

DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED
SCIENCES

FEEDBACK LINEARIZATION, STABILITY AND
CONTROL OF PIECEWISE AFFINE SYSTEMS
EXPLOITING CANONICAL AND
CONVENTIONAL REPRESENTATIONS

by
Aykut KOCAOĞLU

June, 2013
İZMİR

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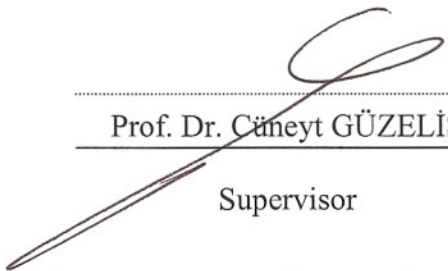
**A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of Dokuz Eylül University
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and Electronics Engineering**

**by
Aykut KOCAOĞLU**

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İZMİR**

PhD THESIS EXAMINATION RESULT FORM

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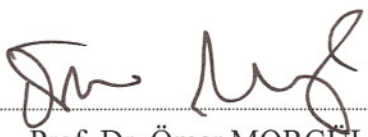
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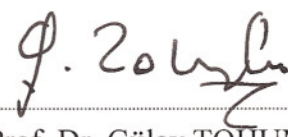
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Aykut KOCAOĞLU

FEEDBACK LINEARIZATION, STABILITY AND CONTROL OF PIECEWISE AFFINE SYSTEMS EXPLOITING CANONICAL AND CONVENTIONAL REPRESENTATIONS

ABSTRACT

Piecewise Affine (PWA) systems constitute a quantitatively simple yet qualitatively rich subclass of nonlinear systems. On the one hand, two-segment PWA ideal diode model, which is the most basic nonlinear circuit element of electrical engineering, can be considered as the simplest example. On the other hand, Chua's circuit, which demonstrates one of the most complicated nonlinear dynamical behaviors, i.e. chaos, contains a three-segment PWA resistor as the unique nonlinear element. PWA system and controller models are becoming attractive in control area since they allow exploiting linear analysis techniques and providing a suitable structure for which the extension of the control system analysis and design methods originally developed for linear systems into this special nonlinear system class is possible.

The thesis has three main contributions to the PWA control systems area. For the PWA control systems, the thesis introduces the development of feedback linearization methods for PWA control systems based on the canonical representation. The second contribution is the proposed robust chaotification method by sliding mode control. For the stability analysis of PWA control systems, a novel quadratic Lyapunov function based on intersection of degenerate ellipsoids and vertex representations of the polyhedral regions is the third contribution of the thesis.

The absolute value based canonical representation employed in feedback linearization of PWA control systems provides parameterizations and compact closed form solutions for the controller design by feedback linearization. These parameterizations and closed form solutions constitute a basis for further theoretical and applied studies on the analysis and design of control systems.

Keywords : Piecewise affine, Lyapunov stability, feedback linearization, chaotification, sliding mode control

PARÇA PARÇA DOĞRUSAL SİSTEMLERİN KANONİK VE KONVANSİYONEL GÖSTERİLİMLER KULLANARAK GERİBESLEMEYLE DOĞRUSALLAŞTIRILMASI, KARARLILIĞI VE KONTROLÜ

ÖZ

Parça Parça Doğrusal (PPD) sistemler, doğrusal olmayan sistemlerin nicel olarak basit nitelik olarak zengin bir alt sınıfını oluştururlar. Bir yandan, elektrik mühendisliğinin en temel doğrusal olmayan devre elemanı olan iki-parçalı PPD ideal diyot modeli, en basit örnek olarak verilebilir. Diğer yandan, tek doğrusal olmayan eleman olarak üç-parçalı PPD bir direnç içeren Chua devresi, basit devre yapısına karşın bilinen en karmaşık doğrusal olmayan dinamik davranışlardan birisi olan kaotik davranışlar gösteren PPD bir dinamik sistem örneği olarak verilebilir. Doğrusal analiz yöntemlerinin kullanılmasına izin vermesi ve doğrusal sistemler için geliştirilen kontrol sistem analiz ve tasarım yöntemlerinin bu özel doğrusal olmayan sistem sınıfı için genişletilmesine uygun bir yapı sağlamasından dolayı, PPD sistem ve kontrolör modelleri kontrol alanında ilgi çekmeye başlamıştır.

Tez, PPD kontrol sistemleri alanına üç ana katkı yapmaktadır. Tez, PPD kontrol sistemlerinin geribeslemeyle doğrusallaştırılması için kanonik gösterilimlere dayalı yöntemlerinin geliştirilmesini sağlamaktadır. İkinci katkı ise önerilen kayan kipli kontrol ile gürbüz kaotikleştirme yöntemidir. PPD kontrol sistemlerinin Liapunov kararlılık analizi için, dejenere elipsoitlerin kesişimi ve verteks gösterilimlerine dayalı yeni dördün Liapunov işlevleri önermesi tezin üçüncü katkısını oluşturmaktadır.

PPD kontrol sistemlerinin geribeslemeyle doğrusallaştırılmasında kullanılan mutlak değer temelli kanonik gösterilim, PPD sistemlerinin kararlılık analizi ve kontrolör tasarımı için bir parametrikleştirme ve sıkı kapalı biçimde çözümler sunmaktadır. Parametrikleştirme ve kapalı çözümler, kontrol sistemlerinin analiz ve tasarımı üzerine gelecekte yapılabilecek kuramsal ve uygulamalı çalışmalara bir taban oluşturmaktadır.

Anahtar sözcükler : Parça-parça doğrusal, Liapunov kararlılığı, geribeslemeyle doğrusallaştırma, kaotikleştirme, kayan kipli kontrol

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CHAPTER ONE

INTRODUCTION

Piecewise Affine (PWA) systems constitute a quantitatively simple yet qualitatively rich subclass of nonlinear systems. PWA dynamical systems have the capability of demonstrating complicated nonlinear dynamical behaviors, i.e. chaos. PWA systems have been studied extensively in control area since they exploit linear behavior in each polyhedral region while capable of exploiting nonlinear behaviors and they can approximate to the nonlinear systems with arbitrary accuracy. The objective of this thesis is to propose qualitative analysis and controller synthesis methods for PWA systems. Canonical representations for PWA mappings are exploited in the thesis to parameterize the feedback linearization so the controller design for PWA systems.

PWA dynamical systems can be mathematically formulated by several representations developed for piecewise affine mappings. Among these representations, the most widely used ones are conventional representation, simplex representation, complementary pivot representation and canonical representation.

Conventional representation expresses the mapping region by region, constituting the most commonly used one of these representations (Hassibi & Boyd, 1998; Johansson, 2003; Johansson & Rantzer, 1998; Li et al., 2006; Rodrigues & Boyd, 2005; Samadi, 2008). Simplex representation (Chien & Kuh, 1977; Fernandez et al., 2008; Julian, 1999; Julian et al., 1999; Julian & Chua, 2002) has regions of arbitrary dimension defined as a convex hull of vertices such that PWA mappings are represented in a specific linear region as the convex combination of the images of the domain space vertices. Complementary pivot representations have linear partitions of the domain determined by an affine equality with linearly complementarity constraints (De Moor et al., 1992; Eaves & Lemke, 1981; Kevenaar & Leenaerts, 1992; Leenaerts & van Bokhoven, 1998; Stevens & Lin, 1981; van Bokhoven, 1981; van Eijndhoven, 1986; Vandenberghe et al., 1989). The canonical representation was first introduced in (Chua & Kang, 1977) which has the capability to represent any

single-valued PWA function in a compact form with using absolute value functions only in addition to linear operations.

The existence of the canonical representation for one-dimensional PWA functions led (Kang & Chua, 1978) to extend the representation for higher dimensional PWA functions with convex polyhedral (polytopic) regions constructed by linear partitions, which have the capability to formulate a quite general class of PWA functions. A set of sufficient conditions and also a set of necessary and sufficient conditions for the canonical representations for PWA continuous functions are introduced by (Chua & Deng, 1988). Güzeliş and Gökner (1991) put forward the idea of nested absolute value where canonical representation was extended to formulate a more general class of PWA functions. The canonical representation has been extended by several studies in the literature (Kahlert & Chua, 1990; Julian et al., 1999; Lin et al., 1994). These representations led to mathematically formulate PWA dynamical systems such as switched affine systems, linear complementarity systems (Brogliato, 2003; Çamlıbel et al., 2002; Çamlıbel et al., 2003; Heemels et al., 2000a, 2000b; Heemels et al., 2002; Heemels et al., 2011; Shen & Pang, 2005, 2007a, 2007b; van der Schaft & Schumacher, 1996, 1998) and conewise linear systems (Arapostathis & Broucke, 2007; Heemels et al., 2000a; Çamlıbel et al., 2006; Çamlıbel et al., 2008; Schumacher, 2004; Shen, 2010).

PWA systems as a special class of nonlinear systems adopts the properties of linear systems in each region while capable of representing nonlinear behaviors. Switched affine systems based on conventional representation are the most widely used one in most of the cases on control system theory (Eghbal et al., 2013; Hassibi & Boyd, 1998; Johansson, 2003; Johansson & Rantzer, 1998; Li et al., 2006; Rodrigues & How, 2003; Rodrigues & Boyd, 2005; Samadi, 2008; Seatzu et al., 2006). PWA systems based on canonical representations are widely applied in nonlinear circuit theory (Chua & Kang, 1977; Chua & Deng, 1986; Güzeliş & Gökner, 1991). Stability analysis of PWA systems and the controller synthesis for these systems are of particular importance due to the fact that they allows to define

the Lyapunov functions in each linear region by employing linear analysis techniques.

Surveys of stability analysis for hybrid and switched linear systems can be found in (DeCarlo et al., 2000; Heemels et al., 2010; Liberzon, 2003; Lin & Antsaklis, 2009; Sun, 2010). A sufficient condition for stability is searching a globally common quadratic Lyapunov function which reveals Linear Matrix Inequalities (LMI) based constraints and can be formulated as convex optimization problems (Hassibi & Boyd 1998; Rodrigues & Boyd, 2005). Searching a globally quadratic Lyapunov function is relaxed with searching a continuous piecewise quadratic Lyapunov function in (Branicky, 1998; Johansson, 2003; Johansson & Rantzer, 1998; Pettersson, 1999; Rodrigues et al., 2000). Searching a Sum Of Squares (SOS), a Lyapunov function is stated in (Papachristodoulou & Prajna, 2005; Samadi & Rodrigues, 2011) where it is achieved by solving a convex problem. Stability of slab systems which constitute a sub-class of PWA systems where the regions are determined by degenerate ellipsoids is described in (Rodrigues & Boyd, 2005). A PWA Lyapunov function (Johansson, 2003) is also considered for stability analysis of PWA systems. Lyapunov based controller synthesis problems are developed in (Daafouz et al., 2002; Hassibi & Boyd, 1998; Lazar & Heemels, 2006; Rodrigues & Boyd, 2005; Rodrigues et al., 2000; Rodrigues & How, 2003; Samadi & Rodrigues, 2009). Controller for special PWA systems such as Chua's circuit is presented in (Barone & Singh, 2002; Bowong & Kagou, 2006; Ge & Wang, 1999; Hwang et al., 1997; Lee & Singh, 2007; Li et al., 2005; Li et al., 2006; Liao & Chen, 1998; Liu & Huang, 2006; Maganti & Singh, 2006; Puebla et al., 2003).

Chua's circuit is of importance in the thesis studies since it is a PWA system exhibiting complex behaviors such as chaos. Chaos is a behavior to be avoided in most applications, thus should be controlled, however it is thought to be useful in the nature and in some engineering applications, so it should not be suppressed even it should be generated. Generation of chaos from a non-chaotic dynamical system is the process of chaotification (or also said anti-control of chaos, or chaotization). Several efficient chaotification methods employing feedback control techniques are

introduced in the literature for both discrete and continuous time systems. Chaotification methods for discrete time systems (Chen & Shi, 2006) are mainly based on a proper feedback law yielding the overall system to exhibit chaos in the sense of Devaney (Devaney, 2003) and/or Li-Yorke (Li & Yorke, 1975). Many chaotification methods for continuous time systems have been developed in the literature (Chen, 1975; Wang, 2003). A part of them can be categorized as Vanecek-Celikovsky method (Vanecek & Celikovsky, 1994), time-delay feedback (Wang et al., 2000; Wang, Chen et al., 2001; Wang, Zhong et al., 2001), impulsive control (Wang, 2003), model based static feedback chaotification (Wang & Chen, 2003; Morgül, 2003). In addition to these methods, sliding mode control based chaotification methods are introduced in (Li & Song, 2008; Xie & Han, 2010). The study in (Li & Song, 2008) can be categorized as a synchronization method because it is designed to follow the states of a reference chaotic system. Xie & Han, (2010) proposed a sliding mode control based chaotification method designed for nonlinear discrete time systems.

This thesis introduces a canonical representation based parametric controller design for PWA systems using feedback linearization. The obtained results extend the work in (Çamlıbel & Ustaoğlu, 2005), where the conditions of full state linearization are presented for cone-wise linear systems, to a more general class of PWA systems having the canonical representation. The thesis introduces also canonical representation based normal forms of PWA systems for both single input and multiple input cases. The conditions on partially feedback linearizability of PWA systems while maximizing the relative degree are introduced as combinatorial problems. A parametric controller for PWA systems using feedback linearization is developed. An approximate linearization based controller design for PWA systems is presented by exploiting the canonical representation. Furthermore, a model based robust chaotification scheme using sliding mode with a dynamical state feedback in order to match all system states to a reference chaotic system is introduced. Herein, a nonlinear sliding surface is chosen such that reaching this surface results the system to match to a reference chaotic system. The chaotification method is also applicable for PWA systems which are partially feedback linearizable with relative degree more

than or equal to three. Finally, a stability analysis of PWA systems with polyhedral regions represented by an intersection of degenerate ellipsoids inspired from (Rodrigues & Boyd, 2005) is introduced. In (Rodrigues & Boyd, 2005), stability problem is defined for PWA systems with slab regions represented in an exact manner by degenerate ellipsoids. Considering the fact that any polyhedral region can be represented by intersections of degenerate ellipsoids, the results of (Rodrigues & Boyd, 2005) for PWA systems with slab regions are extended in the thesis to PWA systems with polyhedral regions represented by intersections of degenerate ellipsoids. Where, the stability problem is formulated as a set of LMIs. Furthermore, a stability analysis of PWA systems over bounded polyhedral regions by using a vertex based representation is presented. In this stability analysis, the polyhedral regions are determined by the vertices and the set of LMIs are obtained with these vertices.

The organization of the chapters of this thesis is as follows. Chapter 2 gives a background on representations of PWA mappings, stability analysis of PWA systems and nonlinear control methods. Chapter 3 introduces the canonical representation based parametric controller design for PWA systems using feedback linearization and approximate feedback linearization. In Chapter 4, a sliding mode control based robust chaotification scheme in which a nonlinear sliding manifold and a dynamical feedback law are determined appropriately to match all states of the controllable linear and input state linearizable nonlinear systems to reference chaotic systems in the normal form is described. In Chapter 5, a stability analysis of PWA systems with polyhedral regions represented by intersection of degenerate ellipsoids and a stability analysis of PWA systems over bounded polyhedral regions by using a vertex based representation are presented.

CHAPTER TWO

BACKGROUND ON PIECEWISE LINEAR REPRESENTATIONS AND NONLINEAR CONTROL SYSTEMS

In this chapter, a brief background on piecewise linear representations most widely used of which are conventional representation, simplex representation, complementary pivot representation and canonical representations; stability analysis of PWA systems and some methods of nonlinear control systems are introduced.

2.1 Piecewise Linear Representations

There are four main representations of piecewise linear mappings. In this section, conventional representation, simplex representation, canonical representations and complementary pivot representation are introduced as the main piecewise linear representations.

2.1.1 Conventional Representation

Conventional representation is the most commonly used one of these representations. Especially, piecewise linear dynamical systems are specified by the conventional representation in most of the cases on control system theory such as (Hassibi & Boyd, 1998; Johansson, 2003; Johansson & Rantzer, 1998; Li et al., 2006; Rodrigues & Boyd, 2005; Samadi, 2008). The conventional representation of a piecewise linear mapping $f : R^n \rightarrow R^m$ can be written in the form:

$$y = f(x) = A_i x + b_i \tag{2.1}$$

with the region R_i for $i = \{1, 2, \dots, l\}$ where $A_i \in R^{m \times n}$ is a matrix, $b_i \in R^m$ is a vector. In most of the cases, the region R_i is polytopic and defined as

$$R_i = \{x \mid h_{ij}^T x - g_{ij} < 0, j = 1, 2, \dots, p_i\} = \{x \mid H_i x - g_i < 0\} \tag{2.2}$$

where $h_{ij} \in R^n$, $g_{ij} \in R$, $H_i \in R^{p_i \times n}$ and $g_i \in R^{p_i}$. The dimensions of $H_i \in R^{p_i \times n}$ and $g_i \in R^{p_i}$ are arbitrary for every region. In most cases, piecewise linear dynamical systems in the state space form are written in the conventional representation as

$$\dot{x} = A_i x + b_i + B_i u \quad (2.3)$$

where $x(t) \in R^n$ is the state and $u \in R^{n_u}$ is the control input.

2.1.2 Simplex Representation

In geometry, a simplex is a region of arbitrary dimension and is generally a convex hull (convex envelope) of vertices. An n -dimensional simplex is an n -dimensional polytope which is the convex combination of its $n + 1$ vertices. Therefore, the region, on which $f(\cdot)$ is defined, can be formulated as:

$$R_i = \text{conv}(v_{i,1}, v_{i,2}, \dots, v_{i,n+1}) = \left\{ x \in R^n \mid x = \sum_{j=1}^{n+1} \mu_j v_{i,j}, \mu_j \in [0,1], \sum_{j=1}^{n+1} \mu_j = 1 \right\} \quad (2.4)$$

The affine mapping in the region R_i maps convex hull of vertices to the convex hull of images of these vertices. Then, the mapping can be formulated as:

$$y = f(x) = \sum_{j=1}^{n+1} \mu_j f(v_{i,j}) \text{ for } R_i = \left\{ x \in R^n \mid x = \sum_{j=1}^{n+1} \mu_j v_{i,j}, \mu_j \in [0,1], \sum_{j=1}^{n+1} \mu_j = 1 \right\} \quad (2.5)$$

Piecewise linear dynamical systems in the state space form can be written in the simplex representation as:

$$\dot{x} = \sum_{j=1}^{n+1} \mu_j f(v_{i,j}) \text{ for } R_i = \left\{ x \in R^n \mid x = \sum_{j=1}^{n+1} \mu_j v_{i,j}, \mu_j \in [0,1], \sum_{j=1}^{n+1} \mu_j = 1 \right\} \quad (2.6)$$

2.1.3 Canonical Representation

The canonical representations have two main advantages. One of them is computer storage amount needed to represent a PWA mapping and other is the global analytic form of the representations, which allows analytically analysis of PWA studies.

The canonical representation was first introduced by (Chua & Kang, 1977) which has the capability to represent any single-valued PWA function $f: R^1 \rightarrow R^1$ with the expression

$$f(x) = a_0 + a_1 x + \sum_{i=1}^l c_i |x - \beta_i| \quad (2.7)$$

where the coefficients a_0, a_1, c_i, β_i are scalars. The coefficients of canonical representation for $i \in \{1, 2, \dots, l\}$ can be written in terms of conventional representation as:

$$\begin{aligned} a_1 &= (A_0 + A_1) / 2 \\ c_i &= (A_i + A_{i-1}) / 2 \\ a_0 &= f(0) - \sum_{i=1}^l c_i |B_i| \end{aligned} \quad (2.8)$$

It is also showed that any single-valued discontinuous PWA function $f: R^1 \rightarrow R^1$ can be represented by the expression

$$f(x) = a_0 + a_1 x + \sum_{i=1}^l \{c_i |x - \beta_i| + b_i \text{sign}(x - \beta_i)\} \quad (2.9)$$

where the coefficients $a_0, a_1, b_i, c_i, \beta_i$ are scalars. The coefficients of canonical representation for $i \in \{1, 2, \dots, l\}$ can be calculated as follows:

$$a_1 = (A_0 + A_1) / 2 \quad (2.10)$$

$$c_i = (A_i + A_{i-1}) / 2 \quad (2.11)$$

$$b_i = \begin{cases} 0, & \text{if } f(\cdot) \text{ is continuous at the breakpoint } x = \beta_i \\ \frac{1}{2} [f(x_i^+) - f(x_i^-)], & \text{otherwise} \end{cases} \quad (2.12)$$

$$a_0 = f(0) - \sum_{i=1}^l [c_i |\beta_i| - b_i \text{sign}(\beta_i)] \quad (2.13)$$

The canonical representation given above is a global and compact representation for one dimensional piecewise linear functions formulated just using absolute value function for continuous case and absolute value and signum functions for discontinuous case. The existence of such compact representation for one dimensional piecewise linear function led (Kang & Chua, 1978) to extend the representation for higher dimensional piecewise linear functions with convex polyhedral (polytopic) regions constructed by linear partitions as defined in (2.14). A general class of n-dimensional m-valued canonical representation of piecewise linear function can be formulated as:

$$f(x) = a + Bx + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| \quad (2.14)$$

where $a, c_i \in R^m$, $B \in R^{m \times n}$, $\alpha_i \in R^n$, and $\beta_i \in R^1$, for $i = 1, 2, \dots, l$.

The canonical representation for multi-valued piecewise linear functions is incapable of formulating whole class of PWA functions with convex polyhedral regions constructed by linear partitions. The sufficient conditions and necessary and sufficient conditions of this representation is introduced by (Chua & Deng, 1988).

Definition 2.1 (Non-degenerate Partition) [Chua & Deng 1988]: A linear partition determined by the hyperplanes:

$$\alpha_i^T x - \beta_i = 0, \quad i \in \{1, 2, \dots, l\} \quad (2.15)$$

is said to be nondegenerate if for every set of linearly dependent vectors $\{\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ik}\}$ with $k \in \{1, 2, \dots, l\}$ the rank of $[\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ik}]$ is strictly less than the rank of the following $(n+1) \times m$ matrix

$$\begin{bmatrix} \alpha_{i1} & \alpha_{i2} & \dots & \alpha_{ik} \\ \beta_{i1} & \beta_{i2} & \dots & \beta_{ik} \end{bmatrix}. \quad (2.16)$$

The following theorem describes the sufficient condition where canonical representation is capable of formulating PWA functions.

Theorem 2.1 (Sufficient condition) [Chua & Deng, 1988]: A continuous PWA function $f(\cdot): R^n \rightarrow R^m$ partitioned by a finite set of $n-1$ dimensional hyperplanes determined by $\alpha_i^T x - \beta_i = 0$ with $i \in \{1, 2, \dots, l\}$ has a canonical representation of the form (2.14) if the linearly partitioned domain space is nondegenerate.

Definition 2.2 (Consistent variation property) [Chua & Deng, 1988; Güzeliş & Gökner, 1991]: A PWL function $f: R^n \rightarrow R^m$ possesses the consistent variation property if and only if

- i) f has a linear partition.
- ii) $J_{R_{i1}^+} - J_{R_{i1}^-} = J_{R_{i2}^+} - J_{R_{i2}^-} = \dots = J_{R_{in_i}^+} - J_{R_{in_i}^-} = c_i \alpha_i^T$, for $i = 1, 2, \dots, l$,

where $J_{R_{ij}^+}$ and $J_{R_{ij}^-}$ denote the Jacobian matrices of the regions R_{ij}^+ and R_{ij}^- , respectively, which are separated by the boundary $\alpha_i^T x - \beta_i = 0$. Here, $j = 1, 2, \dots, n_i$, and n_i pairs of regions are separated by this boundary, such that $\alpha_i^T x - \beta_i \geq 0$ ($\alpha_i^T x - \beta_i \leq 0$, respectively) for $x \in R_{ij}^+$ ($x \in R_{ij}^-$, respectively). Moreover,

the intersection between $R_{i_j}^+$ and $R_{i_j}^-$ must be a subset of an (n-1)-dimensional hyperplane and cannot be covered by any hyperplane of lower dimension. This condition must hold for every pair of neighboring regions separated by a common boundary.

Theorem 2.2 (Necessary and Sufficient Condition) [Chua & Deng, 1988]: A piecewise-linear function $f: R^n \rightarrow R^m$ has a canonical piecewise-linear representation (2.14) if and only if it possesses the consistent variation property.

The canonical representation has been extended by several studies in the literature (Güzeliş & Gökna, 1991; Julian et al., 1999; Kahlert & Chua, 1990; Lin et al., 1994). Güzeliş and Gökna (1991) put forward the idea of nested absolute value where canonical representation was extended to formulate a special class of degenerate linear partitions.

$$f(x) = a + Bx + \sum_i^l c_i \left| \alpha_i^T x + \beta_i \right| + \sum_{j=1}^P b_j \left| \delta_j + \gamma_j^T x + \sum_{i=1}^l d_{ji} \left| \alpha_i^T x + \beta_i \right| \right| \quad (2.17)$$

The representation (2.17) uses both conventional hyper-planes

$$H_i = \left\{ x \in R^n \mid \alpha_i^T x + \beta_i = 0 \right\}, i \in \{1, 2, \dots, l\}, \alpha_i \in R^n, \beta_i \in R \quad (2.18)$$

and PWA hyper-planes

$$S_j = \left\{ x \in R^n \mid \psi_j(x) = 0 \right\}, j \in \{1, 2, \dots, P\} \quad (2.19)$$

where

$$\psi_j(x) = \delta_j + \gamma_j^T x + \sum_{i=1}^l d_{ji} \left| \alpha_i^T x + \beta_i \right| \quad (2.20)$$

For the canonical representation (2.17), the consistent variation property (Chua & Deng, 1988) or, in other words, the consistency of continuity vectors (Güzeliş & Göknaar, 1991) is given by the equations (2.21) and (2.22). For any pair of regions R^i and R^j separated by a conventional hyperplane H_k , the consistency of continuity vectors is the uniqueness of the continuity vectors $q_k^{i,j}$'s for all i, j and k :

$$\left[J^i - J^j, w^i - w^j \right] = q_k^{i,j} \left[\alpha_k^T - \beta_k \right] \quad (2.21)$$

For the pair of regions of R^i and R^j separated by a PWA hyperplane S_k , the consistency of continuity vectors becomes the uniqueness of the following continuity vectors $q_k^{i,j}$'s for all i, j and k :

$$\left[J^i - J^j, w^i - w^j \right] = q_k^{i,j} \left(\left[\gamma_k^T, \delta_k \right] + \sum_{p \in P_1} d_{kp} \left[\alpha_p^T, \beta_p \right] - \sum_{p \in P_2} d_{kp} \left[\alpha_k^T, \beta_k \right] \right) \quad (2.22)$$

The existence of the continuity vectors $q_k^{i,j}$ in the above equations are indeed the necessary and sufficient conditions for the continuity of the PWA function defined over the PWA partitioned domain space.

Theorem 2.3 (Necessary and sufficient condition) [Güzeliş & Göknaar, 1991]: A PWA continuous function $f(\cdot): R^n \rightarrow R^m$ defined over a PWA partition determined by the hyper-planes and PWA hyper-planes given in (2.18) and (2.19) can be represented by the canonical form (2.17) if and only if its continuity vectors are consistent in the sense of expressions given in (2.21) and (2.22). $\square$

The canonical representation (2.17) is quite general; however, they cannot cover the whole set of continuous PWA functions. In the literature (Chua & Deng, 1988; Chua & Kang, 1977; Güzeliş & Göknaar, 1991; Julian et al., 1999; Kahlert & Chua, 1990, 1992; Kang & Chua, 1978; Kevenaar et al., 1994; Leenarts, 1999; Lin et al., 1994, Lin & Unbehauen, 1995), there are many attempts to represent the whole class

of continuous PWA functions using the absolute value function as the unique nonlinear building block. Among these attempts, the work presented in (Lin et al., 1994) may be the most remarkable one as proving that any kind of continuous PWA function defined over a linear partition in R^n can be expressed by n -nested absolute value functions.

