrak{d}_0 \in (0,1)$ ve $\mathfrak{v}_0 > 0$ mevcut ise $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisine $\mathcal{Q}'$da istatistiksel sınırlı dizi denir.

Teorem 3.10. Eğer $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\mathcal{Q}$ NFGMU'da istatistiksel Cauchy dizisi ise, istatistiksel sınırlıdır.

İspat. $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\mathcal{Q}$'da istatistiksel Cauchy dizisi olsun. Bu durumda, her $\mathfrak{d} \in (0,1)$, $\mathfrak{v} > 0$ ve $\mathbb{U} = \mathbb{U}(\mathfrak{d}) \in \mathbb{N}, \mathbb{V} = \mathbb{V}(\mathfrak{d}) \in \mathbb{N}$ için asimptotik yoğunluğu 1 olan

$$\mathbb{K}(\mathfrak{v},\mathfrak{d}) = \Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\big)\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}}: \mathfrak{F}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f}_{\mathbb{U}\mathbb{V}},\mathfrak{v}\Big)$$
$$> 1-\mathfrak{d} \text{ or } \mathfrak{G}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f}_{\mathbb{U}\mathbb{V}},\mathfrak{v}\Big)$$
$$< \mathfrak{d}, \mathfrak{H}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f}_{\mathbb{U}\mathbb{V}},\mathfrak{v}\Big) < \mathfrak{d}\Big\}$$

kümesi vardır. Sabit bir $\mathscr{f}_0 \in \mathcal{Q}$ değeri için

$$\mathfrak{F}\left(\mathscr{f}_{\mathbb{U}\mathbb{V}},\mathscr{f}_{\mathbb{U}\mathbb{V}},\dots,\mathscr{f}_{\mathbb{U}\mathbb{V}},\mathscr{f}_0,\frac{\mathfrak{v}}{2}\right) = \alpha,$$

$$\mathfrak{G}\left(\mathscr{f}_{\mathbb{U}\mathbb{V}},\mathscr{f}_{\mathbb{U}\mathbb{V}},\dots,\mathscr{f}_{\mathbb{U}\mathbb{V}},\mathscr{f}_0,\frac{\mathfrak{v}}{2}\right) = \gamma,$$

$$\mathfrak{H}\left(\mathscr{f}_{\mathbb{U}\mathbb{V}},\mathscr{f}_{\mathbb{U}\mathbb{V}},\dots,\mathscr{f}_{\mathbb{U}\mathbb{V}},\mathscr{f}_0,\frac{\mathfrak{v}}{2}\right) = \psi$$

biçiminde tanımlansın. $\alpha,\gamma,\psi \in (0,1)$ olduğundan $(1-\mathfrak{d}) \therefore \alpha > 1-\beta$, $\mathfrak{d} * \gamma < \delta$ ve $\delta * \psi < \tau$ koşullarını sağlayan $\beta,\delta,\tau \in (0,1)$ değerleri mevcuttur. $\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\big) \in \mathbb{K}(\mathfrak{v},\mathfrak{d})$ alınsın. Bu durumda

$$\mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \geqslant \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_{\mathbb{UV}}, \frac{\mathfrak{v}}{2}\right)$$

$$\therefore \mathfrak{F}\left(\not{f}_{\mathbb{UV}}, \not{f}_{\mathbb{UV}}, \dots, \not{f}_{\mathbb{UV}}, \not{f}_0, \frac{\mathfrak{v}}{2}\right) > (1-\mathfrak{d}) \therefore \alpha > 1-\beta,$$

$$\begin{aligned}\mathfrak{G}&\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right)\\ &\leqslant \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_{\mathbb{UV}}, \frac{\mathfrak{v}}{2}\right) * \mathfrak{G}\left(\not{f}_{\mathbb{UV}}, \not{f}_{\mathbb{UV}}, \dots, \not{f}_{\mathbb{UV}}, \not{f}_0, \frac{\mathfrak{v}}{2}\right) < \mathfrak{d} * \gamma\\ &< \delta,\end{aligned}$$

ve benzer şekilde $\mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) < \tau$ olur. $\mathfrak{d}_0 = \max\{\beta, \delta, \tau\}$ olarak alınırsa

$$\begin{aligned}\mathbb{K}(\mathfrak{v}, \mathfrak{d}) \subseteq \Big\{&\left(\left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q\right), \left(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q\right)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right)\\ &> 1-\mathfrak{d}_0 \text{ yada } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right)\\ &< \mathfrak{d}_0, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) < \mathfrak{d}_0\Big\}\end{aligned}$$

sağlanır ve dolayısıyla

$$\begin{aligned}&\mathrm{d}_q\Big(\left(\left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q\right), \left(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q\right)\right) \in \mathbb{N}^q \times \mathbb{N}^q :\\ &\mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) > 1-\mathfrak{d}_0 \text{ ve}\\ &\mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) < \mathfrak{d}_0, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) < \mathfrak{d}_0\Big) = 1\end{aligned}$$

olur. Sonuç olarak

$$\begin{aligned}&\mathrm{d}_q\Big(\left(\left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q\right), \left(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q\right)\right) \in \mathbb{N}^q \times \mathbb{N}^q :\\ &\mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \leqslant 1-\mathfrak{d}_0 \text{ ya da}\\ &\mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \geqslant \mathfrak{d}_0, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \geqslant \mathfrak{d}_0\Big) = 0\end{aligned}$$

elde edilir. Dolayısıyla, $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ istatistiksel sınırlıdır.

Sonuç 3.2. NFGMU’larda her istatistiksel yakınsak dizi istatistiksel sınırlıdır.

Teorem 3.7'ye ek olarak aşağıdaki teoremi de ifade edebiliriz.

Teorem 3.11. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU ve $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$, $\mathcal{Q}'$da bir dizi olsun. Bu durumda aşağıdaki ifadeler denktir:

(a) $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi istatistiksel Cauchy dizisidir.

(b) $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin istatistiksel yoğun $\left(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ alt dizisi $\mathcal{Q}$'da Cauchy dizisidir.

Şimdi, NFGMU'larda seyrek alt dizi, seyrek olmayan alt dizi, istatistiksel limit ve istatistiksel yığılma noktaları kavramları tanımlanacaktır.

Tanım 3.12. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$, $\mathcal{Q}'$da bir dizi olsun. Eğer $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin $\not{f} \in \mathcal{Q}$ elemanına yakınsayan $\left(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ bir alt dizisi mevcut ise, $\not{f} \in \mathcal{Q}$ noktası $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$'e göre $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin limit noktası olarak adlandırılır. $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin tüm limit noktaları kümesi ${}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ sembolü ile gösterilir.

Tanım 3.13. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $\left(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$, $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin bir alt dizisi olsun. $\mathbb{K} = \{(\mathfrak{y}_m,\mathfrak{z}_n): m,n \in \mathbb{N}\} \subset \mathbb{N}^2$ olarak tanımlansın. Eğer $\mathrm{d}_{\mathfrak{q}}(\mathbb{K}) = 0$ ise $\left(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ dizisine $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin seyrek alt dizisi denir. Eğer $\mathrm{d}_{\mathfrak{q}}(\mathbb{K}) \neq 0$ ise $\left(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ dizisine $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin seyrek olmayan alt dizisi denir.

Tanım 3.14. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. Eğer $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin $\not{f} \in \mathcal{Q}$ elemanına yakınsayan $\left(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ seyrek olmayan bir alt dizisi mevcut ise, $\not{f} \in \mathcal{Q}$ noktası $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$'e göre $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin istatistiksel limit noktası denir. NFGMU'larda tüm istatistiksel limit noktaları kümesi $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ ile gösterilir.

Tanım 3.15. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ ile belirtilen NFGMU'da eğer tüm $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için

$$\begin{aligned}
&\mathrm{d}_{\mathfrak{q}}\Big(\Big(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\Big) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}: \\
&\mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f},\mathfrak{v}\right) > 1-\mathfrak{d} \text{ yada} \\
&\mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f},\mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f},\mathfrak{v}\right) < \mathfrak{d}\Big) \neq 0
\end{aligned}$$

koşulu sağlanırsa $\not{f} \in \mathcal{Q}$ noktasına $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$'e göre $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin istatistiksel yığılma noktası denir. NFGMU'larda tüm istatistiksel yığılma noktaları kümesi $\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ ile gösterilir.

Teorem 3.12. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$, $\mathcal{Q}'$da bir dizi olsun. Bu durumda

$$\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\mathfrak{f}_{\alpha\beta}\right) \subseteq \Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}\right) \subseteq {}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}\right)$$

kapsam ilişkisi geçerlidir.

İspat. $\mathfrak{f} \in \Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}\right)$ ise $\left(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin $\left(\mathfrak{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ alt dizisi mevcuttur, ve bu alt dizinin indeks kümesi $W = \{(\mathfrak{y}_m, \mathfrak{z}_n): m, n \in \mathbb{N}\} \subset \mathbb{N}^2$ sıfırdan farklı $\mathfrak{q}$-boyutlu asimptotik yoğunluğa sahiptir. Dolayısıyla

$$\begin{aligned} d_{\mathfrak{q}}(W) = \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}} \Big| \Big\{ &\left(\left(\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_{\mathfrak{q}}\right), \left(\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_{\mathfrak{q}}\right) \right) \in W^{\mathfrak{q}} \times W^{\mathfrak{q}}: \\ &\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}} \leqslant \mathfrak{y}, \mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}} \leqslant \mathfrak{z} \Big\} \Big| = y > 0, \end{aligned}$$

yazılır ve $\left(\mathfrak{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ dizisi $\mathfrak{f} \in Q$ elemanına yakınsaktır.

Özellikler nedeniyle tüm $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için,

$$\begin{aligned} &\Big\{ \left(\left(\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_{\mathfrak{q}}\right), \left(\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_{\mathfrak{q}}\right) \right) \in W^{\mathfrak{q}} \times W^{\mathfrak{q}}: \mathfrak{F}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) > 1 - \mathfrak{d} \\ &\quad \text{ve } \mathfrak{G}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) < \mathfrak{d} \Big\} \\ &\subseteq \Big\{ \left(\left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}\right), \left(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}\right) \right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}: \mathfrak{F}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) > 1 - \mathfrak{d} \\ &\quad \text{ve } \mathfrak{G}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) < \mathfrak{d} \Big\}, \end{aligned}$$

sağlanır ve

$$\begin{aligned} &\Big\{ \left(\left(\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_{\mathfrak{q}}\right), \left(\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_{\mathfrak{q}}\right) \right) \in W^{\mathfrak{q}} \times W^{\mathfrak{q}}: \mathfrak{y}_u, \mathfrak{z}_v \in \mathbb{N}, u, v = 1,2, \dots, \mathfrak{q} \Big\} \\ &\setminus \Big\{ \left(\left(\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_{\mathfrak{q}}\right), \left(\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_{\mathfrak{q}}\right) \right) \in W^{\mathfrak{q}} \times W^{\mathfrak{q}}: \mathfrak{F}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) \leqslant 1 - \mathfrak{d} \\ &\quad \text{ya da } \mathfrak{G}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) \geqslant \mathfrak{d} \Big\} \\ &\subseteq \Big\{ \left(\left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}\right), \left(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}\right) \right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}: \mathfrak{F}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) > 1 - \mathfrak{d} \\ &\quad \text{ve } \mathfrak{G}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_{\mathfrak{q}}\mathfrak{z}_{\mathfrak{q}}}, \mathfrak{f}, \mathfrak{v}\right) < \mathfrak{d} \Big\} \end{aligned}$$

yazılır.

Bununla birlikte, $\left(\mathfrak{f}_{\mathfrak{y}_m\mathfrak{z}_n}\right)$ alt dizisi $\mathfrak{f} \in Q$ elemanına yakınsak olduğundan

$$\Big\{\Big((\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_\mathfrak{q}),(\mathfrak{z}_1,\mathfrak{z}_2,\dots,\mathfrak{z}_\mathfrak{q})\Big)\in W^\mathfrak{q}\times W^\mathfrak{q}:\mathfrak{F}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)$$
$$\leqslant 1-\mathfrak{d} \text{ ya da } \mathfrak{G}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)$$
$$\geqslant \mathfrak{d},\mathfrak{H}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)\geqslant \mathfrak{d}\Big\}$$

kümesi $\mathbb{N}^\mathfrak{q}$'nun sonlu bir alt kümesi olur.

Sonuç olarak

$$d_\mathfrak{q}\Big(\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q})\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)>1-\mathfrak{d}$$
$$\text{ve } \mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)<\mathfrak{d},\mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)<\mathfrak{d}\Big\}\Big)$$
$$\geqslant d_\mathfrak{q}\Big(\Big((\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_\mathfrak{q}),(\mathfrak{z}_1,\mathfrak{z}_2,\dots,\mathfrak{z}_\mathfrak{q})\Big)\in W^\mathfrak{q}\times W^\mathfrak{q}:\mathfrak{y}_u,\mathfrak{z}_v\in\mathbb{N},u,v=1,2,\dots,\mathfrak{q}\Big)$$
$$-\,d_\mathfrak{q}\Big(\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q})\Big)\in W^\mathfrak{q}\times W^\mathfrak{q}:\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)$$
$$\leqslant 1-\mathfrak{d} \text{ ya da } \mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big)$$
$$\geqslant d_\mathfrak{q}\Big(\Big((\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_\mathfrak{q}),(\mathfrak{z}_1,\mathfrak{z}_2,\dots,\mathfrak{z}_\mathfrak{q})\Big)\in W^\mathfrak{q}\times W^\mathfrak{q}:\mathfrak{y}_u,\mathfrak{z}_v\in\mathbb{N},u,v=1,2,\dots,\mathfrak{q}\Big)-0=y$$
$$>0$$

yazılır. Böylece $\not{f}\in\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$ elde edilir. Dolayısıyla $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(t_{\alpha\beta}\big)\subseteq\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$ sağlanmış olur.

Şimdi ise $y\in\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$ alınsın. Bu durumda tüm $\mathfrak{v}>0$ ve $\mathfrak{d}\in(0,1)$ için

$$d_\mathfrak{q}(\mathcal{S}(\mathfrak{v},\mathfrak{d}))$$
$$=\begin{array}{l} d_\mathfrak{q}\Big(\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q})\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},y,\mathfrak{v}\Big)>1-\mathfrak{d} \\ \text{or } \mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},y,\mathfrak{v}\Big)<\mathfrak{d},\mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},y,\mathfrak{v}\Big)<\mathfrak{d}\Big\}\Big)>0\end{array}$$

yazılır.

$(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_\mathfrak{q}),(\mathfrak{z}_1,\mathfrak{z}_2,\dots,\mathfrak{z}_\mathfrak{q})\in\mathcal{S}(\mathfrak{v},\mathfrak{d})$ alınsın. Bu durumda Önerme 3.2'e göre tüm $\mathfrak{m},\mathfrak{n}\in\{0,1,\dots,\mathfrak{q}\}$ için $\not{f}_{\mathfrak{y}_\mathfrak{m}\mathfrak{z}_\mathfrak{n}}\in\mathfrak{B}_y^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})$ yazılır.

$$B=\Big\{(\mathfrak{y}_\mathfrak{m},\mathfrak{z}_\mathfrak{n})\in\mathbb{N}^2:\not{f}_{\mathfrak{y}_\mathfrak{m}\mathfrak{z}_\mathfrak{n}}\in\mathfrak{B}_y^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})\Big\}$$

kümesi tanımlansın. Buradan $d_{\mathrm{q}}(B) = d_{\mathrm{q}}(\mathcal{S}(\mathfrak{v},\mathfrak{d})) > 0$ olduğu görülür, bu da B içinde $\left(\not{f}_{\mathfrak{y}_{\mathfrak{m}}\mathfrak{z}_{\mathfrak{n}}}\right)$ dizisinin $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ seyrek olmayan bir alt dizisi olduğunu gösterir. B, sonsuz sayıda pozitif tam sayı içerdiğinden $y \in {}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ sonucu elde edilir. Dolayısıyla $\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right) \subseteq {}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ sağlanmış olur.

Teorem 3.13. $\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun ve $st_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlansın. Bu durumda, $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right) = \Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right) = \{\not{f}\}$ elde edilir.

İspat. $st_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlansın. Tanım 3.7 ve Tanım 3.15 birlikte düşünüldüğünde $\not{f} \in \Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ elde edilir. $\not{f} \neq y$ için $y \in \Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ olsun. Bu durumda $\mathfrak{v} > 0$ ve $\mathfrak{d} \in (0,1)$ için

$$d_{\mathrm{q}}(\mathcal{F}(\mathfrak{v},\mathfrak{d})) \\ = \begin{array}{l} d_{\mathrm{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\not{f},\mathfrak{v}\right) > 1-\mathfrak{d}\right.\right. \\ \left.\left. \text{ve } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\not{f},\mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\not{f},\mathfrak{v}\right) < \mathfrak{d}\right\}\right) \neq 0, \end{array}$$

ve

$$d_{\mathrm{q}}(\mathcal{G}(\mathfrak{v},\mathfrak{d})) \\ = \begin{array}{l} d_{\mathrm{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right) > 1-\mathfrak{d}\right.\right. \\ \left.\left. \text{ve } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right) < \mathfrak{d}\right\}\right) \neq 0 \end{array}$$

elde edilir.

$\not{f} \neq y$ olduğundan $\mathcal{F}(\mathfrak{v},\mathfrak{d}) \cap \mathcal{G}(\mathfrak{v},\mathfrak{d}) = \emptyset$ yazılır. Böylece $\mathcal{F}^c(\mathfrak{v},\mathfrak{d}) \supseteq \mathcal{G}(\mathfrak{v},\mathfrak{d})$ kapsam ilişkisi geçerlidir. Dolayısıyla

$$\begin{array}{l} \left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\not{f},\mathfrak{v}\right) \leqslant 1-\mathfrak{d}\right. \\ \left.\text{yada } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\not{f},\mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\not{f},\mathfrak{v}\right) \geqslant \mathfrak{d}\right\} \\ \supseteq \left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right) > 1-\mathfrak{d}\right. \\ \left.\text{ve } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right) < \mathfrak{d}\right\}. \end{array}$$

bulunur. Aynı zamanda

$$d_{\mathrm{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right)\in\mathbb{N}^q\times\mathbb{N}^q : \mathfrak{F}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\cancel{f},\mathfrak{v}\right)\leqslant 1-\mathfrak{d}\right.\right.$$
$$\left.\left.\text{yada } \mathfrak{G}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\cancel{f},\mathfrak{v}\right)\geqslant\mathfrak{d},\mathfrak{H}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\cancel{f},\mathfrak{v}\right)\geqslant\mathfrak{d}\right\}\right)$$
$$\geqslant \begin{array}{l} d_{\mathrm{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right)\in\mathbb{N}^q\times\mathbb{N}^q : \mathfrak{F}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right)\leqslant 1-\mathfrak{d}\right.\right. \\ \left.\left.\text{yada } \mathfrak{G}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right)\geqslant\mathfrak{d},\mathfrak{H}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right)\geqslant\mathfrak{d}\right\}\right). \end{array}$$

elde edilir. $st_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty}\cancel{f}_{\mathfrak{y}\mathfrak{z}} = \cancel{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlandığından

$$d_{\mathrm{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right)\in\mathbb{N}^q\times\mathbb{N}^q : \mathfrak{F}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\cancel{f},\mathfrak{v}\right)\leqslant 1-\mathfrak{d}\right.\right.$$
$$\left.\left.\text{yada } \mathfrak{G}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\cancel{f},\mathfrak{v}\right)\geqslant\mathfrak{d},\mathfrak{H}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\cancel{f},\mathfrak{v}\right)\geqslant\mathfrak{d}\right\}\right)=0$$

olur. Sonuç olarak

$$d_{\mathrm{q}}\left(\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right)\in\mathbb{N}^q\times\mathbb{N}^q : \mathfrak{F}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right)\leqslant 1-\mathfrak{d}\right.\right.$$
$$\left.\left.\text{yada } \mathfrak{G}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right)\geqslant\mathfrak{d},\mathfrak{H}\left(\cancel{f}_{\mathfrak{s}_1\mathfrak{w}_1},\cancel{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\cancel{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\right)\geqslant\mathfrak{d}\right\}\right)=0$$

bulunur. Bu durum, $y \in \Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ olduğu iddiasıyla çelişir. Dolayısıyla, $\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) = \{\cancel{f}\}$ sonucuna varılır.

Şimdi, $st_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty}\cancel{f}_{\mathfrak{y}\mathfrak{z}} = \cancel{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olsun. Tanım 3.14 ve Teorem 3.7'ye göre, $\cancel{f} \in \Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ olduğu görülür. Öyleyse, Teorem 3.12'e göre, $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) = \{\cancel{f}\}$ sonucuna ulaşılır.

Teorem 3.13'nın tersi geçerli değildir. Başka bir deyişle, Q içinde $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) = \Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) = \{\cancel{f}\}$ eşitliğini sağlayan ancak istatistiksel olarak yakınsak olmayan bir $(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi mevcuttur.

Örnek 3.5. Örnek 3.1'de tanımlanan $(\mathbb{R},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ NFGMU ele alınsın. $(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi

$$\cancel{f}_{\mathfrak{y}\mathfrak{z}} = \begin{cases} 1, & \text{eğer } \mathfrak{y} = 2i, \mathfrak{z} = 2j \\ 2, & \text{diğer durumlarda,} \end{cases} (\forall i, j \in \mathbb{N})$$

biçiminde tanımlansın. Bu durumda, $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) = \Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) = \{1,2\}$ olmasına rağmen $(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi istatistiksel yakınsak değildir.

Teorem 3.14. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ ve $(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ $\mathcal{Q}$ içinde iki dizi olsun. Hemen hemen tüm $\mathfrak{y},\mathfrak{z}$ için $\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}$ sağlansın. Bu durumda $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})=\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ ve $\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})=\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ eşitlikleri geçerlidir.

İspat. Hemen hemen tüm $\mathfrak{y},\mathfrak{z}$ için $\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}$ sağlansın. Buna bağlı olarak, $\mathrm{T}_1=\{(\mathfrak{y},\mathfrak{z})\in\mathbb{N}^2:\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}\neq\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}\}$ kümesinin $\mathfrak{q}$-boyutlu asimptotik yoğunluğu sıfırdır, dolayısıyla $d_{\mathfrak{q}}(T_1)=0$ geçerlidir. Eğer $\mathfrak{f}\in\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ ise, $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\mathfrak{f}$ noktasına yakınsayan $(\mathfrak{f}_{\mathfrak{y}_m\mathfrak{z}_n})$ bir alt dizisi vardır ve indeks kümesi $T_2=\{(\mathfrak{y}_m,\mathfrak{z}_n):m,n\in\mathbb{N}\}$ sıfırdan farklı $\mathfrak{q}$-boyutlu asimptotik yoğunluğa sahiptir, yani $d_{\mathfrak{q}}(T_2)\neq 0$ geçerlidir. Şimdi

$$\mathrm{T}_3=\{(\mathfrak{y}_m,\mathfrak{z}_n)\in\mathrm{T}_2:m,n\in\mathbb{N},\mathfrak{f}_{\mathfrak{y}_m\mathfrak{z}_n}\neq\mathfrak{h}_{\mathfrak{y}_m\mathfrak{z}_n}\}$$

kümesi tanımlansın. Açıktır ki $d_{\mathfrak{q}}(T_3)=0$ geçerlidir, yani

$$\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\big((\mathfrak{y}_1,\mathfrak{y}_2,\ldots,\mathfrak{y}_{\mathfrak{q}}),(\mathfrak{z}_1,\mathfrak{z}_2,\ldots,\mathfrak{z}_{\mathfrak{q}})\big)\in\mathrm{T}_2^{\mathfrak{q}}\times\mathrm{T}_2^{\mathfrak{q}}: \mathfrak{y}_1,\mathfrak{y}_2,\ldots,\mathfrak{y}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{z}_1,\mathfrak{z}_2,\ldots,\mathfrak{z}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{f}_{\mathfrak{y}_m\mathfrak{z}_n}\neq\mathfrak{h}_{\mathfrak{y}_m\mathfrak{z}_n}\Big\}\Big|=0$$

sağlanır. Dolayısıyla

$$\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\big((\mathfrak{y}_1,\mathfrak{y}_2,\ldots,\mathfrak{y}_{\mathfrak{q}}),(\mathfrak{z}_1,\mathfrak{z}_2,\ldots,\mathfrak{z}_{\mathfrak{q}})\big)\in\mathrm{T}_2^{\mathfrak{q}}\times\mathrm{T}_2^{\mathfrak{q}}: \mathfrak{y}_1,\mathfrak{y}_2,\ldots,\mathfrak{y}_{\mathfrak{q}}\leqslant\mathfrak{y},\mathfrak{z}_1,\mathfrak{z}_2,\ldots,\mathfrak{z}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{f}_{\mathfrak{y}_m\mathfrak{z}_n}=\mathfrak{h}_{\mathfrak{y}_m\mathfrak{z}_n}\Big\}\Big|>0,$$

veya denk olarak

$$d_{\mathfrak{q}}(Q)=d_{\mathfrak{q}}\big(\{(\mathfrak{y}_m,\mathfrak{z}_n)\in T_2:\mathfrak{f}_{\mathfrak{y}_m\mathfrak{z}_n}=\mathfrak{h}_{\mathfrak{y}_m\mathfrak{z}_n}\}\big)>0$$

geçerlidir.

Şimdi, $(\mathfrak{h}_{\mathfrak{y}_m\mathfrak{z}_n})\in Q$ dizisi alınsın. Bu dizi, $(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ dizisinin seyrek olmayan bir alt dizisi olup, $\mathfrak{f}$'e yakınsaktır ve dolayısıyla $\mathfrak{f}\in\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ yazılır. Bu durum, $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})\subseteq\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ ilişkisinin geçerli olduğunu gösterir. Simetrik olarak $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})\subseteq\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ eşitliği de sağlanır. Sonuç olarak, $\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})=\Lambda^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ olduğu sonucuna varılır. Benzer şekilde, $\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})=\Gamma^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ eşitliği de gösterilebilir.

4. NÖTROSOFİK FUZZY G-METRİK UZAYLARDA LACUNARY İSTATİSTİKSEL YAKINSAKLIK

Bu bölümde NFGMU içinde çift dizilerin lacunary istatistiksel yakınsaklığı ve güçlü lacunary yakınsaklığı incelenerek, bu kavramlar arasındaki ilişkiler sistematik bir şekilde analiz edilecektir. Ayrıca, NFGMU'larda lacunary istatistiksel yakınsaklık ile istatistiksel yakınsaklık arasındaki kapsama ilişkileri ortaya konulacak ve bu bağlamda lacunary istatistiksel Cauchy dizisi tanımlanarak, lacunary istatistiksel yakınsama ile olan bağlantısı detaylı olarak değerlendirilecektir.

Tanım 4.1. $(Q,\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ ile belirtilen NFGMU ve $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ Q'da bir dizi olsun. Eğer $\mathfrak{d}\in(0,1)$ ve $\mathfrak{v}>0$ için

$$\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}=1}^{\mathfrak{z}}\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\right)>1-\mathfrak{d},$$

ve

$$\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}=1}^{\mathfrak{z}}\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\right)<\mathfrak{d},$$

$$\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}=1}^{\mathfrak{z}}\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\right)<\mathfrak{d}$$

koşulları sağlanıyorsa, $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisine $\not{f}\in Q$ elemanına $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre kuvvetli $[C,1,1]$ toplanabilirdir denir ve $[\mathrm{C},1,1]-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ yada $\not{f}_{\mathfrak{y}\mathfrak{z}}\overset{[\mathrm{C},1,1]-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\longrightarrow}\not{f}$ ile gösterilir.

Teorem 4.1. $(Q,\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$, Q'da bir dizi olsun. Bu durumda,

i) Eğer $\not{f}_{\mathfrak{y}\mathfrak{z}}\overset{[\mathrm{C},1,1]-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\longrightarrow}\not{f}$ ise $\not{f}_{\mathfrak{y}\mathfrak{z}}\overset{st_2-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\longrightarrow}\not{f}$ sağlanır.

ii) Eğer $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ sınırlı bir dizi ve $\not{f}_{\mathfrak{y}\mathfrak{z}}\overset{st_2-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\longrightarrow}\not{f}$ ise $\not{f}_{\mathfrak{y}\mathfrak{z}}\overset{[\mathrm{C},1,1]-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\longrightarrow}\not{f}$ geçerlidir.

Tanım 4.2. Eğer her $\mathfrak{d}\in(0,1)$ ve her $\mathfrak{v}>0$ için

$$d_{\mathfrak{q}}^{\theta_2}\left(\left\{\left(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\right)\right)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)\leqslant 1-\mathfrak{d}\right.\right.$$
$$\left.\left.\text{yada }\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)\geqslant\mathfrak{d},\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)\geqslant\mathfrak{d}\right\}\right)=0,$$

veya denk olarak

$$\lim_{u,v\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\Big|\left\{(\mathfrak{s}_t,\mathfrak{w}_t)\in\mathrm{I}_{uv},1\leqslant t\leqslant\mathfrak{q}:\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\right)\leqslant 1-\mathfrak{d}\text{ yada}\right.$$
$$\left.\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\right)\geqslant\mathfrak{d},\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\right)\geqslant\mathfrak{d}\right\}\Big|=0,$$

koşulları sağlanıyorsa $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisine $\not{f}\in\mathcal{Q}$ elemanına $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre lacunary istatistiksel yakınsaktır denir ve $\not{f}_{\mathfrak{y}\mathfrak{z}}\overset{S_{\theta_2}-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\rightarrow}\not{f}$ ya da $S_{\theta_2}-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olarak gösterilir. NFGMU'da tüm lacunary istatistiksel yakınsak diziler uzayı $S_{\theta_2}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$ ile ifade edilir.

Tanım 4.3. Eğer her $\mathfrak{d}\in(0,1)$ ve her $\mathfrak{v}>0$ için

$$\frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\sum_{(\mathfrak{s}_t,\mathfrak{w}_t)\in\mathrm{I}_{uv},1\leqslant t\leqslant\mathfrak{q}}\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\right)>1-\mathfrak{d}$$

ve

$$\frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\sum_{(\mathfrak{s}_t,\mathfrak{w}_t)\in\mathrm{I}_{uv},1\leqslant t\leqslant\mathfrak{q}}\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\right)<\mathfrak{d},$$

$$\frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\sum_{(\mathfrak{s}_t,\mathfrak{w}_t)\in\mathrm{I}_{uv},1\leqslant t\leqslant\mathfrak{q}}\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\right)<\mathfrak{d},$$

koşulları sağlanıyorsa, $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisine $\not{f}\in\mathcal{Q}$ elemanına $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre kuvvetli lacunary yakınsaktır denir ve $[N_{\theta_2}]-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ yada $\not{f}_{\mathfrak{y}\mathfrak{z}}\overset{[N_{\theta_2}]-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\rightarrow}\not{f}$ ile gösterilir.

Teorem 4.2. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\not{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}$'da bir dizi olsun. Bu durumda,

(i) Eğer $\not{f}_{\mathfrak{y}\mathfrak{z}}\overset{[N_{\theta_2}]-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\rightarrow}\not{f}$ ise $\not{f}_{\mathfrak{y}\mathfrak{z}}\overset{S_{\theta_2}-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\rightarrow}\not{f}$ geçerlidir.

(ii) Eğer $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ sınırlı bir dizi ve $\not{f}_{\mathfrak{y}\mathfrak{z}} \xrightarrow{S_{\theta_2}-(\mathfrak{F},\mathfrak{G},\mathfrak{H})} \not{f}$ ise $\not{f}_{\mathfrak{y}\mathfrak{z}} \xrightarrow{[N_{\theta_2}]-(\mathfrak{F},\mathfrak{G},\mathfrak{H})} \not{f}$ sağlanır.

Şimdi, NFGMU'larda çift dizilerin istatistiksel yakınsama ve lacunary istatistiksel yakınsama arasındaki kapsama bağlantıları verilecektir.

Teorem 4.3. Her $\theta_2 = \theta_{u,v}$ lacunary dizi için, eğer $\lim\limits_{u}\inf q_u > 1$ ve $\lim\limits_{v}\inf q_v > 1$ ise, bu durumda

$$\mathrm{st}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H}) \Rightarrow \mathrm{S}_{\theta_2} - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$$

sağlanır.

İspat. $\lim\limits_{u}\inf q_u > 1$ ve $\lim\limits_{v}\inf q_v > 1$ olsun. Bu durumda, yeterince büyük u, v değerleri için $q_u > 1 + \varrho$ ve $q_v > 1 + \varsigma$ eşitsizlikleri sağlanacak $\varrho, \varsigma > 0$ sayıları mevcuttur. Aynı zamanda

$$\frac{\mathfrak{h}_{uv}}{k_{uv}} \geqslant \frac{\varrho\varsigma}{(1+\varrho)(1+\varsigma)}$$

geçerlidir. Eğer $st_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ise her $\mathfrak{d} \in (0,1), \mathfrak{v} > 0$ için ve yeterince büyük u, v değerleri için

$$\begin{aligned}
&\frac{\mathfrak{q}!}{(k_{uv})^{\mathfrak{q}}}\Big|\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}} \leqslant k_u,\\
&\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}} \leqslant l_v, \mathfrak{F}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}\\
&\quad \text{yada } \mathfrak{G}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}\Big\}\Big|\\
&\geqslant \frac{\mathfrak{q}!}{(k_{uv})^{\mathfrak{q}}}\Big|\big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q} : \mathfrak{F}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}\\
&\quad \text{yada } \mathfrak{G}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}\big\}\Big|\\
&\geqslant \left(\frac{\mathfrak{h}_{uv}}{k_{uv}}\right)^{\mathfrak{q}} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\Big|\big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q} : \mathfrak{F}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}\\
&\quad \text{yada } \mathfrak{G}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}\big\}\Big|\\
&\geqslant \left(\frac{\varrho\varsigma}{(1+\varrho)(1+\varsigma)}\right)^{\mathfrak{q}} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\Big|\big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q} : \mathfrak{F}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}\\
&\quad \text{yada } \mathfrak{G}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}\big\}\Big|
\end{aligned}$$

geçerlidir. Dolayısıyla, $\mathrm{S}_{\theta_2} - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlanır.

Teorem 4.4. Her $\theta_2 = \theta_{u,v}$ lacunary dizi için, eğer $\limsup_{u} q_u < \infty, \limsup_{v} q_v < \infty$ geçerli ise

$$S_{\theta_2} - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} f_{\mathfrak{y}\mathfrak{z}} = f(\mathfrak{F}, \mathfrak{G}, \mathfrak{H}) \Rightarrow st_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} f_{\mathfrak{y}\mathfrak{z}} = f(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$$

sağlanır.

İspat. $\text{imsup}_{u} q_u < \infty, \limsup_{v} q_v < \infty$ olsun. Bu durumda, her $u, v \geqslant 1$ için $q_u < \mathcal{M}$ ve $q_v < \mathcal{N}$ olacak biçimde $\mathcal{M}, \mathcal{N} > 0$ sayıları mevcuttur.

$S_{\theta_2} - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} f_{\mathfrak{y}\mathfrak{z}} = f(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ olsun ve

$$\mathrm{T}_{u,v} := \left|\left\{(\mathfrak{s}_t, \mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathrm{q}: \mathfrak{F}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, f_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\right) \leqslant 1 - \mathfrak{d} \text{ or } \mathfrak{G}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, f_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, f_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\right) \geqslant \mathfrak{d}\right\}\right|$$

tanımlansın.

Lacunary istatistiksel yakınsaklık tanımından,

$$S_{\theta_2} - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} f_{\mathfrak{y}\mathfrak{z}} = f(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$$

ise, keyfi $\xi > 0$ sayısı için $u \geqslant u_0, v \geqslant v_0$ olduğunda $\frac{\mathrm{T}_{u,v}\mathrm{q}!}{(\mathfrak{h}_{uv})^{\mathrm{q}}} < \xi$ olacak biçimde $u_0, v_0 \in \mathbb{N}$ mevcuttur.

$$\mathrm{W} = \max\left\{\mathrm{T}_{u,v} : 1 \leqslant u \leqslant u_0, 1 \leqslant v \leqslant v_0\right\},$$

kümesi tanımlansın.

$\mathfrak{y}$ ve $\mathfrak{z}$, $k_{u-1} < \mathfrak{y} \leqslant k_u$ ve $l_{v-1} < \mathfrak{z} \leqslant l_v$ şartlarını sağlayan herhangi birer tam sayı olsun. Bu durumda,

$$
\begin{aligned}
&\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\big)\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\leqslant\mathfrak{y},\\
&\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\leqslant\mathfrak{z},\mathfrak{F}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\Big)\leqslant 1-\mathfrak{d}\\
&\ \text{yada }\ \mathfrak{G}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big|\\
&\leqslant\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\Big|\Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\big)\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\leqslant k_u,\\
&\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\leqslant l_v,\mathfrak{F}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\Big)\leqslant 1-\mathfrak{d}\\
&\ \text{yada }\ \mathfrak{G}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big|\\
&=\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\Big|\Big\{\,\mathrm{T}_{11}+\mathrm{T}_{12}+\mathrm{T}_{21}+\mathrm{T}_{22}+\cdots+\mathrm{T}_{u_0v_0}+\cdots+\mathrm{T}_{u,v}\Big\}\\
&=\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}=1}^{\mathfrak{z}}\mathrm{T}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}}\right\}\\
&\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}=1}^{\mathfrak{z}}\mathrm{T}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}}\right\}\\
&\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}=u_0+1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}=v_0+1}^{\mathfrak{z}}\frac{\mathrm{T}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}}\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}}}{\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}}}\right\}\\
&\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\left(\sup_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\geqslant u_0,\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\geq v_0}\frac{\mathrm{T}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}}}{\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}}}\right)\\
&\times\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}=u_0+1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}=v_0+1}^{\mathfrak{z}}\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}}\right\}\\
&\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\xi\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}=u_0+1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}=v_0+1}^{\mathfrak{z}}\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}}\right\}\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\xi\mathcal{M}\mathcal{N},
\end{aligned}
$$

yazılır.

Burada $\mathfrak{y},\mathfrak{z}\to\infty$ iken $k_{u-1}l_{v-1}\to\infty$ olacağından her $\mathfrak{d}\in(0,1)$ ve $\mathfrak{v}>0$ değerleri için

$$
\begin{aligned}
\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\ &\Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\big)\Big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\leqslant\mathfrak{y},\\
&\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\leqslant\mathfrak{z},\mathfrak{F}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\Big)\leqslant 1-\mathfrak{d}\ \text{or}\\
&\mathfrak{G}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big|=0,
\end{aligned}
$$

elde edilir. Böylece, $st_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\not f_{\mathfrak{y}\mathfrak{z}}=\not f(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlanır.

Teorem 4.5. $\theta_2 = \theta_{u,v}$ lacunary dizi olsun. $\mathrm{S}_{\theta_2}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} = \mathrm{st}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$ olması için gerek ve yeter şart

$$1 < \liminf_{u} \mathrm{q}_u \leqslant \limsup_{u} q_u < \infty \text{ and } 1 < \liminf_{v} \mathrm{q}_v \leqslant \limsup_{v} q_v < \infty$$

sağlanmasıdır.

Teorem 4.6. Eğer $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}) \in \mathrm{st}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} \cap \mathrm{S}_{\theta_2}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$ ise $S_{\theta_2} - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = st_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}}$ sağlanır.

İspat. $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}) \in \mathrm{st}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} \cap \mathrm{S}_{\theta_2}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$ olsun. $st_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{f}_0(\mathfrak{F},\mathfrak{G},\mathfrak{H})$, $S_{\theta_2} - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{f}_1(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $\mathfrak{f}_0 \neq \mathfrak{f}_1$ olarak kabul edilsin.

$\mathfrak{f}_0 \neq \mathfrak{f}_1$ olduğundan, $\mathfrak{d} > 0$ için $\mathfrak{d} < \frac{1}{2}|\mathfrak{f}_0 - \mathfrak{f}_1|$ alınabileceğinden

$$\lim_{\mathfrak{y},\mathfrak{z}\to\infty} \frac{\mathrm{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathrm{q}}} \Big| \{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}} \leqslant \mathfrak{y},$$
$$\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}} \leqslant \mathfrak{z}, \mathfrak{F}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \leqslant 1-\mathfrak{d} \text{ or}$$
$$\mathfrak{G}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}\}\Big| = 1,$$

elde edilir. İstatistiksel limit ifadesinin $k_m l_n$ inci terimini dikkate alınırsa,

$$\frac{\mathrm{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathrm{q}}} \Big| \Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}} \leqslant \mathfrak{y},$$
$$\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}} \leqslant \mathfrak{z}, \mathfrak{F}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}$$
$$\text{yada } \mathfrak{G}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}\Big\}\Big|$$
$$= \frac{\mathrm{q}!}{(k_m l_n)^{\mathrm{q}}} \Big| \Big\{(\mathfrak{s}_{\mathrm{q}},\mathfrak{w}_{\mathrm{q}}) \in \bigcup_{u,v=1,1}^{m,n} I_{uv} : \mathfrak{F}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}$$
$$\text{yada } \mathfrak{G}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}\Big\}\Big|$$
$$\leqslant \frac{1}{\sum_{u,v=1,1}^{m,n} \mathfrak{h}_{uv}} \sum_{u,v=1,1}^{m,n} \mathfrak{h}_{uv}\rho_{u,v}$$
$$= \frac{\mathrm{q}!}{(k_m l_n)^{\mathrm{q}}} \sum_{u,v=1,1}^{m,n} \Big| \Big\{(\mathfrak{s}_{\mathrm{q}},\mathfrak{w}_{\mathrm{q}}) \in I_{uv} : \mathfrak{F}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}$$
$$\text{yada } \mathfrak{G}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\mathfrak{f}_1,\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}\Big\}\Big|,$$

elde edilir.

$S_{\theta_2} - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{f}_1(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğundan,

$$\rho_{u,v} = \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\left|\left\{\left(\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_{\mathfrak{q}}\right) \in I_{uv}: \mathfrak{F}\left(\mathfrak{f}_1, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\right) \leqslant 1 - \mathfrak{d} \right.\right.$$
$$\left.\left. \text{or } \left(\mathfrak{f}_1, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\mathfrak{f}_1, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\right) \geqslant \mathfrak{d}\right\}\right|$$

Pringsheim sıfır dizisidir. Çift lacunary dizi $\theta_{u,v}$ Pringsheim sıfır dizisini başka bir Pringsheim sıfır dizisine dönüştüren dört boyutlu matris dönüşümü için gerekli tüm koşulları sağladığından, son denklem Pringsheim anlamında sıfıra yakınsar. Ayrıca

$$\left\{\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\left| \left\{\left(\left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}\right), \left(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}\right)\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}: \mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}} \leqslant \mathfrak{y}, \mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}} \right.\right.\right.$$
$$\leqslant \mathfrak{z}, \mathfrak{F}\left(\mathfrak{f}_1, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\right)$$
$$\leqslant 1$$
$$- \mathfrak{d} \text{ or } \mathfrak{G}\left(\mathfrak{f}_1, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\mathfrak{f}_1, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\right)$$
$$\left.\left.\left. \geqslant \mathfrak{d}\right\}\right|\right\}.$$

biçiminde bir dizi oluşturur.

Ancak, bu dizi Pringsheim anlamında 1'e yakınsamaz. Bu çelişki, $\mathfrak{f}_0 = \mathfrak{f}_1$ eşitliğini gerektirir.

Tanım 4.4. $(Q, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU ve $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$, Q'da bir dizi olsun. Eğer her $(\mathfrak{y}_u, \mathfrak{z}_v) \in \mathrm{I}_{uv}$ için $(\mathfrak{F},\mathfrak{G},\mathfrak{H}) - \lim\limits_{u,v\to\infty} \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v} = \mathfrak{f}$ sağlayacak biçimde $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $(\mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v})$ şeklinde bir alt dizisi varsa ve her $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için

$$d_{\mathfrak{q}}^{\theta_2}\left(\left(\left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}\right), \left(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}\right)\right) \in \mathbb{N}^q \times \mathbb{N}^q: \mathfrak{F}\left(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) \leqslant 1 - \mathfrak{d} \right.$$
$$\left. \text{yada } \mathfrak{G}\left(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) \geqslant \mathfrak{d}\right) = 0,$$

veya denk olarak

$$\lim_{u,v\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\left| \left\{(\mathfrak{s}_t, \mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\left(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) \leqslant 1 - \mathfrak{d} \right.\right.$$
$$\left.\left. \text{yada } \mathfrak{G}\left(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) \geqslant \mathfrak{d}\right\}\right| = 0$$

sağlanıyorsa, $(\not f_{\mathfrak{y}\mathfrak{z}})$ dizisine $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre lacunary istatistiksel Cauchy (ya da $S_{\theta_2}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$-Cauchy) dizisi denir.

Teorem 4.7. $(Q,\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\not f_{\mathfrak{y}\mathfrak{z}})$, Q'da bir dizi olsun. Eğer $(\not f_{\mathfrak{y}\mathfrak{z}})$ dizisi $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre lacunary istatistiksel yakınsak ise $S_{\theta_2}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$-Cauchy dizisidir.