2.1.4 Complementary Pivot Representation

A general class of piecewise linear mappings can be formulated as the complementary pivot representation proposed by (van Bokhoven, 1981) in the following form.

$$y = Ax + Bu + f \quad (2.23)$$

$$j = Cx + Du + g \quad (2.24)$$

where $A \in R^{m \times n}$, $B \in R^{m \times k}$, $f \in R^m$, $C \in R^{k \times n}$, $D \in R^{k \times k}$, $g \in R^k$ and the variables $u, j \in R^k$ satisfy the linear complementary problem

$$u^T j = 0, \quad u \geq 0, \quad j \geq 0. \quad (2.25)$$

The linear partition of the domain is determined by (2.24) with the constraints in (2.25). According to above inequalities in (2.25), it is obvious that either u_i or j_i is greater than zero which yields maximum 2^k regions. The j_i variable has the form

$$j_i = u_i + c_i^T x + g_i \quad (2.26)$$

where j_i , u_i and g_i are the j th element of the corresponding vector and c_i^T is the j th row of the matrix c hyperplanes of the representation can be defined as follows when both u_i and j_i becomes zero.

$$c_i^T x + g_i = 0 \quad (2.27)$$

As stated in (Julian, 1999; Leenaerts & van Bokhoven, 1998), complementary pivot representation can formulate multi-valued mappings due to the fact that the multiple solutions for vectors u and j may occur for a given x in some cases.

Another complementary pivot representation which is capable of partitioning the input-output space (x, y) simultaneously is also proposed by (van Bokhoven, 1986) in the following form.

$$0 = Iy + Ax + Bu + f \quad (2.28)$$

$$j = Dy + Cx + Iu + g \quad (2.29)$$

where $A \in R^{m \times n}$, $B \in R^{m \times k}$, $f \in R^m$, $C \in R^{k \times n}$, $D \in R^{k \times m}$, $g \in R^k$ and the variables $u, j \in R^k$ satisfy.

$$u^T j \geq 0, \quad u \geq 0, \quad j \geq 0 \quad (2.30)$$

The linear partition of the domain is determined by (2.29) with the constraints in (2.30). According to above inequalities in (2.30), it is obvious that either u_i or j_i is greater than zero which yields maximum 2^k regions. The j_i variable has the form

$$j_i = u_i + d_i^T y + c_i^T x + g_i \quad (2.31)$$

where j_i , u_i and g_i are the j th element of the corresponding vector; c_i^T and d_i^T are the j th row of the matrix C and D respectively. The hyperplanes of the representation can be defined as follows when both u_i and j_i becomes zero.

$$d_i^T y + c_i^T x + g_i = 0 \quad (2.32)$$

It is shown in (Julian, 1999) that the representation (2.28) with (2.29) is a particular case of the representation (2.23) with (2.24) and any representation in the form (2.28) can be reformulated by the form (2.23).

2.2 Stability Analysis of PWA Systems

Stability theory is essential for systems and control theory. Stability of equilibrium points is defined in the sense of Lyapunov. For PWA systems, there exist theorems based on Lyapunov stability theorem characterized as linear matrix inequalities.

Definition 2.3 (Equilibrium Point): $x_0 \in R^n$ is an equilibrium point of $\dot{x} = f(x, t)$ at t_0 if and only if $f(x_0, t) = 0 \quad \forall t \in [t_0, \infty)$.

Definition 2.4 (Stability in the sense of Lyapunov) [Sastry, 1999]: The equilibrium point $x = 0$ is called a stable equilibrium point of $\dot{x} = f(x, t)$ if for all $t_0 \geq 0$ and $\varepsilon > 0$, there exists $\delta(t_0, \varepsilon)$ such that

$$|x_0| < \delta(t_0, \varepsilon) \Rightarrow |x(t)| < \varepsilon \quad \forall t \geq t_0$$

where $x(t)$ is the solution of $\dot{x} = f(x, t)$ starting from x_0 at t_0 .

Definition 2.5 (Global Asymptotic Stability) [Sastry, 1999]: The equilibrium point $x = 0$ is a globally asymptotically stable equilibrium point of $\dot{x} = f(x, t)$ if it is stable and $\lim_{t \rightarrow \infty} x(t) = 0$ for all $x_0 \in R^n$.

Definition 2.6 (Global Uniform Asymptotic Stability) [Sastry, 1999]: The equilibrium point $x = 0$ is a globally, uniformly, asymptotically stable equilibrium point of $\dot{x} = f(x, t)$ if it is globally asymptotically stable and if in addition, the

convergence to the origin of trajectories is uniform in time, that is to say that there is a function $\gamma : R^n \times R_+ \mapsto R$ such that

$$|x(t)| \leq \gamma(x_0, t - t_0) \quad \forall t \geq 0.$$

Definition 2.7 (Exponential Stability, Rate of Convergence) [Sastry, 1999]: The equilibrium point $x = 0$ is an exponentially stable equilibrium point of $\dot{x} = f(x, t)$ if there exist $m, \alpha > 0$ such that

$$|x(t)| \leq m e^{-\alpha(t-t_0)} |x_0|$$

for all $x_0 \in D_0$, $t \geq t_0 \geq 0$. The constant α is called (an estimate of) the rate of convergence.

Theorem 2.4 [Khalil, 2002]: Let $x = 0$ be an equilibrium point for $\dot{x} = f(x, t)$ and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \rightarrow R$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \quad (2.33)$$

$$\text{Let } \dot{V}(x) \leq 0 \text{ in } D \quad (2.34)$$

Then, the equilibrium point $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \quad (2.35)$$

then $x = 0$ is asymptotically stable. The equilibrium point $x = 0$ is globally asymptotically stable if the equations (2.33) and (2.35) are satisfied for all $x \in R^n$.

2.2.1 Lyapunov Stability of Linear Time-Invariant Systems

Consider the linear system $\dot{x} = Ax$ and consider a quadratic Lyapunov function $V(x) = x^T Px$ with a symmetric and positive definite matrix P . The derivative of the Lyapunov function along the system trajectory is as follows:

$$\dot{V}(x) = \dot{x}^T Px + x^T P\dot{x} = x^T [A^T P + PA]x = -x^T Qx \quad (2.36)$$

where

$$Q = -[A^T P + PA] \quad (2.37)$$

For a stable system with appropriately chosen Lyapunov function, the derivative of the Lyapunov function should be negative for all $x \neq 0$. Hence, $A^T P + PA$ should be a negative definite matrix such that $A^T P + PA < 0$. For stable linear time-invariant systems with a quadratic Lyapunov function $V(x) = x^T Px$, there is some positive definite matrix P such that the Lyapunov inequality $A^T P + PA < 0$ holds (Slotine & Li, 1991). However, a chosen positive definite matrix P may not yield a positive definite matrix Q . Determining the positive definite matrix P which yields a positive definite matrix Q is a difficult task which requires the solution of the linear matrix inequalities as follows.

$$\begin{aligned} P &> 0 \\ A^T P + PA &< 0 \end{aligned} \quad (2.38)$$

Instead of solving linear matrix inequalities, choosing any symmetric positive definite matrix Q and then solving the linear equation $A^T P + PA = -Q$ for the matrix P is more efficient. The matrix P determined in such a way is guaranteed to be positive definite for stable linear systems. The following theorem summarizes the necessary and sufficient condition for stability of LTI systems.

Theorem 2.5 [Slotine & Li, 1991]: A necessary and sufficient condition for a LTI system $\dot{x} = Ax$ to be global asymptotically stable is that, for any symmetric positive definite matrix Q , the unique matrix P solution of the Lyapunov equation (2.37) be symmetric positive definite.

2.2.2 Lyapunov Stability of Piecewise Affine Systems

A PWA system can be written in the form:

$$\dot{x} = A_i x + b_i \quad (2.39)$$

with the region R_i for $i = \{1, 2, \dots, l\}$ where $A_i \in R^{n \times n}$ is a matrix, $b_i \in R^n$ is a vector. The region R_i is polytopic and defined as

$$R_i = \{x \mid h_{ij}^T x - g_{ij} < 0, j = 1, 2, \dots, p_i\} = \{x \mid H_i x - g_i < 0\} \quad (2.40)$$

where $h_{ij} \in R^n$, $g_{ij} \in R$, $H_i \in R^{p_i \times n}$ and $g_i \in R^{p_i}$. The dimensions of $H_i \in R^{p_i \times n}$ and $g_i \in R^{p_i}$ are arbitrary for every region. The switching from one affine function to another is defined in terms of states. A more general condition of switching can be a function of both time and state.

Stability of PWA system can be checked by determining a Lyapunov function candidate and checking for every region whether it satisfies the conditions in Theorem 2.4. The Lyapunov function can be chosen as a quadratic function, piecewise quadratic function, sum of squares function or else.

2.2.2.1 Globally Defined Quadratic Lyapunov Function for the Stability of Piecewise Affine Systems

For a linear system, quadratic Lyapunov function is a well-defined tool which determines the necessary and sufficient condition for stability. Therefore, it can be a good estimate for PWA systems which determines the sufficient condition for stability. Hence; although there is not such a globally common quadratic Lyapunov function, the PWA system can still be stable.

A sufficient condition for piecewise affine systems (2.39) with regions (2.40) can be derived via a globally common quadratic Lyapunov function $V(x) = x^T P x$. The derivative of the Lyapunov function along the system trajectory is as follows and should be negative for stability to satisfy Theorem 2.4.

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = (A_i x + b_i)^T P x + x^T P (A_i x + b_i) = x^T [A_i^T P + P A_i] x + b_i^T P x + x^T P b_i < 0$$

$$\dot{V}(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P + P A_i & P b_i \\ b_i^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0 \quad \text{with the region } R_i \text{ defined in (2.40)}$$

$$\text{approximated by } \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} H_i & g_i \\ 0 & 1 \end{bmatrix} \bar{U}_i \begin{bmatrix} H_i & g_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} > 0.$$

Combining the two inequalities by S-procedure yields the following proposition for piecewise affine systems.

Proposition 2.1 [Samadi, 2008]: A sufficient condition for a piecewise affine system in (2.39) to be stable is that there exists a symmetric positive definite matrix P , U_i and $\bar{U}_i$ with nonnegative elements satisfying the following LMIs for $i \in \{1, 2, \dots, l\}$.

$$\begin{cases} PA_i + A_i^T P < 0, & \text{if } b_i = 0 \text{ and } g_i \neq 0 \\ PA_i + A_i^T P + H_i^T U_i H_i < 0, & \text{if } b_i = 0 \text{ and } g_i = 0 \\ \begin{bmatrix} PA_i + A_i^T P & Pb_i \\ b_i^T P & 0 \end{bmatrix} + \begin{bmatrix} H_i & g_i \\ 0 & 1 \end{bmatrix}^T \bar{U}_i \begin{bmatrix} H_i & g_i \\ 0 & 1 \end{bmatrix} < 0, & \text{otherwise} \end{cases} \quad (2.41)$$

2.2.2.2 Globally Common Quadratic Lyapunov Function for the Stability of Piecewise Affine Slab Systems

PWA slab system whose polytopic regions are slabs is a special case of piecewise affine system (2.39). The region of a slab system can be exactly represented by degenerate ellipsoids. Moreover, the region which is in the form of $R_i = \{x \mid d_1^i < c^T x < d_2^i\}$ with $R_i \cap R_j = \emptyset$ for $i \neq j$ can be exactly represented by $\varepsilon_i = \{x \mid \|E_i x + f_i\| \leq 1\}$ where $E_i = 2c^T / (d_2^i - d_1^i)$ and

$f_i = -(d_2^i + d_1^i) / (d_2^i - d_1^i)$. Ellipsoidal covering is beneficial for defining the stabilization problem as a LMI problem. For such systems, the stability can be checked by the following proposition.

Proposition 2.2 [Rodriquez & Boyd, 2005; Samadi, 2008]: All trajectories of the PWA slab system asymptotically converge to $x = 0$ if for a given decay rate $\alpha > 0$, there exist $P \in R^{n \times n}$ and $\tau_i \in R$ for $i \in \{1, 2, \dots, M\}$ such that

$$\begin{aligned} P &> 0, \\ A_i^T P + P A_i &< 0, \quad \forall i \in I(0) \end{aligned} \quad (2.42)$$

$$\begin{cases} \tau_i < 0, \\ \begin{bmatrix} A_i^T P + P A_i + \tau_i E_i^T E_i & Pb_i + \tau_i f_i E_i^T \\ b_i^T P + \tau_i f_i E_i & \tau_i (f_i^2 - 1) \end{bmatrix} < 0, \end{cases} \quad \text{for } i \notin I(0) \quad (2.43)$$

where M is the number of regions and $I(0) = \{i \in \{1, 2, \dots, M\} \mid 0 \in R_i\}$.

2.2.2.3 Piecewise Quadratic Lyapunov Function for the Stability of Piecewise Affine Systems

Piecewise quadratic Lyapunov function relaxes the existence condition of a common quadratic Lyapunov function. Instead, it requires a continuous Lyapunov function which is quadratic in each region as follows:

$$V(x) = \begin{cases} x^T P_i x & \text{for } x \in R_i, \quad i \in I(0) \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = x^T P_i x + 2q_i^T x + r_i & \text{for } x \in R_i, \quad i \notin I(0) \end{cases} \quad (2.44)$$

where $I(0)$ is the index set for region that contain origin such that $I(0) = \{i \in \{1, 2, \dots, M\} \mid 0 \in R_i\}$. It is assumed that $b_i = 0$ for $i \in I(0)$. The following theorem describes a sufficient condition for a PWA system in (2.39) to be stable with a piecewise quadratic Lyapunov function.

Theorem 2.6 [Johansson & Rantzer 1998]: Consider symmetric matrices T and U_i and W_i , such that U_i and W_i have nonnegative entries, while

$$P_i = F_i^T T F_i, \quad i \in I(0) \quad (2.45)$$

$$\bar{P}_i = \bar{F}_i^T T \bar{F}_i, \quad i \notin I(0) \quad (2.46)$$

satisfy

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + H_i^T U_i H_i \\ 0 < P_i - H_i^T W_i H_i \end{cases} \quad i \in I(0) \quad (2.47)$$

$$\begin{cases} 0 > \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{H}_i^T U_i \bar{H}_i \\ 0 < \bar{P}_i - \bar{H}_i^T W_i \bar{H}_i \end{cases} \quad i \notin I(0) \quad (2.48)$$

Then every continuous piecewise C^1 trajectory $x(t) \in UR_i$, satisfying for $t \geq 0$, tends to zero exponentially where $\bar{A}_i = \begin{bmatrix} A_i & b_i \\ 0 & 0 \end{bmatrix}$, $\bar{H}_i = [H_i \ g_i]$, $\bar{F}_i = [F_i \ f_i]$ with $\bar{H}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0$ for $x \in R_i$ and $\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix}$ for $x \in R_i \cap R_j$. M is the number of regions and $I(0) = \{i \in \{1, 2, \dots, M\} \mid 0 \in R_i\}$.

2.2.2.4 Piecewise Linear Lyapunov Function for the Stability of Piecewise Affine Systems

A piecewise linear Lyapunov function is an alternative Lyapunov function candidate. Searching such a Lyapunov function can be solved by linear programming instead solving LMI's which is the case in piecewise quadratic Lyapunov function. A continuous piecewise linear Lyapunov function can be defined as follows.

$$V(x) = \begin{cases} p_i^T x & x \in X_i, \ i \in I_0 \\ \bar{p}_i^T x = p_i^T \bar{x} + q_i & x \in X_i, \ i \in I_1 \end{cases} \quad (2.49)$$

A sufficient condition for stability with piecewise linear Lyapunov function is as follows.

Theorem 2.7 [Johansson, 2003]: Let the polyhedral partition (2.40) with matrices

$\bar{F}_i$, satisfying $\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix}$ for $x \in R_i \cap R_j$ with $f_i = 0$ for $i \in I(0)$ and matrices

$\bar{E}_i$ satisfying $e_i = 0$ for $i \in I(0)$. Assume furthermore that $\bar{H}_i \bar{x} \neq 0$ for every $x \in R_i$ with $x \neq 0$. If there exists a vector t and non-negative vectors $u_i \succ 0$ and $\omega_i \succ 0$ while

$$p_i = F_i^T t \quad i \in I(0) \quad (2.50)$$

$$\bar{p}_i = \bar{F}_i^T t \quad i \notin I(0) \quad (2.51)$$

satisfy

$$\begin{cases} 0 = p_i^T A_i - u_i H_i \\ 0 = p_i^T + \omega_i H_i \end{cases} \quad i \in I(0) \quad (2.52)$$

$$\begin{cases} 0 = \bar{p}_i^T \bar{A}_i - u_i \bar{H}_i \\ 0 = \bar{p}_i^T + \omega_i \bar{H}_i \end{cases} \quad i \notin I(0) \quad (2.53)$$

then every trajectory $x(t) \in \cup R_i$ for $t \geq 0$ tends to zero exponentially where

$$\bar{A}_i = \begin{bmatrix} A_i & b_i \\ 0 & 0 \end{bmatrix}, \quad \bar{H}_i = [H_i \quad g_i], \quad \bar{F}_i = [F_i \quad f_i] \quad \text{with} \quad \bar{H}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \quad \text{for} \quad x \in R_i \quad \text{and}$$

$$\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix} \quad \text{for} \quad x \in R_i \cap R_j. \quad M \quad \text{is the number of regions and}$$

$$I(0) = \{i \in \{1, 2, \dots, M\} \mid 0 \in R_i\}.$$

2.3 Nonlinear Control System Design

In real world applications, there exist nonlinearities in plant dynamics, sensors and/or actuators etc. Linear controllers cannot always deal with such nonlinear systems. This leads the evaluation of nonlinear controllers two of which are feedback linearization and sliding mode control. Feedback linearization and sliding mode control are two essential methods widely used in nonlinear control (Isidori, 1995; Khalil, 1996; Sastry, 1999; Slotine & Li, 1991; Vidyasagar, 1993).

2.3.1 Feedback Linearization

The aim of feedback linearization is transforming the nonlinear system into a linear system with a state transformation and a convenient static state feedback in order to enable applying linear control methods.

2.3.1.1 Input-State Feedback Linearization

An n -dimensional single input observable system in the following form

$$\dot{x} = f(x) + g(x) \cdot u \quad (2.54)$$

where $x \in R^n$ is the state, $u \in R$ is the scalar control input, $f: R^n \rightarrow R^n$ and $g: R^n \rightarrow R^n$ are sufficiently smooth functions. The system (2.54) can be transformed into a simpler form with a state transformation. The aim of input-state feedback linearization is transforming the nonlinear system (2.54) into a linear system with a state transformation and a convenient static state feedback.

Theorem 2.8 [Slotine & Li, 1991]: The nonlinear system (2.54), with $\hat{f}(x)$ and $\hat{g}(x)$ being smooth vector fields is input-state linearizable if and only if there exists a region D_0 such that the following conditions hold:

- 1) The vector fields $\{\hat{g}, ad_{\hat{f}}\hat{g}, \dots, ad_{\hat{f}}^{n-1}\hat{g}\}$ are linearly independent in D_0 i.e.

$$rank \begin{bmatrix} \hat{g} & ad_{\hat{f}}\hat{g} & \dots & ad_{\hat{f}}^{n-1}\hat{g} \end{bmatrix} = n$$

- 2) The set $\{\hat{g}, ad_{\hat{f}}\hat{g}, \dots, ad_{\hat{f}}^{n-2}\hat{g}\}$ is involutive in D_0 i.e. the Lie bracket of any pair

of vector fields belonging to the set $\{\hat{g}, ad_{\hat{f}}\hat{g}, \dots, ad_{\hat{f}}^{n-2}\hat{g}\}$ is also a vector field this set.

If the system (2.54) is input-state linearizable then there should exist a smooth scalar function $h(x)$ satisfying

$$L_{\hat{g}} L_{\hat{f}}^i h(x) = 0 \text{ for } i \in \{0, \dots, n-2\} \quad (2.55)$$

and

$$L_{\hat{g}}L_{\hat{f}}^{n-1}h(x) \neq 0 \quad (2.56)$$

for all $x \in D_0$ (Sastry, 1999).

Under the above input-state linearizability assumption, the system in (2.54) can be transformed with the state transformation $z = T(x) = \begin{bmatrix} h(x) & L_{\hat{f}}h(x) & \cdots & L_{\hat{f}}^{n-1}h(x) \end{bmatrix}^T$, which is a diffeomorphism over $D_0 \in R^n$, (Sastry, 1999) into the normal form as:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= L_{\hat{f}}^n h(x) + L_{\hat{g}}L_{\hat{f}}^{n-1}h(x)u \end{aligned} \quad (2.57)$$

Choosing the control law (2.58) linearizes the system

$$u = \frac{1}{L_{\hat{g}}L_{\hat{f}}^{n-1}h(x)}v - \frac{L_{\hat{f}}^n h}{L_{\hat{g}}L_{\hat{f}}^{n-1}h(x)} \quad (2.58)$$

where v is the new control input. The linearized system has the following form.

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v \end{aligned} \quad (2.59)$$

Then, a linear control law can be applied as shown in Figure 2.1.

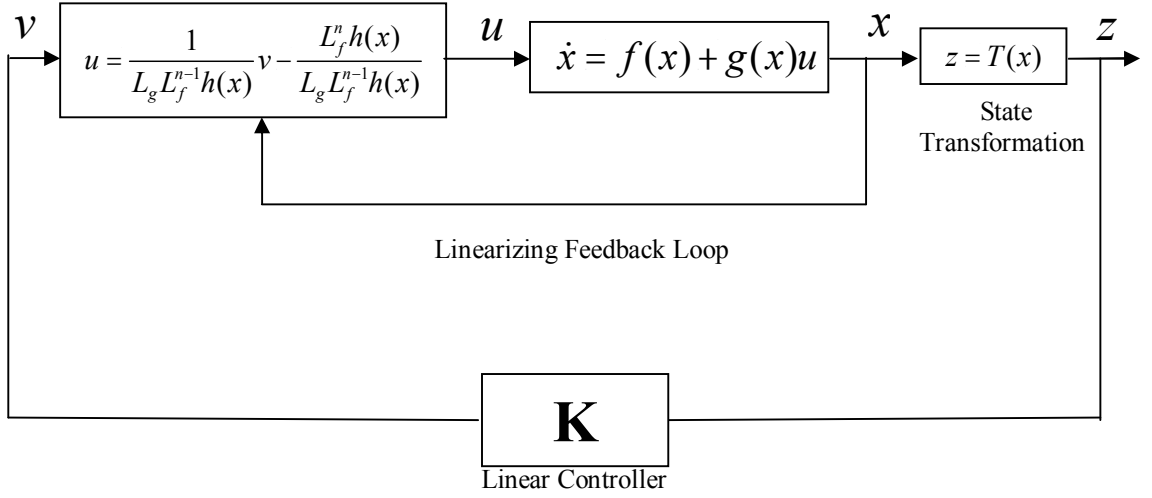


Figure 2.1 Input state feedback linearization with a linear control loop

2.3.1.2 Input-Output Feedback Linearization

The system (2.54) with an output equation $y = h(x)$ where $h: R^n \rightarrow R$ is input-output linearizable with relative degree r if it satisfies

$$L_{\hat{g}} L_{\hat{f}}^i h(x) = 0 \text{ for } i \in \{0, \dots, r-2\}$$

and

$$L_{\hat{g}} L_{\hat{f}}^{r-1} h(x) \neq 0$$

for all $x \in D_0$ (Sastry 1999).

The system in (2.54) can be transformed with the state transformation

$$\begin{aligned} z = T(x) &= \begin{bmatrix} \mu & \psi \end{bmatrix} = \begin{bmatrix} y & \dot{y} & \dots & y^{(r-1)} & \psi_1 & \psi_2 & \dots & \psi_{n-r} \end{bmatrix}^T \\ &= \begin{bmatrix} h(x) & L_{\hat{f}} h(x) & \dots & L_{\hat{f}}^{r-1} h(x) & \psi_1 & \psi_2 & \dots & \psi_{n-r} \end{bmatrix}^T \end{aligned}$$

which is a diffeomorphism over $D_0 \in R^n$, (Sastry, 1999) into the normal form as

$$\begin{aligned}
\dot{\mu}_1 &= \mu_2 \\
\dot{\mu}_2 &= \mu_3 \\
&\vdots \\
\dot{\mu}_r &= L_f^r h(x) + L_g L_f^{r-1} h(x) u \\
\dot{\psi} &= w(\mu, \psi)
\end{aligned} \tag{2.60}$$

where $\mu = [\mu_1 \ \mu_2 \ \cdots \ \mu_r]^T = [z_1 \ z_2 \ \cdots \ z_r]^T$ and

$\psi = [\psi_1 \ \psi_2 \ \cdots \ \psi_{n-r}]^T = [z_{r+1} \ z_{r+2} \ \cdots \ z_n]^T$.

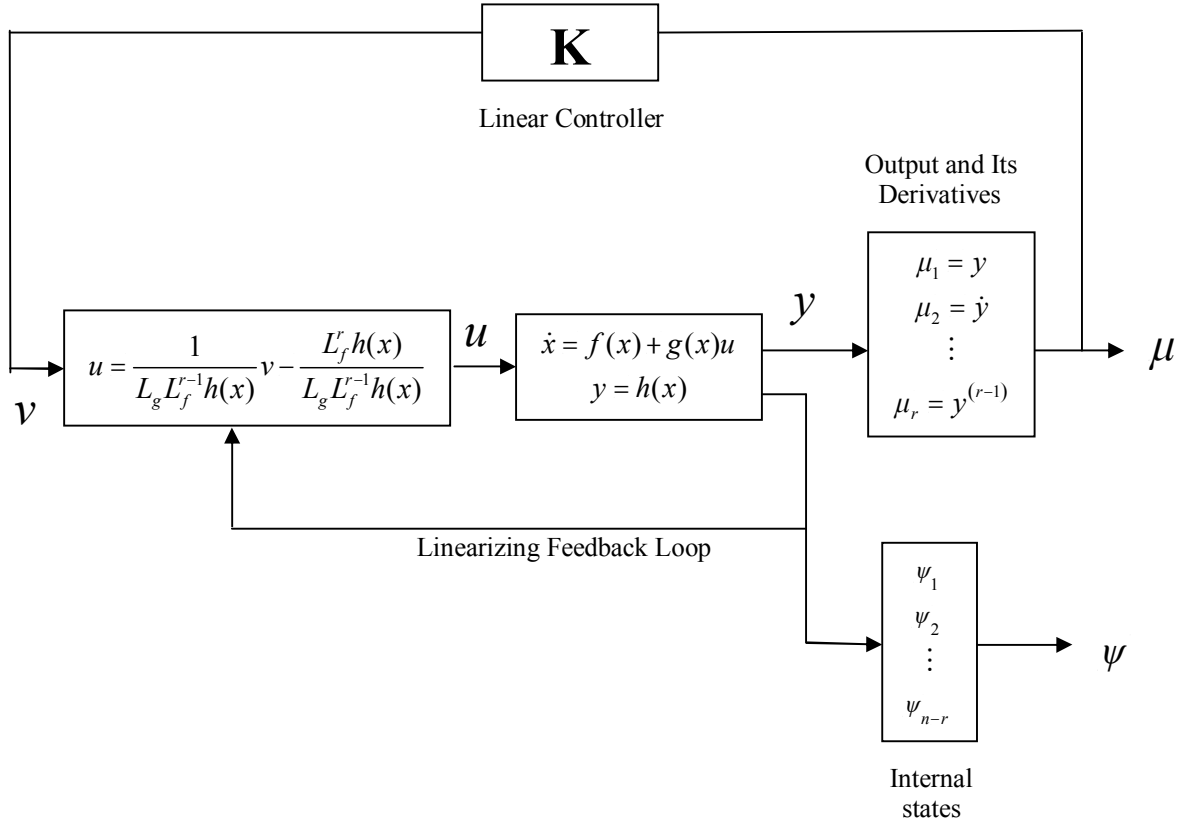


Figure 2.2 Input-output feedback linearization with a linear control loop

Choosing the control law (2.61) partially linearizes the system (2.60) where the first r dynamics of (2.60) become linear.

$$u = \frac{1}{L_{\hat{g}} L_{\hat{f}}^{r-1} h(x)} v - \frac{L_{\hat{f}}^r h}{L_{\hat{g}} L_{\hat{f}}^{r-1} h(x)} \quad (2.61)$$

where v is the new control input. A linear control law can be applied as shown in Figure 2.2 in order to control the trajectory of the output. However, the internal states should be considered, since they can be unstable which indicates the instability of the system's internal states.

CHAPTER THREE

CANONICAL REPRESENTATION BASED PARAMETRIC CONTROLLER DESIGN FOR PIECEWISE AFFINE SYSTEMS USING FEEDBACK LINEARIZATION

In this section, feedback linearization and approximate feedback linearization of PWA systems are introduced. The feedback linearizing control is parameterized in terms of the PWA system parameters.

3.1 Feedback Linearization of PWA Systems in Canonical Form Using Parameters

In this subsection, normal forms of PWA systems based on canonical representation are obtained for both single input and multiple input cases. The transformation and the linearizing input are parameterized in terms of system parameters. Moreover, the conditions for full state linearization are introduced and the equivalence of the conditions for full state linearization to the conditions for input to state linearizability is derived. Simulation results are presented in order to show the effectiveness of the results.