İspat. $S_{\theta_2}-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\not f_{\mathfrak{y}\mathfrak{z}}=\not f(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olsun ve her $\mathfrak{v}>0$ ve $\mathfrak{f},\mathfrak{g}>0$ için

$$K(\mathfrak{f},\mathfrak{g}):=\Big\{(\mathfrak{s}_t,\mathfrak{w}_t)\in \mathrm{I}_{uv},1\leqslant t\leqslant \mathfrak{q}:\mathfrak{F}\big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big)>1-\frac{1}{\mathfrak{f}\mathfrak{g}}\text{ ve }$$
$$\mathfrak{G}\big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big)<\frac{1}{\mathfrak{f}\mathfrak{g}},\mathfrak{H}\big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big)<\frac{1}{\mathfrak{f}\mathfrak{g}}\Big\}$$

olarak belirlensin. Bu durumda

(a) $K(\mathfrak{f}+1,\mathfrak{g}+1)\subset K(\mathfrak{f},\mathfrak{g})$; ve

(b) $\frac{|K(\mathfrak{f},\mathfrak{g})\cap \mathrm{I}_{uv}|}{\mathfrak{h}_{uv}}\to 1,u,v\to\infty$

ilişkileri geçerlidir.

Şimdi $u\geqslant k(1)$ ve $v\geqslant l(1)$ iken $\frac{|K(1,1)\cap \mathrm{I}_{uv}|}{\mathfrak{h}_{uv}}>0$, yani $K(1,1)\cap \mathrm{I}_{uv}\neq\emptyset$ olacak şekilde $k(1)$ ve $l(1)$ seçilsin. Bir sonraki seçimde, $u\geqslant k(2)$ ve $v\geqslant l(2)$ iken $K(2,2)\cap \mathrm{I}_{uv}\neq\emptyset$ olacak biçimde $k(2)>k(1)$ ve $l(2)>l(1)$ olsun. Daha sonra $k(1)\leqslant u\leqslant k(2)$ ve $l(1)\leqslant v\leqslant l(2)$ koşulunu sağlayan her (u,v) için $(\mathfrak{y}_u,\mathfrak{z}_v)\in \mathrm{K}(1,1)\cap \mathrm{I}_{uv}$, yani

$$\mathfrak{F}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not f_{\mathfrak{y}_u\mathfrak{z}_v},\mathfrak{v}\Big)>1,$$

$$\mathfrak{G}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not f_{\mathfrak{y}_u\mathfrak{z}_v},\mathfrak{v}\Big)<0,$$

$$\mathfrak{H}\Big(\not f,\not f_{\mathfrak{s}_1\mathfrak{w}_1},\not f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not f_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not f_{\mathfrak{y}_u\mathfrak{z}_v},\mathfrak{v}\Big)<0,$$

sağlayacak biçimde $(\mathfrak{y}_u,\mathfrak{z}_v)\in \mathrm{I}_{uv}$ seçilsin. Tüm bu seçimlerin sonunda, $u>k(\mathfrak{f}+1)$ ve $v>l(\mathfrak{g}+1)$ iken $K(\mathfrak{f}+1,\mathfrak{g}+1)\cap \mathrm{I}_{uv}\neq\emptyset$ olacak biçimde $k(\mathfrak{f}+1)>k(\mathfrak{f})$ ve $l(\mathfrak{g}+1)>l(\mathfrak{g})$ alınsın. Bu durumda, $k(\mathfrak{f})\leqslant u\leqslant k(\mathfrak{f}+1)$ ve $l(\mathfrak{g})\leqslant v\leqslant l(\mathfrak{g}+1)$ sağlayan (uv) çifti için

$$\mathfrak{F}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) > 1 - \frac{1}{\mathfrak{f}\mathfrak{g}} \text{ ve}$$

$$\mathfrak{G}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) < \frac{1}{\mathfrak{f}\mathfrak{g}},$$

$$\mathfrak{H}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) < \frac{1}{\mathfrak{f}\mathfrak{g}},$$

eşitsizlikleri sağlayacak biçimde $(\mathfrak{y}_u, \mathfrak{z}_v) \in \mathrm{I}_{uv}$ seçilebilir.

Böylece, her bir u, v için $(\mathfrak{y}_u, \mathfrak{z}_v) \in \mathrm{I}_{uv}$ olur ve son eşitsizliklerden dolayı $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H}) - \lim \not{f}_{\mathfrak{y}_u\mathfrak{z}_v} = \not{f}$ elde edilir. Her bir $\mathfrak{d} > 0$ için $(1 - \mathfrak{m}) \therefore (1 - \mathfrak{m}) > 1 - \mathfrak{d}$ ve $m * m < \mathfrak{d}$ koşullarını sağlayan $\mathfrak{m} > 0$ sayısı seçilebilir. Her $\mathfrak{v} > 0$ için

$$\begin{aligned}
A &:= \Big\{(\mathfrak{s}_t, \mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) > 1 - \mathfrak{d} \\
&\quad \text{yada } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) < \mathfrak{d}\Big\}, \\
B &:= \big\{(\mathfrak{s}_t, \mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\big) > 1 - \mathfrak{m} \\
&\quad \text{yada } \mathfrak{G}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) < \mathfrak{m}, \mathfrak{H}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) < \mathfrak{m}\Big\}, \\
C &:= \big\{(\mathfrak{y}_t, \mathfrak{z}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\big(\not{f}, \not{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \not{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \not{f}_{\mathfrak{y}_t\mathfrak{z}_t}, \mathfrak{v}\big) > 1 - \mathfrak{m} \\
&\quad \text{yada } \mathfrak{G}\big(\not{f}, \not{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \not{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \not{f}_{\mathfrak{y}_t\mathfrak{z}_t}, \mathfrak{v}\big) < \mathfrak{m}, \mathfrak{H}\big(\not{f}, \not{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \not{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \not{f}_{\mathfrak{y}_t\mathfrak{z}_t}, \mathfrak{v}\big) < \mathfrak{m}\big\},
\end{aligned}$$

alınırsa $(B \cap C) \subset A$ bulunur, böylece $A^c \subset (B^c \cup C^c)$ elde edilir. Sonuç olarak

$$\begin{aligned}
&\lim_{u,v\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^\mathfrak{q}} \Big| \Big\{(\mathfrak{s}_t, \mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) \leqslant 1 - \mathfrak{d} \\
&\quad \text{yada } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \not{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\right) \geqslant \mathfrak{d}\Big\}\Big| \\
\leq \; &\lim_{u,v\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^\mathfrak{q}} \Big| \big\{(\mathfrak{s}_t, \mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\big) \leqslant 1 - \mathfrak{m} \\
&\quad \text{yada } \mathfrak{G}\big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\big) \geqslant \mathfrak{m}, \mathfrak{H}\big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\big) \geqslant \mathfrak{m}\big\}\Big| \\
+ \; &\lim_{u,v\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^\mathfrak{q}} \Big| \big\{(\mathfrak{y}_t, \mathfrak{z}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\big(\not{f}, \not{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \not{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \not{f}_{\mathfrak{y}_t\mathfrak{z}_t}, \mathfrak{v}\big) \leqslant 1 - \mathfrak{m} \\
&\quad \text{yada } \mathfrak{G}\big(\not{f}, \not{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \not{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \not{f}_{\mathfrak{y}_t\mathfrak{z}_t}, \mathfrak{v}\big) \geqslant \mathfrak{m}, \mathfrak{H}\big(\not{f}, \not{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \not{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \not{f}_{\mathfrak{y}_t\mathfrak{z}_t}, \mathfrak{v}\big) \geqslant \mathfrak{m}\big\}\Big|,
\end{aligned}$$

yazılır.

$S_{\theta_2} - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ ve $S_{\theta_2} - \lim\limits_{u,v\to\infty} \not{f}_{\mathfrak{y}_u\mathfrak{z}_v} = \not{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ olduğundan $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi

$S_{\theta_2}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$-Cauchy dizisi olduğu gösterilmiş olur.

Tersine $\left(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre $S_{\theta_2}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$-Cauchy dizisi olsun. Tanıma göre, her $(\mathfrak{y}_u,\mathfrak{z}_v) \in \mathrm{I}_{uv}$ için $(\mathfrak{F},\mathfrak{G},\mathfrak{H}) - \lim_{u,v\to\infty} \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v} = \mathfrak{f}$ sağlayacak biçimde $\left(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin $\left(\mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}\right)$ şeklinde bir alt dizisi vardır ve her $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için

$$\lim_{u,v\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}} \Big| \Big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\Big) \leqslant 1-\mathfrak{d} \\ \text{or } \mathfrak{G}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\Big) \geqslant \mathfrak{d}, \mathfrak{H}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\Big) \geqslant \mathfrak{d}\Big\}\Big| = 0$$

sağlanır.

Daha önce gösterildiği gibi

$$\begin{aligned} &\lim_{u,v\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}} \Big| \big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\big(\mathfrak{f}, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\big) \leqslant 1-\mathfrak{d} \\ &\quad \text{yada } \mathfrak{G}\big(\mathfrak{f}, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\mathfrak{f}, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_t\mathfrak{w}_t}, \mathfrak{v}\big) \geqslant \mathfrak{d}\big\}\Big| \\ \leqslant \; &\lim_{u,v\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}} \Big| \Big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\Big) \leqslant 1-\frac{\mathfrak{d}}{2} \\ &\quad \text{yada } \mathfrak{G}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\Big) \geqslant \frac{\mathfrak{d}}{2}, \mathfrak{H}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\Big) \geqslant \frac{\mathfrak{d}}{2}\Big\}\Big| \\ + \; &\lim_{u,v\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}} \Big| \Big\{(\mathfrak{y}_t,\mathfrak{z}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\big(\mathfrak{f}, \mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_t\mathfrak{z}_t}, \mathfrak{v}\big) \leqslant 1-\frac{\mathfrak{d}}{2} \\ &\quad \text{yada } \mathfrak{G}\big(\mathfrak{f}, \mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_t\mathfrak{z}_t}, \mathfrak{v}\big) \geqslant \frac{\mathfrak{d}}{2}, \mathfrak{H}\big(\mathfrak{f}, \mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \dots, \mathfrak{f}_{\mathfrak{y}_t\mathfrak{z}_t}, \mathfrak{v}\big) \geqslant \frac{\mathfrak{d}}{2}\Big\}\Big|, \end{aligned}$$

eşitsizliği vardır. $(\mathfrak{F},\mathfrak{G},\mathfrak{H}) - \lim_{u,v\to\infty} \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v} = \mathfrak{f}$ olduğundan

$$\lim_{u,v\to\infty} \frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}} \Big| \Big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathfrak{q}, \mathfrak{F}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\Big) \leqslant 1-\mathfrak{d} \\ \text{or } \mathfrak{G}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\Big) \geqslant \mathfrak{d}, \mathfrak{H}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}, \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v}, \mathfrak{v}\Big) \geqslant \mathfrak{d}\Big\}\Big| = 0$$

göz önünde bulundurulduğunda $\mathrm{S}_{\theta_2} - \lim_{u,v\to\infty} \mathfrak{f}_{\mathfrak{y}_u\mathfrak{z}_v} = \mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ elde edilir.

5. NÖTROSOFİK FUZZY G-METRİK UZAYLARDA İDEAL YAKINSAKLIK

Bu bölümde, NFGMU'larda çift diziler için $\mathfrak{q}$. dereceden ideal yakınsama kavramı tanımlanmaktadır. İdeal yakınsamanın temel özellikleri incelenmekte, $\mathcal{I}_2$-Cauchy çift diziler analiz edilmekte ve NFGMU'larda $\mathcal{I}_2$-tamlık kavramı araştırılmaktadır. Ayrıca, ideal yakınsamaya sahip dizi örnekleri verilerek bu yapıların cebirsel ve topolojik özellikleri tartışılmaktadır. Çalışmada ayrıca, dizilerin $\mathcal{I}_2$-limit ve $\mathcal{I}_2$-yığılma noktaları ele alınmaktadır.

Tanım 5.1. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU, $\mathcal{I}_2$ uygun ideal ve $(\not{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}'$da bir dizi olsun. Eğer her $\mathfrak{d} \in (0,1)$ ve her $\mathfrak{v} > 0$ için

$$\Big\{\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}})\Big) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\Big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \leqslant 1 - \mathfrak{d} \text{ yada } \mathfrak{G}\Big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \geqslant \mathfrak{d}, \mathfrak{H}\Big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \geqslant \mathfrak{d}\Big\} \in \mathcal{I}_2,$$

veya denk olarak

$$d_{\mathfrak{q}}^{\mathcal{I}_2}\Big(\Big\{\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}})\Big) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\Big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \leqslant 1 - \mathfrak{d} \text{ yada } \mathfrak{G}\Big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \geqslant \mathfrak{d}, \mathfrak{H}\Big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \geqslant \mathfrak{d}\Big\}\Big) = 0$$

koşulları sağlanıyorsa $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisine $\not{f} \in \mathcal{Q}$ elemanına $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ normuna göre ideal yakınsaktır denir, ve $\not{f}_{\mathfrak{y}\mathfrak{z}} \overset{\mathcal{I}_2-(\mathfrak{F},\mathfrak{G},\mathfrak{H})}{\rightarrow} \not{f}$ ya da $\mathcal{I}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ olarak gösterilir. NFGMU'da tüm ideal yakınsak diziler uzayı $\mathcal{I}_2^{(\Theta,\Omega,\Xi)}$ ile ifade edilir.

Bu bölümde, NFGMU uzaylarında $c_0^{\mathcal{I}_2}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H}), c^{\mathcal{I}_2}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ ve $l_\infty^{\mathcal{I}_2}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ dizi uzayları tanımlanacaktır. Bu uzayların cebirsel ve topolojik özellikleri incelenecektir. Bu uzaylar,

$$c_0^{\mathcal{I}_2}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H}) \coloneqq \Big\{\not{f} = (\not{f}_{\mathfrak{y}\mathfrak{z}}) \in \mathcal{Q} : \Big\{\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}})\Big) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \leqslant 1 - \mathfrak{d}, \mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \geqslant \mathfrak{d}, \mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \geqslant \mathfrak{d}\Big\} \in \mathcal{I}_2\Big\},$$

$$
\begin{aligned}
c^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H}) \coloneqq \Big\{ \not{f} = (\not{f}_{\mathfrak{y}\mathfrak{z}}) \\
&\in \mathcal{Q}\colon \Big\{ \Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}) \Big) \\
&\in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}\colon \mathfrak{F}\Big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big) \\
&\leqslant 1-\mathfrak{d}, \mathfrak{G}\Big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big) \geqslant \mathfrak{d}, \mathfrak{H}\Big(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big) \\
&\geqslant \mathfrak{d}\Big\} \in \mathcal{J}_2 \Big\},
\end{aligned}
$$

$$
\begin{aligned}
l_{\infty}^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H}) \coloneqq \Big\{ \not{f} = (\not{f}_{\mathfrak{y}\mathfrak{z}}) \in \mathcal{Q}, \exists \mathbb{T} \\
&> 0\colon \Big\{ \Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}) \Big) \\
&\in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}\colon \mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big) \\
&\leqslant 1-\mathbb{T}, \mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big) \geqslant \mathbb{T}, \mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big) \geqslant \mathbb{T}\Big\} \\
&\in \mathcal{J}_2 \Big\}.
\end{aligned}
$$

Teorem 5.1. $c_0^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $c^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ uzayları lineer uzaylardır.

İspat. Bilindiği üzere $c_0^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H}) \subset c^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ bağıntısı sağlanmaktadır. Şimdi, $c^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ için ilgili sonucu göstermek amaçlanmaktadır. $c_0^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ uzayının lineerliği ise benzer yöntemle ispatlanabilir. $\not{f} = (\not{f}_{\mathfrak{y}\mathfrak{z}}) \in c^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$, $\mathfrak{p} = (\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}) \in c^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ dizileri alınsın. Bu durumda, $\mathcal{J}_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \rho_1(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $\mathcal{J}_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{p}_{\mathfrak{y}\mathfrak{z}} = \rho_2(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlanmaktadır. Keyfi γ ve η skaler değerleri için $\mathcal{J}_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} (\gamma\not{f}_{\mathfrak{y}\mathfrak{z}} + \eta\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}) = \gamma\rho_1 + \eta\rho_2(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğu gösterilecektir.

$$
\begin{aligned}
\mathcal{A}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} := \Big\{ \Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}) \Big) \\
&\in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}}\colon \mathfrak{F}\left(\rho_1, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\frac{\mathfrak{v}}{2|\gamma|}\right) \\
&\leqslant 1-\mathfrak{d}, \mathfrak{G}\left(\rho_1, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\frac{\mathfrak{v}}{2|\gamma|}\right) \\
&\geqslant \mathfrak{d}, \mathfrak{H}\left(\rho_1, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\frac{\mathfrak{v}}{2|\gamma|}\right) \geqslant \mathfrak{d}\Big\} \in \mathcal{J}_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} := \Bigg\{\Big(\big(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}\big), \big(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}\big)\Big) \\
\in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\left(\rho_2, \mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{p}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) \\
\leqslant 1 - \mathfrak{d}, \mathfrak{G}\left(\rho_2, \mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{p}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) \\
\geqslant \mathfrak{d}, \mathfrak{H}\left(\rho_2, \mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{p}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) \geqslant \mathfrak{d}\Bigg\} \in \mathcal{I}_2
\end{aligned}
$$

kümeleri ele alınsın. Bu kümelerin tümleyenleri sırasıyla

$$
\begin{aligned}
\mathcal{A}^{c}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} := \Bigg\{\Big(\big(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}\big), \big(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}\big)\Big) \\
\in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\left(\rho_1, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\gamma|}\right) \\
> 1 - \mathfrak{d}, \mathfrak{G}\left(\rho_1, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\gamma|}\right) \\
< \mathfrak{d}, \mathfrak{H}\left(\rho_1, \mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\gamma|}\right) < \mathfrak{d}\Bigg\} \in \mathcal{F}(\mathcal{I}_2),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}^{c}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} := \Bigg\{\Big(\big(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}\big), \big(\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}\big)\Big) \\
\in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\left(\rho_2, \mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{p}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) \\
> 1 - \mathfrak{d}, \mathfrak{G}\left(\rho_2, \mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{p}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) \\
< \mathfrak{d}, \mathfrak{H}\left(\rho_2, \mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{p}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) < \mathfrak{d}\Bigg\} \in \mathcal{F}(\mathcal{I}_2)
\end{aligned}
$$

biçimindedir.

Buna bağlı olarak, $\mathcal{C}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} = \mathcal{A}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} \cup \mathcal{B}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}$ kümesi tanımlanmakta olup, $\mathcal{C}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} \in \mathcal{I}_2$ koşulunu sağlamaktadır. Sonuç olarak $\mathcal{C}^{c}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})} \in \mathcal{F}(\mathcal{I}_2)$ olur. $\mathfrak{f} = \left(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}\right) \in c^{\mathcal{I}_2}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$, $\mathfrak{p} = \left(\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}\right) \in c^{\mathcal{I}_2}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ için

$$
\begin{aligned}
\mathcal{C}^{c}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d}) \subset \Big\{ & \Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}}) \Big) \\
& \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\Big(\gamma\rho_1 + \eta\rho_2, \gamma\mathfrak{x}_{\mathfrak{s}_1\mathfrak{w}_1} + \eta\mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1}, \gamma\mathfrak{x}_{\mathfrak{s}_2\mathfrak{w}_2} + \eta\mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \gamma\mathfrak{x}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}} \\
& + \eta\mathfrak{p}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \\
& > 1 \\
& - \mathfrak{d}, \mathfrak{G}\Big(\gamma\rho_1 + \eta\rho_2, \gamma\mathfrak{x}_{\mathfrak{s}_1\mathfrak{w}_1} + \eta\mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1}, \gamma\mathfrak{x}_{\mathfrak{s}_2\mathfrak{w}_2} + \eta\mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \gamma\mathfrak{x}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}} + \eta\mathfrak{p}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \\
& < \mathfrak{d}, \Xi\Big(\gamma\rho_1 + \eta\rho_2, \gamma\mathfrak{x}_{\mathfrak{s}_1\mathfrak{w}_1} + \eta\mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1}, \gamma\mathfrak{x}_{\mathfrak{s}_2\mathfrak{w}_2} + \eta\mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \gamma\mathfrak{x}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}} + \eta\mathfrak{p}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{v}\Big) \\
& < \mathfrak{d} \Big\}
\end{aligned}
$$

gösterilecektir.

Bunun için $\mathfrak{f}, \mathfrak{g} \in \mathcal{C}^{c}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})$ alınsın. Bu durumda,

$$
\begin{aligned}
&\mathfrak{F}\left(\rho_1, \mathfrak{x}_{\mathfrak{f}_1\mathfrak{g}_1}, \mathfrak{x}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \mathfrak{x}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\gamma|}\right) > 1 - \mathfrak{d} \text{ ve} \\
&\mathfrak{G}\left(\rho_1, \mathfrak{x}_{\mathfrak{f}_1\mathfrak{g}_1}, \mathfrak{x}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \mathfrak{x}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\gamma|}\right) < \mathfrak{d}, \\
&\mathfrak{H}\left(\rho_1, \mathfrak{x}_{\mathfrak{f}_1\mathfrak{g}_1}, \mathfrak{x}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \mathfrak{x}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\gamma|}\right) < \mathfrak{d},
\end{aligned}
$$

ve

$$
\begin{aligned}
&\mathfrak{F}\left(\rho_2, \mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1}, \mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \mathfrak{p}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) > 1 - \mathfrak{d} \text{ ve} \\
&\mathfrak{G}\left(\rho_2, \mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1}, \mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \mathfrak{p}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) < \mathfrak{d}, \\
&\mathfrak{H}\left(\rho_2, \mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1}, \mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \mathfrak{p}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) < \mathfrak{d},
\end{aligned}
$$

sağlanmaktadır.