3.1.1 Feedback Linearization of PWA systems with Relative Degree r for Single Input Case

An n -dimensional single input system is considered in the following form.

$$\dot{x} = a + Ax + \sum_{i=1}^l c_i \left| \alpha_i^T x - \beta_i \right| + bu \quad (3.1)$$

where $a \in R^n$, $A \in R^{n \times n}$, $c_i \in R^n$, $\alpha_i \in R^n$, $\beta_i \in R$, $b \in R^n$, u is a scalar control input and $x \in R^n$ is the state.

Theorem 3.1: Any PWA system (3.1) in the canonical representation can be transformed into the normal form as

$$\begin{aligned}
\dot{\mu}_1 &= \mu_2 \\
\dot{\mu}_2 &= \mu_3 \\
&\vdots \\
\dot{\mu}_r &= \rho(\mu, \psi) + u \\
\dot{\psi} &= w(\mu, \psi)
\end{aligned} \tag{3.2}$$

with the diffeomorphic state transformation

$$z = \begin{bmatrix} \mu \\ \psi \end{bmatrix} = Tx + m = \begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r-1} \\ h_r^T \\ \vdots \\ h_n^T \end{pmatrix} x + \begin{pmatrix} 0 \\ h_1^T a \\ \vdots \\ h_1^T A^{r-2} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{with } h_1^T \text{ is the last row of the matrix}$$

$$C^T (CC^T)^{-1} \quad \text{if and only if} \quad \text{rank}[C] = \text{rank} \begin{bmatrix} C \\ e \end{bmatrix} = n \quad \text{where}$$

$$\mu = [\mu_1 \quad \mu_2 \quad \cdots \quad \mu_r]^T = [z_1 \quad z_2 \quad \cdots \quad z_r]^T,$$

$$\psi = [\psi_1 \quad \psi_2 \quad \cdots \quad \psi_{n-r}]^T = [z_{r+1} \quad z_{r+2} \quad \cdots \quad z_n]^T, \quad C = [D \quad AD \quad \cdots \quad A^{r-2}D \quad A^{r-1}b],$$

$$e = [0 \quad 0 \quad \cdots \quad 0 \quad 1], \quad D = [b \quad c_1 \quad \cdots \quad c_l] \quad \text{and}$$

$$\rho(\mu, \psi) = h_1^T A^{r-1} a - h_1^T A^r T^{-1} m + h_1^T A^r T^{-1} \begin{bmatrix} \mu \\ \psi \end{bmatrix} + \sum_{i=1}^l h_1^T A^{r-1} c_i \mid \alpha_i^T T^{-1} \begin{bmatrix} \mu \\ \psi \end{bmatrix} - \alpha_i^T T^{-1} m - \beta_i \mid.$$

Proof:

The affine transformation is defined as

$$z = Tx + m = \begin{bmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_n^T \end{bmatrix} x + \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}. \quad (3.3)$$

The transformation of the first state $z_1 = h_1^T x + m_1$ yields the following state equation.

$$\dot{z}_1 = h_1^T \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| + bu \right). \quad (3.4)$$

(3.2) states that $z_2 = \dot{z}_1$; therefore, to obtain an affine transformation $z_2 = h_2^T x + m_2$ it must be satisfied that $h_1^T b = 0$ and $h_1^T c_i = 0$ for $i \in \{1, 2, \dots, l\}$. Then the transformation for the second state becomes $z_2 = h_1^T Ax + h_1^T a$ and this transformation yields the following state equation.

$$\dot{z}_2 = h_1^T A \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| + bu \right). \quad (3.5)$$

(3.2) states that $z_3 = \dot{z}_2$; therefore, to obtain a linear transformation $z_3 = h_3^T x + m_3$ it must be satisfied that $h_1^T Ab = 0$ and $h_1^T Ac_i = 0$ for $i \in \{1, 2, \dots, l\}$. Then the transformation for the third state becomes $z_3 = h_1^T A^2 x + h_1^T Aa$. Repeating the procedure yields

$$\dot{z}_{r-1} = z_r = h_1^T A^{r-2} \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| + bu \right) = h_1^T A^{r-1} x \quad (3.6)$$

with $h_1^T A^{r-2} b = 0$ and $h_1^T A^{r-2} c_i = 0$ for $i \in \{1, 2, \dots, l\}$. The r th equation of (3.2) takes the following form

$$\dot{z}_r = h_1^T A^{r-1} \left(a + Ax + \sum_{i=1}^l c_i |a_i^T x - \beta_i| \right) + h_1^T A^{r-1} b u \quad (3.7)$$

with $h_1^T A^{r-1} b \neq 0$ or $h_1^T A^{r-1} b = 1$ can be chosen without loss of generality.

Substantially, for a PWA system (3.1) to take the normal form (3.2) it must satisfy the following conditions

$$h_1^T A^j c_i = 0 \text{ for all } j \in \{0, 1, \dots, r-2\} \text{ and } i \in \{1, 2, \dots, l\} \quad (3.8)$$

$$h_1^T A^j b = 0 \text{ for all } j \in \{0, 1, \dots, r-2\} \quad (3.9)$$

$$h_1^T A^{r-1} b = 1 \text{ and} \quad (3.10)$$

which forms the transformation $\mu = \begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r-1} \end{pmatrix} x + \begin{pmatrix} m_1 \\ h_1^T a \\ \vdots \\ h_1^T A^{r-2} a \end{pmatrix}$ with arbitrary m_1 .

Without loss of generality m_1 can be chosen to be equal to zero. The conditions (3.8), (3.9) and (3.10) can be rewritten as a linear system of equation such that

$$h_1^T \begin{bmatrix} D & AD & \dots & A^{r-2}D & A^{r-1}b \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad (3.11)$$

where $D = \begin{bmatrix} b & c_1 & \dots & c_l \end{bmatrix}$. In order to determine a solution for h_1 , satisfying (3.11) the rank of the generalized controllability matrix for PWA systems must be n for some $r \in \{1, 2, \dots, n\}$ and satisfy (3.12)

$$\text{rank}[C] = \text{rank} \begin{bmatrix} C \\ e \end{bmatrix} = n \quad (3.12)$$

where $C = \begin{bmatrix} D & AD & \dots & A^{r-2}D & A^{r-1}b \end{bmatrix}$ and $e = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}$.

The proof concludes if $z = Tx + m$ is a diffeomorphic transformation. The transformation $z = Tx + m$ is a diffeomorphism if and only if $\text{rank}[T] = n$. The linear

independence of the first r rows $\begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r-1} \end{pmatrix}$ of the transformation matrix can be

determined using the conditions (3.9) and (3.10).

The first r rows of the transformation matrix is linear independent if all the coefficients α_j of (3.13) equals zero for $j = 1, 2, \dots, r$.

$$\alpha_1 h_1^T + \alpha_2 h_1^T A + \dots + \alpha_r h_1^T A^{r-1} = 0 \quad (3.13)$$

Multiplying (3.13) by b gives

$$\alpha_1 h_1^T b + \alpha_2 h_1^T Ab + \dots + \alpha_r h_1^T A^{r-1}b = 0. \quad (3.14)$$

(3.14) with (3.9) and (3.10) implies $\alpha_r = 0$. Substituting $\alpha_r = 0$ in (3.13) yields

$$\alpha_1 h_1^T + \alpha_2 h_1^T A + \alpha_3 h_1^T A^2 + \dots + \alpha_{r-1} h_1^T A^{r-2} = 0 \quad (3.15)$$

multiplying (3.15) by Ab

$$\alpha_1 h_1^T Ab + \dots + \alpha_{r-1} h_1^T A^{r-1}b = 0 \quad (3.16)$$

(3.16) with (3.9) and (3.10) implies $\alpha_{r-1} = 0$. Substituting $\alpha_{r-1} = 0$ in (3.13) yields

$$\alpha_1 h_1^T + \alpha_2 h_1^T A + \alpha_3 h_1^T A^2 + \dots + \alpha_{r-2} h_1^T A^{r-3} = 0 \quad (3.17)$$

multiplying (3.17) by A^2b

$$\alpha_1 h_1^T A^2 b + \dots + \alpha_{r-2} h_1^T A^{r-1} b = 0 \quad (3.18)$$

(3.18) with (3.9) and (3.10) implies $\alpha_{r-2} = 0$. Substituting $\alpha_{r-2} = 0$ in (3.13) and repeating the procedure yields $\alpha_i = 0$ for $i = 1, \dots, r$ which shows the linear independency of the first r rows of the transformation matrix T .

There exist $n - r$ more functions $\psi = [h_{r+1} \ h_{r+2} \ \dots \ h_n]^T x$ which satisfies

$$h_j^T b = 0 \text{ for } j \in \{r+1, r+2, \dots, n\} \quad (3.19)$$

to obtain the normal form (3.2) where the $n - r$ dynamics $\dot{\psi}$ are not a function of u . There also exist h_j for $j \in \{r+1, r+2, \dots, n\}$ which yields a diffeomorphism such that

$$z = \begin{bmatrix} \mu \\ \psi \end{bmatrix} = Tx + m = \begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r-1} \\ h_r^T \\ \vdots \\ h_n^T \end{pmatrix} x + \begin{pmatrix} 0 \\ h_1^T a \\ \vdots \\ h_1^T A^{r-2} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (3.20)$$

The transformation is a diffeomorphism if the transformation matrix

$$T = \begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r-1} \\ h_r^T \\ \vdots \\ h_n^T \end{pmatrix} \quad (3.21)$$

has rank n . One can find $n-r$ vectors h_j for $j \in \{r, r+1, \dots, n\}$ which satisfies the linear system of equation (3.22) including the conditions (3.19), (3.9) and (3.10) obtained by multiplying transformation matrix (3.21) by b .

$$\begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r-1} \\ h_r^T \\ \vdots \\ h_n^T \end{pmatrix} b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.22)$$

The existence of such $n-r$ vectors h_j , which yields a diffeomorphic transformation, is obvious.

Theorem 3.1 also states that h_1^T is the last row of the matrix $C^T (CC^T)^{-1}$. It is obviously seen from the solution of the linear system of equation (3.11) which can be rewritten as

$$h_1^T C = [0 \ 0 \ \dots \ 0 \ 1]. \quad (3.23)$$

There exists a solution for the linear system of equation since it satisfies equation (3.12) and it can be solved by generalized inverse

$$h_1^T = [0 \ 0 \ \dots \ 0 \ 1] C^T (C C^T)^{-1} \quad (3.24)$$

which is the last row of the matrix $C^T (C C^T)^{-1}$. ■

Theorem 3.2: Any PWA system (3.1) in the canonical representation is partially feedback linearizable with relative degree r via the diffeomorphic state transformation $z = Tx + m$ if and only if it can be transformed to the normal form

$$(3.2) \quad \text{i.e.} \quad \text{rank}[C] = \text{rank} \begin{bmatrix} C \\ e \end{bmatrix} = n \quad \text{where} \quad C = [D \ AD \ \dots \ A^{r-2}D \ A^{r-1}b],$$

$$e = [0 \ 0 \ \dots \ 0 \ 1] \text{ and } D = [b \ c_1 \ \dots \ c_l].$$

Proof:

If the PWA system (3.1) can be transformed to the normal form, it takes the form (3.2) with the state transformation (3.20) and (3.1) can be rewritten as

$$\begin{aligned} \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= \mu_3 \\ &\vdots \\ \dot{\mu}_r &= h_1^T A^{r-1} a - h_1^T A^r T^{-1} m + h_1^T A^r T^{-1} \begin{bmatrix} \mu \\ \psi \end{bmatrix} + \sum_{i=1}^l h_1^T A^{r-1} c_i | \alpha_i^T T^{-1} \begin{bmatrix} \mu \\ \psi \end{bmatrix} - \alpha_i^T T^{-1} m - \beta_i | + u \\ \dot{\psi} &= w(\mu, \psi) \end{aligned} \quad (3.25)$$

The static feedback

$$\begin{aligned} u &= -h_1^T A^{r-1} a + h_1^T A^r T^{-1} m - h_1^T A^r T^{-1} \begin{bmatrix} \mu \\ \psi \end{bmatrix} \\ &\quad - \sum_{i=1}^l h_1^T A^{r-1} c_i | \alpha_i^T T^{-1} \begin{bmatrix} \mu \\ \psi \end{bmatrix} - \alpha_i^T T^{-1} m - \beta_i | + v \end{aligned} \quad (3.26)$$

partially linearizes the system (3.25) where v is the new control input. The system (3.25) becomes as

$$\begin{aligned}\dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= \mu_3 \\ &\vdots \\ \dot{\mu}_r &= v \\ \dot{\psi} &= w(\mu, \psi)\end{aligned}\tag{3.27}$$

where the subsystem with the first r states becomes linear. ■

Lemma 3.1: Any PWA system (3.1) in the canonical representation is partially feedback linearizable with the following input

$$u = -h_1^T A^{r-1} a - h_1^T A^r x - \sum_{i=1}^l h_1^T A^{r-1} c_i |\alpha_i^T x - \beta_i| + v\tag{3.28}$$

parameterized in terms of system parameters where h_1^T is the last row of the matrix $C^T (CC^T)^{-1}$, $C = \begin{bmatrix} D & AD & \dots & A^{r-2}D & A^{r-1}b \end{bmatrix}$ and $D = \begin{bmatrix} b & c_1 & \dots & c_l \end{bmatrix}$.

Proof:

It is trivial that the static feedback (3.26) partially linearizes the system (3.25) which is the conjugate transformation of the system (3.1) with the transformation (3.20). Substituting (3.20) in (3.26), the linearizing control input (3.26) can be written in terms of the states $x \in R^n$ as (3.28). The $h_1^T \in R^n$ can be also written in terms of the system parameters and is the last row of the matrix $C^T (CC^T)^{-1}$ as stated in (3.24). ■

3.1.2 Full State Feedback Linearization for Single Input Case

Theorem 3.3 : Any PWA system (3.1) in the canonical representation is full state feedback linearizable with the state transformation $z = Tx + m$ if and only if $c_i \in \text{Range}(b)$ for all $i \in \{1, 2, \dots, l\}$ and the controllability matrix $C^{n \times n} = [b \quad Ab \quad \dots \quad A^{n-1}b]$ has rank n .

Proof:

It is assumed that $c_i \in \text{Range}(b)$ where $\text{Range}(b)$ denotes the range of b , i.e. $\text{Range}(b) = \{y \mid \text{there exists at least one } x \in R \text{ such that } y = bx\}$. Therefore, $c_i = f_i \cdot b$ for all $i \in \{1, 2, \dots, l\}$ where $f_i \in R$. The system (3.1) can be rewritten as

$$\dot{x} = a + Ax + b \left[\sum_{i=1}^l f_i |\alpha_i^T x - \beta_i| + u \right] \quad (3.29)$$

If the controllability matrix $C^{n \times n} = [b \quad Ab \quad \dots \quad A^{n-1}b]$ has rank n , hence it is invertible, any system in the form of (3.29) can be transformed into the form (3.31) via the affine transformation

$$z = Tx + m = \begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{n-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ h_1^T a \\ \vdots \\ h_1^T A^{n-2} a \end{pmatrix} \quad (3.30)$$

where h_1 is the n th row of $(C^{n \times n})^{-1}$.

$$\dot{z} = b_c (h_1^T A^{n-1} a - b_c^T A_c m) + A_c z + b_c \left[\sum_{i=1}^l f_i |\alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i| + u \right] \quad (3.31)$$

where $A_c \in R^{n \times n}$ is a constant matrix and $b_c \in R^n$ is a constant vector. A_c and b_c are in the following controllable canonical form:

$$A_c = TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & \cdots & a_{n-1} \end{bmatrix}, \quad b_c = Tb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (3.32)$$

with $[a_0 \ a_1 \ \cdots \ a_{n-1}] = h_1^T A^n T^{-1}$.

The feedback law as

$$u = -h_1^T A^{n-1} a + h_1^T A^n T^{-1} m - h_1^T A^n T^{-1} z - \sum_{i=1}^l f_i |\alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i| + v$$

which can be written in terms of the states x as

$$u = -h_1^T A^{n-1} a - h_1^T A^n x - \sum_{i=1}^l f_i |\alpha_i^T x - \beta_i| + v \quad (3.33)$$

linearizes the system as follows.

$$\dot{z} = A_c z + b_c v \quad (3.34)$$

The proof concludes with the following theorem which reveals the equivalence of Theorem 3.2 and Theorem 3.3 for full state linearization.

Theorem 3.4: The conditions in Theorem 3.2 and the conditions in Theorem 3.3 are equivalent for relative degree n i.e.

$$\text{rank}[C] = \text{rank} \begin{bmatrix} C \\ e \end{bmatrix} = n \Leftrightarrow c_i \in \text{Range}(b) \text{ and } \text{rank}[C^{n \times n}] = n$$

where $C = \begin{bmatrix} D & AD & \dots & A^{n-2}D & A^{n-1}b \end{bmatrix}$, $e = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} b & c_1 & \dots & c_l \end{bmatrix}$ and $C^{n \times n} = \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}$.

Proof:

For full state feedback linearization, Theorem 3.3 introduces $c_i \in \text{Range}(b)$ and the controllability matrix $C^{n \times n} = \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}$ has rank n . The $\text{rank} \begin{bmatrix} D & AD & \dots & A^{n-2}D & A^{n-1}b \end{bmatrix}$ is equal to the $\text{rank} \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}$, since $c_i \in \text{Range}(b)$ and $D = \begin{bmatrix} b & c_1 & \dots & c_l \end{bmatrix}$ has rank 1 with linearly dependent columns which also gives linearly dependent columns for each $A^j D$ $j \in \{1, 2, \dots, n-1\}$. Moreover, eliminating the linearly dependent columns of the matrix $\begin{bmatrix} C \\ e \end{bmatrix}$, $\text{rank} \begin{bmatrix} C \\ e \end{bmatrix}$ is also equal to $\text{rank} \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}$. Therefore, the conditions $c_i \in \text{Range}(b)$ and $a \in \text{Range}(b)$ with $\text{rank} \begin{bmatrix} C^{n \times n} \end{bmatrix} = n$ implies

$$\text{rank} \begin{bmatrix} D & AD & \dots & A^{n-2}D & A^{n-1}b \end{bmatrix} = \text{rank} \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ e \end{bmatrix} = n \quad \text{i.e.}$$

Theorem 3.3 implies Theorem 3.2.

Theorem 3.2 introduces the following condition for full state linearization i.e. relative degree n .

$$\text{rank} \begin{bmatrix} C \\ e \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ e \end{bmatrix} = n \quad (3.35)$$

The condition (3.35) is equivalent to the conditions (3.11) which can be rearranged as follows.

$$\begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{n-1} \end{pmatrix} D = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \\ 1 & h_1^T A^{n-1} c_1 & \cdots & h_1^T A^{n-1} c_l \end{pmatrix} \quad (3.36)$$

It is well known that the rank of the multiplication of two matrices is equal to the rank of the second matrix if the first matrix has full column rank i.e. $\text{rank}(MN) = \text{rank } N$ if the matrix M has full column rank. Then (3.36) implies that $\text{rank} D = \text{rank} [b \ c_1 \ \cdots \ c_l] = 1$. Furthermore; $c_i \in \text{Range}(b)$ and $a \in \text{Range}(b)$, since $\text{rank} [b \ c_1 \ \cdots \ c_l] = 1$. Eliminating the linearly dependent columns of the matrix C , $\text{rank} [D \ AD \ \cdots \ A^{n-2}D \ A^{n-1}b]$ is also equal to $\text{rank} [b \ Ab \ \cdots \ A^{n-1}b]$. Theorem 3.2 also implies Theorem 3.3 which derives the equivalence of the theorems for relative degree n . ■

Example 3.1:

Chua's circuit is a simple electronic circuit as shown in Figure 3.1 that exhibits complex nonlinear dynamics such as chaos. The circuit has a nonlinear (piecewise linear) resistor which results a piecewise linear system in the state space form of

$$\begin{aligned} \dot{V}_{c1} &= C_1^{-1} [R^{-1}(V_{c2} - V_{c1}) - \sigma(V_{c1})] \\ \dot{V}_{c2} &= C_2^{-1} [R^{-1}(V_{c2} - V_{c1}) + I_L] \\ \dot{I}_L &= -L^{-1}V_{c2} \end{aligned} \quad (3.37)$$

where $\sigma(V_{c1})$ is a piecewise linear function describing electrical response of the nonlinear resistor.

The control input is added to the first state by using a current source as shown in Figure 3.1.

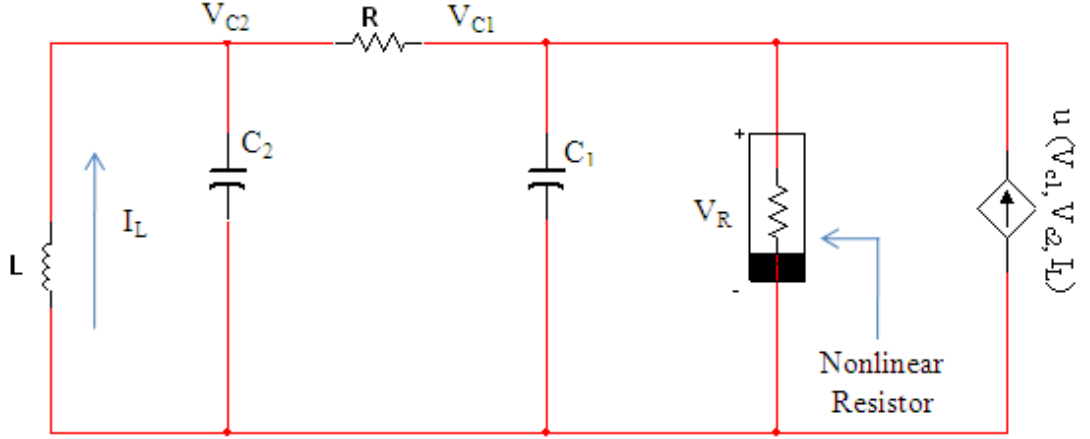


Figure 3.1 Chua's Circuit with a dependent current source parallel to the nonlinear resistor

The system in (3.37) with control input can be written in the dimensionless form (Bilotto & Pantano, 2008) as

$$\begin{aligned}\dot{x}_1 &= \alpha [x_2 - x_1 - \sigma(x_1) + u] \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta x_2 - \gamma x_3\end{aligned}\tag{3.38}$$

where $\alpha = \frac{C_2}{C_1}$, $\beta = \frac{R^2 C_2}{L}$, $\gamma = \frac{r_0 R C_2}{L}$ and the piecewise linear function

$$\sigma(x_1) = m_1 x_1 + \frac{1}{2}(m_0 - m_1)[|x_1 + 1| - |x_1 - 1|].\tag{3.39}$$

For simulation the parameters are fixed as $\alpha = 9$, $\beta = 100/7$, $\gamma = 0.016$, $m_0 = -8/7$ and $m_1 = -5/7$ with regard to previous works (Bowong & Kagou, 2006). The system is a piecewise linear system and can be written in the canonical form as follows

$$\dot{x} = a + Ax + \sum_{i=1}^2 c_i |\alpha_i^T x - \beta_i| + bu\tag{3.40}$$

with the parameters $A = \begin{bmatrix} -\frac{2}{7}\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}$, $c_1 = \begin{bmatrix} 3\alpha/14 \\ 0 \\ 0 \end{bmatrix}$, $c_2 = \begin{bmatrix} -3\alpha/14 \\ 0 \\ 0 \end{bmatrix}$,

$$\alpha_1 = \alpha_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \beta_1 = -1, \beta_2 = 1, b = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}, a = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The PWA system (3.40) in the canonical representation is full state feedback linearizable, since it satisfies Theorem 3.3 such that $c_1 \in R(b)$, $c_2 \in R(b)$ and the

controllability matrix $C^{n \times n} = [b \quad Ab \quad A^2b] = \begin{bmatrix} \alpha & -\frac{2\alpha^2}{7} & \alpha\left(\frac{2\alpha}{7} + \alpha\right)^2 \\ 0 & \alpha & -\frac{2\alpha^2}{7} - \alpha \\ 0 & 0 & -\alpha\beta \end{bmatrix}$ has rank 3.

It also satisfies Theorem 3.2 which is equivalent to Theorem 3.2 for relative degree $r = 3$ such that

$$\text{rank}[C] = \text{rank} \begin{bmatrix} C \\ e \end{bmatrix} = 3$$

where

$$C = [D \quad AD \quad A^2b] = \begin{bmatrix} 9 & 1.9286 & -1.9286 & -23.1429 & -4.9592 & 4.9592 & 140.5102 \\ 0 & 0 & 0 & 9 & 1.9286 & -1.9286 & -32.1429 \\ 0 & 0 & 0 & 0 & 0 & 0 & -128.5714 \end{bmatrix},$$

$$e = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1] \text{ with } D = [b \quad c_1 \quad \dots \quad c_l].$$

The system (3.40) can be transformed into the normal form (3.42) by an affine transform as:

$$z = Tx + m = \begin{pmatrix} h_1^T \\ h_1^T A \\ h_1^T A^2 \end{pmatrix} x + \begin{pmatrix} 0 \\ h_1^T a \\ h_1^T Aa \end{pmatrix} = \begin{pmatrix} 0 & 0 & -0.078 \\ 0 & 0.1111 & 0.0001 \\ 0.1111 & -0.1129 & 0.1111 \end{pmatrix} x \quad (3.41)$$

where h_1 is the n th row of $(C^{n \times n})^{-1}$ such that $h_1 = [0 \ 0 \ -0.078]^T$.

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= -0.3986x_1 - 0.4744x_2 - 0.1147x_3 - \sum_{i=1}^l f_i |\alpha_i^T x - \beta_i| + u\end{aligned}\tag{3.42}$$

It is obvious that $c_1 = f_1 \cdot b$ and $c_2 = f_2 \cdot b$ with $f_1 = 3/14$ and $f_2 = -3/14$. The feedback law (3.43)

$$\begin{aligned}u &= -h_1^T A^2 a - h_1^T A^3 x - \sum_{i=1}^2 f_i |\alpha_i^T x - \beta_i| + v \\ &= 0.3986x_1 + 0.4744x_2 + 0.1147x_3 - \frac{3}{14}|x+1| + \frac{3}{14}|x-1| + v\end{aligned}\tag{3.43}$$

linearizes the system (3.42) as follows.

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= v\end{aligned}\tag{3.44}$$

Then, one can choose a linear controller to stabilize the system (3.44) or to track a desired trajectory. A simple proportional controller is chosen to stabilize the system as $v = -z_1 - 3z_2 - 3z_3 = -3x_1 + 0.048x_2 - 2.9333x_3$.

The linearizing control law for the system (3.40) can be directly obtained by (3.28) for relative degree $r = 3$ as follows.

$$\begin{aligned}
u &= -h_1^T A^2 a - h_1^T A^3 x - \sum_{i=1}^2 h_1^T A^2 c_i |\alpha_i^T x - \beta_i| + v \\
&= 0.3986x_1 + 0.4744x_2 + 0.1147x_3 - \frac{3}{14}|x+1| + \frac{3}{14}|x-1| + v
\end{aligned} \tag{3.45}$$

parameterized in terms of system parameters where h_1^T is the last row of the matrix $C^T (CC^T)^{-1}$ such that $h_1 = [0 \quad 0 \quad -0.078]^T$.

As seen in Figure 3.2 the control input (3.45) and a linear controller stabilizes the system. In Figure 3.3 the states of the linearized system (3.44) is shown. The control input as a function of time is presented in Figure 3.4. For the simulation initial states are chosen as $x_0 = [0.5 \quad 0.1 \quad -0.5]^T$.

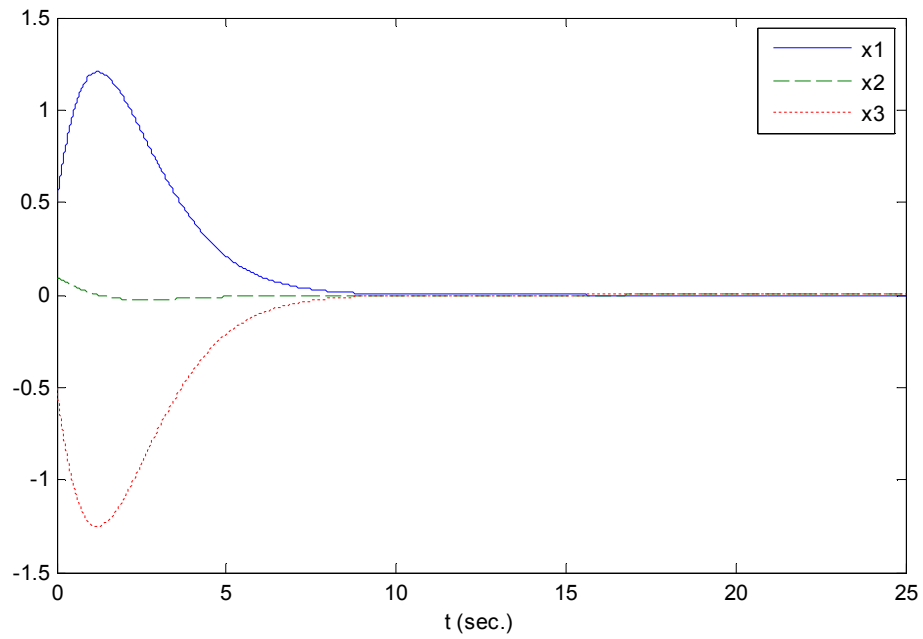


Figure 3.2 x_1 , x_2 and x_3 versus time of the controlled system (3.40)

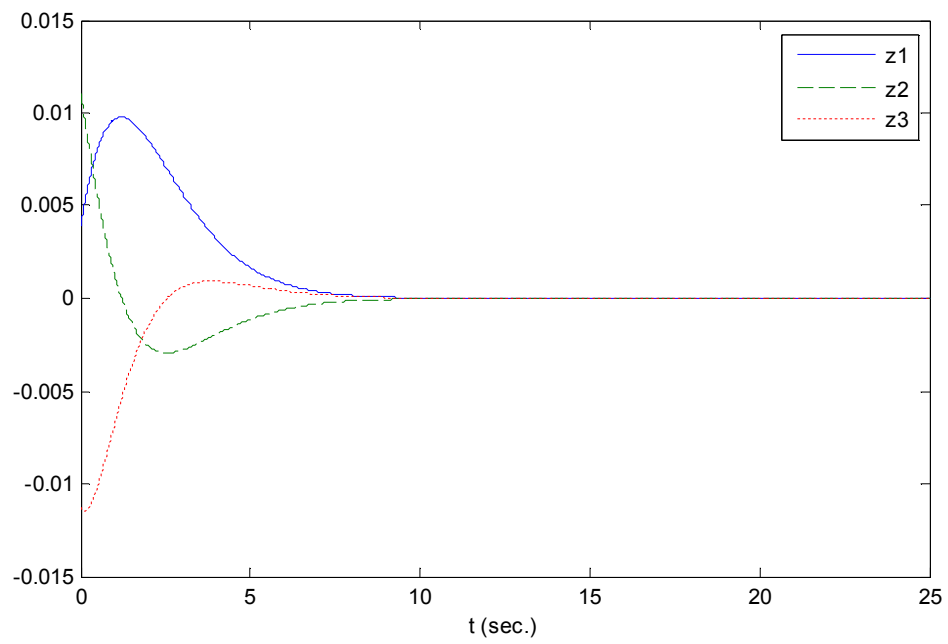


Figure 3.3 z_1 , z_2 and z_3 versus time of the (3.44)

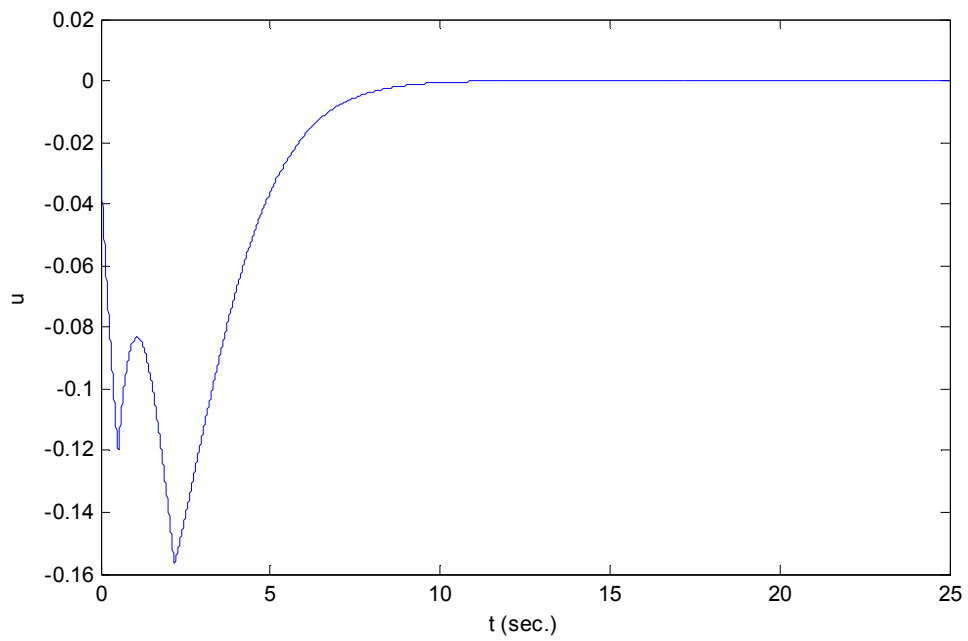


Figure 3.4 Control input (3.45) with a linear controller which stabilizes the system (3.40)

Example 3.2:

The Chua's circuit with control input added to the third state by using a voltage source as shown in Figure 3.5. The system with control input can be written in the dimensionless form (Bilotto & Pantano, 2008) as

$$\begin{aligned}\dot{x}_1 &= \alpha [x_2 - x_1 - \sigma(x_1)] \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta x_2 - \gamma x_3 + u\end{aligned}\tag{3.46}$$

with the piecewise linear function

$$\sigma(x_1) = m_1 x_1 + \frac{1}{2}(m_0 - m_1)[|x_1 + 1| - |x_1 - 1|].\tag{3.47}$$

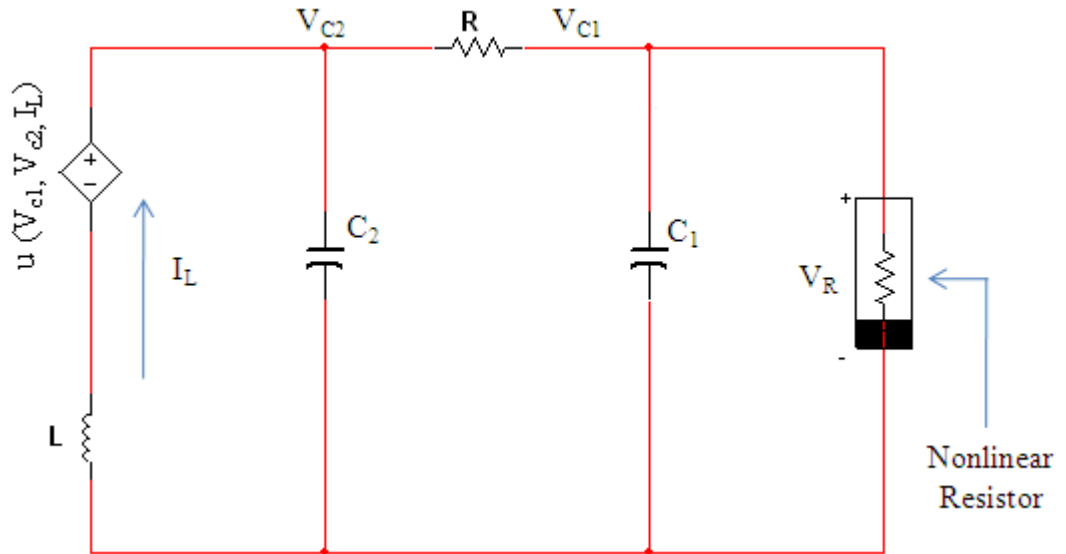


Figure 3.5 Chua's Circuit with a dependent voltage source series to the inductor

For simulation the parameters are fixed as $\alpha = 9$, $\beta = 100/7$, $\gamma = 0.016$, $m_0 = -8/7$ and $m_1 = -5/7$ with regard to previous works (Bowong & Kagou, 2006). The system is a piecewise linear system and can be written in the canonical form as follows

$$\dot{x} = a + Ax + \sum_{i=1}^2 c_i |\alpha_i^T x - \beta_i| + bu \quad (3.48)$$

$$\text{with the parameters } A = \begin{bmatrix} -2/7\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}, \quad c_1 = \begin{bmatrix} 3\alpha/14 \\ 0 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -3\alpha/14 \\ 0 \\ 0 \end{bmatrix},$$

$$\alpha_1 = \alpha_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \beta_1 = -1, \quad \beta_2 = 1, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad a = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (3.49)$$

The PWA system (3.48) in the canonical representation has relative degree 2, since it satisfies Theorem 3.2 for $r = 2$ such that

$$\text{rank}[C] = \text{rank} \begin{bmatrix} C \\ e \end{bmatrix} = 3$$

$$\text{with } C = [D \quad Ab] = \begin{bmatrix} 0 & 3\alpha/14 & -3\alpha/14 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -\gamma \end{bmatrix} = \begin{bmatrix} 0 & 1.9286 & -1.9286 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -0.016 \end{bmatrix},$$

$$e = [0 \quad 0 \quad 0 \quad 1] \text{ and } D = [b \quad c_1 \quad \dots \quad c_l].$$

The system (3.48) can be transformed into the normal form (3.51) by an affine transform as:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} h_1^T \\ h_1^T A \\ h_3 \end{pmatrix} x + \begin{pmatrix} 0 \\ h_1^T a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} x \quad (3.50)$$

where h_1 is the last row of the matrix $C^T (CC^T)^{-1}$ such that $h_1 = [0 \quad 1 \quad 0]^T$ and h_3 satisfies (3.19) such that $L_g z_3 = 0 \Rightarrow h_3^T g = 0$. In order to obtain a diffeomorphic state transformation, it can be chosen that $h_3 = [1 \quad 0 \quad 0]$.

$$\begin{aligned}\dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= a(\mu, \psi) + u\end{aligned}\tag{3.51}$$

$$\dot{\psi}_1 = \alpha\mu_1 - \frac{2}{7}\alpha\psi_1 + \frac{3\alpha}{14}|\psi_1 + 1| - \frac{3\alpha}{14}|\psi_1 - 1|$$

$$\text{where } a(\mu, \psi) = (\alpha - \beta - \gamma)\mu_1 - (\gamma + 1)\mu_2 + \left(\gamma - \frac{2}{7}\alpha\right)\psi_1 + \frac{3\alpha}{14}|\psi_1 + 1| - \frac{3\alpha}{14}|\psi_1 - 1|$$

It is obvious that the feedback law (3.52)

$$\begin{aligned}u &= -(\alpha - \beta - \gamma)\mu_1 + (\gamma + 1)\mu_2 - \left(\gamma - \frac{2}{7}\alpha\right)\psi_1 - \frac{3\alpha}{14}|\psi_1 + 1| + \frac{3\alpha}{14}|\psi_1 - 1| + v \\ &= 5.3017\mu_1 + 1.016\mu_2 + 2.5554\psi_1 - 1.9286|\psi_1 + 1| + 1.9286|\psi_1 - 1| + v\end{aligned}\tag{3.52}$$

partially linearizes the system (3.48) as follows.

$$\begin{aligned}\dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= v\end{aligned}\tag{3.53}$$

$$\dot{\psi}_1 = \alpha\mu_1 - \frac{2}{7}\alpha\psi_1 + \frac{3}{14}|\psi_1 + 1| - \frac{3}{14}|\psi_1 - 1|$$

The linearizing control law for the system (3.48) can be directly obtained by (3.28) for relative degree $r = 2$ as follows

$$\begin{aligned}u &= -h_1^T Aa - h_1^T A^2x - \sum_{i=1}^2 h_1^T Ac_i |\alpha_i^T x - \beta_i| + v \\ &= 3.5714x_1 + 4.2857x_2 + 1.016x_3 - 1.9286|x + 1| + 1.9286|x - 1| + v\end{aligned}\tag{3.54}$$

which is parameterized in terms of system parameters where h_1^T is the last row of the matrix $C^T (CC^T)^{-1}$ such that $h_1 = [0 \ 1 \ 0]^T$.

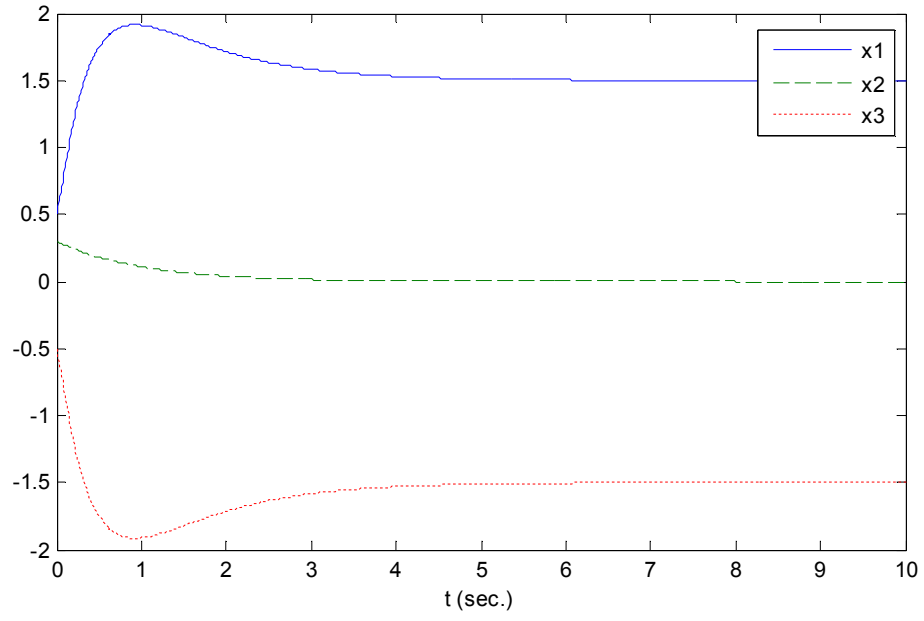


Figure 3.6 x_1 , x_2 and x_3 versus time of the controlled system (3.48) for initial states $x_0 = [0.5 \ 0.3 \ -0.5]^T$

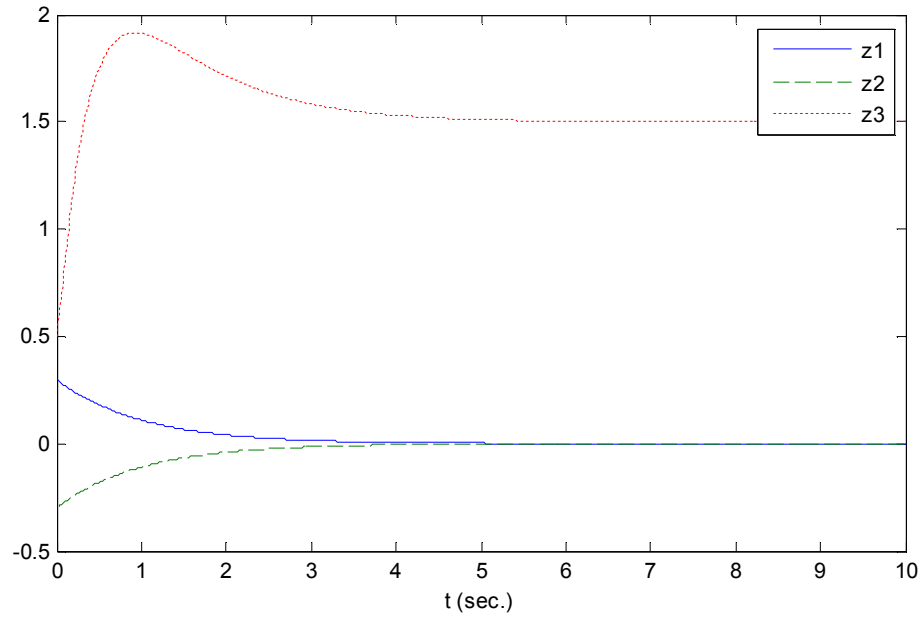


Figure 3.7 z_1 , z_2 and z_3 versus time of the (3.51) for initial states $x_0 = [0.5 \ 0.3 \ -0.5]^T$

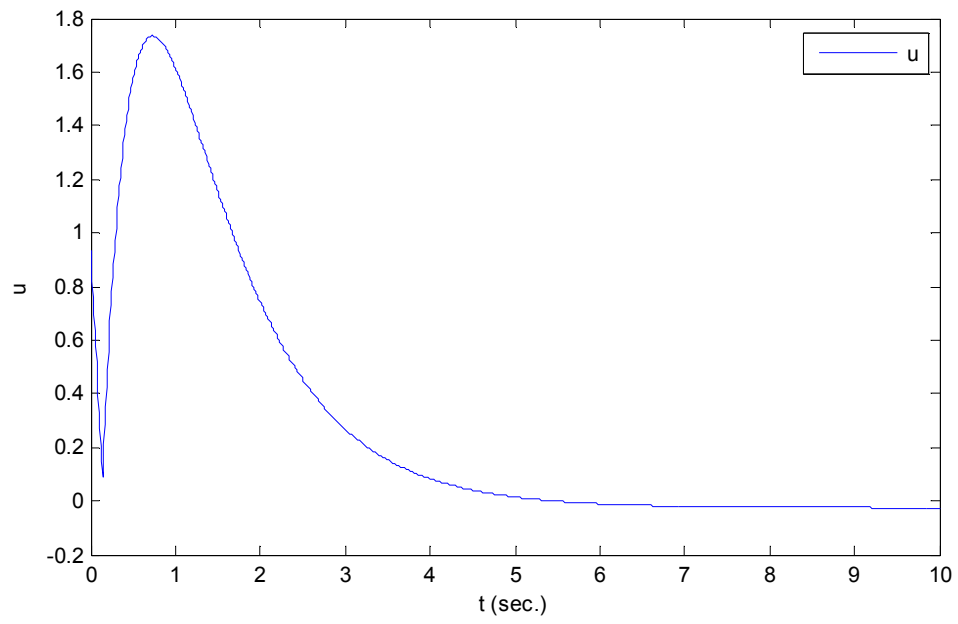


Figure 3.8 Control input (3.54) with a linear controller which stabilizes the system (3.48) for initial states $x_0 = [0.5 \ 0.3 \ -0.5]^T$

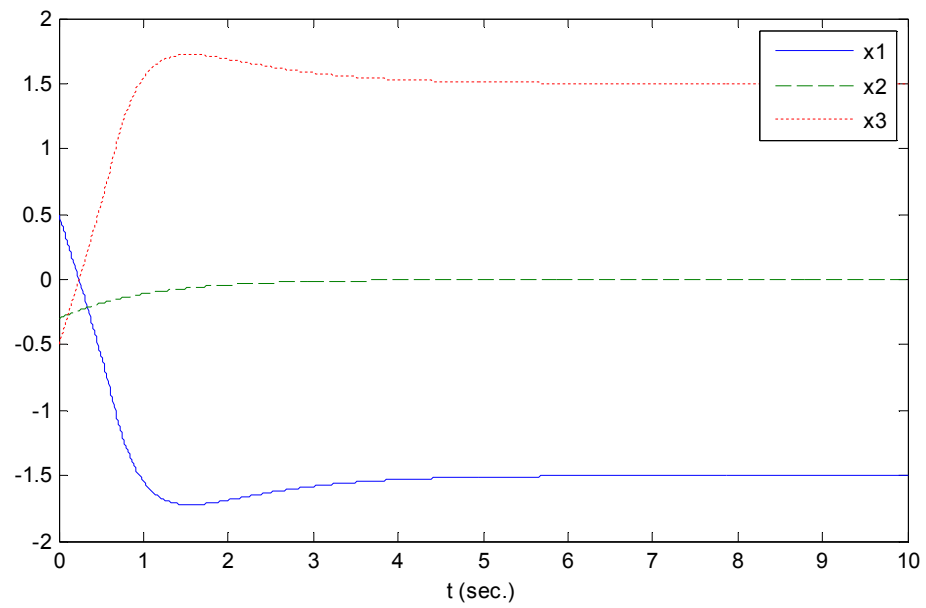


Figure 3.9 x_1 , x_2 and x_3 versus time of the controlled system (3.48) for initial states $x_0 = [0.5 \ -0.3 \ -0.5]^T$

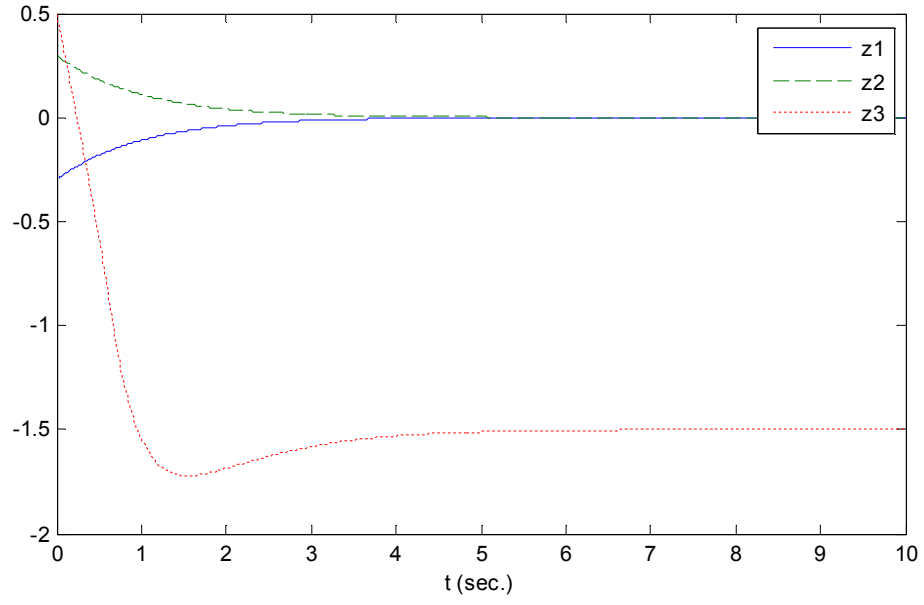


Figure 3.10 z_1 , z_2 and z_3 versus time of the (3.51) for initial states $x_0 = [0.5 \ -0.3 \ -0.5]^T$

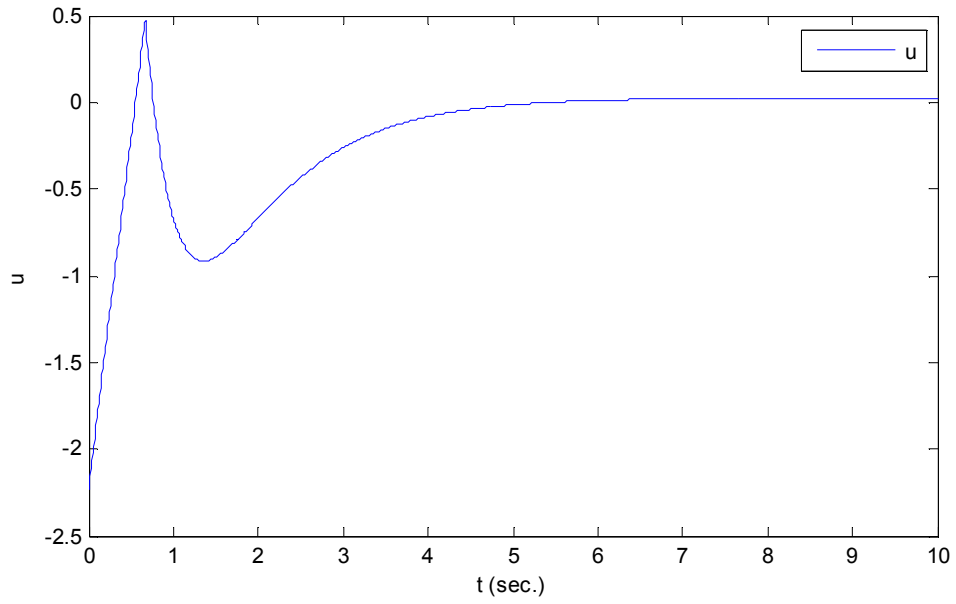


Figure 3.11 Control input (3.54) with a linear controller which stabilizes the system (3.48) for initial states $x_0 = [0.5 \ -0.3 \ -0.5]^T$

The linear controller $v = -z_1 - 2z_2$ stabilizes the first two states of (3.53) which yields zero dynamics such that

$$\dot{\psi}_1 = -\frac{2}{7}\alpha\psi_1 + \frac{3}{14}|\psi_1 + 1| - \frac{3}{14}|\psi_1 - 1|.$$

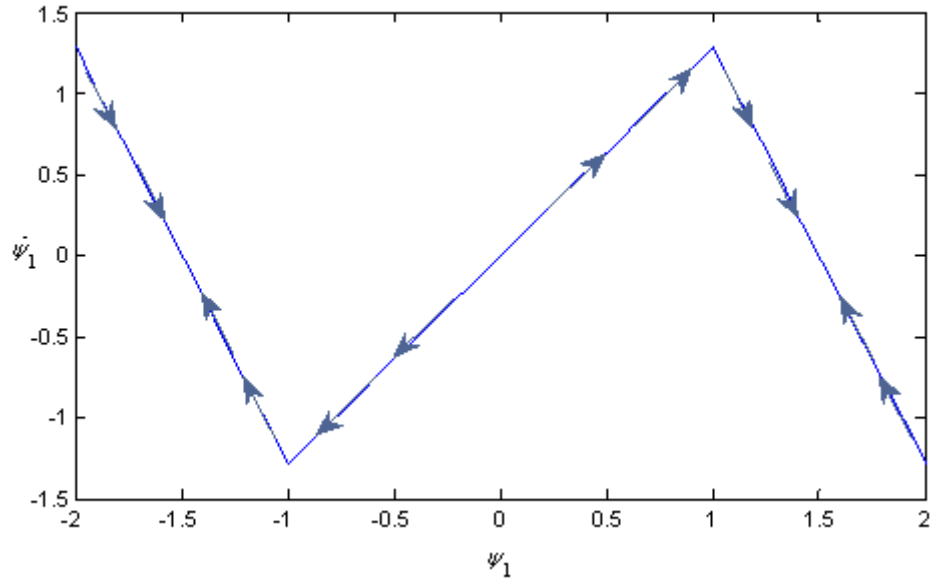


Figure 3.12 Phase portrait of the zero dynamic of (3.51)

As seen in Figure 3.12, phase portrait shows that the zero dynamics results either $\psi_1 = 1.5$ or $\psi_1 = -1.5$. The behavior of the internal state ψ_1 for $x_0 = [0.5 \ 0.3 \ -0.5]^T$ is presented in Figure 3.7 and the behavior of the internal state ψ_1 for $x_0 = [0.5 \ -0.3 \ -0.5]^T$ is presented in Figure 3.10. As seen in Figure 3.6 and Figure 3.9, the system states respectively converge to either $x = [1.5 \ 0 \ -1.5]^T$ or $x = [-1.5 \ 0 \ 1.5]^T$ for different initial states. The control input as a function of time is presented in Figure 3.8 and Figure 3.11 for different initial states.

3.1.3 Feedback Linearization of PWA systems with Relative Degree r for Multiple Input Case

An n -dimensional multiple input system is considered in the following form

$$\dot{x} = a + Ax + \sum_{i=1}^l c_i \left| \alpha_i^T x - \beta_i \right| + Bu \quad (3.55)$$

where $a \in R^n$, $A \in R^{n \times n}$, $c_i \in R^n$, $\alpha_i \in R^n$, $\beta_i \in R$, $B \in R^{n \times m}$, $u \in R^m$ is the control input and $x \in R^n$ is the state. It is assumed that $\text{rank}[B] = m$.