Dolayısıyla

$$
\begin{aligned}
&\mathfrak{F}\Big(\gamma\rho_1 + \eta\rho_2, \gamma\mathfrak{x}_{\mathfrak{f}_1\mathfrak{g}_1} + \eta\mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1}, \gamma\mathfrak{x}_{\mathfrak{f}_2\mathfrak{g}_2} + \eta\mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \gamma\mathfrak{x}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}} + \eta\mathfrak{p}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \mathfrak{v}\Big) \\
&\geq \mathfrak{F}\left(\gamma\rho_1, \gamma\mathfrak{x}_{\mathfrak{f}_1\mathfrak{g}_1}, \gamma\mathfrak{x}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \gamma\mathfrak{x}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2}\right) \therefore \mathfrak{F}\left(\eta\rho_2, \eta\mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1}, \eta\mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \eta\mathfrak{p}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2}\right) \\
&= \mathfrak{F}\left(\rho_1, \mathfrak{x}_{\mathfrak{f}_1\mathfrak{g}_1}, \mathfrak{x}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \mathfrak{x}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\gamma|}\right) \therefore \mathfrak{F}\left(\rho_2, \mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1}, \mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2}, \dots, \mathfrak{p}_{\mathfrak{f}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}}, \frac{\mathfrak{v}}{2|\eta|}\right) \\
&> (1 - \mathfrak{d}) \therefore (1 - \mathfrak{d}) = 1 - \mathfrak{d}
\end{aligned}
$$

yazılır. Sonuç olarak

$$\mathfrak{F}\left(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{f}_1\mathfrak{g}_1}+\eta\mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1},\gamma\not{f}_{\mathfrak{f}_2\mathfrak{g}_2}+\eta\mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2},\dots,\gamma\not{f}_{\mathfrak{f}_q\mathfrak{g}_q}+\eta\mathfrak{p}_{\mathfrak{f}_q\mathfrak{g}_q},\mathfrak{v}\right)>1-\mathfrak{d}$$

elde edilir.

Bununla birlikte,

$$\begin{aligned}
&\mathfrak{G}\left(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{f}_1\mathfrak{g}_1}+\eta\mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1},\gamma\not{f}_{\mathfrak{f}_2\mathfrak{g}_2}+\eta\mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2},\dots,\gamma\not{f}_{\mathfrak{f}_q\mathfrak{g}_q}+\eta\mathfrak{p}_{\mathfrak{f}_q\mathfrak{g}_q},\mathfrak{v}\right)\\
&\le\mathfrak{G}\left(\gamma\rho_1,\gamma\not{f}_{\mathfrak{f}_1\mathfrak{g}_1},\gamma\not{f}_{\mathfrak{f}_2\mathfrak{g}_2},\dots,\gamma\not{f}_{\mathfrak{f}_q\mathfrak{g}_q},\frac{\mathfrak{v}}{2}\right)*\mathfrak{G}\left(\eta\rho_2,\eta\mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1},\eta\mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2},\dots,\eta\mathfrak{p}_{\mathfrak{f}_q\mathfrak{g}_q},\frac{\mathfrak{v}}{2}\right)\\
&=\mathfrak{G}\left(\rho_1,\not{f}_{\mathfrak{f}_1\mathfrak{g}_1},\not{f}_{\mathfrak{f}_2\mathfrak{g}_2},\dots,\not{f}_{\mathfrak{f}_q\mathfrak{g}_q},\frac{\mathfrak{v}}{2|\gamma|}\right)*\mathfrak{G}\left(\rho_2,\mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1},\mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2},\dots,\mathfrak{p}_{\mathfrak{f}_q\mathfrak{g}_q},\frac{\mathfrak{v}}{2|\eta|}\right)\\
&<\mathfrak{d}*\mathfrak{d}=\mathfrak{d}
\end{aligned}$$

yazılır. Dolayısıyla

$$\mathfrak{G}\left(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{f}_1\mathfrak{g}_1}+\eta\mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1},\gamma\not{f}_{\mathfrak{f}_2\mathfrak{g}_2}+\eta\mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2},\dots,\gamma\not{f}_{\mathfrak{f}_q\mathfrak{g}_q}+\eta\mathfrak{p}_{\mathfrak{f}_q\mathfrak{g}_q},\mathfrak{v}\right)<\mathfrak{d}$$

elde edilir.

Benzer şekilde

$$\mathfrak{H}\left(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{f}_1\mathfrak{g}_1}+\eta\mathfrak{p}_{\mathfrak{f}_1\mathfrak{g}_1},\gamma\not{f}_{\mathfrak{f}_2\mathfrak{g}_2}+\eta\mathfrak{p}_{\mathfrak{f}_2\mathfrak{g}_2},\dots,\gamma\not{f}_{\mathfrak{f}_q\mathfrak{g}_q}+\eta\mathfrak{p}_{\mathfrak{f}_q\mathfrak{g}_q},\mathfrak{v}\right)<\mathfrak{d}$$

bulunur. Sonuç olarak

$$\begin{aligned}
\mathcal{C}^{c}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})\subset\Big\{&\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\Big)\\
&\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\eta\mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1},\gamma\not{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\eta\mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\gamma\not{f}_{\mathfrak{s}_q\mathfrak{w}_q}\\
&+\eta\mathfrak{p}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\Big)\\
&>1\\
&-\mathfrak{d},\mathfrak{G}\left(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\eta\mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1},\gamma\not{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\eta\mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\gamma\not{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\eta\mathfrak{p}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)\\
&<\mathfrak{d},\Xi\left(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\eta\mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1},\gamma\not{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\eta\mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\gamma\not{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\eta\mathfrak{p}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)\\
&<\mathfrak{d}\Big\}
\end{aligned}$$

gösterilmiş olur. $\mathcal{C}^{c}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})\in\mathcal{F}(\mathcal{I}_2)$ olduğundan

$$\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\big) \in \mathbb{N}^q\times\mathbb{N}^q : \mathfrak{F}\big(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\eta\mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1},\gamma\not{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\eta\mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\gamma\not{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\eta\mathfrak{p}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\big) > 1-\mathfrak{d}, \mathfrak{G}\big(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\eta\mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1},\gamma\not{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\eta\mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\gamma\not{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\eta\mathfrak{p}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\big) < \mathfrak{d}, \Xi\big(\gamma\rho_1+\eta\rho_2,\gamma\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\eta\mathfrak{p}_{\mathfrak{s}_1\mathfrak{w}_1},\gamma\not{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\eta\mathfrak{p}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\gamma\not{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\eta\mathfrak{p}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\big) < \mathfrak{d}\Big\} \in \mathcal{F}(\mathcal{J}_2)$$

yazılır. Bu ise $\mathcal{J}_2-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\big(\gamma\not{f}_{\mathfrak{y}\mathfrak{z}}+\eta\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}\big)=\gamma\rho_1+\eta\rho_2(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğunu ifade eder. $\big(\gamma\not{f}_{\mathfrak{y}\mathfrak{z}}+\eta\mathfrak{p}_{\mathfrak{y}\mathfrak{z}}\big)\in c^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ elde edilir. Sonuç olarak $c^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ lineerliği gösterilmiş olur.

Teorem 5.2. $(Q,\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. Bu durumda

$$c_0^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})\subset c^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})\subset l_\infty^{\mathcal{J}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$$

kapsam ilişkisi geçerlidir.

Kapsama ilişkisinin tersi geçerli değildir. Bu durumu desteklemek için aşağıdaki örneği sunuyoruz.

Örnek 5.1. $Q=\mathbb{R}$, $\|\not{f}\|=\sup_{\mathfrak{y},\mathfrak{z}}\big|\not{f}_{\mathfrak{y}\mathfrak{z}}\big|$ koşulunu sağlayan bir NFGMU olsun. $\mathfrak{d}_1\therefore\mathfrak{d}_2=\min\{\mathfrak{d}_1,\mathfrak{d}_2\}$ ve $\mathfrak{d}_1\circledast\mathfrak{d}_2=\max\{\mathfrak{d}_1,\mathfrak{d}_2\}$, $\forall\mathfrak{d}_1,\mathfrak{d}_2\in[0,1]$ işlemleri tanımlansın. $\mathbb{R}^2\times(0,\infty)$ üzerinde $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normu

$$\mathfrak{F}\big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\big):=\frac{\mathfrak{v}}{\mathfrak{v}+\|\not{f}\|},$$
$$\mathfrak{G}\big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\big):=\frac{\|\not{f}\|\|}{\mathfrak{v}+\|\not{f}\|\|},$$
$$\mathfrak{H}\big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\big):=\frac{\mathfrak{v}}{\|\not{f}\|\|}$$

şeklinde tanımlansın. Bu durumda, $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ NFGMU oluşturmaktadır. $(\not{f}_{\mathfrak{y}\mathfrak{z}}) = (1,0,0,\dots)$ dizisi için $(\not{f}_{\mathfrak{y}\mathfrak{z}}) \in c^{\mathcal{I}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $\mathcal{I}_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = 1(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlanır, fakat $(\not{f}_{\mathfrak{y}\mathfrak{z}}) \notin c_0^{\mathcal{I}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olur.

Örnek 5.2. $\mathcal{Q} = \mathbb{R}$ olmak üzere, $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normu Örnek 5.1.'de tanımlanan norm ile birlikte $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ NFGMU tanımlasın. $(\not{f}_{\mathfrak{y}\mathfrak{z}}) = (-1)^{\mathfrak{y}+\mathfrak{z}}$ dizisi için, $(\not{f}_{\mathfrak{y}\mathfrak{z}}) \in l_\infty^{\mathcal{I}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlanırken $(\not{f}_{\mathfrak{y}\mathfrak{z}}) \notin c^{\mathcal{I}_2}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olur.

Lemma 5.1. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. Her $\mathfrak{d} \in (0,1)$ ve her $\mathfrak{v} > 0$ için aşağıdaki ifadeler denktir:

a) $\mathcal{I}_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$.

b) $\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) \le 1 - \mathfrak{d}\right\} \in \mathcal{I}_2$,

$$\left\{\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) \geqslant \mathfrak{d}\right\}\right\} \in \mathcal{I}_2,$$

$$\left\{\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) \geqslant \mathfrak{d}\right\}\right\} \in \mathcal{I}_2.$$

c) $\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) > 1 - \mathfrak{d}, \mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) < \mathfrak{d}, \mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) < \mathfrak{d}\right\} \in \mathcal{F}(\mathcal{I}_2)$.

d) $\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) > 1 - \mathfrak{d}\right\} \in \mathcal{F}(\mathcal{I}_2)$,

$$\left\{\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) < \mathfrak{d}\right\}\right\} \in \mathcal{F}(\mathcal{I}_2),$$

$$\left\{\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\right) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} : \mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right) < \mathfrak{d}\right\}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

(e)

$$\mathcal{I}_2 - \lim \mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) = 1 \text{ ve}$$
$$\mathcal{I}_2 - \lim \mathfrak{G}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) = 0,$$
$$\mathcal{I}_2 - \lim \mathfrak{H}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) = 0.$$

İspat. (a), (b), (c) ve (d) ifadelerinin eşdeğerliği açık bir şekilde gösterilebilir. Burada, (b) ile (e) ifadelerinin eşdeğerliğini ispat etmeye odaklanılacaktır.

(b) ifadesinin geçerli olduğunu kabul edilsin. Her $\mathfrak{d} \in (0,1)$, $\mathfrak{v} > 0$ için

$$\begin{aligned}
&\left\{\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \left|\mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) - 1\right| \geq \mathfrak{d}\right\} \\
&= \left\{\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) \geq 1 + \mathfrak{d}\right\} \\
&\cup \left\{\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) \leq 1 - \mathfrak{d}\right\}
\end{aligned}$$

ve her $\mathfrak{d} \in (0,1)$ için

$$\left\{\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) \geq 1 + \mathfrak{d}\right\} = \emptyset \in \mathcal{I}_2,$$

şartı sağlandığından, (b) ifadesiyle birlikte

$$\left\{\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \left|\mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) - 1\right| \geq \mathfrak{d}\right\} \in \mathcal{I}_2$$

sonucu elde edilir. Dolayısıyla

$$\mathcal{I}_2 - \lim \mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) = 1$$

bulunur.

Benzer şekilde, her $\mathfrak{d} \in (0,1)$, $\mathfrak{v} > 0$ için

$$\begin{aligned}
&\left\{\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \left|\mathfrak{G}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) - 0\right| \geq \mathfrak{d}\right\} \\
&= \left\{\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{G}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) \geq \mathfrak{d}\right\} \\
&\cup \left\{\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{G}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\right) \leq -\mathfrak{d}\right\}
\end{aligned}$$

ve

$$\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right)\in\mathbb{N}^q\times\mathbb{N}^q : \mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)\leq -\mathfrak{d}\right\}=\emptyset \in \mathcal{J}_2,$$

şartı sağlandığından

$$\mathcal{J}_2-\lim\mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)=0$$

bulunur.

Bununla birlikte, $\mathcal{J}_2-\lim\mathfrak{H}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)=0$ sağlanmaktadır. (e) ifadesinin (b) ifadesini sağladığı açıktır.

Teorem 5.3. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. Eğer $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\mathcal{Q}$'da $\mathcal{J}_2$-yakınsak ise $\mathcal{J}_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}$ limiti tektir.

İspat. Kabul edelim ki $\mathscr{f},\mathscr{g}\in\mathcal{Q}$ için $\mathcal{J}_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $\mathcal{J}_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{g}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olsun. $\mathscr{f}=\mathscr{g}$ olduğu gösterilecektir. $\mathfrak{d}\in(0,1)$ için $(1-\mathfrak{d}_1)\therefore(1-\mathfrak{d}_1)>1-\mathfrak{d}$ ve $\mathfrak{d}_1*\mathfrak{d}_1<\mathfrak{d}$ koşullarını sağlayan $\mathfrak{d}_1\in(0,1)$ seçilsin. Şimdi, $\mathfrak{v}>0$ için

$$\mathbb{A}_1(\mathfrak{v},\mathfrak{d}_1):=\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right)\in\mathbb{N}^q\times\mathbb{N}^q : \mathfrak{F}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right)\leqslant 1-\mathfrak{d}_1\right\},$$

$$\mathbb{A}_2(\mathfrak{v},\mathfrak{d}_1):=\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right)\in\mathbb{N}^q\times\mathbb{N}^q : \mathfrak{G}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right)\geq \mathfrak{d}_1\right\},$$

$$\mathbb{A}_3(\mathfrak{v},\mathfrak{d}_1):=\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right)\in\mathbb{N}^q\times\mathbb{N}^q : \mathfrak{H}\left(\mathscr{f},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right)\geq \mathfrak{d}_1\right\},$$

$$\mathbb{B}_1(\mathfrak{v},\mathfrak{d}_1):=\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right)\in\mathbb{N}^q\times\mathbb{N}^q : \mathfrak{F}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right)\leqslant 1-\mathfrak{d}_1\right\},$$

$$\mathbb{B}_{\mathbf{2}}(\mathfrak{v},\mathfrak{d}_1) := \left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{G}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) \geq \mathfrak{d}_1\right\},$$

$$\mathbb{B}_{\mathbf{3}}(\mathfrak{v},\mathfrak{d}_1) := \left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{H}\left(\mathscr{g},\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) \geq \mathfrak{d}_1\right\}$$

kümeleri tanımlansın.

Bununla birlikte,

$$\mathbb{U}(\mathfrak{v},\mathfrak{d}_1) = \left(\mathbb{A}_{\mathbf{1}}(\mathfrak{v},\mathfrak{d}_1) \cup \mathbb{B}_{\mathbf{1}}(\mathfrak{v},\mathfrak{d}_1)\right) \cap \left(\mathbb{A}_{\mathbf{2}}(\mathfrak{v},\mathfrak{d}_1) \cup \mathbb{B}_{\mathbf{2}}(\mathfrak{v},\mathfrak{d}_1)\right) \cap \left(\mathbb{A}_{\mathbf{3}}(\mathfrak{v},\mathfrak{d}_1) \cup \mathbb{B}_{\mathbf{3}}(\mathfrak{v},\mathfrak{d}_1)\right)$$

tanımlansın.

$\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathscr{f}_{\mathfrak{y}\mathfrak{z}} = \mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathscr{f}_{\mathfrak{y}\mathfrak{z}} = \mathscr{g}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğundan $\mathbb{A}_{\mathbf{1}}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{J}_2$, $\mathbb{A}_{\mathbf{2}}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{J}_2$, $\mathbb{A}_{\mathbf{3}}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{J}_2$, $\mathbb{B}_{\mathbf{1}}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{J}_2$, $\mathbb{B}_{\mathbf{2}}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{J}_2$, $\mathbb{B}_{\mathbf{3}}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{J}_2$ şartları sağlanmaktadır. Dolayısıyla, $\mathbb{U}^c(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{F}(\mathcal{J}_2)$ elde edilir ki bu da $\mathbb{U}^c(\mathfrak{v},\mathfrak{d}_1) \neq \emptyset$ sonucunu verir.

$\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right) \in \mathbb{U}^c(\mathfrak{v},\mathfrak{d}_1)$ olsun. O halde aşağıdaki durumlardan biri sağlanır:

$$\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right) \in \mathbb{A}_1^c \cap \mathbb{B}_1^c,$$

$$\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right) \in \mathbb{A}_2^c \cap \mathbb{B}_2^c$$

ve

$$\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right) \in \mathbb{A}_3^c \cap \mathbb{B}_3^c.$$

$\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right) \in \mathbb{A}_1^c \cap \mathbb{B}_1^c$ olsun. (NFGMU-3), (NFGMU-6) ve Tanım 2.1'nin (3) numaralı kısmını kullanılırsa

$$
\begin{aligned}
\mathfrak{F}(f, g, \dots, g, \mathfrak{v}) &\geq \mathfrak{F}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, f_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) \therefore \mathfrak{F}\left(f_{\mathfrak{s}_q\mathfrak{w}_q}, g, \dots, g, \frac{\mathfrak{v}}{2}\right) \\
&\geq \mathfrak{F}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, f_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) \therefore \mathfrak{F}\left(g, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, f_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) \\
&\geq (1 - \mathfrak{d}_1) \therefore (1 - \mathfrak{d}_1) > 1 - \mathfrak{d}
\end{aligned}
$$

ifadesi elde edilir.

$\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{A}_2^c \cap \mathbb{B}_2^c$ olsun. (NFGMU-10), (NFGMU-13) ve Tanım 3.1'nin (3') numaralı kısmını göz önünde bulundulursa

$$
\begin{aligned}
\mathfrak{G}((f, g, \dots, g, \mathfrak{v}) &\leqslant \mathfrak{G}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, f_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) * \mathfrak{G}\left(f_{\mathfrak{s}_q\mathfrak{w}_q}, g, \dots, g, \frac{\mathfrak{v}}{2}\right) \\
&\leqslant \mathfrak{G}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, f_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) * \mathfrak{G}\left(g, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, f_{\mathfrak{s}_q\mathfrak{w}_q}, \frac{\mathfrak{v}}{2}\right) \leqslant \mathfrak{d}_1 * \mathfrak{d}_1 \\
&< \mathfrak{d}
\end{aligned}
$$

ifadesi elde edilir.

$\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\right) \in \mathbb{A}_3^c \cap \mathbb{B}_3^c$ olsun. (NFGMU-17), (NFGMU-20) ve Tanım 3.1'nin (3') numaralı kısmını kullanarak

$$\mathfrak{H}((f, g, \dots, g, \mathfrak{v}) < \mathfrak{d}$$

bulunur.

$\mathfrak{d} \in (0,1)$ rastgele seçildiğinden, her $\mathfrak{v} > 0$ için $\mathfrak{F}(f, g, \dots, g, \mathfrak{v}) = 1$, $\mathfrak{G}((f, g, \dots, g, \mathfrak{v}) = 0$ ve $\mathfrak{H}((f, g, \dots, g, \mathfrak{v}) = 0$ elde edilir. Dolayısıyla, her durumda $f = g$ olduğu görülür. Bu da $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} f_{\mathfrak{y}\mathfrak{z}}$ limitinin tekliğini gösterdiğini kanıtlar.

Teorem 5.4. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU ve $(f_{\mathfrak{y}\mathfrak{z}}), (\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}) \in \mathcal{Q}$ olsunlar. Bu durumda,

a) Eğer $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H}) - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} f_{\mathfrak{y}\mathfrak{z}} = f$ ise $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} f_{\mathfrak{y}\mathfrak{z}} = f(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ geçerlidir.

b) Eğer $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} f_{\mathfrak{y}\mathfrak{z}} = f(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ ve $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} \mathfrak{h}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{h}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ ise $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} \left(f_{\mathfrak{y}\mathfrak{z}} + \mathfrak{h}_{\mathfrak{y}\mathfrak{z}}\right) = f + \mathfrak{h}$ geçerlidir.

c) Eğer $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} f_{\mathfrak{y}\mathfrak{z}} = f(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ ve $\kappa \in \mathbb{R}$ ise $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} \kappa f_{\mathfrak{y}\mathfrak{z}} = \kappa f(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ geçerlidir.

İspat. Eğer $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H}) - \lim_{\mathfrak{y},\mathfrak{z} \to \infty} f_{\mathfrak{y}\mathfrak{z}} = \alpha$ ise her $\mathfrak{d} \in (0,1)$, $\mathfrak{v} > 0$ için $\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q \geqslant \hbar_0$ ve

$\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathrm{q}} \geqslant \hbar_0$ olduğunda

$$\mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\right) > 1 - \mathfrak{d}, \mathfrak{G}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\right) < \mathfrak{d}$$

ve

$$\mathfrak{H}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\right) < \mathfrak{d}$$

koşullarını sağlayan bir $\hbar_0 \in \mathbb{N}$ sayısı mevcuttur.

$$\begin{aligned}\mathcal{A} = \Big\{\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathrm{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathrm{q}})\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} &: \mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\right) \\ &\leqslant 1 - \mathfrak{d}, \mathfrak{G}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\right) \\ &\geqslant \mathfrak{d}\Big\} \subseteq \{1,2, \dots, \hbar_0 - 1\} \times \{1,2, \dots, \hbar_0 - 1\}\end{aligned}$$

kapsam ilişkisi geçerlidir.