Theorem 3.5: Any PWA system (3.55) in the canonical representation can be transformed into the normal form as

$$\begin{bmatrix} \dot{z}_1^1 \\ \dot{z}_2^1 \\ \vdots \\ \dot{z}_{r_1}^1 \\ \dot{z}_1^2 \\ \dot{z}_2^2 \\ \vdots \\ \dot{z}_{r_2}^2 \\ \vdots \\ \dot{z}_1^m \\ \dot{z}_2^m \\ \vdots \\ \dot{z}_{r_m}^m \\ \dot{z}_{r_T+1} \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2^1 \\ z_3^1 \\ \vdots \\ \rho_1(z) + q_1^T u \\ z_2^2 \\ z_3^2 \\ \vdots \\ \rho_2(z) + q_2^T u \\ \vdots \\ z_2^k \\ z_3^k \\ \vdots \\ \rho_m(z) + q_m^T u \\ w_1(z) + q_{r_T+1}^T u \\ \vdots \\ w_{n-r_T}(z) + q_n^T u \end{bmatrix} \quad (3.56)$$

with

$$\text{rank} \begin{pmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_m^T \end{pmatrix} = m \quad (3.57)$$

via the diffeomorphic state transformation

$$\begin{aligned}
z = \begin{bmatrix} z_1^1 \\ z_2^1 \\ \vdots \\ z_{r_1}^1 \\ z_1^2 \\ z_2^2 \\ \vdots \\ z_{r_2}^2 \\ \vdots \\ z_1^m \\ z_2^m \\ \vdots \\ z_{r_m}^m \\ z_{r_T+1} \\ \vdots \\ z_n \end{bmatrix} &= Tx + m = \begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r_1-1} \\ h_2^T \\ h_2^T A \\ \vdots \\ h_2^T A^{r_2-1} \\ \vdots \\ h_m^T \\ h_m^T A \\ \vdots \\ h_m^T A^{r_m-1} \\ h_{r_T+1} \\ \vdots \\ h_n \end{pmatrix} x + \begin{pmatrix} 0 \\ h_1^T a \\ \vdots \\ h_1^T A^{r_1-2} a \\ 0 \\ h_2^T a \\ \vdots \\ h_2^T A^{r_2-2} a \\ \vdots \\ 0 \\ h_m^T a \\ \vdots \\ h_m^T A^{r_m-2} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\end{aligned} \tag{3.58}$$

with $h_j^T = e_j$ for $j \in \{1, 2, \dots, m\}$ if and only if there exists an $m \times m$ matrix

$$E = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix} \text{ having rank } m \text{ with } e_i \in \{p_1^j A^{r_j-1} B, p_2^j A^{r_j-1} B, \dots, p_{k_j}^j A^{r_j-1} B\} \text{ for some } r_j \text{ s}$$

maximizing $r_T = r_1 + r_2 + \dots + r_m \leq n$ where $\text{span}(p_1^j, p_2^j, \dots, p_{k_j}^j)$ is the null space of

$$C_j^T, \quad C_j = \begin{bmatrix} D & AD & \dots & A^{r_j-2} D & A^{r_j-1} b \end{bmatrix}, D = \begin{bmatrix} B & c_1 & \dots & c_l \end{bmatrix},$$

$$\rho_j(z) = h_j^T A^{r_j-1} a - h_j^T A^{r_j} T^{-1} m + h_j^T A^{r_j} T^{-1} z + \sum_{i=1}^l h_j^T A^{r_j-1} c_i | \alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i | \quad \text{and}$$

$$q_j = h_j^T A^{r_j-1} B \text{ for } j \in \{1, 2, \dots, m\}.$$

Proof:

The transformation of the first state $z_1^j = h_j^T x$ in (3.58) yields the following state equation.

$$\dot{z}_1^j = h_j^T \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| + Bu \right). \quad (3.59)$$

(3.56) states that $z_2^j = \dot{z}_1^j$; therefore, to obtain an affine transformation as in (3.58) it must be satisfied that $h_j^T B = 0$ and $h_j^T c_i = 0$ for $i \in \{1, 2, \dots, l\}$. Then the transformation for the second state becomes $z_2^j = h_j^T Ax + h_j^T a$ and this transformation yields the following state equation.

$$\dot{z}_2^j = h_j^T A \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| + Bu \right). \quad (3.60)$$

(3.56) states that $z_3^j = \dot{z}_2^j$; therefore, to obtain a affine transformation as in (3.58) it must be satisfied that $h_j^T AB = 0$ and $h_j^T Ac_i = 0$ for $i \in \{1, 2, \dots, l\}$. Then the transformation for the third state becomes $z_3^j = h_j^T A^2 x + h_j^T Aa$. Repeating the procedure yields

$$\dot{z}_{r_j-1} = z_{r_j} = h_j^T A^{r_j-2} \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| + Bu \right) = h_j^T A^{r_j-1} x + h_j^T A^{r_j-2} a \quad (3.61)$$

with $h_j^T A^{r_j-2} B = 0$ and $h_j^T A^{r_j-2} c_i = 0$ for $i \in \{1, 2, \dots, l\}$. The r th equation of (3.56) takes the following form

$$\dot{z}_r^j = h_j^T A^{r_j-1} \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| \right) + h_j^T A^{r_j-1} Bu \quad (3.62)$$

with

$$h_j^T A^{r_j-1} B \neq 0 \text{ for } j \in \{1, 2, \dots, m\}. \quad (3.63)$$

The normal form (3.56) should satisfy (3.57) which also implies the condition (3.63).

Substantially, for a PWA system (3.55) to take the normal form (3.56) it must satisfy the conditions

$$h_j^T A^{k_j} c_i = 0 \text{ for all } k_j \in \{1, 2, \dots, r_j - 2\}, i \in \{1, 2, \dots, l\} \text{ and } j \in \{1, 2, \dots, m\} \quad (3.64)$$

$$h_j^T A^{k_j} B = 0 \text{ for all } k_j \in \{1, 2, \dots, r_j - 2\} \text{ and } j \in \{1, 2, \dots, m\} \quad (3.65)$$

$$\text{rank} \begin{pmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_m^T \end{pmatrix} = \text{rank} \begin{pmatrix} h_1^T A^{r_1-1} B \\ h_2^T A^{r_2-1} B \\ \vdots \\ h_m^T A^{r_m-1} B \end{pmatrix} = m \quad (3.66)$$

These conditions yields the normal form (3.56) with (3.57) where the first r_T transformation becomes as

$$\begin{pmatrix} z_1^1 \\ z_2^1 \\ \vdots \\ z_{r_1}^1 \\ z_1^2 \\ z_2^2 \\ \vdots \\ z_{r_2}^2 \\ \vdots \\ z_1^m \\ z_2^m \\ \vdots \\ z_{r_m}^m \end{pmatrix} = \begin{pmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r_1-1} \\ h_2^T \\ h_2^T A \\ \vdots \\ h_2^T A^{r_2-1} \\ \vdots \\ h_m^T \\ h_m^T A \\ \vdots \\ h_m^T A^{r_m-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ h_1^T a \\ \vdots \\ h_1^T A^{r_1-2} a \\ 0 \\ h_1^T a \\ \vdots \\ h_2^T A^{r_2-2} a \\ \vdots \\ 0 \\ h_m^T a \\ \vdots \\ h_m^T A^{r_m-2} a \end{pmatrix}. \quad (3.67)$$

The proof concludes if $z = Tx + m$ is a diffeomorphic transformation. The transformation $z = Tx + m$ is a diffeomorphism if and only if $\text{rank}[T] = n$. The linear independence of first r_T rows of the transformation matrix T can be determined using the conditions (3.65) and (3.66).

The first r_T rows of the transformation matrix T is linear independent if (3.68) implies all the coefficients α_{ij} of (3.68) equals zero.

$$\sum_{i=1}^k \sum_{j=1}^{r_j} \alpha_{ij} h_i^T A^{j-1} = 0 \quad (3.68)$$

Multiplying (3.68) by B with (3.65) gives

$$\alpha_{1r_1} h_1^T A^{r_1-1} B + \alpha_{2r_2} h_2^T A^{r_2-1} B + \dots + \alpha_{kr_k} h_k^T A^{r_k-1} B = 0. \quad (3.69)$$

(3.69) with (3.66) implies $\alpha_{1r_k} = 0$ for $k \in \{1, 2, \dots, m\}$. Substituting $\alpha_{1r_k} = 0$ for $k \in \{1, 2, \dots, m\}$ in (3.68) yields

$$\sum_{i=1}^k \sum_{j=1}^{r_j-1} \alpha_{ij} h_i^T A^{j-1} = 0 \quad (3.70)$$

multiplying (3.70) by AB with (3.65) gives

$$\alpha_{1(r_1-1)} h_1^T A^{r_1-1} B + \alpha_{2(r_2-1)} h_2^T A^{r_2-1} B + \dots + \alpha_{k(r_k-1)} h_k^T A^{r_k-1} B = 0. \quad (3.71)$$

(3.71) with (3.66) implies $\alpha_{1r_k-1} = 0$ for $k \in \{1, 2, \dots, m\}$. Substituting $\alpha_{1r_k-1} = 0$ for $k \in \{1, 2, \dots, m\}$ in (3.70) and repeating the procedure one can see that all the α_{ij} s

become zero which shows the linear independency of first r_T rows of the transformation matrix T .

One can find $n - r_T$ vectors h_j s forming the last $n - r_T$ rows of the transformation matrix T for $j \in \{r_T + 1, r_T + 2, \dots, n\}$ which satisfies $\text{rank}[T] = n$.

The conditions (3.64) and (3.65) can be rewritten as a linear system of equation such that

$$h_j^T \begin{bmatrix} D & AD & \dots & A^{r_j-2}D \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix} \quad \text{for } j \in \{1, 2, \dots, m\} \quad (3.72)$$

where $D = \begin{bmatrix} B & c_1 & \dots & c_l \end{bmatrix}$. In order to determine the solution for h_j s satisfying (3.72) and (3.66), the null spaces of $\begin{bmatrix} D & AD & \dots & A^{r_j-2}D \end{bmatrix}^T$ should satisfy (3.66) for $j \in \{1, 2, \dots, m\}$. Assume that the null space of $\begin{bmatrix} D & AD & \dots & A^{r_j-2}D \end{bmatrix}^T$ is $\text{span}(p_1^j, p_2^j, \dots, p_{k_j}^j)$. The condition (3.66) is satisfied if

$$\alpha_1 h_1^T A^{r_1-1} B + \alpha_2 h_2^T A^{r_2-1} B + \dots + \alpha_m h_m^T A^{r_m-1} B = 0 \quad (3.73)$$

implies

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_m \end{bmatrix} = 0. \quad (3.74)$$

Solutions for h_j s also lie in the null space of $\begin{bmatrix} D & AD & \dots & A^{r_j-2}D \end{bmatrix}^T$. Then (3.73) can be rewritten as

$$\begin{aligned} & \alpha_1 \left[c_1^1 (p_1^1)^T A^{r_1-1} B + c_2^1 (p_2^1)^T A^{r_2-1} B + c_{k_1}^1 (p_{k_1}^1)^T A^{r_1-1} B \right] + \dots + \alpha_m \left[c_1^m (p_1^m)^T A^{r_m-1} B + \dots \right. \\ & \left. + c_{k_m}^m (p_{k_m}^m)^T A^{r_m-1} B \right] = 0. \end{aligned} \quad (3.75)$$

With Lemma 3.2, it is easily seen that (3.75) implies (3.74) if and only if

$$\text{rank}[E] = m \quad (3.76)$$

$$\text{with } E = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix} \text{ and } e_i \in \{p_1^j A^{r_j-1} B, p_2^j A^{r_j-1} B, \dots, p_{k_j}^j A^{r_j-1} B\}. \text{ If there exists such } E$$

satisfying (3.76), then h_j s can be chosen as

$$h_j^T = e_j \quad (3.77)$$

for $j \in \{1, 2, \dots, m\}$ which satisfies (3.64), (3.65) and (3.66). There can be more than one solution for h_j s however one of the solution is $h_j^T = e_j$. ■

Fact 3.1: For maximum relative degree $r_T = r_1 + r_2 + \dots + r_m \leq n$, a combinatoric problem should be solved such that

$$\text{Find } r_j \text{ s maximizing } r_T \text{ which satisfies } \text{rank} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix} = m$$

where $e_i \in \{p_1^j A^{r_j-1} B, p_2^j A^{r_j-1} B, \dots, p_{k_j}^j A^{r_j-1} B\}$ and $\text{span}(p_1^j, p_2^j, \dots, p_{k_j}^j)$ is the null space of $C_j^T = [D \quad AD \quad \dots \quad A^{r_j-2} D \quad A^{r_j-1} b]^T$.

Lemma 3.2:

Linearly independent vectors $\{(s_1 + \bar{s}_1), s_2, \dots, s_m\}$ implies the linearly independence of either $\{s_1, s_2, \dots, s_m\}$ or $\{\bar{s}_1, s_2, \dots, s_m\}$ i.e. if the summation of two vectors satisfy

linear independence, then one of the vectors forming the sum also satisfy the linear independence.

Proof:

Assume that both of the vector sets $\{s_1, s_2, \dots, s_m\}$ and $\{\bar{s}_1, s_2, \dots, s_m\}$ are linearly dependent. Then one can write

$$s_1 = k_1 s_2 + k_2 s_3 + \dots + k_{m-1} s_m \quad (3.78)$$

and

$$\bar{s}_1 = n_1 s_2 + n_2 s_3 + \dots + n_{m-1} s_m \quad (3.79)$$

since $\{s_2, \dots, s_m\}$ are linearly independent.

The vectors $\{(s_1 + \bar{s}_1), s_2, \dots, s_m\}$ are linear independent such that

$$\alpha_1 (s_1 + \bar{s}_1) + \alpha_2 s_2 + \dots + \alpha_m s_m = 0 \quad (3.80)$$

implies

$$\begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_m \end{pmatrix} = 0. \quad (3.81)$$

Substituting (3.78) and (3.79) in (3.80) yields

$$(\alpha_1 (k_1 + n_1) + \alpha_2) s_2 + (\alpha_1 (k_2 + n_2) + \alpha_3) s_3 + \dots + (\alpha_1 (k_{m-1} + n_{m-1}) + \alpha_m) s_m = 0. \quad (3.82)$$

One can find $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} \neq 0$ satisfying (3.82) such that

$$\frac{\alpha_j}{\alpha_1} = -k_{j-1} - n_{j-1} \text{ for } j \in \{1, 2, \dots, m\} \text{ which contradicts with (3.82). Then both of the}$$

vector sets $\{s_1, s_2, \dots, s_m\}$ and $\{\bar{s}_1, s_2, \dots, s_m\}$ can not be linearly dependent.

Theorem 3.6: Any PWA system (3.55) in the canonical representation is partially feedback linearizable with relative degree r_T via the diffeomorphic state transformation $z = Tx + m$ if and only if it can be transformed to the normal form

$$(3.56) \text{ i.e. there exists an } m \times m \text{ matrix } E = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix} \text{ has rank } m \text{ with}$$

$$e_i \in \{p_1^j A^{r_j-1} B, p_2^j A^{r_j-1} B, \dots, p_{k_j}^j A^{r_j-1} B\} \text{ for some } r_j \text{ s maximizing}$$

$$r_T = r_1 + r_2 + \dots + r_m \leq n \text{ where } \text{span}(p_1^j, p_2^j, \dots, p_{k_j}^j) \text{ is the null space of } C_j^T,$$

$$C_j = \begin{bmatrix} D & AD & \dots & A^{r_j-2} D & A^{r_j-1} b \end{bmatrix} \text{ and } D = \begin{bmatrix} B & c_1 & \dots & c_l \end{bmatrix}.$$

Proof:

If the PWA system (3.55) can be transformed to the normal form, it takes the form (3.56) with the state transformation (3.58) and (3.55) can be rewritten as

$$\begin{bmatrix} \dot{z}_1^1 \\ \dot{z}_2^1 \\ \vdots \\ \dot{z}_{r_1}^1 \\ \dot{z}_1^2 \\ \dot{z}_2^2 \\ \vdots \\ \dot{z}_{r_2}^2 \\ \vdots \\ \dot{z}_1^m \\ \dot{z}_2^m \\ \vdots \\ \dot{z}_{r_m}^m \\ \dot{z}_{r_r+1} \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2^1 \\ z_3^1 \\ \vdots \\ \rho_1(z) + q_1^T u \\ z_2^2 \\ z_3^2 \\ \vdots \\ \rho_2(z) + q_2^T u \\ \vdots \\ z_2^k \\ z_3^k \\ \vdots \\ \rho_m(z) + q_m^T u \\ w_1(z) + q_{r_r+1}^T u \\ \vdots \\ w_{n-r_r}(z) + q_n^T u \end{bmatrix} \quad (3.83)$$

with $\rho_j(z) = h_j^T A^{r_j-1} a - h_j^T A^{r_j} T^{-1} m + h_j^T A^{r_j} T^{-1} z + \sum_{i=1}^l h_j^T A^{r_j-1} c_i | \alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i |$
, $q_j = h_j^T A^{r_j-1} B$ for $j \in \{1, 2, \dots, m\}$.

The r_j th rows for $j \in \{1, 2, \dots, m\}$ of the dynamic (3.83) forms the following equation.

$$\begin{bmatrix} h_1^T A^{r_1-1} a - h_1^T A^{r_1} T^{-1} m + h_1^T A^{r_1} T^{-1} z + \sum_{i=1}^l h_1^T A^{r_1-1} c_i | \alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i | \\ h_2^T A^{r_2-1} a - h_2^T A^{r_2} T^{-1} m + h_2^T A^{r_2} T^{-1} z + \sum_{i=1}^l h_2^T A^{r_2-1} c_i | \alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i | \\ \vdots \\ h_m^T A^{r_m-1} a - h_m^T A^{r_m} T^{-1} m + h_m^T A^{r_m} T^{-1} z + \sum_{i=1}^l h_m^T A^{r_m-1} c_i | \alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i | \end{bmatrix} + \begin{bmatrix} h_1^T A^{r_1-1} B \\ h_2^T A^{r_2-1} B \\ \vdots \\ h_m^T A^{r_m-1} B \end{bmatrix} u = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \quad (3.84)$$

The static feedback $u \in R^m$ which linearizes the first r_T dynamics of the (3.25) can be chosen solving equation (3.84) which yields

$$u = \begin{bmatrix} h_1^T A^{r_1-1} B \\ h_2^T A^{r_2-1} B \\ \vdots \\ h_m^T A^{r_m-1} B \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} - \begin{bmatrix} h_1^T A^{r_1-1} a - h_1^T A^{r_1} T^{-1} m + h_1^T A^{r_1} T^{-1} z + \sum_{i=1}^l h_1^T A^{r_1-1} c_i | \alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i | \\ h_2^T A^{r_2-1} a - h_2^T A^{r_2} T^{-1} m + h_2^T A^{r_2} T^{-1} z + \sum_{i=1}^l h_2^T A^{r_2-1} c_i | \alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i | \\ \vdots \\ h_m^T A^{r_m-1} a - h_m^T A^{r_m} T^{-1} m + h_m^T A^{r_m} T^{-1} z + \sum_{i=1}^l h_m^T A^{r_m-1} c_i | \alpha_i^T T^{-1} z - \alpha_i^T T^{-1} m - \beta_i | \end{bmatrix} \quad (3.85)$$

where v_j s for $j \in \{1, 2, \dots, m\}$ are the new control inputs. The system (3.83) becomes as

$$\begin{bmatrix} \dot{z}_1^1 \\ \dot{z}_2^1 \\ \vdots \\ \dot{z}_{r_1}^1 \\ \dot{z}_1^2 \\ \dot{z}_2^2 \\ \vdots \\ \dot{z}_{r_2}^2 \\ \vdots \\ \dot{z}_1^m \\ \dot{z}_2^m \\ \vdots \\ \dot{z}_{r_m}^m \\ \dot{z}_{r_T+1} \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2^1 \\ z_3^1 \\ \vdots \\ v_1 \\ z_2^2 \\ z_3^2 \\ \vdots \\ v_2 \\ \vdots \\ z_2^k \\ z_3^k \\ \vdots \\ v_m \\ w_1(z) + q_{r_T+1}^T u \\ \vdots \\ w_{n-r_T}(z) + q_n^T u \end{bmatrix} \quad (3.86)$$

where the subsystem with first r_T dynamics becomes linear. ■

Lemma 3.3: Any PWA system (3.55) in the canonical representation is partially feedback linearizable with the following input

$$u = \begin{bmatrix} h_1^T A^{r_1-1} B \\ h_2^T A^{r_2-1} B \\ \vdots \\ h_m^T A^{r_m-1} B \end{bmatrix}^{-1} \left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} - \begin{bmatrix} h_1^T A^{r_1-1} \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| \right) \\ h_2^T A^{r_2-1} \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| \right) \\ \vdots \\ h_m^T A^{r_m-1} \left(a + Ax + \sum_{i=1}^l c_i |\alpha_i^T x - \beta_i| \right) \end{bmatrix} \right) \quad (3.87)$$

parameterized in terms of system parameters where $h_j^T = e_j$ with

$$e_i \in \{p_1^j A^{r_j-1} B, p_2^j A^{r_j-1} B, \dots, p_{k_j}^j A^{r_j-1} B\} \quad \text{satisfying} \quad \text{rank} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix} = m \quad \text{and}$$

$\text{span}(p_1^j, p_2^j, \dots, p_{k_j}^j)$ is the null space of $C_j^T = \begin{bmatrix} D & AD & \dots & A^{r_j-2} D & A^{r_j-1} b \end{bmatrix}^T$.

Proof:

It is trivial that the static feedback (3.85) partially linearizes the system (3.83) which is the conjugate transformation of the system (3.55) with the transformation (3.58). Substituting (3.58) in (3.84), the linearizing control input (3.85) can be written in terms of the states $x \in R^n$ as (3.87). The $h_1^T \in R^n$ can be also written in terms of the system parameters as (3.77) which satisfies (3.76). ■

3.1.4 Full State Feedback Linearization for Multiple Input Case

Theorem 3.7: Any PWA system (3.55) in the canonical representation is full state feedback linearizable with the state transformation $z = Tx + m$ if and only if $c_i \in \text{Range}(B)$ and the controllability matrix $C = \begin{bmatrix} B & AB & \dots & A^{n-1} B \end{bmatrix}$ has rank n .

Proof:

The proof concludes with the following theorem which reveals the equivalence of Theorem 3.6 and Theorem 3.7 for full state linearization

Theorem 3.8: Theorem 3.6 and Theorem 3.7 are equivalent for relative degree n .

Proof:

It is assumed that $c_i \in \text{Range}(B)$ where $\text{Range}(B)$ denotes the range of B . Therefore, $c_i = Bf_i$ for $i=1,2,\dots,l$ and $a = Bf_0$ where $f_i \in R^m$ and $f_0 \in R^m$. The system can be rewritten as

$$\dot{x} = a + Ax + B \left[\sum_{i=1}^l f_i \left| \alpha_i^T x - \beta_i \right| + u \right] \quad (3.88)$$

The feedback law as $u = -\sum_{i=1}^l f_i \left| \alpha_i^T x - \beta_i \right| + v$ linearizes the system as follows.

$$\dot{x} = a + Ax + Bv \quad (3.89)$$

The system (3.89) with the affine transform (3.58) for relative degree n can be transformed into the normal form (3.56) if there exist h_j s satisfying the conditions in (3.64), (3.65) and (3.66). The condition $h_j^T A^{k_j} c_i = 0$ is equal to $h_j^T A^{k_j} B = 0$ for all $k_j \in \{1, 2, \dots, r_j - 2\}$ and $i \in \{1, 2, \dots, l\}$, since $c_i \in \text{Range}(B)$. Then one can only

check $h_j^T A^{k_j} B = 0$. The conditions $\text{rank} \begin{pmatrix} h_1^T A^{r_1-1} B \\ h_2^T A^{r_2-1} B \\ \vdots \\ h_m^T A^{r_m-1} B \end{pmatrix} = m$ and $h_j^T A^{k_j} B = 0$ can be

joined to form following equations for $j \in \{1, 2, \dots, m\}$.

$$h_j \begin{bmatrix} B & AB & \dots & A^{r_j-1} B & \dots & A^{n-1} B \end{bmatrix} = F_j \quad (3.90)$$

where F is chosen to satisfy $\text{rank} \begin{pmatrix} h_1^T A^{r_1-1} B \\ h_2^T A^{r_2-1} B \\ \vdots \\ h_m^T A^{r_m-1} B \end{pmatrix} = m$ and relative degree

$r_T = r_1 + r_2 + \dots + r_m = n$. There are not always unique F_j s. There exist h_j s satisfying

(3.90) since $\text{rank} \begin{bmatrix} B & AB & \dots & A^{n-1} B \end{bmatrix} = n$ and the solution for h_j s becomes

$h_j = F_j C^T (C C^T)^{-1}$ with $C = \begin{bmatrix} B & AB & \dots & A^{n-1} B \end{bmatrix}$. The conditions $c_i \in \text{Range}(B)$

and $\text{rank} \begin{bmatrix} B & AB & \dots & A^{n-1} B \end{bmatrix} = n$ reveals the existence of the normal form in

(3.56) i.e. Theorem 3.7 implies Theorem 3.6

Theorem 3.6 also implies Theorem 3.7 such that (3.66) implies

$$\text{rank} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ h_1^T A^{r_1-1} D \\ 0 \\ 0 \\ \vdots \\ h_2^T A^{r_2-1} D \\ \vdots \\ 0 \\ 0 \\ \vdots \\ h_m^T A^{r_m-1} D \end{bmatrix} = m \text{ and the following equation}$$

$$\begin{bmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r_1-1} \\ h_2^T \\ h_2^T A \\ \vdots \\ h_2^T A^{r_2-1} \\ \vdots \\ h_k^T \\ h_m^T A \\ \vdots \\ h_m^T A^{r_m-1} \end{bmatrix} \cdot D = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ h_1^T A^{r_1-1} D \\ 0 \\ 0 \\ \vdots \\ h_2^T A^{r_2-1} D \\ \vdots \\ 0 \\ 0 \\ \vdots \\ h_m^T A^{r_m-1} D \end{bmatrix} \quad (3.91)$$

implies $\text{rank}[D] = m$ since

$$\text{rank} \begin{bmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r_1-1} \\ h_2^T \\ h_2^T A \\ \vdots \\ h_2^T A^{r_2-1} \\ \vdots \\ h_k^T \\ h_m^T A \\ \vdots \\ h_m^T A^{r_m-1} \end{bmatrix} = n \quad (3.92)$$

and it is well known that the rank of the multiplication of two matrices is equal to the rank of the second matrix if the first matrix has full column rank i.e. $\text{rank}(MN) = \text{rank } N$ if the matrix M has full column rank. If $\text{rank}[D] = m$ i.e. $\text{rank} \begin{bmatrix} B & c_1 & \dots & c_l \end{bmatrix} = m$, then $c_i \in \text{Range}(B)$ since $\text{rank}[B] = m$.

Using the conditions $\text{rank} \begin{pmatrix} h_1^T A^{r_1-1} B \\ h_2^T A^{r_2-1} B \\ \vdots \\ h_m^T A^{r_m-1} B \end{pmatrix} = m$ and $h_j^T A^{k_j} B = 0$ for all

$k_j \in \{1, 2, \dots, r_j - 2\}$, the following equation can be written.

$$\begin{bmatrix} h_1^T \\ h_1^T A \\ \vdots \\ h_1^T A^{r_1-1} \\ h_2^T \\ h_2^T A \\ \vdots \\ h_2^T A^{r_2-1} \\ \vdots \\ h_m^T \\ h_m^T A \\ \vdots \\ h_m^T A^{r_m-1} \end{bmatrix} [B \quad AB \quad \cdots \quad A^{n-1}B] = \begin{bmatrix} 0 & \cdots & 0 & h_1^T A^{r_1-1}B & h_1^T A^{r_1}B & \cdots & h_1^T A^{n-1}B \\ \vdots & \ddots & h_1^T A^{r_1-1}B & h_1^T A^{r_1}B & \cdots & h_1^T A^{n-1}B & \cdots \\ 0 & \ddots & \ddots & \cdots & \ddots & \cdots & \cdots \\ h_1^T A^{r_1-1}B & h_1^T A^{r_1}B & \cdots & h_1^T A^{n-1}B & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & h_2^T A^{r_1-1}B & h_2^T A^{r_1}B & \cdots & h_2^T A^{n-1}B \\ \vdots & \ddots & h_2^T A^{r_1-1}B & h_2^T A^{r_1}B & \cdots & h_2^T A^{n-1}B & \cdots \\ 0 & \ddots & \ddots & \cdots & \ddots & \cdots & \cdots \\ h_2^T A^{r_1-1}B & h_2^T A^{r_1}B & \cdots & h_2^T A^{n-1}B & \cdots & \cdots & \cdots \\ \vdots & & & & & & \\ 0 & \cdots & 0 & h_m^T A^{r_1-1}B & h_m^T A^{r_1}B & \cdots & h_m^T A^{n-1}B \\ \vdots & \ddots & h_m^T A^{r_1-1}B & h_m^T A^{r_1}B & \cdots & h_m^T A^{n-1}B & \cdots \\ 0 & \ddots & \ddots & \cdots & \ddots & \cdots & \cdots \\ h_m^T A^{r_1-1}B & h_m^T A^{r_1}B & \cdots & h_m^T A^{n-1}B & \cdots & \cdots & \cdots \end{bmatrix} \quad (3.93)$$

$$\text{Rank} \begin{bmatrix} 0 & \cdots & 0 & h_1^T A^{r_1-1}B & h_1^T A^{r_1}B & \cdots & h_1^T A^{n-1}B \\ \vdots & \ddots & h_1^T A^{r_1-1}B & h_1^T A^{r_1}B & \cdots & h_1^T A^{n-1}B & \cdots \\ 0 & \ddots & \ddots & \cdots & \ddots & \cdots & \cdots \\ h_1^T A^{r_1-1}B & h_1^T A^{r_1}B & \cdots & h_1^T A^{n-1}B & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & h_2^T A^{r_1-1}B & h_2^T A^{r_1}B & \cdots & h_2^T A^{n-1}B \\ \vdots & \ddots & h_2^T A^{r_1-1}B & h_2^T A^{r_1}B & \cdots & h_2^T A^{n-1}B & \cdots \\ 0 & \ddots & \ddots & \cdots & \ddots & \cdots & \cdots \\ h_2^T A^{r_1-1}B & h_2^T A^{r_1}B & \cdots & h_2^T A^{n-1}B & \cdots & \cdots & \cdots \\ \vdots & & & & & & \\ 0 & \cdots & 0 & h_m^T A^{r_1-1}B & h_m^T A^{r_1}B & \cdots & h_m^T A^{n-1}B \\ \vdots & \ddots & h_m^T A^{r_1-1}B & h_m^T A^{r_1}B & \cdots & h_m^T A^{n-1}B & \cdots \\ 0 & \ddots & \ddots & \cdots & \ddots & \cdots & \cdots \\ h_m^T A^{r_1-1}B & h_m^T A^{r_1}B & \cdots & h_m^T A^{n-1}B & \cdots & \cdots & \cdots \end{bmatrix} = n$$

with (3.92) implies $\text{rank} [B \quad AB \quad \cdots \quad A^{n-1}B] = n$.