Bu durumda, $\mathcal{I}_2$ uygun ideal olduğundan

$$\begin{aligned}\Big\{\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathrm{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathrm{q}})\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} &: \mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\right) \\ &\leqslant 1 - \mathfrak{d}, \mathfrak{G}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\right) \geqslant \mathfrak{d}, \mathfrak{H}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\right) \\ &\geqslant \mathfrak{d}\Big\} \in \mathcal{I}_2\end{aligned}$$

sağlanır. Dolayısıyla $\mathcal{I}_2 - \lim\limits_{\mathfrak{y},\mathfrak{z} \to \infty} \mathscr{f}_{\mathfrak{y}\mathfrak{z}} = \mathscr{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ elde edilir.

b) $\mathfrak{d} \in (0,1)$ için $(1 - \mathfrak{d}_1) \therefore (1 - \mathfrak{d}_1) > 1 - \mathfrak{d}$ ve $\mathfrak{d}_1 * \mathfrak{d}_1 < \mathfrak{d}$ koşullarını sağlayan $\mathfrak{d}_1 \in (0,1)$ seçilsin. Her $\mathfrak{v} > 0$ için

$$\begin{aligned}\mathbb{K}_{\mathfrak{F},1}(\mathfrak{v}, \mathfrak{d}_1) \coloneqq \Big\{&\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathrm{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathrm{q}})\Big) \\ &\in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}}: \mathfrak{F}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \frac{\mathfrak{v}}{2}\right) \leq 1 - \mathfrak{d}_1\Big\},\end{aligned}$$

$$\begin{aligned}\mathbb{K}_{\mathfrak{G},1}(\mathfrak{v}, \mathfrak{d}_1) \coloneqq \Big\{&\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathrm{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathrm{q}})\Big) \\ &\in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}}: \mathfrak{G}\left(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \frac{\mathfrak{v}}{2}\right) \geq \mathfrak{d}_1\Big\},\end{aligned}$$

$$\mathbb{K}_{\mathfrak{H},1}(\mathfrak{v},\mathfrak{d}_1) := \Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\big)\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{H}\left(\not{f}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \frac{\mathfrak{v}}{2}\right) \geq \mathfrak{d}_1\Big\},$$

$$\mathbb{K}_{\mathfrak{F},2}(\mathfrak{v},\mathfrak{d}_1) := \Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\big)\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{F}\left(\mathfrak{h}, \mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{h}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \frac{\mathfrak{v}}{2}\right) \leq 1 - \mathfrak{d}_1\Big\},$$

$$\mathbb{K}_{\mathfrak{G},2}(\mathfrak{v},\mathfrak{d}_1) := \Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\big)\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{G}\left(\mathfrak{h}, \mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{h}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \frac{\mathfrak{v}}{2}\right) \geq \mathfrak{d}_1\Big\},$$

$$\mathbb{K}_{\mathfrak{H},2}(\mathfrak{v},\mathfrak{d}_1) := \Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\big)\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{H}\left(\mathfrak{h}, \mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{h}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \frac{\mathfrak{v}}{2}\right) \geq \mathfrak{d}_1\Big\}$$

ve

$$\mathbb{K}_{\mathfrak{F},\mathfrak{G},\mathfrak{H}}(\mathfrak{v},\mathfrak{d}_1) = \Big(\mathbb{K}_{\mathfrak{F},1}(\mathfrak{v},\mathfrak{d}_1) \cup \mathbb{K}_{\mathfrak{F},2}(\mathfrak{v},\mathfrak{d}_1)\Big) \cup \Big(\mathbb{K}_{\mathfrak{G},1}(\mathfrak{v},\mathfrak{d}_1) \cup \mathbb{K}_{\mathfrak{G},2}(\mathfrak{v},\mathfrak{d}_1)\Big) \cup \Big(\mathbb{K}_{\mathfrak{H},1}(\mathfrak{v},\mathfrak{d}_1) \cup \mathbb{K}_{\mathfrak{H},2}(\mathfrak{v},\mathfrak{d}_1)\Big)$$

kümeleri tanımlansın.

$\mathcal{I}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ve $\mathcal{I}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{h}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{h}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğundan her $\mathfrak{d}_1 \in (0,1)$ ve $\mathfrak{v} > 0$ için $\mathbb{K}_{\mathfrak{F},1}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{I}_2$, $\mathbb{K}_{\mathfrak{F},2}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{I}_2$, $\mathbb{K}_{\mathfrak{G},1}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{I}_2$, $\mathbb{K}_{\mathfrak{G},2}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{I}_2$, $\mathbb{K}_{\mathfrak{H},1}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{I}_2$ ve $\mathbb{K}_{\mathfrak{H},2}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{I}_2$ sağlanmaktadır. Bu durumda $\emptyset \neq \mathbb{K}^{c}_{\mathfrak{F},\mathfrak{G},\mathfrak{H}}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{F}(\mathcal{I}_2)$ olur.

$$\mathbb{K}^{c}_{\mathfrak{F},\mathfrak{G},\mathfrak{H}}(\mathfrak{v},\mathfrak{d}_1) \subset \Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\big)\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{F}\Big(\not{f} + \mathfrak{h}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1} + \mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2} + \mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}} + \mathfrak{h}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\Big) > 1 - \mathfrak{d}, \mathfrak{G}\Big(\not{f} + \mathfrak{h}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1} + \mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2} + \mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}} + \mathfrak{h}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\Big) < \mathfrak{d}, \mathfrak{H}\Big(\not{f} + \mathfrak{h}, \not{f}_{\mathfrak{s}_1\mathfrak{w}_1} + \mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2} + \mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}} + \mathfrak{h}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{v}\Big) < \mathfrak{d}\Big\}$$

olduğu ispatlanacaktır.

Bunun için $\mathfrak{s},\mathfrak{w} \in \mathbb{K}^{c}_{\mathfrak{F},\mathfrak{G},\mathfrak{H}}(\mathfrak{v},\mathfrak{d}_1)$ alınsın. Bu durumda

$$\mathfrak{F}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) > 1-\mathfrak{d}_1, \mathfrak{F}\left(\mathfrak{h},\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) > 1-\mathfrak{d}_1,$$
$$\mathfrak{G}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1, \mathfrak{G}\left(\mathfrak{h},\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1,$$
$$\mathfrak{H}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1, \mathfrak{H}\left(\mathfrak{h},\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1,$$

elde edilir.

$$\begin{aligned}&\mathfrak{F}\left(\mathfrak{f}+\mathfrak{h},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)\\ &\qquad \geq \mathfrak{F}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) \therefore \mathfrak{F}\left(\mathfrak{h},\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right)\\ &\qquad > (1-\mathfrak{d}_1) \therefore (1-\mathfrak{d}_1) > 1-\mathfrak{d},\end{aligned}$$

$$\begin{aligned}&\mathfrak{G}\left(\mathfrak{f}+\mathfrak{h},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)\\ &\qquad \leq \mathfrak{G}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) * \mathfrak{G}\left(\mathfrak{h},\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1 * \mathfrak{d}_1\\ &\qquad < \mathfrak{d},\end{aligned}$$

$$\begin{aligned}&\mathfrak{H}\left(\mathfrak{f}+\mathfrak{h},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right)\\ &\qquad \leq \mathfrak{H}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q},\frac{\mathfrak{v}}{2}\right) * \mathfrak{H}\left(\mathfrak{h},\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\omega_{r_2u_2},\dots,\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1 * \mathfrak{d}_1\\ &\qquad < \mathfrak{d}\end{aligned}$$

bulunur.

Sonuç olarak,

$$\mathfrak{F}\left(\mathfrak{f}+\mathfrak{h},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right) > 1-\mathfrak{d},$$

$$\mathfrak{G}\left(\mathfrak{f}+\mathfrak{h},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right) < \mathfrak{d},$$

$$\mathfrak{H}\left(\mathfrak{f}+\mathfrak{h},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\right) < \mathfrak{d}$$

sağlanır.

$$\mathbb{K}^{c}_{\mathfrak{F},\mathfrak{G},\mathfrak{H}}(\mathfrak{v},\mathfrak{d}_1) \subset \Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\Big) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\Big(\mathfrak{f}+\mathfrak{h},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\Big) > 1-\mathfrak{d}, \mathfrak{G}\Big(\mathfrak{f}+\mathfrak{h},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\Big) < \mathfrak{d}, \mathfrak{H}\Big(\mathfrak{f}+\mathfrak{h},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}+\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}+\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q}+\mathfrak{h}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{v}\Big) < \mathfrak{d}\Big\}$$

kapsam ilişkisi geçerlidir. $\mathbb{K}^{c}_{\mathfrak{F},\mathfrak{G},\mathfrak{H}}(\mathfrak{v},\mathfrak{d}_1) \in \mathcal{F}(\mathcal{I}_2)$ olduğundan $\mathcal{I}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} (\mathfrak{f}_{\mathfrak{y}\mathfrak{z}} + \mathfrak{h}_{\mathfrak{y}\mathfrak{z}}) = \mathfrak{f} + \mathfrak{h}$ elde edilir.

c) Açıktır, ve bu nedenle ispat ihmal edilmiştir.

Aşağıdaki örnek, Teorem 5.4'ün (a) ifadesinin tersi'nin geçerli olmadığını göstermek için verilmiştir.

Örnek 5.3. Örnek 3.1'de tanımlandığı gibi $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ NFGMU alınsın. Burada, $\mathcal{Q} = \mathbb{R}$ olsun ve $g: \mathbb{R}^{q+1} \longrightarrow \mathbb{R}^{+}$ fonksiyonu $g(\mathfrak{f}_0,\mathfrak{f}_1,\dots,\mathfrak{f}_q) = \max_{0\leqslant u,v\leqslant q} \{|\mathfrak{f}_u - \mathfrak{f}_v|\}$ şeklinde tanımlansın. $\mathbb{R}$'de $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi, $u, v \in \mathbb{N}$ için

$$\mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = \begin{cases} \mathfrak{y}\mathfrak{z}, & \text{eğer } \mathfrak{y} = u^3, \mathfrak{z} = v^3, \\ 1, & \text{durumlarda,} \end{cases}$$

alınsın. Bu durumda, $\mathcal{I}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = 1(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğu halde yakınsak değildir.

Tanım 5.2. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}$'da bir dizi olsun. Eğer her $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için

$$\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\Big) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{f}_{\mathbb{U}\mathbb{V}},\mathfrak{v}\Big) \leqslant 1-\mathfrak{d} \text{ or } \mathfrak{G}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{f}_{\mathbb{U}\mathbb{V}},\mathfrak{v}\Big) \geqslant \mathfrak{d}, \mathfrak{H}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathfrak{f}_{\mathbb{U}\mathbb{V}},\mathfrak{v}\Big) \geqslant \mathfrak{d}\Big\} \in \mathcal{I}_2,$$

koşulunu sağlayan $\mathbb{U} = \mathbb{U}(\mathfrak{d}) \in \mathbb{N}, \mathbb{V} = \mathbb{V}(\mathfrak{d}) \in \mathbb{N}$ varsa $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ dizisine $\mathcal{Q}'$da $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre $\mathcal{I}_2$-Cauchy dizisi denir.

Teorem 5.5. Eğer $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}'$da $\mathcal{I}_2$-yakınsak dizi ise $\mathcal{I}_2$-Cauchy dizisidir.

İspat. $\mathcal{I}_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olsun. $\mathfrak{d} \in (0,1)$ için $(1-\mathfrak{d}_1) \therefore (1-\mathfrak{d}_1) > 1-\mathfrak{d}$ ve $\mathfrak{d}_1 * \mathfrak{d}_1 < \mathfrak{d}$ koşullarını sağlayan $\mathfrak{d}_1 \in (0,1)$ seçilsin. Şimdi, $\mathfrak{v} > 0$ için

$$P(\mathfrak{d}_1) = \Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q})\big) \in \mathbb{N}^\mathfrak{q} \times \mathbb{N}^\mathfrak{q} : \mathfrak{F}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\frac{\mathfrak{v}}{2}\right) \leqslant 1-\mathfrak{d}_1, \mathfrak{G}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\frac{\mathfrak{v}}{2}\right) \geqslant \mathfrak{d}_1, \mathfrak{H}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\frac{\mathfrak{v}}{2}\right) \geqslant \mathfrak{d}_1\Big\}$$

ve

$$P^c(\mathfrak{d}_1) = \Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q})\big) \in \mathbb{N}^\mathfrak{q} \times \mathbb{N}^\mathfrak{q} : \mathfrak{F}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\frac{\mathfrak{v}}{2}\right) > 1-\mathfrak{d}_1, \mathfrak{G}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\frac{\mathfrak{v}}{2}\right) \text{e} < \mathfrak{d}_1, \mathfrak{H}\left(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1\Big\}$$

kümeleri ele alınsın.

$\mathcal{I}_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğundan $P(\mathfrak{d}_1) \in \mathcal{I}_2$ ve $P^c(\mathfrak{d}_1) \in \mathcal{F}(\mathcal{I}_2)$ sağlanır.

$$(\mathcal{k}_1,\mathcal{k}_2,\dots,\mathcal{k}_\mathfrak{q}),(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_\mathfrak{q}) \in P^c(\mathfrak{d}_1)$$

alınsın.

Bu durumda,

$$\mathfrak{F}\left(\mathfrak{f}_{\mathcal{k}_1\mathfrak{h}_1},\mathfrak{f}_{\mathcal{k}_2\mathfrak{h}_2},\dots,\mathfrak{f}_{\mathcal{k}_\mathfrak{q}\mathfrak{h}_\mathfrak{q}},\mathfrak{f},\frac{\mathfrak{v}}{2}\right) > 1-\mathfrak{d}_1,$$

ve

$$\mathfrak{G}\left(\mathfrak{f}_{\mathcal{k}_1\mathfrak{h}_1},\mathfrak{f}_{\mathcal{k}_2\mathfrak{h}_2},\dots,\mathfrak{f}_{\mathcal{k}_\mathfrak{q}\mathfrak{h}_\mathfrak{q}},\mathfrak{f},\frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1, \mathfrak{H}\left(\mathfrak{f}_{\mathcal{k}_1\mathfrak{h}_1},\mathfrak{f}_{\mathcal{k}_2\mathfrak{h}_2},\dots,\mathfrak{f}_{\mathcal{k}_\mathfrak{q}\mathfrak{h}_\mathfrak{q}},\mathfrak{f},\frac{\mathfrak{v}}{2}\right) < \mathfrak{d}_1$$

elde edilir.

Bazı $\mathfrak{y},\mathfrak{z} \in \{1,2,\dots,\mathfrak{q}\}$ için sabit $\mathcal{k}_\mathfrak{y},\mathfrak{h}_\mathfrak{z} \in \mathbb{N}$ değerleri için,

$$\mathfrak{F}\left(\mathfrak{f}_{\mathcal{k}_\mathfrak{y}\mathfrak{h}_\mathfrak{z}},\mathfrak{f},\dots,\mathfrak{f},\frac{\mathfrak{v}}{2}\right) \geqslant \mathfrak{F}\left(\mathfrak{f}_{\mathcal{k}_1\mathfrak{h}_1},\mathfrak{f}_{\mathcal{k}_2\mathfrak{h}_2},\dots,\mathfrak{f}_{\mathcal{k}_\mathfrak{q}\mathfrak{h}_\mathfrak{q}},\mathfrak{f},\frac{\mathfrak{v}}{2}\right) > 1-\mathfrak{d}_1$$

ve

$$\mathfrak{G}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathscr{f},\dots,\mathscr{f},\frac{\mathfrak{v}}{2}\right)\leqslant\mathfrak{G}\left(\mathscr{f}_{\mathscr{k}_1\mathfrak{h}_1},\mathscr{f}_{\mathscr{k}_2\mathfrak{h}_2},\dots,\mathscr{f}_{\mathscr{k}_{\mathfrak{q}}\mathfrak{h}_{\mathfrak{q}}},\mathscr{f},\frac{\mathfrak{v}}{2}\right)<\mathfrak{d}_1,$$
$$\mathfrak{H}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathscr{f},\dots,\mathscr{f},\frac{\mathfrak{v}}{2}\right)\leqslant\mathfrak{H}\left(\mathscr{f}_{\mathscr{k}_1\mathfrak{h}_1},\mathscr{f}_{\mathscr{k}_2\mathfrak{h}_2},\dots,\mathscr{f}_{\mathscr{k}_{\mathfrak{q}}\mathfrak{h}_{\mathfrak{q}}},\mathscr{f},\frac{\mathfrak{v}}{2}\right)<\mathfrak{d}_1$$

elde edilir.

$(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\in P^c(\mathfrak{d}_1)$ için

$$\mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\right)\geqslant\mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f},\frac{\mathfrak{v}}{2}\right)\therefore\mathfrak{F}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathscr{f},\dots,\mathscr{f},\frac{\mathfrak{v}}{2}\right)$$
$$>(1-\mathfrak{d}_1)\therefore(1-\mathfrak{d}_1)>1-\mathfrak{d}$$

ve

$$\mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\right)\leqslant\mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f},\frac{\mathfrak{v}}{2}\right)\therefore\mathfrak{F}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathscr{f},\dots,\mathscr{f},\frac{\mathfrak{v}}{2}\right)$$
$$<\mathfrak{d}_1*\mathfrak{d}_1<\mathfrak{d}$$

$$\mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\right)\leqslant\mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f},\frac{\mathfrak{v}}{2}\right)\therefore\mathfrak{H}\left(\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathscr{f},\dots,\mathscr{f},\frac{\mathfrak{v}}{2}\right)$$
$$<\mathfrak{d}_1*\mathfrak{d}_1<\mathfrak{d}$$

sağlanır. Böylece

$$P^c(\mathfrak{d}_1)\subseteq\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\right)$$
$$>1-\mathfrak{d}\text{ ve }\mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\right)$$
$$<\mathfrak{d},\mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\right)<\mathfrak{d}\Big\}$$

yazılır.

Sonuç olarak

$$\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\right)$$
$$>1-\mathfrak{d}\text{ ve }\mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\right)$$
$$<\mathfrak{d},\mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathscr{f}_{\mathscr{k}_{\mathfrak{y}}\mathfrak{h}_{\mathfrak{z}}},\mathfrak{v}\right)<\mathfrak{d}\Big\}\in\mathcal{F}(\mathcal{I}_2),$$

veya

$$\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\ldots,\mathfrak{s}_\mathfrak{q}),(\mathfrak{w}_1,\mathfrak{w}_2,\ldots,\mathfrak{w}_\mathfrak{q})\big)\in\mathbb{N}^\mathfrak{q}\times\mathbb{N}^\mathfrak{q}:\mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\ldots,\mathscr{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathscr{f}_{\mathscr{k}_\mathfrak{y}\mathfrak{h}_\mathfrak{z}},\mathfrak{v}\right)$$
$$\leq 1-\mathfrak{d} \text{ ve } \mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\ldots,\mathscr{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathscr{f}_{\mathscr{k}_\mathfrak{y}\mathfrak{h}_\mathfrak{z}},\mathfrak{v}\right)$$
$$\geq \mathfrak{d},\mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\ldots,\mathscr{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathscr{f}_{\mathscr{k}_\mathfrak{y}\mathfrak{h}_\mathfrak{z}},\mathfrak{v}\right)\geq\mathfrak{d}\Big\}\in\mathcal{I}_2$$

elde edilir. İspat tamamlanmış olur.

Teorem 5.5'in tersi genellikle geçerli değildir. Bunu göstermek için aşağıdaki örneği ele alınsın.

Örnek 5.4. $\mathcal{Q}=(0,1]$ olsun. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ uzayı Örnek 3.2'de tanımlanan NFGMU olsun. $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi

$$\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\begin{cases}1, & \text{eğer } \mathfrak{y}=k^3,\mathfrak{z}=l^3,\\ \dfrac{1}{\mathfrak{y}\mathfrak{z}}, & \text{diğer durumlarda, } (\forall k,l\in\mathbb{N})\end{cases}$$

şeklinde tanımlansın. Bu durumda, $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\mathcal{Q}$'da ideal Cauchy dizisidir ancak ideal yakınsak değildir.

Tanım 5.3. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ uzayında her $\mathcal{I}_2$-Cauchy dizisi $\mathcal{I}_2$-yakınsak ise uzaya $\mathcal{I}_2$-tam uzay denir.

Teorem 5.6. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}'$da bir dizi olsun. Her $\mathfrak{d}\in(0,1)$ ve her $\mathfrak{v}>0$ için aşağıdaki ifadeler denktir:

(a) $\mathcal{I}_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$.

(b) $\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\ldots,\mathfrak{s}_\mathfrak{q}),(\mathfrak{w}_1,\mathfrak{w}_2,\ldots,\mathfrak{w}_\mathfrak{q})\big)\in\mathbb{N}^\mathfrak{q}\times\mathbb{N}^\mathfrak{q}:\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\ldots,\mathscr{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}}\right)\notin \mathcal{V}_{\mathscr{f}}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})\Big\}\in\mathcal{I}_2$.

İspat. İspat, doğrudan $\mathcal{I}_2$-yakınsaklık tanımından elde edilir.

Teorem 5.7. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun. $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\mathcal{Q}$'da $\mathcal{I}_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olsun. Bu durumda, her $\mathfrak{d}\in(0,1)$, $\mathfrak{v}>0$ ve her $m\in\{1,2,\ldots,\mathfrak{q}\}$ için

$$\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}: \not{f}_{\mathfrak{s}_m\mathfrak{w}_m}\notin\mathcal{B}_{\not{f}}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})\right\}\in\mathcal{I}_2$$

sağlanır.

İspat. Her $\mathfrak{d}\in(0,1)$, $\mathfrak{v}>0$ ve her $m\in\{1,2,\dots,\mathrm{q}\}$ için

$$\mathbb{K}(\mathfrak{v},\mathfrak{d})=\Big\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}: \mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)>1-\mathfrak{d}$$
$$\text{ve } \mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)<\mathfrak{d}, \mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)<\mathfrak{d}\Big\},$$

ve

$$\mathbb{L}(\mathfrak{v},\mathfrak{d})=\left\{\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\right)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}: \not{f}_{\mathfrak{s}_m\mathfrak{w}_m}\in\mathcal{B}_{\not{f}}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})\right\}$$

tanımlansın.

$$\mathcal{I}_2-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$$

olduğundan $\mathbb{K}(\mathfrak{v},\mathfrak{d})\in\mathcal{F}(\mathcal{I}_2)$ yazılır. $(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\in\mathbb{K}(\mathfrak{v},\mathfrak{d})$ alınsın.

Bu durumda

$$\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)>1-\mathfrak{d} \text{ ve } \mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)<\mathfrak{d}$$

$$\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)<\mathfrak{d}$$

olur. Önerme 3.2'e göre her $m\in\{1,2,\dots,\mathrm{q}\}$ için $\not{f}_{\mathfrak{s}_m\mathfrak{w}_m}\in\mathcal{B}_{\not{f}}^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{v},\mathfrak{d})$ elde edilir.

Dolayısıyla, $(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\in\mathbb{L}(\mathfrak{v},\mathfrak{d})$, ve bu nedenle $\mathbb{K}(\mathfrak{v},\mathfrak{d})\subseteq\mathbb{L}(\mathfrak{v},\mathfrak{d})$ olur. Bu durum, $\mathbb{L}(\mathfrak{v},\mathfrak{d})\in\mathcal{F}(\mathcal{I}_2)$ eşitliğini sağlar ve istenen sonucu verir

Tanım 5.4. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\not{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}'$da bir dizi olsun. Eğer bazı $\not{f}_0\in\mathcal{Q}$ için,

$$\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\Big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_0,\mathfrak{v}\Big)$$
$$\le 1-\mathfrak{d}_0,\mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_0,\mathfrak{v}\Big)$$
$$\ge \mathfrak{d}_0,\mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_0,\mathfrak{v}\Big)\ge\mathfrak{d}_0\Big\}\in\mathcal{I}_2$$

koşunu sağlayan $\mathfrak{d}_0\in(0,1)$ ve $\mathfrak{v}>0$ mevcut ise $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisine $\mathcal{Q}'$da $\mathcal{I}_2$-sınırlı dizi denir.

Teorem 5.8. Eğer $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\mathcal{Q}$ NFGMU'da $\mathcal{I}_2$-Cauchy dizisi ise, $\mathcal{I}_2$-sınırlıdır.

İspat. $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\mathcal{Q}$'da $\mathcal{I}_2$-Cauchy dizisi olsun. Bu durumda, her $\mathfrak{d}\in(0,1)$ ve $\mathfrak{v}>0$ için

$$\mathbb{K}(\mathfrak{v},\mathfrak{d})=\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\Big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{\mathbb{U}\mathbb{V}},\mathfrak{v}\Big)$$
$$>1-\mathfrak{d}\text{ ya da }\mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{\mathbb{U}\mathbb{V}},\mathfrak{v}\Big)$$
$$<\mathfrak{d},\mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{\mathbb{U}\mathbb{V}},\mathfrak{v}\Big)<\mathfrak{d}\Big\}\in\mathcal{F}(\mathcal{I}_2)$$

kümesi için $\mathbb{U}=\mathbb{U}(\mathfrak{d})\in\mathbb{N},\mathbb{V}=\mathbb{V}(\mathfrak{d})\in\mathbb{N}$ vardır.

Sabit bir $\not{f}_0\in\mathcal{Q}$ değeri için

$$\mathfrak{F}\left(\not{f}_{\mathbb{U}\mathbb{V}},\not{f}_{\mathbb{U}\mathbb{V}},\dots,\not{f}_{\mathbb{U}\mathbb{V}},\not{f}_0,\frac{\mathfrak{v}}{2}\right)=\alpha,$$

$$\mathfrak{G}\left(\not{f}_{\mathbb{U}\mathbb{V}},\not{f}_{\mathbb{U}\mathbb{V}},\dots,\not{f}_{\mathbb{U}\mathbb{V}},\not{f}_0,\frac{\mathfrak{v}}{2}\right)=\gamma,$$

$$\mathfrak{H}\left(\not{f}_{\mathbb{U}\mathbb{V}},\not{f}_{\mathbb{U}\mathbb{V}},\dots,\not{f}_{\mathbb{U}\mathbb{V}},\not{f}_0,\frac{\mathfrak{v}}{2}\right)=\psi$$

biçiminde tanımlansın.

$\alpha,\gamma,\psi\in(0,1)$ olduğundan $(1-\mathfrak{d})\therefore\alpha>1-\beta$, $\mathfrak{d}*\gamma<\delta$ ve $\delta*\psi<\tau$ koşullarını sağlayan $\beta,\delta,\tau\in(0,1)$ değerleri mevcuttur. $(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\in\mathbb{K}(\mathfrak{v},\mathfrak{d})$ alınsın. Bu durumda

$$\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_0,\mathfrak{v}\Big)\geqslant\mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\not{f}_{\mathbb{U}\mathbb{V}},\frac{\mathfrak{v}}{2}\right)$$
$$\therefore\mathfrak{F}\left(\not{f}_{\mathbb{U}\mathbb{V}},\not{f}_{\mathbb{U}\mathbb{V}},\dots,\not{f}_{\mathbb{U}\mathbb{V}},\not{f}_0,\frac{\mathfrak{v}}{2}\right)>(1-\mathfrak{d})\therefore\alpha>1-\beta,$$

$$
\begin{aligned}
&\mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \\
&\qquad\qquad \leqslant \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_{\mathbb{UV}}, \frac{\mathfrak{v}}{2}\right) * \mathfrak{G}\left(\not{f}_{\mathbb{UV}}, \not{f}_{\mathbb{UV}}, \ldots, \not{f}_{\mathbb{UV}}, \not{f}_0, \frac{\mathfrak{v}}{2}\right) < \mathfrak{d} * \gamma \\
&\qquad\qquad < \delta,
\end{aligned}
$$

ve benzer şekilde $\mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) < \tau$ olsun. $\mathfrak{d}_0 = \max\{\beta, \delta, \tau\}$ olarak alınırsa

$$
\begin{aligned}
\mathbb{K}(\mathfrak{v}, \mathfrak{d}) \subseteq \Big\{ &\left((\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \\
&> 1 - \mathfrak{d}_0 \text{ yada } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \\
&< \mathfrak{d}_0, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) < \mathfrak{d}_0 \Big\},
\end{aligned}
$$

ve dolayısıyla

$$
\begin{aligned}
\Big\{ &\left((\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \\
&> 1 - \mathfrak{d}_0 \text{ yada } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \\
&< \mathfrak{d}_0, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) < \mathfrak{d}_0 \Big\} \in \mathcal{F}(\mathcal{I}_2),
\end{aligned}
$$

yazılır. Sonuç olarak

$$
\begin{aligned}
\Big\{ &\left((\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_q)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \\
&\leqslant 1 - \mathfrak{d}_0 \text{ yada } \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \\
&\geqslant \mathfrak{d}_0, \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}_0, \mathfrak{v}\right) \geqslant \mathfrak{d}_0 \Big\} \in \mathcal{I}_2
\end{aligned}
$$

elde edilir. Dolayısıyla, $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ $\mathcal{I}_2$-sınırlıdır.

Sonuç 5.1. Bir NFGMU’larda her $\mathcal{I}_2$-yakınsak dizi $\mathcal{I}_2$-sınırlıdır.

Tanım 5.5. Eğer $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisinin $\mathcal{Q}$’da

$$
\lim_{\substack{\mathfrak{s}_i, \mathfrak{w}_i \to \infty \\ (\mathfrak{s}_i, \mathfrak{w}_i) \in \mathbb{M}, (i=1,2,\ldots,q)}} \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \not{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \not{f}, \mathfrak{v}\right) = 1,
$$

$$\lim_{\substack{\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i}\to\infty \\ (\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i})\in\mathbb{M},(\mathfrak{i}=1,2,\dots,\mathfrak{q})}} \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\right)=0,$$

$$\lim_{\substack{\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i}\to\infty \\ (\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i})\in\mathbb{M},(\mathfrak{i}=1,2,\dots,\mathfrak{q})}} \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\right)=0,$$

olacak şekilde bir alt dizisinin indeks kümesi $\mathbb{M}\in\mathcal{F}(\mathcal{J}_2)$ $(\mathbb{H}=\mathbb{N}^2\setminus\mathbb{M}\in\mathcal{J}_2)$ mevcut ise $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\not{f}$'e $\mathcal{J}_2^*$-yakınsaktır denir, $\mathcal{J}_2^*-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ile gösterilir.

Teorem 5.9. $\mathcal{J}_2$ kuvvetli uygun ideal olsun. Eğer $\mathcal{J}_2^*-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ise $\mathcal{J}_2-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlanmaktadır.

İspat. $\mathcal{J}_2^*-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olduğundan

$$\lim_{\substack{\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i}\to\infty \\ (\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i})\in\mathbb{M},(\mathfrak{i}=1,2,\dots,\mathfrak{q})}} \mathfrak{F}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\right)=1,$$

$$\lim_{\substack{\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i}\to\infty \\ (\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i})\in\mathbb{M},(\mathfrak{i}=1,2,\dots,\mathfrak{q})}} \mathfrak{G}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\right)=0,$$

$$\lim_{\substack{\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i}\to\infty \\ (\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i})\in\mathbb{M},(\mathfrak{i}=1,2,\dots,\mathfrak{q})}} \mathfrak{H}\left(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\not{f},\mathfrak{v}\right)=0,$$

olacak şekilde bir alt dizisinin indeks kümesi $\mathbb{M}\in\mathcal{F}(\mathcal{J}_2)$ $(\mathbb{H}=\mathbb{N}^2\setminus\mathbb{M}\in\mathcal{J}_2)$ mevcuttur.

Bu durumda, her $\mathfrak{d}\in(0,1)$, her $(\mathfrak{s}_\mathfrak{i},\mathfrak{w}_\mathfrak{i})\in\mathbb{M},(\mathfrak{i}=1,2,\dots,\mathfrak{q})$ ve her $\mathfrak{v}>0$ için $\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\geqslant\hbar_0$ ve $\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\geqslant\hbar_0$ olduğunda

$$\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\right)>1-\mathfrak{d},\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\right)<\mathfrak{d}$$

ve

$$\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},\mathfrak{v}\right)<\mathfrak{d}$$

koşullarını sağlayan bir $\hbar_0\in\mathbb{N}$ sayısı mevcuttur.

Bu durumda,

$$A(\mathfrak{v}) = \Big\{\big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_q), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_q)\big) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\big(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\big) \leqslant 1 - \mathfrak{d} \text{ or } \mathfrak{G}\big(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(\mathscr{f}, \mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathfrak{v}\big) \geqslant \mathfrak{d}\Big\} \subset \mathbb{H} \cup \Big(\mathbb{M} \cap \big((\{1,2,\dots,(\hbar_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\dots,(\hbar_0 - 1)\})\big)\Big)$$

kapsam ilişkisi geçerlidir.

$\mathcal{J}_2$ kuvvetli uygun ideal olduğundan

$$\mathbb{H} \cup \Big(\mathbb{M} \cap \big((\{1,2,\dots,(\hbar_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\dots,(\hbar_0 - 1)\})\big)\Big) \in \mathcal{J}_2,$$

elde edilir, ve $A(\mathfrak{v}) \in \mathcal{J}_2$ olur. Böylece $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathscr{f}_{\mathfrak{y}\mathfrak{z}} = \mathscr{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ elde edilir.

Teorem 5.10. Eğer $\mathcal{J}_2$ uygun ideali $(AP2)$ özelliğine sahipse keyfi $\big(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\big) \in Q$ dizisi için $\mathcal{J}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathscr{f}_{\mathfrak{y}\mathfrak{z}} = \mathscr{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ olduğunda $\mathcal{J}_2^* - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathscr{f}_{\mathfrak{y}\mathfrak{z}} = \mathscr{f}(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ sağlanmaktadır.

Tanım 5.6. $\mathcal{J}_2$ kuvvetli uygun ideal olsun. Eğer her $\mathfrak{d} \in (0,1)$, her $(\mathfrak{s}, \mathfrak{w}), (\mathfrak{u}, \mathfrak{v}) \in \mathbb{M}$ ve her $\mathfrak{v} > 0$ için $\mathfrak{s}, \mathfrak{w}, \mathfrak{u}, \mathfrak{v} > \hbar_0(\mathfrak{v})$ olduğunda

$$\mathfrak{F}\big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathscr{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\big) > 1 - \mathfrak{d} \text{ or } \mathfrak{G}\big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathscr{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\big) < \mathfrak{d}, \mathfrak{H}\big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathscr{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\big) < \mathfrak{d}$$

sağlayacak şekilde $\hbar_0(\mathfrak{v}) \in \mathbb{N}$ sayısı ve $\mathbb{M} \in \mathcal{F}(\mathcal{J}_2)$ (yani $\mathbb{H} = \mathbb{N}^2 \setminus \mathbb{M} \in \mathcal{J}_2$) olacak şekilde bir indeks küme mevcutsa $\big(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\big) \in Q$ dizisine NFGMU'larda $\mathcal{J}_2^*$-Cauchy dizisi denir.

Bu durum

$$\lim_{\substack{\mathfrak{s}_i,\mathfrak{w}_i,\mathfrak{u},\mathfrak{v}\to\infty \\ (\mathfrak{s}_i,\mathfrak{w}_i),(\mathfrak{u},\mathfrak{v})\in\mathbb{M},(i=1,2,\dots,q)}} \mathfrak{F}\big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathscr{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\big) = 1,$$

$$\lim_{\substack{\mathfrak{s}_i,\mathfrak{w}_i,\mathfrak{u},\mathfrak{v}\to\infty \\ (\mathfrak{s}_i,\mathfrak{w}_i),(\mathfrak{u},\mathfrak{v})\in\mathbb{M},(i=1,2,\dots,q)}} \mathfrak{G}\big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q}, \mathscr{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\big) = 0,$$

$$\lim_{\substack{\mathfrak{s}_i,\mathfrak{w}_i,\mathfrak{u},\mathfrak{v}\to\infty \\ (\mathfrak{s}_i,\mathfrak{w}_i),(\mathfrak{u},\mathfrak{v})\in\mathbb{M},(i=1,2,\dots,q)}} \mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathscr{f}_{\mathfrak{u}\mathfrak{v}},\mathfrak{v}\right)=0,$$

biçiminde de ifade edilebilir.

Teorem 5.11. Eğer $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)\in Q$ dizisi $\mathcal{I}_2^*$-Cauchy dizisi ise $\mathcal{I}_2$-Cauchy dizisidir.

İspat. $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)\in Q$ dizisi $\mathcal{I}_2^*$-Cauchy dizisi olsun. Bu durumda her $\mathfrak{d}\in(0,1)$, her $(\mathfrak{s},\mathfrak{w}),(\mathfrak{u},\mathfrak{v})\in\mathbb{M}$ ve her $\mathfrak{v}>0$ için $\mathfrak{s},\mathfrak{w},\mathfrak{u},\mathfrak{v}>\hbar_0(\mathfrak{v})$ olduğunda

$$\mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathscr{f}_{\mathfrak{u}\mathfrak{v}},\mathfrak{v}\right)>1-\mathfrak{d} \text{ ya da } \mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathscr{f}_{\mathfrak{u}\mathfrak{v}},\mathfrak{v}\right)$$
$$<\mathfrak{d},\mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathscr{f}_{\mathfrak{u}\mathfrak{v}},\mathfrak{v}\right)<\mathfrak{d}$$

sağlayacak şekilde $\hbar_0(\mathfrak{v})\in\mathbb{N}$ sayısı ve $\mathbb{M}\in\mathcal{F}(\mathcal{I}_2)$ (yani $\mathbb{H}=\mathbb{N}^2\setminus\mathbb{M}\in\mathcal{I}_2$) olacak şekilde bir indeks küme mevcuttur.

$$\begin{aligned} A(\mathfrak{v})=\Big\{&\left((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_q),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_q)\right)\in\mathbb{N}^q\times\mathbb{N}^q\colon \mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathscr{f}_{\mathfrak{u}\mathfrak{v}},\mathfrak{v}\right)\\ &\leqslant 1-\mathfrak{d} \text{ ya da } \mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathscr{f}_{\mathfrak{u}\mathfrak{v}},\mathfrak{v}\right)\\ &\geqslant\mathfrak{d},\mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_q\mathfrak{w}_q},\mathscr{f}_{\mathfrak{u}\mathfrak{v}},\mathfrak{v}\right)\geqslant\mathfrak{d}\Big\}\\ &\subset\mathbb{H}\cup\left(\mathbb{M}\cap\left((\{1,2,\dots,(\hbar_0-1)\}\times\mathbb{N})\cup(\mathbb{N}\times\{1,2,\dots,(\hbar_0-1)\})\right)\right)\end{aligned}$$

kapsam ilişkisi geçerlidir.

$\mathcal{I}_2$ kuvvetli uygun ideal olduğundan

$$\mathbb{H}\cup\left(\mathbb{M}\cap\left((\{1,2,\dots,(\hbar_0-1)\}\times\mathbb{N})\cup(\mathbb{N}\times\{1,2,\dots,(\hbar_0-1)\})\right)\right)\in\mathcal{I}_2,$$

elde edilir, ve $A(\mathfrak{v})\in\mathcal{I}_2$ olur. Dolayısıyla $\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right)\in Q$ dizisi $\mathcal{I}_2$–Cauchy dizisidir.

Lemma 5.2. $\{\mathbb{P}_i\}_{i\in\mathbb{N}}$, $\mathbb{N}^2$ kümesinin sayılabilir alt kümelerinden oluşan bir ailesi olsun ve $\mathcal{F}(\mathcal{I}_2)$, (AP2) özelliğine sahip güçlü uygun bir $\mathcal{I}_2$ ideali için bir süzgeç olsun. Her $i\in\mathbb{N}$ için $\mathbb{P}_i\in\mathcal{F}(\mathcal{I}_2)$ sağlanıyorsa, $P\in\mathcal{F}(\mathcal{I}_2)$ olan bir küme mevcuttur ve her $i\in\mathbb{N}$ için $P\setminus\mathbb{P}_i$ sonludur.

Teorem 5.12. $\mathcal{J}_2$, $(AP2)$ özelliğine sahip kuvvetli uygun ideal ise $\mathcal{J}_2^*$-Cauchy dizisi ve $\mathcal{J}_2$-Cauchy dizisi çakışmaktadır.

İspat. Teorem 5.11'de $\mathcal{J}_2$ idealinin $(AP2)$ özelliğini taşımadığı durumda, her $\mathcal{J}_2^*$-Cauchy dizisinin aynı zamanda bir $\mathcal{J}_2$-Cauchy dizisi olduğu kanıtlanmıştır. Şimdi, $\mathcal{J}_2$ idealinin $(AP2)$ özelliğine sahip olduğu varsayımı altında, her $\mathcal{J}_2$-Cauchy dizisinin aynı zamanda bir $\mathcal{J}_2^*$-Cauchy dizisi olduğu gösterilecektir. $\left(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}\right) \in Q$ dizisi $\mathcal{J}_2$-Cauchy dizisi olsun. Bu durumda, her $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için

$$\mathbb{K}(\mathfrak{d}, \mathfrak{v}) = \Big\{\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}})\Big) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} \colon \mathfrak{F}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\Big)$$
$$\leqslant 1 - \mathfrak{d},\ \ \mathfrak{G}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\Big)$$
$$\geqslant \mathfrak{d}, \mathfrak{H}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\Big) \geqslant \mathfrak{d}\Big\} \in \mathcal{J}_2,$$

koşulunu sağlayan $\mathfrak{u} = \mathfrak{u}(\mathfrak{d}) \in \mathbb{N}, \mathfrak{v} = \mathfrak{v}(\mathfrak{d}) \in \mathbb{N}$ vardır.

$\mathfrak{u}_{\mathfrak{i}} = \mathfrak{u}\left(\frac{1}{\mathfrak{i}}\right)$, $\mathfrak{v}_{\mathfrak{i}} = \mathfrak{v}\left(\frac{1}{\mathfrak{i}}\right)$ olmak üzere

$$\mathbb{P}_{\mathfrak{i}} = \Big\{\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathfrak{q}})\Big) \in \mathbb{N}^{\mathfrak{q}} \times \mathbb{N}^{\mathfrak{q}} \colon \mathfrak{F}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{u}_{\mathfrak{i}}\mathfrak{v}_{\mathfrak{i}}}, \mathfrak{v}\Big)$$
$$> 1 - \frac{1}{\mathfrak{i}},\ \ \mathfrak{G}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{u}_{\mathfrak{i}}\mathfrak{v}_{\mathfrak{i}}}, \mathfrak{v}\Big)$$
$$< \frac{1}{\mathfrak{i}}, \mathfrak{H}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{u}_{\mathfrak{i}}\mathfrak{v}_{\mathfrak{i}}}, \mathfrak{v}\Big) < \frac{1}{\mathfrak{i}}\Big\},$$

tanımlansın.

Her $\mathfrak{i} \in \mathbb{N}$ için $\mathbb{P}_{\mathfrak{i}} \in \mathcal{F}(\mathcal{J}_2)$ olduğu açıktır. $\mathcal{J}_2$, $(AP2)$ özelliğine sahip olduğundan, her $\mathfrak{i} \in \mathbb{N}$ için $P \setminus \mathbb{P}_{\mathfrak{i}}$ sonlu ve $P \in \mathcal{F}(\mathcal{J}_2)$ olan bir küme mevcuttur.

Şimdi, tüm $(\mathfrak{s}_{\mathfrak{i}}, \mathfrak{w}_{\mathfrak{i}}), (\mathfrak{u}, \mathfrak{v}) \in P$ için

$$\lim_{\mathfrak{s}_{\mathfrak{i}}, \mathfrak{w}_{\mathfrak{i}}, \mathfrak{u}, \mathfrak{v} \to \infty, (\mathfrak{i}=1,2,\dots,\mathfrak{q})} \mathfrak{F}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\Big) = 1,$$

$$\lim_{\mathfrak{s}_{\mathfrak{i}}, \mathfrak{w}_{\mathfrak{i}}, \mathfrak{u}, \mathfrak{v} \to \infty, (\mathfrak{i}=1,2,\dots,\mathfrak{q})} \mathfrak{G}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\Big) = 0,$$

$$\lim_{\mathfrak{s}_{\mathfrak{i}}, \mathfrak{w}_{\mathfrak{i}}, \mathfrak{u}, \mathfrak{v} \to \infty, (\mathfrak{i}=1,2,\dots,\mathfrak{q})} \mathfrak{H}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}}, \mathfrak{f}_{\mathfrak{u}\mathfrak{v}}, \mathfrak{v}\Big) = 0,$$

olduğu gösterilecektir.

$$\mathfrak{F}\left(\mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \dots, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \frac{\mathfrak{v}}{2}\right) = \alpha \text{ ve } \mathfrak{G}\left(\mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \dots, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \frac{\mathfrak{v}}{2}\right) = \gamma,$$
$$\mathfrak{H}\left(\mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \dots, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \frac{\mathfrak{v}}{2}\right) = \psi.$$

olarak iade edilsin.

$\alpha, \gamma, \psi \in (0,1)$ olduğundan $(1-\mathfrak{v}) \therefore \alpha > 1-\beta, \mathfrak{v} * \gamma < \delta$ ve $\mathfrak{v} * \psi < \tau$ sağlayan $\beta, \delta, \tau \in (0,1)$ sayıları mevcuttur.

$\left((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_\mathrm{q}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_\mathrm{q})\right) \in \mathbb{K}^c(\mathfrak{d}, \mathfrak{v})$ alınsın. Bu durumda

$$\begin{aligned}
&\Theta\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \mathfrak{v}\right)\\
&\geq \mathfrak{F}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathfrak{v}\right) \therefore \mathfrak{F}\left(\mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \dots, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \frac{\mathfrak{v}}{2}\right)\\
&> (1-\mathfrak{v}) \therefore \alpha > 1-\beta,
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \mathfrak{v}\right)\\
&\leq \mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathfrak{v}\right) * \mathfrak{G}\left(\mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \dots, \mathscr{f}_{\mathrm{u_i}\mathfrak{v}_\mathrm{i}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \frac{\mathfrak{v}}{2}\right)\\
&< \mathfrak{v} * \gamma < \delta,
\end{aligned}$$

ve benzer şekilde

$$\mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \mathfrak{v}\right) < \tau$$

bulunur.