The conditions in Theorem 3.6 also implies Theorem 3.7. ■

3.2 Approximate Feedback Linearization of PWA Systems in Canonical Form

In this subsection, feedback linearization of single input PWA system in (3.94) is considered. The $f: R^n \rightarrow R^n$ and $g: R^n \rightarrow R^n$ in system (3.94) are not smooth

functions; therefore well-known methods for feedback linearization cannot be applied. However, the functions can be approximated by smooth functions to obtain an approximate feedback control. The approximation is chosen such that it becomes an exact representation by adjusting a parameter. Numerical simulations are presented in order to show the effectiveness of the method.

An n -dimensional single input system in the form (3.94) is considered.

$$\dot{x} = f(x) + g(x)u \quad (3.94)$$

where $x \in R^n$ is the state, $u \in R$ is the scalar control input, $f: R^n \rightarrow R^n$ and $g: R^n \rightarrow R^n$. The $f: R^n \rightarrow R^n$ and $g: R^n \rightarrow R^n$ are PWA functions in canonical representation as:

$$f(x) = a^f + A^f x + \sum_{i=1}^{l_f} c_i^f |(\alpha_i^f)^T x - \beta_i^f| \quad (3.95)$$

$$g(x) = a^g + A^g x + \sum_{i=1}^{l_g} c_i^g |(\alpha_i^g)^T x - \beta_i^g| \quad (3.96)$$

The non-smoothness of functions $f(\cdot)$ and $g(\cdot)$ is caused absolute value which can be approximated by $|(\alpha_i^f)^T x - \beta_i^f| \approx \frac{1}{B} \ln \cosh \left[B \cdot [(\alpha_i^f)^T x - \beta_i^f] \right]$ with adjusting B . The approximation becomes an exact representation when $B \rightarrow \infty$. For sufficiently large B , it is obvious that the functions $f(\cdot): R^n \rightarrow R^n$ and $g(\cdot): R^n \rightarrow R^n$ can be approximated with sufficiently small error and approximated functions can be written as

$$\hat{f}(x) = a^f + A^f x + \sum_{i=1}^{l_f} c_i^f \cdot \frac{1}{B} \ln \cosh \left[B \cdot [(\alpha_i^f)^T x - \beta_i^f] \right] \quad (3.97)$$

$$\hat{g}(x) = a^g + A^g x + \sum_{i=1}^{l_g} c_i^g \cdot \frac{1}{B} \ln \cosh \left[B \cdot [(\alpha_i^g)^T x - \beta_i^g] \right] \quad (3.98)$$

The system (3.94) can be also approximated with (3.97) and (3.98) such that

$$\dot{x} = \hat{f}(x) + \hat{g}(x)u \quad (3.99)$$

The proposed method is design a control law by feedback linearization method for the system (3.99) which results to obtain an approximate feedback control law for the system (3.94). The system (3.99) is input-state linearizable if it satisfies the conditions in Theorem 3.9.

Theorem 3.9 [Slotine & Li, 1991]: The nonlinear system (3.99), with $\hat{f}(x)$ and $\hat{g}(x)$ being smooth vector fields is input-state linearizable if and only if there exists a region D_0 such that the following conditions hold:

- 1) The vector fields $\{\hat{g}, ad_{\hat{f}}\hat{g}, \dots, ad_{\hat{f}}^{n-1}\hat{g}\}$ are linearly independent in D_0 i.e.

$$rank \begin{bmatrix} \hat{g} & ad_{\hat{f}}\hat{g} & \dots & ad_{\hat{f}}^{n-1}\hat{g} \end{bmatrix} = n$$

- 2) The set $\{\hat{g}, ad_{\hat{f}}\hat{g}, \dots, ad_{\hat{f}}^{n-2}\hat{g}\}$ is involutive in D_0 i.e. the Lie bracket of any pair of vector fields belonging to the set $\{\hat{g}, ad_{\hat{f}}\hat{g}, \dots, ad_{\hat{f}}^{n-2}\hat{g}\}$ is also a vector field this set.

If the system (3.99) is input-state linearizable then there should exist a smooth scalar function $h(x)$ satisfying

$$L_{\hat{g}}L_{\hat{f}}^i h(x) = 0 \text{ for } i \in \{0, \dots, n-2\} \quad (3.100)$$

and

$$L_{\hat{g}}L_{\hat{f}}^{n-1}h(x) \neq 0 \quad (3.101)$$

for all $x \in D_0$ (Sastry, 1991).

Under the above input-state linearizability assumption, the system in (3.99) can be transformed with the state transformation $z = T(x) = \begin{bmatrix} h(x) & L_{\hat{f}}h(x) & \cdots & L_{\hat{f}}^{n-1}h(x) \end{bmatrix}^T$, which is a diffeomorphism over $D_0 \in R^n$ (Sastry, 1999) into the normal form as:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= L_{\hat{f}}^n h(x) + L_{\hat{g}} L_{\hat{f}}^{n-1} h(x) u \end{aligned} \quad (3.102)$$

Choosing the control law (3.103) linearizes the system

$$u = \frac{1}{L_{\hat{g}} L_{\hat{f}}^{n-1} h(x)} v - \frac{L_{\hat{f}}^n h}{L_{\hat{g}} L_{\hat{f}}^{n-1} h(x)} \quad (3.103)$$

where v is the new control input.

The system (3.99) with an output equation $y = h(x)$ where $h: R^n \rightarrow R$ is input-output linearizable with relative degree r if it satisfies

$$L_{\hat{g}} L_{\hat{f}}^i h(x) = 0 \text{ for } i \in \{0, \dots, r-2\} \quad (3.104)$$

and

$$L_{\hat{g}} L_{\hat{f}}^{r-1} h(x) \neq 0 \quad (3.105)$$

for all $x \in D_0$ (Sastry 1991).

The system in (3.99) can be transformed with the state transformation

$$\begin{aligned} z = T(x) &= \begin{bmatrix} \mu & \psi \end{bmatrix} = \begin{bmatrix} y & \dot{y} & \cdots & y^{(r-1)} & \psi_1 & \psi_2 & \cdots & \psi_{n-r} \end{bmatrix}^T \\ &= \begin{bmatrix} h(x) & L_{\hat{f}}h(x) & \cdots & L_{\hat{f}}^{r-1}h(x) & \psi_1 & \psi_2 & \cdots & \psi_{n-r} \end{bmatrix}^T \end{aligned} \quad (3.106)$$

which is a diffeomorphism over $D_0 \in R^n$, (Sastry, 1999) into the normal form as

$$\begin{aligned}
 \dot{\mu}_1 &= \mu_2 \\
 \dot{\mu}_2 &= \mu_3 \\
 &\vdots \\
 \dot{\mu}_r &= L_f^r h(x) + L_g L_f^{r-1} h(x) u \\
 \dot{\psi} &= w(\mu, \psi)
 \end{aligned} \tag{3.107}$$

where $\mu = [\mu_1 \ \mu_2 \ \cdots \ \mu_r]^T = [z_1 \ z_2 \ \cdots \ z_r]^T$ and

$$\psi = [\psi_1 \ \psi_2 \ \cdots \ \psi_{n-r}]^T = [z_{r+1} \ z_{r+2} \ \cdots \ z_n]^T.$$

Choosing the control law (3.108) partially linearizes the system (3.107) where the first r dynamics of (3.107) become linear.

$$u = \frac{1}{L_g L_f^{r-1} h(x)} v - \frac{L_f^r h}{L_g L_f^{r-1} h(x)} \tag{3.108}$$

where v is the new control input.

(3.103) for input-state linearization and (3.108) for input-output linearization are the approximate feedback controls for the PWA system (3.94). The proposed method for obtaining an approximate feedback control can be also extended for multiple input PWA systems in canonical representation.

Example 3.3:

Chua's circuit with an input additive to the third state is a PWA system in the dimensionless state space form (3.109).

$$\begin{aligned}
\dot{x}_1 &= \alpha [x_2 - x_1 - \sigma(x_1)] \\
\dot{x}_2 &= x_1 - x_2 + x_3 \\
\dot{x}_3 &= -\beta x_2 - \gamma x_3 + u
\end{aligned} \tag{3.109}$$

For simulation the parameters are fixed as $\alpha = 9$, $\beta = 100/7$, $\gamma = 0.016$, $m_0 = -8/7$ and $m_1 = -5/7$ with regard to previous works (Bowong & Kagou, 2006) where the piecewise linear function is

$$\sigma(x_1) = m_1 x_1 + \frac{1}{2}(m_0 - m_1)[|x_1 + 1| - |x_1 - 1|]. \tag{3.110}$$

The system (3.109) can be written in the canonical form as follows

$$\dot{x} = a^f + A^f x + \sum_{i=1}^2 c_i^f |(a_i^f)^T x - \beta_i^f| + a^g u \tag{3.111}$$

with the parameters $A^f = \begin{bmatrix} -\frac{2}{7}\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}$, $c_1^f = \begin{bmatrix} 3\alpha/14 \\ 0 \\ 0 \end{bmatrix}$, $c_2^f = \begin{bmatrix} -3\alpha/14 \\ 0 \\ 0 \end{bmatrix}$,

$$a_1^f = a_2^f = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \beta_1^f = -1, \beta_2^f = 1, a^f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, a^g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The system (3.111) can be approximated as

$$\dot{x} = \hat{f}(x) + \hat{g}(x)u. \tag{3.112}$$

with smooth functions $\hat{f}(x) = a^f + A^f x + \sum_{i=1}^2 c_i^f \cdot \frac{1}{B} \ln \cosh[B \cdot ((a_i^f)^T x - \beta_i^f)]$ and

$g(x) = a^g$. In order to check the input-state linearizability, $ad_{\hat{f}} \hat{g}$ and $ad_{\hat{f}}^2 \hat{g}$ are calculated such that

$$ad_{\hat{f}}\hat{g} = \frac{\partial \hat{g}}{\partial x} \cdot \hat{f} - \frac{\partial \hat{f}}{\partial x} \cdot \hat{g} = - \begin{bmatrix} \frac{\partial \hat{f}_1}{\partial x_3} \\ \frac{\partial \hat{f}_2}{\partial x_2} \\ \frac{\partial \hat{f}_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ \gamma \end{bmatrix} \quad (3.113)$$

$$ad_{\hat{f}}^2\hat{g} = \frac{\partial ad_{\hat{f}}\hat{g}}{\partial x} \cdot \hat{f} - \frac{\partial \hat{f}}{\partial x} \cdot ad_{\hat{f}}\hat{g} = \begin{bmatrix} \frac{\partial \hat{f}_1}{\partial x_2} - \gamma \frac{\partial \hat{f}_1}{\partial x_3} \\ \frac{\partial \hat{f}_2}{\partial x_2} - \gamma \frac{\partial \hat{f}_2}{\partial x_3} \\ \frac{\partial \hat{f}_3}{\partial x_2} - \gamma \frac{\partial \hat{f}_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \alpha \\ -1 - \gamma \\ -\beta + \gamma^2 \end{bmatrix}. \quad (3.114)$$

The matrix $\begin{bmatrix} \hat{g} & ad_{\hat{f}}\hat{g} & ad_{\hat{f}}^2\hat{g} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & -1 & -1 - \gamma \\ 1 & \gamma & -\beta + \gamma^2 \end{bmatrix}$ has full rank for all $x \in R^3$, since

$$\det \begin{bmatrix} \hat{g} & ad_{\hat{f}}\hat{g} & ad_{\hat{f}}^2\hat{g} \end{bmatrix} = \det \begin{bmatrix} 0 & 0 & \alpha \\ 0 & -1 & -1 - \gamma \\ 1 & \gamma & -\beta + \gamma^2 \end{bmatrix} \begin{bmatrix} -\alpha \\ 1 \\ \beta \end{bmatrix} = \alpha \neq 0 \text{ i.e.} \quad (3.115)$$

$rank \begin{bmatrix} \hat{g} & ad_{\hat{f}}\hat{g} & ad_{\hat{f}}^2\hat{g} \end{bmatrix} = 3$. The set $\{\hat{g}, ad_{\hat{f}}\hat{g}, \dots, ad_{\hat{f}}^{n-2}\hat{g}\}$ is involutive since $\hat{g}$ and $ad_{\hat{f}}\hat{g}$ are constant vector fields. Therefore, the conditions in Theorem 3.9 are satisfied for all $x \in R^3$. Then the system (3.112) is input-state linearizable and there exists a scalar function $h(x)$ satisfying (3.101) and (3.102) such that

$$L_{\hat{g}}L_{\hat{f}}^0 h = 0 \Rightarrow \frac{\partial h}{\partial x} \cdot \hat{g} = 0 \Rightarrow \frac{\partial h}{\partial x_3} = 0 \text{ which implies that } h \text{ is independent of } x_3,$$

$$\begin{aligned}
L_{\hat{g}}L_{\hat{f}}^1h &= \frac{\partial L_{\hat{f}}h}{\partial x} \cdot g = \frac{\partial L_{\hat{f}}h}{\partial x_3} = \frac{\partial \left(\frac{\partial h}{\partial x_1} \cdot \hat{f}_1 + \frac{\partial h}{\partial x_2} \cdot \hat{f}_2 \right)}{\partial x_3} = \frac{\partial \left(\frac{\partial h}{\partial x_2} \cdot \hat{f}_2 \right)}{\partial x_3} \\
&= \frac{\partial \left(\frac{\partial h}{\partial x_2} \right)}{\partial x_3} \cdot \hat{f}_2 + \frac{\partial h}{\partial x_2} \cdot \frac{\partial \hat{f}_2}{\partial x_3} = 0 \Rightarrow \frac{\partial h}{\partial x_2} = 0 \text{ which implies that } h \text{ is independent of} \\
x_2 \text{ and } L_{\hat{g}}L_{\hat{f}}^2h &\neq 0 \Rightarrow \frac{\partial L_{\hat{f}}^2h}{\partial x_3} \neq 0 \Rightarrow \frac{\partial h}{\partial x_1} \frac{\partial \hat{f}_1}{\partial x_2} \neq 0 \Rightarrow \alpha \frac{\partial h}{\partial x_1} \neq 0; \text{ therefore, it can be} \\
\text{chosen that } h(x) &= x_1. \text{ The transformation can be calculated as}
\end{aligned}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} h \\ L_{\hat{f}}h \\ L_{\hat{f}}^2h \end{bmatrix} = \begin{bmatrix} x_1 \\ \alpha [x_2 - x_1 - \sigma(x_1)] \\ -\alpha^2 \left[1 + \frac{\partial \sigma(x_1)}{\partial (x_1)} \right] \cdot [x_2 - x_1 - \sigma(x_1)] + \alpha (x_1 - x_2 + x_3) \end{bmatrix} \quad (3.116)$$

which transforms the system (3.112) into the normal form (3.117).

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\vdots & \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= L_{\hat{f}}^n h(x) + L_{\hat{g}}L_{\hat{f}}^{n-1} h(x)u
\end{aligned} \quad (3.117)$$

The transformation is global and the inverse of the state transformation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ \frac{1}{\alpha} z_2 + z_1 + \sigma(z_1) \\ \frac{z_2}{\alpha} + \sigma(z_1) + \left[1 + \frac{\partial \sigma(z_1)}{\partial (z_1)} \right] \cdot z_2 \end{bmatrix}. \quad (3.118)$$

The control law (3.119) linearizes the system (3.112) and (3.119) is also an approximate linearizing feedback for the PWA system (3.109).

$$u = \frac{1}{L_g L_f^2 h(x)} (v - L_f^3 h(x)) \quad (3.119)$$

Then, one can choose a linear controller to stabilize the system or to track a desired trajectory. A simple proportional controller is chosen to stabilize the system as $v = -z_1 - 3z_2 - 3z_3 = -h(x) - 3L_f h(x) - 3L_f^2 h(x)$. The control input as a function of time in Figure 3.14 stabilizes the system (3.111) as shown in Figure 3.13.

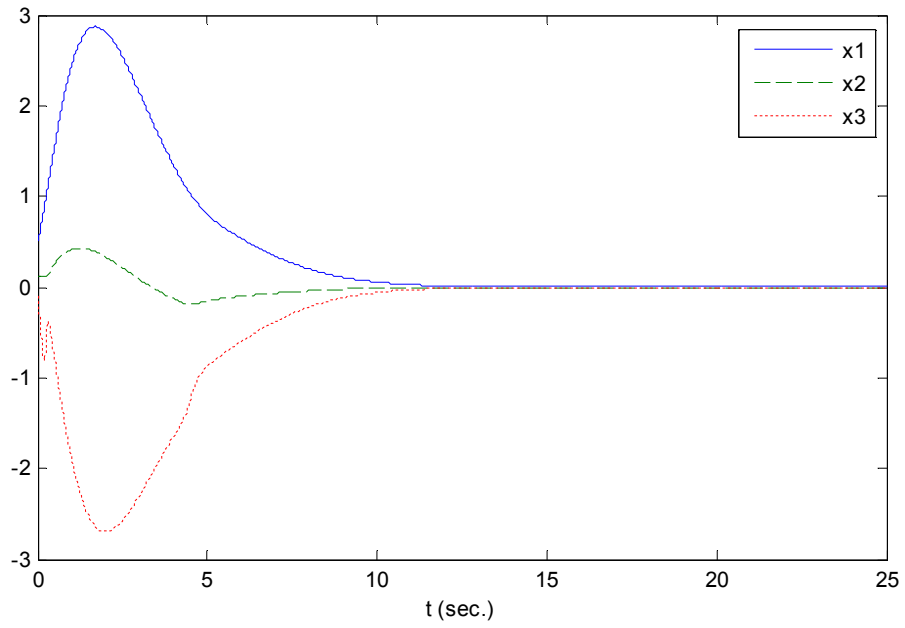


Figure 3.13 x_1 , x_2 and x_3 versus time of the controlled system (3.111) for initial states

$$x_0 = [0.5 \quad 0.1 \quad -0.1]^T \text{ and } B = 10$$

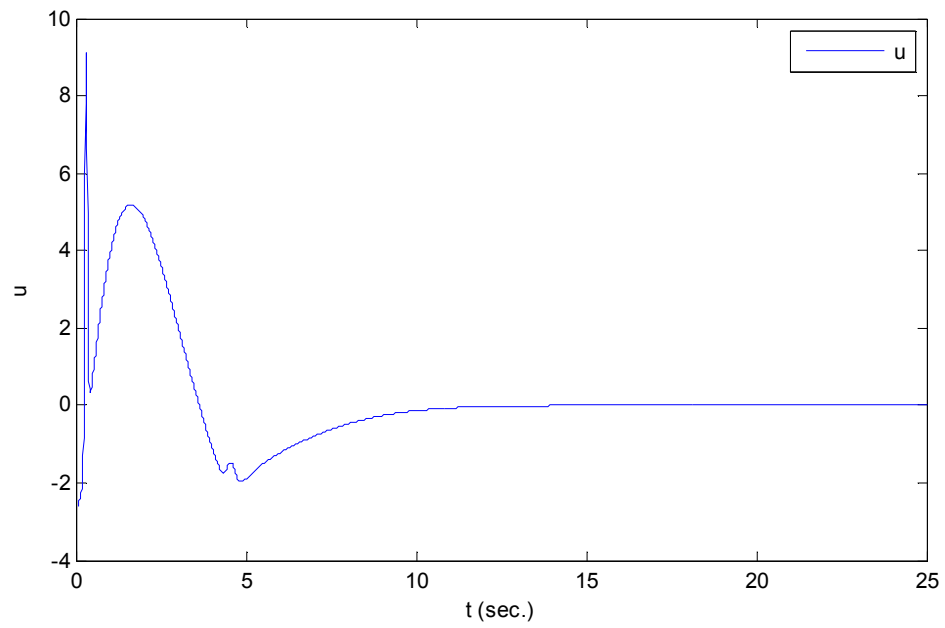


Figure 3.14 Control input (3.119) with a linear controller stabilizes the system (3.111) for initial states $x_0 = [0.5 \quad 0.1 \quad -0.1]^T$ and $B = 10$

CHAPTER FOUR

MODEL BASED ROBUST CHAOTIFICATION USING SLIDING MODE CONTROL

This section describes a model based robust chaotification scheme using sliding mode control. The proposed chaotification method yields a dynamical state feedback in order to match all system states to a reference chaotic system. In contrast to this, other model based dynamical feedback method (Savaş & Güzeliş, 2010) introduces extra states and matches to a higher dimensional reference chaotic system. The proposed method can be applied to any single input, observable and input state linearizable system subject to parameter uncertainties, nonlinearities, noises and disturbances. Input state linearizable PWA systems as a special case also allow the application of the chaotification scheme. It is assumed that parameter uncertainties, nonlinearities, noises and disturbances are all additive to the input and they can be modeled as an unknown function having a bound specified by a known function. Discontinuous feedback control law of sliding mode chaotification method provides robustness against noise and disturbances. The robustness of the proposed method makes the chaotification immune in the sense that the resulting system remains in chaos for a wide range of system parameters and also under the noise and disturbances. The matching of the considered system to the reference chaotic system is always achieved in finite time which can be made arbitrarily small by modifying a parameter changing the control input. The proposed method needs reference chaotic systems in the normal form.

4.1 Normal Form of Reference Chaotic Systems

A great number of chaotic systems are in the normal form (Spratt, 1997; Spratt & Linz, 2000; Wang & Chen, 2003; Morgül, 2003) given as

$$\dot{z} = A_c z + b_c g_c(z) \tag{4.1}$$

where $g_c = R^n \rightarrow R$ is a nonlinear function, $A_c \in R^{n \times n}$ is a constant matrix and $b_c \in R^n$ is a constant vector. A_c and b_c are in the following controllable canonical form:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & \cdots & a_{n-1} \end{bmatrix}, \quad b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (4.2)$$

Including the linear terms weighted by a_i 's into $G_c(z)$, the system in (4.1) can be reformulated as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= G_c(z) \end{aligned} \quad (4.3)$$

where $G_c(z) = g_c(z) + a_0 z_1 + a_1 z_2 + \dots + a_{n-1} z_n$.

For $n = 3$, $G_c(z)$ represents the jerk function and $g_c(z)$ represents the nonlinearity in the jerk function. In the rest of the paper, $G_c(z)$ will be called as jerk function for arbitrary dimension $n \geq 3$. There exists a large class of systems which are in the form of jerk equations (Sprott, 1997; Sprott & Linz, 2000).

In addition, a great number of chaotic systems can be formulated as

$$\dot{x} = Ax + bg_c(x) \quad (4.4)$$

where $A \in R^{n \times n}$ is an arbitrary constant matrix, $b \in R^n$ is an arbitrary constant vector and $g_c = R^n \rightarrow R$ represents the nonlinear part of the chaotic system. Any system in

the form of (4.4) can be transformed into the normal form as in (4.1) via the linear transformation $z = Tx$ if the controllability matrix $C^{n \times n}$ has rank n , hence it is invertible (Wang & Chen, 2003).

$$C = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

The transformation matrix is given as

$$T = \begin{pmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{pmatrix} \quad (4.5)$$

where q^T is the n th row of C^{-1} , and in the transformed system $A_c = TAT^{-1}$ and $b_c = Tb$ have the form in (4.2).

Example 4.1 (Transformation of Chua's circuit with cubic nonlinearity into the normal form): State space of Chua's circuit with cubic nonlinearity (Hirsch et al., 2003) is in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha [x_2 - f(x_1)] \\ x_1 - x_2 + x_3 \\ -\beta x_2 \end{pmatrix} \quad (4.6)$$

where $f(x_1) = m_0 x_1^3 + m_1 x_1$ is the cubic nonlinearity of the Chua's circuit. This system can be written in the form of (4.4) as follows.

$$\dot{x} = \begin{pmatrix} 0 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} [-\alpha f(x_1)] \quad (4.7)$$

There exists an invertible linear transformation

$$z = Tx = \begin{pmatrix} 0 & 0 & -\beta^{-1} \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} x \quad (4.8)$$

yielding a normal form as in (4.1),

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & (\alpha - \beta) & -1 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} [-\alpha f(\beta z_1 + z_2 + z_3)] \quad (4.9)$$

and can be reformulated as in (4.3) for $n = 3$ where

$$G_c(z) = (\alpha - \beta)z_2 - z_3 - \alpha f(\beta z_1 + z_2 + z_3). \quad (4.10)$$

(Eichhorn et al., 1998) introduced the conditions to transform a three dimensional nonlinear system into at least one equivalent jerk equation. (Wang et al., 2004) proved that chaotic systems in the strict-feedback form can be transformed into the normal form. There exists a great number of three dimensional chaotic systems which either exist in the normal form (Sprott, 1997) or can be transformed into the normal form via global diffeomorphisms on R^n as described above.

A method to generate chaotic systems for $n > 3$ in the form of (4.1), which have chaotic attractors qualitatively similar to lower dimensional chaotic systems of the same form, is described in (Morgül, 2003). In order to obtain a four dimensional chaotic system via modifying a three dimensional chaotic system in the form of (4.3), the following system is considered

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \hat{G}_c(z_1, z_2, z_3) \end{aligned} \quad (4.11)$$

The system in (4.11) is modified in order to obtain a higher dimensional system in the following form

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= \hat{G}_c(z_1, z_2, z_3) + z_4 \\
\dot{z}_4 &= -\gamma z_4
\end{aligned} \tag{4.12}$$

where $\gamma > 0$ is an arbitrary constant. It is well known that $z_4(t) = z_4(0)e^{-\gamma t} \rightarrow 0$ as $t \rightarrow \infty$. The system in (4.12) exhibits a chaotic attractor qualitatively similar to the lower dimensional chaotic model in (4.11). By the procedure in (Morgül, 2003), the system in (4.12) can be reformulated into the normal form as follows

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= G_c(z)
\end{aligned} \tag{4.13}$$

where

$$G_c(z_1, z_2, z_3, z_4) = \frac{d}{dt}(\hat{G}_c(z_1, z_2, z_3)) - \gamma [z_4 - \hat{G}_c(z_1, z_2, z_3)]. \tag{4.14}$$

Furthermore, an arbitrary dimensional chaotic system can be obtained by introducing a new state variable at each step, provided that $\hat{G}_c$ is sufficiently smooth. The details of the procedure may be found in (Morgül, 2003).