$\mathfrak{d} = \max\{\beta, \delta, \tau\}$ olarak seçilirse,

$$\Theta\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \mathfrak{v}\right) > 1-\mathfrak{d}, \mathfrak{G}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \mathfrak{v}\right) < \mathfrak{d}$$
$$\text{ve } \mathfrak{H}\left(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \mathscr{f}_{\mathfrak{s}_\mathrm{q}\mathfrak{w}_\mathrm{q}}, \mathscr{f}_{\mathrm{u}\mathfrak{v}}, \mathfrak{v}\right) < \mathfrak{d}$$

elde edilir Dolayısıyla, $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}) \in \mathcal{Q}$ dizisi $\mathcal{I}_2^*$-Cauchy'dir.

Tanım 5.7. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU ve $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$, $\mathcal{Q}'$da bir dizi olsun. Eğer $(\mathscr{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\mathscr{f} \in \mathcal{Q}$ elemanına yakınsayan $(\mathscr{f}_{\mathfrak{y}_m\mathfrak{z}_n})$ bir alt dizisi mevcut ise, $\mathscr{f} \in \mathcal{Q}$ noktası $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$'e göre $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin limit noktası olarak adlandırılır. $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin tüm limit noktaları kümesi $£^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}})$ sembolü ile gösterilir.

Tanım 5.8. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU ve $(\not{f}_{\mathfrak{y}_m \mathfrak{z}_n})$, $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin bir alt dizisi olsun. $\mathbb{K} = \{(\mathfrak{y}_m, \mathfrak{z}_n): m, n \in \mathbb{N}\} \subset \mathbb{N}^2$ olarak tanımlansın. Eğer $d_{\mathrm{q}}^{\mathcal{I}_2}(\mathbb{K}) = 0$ ise $(\not{f}_{\mathfrak{y}_m \mathfrak{z}_n})$ dizisine $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\mathcal{I}_2$-seyrek alt dizisi denir. Eğer $d_{\mathrm{q}}^{\mathcal{I}_2}(\mathbb{K}) \neq 0$ ise $(\not{f}_{\mathfrak{y}_m \mathfrak{z}_n})$ dizisine $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\mathcal{I}_2$-seyrek olmayan alt dizisi denir.

Tanım 5.9. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU olsun. Eğer $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\not{f} \in \mathcal{Q}$ elemanına yakınsayan $(\not{f}_{\mathfrak{y}_m \mathfrak{z}_n})$ $\mathcal{I}_2$-seyrek olmayan bir alt dizisi mevcut ise, $\not{f} \in \mathcal{Q}$ noktası $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$'e göre $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\mathcal{I}_2$-limit noktası denir. NFGMU'larda tüm $\mathcal{I}_2$-limit noktaları kümesi $\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}})$ ile gösterilir.

Örnek 5.5. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU olsun. $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi

$$\not{f}_{\mathfrak{y}\mathfrak{z}} = \begin{cases} 1, & \text{eğer } \mathfrak{y}, \mathfrak{z} \text{ tek ise} \\ 0, & \text{eğer } \mathfrak{y}, \mathfrak{z} \text{ çift ise} \end{cases}$$

şeklinde tanımlansın. $\mathcal{R}$, tüm tek sayılardan oluşan küme olsun. Bu durumda, $d(\mathcal{R}) = \frac{1}{2}$ olur. $\mathcal{I}_2$ kuvvetli uygun ideal olduğundan $d_{\mathrm{q}}^{\mathcal{I}_2}(\mathcal{R}) = \frac{1}{2}$ elde edilir. Böylece, $(\not{f}_{\mathfrak{y}_m \mathfrak{z}_n})$, $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\mathcal{I}_2$-seyrek bir alt dizisidir. Aynı zamanda, dizi $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ normuna göre 1'e yakınsaktır. Dolayısıyla, 1, $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\mathcal{I}_2$-limit noktasıdır.

Tanım 5.10. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ ile belirtilen NFGMU'da eğer tüm $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için

$$\begin{aligned} & d_{\mathrm{q}}^{\mathcal{I}_2}\Big(\big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathrm{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathrm{q}})\big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}}: \\ & \mathfrak{F}\big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \not{f}, \mathfrak{v}\big) > 1 - \mathfrak{d} \text{ or} \\ & \mathfrak{G}\big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \not{f}, \mathfrak{v}\big) < \mathfrak{d}, \mathfrak{H}\big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \not{f}, \mathfrak{v}\big) < \mathfrak{d}\Big) \neq 0, \end{aligned}$$

koşulu sağlanırsa $\not{f} \in \mathcal{Q}$ noktasına $(\mathfrak{F}, \mathfrak{G}, \mathfrak{H})$'e göre $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\mathcal{I}_2$-yığılma noktası denir. NFGMU'larda tüm $\mathcal{I}_2$-yığılma noktaları kümesi $\Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}})$ ile gösterilir.

Örnek 5.6. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU olsun. $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi

$$\not{f}_{\mathfrak{y}\mathfrak{z}} = \begin{cases} 1, & \text{eğer } \mathfrak{y}, \mathfrak{z} \text{ tam kare ise} \\ 0, & \text{aksi halde} \end{cases}$$

şeklinde tanımlansın. $\mathcal{R}$, tüm tam kare sayılardan oluşan küme olsun. Bu durumda, $d(\mathcal{R}) = 0$ olur. $\mathcal{I}_2$ kuvvetli uygun ideal olduğundan $d_{\mathrm{q}}^{\mathcal{I}_2}(\mathcal{R}) = 0$ elde edilir.

Şimdi, her $\mathfrak{d} \in (0,1)$ ve her $\mathfrak{v} > 0$ için

$$\Big\{\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathrm{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathrm{q}})\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, 0, \mathfrak{v}\Big) > 1 - \mathfrak{d}, \mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, 0, \mathfrak{v}\Big) < \mathfrak{d}, \mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, 0, \mathfrak{v}\Big) < \mathfrak{d}\Big\} = (\mathbb{N}^q \times \mathbb{N}^q) \setminus \mathcal{R}$$

olduğu açıktır. $d_{\mathrm{q}}^{\mathcal{I}_2}\big((\mathbb{N}^q \times \mathbb{N}^q) \setminus Q\big) = 1$ olduğundan

$$d_{\mathrm{q}}^{\mathcal{I}_2}\Big(\Big((\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathrm{q}}), (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_{\mathrm{q}})\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, 0, \mathfrak{v}\Big) > 1 - \mathfrak{d}, \mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, 0, \mathfrak{v}\Big) < \mathfrak{d}, \mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \not{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \dots, \not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, 0, \mathfrak{v}\Big) < \mathfrak{d}\Big) \neq 0$$

olur. Böylece, $0 \in \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$.

Teorem 5.13. $(Q, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU ve $\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$, Q'da bir dizi olsun. Bu durumda

$$\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big) \subseteq \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big) \subseteq \pounds^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$$

kapsam ilişkisi geçerlidir.

İspat. NFGMU $(Q, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ çerçevesinde $\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$ dizisi ele alınsın. $\not{f} \in \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$ ise, $\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$ dizisinin $\big(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n}\big)$ alt dizisi mevcuttur ve $\big(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n}\big)$ dizisi $\not{f}$'e yakınsaktır.

$\big(\not{f}_{\mathfrak{y}_m\mathfrak{z}_n}\big)$ dizisi $\not{f}$'e yakınsak olduğundan, tüm $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için

$$K(\mathfrak{d},\mathfrak{v})=\Big\{\Big(\big(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_\mathfrak{q}\big),\big(\mathfrak{z}_1,\mathfrak{z}_2,\dots,\mathfrak{z}_\mathfrak{q}\big)\Big)\in\mathbb{N}^\mathfrak{q}\times\mathbb{N}^\mathfrak{q}:\mathfrak{F}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)$$
$$\leqslant 1-\mathfrak{d}\text{ ya da }\mathfrak{G}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)$$
$$\geqslant\mathfrak{d},\mathfrak{H}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}$$

kümesi sonludur. Aynı zaman da

$$\Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\big)\Big)\in\mathbb{N}^\mathfrak{q}\times\mathbb{N}^\mathfrak{q}:\mathfrak{F}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)$$
$$>1-\mathfrak{d}\text{ ve }\mathfrak{G}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)<\mathfrak{d},\mathfrak{H}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)$$
$$<\mathfrak{d}\Big\}$$
$$\subseteq\Big\{\Big(\Big(\big(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_\mathfrak{q}\big),\big(\mathfrak{z}_1,\mathfrak{z}_2,\dots,\mathfrak{z}_\mathfrak{q}\big)\Big)\in K^\mathfrak{q}\times K^\mathfrak{q}:\mathfrak{y}_u,\mathfrak{z}_v\in\mathbb{N},u,v$$
$$=1,2,\dots,\mathfrak{q}\Big)\Big\}-K(\mathfrak{d},\mathfrak{v})$$

kapsam ilişkisi geçerlidir.

Buradan

$$T(\mathfrak{d},\mathfrak{v})=\Big\{\Big(\Big(\big(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_\mathfrak{q}\big),\big(\mathfrak{z}_1,\mathfrak{z}_2,\dots,\mathfrak{z}_\mathfrak{q}\big)\Big)\in K^\mathfrak{q}\times K^\mathfrak{q}:\mathfrak{y}_u,\mathfrak{z}_v\in\mathbb{N},u,v=1,2,\dots,\mathfrak{q}\Big)\Big\}$$
$$=\Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\big)\Big)$$
$$\in\mathbb{N}^\mathfrak{q}\times\mathbb{N}^\mathfrak{q}:\mathfrak{F}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)$$
$$>1-\mathfrak{d}\text{ ve }\mathfrak{G}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)<\mathfrak{d},\mathfrak{H}\Big(\not{f}_{\mathfrak{y}_1\mathfrak{z}_1},\not{f}_{\mathfrak{y}_2\mathfrak{z}_2},\dots,\not{f}_{\mathfrak{y}_\mathfrak{q}\mathfrak{z}_\mathfrak{q}},\not{f},\mathfrak{v}\Big)$$
$$<\mathfrak{d}\Big\}\cup K(\mathfrak{d},\mathfrak{v}).$$

Eğer

$$d_\mathfrak{q}^{\mathcal{I}_2}\Big(\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_\mathfrak{q}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_\mathfrak{q}\big)\Big)\in\mathbb{N}^\mathfrak{q}\times\mathbb{N}^\mathfrak{q}:\mathfrak{F}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},0,\mathfrak{v}\Big)$$
$$>1-\mathfrak{d},\mathfrak{G}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},0,\mathfrak{v}\Big)<\mathfrak{d},\mathfrak{H}\Big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_\mathfrak{q}\mathfrak{w}_\mathfrak{q}},0,\mathfrak{v}\Big)$$
$$<\mathfrak{d}\Big)=0$$

ise $d_\mathfrak{q}^{\mathcal{I}_2}\big(T(\mathfrak{d},\mathfrak{v})\big)=0$ elde edilir, bu ise bir çelişkidir. Dolayısıyla $\not{f}\in\Gamma_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}^{\mathcal{I}_2}\big(\not{f}_{\mathfrak{y}\mathfrak{z}}\big)$ olur.

Dolayısıyla $\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}) \subseteq \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ sağlanmış olur.

Şimdi ise $\Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}) \subseteq £^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ kapsam ilişkisi gösterilecektir. Bunun için $\mathfrak{f} \in \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ alınsın. Keyfi $\mathfrak{v} > 0$ ve $\mathfrak{d} \in (0,1)$ için

$$\mathcal{R}(\mathfrak{d},\mathfrak{v}) = \Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\ldots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\ldots,\mathfrak{w}_{\mathrm{q}})\Big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{F}\Big(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \ldots, \mathfrak{f}_{\mathfrak{y}_{\mathrm{q}}\mathfrak{z}_{\mathrm{q}}}, \mathfrak{f}, \mathfrak{v}\Big) > 1 - \mathfrak{d} \text{ ve } \mathfrak{G}\Big(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \ldots, \mathfrak{f}_{\mathfrak{y}_{\mathrm{q}}\mathfrak{z}_{\mathrm{q}}}, \mathfrak{f}, \mathfrak{v}\Big) < \mathfrak{d}, \mathfrak{H}\Big(\mathfrak{f}_{\mathfrak{y}_1\mathfrak{z}_1}, \mathfrak{f}_{\mathfrak{y}_2\mathfrak{z}_2}, \ldots, \mathfrak{f}_{\mathfrak{y}_{\mathrm{q}}\mathfrak{z}_{\mathrm{q}}}, \mathfrak{f}, \mathfrak{v}\Big) < \mathfrak{d}\Big\}$$

tanımlansın. $d^{\mathcal{I}_2}_{\mathrm{q}}(\mathcal{R}(\mathfrak{d},\mathfrak{v})) \neq 0$ olduğundan, $\mathcal{R}(\mathfrak{d},\mathfrak{v})$ sonsuz bir alt kümedir ve $\mathcal{R}(\mathfrak{d},\mathfrak{v}) = \{(\mathfrak{y}_m,\mathfrak{z}_n) : m,n \in \mathbb{N}\}$ olur. $(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\{\mathfrak{f}\}_{\mathcal{R}}$ alt dizisi $(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ normuna göre $\mathfrak{f}$'e yakınsaktır ve böylece $\mathfrak{f} \in £^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ elde edilir. Bu kapsam ilişkileri birlikte düşünüldüğünde

$$\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}) \subseteq \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}) \subseteq £^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$$

bulunur.

Teorem 5.14. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU olsun ve $\mathcal{I}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlansın. Bu durumda, $\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}) = \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}) = \{\mathfrak{f}\}$ elde edilir.

İspat. $\mathcal{I}_2 - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} \mathfrak{f}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlansın. Tanım 5.1 ve Tanım 5.10 birlikte düşünüldüğünde $\mathfrak{f} \in \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ elde edilir. $\mathfrak{f} \neq y$ için $y \in \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{f}_{\mathfrak{y}\mathfrak{z}})$ olsun. Bu durumda $\mathfrak{v} > 0$ ve $\mathfrak{d} \in (0,1)$ için

$$d^{\mathcal{I}_2}_{\mathrm{q}}(\mathcal{F}(\mathfrak{v},\mathfrak{d})) = d^{\mathcal{I}_2}_{\mathrm{q}}\left(\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\ldots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\ldots,\mathfrak{w}_{\mathrm{q}})\Big) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{F}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{f}, \mathfrak{v}\Big) > 1 - \mathfrak{d} \; \mathfrak{G}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{f}, \mathfrak{v}\Big) < \mathfrak{d}, \mathfrak{H}\Big(\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1}, \mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2}, \ldots, \mathfrak{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}}, \mathfrak{f}, \mathfrak{v}\Big) < \mathfrak{d}\Big\}\right) \neq 0$$

ve

$d_{\mathrm{q}}^{\mathcal{I}_2}(\mathcal{G}(\mathfrak{v},\mathfrak{d}))$

$$= \begin{array}{l} d_{\mathrm{q}}^{\mathcal{I}_2}\Big(\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)>1-\mathfrak{d} \\ \mathfrak{G}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)<\mathfrak{d},\mathfrak{H}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)<\mathfrak{d}\Big\}\Big)\neq 0 \end{array}$$

elde edilir.

$\mathscr{f}\neq y$ olduğundan $\mathcal{F}(\mathfrak{v},\mathfrak{d})\cap\mathcal{G}(\mathfrak{v},\mathfrak{d})=\emptyset$ yazılır. Böylece $\mathcal{F}(\mathfrak{v},\mathfrak{d})^c\supseteq\mathcal{G}(\mathfrak{v},\mathfrak{d})$ ilişkisi geçerlidir. Dolayısıyla

$$\begin{array}{l} \Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f},\mathfrak{v}\Big)\leqslant 1-\mathfrak{d} \\ \text{yada } \mathfrak{G}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\} \\ \supseteq\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)>1-\mathfrak{d} \\ \text{ve } \mathfrak{G}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)<\mathfrak{d},\mathfrak{H}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)<\mathfrak{d}\Big\}. \end{array}$$

bulunur. Böylece

$$\begin{array}{l} d_{\mathrm{q}}^{\mathcal{I}_2}\Big(\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f},\mathfrak{v}\Big)\leqslant 1-\mathfrak{d} \\ \text{yada } \mathfrak{G}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big) \end{array}$$

$$\geqslant \begin{array}{l} d_{\mathrm{q}}^{\mathcal{I}_2}\Big(\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)\leqslant 1-\mathfrak{d} \\ \text{yada } \mathfrak{G}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big). \end{array}$$

elde edilir. $\mathcal{I}_2-\lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty}\mathscr{f}_{\mathfrak{y}\mathfrak{z}}=\mathscr{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlandığından

$$\begin{array}{l} d_{\mathrm{q}}^{\mathcal{I}_2}\Big(\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f},\mathfrak{v}\Big)\leqslant 1-\mathfrak{d} \\ \text{yada } \mathfrak{G}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathscr{f},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big)=0 \end{array}$$

olur. Dolayısıyla

$$\begin{array}{l} d_{\mathrm{q}}^{\mathcal{I}_2}\Big(\Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{F}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)\leqslant 1-\mathfrak{d} \\ \text{yada } \mathfrak{G}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\mathscr{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathscr{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathscr{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},y,\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big)=0 \end{array}$$

bulunur. Bu durum, $y\in\Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\big)$ olduğu iddiasıyla çelişir. Dolayısıyla, $\Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\big(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\big)=\{\mathscr{f}\}$ sonucuna varılır.

$\mathcal{I}_2 - \lim\limits_{\mathfrak{y},\mathfrak{z}\to\infty} \not{f}_{\mathfrak{y}\mathfrak{z}} = \not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olsun. Tanım 5.9'a göre $\not{f} \in \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}})$ sağlanır. Teorem 5.13'e göre $\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}}) = \{\not{f}\}$ sonucuna varılır.

Bununla birlikte, Teorem 5.14'in tersi geçerli değildir. Daha spesifik olarak $\Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}}) = \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}}) = \{\not{f}\}$ olmasına rağmen $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\not{f}$'e $\mathcal{I}_2$-yakınsak değildir.

Örnek 5.7. Örnek 3.1'de tanımlanan $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ yapısı bir NFGMU olarak ele alınsın. $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi

$$\not{f}_{\mathfrak{y}\mathfrak{z}} = \begin{cases} 1, & \text{eğer } \mathfrak{y} = 2m, \mathfrak{z} = 2n \text{ ise,} \\ 2, & \text{aksi halde} \end{cases} \quad m,n \in \mathbb{N}$$

şeklinde tanımlansın.

Bu durumda $\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}}) = \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}}) = \{1,2\}$ sağlanırken, $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\mathcal{I}_2$-yakınsak değildir.

Teorem 5.15. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\not{f}_{\mathfrak{y}\mathfrak{z}}),(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}) \in \mathcal{Q}$ dizi olsunlar. Hemen hemen tüm $\mathfrak{y},\mathfrak{z}$ için $d_{\mathrm{q}}^{\mathcal{I}_2}(\{(\mathfrak{y},\mathfrak{z}): \not{f}_{\mathfrak{y}\mathfrak{z}} \neq \mathfrak{h}_{\mathfrak{y}\mathfrak{z}}\}) = 0$ sağlansın. Bu durumda $\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}}) = \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ ve $\Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}}) = \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ eşitlikleri geçerlidir.

İspat. $\varrho \in \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\not{f}_{\mathfrak{y}\mathfrak{z}})$ alınsın. Bu durumda, tüm $\mathfrak{d} \in (0,1)$ ve $\mathfrak{v} > 0$ için

$$\begin{aligned}
&d_{\mathrm{q}}^{\mathcal{I}_2}\Big(\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}}: \\
&\mathfrak{F}\big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\varrho,\mathfrak{v}\big) > 1-\mathfrak{d} \text{ ve} \\
&\mathfrak{G}\big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\varrho,\mathfrak{v}\big) < \mathfrak{d}, \mathfrak{H}\big(\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\varrho,\mathfrak{v}\big) < \mathfrak{d}\Big) \neq 0,
\end{aligned}$$

sağlanmaktadır.

$W = \{(\mathfrak{y},\mathfrak{z}): \not{f}_{\mathfrak{y}\mathfrak{z}} = \mathfrak{h}_{\mathfrak{y}\mathfrak{z}}\}$ olduğunda $d_{\mathrm{q}}^{\mathcal{I}_2}(W) = 1$ olur. Dolayısıyla

$$\begin{gathered}
d_q^{\mathcal{I}_2}\Big(\Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}}: \\
\mathfrak{F}\big(\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{h}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\varrho,\mathfrak{v}\big) > 1-\mathfrak{d} \text{ ve} \\
\mathfrak{G}\big(\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{h}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\varrho,\mathfrak{v}\big) < \mathfrak{d}, \mathfrak{H}\big(\mathfrak{h}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{h}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{h}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\varrho,\mathfrak{v}\big) < \mathfrak{d}\Big\} \cap W\Big) \neq 0
\end{gathered}$$

yazılır.

Sonuç olarak $\varrho \in \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ olur. Böylece $\Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) \subset \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ sağlanır. Benzer şekilde, $\Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}) \subset \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ kapsam ilişkisi de geçerlidir. $\Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) = \Gamma^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ elde edilir.

$\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) = \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ eşitliği gösterilecektir. $\varpi \in \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ alınsın. $(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ dizisinin $\varpi \in \mathcal{Q}$ elemanına yakınsayan $(\cancel{f}_{\mathfrak{y}_m\mathfrak{z}_n})$ $\mathcal{I}_2$-seyrek olmayan bir alt dizisi mevcuttur. $W = \{(\mathfrak{y}_m, \mathfrak{z}_n): m, n \in \mathbb{N}\}$ alınsın.

$$d_{\mathrm{q}}^{\mathcal{I}_2}\left(\{(\mathfrak{y},\mathfrak{z}): \cancel{f}_{\mathfrak{y}_m\mathfrak{z}_n} \neq \mathfrak{h}_{\mathfrak{y}_m\mathfrak{z}_n}\}\right) = 0$$

olduğundan

$$d_{\mathrm{q}}^{\mathcal{I}_2}\left(\{(\mathfrak{y},\mathfrak{z}): \cancel{f}_{\mathfrak{y}_m\mathfrak{z}_n} = \mathfrak{h}_{\mathfrak{y}_m\mathfrak{z}_n}\}\right) \neq 0$$

yazılır.

Sonuç olarak, $\{\cancel{f}\}_{W'}$ şeklinde $\{\cancel{f}\}_W$'nin $\mathcal{I}_2$-seyrek olmayan bir alt dizisi bulunur. Sonuç olarak, $\varpi \in \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ elde edilir. Dolayısıyla $\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}}) \subset \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ bulunur. Benzer şekilde $\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}) \subset \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ elde edilir. $\Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}) = \Lambda^{\mathcal{I}_2}_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ gösterilmiş olur.