Example 4.2 (Obtaining a four dimensional chaotic system by modifying a three dimensional chaotic system with a quadratic nonlinearity): A three dimensional chaotic system with a quadratic nonlinearity is considered here in order to obtain a four dimensional chaotic system as described above. Three dimensional chaotic system with quadratic nonlinearity is given in the state space form as in (4.11) where $\hat{G}_c = \alpha z_3 + \beta z_2 + z_1^2 - 1$ (Sprott & Linz, 2000). This chaotic system can be modified in

order to obtain a four dimensional chaotic system as described in (4.12) and it can be written into the normal form as in (4.13) by choosing $\gamma = 1$:

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= G_c(z)\end{aligned}\tag{4.15}$$

where

$$G_c(z) = (\alpha - 1)z_4 + (\alpha + \beta)z_3 + (\beta + 2z_1)z_2 + z_1^2 - 1.\tag{4.16}$$

4.2 Sliding Mode Chaotifying Control Laws for Matching Input State Linearizable Systems to Reference Chaotic Systems

Continuous time single input systems subject to uncertainties, nonlinearities, noises and disturbances are considered in the paper as the systems to be chaotified

$$\dot{x} = f(x) + g(x)[u + \delta(t, x, u)]\tag{4.17}$$

where, $x \in R^n$ is the state, $u \in R$ is the scalar control input, $f: R^n \rightarrow R^n$ and $g: R^n \rightarrow R^n$ are sufficiently smooth functions, and the function $\delta(t, x, u)$ with $\delta: R \times R^n \times R \rightarrow R$ is an unknown real valued function describing the uncertainties, nonlinearities, noises and disturbances additive to the input. It is assumed that the unknown function $\delta(t, x, u)$ is bounded by a known function. For this system, a feedback law such that the resulting closed-loop system exhibits the chaotic behavior after a finite time under uncertainties, nonlinearities, noises and disturbances additive to the input is proposed. All the states of the systems of (4.17) are assumed available or can be obtained in an indirect way since the systems are considered as observable.

A sliding mode feedback control law yielding the desired chaotic behavior for the closed loop system is provided in Section 4.2.1 for the controllable linear system case of (4.17). Section 4.2.2 provides the feedback law in the same way for the input state feedback linearizable case of (4.17).

4.2.1 Linear System Case

An n -dimensional single input observable linear system subject to parameter uncertainties, nonlinearities, noises and disturbances additive to the input is considered in the following form.

$$\dot{x} = Ax + b[u + \delta(t, x, u)] \quad (4.18)$$

Assuming that the linear system is controllable, the system can be transformed via a linear transformation $z = Tx$, similarly given in (4.5), into the following normal form

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & \cdots & a_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [u + \delta(t, T^{-1}z, u)] \quad (4.19)$$

where $\delta(t, T^{-1}z, u)$ represents uncertainties, nonlinearities, noises and disturbances additive to the input. The dynamical feedback law is chosen as

$$u = -a_0 z_1 - a_1 z_2 - \cdots - a_{n-2} z_{n-1} - a_{n-1} z_n + \dot{z}_n - \ddot{z}_n + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} \dot{z}_n + v \quad (4.20)$$

where $G_c(z_1, z_2, \dots, z_n): R^n \rightarrow R$ is a nonlinear function chosen to be the jerk function of the reference chaotic system in the normal form of (4.3). Then, the

system becomes

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \left[v + \dot{z}_n - \ddot{z}_n + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} \dot{z}_n + \hat{\delta}(t, z, v) \right] \quad (4.21)$$

where v is the new control input and $\hat{\delta}(t, z, v)$ is the uncertainty, nonlinearity and noise rewritten in terms of z and v . By defining a new state variable $\dot{z}_n = z_{n+1}$ with

$$\ddot{z}_n = \dot{z}_{n+1} = \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} z_{n+1} + v + \hat{\delta}(t, z, v), \text{ the } n + 1 \text{ dimensional state}$$

space form of the system can be obtained as in (4.22). The amplitude of $\hat{\delta}(t, z, v)$ is assumed to be bounded by a known function: $|\hat{\delta}(t, z, v)| \leq \hat{p}(t, z) + k \|v\|_\infty$ for $\hat{p}(t, z) > 0$ and $0 \leq k < 1$.

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= z_{n+1} \\ \dot{z}_{n+1} &= \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} z_{n+1} + v + \hat{\delta}(t, z, v) \end{aligned} \quad (4.22)$$

Now, the sliding manifold is specified as $s = z_{n+1} - G_c(z_1, z_2, \dots, z_n)$ where $G_c(z_1, z_2, \dots, z_n)$ is chosen to be the jerk function of the reference chaotic system in the normal form of (4.3). Then, one can apply the sliding mode control where $n > 0$ is a scalar to adjust finite reaching time to the sliding manifold.

$$v = -\frac{\eta + \hat{p}(t, z)}{1 - k} \text{sign}(s) \quad (4.23)$$

After reaching the sliding manifold, s becomes zero, so $z_{n+1} = G_c(z_1, z_2, \dots, z_n)$. Therefore, the first n states of the system (4.22) can be seen to be matched to the reference chaotic system (4.3) with the jerk function $G_c(z_1, z_2, \dots, z_n)$.

In order to show that the matching can be achieved in finite time, one can choose the Lyapunov function as $V = \frac{1}{2}s^2$. It is shown below that time derivative $\dot{V}$ of this Lyapunov function along trajectories of the system in (4.22) is not greater than $-\eta|s|$. So, reaching to the sliding manifold is of finite duration rather than asymptotical (Perruquetti & Barbot, 2002).

$$\begin{aligned} \dot{V} &= s\dot{s} = s(\nabla_{\bar{z}}s)^T \dot{\bar{z}} \\ &= s \begin{bmatrix} -\frac{\partial G_c}{\partial z_1} \\ -\frac{\partial G_c}{\partial z_2} \\ \vdots \\ -\frac{\partial G_c}{\partial z_n} \\ 1 \end{bmatrix}^T \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_{n+1} \\ \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \dots + \frac{\partial G_c}{\partial z_n} z_{n+1} + v + \hat{\delta}(t, z, v) \end{bmatrix} = s \left[v + \hat{\delta}(t, z, v) \right] \end{aligned} \quad (4.24)$$

where $\nabla_{\bar{z}}s$ is the gradient of s manifold with respect to $\bar{z} = [z^T z_{n+1}]^T$.

$$\dot{V} \leq sv + |s| |\hat{\delta}(t, z, v)| \quad (4.25)$$

Under the assumption of $|\hat{\delta}(t, z, v)| \leq \hat{p}(t, z) + k \|v\|_\infty$, (4.25) becomes as follows.

$$\dot{V} \leq sv + [\hat{p}(t, z) + k \|v\|_\infty] |s| \quad (4.26)$$

Now, substituting v in (4.23) into (4.26), an upper bound for $\dot{V}$ is obtained as follows.

$$\dot{V} \leq -\frac{\eta + \hat{p}(t, z)}{1-k} |s| + \left[\hat{p}(t, z) + k \frac{\eta + \hat{p}(t, z)}{1-k} \right] |s| = -\eta |s| \quad (4.27)$$

The differential inequality in (4.27) can be solved first by dividing both sides with $|s|$ and then integrating them. So, one can get $|s(t)| - |s(0)| \leq -\eta t$ for the initial time, i.e. $t_0 = 0$. It means that the time needed to reach sliding manifold $s = 0$ should be finite and has an upper bound as $t_{reach} \leq \frac{|s(0)|}{\eta}$, i.e. $t_0 = 0$ (Slotine & Li, 1991). So, (4.22) with (4.23) becomes (4.28) when $s = 0 \Rightarrow z_{n+1} = G_c(z_1, z_2, \dots, z_n)$.

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= G_c(z_1, z_2, \dots, z_n) \\ \dot{z}_{n+1} &= \dot{G}_c(z_1, z_2, \dots, z_n) \end{aligned} \quad (4.28)$$

The first n states of the system (4.28) can be seen to be matched to the reference chaotic system (4.3) with the jerk function $G_c(z_1, z_2, \dots, z_n)$. $|z_{n+1}| = |G_c(z_1, z_2, \dots, z_n)| < \infty$ since z_i 's are bounded for a chaotic trajectory and the continuous function $G_c(z_1, z_2, \dots, z_n)$ maps a bounded set into a bounded set.

4.2.2 Input State Linearizable Nonlinear System Case

An n -dimensional single input observable and input state linearizable system subject to parameter uncertainties, nonlinearities, noises and disturbances additive to the input is considered in the form given below

$$\dot{x} = f(x) + g(x)[u + \delta(t, x, u)] \quad (4.29)$$

where $x \in R^n$ is the state, $u \in R$ is the scalar control input, $f: R^n \rightarrow R^n$ and $g: R^n \rightarrow R^n$ sufficiently smooth functions, and $\delta(t, x, u)$ with $\delta: R \times R^n \times R$ is an unknown real valued function describing the uncertainties, nonlinearities, noises and disturbances additive to the input. The $\delta(t, x, u)$ is assumed to be bounded by a known function. Since the system (4.29) is assumed to be input state linearizable then there should exist a smooth scalar function $h(x)$ satisfying $L_g L_f^i h(x) = 0$ for $i = 0, \dots, n-2$ and $L_g L_f^{n-1} h(x) \neq 0$ for all $x \in D$ (Sastry, 1999).

Under the above input state linearizability assumption, the system in (4.29) can be transformed with the state transformation $z = T(x) = \left(h(x) L_f h(x) \cdots L_f^{n-1} h(x) \right)^T$, which is a diffeomorphism over $D \in R^n$, (Sastry, 1999) into the normal form as:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z)) \left[u + \delta(t, T^{-1}(z), u) \right] \end{aligned} \quad (4.30)$$

The dynamical feedback law is chosen as

$$u = \frac{1}{L_g L_f^{n-1} h(T^{-1}(z))} \left[\dot{z}_n - \ddot{z}_n - L_f^n h(T^{-1}(z)) + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} \dot{z}_n + v \right] \quad (4.31)$$

where $G_c(z_1, z_2, \dots, z_n): R^n \rightarrow R$ is a nonlinear function chosen to be the jerk function of the reference chaotic system in the normal form of (4.3). Then, the system in (4.30) becomes

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= \dot{z}_n - \ddot{z}_n + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} \dot{z}_n + v + L_g L_f^{n-1} h(T^{-1}(z)) \left[\hat{\delta}(t, z, v) \right]
\end{aligned} \tag{4.32}$$

where v is the new control input and $\hat{\delta}(t, x, v)$ is the uncertainty, nonlinearity and noise rewritten in terms of z and v . As done in the linear case, by defining a new state variable $\dot{z}_n = z_{n+1}$ with

$$\ddot{z}_n = \dot{z}_{n+1} = \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} z_{n+1} + v + L_g L_f^{n-1} h(T^{-1}(z)) \left[\hat{\delta}(t, z, v) \right], \quad \text{an } n+1$$

dimensional state space form of the system can be obtained as in (4.33). The amplitude of $\hat{\delta}(t, z, v)$ with $L_g L_f^{n-1} h(T^{-1}(z))$ is assumed to be bounded by a known function: $\left| L_g L_f^{n-1} h(T^{-1}(z)) \left[\hat{\delta}(t, z, v) \right] \right| \leq \hat{p}(t, z) + k \|v\|_\infty$ for $\hat{p}(t, z) > 0$ and $0 \leq k < 1$.

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_n &= z_{n+1} \\
\dot{z}_{n+1} &= \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} z_{n+1} + v + L_g L_f^{n-1} h(T^{-1}(z)) \left[\hat{\delta}(t, z, v) \right]
\end{aligned} \tag{4.33}$$

The sliding manifold is chosen as $s = z_{n+1} - G_c(z_1, z_2, \dots, z_n)$ where $G_c(z_1, z_2, \dots, z_n)$ is the jerk function of the reference chaotic system in the normal form of (4.3). Then, one can apply the sliding mode control in (4.34) where $\eta > 0$ is a scalar which is used to adjust finite reaching time to the sliding manifold. As in a similar way to the linear case, the first n states of the system (4.33) matches to the reference chaotic system (4.3) with the jerk function $G_c(z_1, z_2, \dots, z_n)$ when reaching phase is over.

$$v = -\frac{\eta + \hat{p}(t, z)}{1 - k} \text{sign}(s) \tag{4.34}$$

After reaching the sliding manifold, s becomes zero, so $z_{n+1} = G_c(z_1, z_2, \dots, z_n)$. Therefore, the first n states of the system (4.33) can be seen to be matched to the reference chaotic system (4.3) with the jerk function $G_c(z_1, z_2, \dots, z_n)$.

As done in the linear controllable case, $V = \frac{1}{2}s^2$ can be chosen as the Lyapunov function and its time derivative along the trajectories of the system in (4.33) is shown not to be greater than $-\eta|s|$ to ensure that the system (4.33) reaches to the sliding manifold in finite time.

$$\dot{V} = s\dot{s} = s(\nabla_{\bar{z}}s)^T \dot{\bar{z}} = s \left[v + L_g L_f^{n-1} h(T^{-1}(z)) \left[\hat{\delta}(t, z, v) \right] \right] \quad (4.35)$$

where $\nabla_{\bar{z}}s$ is the gradient of s manifold with respect to $\bar{z} = \begin{bmatrix} z^T & z_{n+1} \end{bmatrix}^T$.

$$\dot{V} \leq sv + |s| \left\| L_g L_f^{n-1} h(T^{-1}(z)) \left[\hat{\delta}(t, z, v) \right] \right\| \quad (4.36)$$

Under the assumption of $\left\| L_g L_f^{n-1} h(T^{-1}(z)) \left[\hat{\delta}(t, z, v) \right] \right\| \leq \hat{p}(t, z) + k\|v\|_\infty$, it becomes as follows.

$$\dot{V} \leq sv + \left[p(x, t) + k\|v\|_\infty \right] |s| \quad (4.37)$$

Now, substituting v in (4.34) into (4.37), an upper bound for $\dot{V}$ is obtained as follows.

$$\dot{V} \leq -\frac{\eta + p(x, t)}{1 - k} |s| + \left[p(x, t) + k \frac{\eta + p(x, t)}{1 - k} \right] |s| = -\eta |s| \quad (4.38)$$

The differential inequality in (4.38) means that the time needed to reach sliding manifold $s = 0$ should be finite and has an upper bound $t_{reach} \leq \frac{|s(0)|}{\eta}$, i.e. $t_0 = 0$

(Slotine & Li, 1991). So, (4.33) with (4.34) becomes (4.39) when $s = 0 \Rightarrow z_{n+1} = G_c(z_1, z_2, \dots, z_n)$.

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= G_c(z_1, z_2, \dots, z_n) \\ \dot{z}_{n+1} &= \dot{G}_c(z_1, z_2, \dots, z_n)\end{aligned}\tag{4.39}$$

The first n states of the system (4.39) can be seen to be matched to the reference chaotic system (4.3) with the jerk function $G_c(z_1, z_2, \dots, z_n)$. $|z_{n+1}| = |G_c(z_1, z_2, \dots, z_n)| < \infty$ since z_i 's are bounded for a chaotic trajectory and the continuous function $G_c(z_1, z_2, \dots, z_n)$ maps a bounded set into a bounded set.

4.3 Simulation Results

4.3.1 Linear System Application

A single input linear system with parameter uncertainty is considered in the form of (4.40)

$$\dot{x} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (u + \delta(x))\tag{4.40}$$

where $\delta(x)$ is the parameter uncertainty and defined as $\delta(x) = \Delta a_0 x_1 + \Delta a_1 x_2 + \Delta a_2 x_3$. $\delta(x)$ is obviously bounded by a known function $|\delta(x)| \leq p(x) = |\Delta a_0| |x_1| + |\Delta a_1| |x_2| + |\Delta a_2| |x_3|$. System can be transformed into the controllable canonical form by a linear transform as in (4.5):

$$z = Tx = \begin{pmatrix} 0 & 0.5 & -0.25 \\ 0 & -0.5 & 0.75 \\ 1 & 0.5 & -1.25 \end{pmatrix} x \quad (4.41)$$

The transformed system is in the normal form as in (4.19):

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -3 \end{pmatrix} \dot{z} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (u + \delta(T^{-1}z)) \quad (4.42)$$

To focus on the effect of parameter uncertainty, the system can also be written as

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 + \Delta\omega_0 & -4 + \Delta\omega_1 & -3 + \Delta\omega_2 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \quad (4.43)$$

where $\Delta\omega_0 = \Delta a_0 + 3\Delta a_1 + 2\Delta a_2$, $\Delta\omega_1 = 2\Delta a_0 + \Delta a_1 + 2\Delta a_2$ and $\Delta\omega_2 = \Delta a_0$.

In order to chaotify the system in (Slotine & Li, 1991), $G_c(z_1, z_2, z_3)$ is taken to be the jerk function of Chua's circuit with cubic nonlinearity as given in (4.10) and the dynamical feedback law is chosen as follows with $\alpha = 15.6$, $\beta = 28.58$, $m_0 = 0.0659179490$, $m_1 = -0.1671315463$ where the parameters are taken from (Bilotto & Pantano, 2008).

$$\begin{aligned} u &= 4z_1 + 4z_2 + 3z_3 + \dot{z}_3 - \ddot{z}_3 + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} \dot{z}_3 + v \\ &= 4z_1 + 4z_2 + 3z_3 + \dot{z}_3 - \ddot{z}_3 - \alpha 3m_0 (\beta z_1 + z_2 + z_3)^2 (\beta z_2 + z_3 + \dot{z}_3) \\ &\quad - \alpha m_1 \beta z_2 + (\alpha - \beta - \alpha m_1) z_3 - (1 + \alpha m_1) \dot{z}_3 + v \end{aligned} \quad (4.44)$$

By inserting the control input in (4.44) to the system in (4.42) and by defining a new state variable $\dot{z}_3 = z_4$ with

$\ddot{z}_3 = \dot{z}_4 = \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + v + \hat{\delta}(t, z, v)$, the 4 dimensional state space form

of the system can be obtained as:

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + v + \hat{\delta}(t, z, v)\end{aligned}\tag{4.45}$$

The sliding manifold is specified as $s = z_4 - G_c(z_1, z_2, z_3)$ and switching control input (v) described in (4.23) with setting the parameters of it as $k=0, \eta=1$ and $\hat{p}(z) = p(T^{-1}z) = |\Delta a_0||z_1 + 2z_2 + z_3| + |\Delta a_1||3z_1 + z_2| + |\Delta a_2||2z_2 + 2z_3|$ is defined as follows

$$v = -[1 + \hat{p}(z)]\text{sign}(s)\tag{4.46}$$

After the finite time $t_{reach} < |s(t=0)|/\eta$ (Slotine & Li, 1991), the system in (4.45) reaches to the sliding manifold and s becomes zero, so $z_4 = G_c(z_1, z_2, z_3)$. Then, the system in (4.45) with its first three states matches to reference chaos model in (4.3) as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= G_c(z_1, z_2, z_3) \\ \dot{z}_4 &= G_c(z_1, z_2, z_3)\end{aligned}\tag{4.47}$$

and the system in (4.40) becomes topologically conjugate of the reference chaotic system (Wang & Chen, 2003).

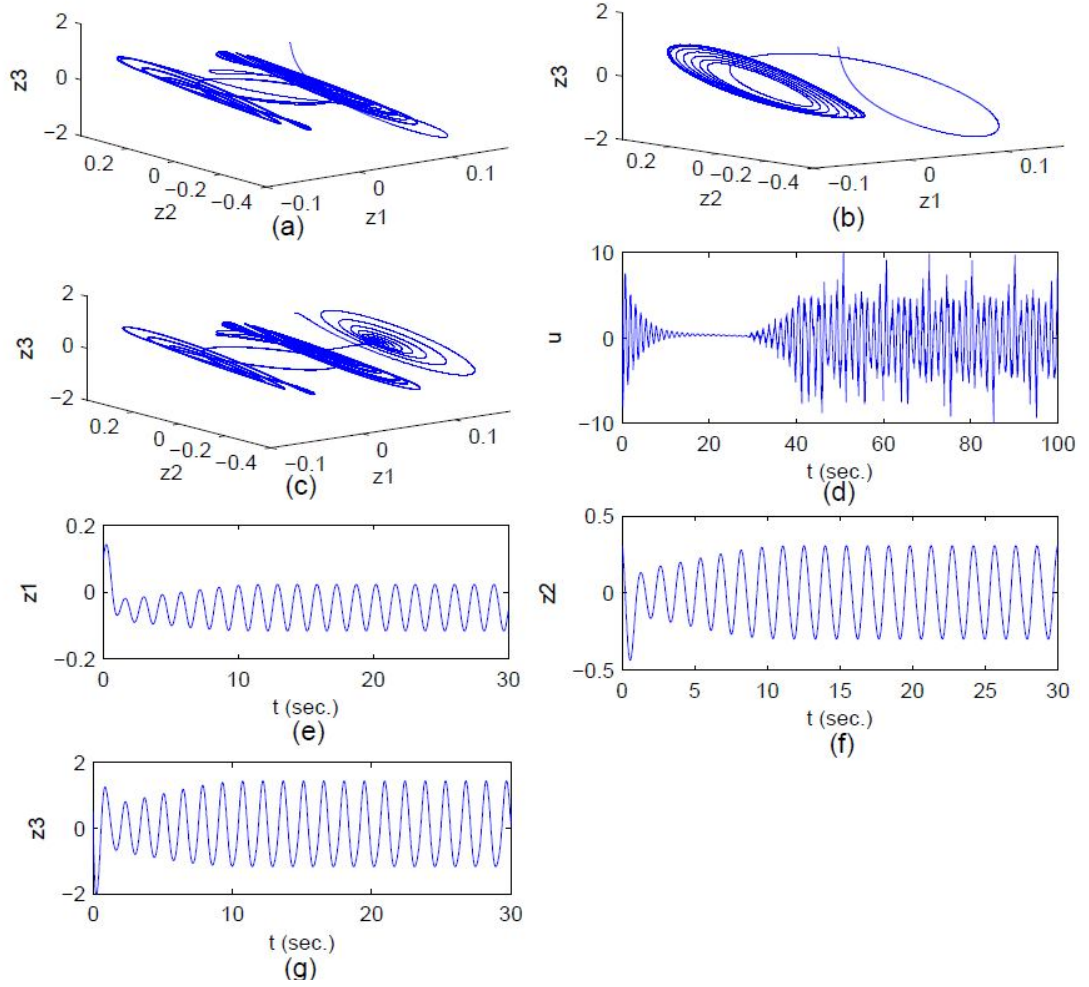


Figure 4.1 Chaotic attractor of (a) reference chaotic system, i.e. the cubic Chua's circuit (4.9) (b) the chaotified system with the model based methods in (Wang & Chen, 2003; Morgül, 2003) which cause limit cycle with lyapunov exponents $\lambda_1 \cong 0(-0.0027)$, $\lambda_2 = -2.428$, $\lambda_3 = -2.4893$ (c) the chaotified system with the proposed sliding mode control method (d) chaotifying control input in (4.44) for the proposed method (e) z_1 versus time (f) z_2 versus time (g) z_3 versus time for $t \leq 30s$ of the chaotified system with the model based methods in (Wang & Chen, 2003; Morgül, 2003).

In Figure 4.1, the simulation result for $\Delta a_0 = -0.4596$, $\Delta a_1 = 0.8103$ and $\Delta a_2 = -0.8812$ is presented. Chaotic attractor of reference chaotic system given in (4.9) is shown in Figure 4.1a. Chaotic attractor of the chaotified system with the model based methods in (Wang & Chen, 2003; Morgül, 2003) and chaotic attractor of the chaotified system with the proposed sliding mode control method are shown in Figure 4.1b, 4.1c. In Figure 4.1b, chaotified system with the model based method in (Wang & Chen, 2003; Morgül, 2003) is observed to exhibit limit cycle due to the parameter uncertainty with Lyapunov exponents $\lambda_1 \cong 0(-0.0027)$, $\lambda_2 = -2.428$, $\lambda_3 = -2.4893$ calculated by (Govorukhin, 2008). As seen in Figure 4.1c, chaotified system with the proposed method matches to the chaotic system despite the uncertainty and after a finite transient time slides on it. In Figure 4.1d, the chaotifying control input in (4.44) for the proposed method is presented. In order to observe limit cycle behavior of the chaotified system with the model based method in (Wang & Chen, 2003; Morgül, 2003), the states z_1 , z_2 and z_3 versus time for $t \leq 30s$ are shown in Figure 4.1e-4.1g.

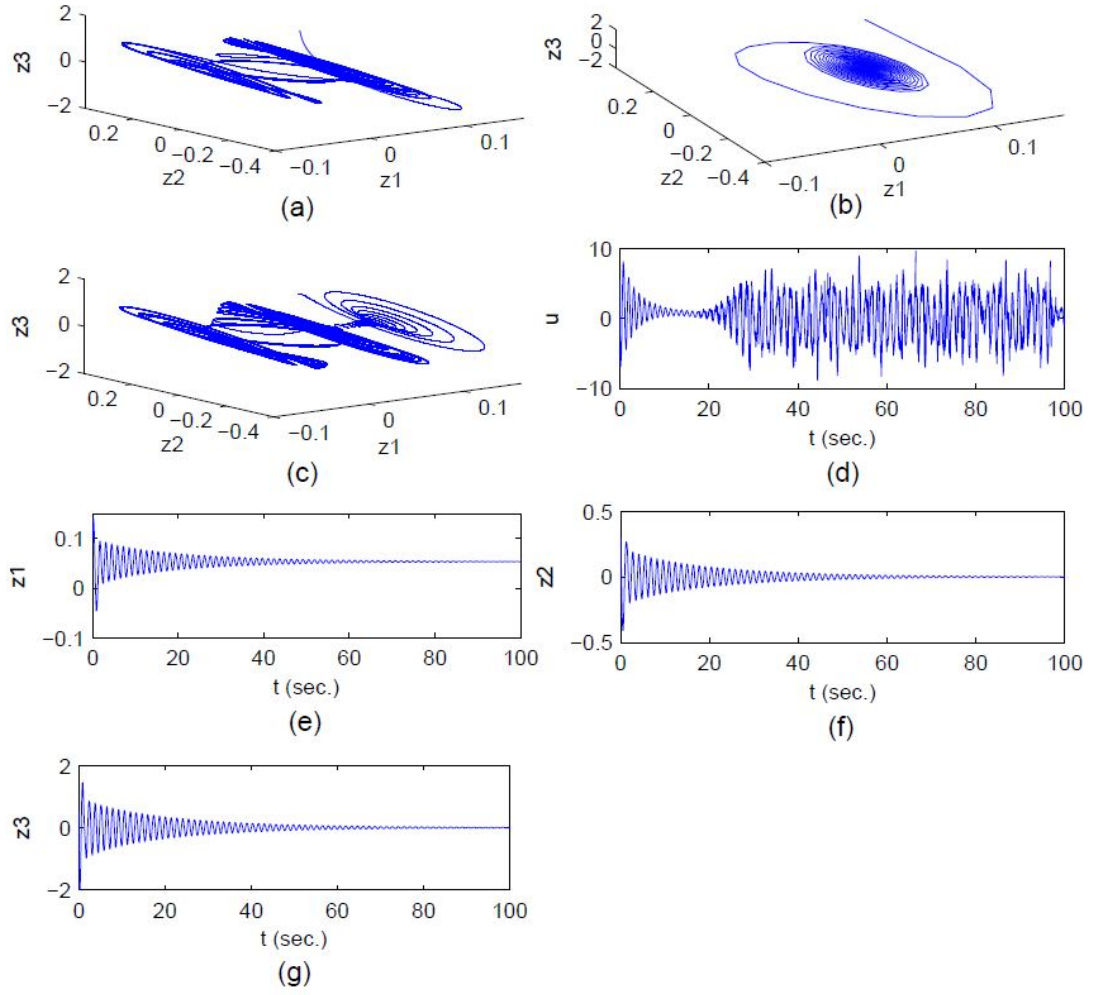


Figure 4.2 Chaotic attractor of (a) reference chaotic system, i.e. the cubic Chua's circuit (4.9) (b) the chaotified system with the model based methods in (Wang & Chen, 2003; Morgül, 2003) which tends toward an equilibrium point with Lyapunov exponents $\lambda_1=-0.0432$, $\lambda_2=-0.0446$, $\lambda_3=-6.3534$ (c) the chaotified system with the proposed sliding mode control method (d) chaotifying control input in (4.44) for the proposed method (e) z_1 versus time (f) z_2 versus time (g) z_3 versus time of the chaotified system with the model based methods in (Wang & Chen, 2003; Morgül, 2003).

In Figure 4.2, the simulation result for $\Delta a_0 = -0.8359$, $\Delta a_1 = -0.9793$ and $\Delta a_2 = -0.9844$ is presented. Chaotic attractor of reference chaotic system given in (4.9) is shown in Figure 4.2a. Chaotic attractor of the chaotified system with the model based methods in (Wang & Chen, 2003; Morgül, 2003) and chaotic attractor of the chaotified system with the proposed sliding mode control method are shown in Figure 4.2b, 4.2c. In Figure 4.2b, chaotified system with the model based method in (Wang & Chen, 2003; Morgül, 2003) is observed to tend towards an equilibrium

point due to the parameter uncertainty with Lyapunov exponents $\lambda_1 = -0.0432$, $\lambda_2 = -0.0446$, $\lambda_3 = -6.3534$ calculated by (Govorukhin, 2008). As seen in Figure 4.2c, chaotified system with the proposed method matches to the chaotic system despite the uncertainty and after a finite transient time slides on it. In Figure 4.2d, the chaotifying control input in (4.44) for the proposed method is presented. In order to observe the asymptotically stability of the chaotified system with the model based method in (Wang & Chen, 2003; Morgül, 2003), the states z_1 , z_2 and z_3 versus time are shown in Figure 4.2e-4.2g.

Furthermore, in order to show the effectiveness of the proposed method, the system in (4.40) is subjected to randomly chosen Δa_i 's in the range $[-1,1]$ for one hundred trials. For the model based method (Wang & Chen, 2003; Morgül, 2003) just 32 of 100 trials have Lyapunov exponents $\lambda_1 > 0$ in the interval of $[0.0468, 0.5529]$, $\lambda_2 \cong 0$ in the interval of $[-0.0071, 0.0015]$ and $\lambda_3 < 0$ in the interval of $[-6.8093, -3.4982]$ which is the sign of chaotic behavior for 3 dimensional systems (Sprott, 2003), whereas proposed method copes with uncertainties and after a finite transient, it exhibits chaotic behavior for all trials.

4.3.2 Nonlinear System Application

A link driven by a motor through a torsional spring (a single-link flexible-joint robot arm) with unit values of coefficients is considered. The details of the system may be found in (Slotine & Li, 1991). A state space form of the system is given as

$$\dot{x} = \begin{pmatrix} x_2 \\ -\sin x_1 - (x_1 - x_3) \\ x_4 \\ x_1 - x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} (u + \delta(t)) \quad (4.48)$$

where $\delta(t)$ is a uniformly distributed random noise which is added to see the effect of noise and it is in the interval $[d_0, d_1]$. $\delta(t)$ is bounded by a known scalar: $|\delta(t)| \leq p = \max(|d_0|, |d_1|)$.

By the change of the variables, the feedback linearizable system in (4.48) can be transformed with the global diffeomorphism $z = (x_1 x_2 - \sin x_1 - (x_1 - x_3) - x_2 \cos x_1 - (x_2 - x_4))^T$ into the normal form

$$\dot{z} = \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ \sin z_1 (z_2^2 + \cos z_1 + 1) - (z_3 + \sin z_1)(2 + \cos z_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} (u + \delta(t)) \quad (4.49)$$

In order to chaotify the system in (4.49), $G_c(z_1, z_2, z_3, z_4)$ is taken to be the jerk function of four dimensional chaotic system defined in (4.15) with quadratic nonlinearity as given in (4.16) and the dynamical feedback law is chosen as

$$\begin{aligned} u &= \dot{z}_4 - \ddot{z}_4 - \sin z_1 (z_2^2 + \cos z_1 + 1) + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + \frac{\partial G_c}{\partial z_4} \dot{z}_4 + v \\ &= \dot{z}_4 - \ddot{z}_4 - \sin z_1 (z_2^2 + \cos z_1 + 1) + 2z_2^2 + 2z_1(z_2 + z_3) \\ &\quad + \beta z_3 + (\alpha + \beta)z_4 + (\alpha - 1)\dot{z}_4 + v \end{aligned} \quad (4.50)$$

By inserting the control input in (4.50) to the system in (4.49) and by defining a new state variable $\dot{z}_4 = z_5$ with

$$\ddot{z}_4 = \dot{z}_5 = \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + \frac{\partial G_c}{\partial z_4} z_5 + v + \hat{\delta}(t),$$

the 5 dimensional state space form of the system can be obtained as:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= z_5 \\
\dot{z}_5 &= \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + \frac{\partial G_c}{\partial z_4} z_5 + v + \hat{\delta}(t)
\end{aligned} \tag{4.51}$$

The sliding manifold is specified as $s = z_5 - G_c(z_1, z_2, z_3, z_4)$ and the switching control input v described in (4.34) with setting the parameters of it $k = 0, \eta = 1$ and $\hat{p}(t) = p$ is defined as follows

$$v = -(1 + p)\text{sign}(s) \tag{4.52}$$

After the finite time $t_{reach} < |s(t=0)|/\eta$ (Slotine & Li, 1991), the system reaches to the sliding manifold and s becomes zero, so $z_5 = G_c(z_1, z_2, z_3, z_4)$ and the system in (4.51) with its four states matches to reference chaos model in (4.3) as

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= G_c(z_1, z_2, z_3, z_4) \\
\dot{z}_5 &= \dot{G}_c(z_1, z_2, z_3, z_4)
\end{aligned} \tag{4.53}$$

and the system in (4.48) becomes topologically conjugate of the reference chaotic system (Wang & Chen, 2003).

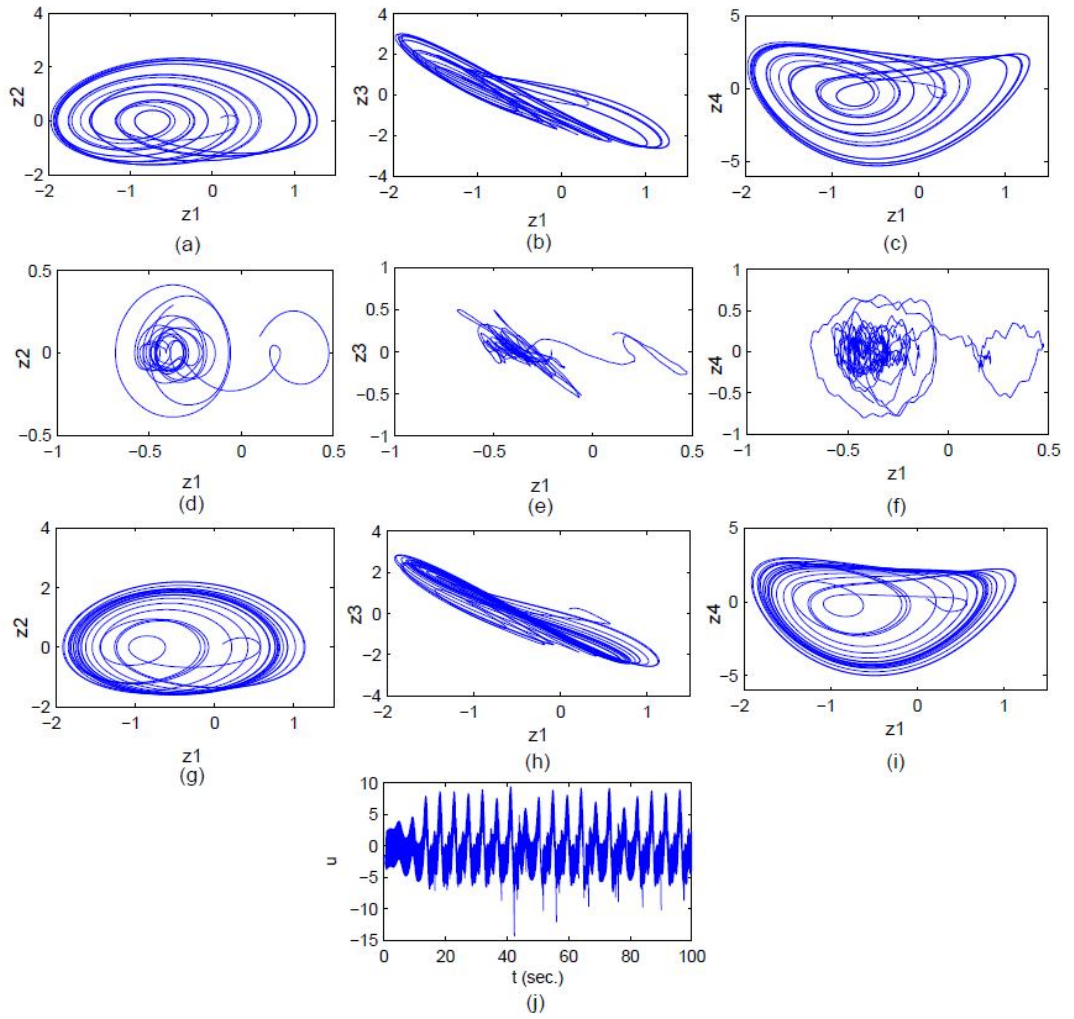


Figure 4.3 (a) z_1 versus z_2 (b) z_1 versus z_3 (c) z_1 versus z_4 of reference chaotic system in (4.15), (d) z_1 versus z_2 (e) z_1 versus z_3 (f) z_1 versus z_4 of the chaotic system with the model based method in (Wang & Chen, 2003; Morgül, 2003), (g) z_1 versus z_2 (h) z_1 versus z_3 (i) z_1 versus z_4 of the chaotic system with the proposed sliding mode control method and (j) chaotifying control input in (4.50) for the proposed method.

In Figure 4.3, the simulation result for $d_0 = -0.18$ and $d_1 = 1.93$ is presented. In Figure 4.3a-4.3c, z_1 versus z_2 , z_3 and z_4 of reference chaotic system in (4.15) are shown. In Figure 4.3d-4.3f, z_1 versus z_2 , z_3 and z_4 of the chaotified system with the model based method in (Wang & Chen, 2003; Morgül, 2003) are shown. In Figure 4.3g-4.3i, z_1 versus z_2 , z_3 and z_4 of the chaotified system with the proposed sliding

mode control method are shown. In Figure 4.3j, chaotifying control input in (4.50) for the proposed method is presented. In Figure 4.3d-4.3f, it is observed that the chaotified system with the model based methods in (Wang & Chen, 2003; Morgül, 2003) exhibits different behavior than reference chaotic system due to the effect of the uniformly distributed noise. As seen in Fig. 4.3g-4.3i, chaotified system with proposed method reaches chaotic manifold despite the noise and slides on it thereafter.

Furthermore, in order to show the effectiveness of the proposed method, the system in (4.48) is subjected to uniformly distributed random noise $\delta(x)$ in the interval $[d_0, d_1]$ where d_0 and d_1 are chosen randomly between $[-2, 2]$ for one hundred trials. For the model based method (Wang & Chen, 2003; Morgül, 2003) just 7 of 100 trials have Lyapunov exponents $\lambda_1 > 0$ in the interval $[0.0145, 0.1543]$, $\lambda_2 \cong 0$ in the interval $[-0.0049, 0.0061]$, $\lambda_3 < 0$ in the interval $[-0.6464, 0.5121]$ and $\lambda_4 < 0$ in the interval $[-1.014, -1.0018]$ which is the sign of chaotic behavior for 4 dimensional systems (Sprott, 2003), whereas the proposed method copes with noises and after a finite transient time, it exhibits chaotic behavior for all trials.

CHAPTER FIVE

STABILITY ANALYSIS OF PIECEWISE AFFINE SYSTEMS AND LYAPUNOV BASED CONTROLLER DESIGN

In this section, a stability analysis of PWA systems with polyhedral regions represented by intersection of degenerate ellipsoids inspired from (Rodrigues & Boyd, 2005) is introduced. The results of (Rodrigues & Boyd, 2005) for PWA systems with slab regions are extended to PWA systems with polyhedral regions represented by intersections of degenerate ellipsoids. The stability problem is formulated as a set of LMIs. Furthermore, a stability analysis of PWA systems over bounded polyhedral regions by using a vertex based representation is presented. In this stability analysis, the polyhedral regions are determined by vertices and the set of LMIs are obtained based on these vertices.

5.1 Stability Analysis of PWA Systems with Polyhedral Regions Represented by Intersection of Degenerate Ellipsoids

Polyhedral regions in PWA systems can be outer approximated by a quadratic curve or union of ellipsoids (Hassibi & Boyd, 1998; Johansson, 2003; Johansson & Rantzer, 1998; Samadi & Rodrigues, 2011). On the other hand, piecewise slab system which is an important special form of PWA system can be exactly represented by degenerate ellipsoid (Rodrigues & Boyd, 2005). The outer approximation is useful to formulate the stabilization problem as a LMI problem (Rodrigues & Boyd, 2005). In this subsection, a stability analysis for a linear partition consisting of the polyhedral regions which are represented by intersection of degenerate ellipsoids is introduced.

A degenerate ellipsoid is a special form of ellipsoid which can exactly represent polyhedral regions in the form of $R_i = \{x \mid d_1^i < c^T x < d_2^i\}$ with $R_i \cap R_j = \emptyset$ for $i \neq j$ and can be formulated as $\varepsilon_i = \{x \mid \|E_i x + f_i\| \leq 1\}$ where $E_i = 2c^T / (d_2^i - d_1^i)$ and

$f_i = -(d_2^i + d_1^i) / (d_2^i - d_1^i)$. Any polyhedral region defined as

$$R_i = \{x | h_{ij}^T x - g_{ij} < 0, j = 1, 2, \dots, p_i\} = \{x | H_i x - g_i < 0\}.$$

where $h_{ij} \in R^n$, $g_{ij} \in R$, $H_i \in R^{p_i \times n}$ and $g_i \in R^{p_i}$ can be exactly represented by the intersection of uttermost p_i degenerate ellipsoids as

$$R_i = \mathcal{E}_{i1} \cap \mathcal{E}_{i2} \cap \dots \mathcal{E}_{ip_i} \quad (5.1)$$

with

$$\mathcal{E}_{ij} = \{x | \|E_{ij}x + f_{ij}\| \leq 1\} \quad (5.2)$$

for $j = 1, 2, \dots, p_i$. As shown in Figure 5.1, using sufficiently large degenerate ellipsoids for each hyperplane, any bounded polyhedral region can be exactly represented by the intersection of p_i degenerate ellipsoids.

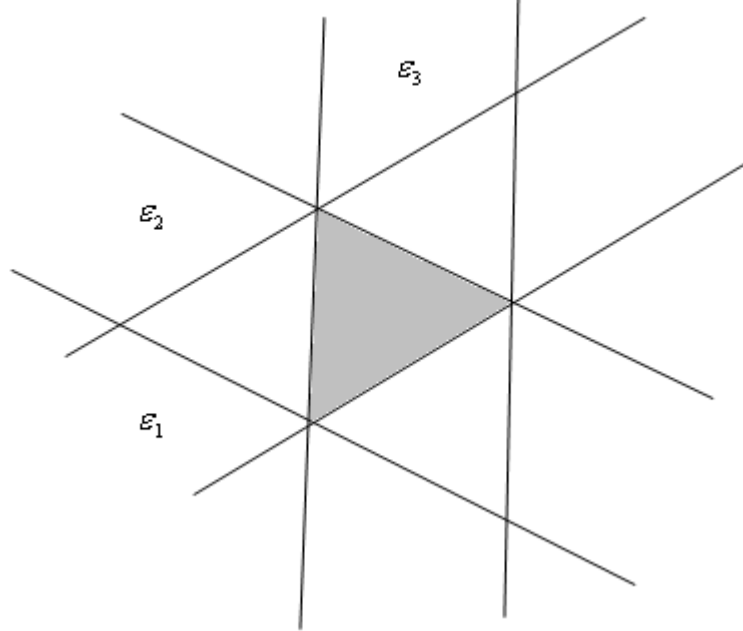


Figure 5.1 A polyhedral region (gray) represented by intersection of three degenerate ellipsoids

A sufficient condition for stability is obtained by finding a Lyapunov function in a special form. For the stability analysis with the bounded polyhedral regions

represented by intersection of degenerate ellipsoids, a set of LMIs for searching a globally defined quadratic Lyapunov function is introduced. The unbounded polyhedral regions in the system can be represented by sufficiently large bounded polyhedral regions in order to obtain meaningful results for practical applications.

In this subsection, an n -dimensional PWA system is considered in the following form

$$\dot{x} = A_i x + b_i \quad (5.3)$$

with the region R_i for $i = \{1, 2, \dots, l\}$ where $A_i \in R^{n \times n}$ is a matrix, $b_i \in R^n$ is a vector.

The region R_i is polytopic and defined as

$$R_i = \{x | h_{ij}^T x - g_{ij} < 0, j = 1, 2, \dots, p_i\} = \{x | H_i x - g_i < 0\} \quad (5.4)$$

where $h_{ij} \in R^n$, $g_{ij} \in R$, $H_i \in R^{p_i \times n}$ and $g_i \in R^{p_i}$. The dimensions of $H_i \in R^{p_i \times n}$ and $g_i \in R^{p_i}$ are arbitrary for every region. The following theorem gives a sufficient condition for the asymptotic stability.

Theorem 5.1: All trajectories of the PWA system (5.3) converge to $x = 0$ if there exist $P \in R^{n \times n}$ and $\tau_{ij} > 0$ for $i \in \{1, 2, \dots, M\}$ and $j \in \{1, 2, \dots, p_i\}$ such that

$$\begin{aligned} P &> 0, \\ A_i^T P + P A_i &< 0, \quad \forall i \in I(0), \end{aligned} \quad (5.5)$$

$$\begin{bmatrix} A_i^T P + P A_i - \sum_{j=1}^{p_i} \tau_{ij} E_{ij}^T E_{ij} & P b_i - \sum_{j=1}^{p_i} \tau_{ij} E_{ij}^T f_{ij} \\ b_i^T P - \sum_{j=1}^{p_i} \tau_{ij} f_{ij}^T E_{ij} & - \sum_{j=1}^{p_i} \tau_{ij} (f_{ij}^T f_{ij} - 1) \end{bmatrix} < 0, \quad \forall i \notin I(0) \quad (5.6)$$

where M is the number of regions and $I(0) = \{i \in \{1, 2, \dots, M\} \mid 0 \in R_i\}$ is the index set for the region that contain origin.

Proof:

The globally quadratic function in (5.7) is chosen as a Lyapunov function candidate.

$$V(x) = x^T P x \quad \text{with} \quad P = P^T > 0 \quad (5.7)$$

Since (5.5) is a negative definite matrix, it can be written that

$$x^T (A_i^T P + P A_i) x < 0 \quad (5.8)$$

which implies $\dot{V} < 0$ the derivative of the Lyapunov function is strictly negative for $i \in I(0)$ such that $\dot{V} < 0$. Moreover, since (5.6) defines a negative definite matrix, it can be written that

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P + P A_i - \sum_{j=1}^{p_i} \tau_{ij} E_{ij}^T E_{ij} & P b_i - \sum_{j=1}^{p_i} \tau_{ij} E_{ij}^T f_{ij} \\ b_i^T P - \sum_{j=1}^{p_i} \tau_{ij} f_{ij}^T E_{ij} & -\sum_{j=1}^{p_i} \tau_{ij} (f_{ij}^T f_{ij} - 1) \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0 \quad (5.9)$$

which implies

$$\frac{\partial V}{\partial x} \cdot (A_i x + b_i) + \sum_{j=1}^{p_i} \tau_{ij} (\|E_{ij} x + f_{ij}\| - 1) < 0. \quad (5.10)$$

(5.10) with (5.2) and (5.1) implies that $\dot{V} < 0$. ■

The system (5.3) with control input is as follows

$$\dot{x} = A_i x + b_i + Bu \quad (5.11)$$

with the region R_i for $i = \{1, 2, \dots, l\}$ where $A_i \in R^{n \times n}$, $B \in R^{n \times m}$, $b_i \in R^n$ and $u \in R^m$ is the control input. The following theorem gives a set of LMIs to design a linear controller for the system (5.11). The system (5.11) with $u = Kx$ forms the following closed loop system.

$$\dot{x} = (A_i + BK)x + b_i \quad (5.12)$$

Theorem 5.2: There exist a linear controller $u = Kx$ with $K = Y^{-1}Q$ which asymptotically stabilizes the system (5.11) if there exist a symmetric matrix $P \in R^{n \times n}$, $Q \in R^{m \times n}$, $Y \in R^{m \times m}$ and $\tau_{ij} > 0$ for $i \in \{1, 2, \dots, M\}$ and $j \in \{1, 2, \dots, p_i\}$ such that

$$P > 0, \quad (5.13)$$

$$A_i^T P + Q^T B^T + P A_i + B Q < 0, \quad \forall i \in I(0), \quad (5.14)$$

$$\begin{bmatrix} A_i^T P + P A_i + Q^T B^T + B Q - \sum_{j=1}^{p_i} \tau_{ij} E_{ij}^T E_{ij} & P b_i - \sum_{j=1}^{p_i} \tau_{ij} E_{ij}^T f_{ij} \\ b_i^T P - \sum_{j=1}^{p_i} \tau_{ij} f_{ij}^T E_{ij} & -\sum_{j=1}^{p_i} \tau_{ij} (f_{ij}^T f_{ij} - 1) \end{bmatrix} < 0, \quad \forall i \notin I(0) \quad (5.15)$$

$$PB = BY \quad (5.16)$$

$$Y > 0 \quad \text{or} \quad Y < 0 \quad (5.17)$$

where M is the number of regions and $I(0) = \{i \in \{1, 2, \dots, M\} \mid 0 \in R_i\}$ is the index set for the region that contain origin.

Proof:

The globally quadratic function in (5.18) is chosen as a Lyapunov function candidate for the closed loop system (5.12).

$$V(x) = x^T P x \quad \text{with} \quad P = P^T > 0 \quad (5.18)$$

Substituting $Q = YK$ and using (5.16), (5.14) and (5.15) can be rewritten as

$$(A_i + BK)^T P + P(A_i + BK) < 0 \quad (5.19)$$

$$\begin{bmatrix} (A_i + BK)^T P + P(A_i + BK) - \sum_{j=1}^{p_i} \tau_{ij} E_{ij}^T E_{ij} & P b_i - \sum_{j=1}^{p_i} \tau_{ij} E_{ij}^T f_{ij} \\ b_i^T P - \sum_{j=1}^{p_i} \tau_{ij} f_{ij}^T E_{ij} & -\sum_{j=1}^{p_i} \tau_{ij} (f_{ij}^T f_{ij} - 1) \end{bmatrix} < 0, \quad (5.20)$$

Multiplying (5.19) and (5.20) from left and right by $\begin{bmatrix} x \\ 1 \end{bmatrix}^T$ and $\begin{bmatrix} x \\ 1 \end{bmatrix}$ yields

$$x^T [(A_i + BK)^T P + P(A_i + BK)] x < 0 \quad (5.21)$$

$$x^T [(A_i + BK)^T P + P(A_i + BK)] x + x^T P b_i + b_i^T P x - \left(\sum_{j=1}^{p_i} (\|E_{ij} x + f_{ij}\| - 1) \right) < 0 \quad (5.22)$$

(5.22) with (5.2) implies

$$x^T [(A_i + BK)^T P + P(A_i + BK)] x + x^T P b_i + b_i^T P x < 0 \quad (5.23)$$

A quadratic Lyapunov function $V = x^T P x$ with (5.21) for all $i \in I(0)$ and (5.23) for all $i \notin I(0)$ implies the asymptotic stability of the closed loop system (5.12). ■

5.2 Stability Analysis of PWA Systems over Bounded Polyhedral Regions by Using a Vertex Based Representation

In this subsection, PWA system in the form

$$\dot{x} = A_i x + b_i \quad (5.24)$$

with $R_i^x = \{x \mid h_{ij}^T x - g_{ij} < 0, j = 1, 2, \dots, p_i\} = \{x \mid H_i^T x - g_i < 0\}$ with

$H_i = [h_{i1}, h_{i2}, \dots, h_{ip_i}]$ is considered. Any bounded polyhedral region R_i^x can be formulated with the convex combination of vertices of this region as:

$$R_i^x = \text{conv}(v_{i,1}^x, v_{i,2}^x, \dots, v_{i,K_i}^x) = \left\{ x \in R^n \mid x = \sum_{k=1}^{K_i} \mu_k^x v_{i,k}^x, \mu_k^x \in [0, 1], \sum_{k=1}^{K_i} \mu_k^x = 1 \right\} \quad (5.25)$$

Any affine mapping $f(x): R \rightarrow R$ in the region R_i^x maps a convex hull of vertices into the convex hull of the images of these vertices. Then, the mapping can be formulated as:

$$y = f_i(x) = \sum_{k=1}^{K_i} \mu_k^x f(v_{i,k}^x) \text{ for } R_i^x = \left\{ x \in R^n \mid x = \sum_{k=1}^{K_i} \mu_k^x v_{i,k}^x, \mu_k^x \in [0, 1], \sum_{k=1}^{K_i} \mu_k^x = 1 \right\}$$

$$\text{and the image of the region } R_i^x \text{ is } R_i^y = \left\{ x \in R^n \mid x = \sum_{k=1}^{K_i} \mu_k^y v_{i,k}^y, \mu_k^y \in [0, 1], \sum_{k=1}^{K_i} \mu_k^y = 1 \right\}$$

where $v_{i,k}^y = f(v_{i,k}^x)$.

The following theorem gives a sufficient condition for the exponential stability.

Theorem 5.3: All trajectories of the PWA system (5.24) exponentially converge to $x = 0$ if there exist $P \in R^{n \times n}$ for $i \in \{1, 2, \dots, M\}$ such that

$$P > 0 \quad (5.26)$$

$$\begin{bmatrix} (V_i^x)^T (A_i^T P + P A_i + \alpha P) V_i^x & (V_i^x)^T P b_i \\ b_i^T P V_i^x & 0 \end{bmatrix} \leq 0 \quad (5.27)$$

where $V_i^x = \begin{bmatrix} v_{i,1}^x & v_{i,2}^x & \cdots & v_{i,K_i}^x \end{bmatrix}$, M is the number of regions.

Proof:

Multiplying (5.27) by $\begin{bmatrix} (\eta^x)^T & 1 \end{bmatrix}$ and $\begin{bmatrix} (\eta^x)^T & 1 \end{bmatrix}^T$ from left and right respectively yields

$$(V_i^x \eta^x)^T (A_i^T P + P A_i + \alpha P) V_i^x \eta^x + b_i^T P V_i^x \eta^x + (V_i^x \eta^x)^T P b_i \leq 0 \quad (5.28)$$

where $\eta^x = \begin{bmatrix} \mu_1^x & \mu_2^x & \cdots & \mu_{K_i}^x \end{bmatrix}^T$.

substituting $x = V_i^x \eta^x = \sum_{k=1}^{K_i} \mu_k^x v_{i,k}$, (5.28) can be written that

$$x^T (A_i^T P + P A_i + \alpha P) x + b_i^T P x + x^T P b_i \leq 0. \quad (5.29)$$

(5.29) implies that

$$\dot{V} = x^T (A_i^T P + P A_i) x + b_i^T P x + x P b_i \leq -\alpha x^T P x \text{ for } x \in R_i^x \quad (5.30)$$

A quadratic Lyapunov function $V = x^T P x$ with (5.30) implies the exponential stability of the system (5.24). ■

CHAPTER SIX

CONCLUSION

The thesis introduces several methods for the analysis and design of PWA controller systems. The contributions can be summarized under the three main groups, namely on Lyapunov stability, feedback linearization and robust chaotification.

The first main group of contributions is on the Lyapunov stability of PWA control systems. Quadratic Lyapunov function is developed in order to derive stability conditions for PWA dynamical systems. For the Lyapunov stability of PWA systems, two sets of sufficient conditions different from the ones available in the literature are developed in the thesis. In one of the approaches followed in the derivations to prove the stability for the proposed sufficient conditions, the polyhedral regions are considered as the intersections of degenerate ellipsoids. In the other approach, the regions are expressed in terms of vertices. The conditions are obtained in terms of linear matrix inequalities, so they give the possibility of applying efficient convex optimization techniques to solve the stability problems. The results are useful particularly in the controller design ensuring the closed loop stability and have potential applications particularly in the analysis and design of non-smooth control systems.

The second main group of contributions proposes the canonical representation based feedback linearization of PWA control systems. Canonical representations for normal forms of PWA systems are obtained for both single-input and multi-input cases. Input-state feedback linearization methods for PWA systems are established based on the canonical representation of PWA functions. The conditions on partially feedback linearizability of PWA systems while maximizing the relative degree are determined in this context as a combinatorial problem. An approximate linearization based controller design for PWA systems is proposed in the thesis also by using the canonical representation. The canonical representation approach exploited in the feedback linearization provides a parameterization of the controller design for PWA

systems and also provides compact closed form representations for the control systems and controllers simplifying the analysis of non-smooth systems. The results obtained in the thesis for the canonical representations defined over non-degenerate polyhedral partitions resulting in 1-nested absolute value canonical forms can further be extended to other canonical representations of PWA systems employing higher degrees of nesting of absolute value operation. These parameterizations and closed form solutions have a potential to be used as mathematical models for further theoretical and applied studies on the analysis and design of control systems.

The third main group of contributions of the thesis is on the robust chaotification of input-state linearizable systems. A model based robust chaotification scheme using sliding mode control which, indeed, defines a special type of discontinuous nonlinear controller is developed. The proposed PWA controller applies a dynamical state feedback in order to match the system states to a reference chaotic system. The sliding mode control makes the chaotification to be robust against to noise/disturbances including the model uncertainties. The chaotification method is valid for any input-state linearizable system and also applicable for PWA systems which are partially feedback linearizable with relative degree more than or equal to three.

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