Örnek 5.8. $(\mathcal{Q}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \therefore, *)$ bir NFGMU olsun. $(\cancel{f}_{\mathfrak{y}\mathfrak{z}})$ ve $(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}})$ dizileri sırasıyla

$$\cancel{f}_{\mathfrak{y}\mathfrak{z}} = \begin{cases} 1, & \text{eğer } \mathfrak{y}, \mathfrak{z} \text{ tam kare sayılar ise,} \\ 0, & \text{aksi halde} \end{cases}$$

ve

$$\mathfrak{h}_{\mathfrak{y}\mathfrak{z}} = \begin{cases} 2, & \text{eğer } \mathfrak{y}, \mathfrak{z} \text{ tam kare sayılar ise,} \\ 0, & \text{aksi halde} \end{cases}$$

biçiminde tanımlansın.

T tüm tam kare sayıların kümesi olarak tanımlansın. Bu durumda $d(T) = 0$ olur. $\mathcal{I}_2$ uygun ideal olduğundan $d_{\mathrm{q}}^{\mathcal{I}_2}(T) = 0$ elde edilir. Sonuç olarak

$$d_{\mathfrak{q}}^{\mathcal{J}_2}\left(\left\{(\mathfrak{y},\mathfrak{z}) : \mathscr{f}_{\mathfrak{y}\mathfrak{z}} \neq \mathfrak{h}_{\mathfrak{y}\mathfrak{z}}\right\}\right) = d_{\mathfrak{q}}^{\mathcal{J}_2}(T) = 0$$

elde edilir.

Sonuç olarak, $\Gamma_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}^{\mathcal{J}_2}\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right) = \Gamma_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}^{\mathcal{J}_2}\left(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}\right) = \{0\}$ ve $\Lambda_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}^{\mathcal{J}_2}\left(\mathfrak{h}_{\mathfrak{y}\mathfrak{z}}\right) = \Lambda_{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}^{\mathcal{J}_2}\left(\mathscr{f}_{\mathfrak{y}\mathfrak{z}}\right) = \{0\}$ yazılır.

6 NFGMU’LARDA IDEAL YARDIMIYLA ELDE EDİLEN GENELLEŞTİRİLMİŞ İSTATİSTİKSEL YAKINSAKLIK

Bu bölümde, NFGMS içinde $\mathcal{I}_2$-istatistiksel yakınsaklık, $\mathcal{I}_2$-lacunary istatistiksel yakınsaklık, kuvvetli $\mathcal{I}_2$-Cesàro toplanabilirlilik, kuvvetli $\mathcal{I}_2$-lacunary toplanabilirlilik kavramları ele alınmaktadır. Ayrıca, bu kavramlarla ilgili çeşitli özellikler incelenmektedir.

Tanım 6.1. $(\mathcal{Q},\mathfrak{F},\mathfrak{G},\mathfrak{H},\therefore,*)$ bir NFGMU ve $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ $\mathcal{Q}$’da bir dizi olsun. Eğer $\mathfrak{v}>0$, $\mathfrak{d}\in(0,1)$ ve $\varrho>0$ için

$$\Big\{(\mathfrak{y},\mathfrak{z})\in\mathbb{N}^2\colon \frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\ \Big\{\Big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}})\Big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}\colon \mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\le\mathfrak{y},$$
$$\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{F}\Big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\le 1-\mathfrak{d},\text{ yada}$$
$$\mathfrak{G}\Big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\ge\mathfrak{d},\mathfrak{H}\Big(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\ge\mathfrak{d}\Big\}\ |\ge\varrho\Big\}\in\mathcal{I}_2,$$

sağlanıyorsa $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\not{f}'$e $\mathcal{I}_2$-istatistiksel yakınsaktır denir. $\not{f}_{\mathfrak{y}\mathfrak{z}}\xrightarrow{S\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)}\not{f}$ veya $S\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olarak gösterilir. NFGMU’da tüm $\mathcal{I}_2$-istatistiksel yakınsak diziler uzayı $S\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)$ ile ifade edilir.

Tanım 6.2. Eğer $\mathfrak{v}>0$, $\mathfrak{d}\in(0,1)$ ve $\varrho>0$ için

$$\Big\{(u,v)\in\mathbb{N}^2\colon\frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\Big|\ \{(\mathfrak{s}_t,\mathfrak{w}_t)\in \mathrm{I}_{uv},1\leqslant t\leqslant\mathfrak{q}\colon\mathfrak{F}(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v})\leqslant 1-\mathfrak{d}\text{ or}$$
$$\mathfrak{G}(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v})\geqslant\mathfrak{d},\mathfrak{H}(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v})\geqslant\mathfrak{d}\}\ |\ge\varrho\Big\}\in\mathcal{I}_2.$$

sağlanıyorsa $(\not{f}_{\mathfrak{y}\mathfrak{z}})$ dizisi $\not{f}'$e $\mathcal{I}_2$-lacunary istatistiksel yakınsaktır denir. $\not{f}_{\mathfrak{y}\mathfrak{z}}\xrightarrow{S_{\theta_2}\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)}\not{f}$ veya $S_{\theta_2}\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olarak gösterilir. NFGMU’da tüm $\mathcal{I}_2$-lacunary istatistiksel yakınsak diziler uzayı $S_{\theta_2}\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)$ ile ifade edilir.

Tanım 6.3. Eğer $\mathfrak{v}>0$, $\mathfrak{d}\in(0,1)$ için

$$\left\{(\mathfrak{y},\mathfrak{z})\in\mathbb{N}^2:\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}=1}^{\mathfrak{z}}\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)\leqslant 1-\mathfrak{d}\ \text{yada}\right.$$
$$\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}=1}^{\mathfrak{z}}\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)\geq\mathfrak{d}$$
$$\left.\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}=1}^{\mathfrak{z}}\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\right)\geq\mathfrak{d}\right\}\in\mathcal{I}_2,$$

sağlanıyorsa $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\not{f}'$e kuvvetli $\mathcal{I}_2$-Cesàro yakınsaktır denir. $\not{f}_{\mathfrak{y}\mathfrak{z}}\xrightarrow{C_1\left[\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]}\not{f}$ veya $C_1\left[\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olarak gösterilir. NFGMU'da tüm kuvvetli $\mathcal{I}_2$- Cesàro yakınsak diziler uzayı $C_1\left[\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]$ ile ifade edilir.

Tanım 6.4. Eğer $\mathfrak{v}>0$, $\mathfrak{d}\in(0,1)$ için

$$\left\{(u,v)\in\mathbb{N}^2:\frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\sum_{(\mathfrak{s}_t,\mathfrak{w}_t)\in\mathrm{I}_{uv},1\leqslant t\leqslant\mathfrak{q}}\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\right)\leq 1-\varkappa\ \text{or}\right.$$
$$\frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\sum_{(\mathfrak{s}_t,\mathfrak{w}_t)\in\mathrm{I}_{uv},1\leqslant t\leqslant\mathfrak{q}}\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\right)\geq\mathfrak{d}$$
$$\left.\frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\sum_{(\mathfrak{s}_t,\mathfrak{w}_t)\in\mathrm{I}_{uv},1\leqslant t\leqslant\mathfrak{q}}\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\right)\mathfrak{d}\right\}\in\mathcal{I}_2.$$

sağlanıyorsa $\left(\not{f}_{\mathfrak{y}\mathfrak{z}}\right)$ dizisi $\not{f}'$e kuvvetli $\mathcal{I}_2$-lacunary yakınsaktır denir. $\not{f}_{\mathfrak{y}\mathfrak{z}}\xrightarrow{N_{\theta_2}\left[\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]}\not{f}$ veya $N_{\theta_2}\left[\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olarak gösterilir. NFGMU'da tüm kuvvetli $\mathcal{I}_2$-lacunary yakınsak diziler uzayı $N_{\theta_2}\left[\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]$ ile ifade edilir.

Teorem 6.1. Eğer $\lim_{u}\inf q_u>1$ ve $\lim_{v}\inf q_v>1$ ise, bu durumda

$$S\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})\Rightarrow S_{\theta_2}\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$$

sağlanır.

İspat. $\lim_{u}\inf q_u>1$ ve $\lim_{v}\inf q_v>1$ olsun. Bu durumda, yeterince büyük u,v değerleri için $q_u > 1 + \varrho$ ve $q_v > 1 + \varsigma$ olacak şekilde $\varrho, \varsigma > 0$ sayıları mevcuttur. Bu durumda,

$$\frac{\mathfrak{h}_{uv}}{k_{uv}} \geqslant \frac{\varrho\varsigma}{(1+\varrho)(1+\varsigma)}$$

geçerlidir. Eğer $S\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right) - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} f_{\mathfrak{y}\mathfrak{z}} = f(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ ise her $\mathfrak{d} \in (0,1), \mathfrak{v} > 0$ için ve yeterince büyük u, v değerleri için

$$\begin{aligned}
&\frac{\mathrm{q}!}{(k_{uv})^{\mathrm{q}}}\Big| \Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big) \in \mathbb{N}^q \times \mathbb{N}^q : \mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}} \leqslant k_u,\\
&\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}} \leqslant l_v, \mathfrak{F}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}\\
&\text{ yada } \mathfrak{G}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\big) \geqslant \mathfrak{d}\Big\}\Big|\\
&\geqslant \frac{\mathrm{q}!}{(k_{uv})^{\mathrm{q}}}\Big| \big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathrm{q} : \mathfrak{F}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}\\
&\text{ yada } \mathfrak{G}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}\big\}\Big|\\
&\geqslant \left(\frac{\mathfrak{h}_{uv}}{k_{uv}}\right)^{\mathrm{q}} \frac{\mathrm{q}!}{(\mathfrak{h}_{uv})^{\mathrm{q}}}\Big| \big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathrm{q} : \mathfrak{F}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}\\
&\text{ yada } \mathfrak{G}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}\big\}\Big|\\
&\geqslant \left(\frac{\varrho\varsigma}{(1+\varrho)(1+\varsigma)}\right)^{\mathrm{q}} \frac{\mathrm{q}!}{(\mathfrak{h}_{uv})^{\mathrm{q}}}\Big| \big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathrm{q} : \mathfrak{F}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \leqslant 1-\mathfrak{d}\\
&\text{ yada } \mathfrak{G}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}\big\}\Big|,
\end{aligned}$$

geçerlidir. Bu durumda, $\varrho > 0$ için

$$\begin{aligned}
&\Big\{(u,v) \in \mathbb{N}^2 : \frac{\mathrm{q}!}{(\mathfrak{h}_{uv})^{\mathrm{q}}}\Big| \big\{(\mathfrak{s}_t,\mathfrak{w}_t) \in \mathrm{I}_{uv}, 1 \leqslant t \leqslant \mathrm{q} : \mathfrak{F}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \leqslant 1-\mathfrak{d} \text{ or}\\
&\qquad \mathfrak{G}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}, \mathfrak{H}\big(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big) \geqslant \mathfrak{d}\big\} \Big| \geq \varrho\Big\}\\
&\subseteq \Big\{(\mathfrak{y},\mathfrak{z}) \in \mathbb{N}^2 : \frac{\mathrm{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathrm{q}}}\Big| \Big\{\big((\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}),(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}})\big) \in \mathbb{N}^{\mathrm{q}} \times \mathbb{N}^{\mathrm{q}} : \mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}} \leq \mathfrak{y},\\
&\qquad \mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}} \leqslant \mathfrak{z}, \mathfrak{F}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right) \leq 1-\mathfrak{d}, \text{ yada}\\
&\mathfrak{G}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right) \geq \mathfrak{d}, \mathfrak{H}\left(f, f_{\mathfrak{s}_1\mathfrak{w}_1}, f_{\mathfrak{s}_2\mathfrak{w}_2},\dots,f_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right) \geq \mathfrak{d}\Big\} \Big| \geq \varrho\left(\frac{\varrho\varsigma}{(1+\varrho)(1+\varsigma)}\right)^{\mathrm{q}}\Big\} \in \mathcal{I}_2,
\end{aligned}$$

elde edilir. Dolayısıyla $S_{\theta_2}\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right) - \lim_{\mathfrak{y},\mathfrak{z}\to\infty} f_{\mathfrak{y}\mathfrak{z}} = f(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ sağlanır.

Teorem 6.2. Eğer $\limsup_u q_u < \infty, \limsup_v q_v < \infty$ geçerli ise

$$S_{\theta_2}\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})\Longrightarrow S\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$$

sağlanır.

İspat. $\limsup_{u} q_u < \infty, \limsup_{v} q_v < \infty$ olsun. Bu durumda, her $u, v \geqslant 1$ için $q_u < \mathcal{M}$ ve $q_v < \mathcal{N}$ olacak biçimde $\mathcal{M}, \mathcal{N} > 0$ sayıları mevcuttur.

$S_{\theta_2}\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ olsun ve

$$\begin{aligned}\mathrm{T}_{u,v}:=\Big|\big\{(\mathfrak{s}_t,\mathfrak{w}_t)\in \mathrm{I}_{uv}, 1\leqslant t\leqslant \mathfrak{q}: \mathfrak{F}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big)\leqslant 1-\mathfrak{d} \text{ or}\\ \mathfrak{G}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big)\geqslant\mathfrak{d}, \mathfrak{H}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big)\geqslant\mathfrak{d}\big\}\Big|\end{aligned}$$

tanımlansın.

$$S_{\theta_2}\left(\mathcal{I}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\mathfrak{f}_{\mathfrak{y}\mathfrak{z}}=\mathfrak{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$$

ise her $\mathfrak{v}>0$, $\mathfrak{d}\in(0,1)$ ve $\varrho>0$ için

$$\begin{aligned}&\Big\{(u,v)\in\mathbb{N}^2:\frac{\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\Big|\,\big\{(\mathfrak{s}_t,\mathfrak{w}_t)\in \mathrm{I}_{uv}, 1\leqslant t\leqslant \mathfrak{q}: \mathfrak{F}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big)\leqslant 1-\mathfrak{d} \text{ or}\\ &\quad\mathfrak{G}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big)\geqslant\mathfrak{d}, \mathfrak{H}\big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_t\mathfrak{w}_t},\mathfrak{v}\big)\geqslant\mathfrak{d}\big\}\,|\geq\varrho\Big\}\\ &=\Big\{(u,v)\in\mathbb{N}^2:\frac{\mathrm{T}_{u,v}\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}\geq\varrho\Big\}\in\mathcal{I}_2\end{aligned}$$

yazılır.

Böylece, $u_0, v_0 \in \mathbb{N}$ olmak üzere, tüm $u \geq u_0, v \geq v_0$ için

$$\frac{\mathrm{T}_{u,v}\mathfrak{q}!}{(\mathfrak{h}_{uv})^{\mathfrak{q}}}<\varrho$$

eşitsizliğini sağlayan pozitif tamsayılar seçilebilir. Şimdi

$$H:=\max\{T_{u,v}: 1\leq u\leq u_0, 1\leq v\leq v_0\}$$

tanımlansın ve $k_{u-1}<\mathfrak{y}\leq k_u$ ve $l_{v-1}<\mathfrak{z}\leq l_v$ sağlayan $\mathfrak{y},\mathfrak{z}$ tamsayıları seçilsin.

Bu durumda, $\mathfrak{v}>0$, $\mathfrak{d}\in(0,1)$ ve $\varrho>0$ için

$$\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\big)\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},$$

$$\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{F}\Big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\leqslant 1-\mathfrak{d}$$

$$\text{yada }\mathfrak{G}\Big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big|$$

$$\leqslant\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\Big|\Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\big)\Big)\in\mathbb{N}^q\times\mathbb{N}^q:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant k_u,$$

$$\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant l_v,\mathfrak{F}\Big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\leqslant 1-\mathfrak{d}$$

$$\text{yada }\mathfrak{G}\Big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big|$$

$$=\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\Big|\Big\{\mathrm{T}_{11}+\mathrm{T}_{12}+\mathrm{T}_{21}+\mathrm{T}_{22}+\cdots+\mathrm{T}_{u_0v_0}+\cdots+\mathrm{T}_{u,v}\Big\}$$

$$=\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}=1}^{\mathfrak{z}}\mathrm{T}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}}\right\}$$

$$\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}=1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}=1}^{\mathfrak{z}}\mathrm{T}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}}\right\}$$

$$\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}=u_0+1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}=v_0+1}^{\mathfrak{z}}\frac{\mathrm{T}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}}\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}}}{\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}}}\right\}$$

$$\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\frac{\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}\left(\sup_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\geqslant u_0,\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\geq v_0}\frac{\mathrm{T}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}}}{\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}}}\right)$$

$$\times\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}=u_0+1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}=v_0+1}^{\mathfrak{z}}\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}}\right\}$$

$$\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\xi\left\{\sum_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}=u_0+1}^{\mathfrak{y}}\ \sum_{\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}=v_0+1}^{\mathfrak{z}}\mathfrak{h}_{\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}},\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}}\right\}\leqslant\frac{Wu_0^2\mathfrak{q}!}{(k_{u-1}l_{v-1})^{\mathfrak{q}}}+\xi\mathcal{M}\mathcal{N},$$

yazılır.

Burada $\mathfrak{y},\mathfrak{z}\to\infty$ iken $k_{u-1}l_{v-1}\to\infty$ olacağından her $\mathfrak{d}\in(0,1)$ ve $\mathfrak{v}>0$ değerleri için

$$\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\frac{\mathfrak{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathfrak{q}}}\Big|\Big\{\Big(\big(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\big),\big(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\big)\Big)\in\mathbb{N}^{\mathfrak{q}}\times\mathbb{N}^{\mathfrak{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathfrak{q}}\leqslant\mathfrak{y},$$

$$\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathfrak{q}}\leqslant\mathfrak{z},\mathfrak{F}\Big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\leqslant 1-\mathfrak{d}\text{ or}$$

$$\mathfrak{G}\Big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\geqslant\mathfrak{d},\mathfrak{H}\Big(\mathfrak{f},\mathfrak{f}_{\mathfrak{s}_1\mathfrak{w}_1},\mathfrak{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\mathfrak{f}_{\mathfrak{s}_{\mathfrak{q}}\mathfrak{w}_{\mathfrak{q}}},\mathfrak{v}\Big)\geqslant\mathfrak{d}\Big\}\Big|=0,$$

elde edilir.

$$\left\{(\mathfrak{y},\mathfrak{z})\in\mathbb{N}^2:\frac{\mathrm{q}!}{(\mathfrak{y}\mathfrak{z})^{\mathrm{q}}}\middle|\left\{\left(\left(\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\right),\left(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\right)\right)\in\mathbb{N}^{\mathrm{q}}\times\mathbb{N}^{\mathrm{q}}:\mathfrak{s}_1,\mathfrak{s}_2,\dots,\mathfrak{s}_{\mathrm{q}}\leq\mathfrak{y},\right.\right.$$
$$\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_{\mathrm{q}}\leqslant\mathfrak{z},\mathfrak{F}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)\leq 1-\mathfrak{d},\text{ yada}$$
$$\left.\left.\mathfrak{G}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)\geq\mathfrak{d},\mathfrak{H}\left(\not{f},\not{f}_{\mathfrak{s}_1\mathfrak{w}_1},\not{f}_{\mathfrak{s}_2\mathfrak{w}_2},\dots,\not{f}_{\mathfrak{s}_{\mathrm{q}}\mathfrak{w}_{\mathrm{q}}},\mathfrak{v}\right)\geq\mathfrak{d}\right\}\middle|\geq\varrho\right\}\in\mathcal{J}_2,$$

Böylece, $S\left(\mathcal{J}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)-\lim_{\mathfrak{y},\mathfrak{z}\to\infty}\not{f}_{\mathfrak{y}\mathfrak{z}}=\not{f}(\mathfrak{F},\mathfrak{G},\mathfrak{H})$ elde edilmiş olur.

Teorem 6.3. $S_{\theta_2}\left(\mathcal{J}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)=S\left(\mathcal{J}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right)$ olması için gerek ve yeter şart

$$1<\liminf_{u}\mathrm{q}_u\leqslant\limsup_{u}q_u<\infty\text{ ve }1<\liminf_{v}\mathrm{q}_v\leqslant\limsup_{v}q_v<\infty$$

sağlanmasıdır.

İspat. Teorem 6.2 ve Teorem 6.3'den doğrudan açıktır.

Teorem 6.4. Eğer $\liminf_{u}\mathrm{q}_u>1$ ve $\liminf_{v}\mathrm{q}_v>1$ ise $C_1\left[\mathcal{J}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]\subseteq N_{\theta_2}\left[\mathcal{J}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]$ sağlanmaktadır.

Teorem 6.5. Eğer $\limsup_{u}q_u<\infty$ ve $\limsup_{v}q_v<\infty$ ise $N_{\theta_2}\left[\mathcal{J}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]\subseteq C_1\left[\mathcal{J}_2^{(\mathfrak{F},\mathfrak{G},\mathfrak{H})}\right]$ sağlanmaktadır.

6. SONUÇ VE ÖNERİLER

Bu çalışma, NFGMS içinde çift dizilerin yakınsama teorisine yönelik kapsamlı bir inceleme sunmaktadır. Öncelikle, istatistiksel yakınsama, istatistiksel limit noktaları ve istatistiksel yığılma noktaları analiz edilerek, NFGMS bağlamında bu kavramların temel özellikleri ortaya konulmuştur. Daha sonra, çift dizilerin lacunary istatistiksel yakınsaması ve güçlü lacunary yakınsaması ele alınarak, aralarındaki ilişkiler değerlendirilmiştir.

Çalışmanın önemli katkılarından biri, NFGMS'lerde ideal yakınsama kavramının tanıtılması ve bu bağlamda $\mathcal{I}_2$-Cauchy çift dizileri ile $\mathcal{I}_2$-tamlık kavramlarının detaylı bir şekilde incelenmesidir. İdeal yakınsamanın temel özellikleri analiz edilerek, $\mathcal{I}_2$-limitler ve $\mathcal{I}_2$-yığılma noktaları arasındaki ilişkiler açıklığa kavuşturulmuştur. Ayrıca, $\mathcal{I}_2$-istatistiksel yakınsama, $\mathcal{I}_2$-lacunary istatistiksel yakınsama, güçlü $\mathcal{I}_2$-Cesàro toplanabilirlik ve güçlü $\mathcal{I}_2$-lacunary toplanabilirlik kavramları tanımlanmış ve bunların özellikleri değerlendirilmiştir.

Elde edilen bulgular, NFGMS bağlamında çift dizilerin yakınsama teorisini genişletmekte ve bu uzaylarda yakınsamanın cebirsel ve topolojik yapısına dair yeni perspektifler sunmaktadır. Bu çalışma, gelecekte yapılacak araştırmalar için bir temel oluşturmakta olup, NFGMS'lerde farklı yakınsama türlerinin daha ileri düzeyde incelenmesine olanak sağlamaktadır.

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ÖZGEÇMİŞ

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