# Modeling and Analyzing the Effect of Financial Flows in Inventory Management

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

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Date:

## ABSTRACT

In the first part of this study, we examine two different financially constrained inventory management problems. First, we insert a payment constraint into the newsvendor problem with pricing. A newsvendor decides on how many items to order and also how to price them, since demand is random but a function of price. We numerically compare two cases; with and without payment constraints and analyze how financial constraints affect newsvendor's decisions. Second, we investigate a dynamic pricing problem with shortfall minimization objective. The seller wants to minimize a possible positive gap between a target level and his total profit. We perform a numerical comparison between the classical expected profit maximization dynamic pricing problem and our version.

In the second part, we investigate a simple dynamic inventory problem with random supply. A seller decides whether to order or not in a given period when the probability of receiving the ordered item is less than 1. We investigate this problem with two kinds of objectives: the expected profit maximization and the expected utility maximization.

In the third part, we consider the dynamic pricing problem of inventories with a risk-averse approach. Risk-aversion manifests itself in our problem as a utility maximization formulation. In the classical dynamic pricing problem, the objective of the seller is purely to maximize the expected total profit obtained by selling his inventories. In our problem, the objective of the seller is to maximize the expected utility. We use an exponential utility function where the risk tolerance constant reflects the risk-aversion degree of the seller. For this formulation, we establish some structural properties of the optimal solution. A numerical study investigating the influence of risk-aversion on pricing decisions is also presented.

## ÖZETÇE

Bu çalışmanın ilk bölümünde, iki farklı finansal kısıtlı envanter yönetimi problemi incelenmiştir. Birincisi, gazate dağıtıcısı problemine fiyat kararının da eklenmiş halidir. Gazete dağıtıcısı sipariş miktarı ile birlikte satış fiyatına da karar vermektedir ve talep fiyata bağlı bir fonksiyondur. Bu problem ile ilgili, finansal kısıtın etkisini görmek amacıyla, finansal kısıtlı ve finansal kısıtsız olmak üzere iki model sayısal örnek üzerinden karşılaştırılmış ve analiz edilmiştir. Bu bölümde çalışılan ikinci problem ise dinamik fiyatlandırma probleminin farklı bir versiyonudur. Bu versiyonda satıcının amacı beklenen kârı ençoklamak değil, hedeflenen kâr ile elde edilen kâr arasındaki farkı enazlamaktır. Bu problem ve klasik dinamik fiyatlandırma problemini karşılaştıran bir sayısal çalışma da bölümün sonuna eklenmiştir.

Çalışmanın ikinci bölümünde, arzın rassal olduğu dinamik envanter problemi incelenmiştir. Her dönemin başında, karar verici sipariş verip vermemek arasında karar verir ve eğer sipariş verirse, belli bir olasılıkla siparişi eline geçer. Bu problem, beklenen karı ençoklama ve beklenen faydayı (utility) ençoklama olmak üzere iki farklı amaç fonksiyonu ile incelenmiştir.

Son bölümde ise riske duyarlı karar verici ile dinamik fiyatlandırma problemi incelenmiştir. Riske duyarlılık modele fayda ençoklaması olarak yansıtılmıştır. Karar vericinin amacı, satış sezonu sonu beklenen faydayı ençoklamaktır. Bunun için eksponansiyel fayda fonksiyonu kullanılmıştır. Risk tolerans katsayısı satıcının riske duyarlılık derecesini yansıtır. Bu problem için, eniyi çözümün bazı yapısal özellikleri ve riske duyarlılık derecesinin fiyat kararları üzerindeki etkisini inceleyen bir sayısal örnek sunulmuştur.

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# NOMENCLATURE

| | |
|---|---|
| $p$ | :Selling price |
| $w$ | :Unit purchasing cost |
| $s$ | :Salvage value |
| $Q$ | :Quantity ordered |
| $D$ | :Demand |
| $e$ | :Random disturbance on demand |
| $\bar{e}$ | :Upper level of random disturbance on demand |
| $a$ | :Certain demand |
| $b$ | :Decrease on demand per unit price change |
| $\Pi(p, Q)$ | :Random profit to be gained when setting price $p$ and ordering $Q$ items |
| $K$ | :Payment amount |
| $\alpha$ | :Probability not to satisfy the payment constraint |
| $\lambda_p$ | :Probability a customer arrives (buys an item) when current price is $p$ |
| $I$ | :Inventory level |
| $J_n(I)$ | :Optimal expected profit to be gained from period $n$ to the end when the current inventory level is $I$ |
| $T$ | :Target profit level |
| $n$ | :Periods from 1 to $N$ |
| $X_n$ | :Current cash level at period $n$ |
| $J_n^{sh}(X, I)$ | :Optimal value of having $X$ amount of cash and $I$ inventory in period $n$ for the expected shortfall minimization objective |
| $ETP$ | :Expected total profit |
| $ESR$ | :Expected shortfall ratio |

| | |
|---|---|
| $DiETP$ | :Difference in the expected total profits |
| $DiEsR$ | :Difference in the expected shortfall ratios |
| $q$ | :Probability that ordered item arrives |
| $h$ | :Holding cost per item per period |
| $\gamma$ | :Degree of risk-aversion |
| $R_n(I;p)$ | :Expected profit to be gained from period $n$ to the end when the current inventory level is $I$ and current given price is $p$ |
| $\omega$ | :Price policy (set of given prices) |
| $G_n(I)$ | :Optimal expected utility to be gained from period $n$ to the end when the current inventory level is $I$ |
| $U_n(I;p)$ | :Expected utility to be gained from period $n$ to the end when the current inventory level is $I$ and current given price is $p$ |

# Chapter 1

# INTRODUCTION

Demand randomness makes inventory management a challenging problem. To manage the uncertainty, one may evaluate and weigh potential risks. This is where the risk attitude of decision maker appears. Classical models in inventory management assumes that the decision maker is risk-neutral but in real life, it is a well-known fact that majority of the decision makers show risk-averse behaviors.

## *1.1 Motivation*

There have been many studies on inventory management and the dynamic pricing problem. An extensive literature has been presented in both deterministic and stochastic models examining the newsvendor problem, dynamic inventory and pricing problems. However, most of the research on this field assumes that the decision makers can practically implement theoretical optimal decisions without considering financial/budgetary availabilities. In real life, companies trying to optimize inventory and pricing operations without considering their financial position may fall into economical infeasibility and hold risk of bankruptcy. Especially start-up and/or growing companies are relatively more sensitive to financial fluctuations than mature ones. Thus, optimizing inventory and pricing decisions in accordance with financial constraints has a recent increase in popularity, especially in such an economical recession periods.

A limited number of the works in the literature takes the financial considerations into account. This gap leaded us to point our research to this field. In this thesis, we involve financial considerations in inventory management in basically two ways:

adding financially binding constraint and setting a risk-averse objective.

## 1.2 Some Explanatory Words on Risk-Aversion

The main-flow studies on the inventory and/or pricing management provide solutions for the risk-neutral decision makers. The major characteristics that makes a decision maker risk-neutral is that he aims to maximize the expected profit or to minimize the expected cost, insensitively to the variations on profit or cost. On the other hand, a risk-averse (risk-sensitive) decision maker prefers relatively lower profits (or higher costs) in order to avoid the risk of extreme losses.

A simple example may help to capture the idea behind risk-aversion. Assume a lottery game with two options: The first option promises a hundred units of a prize with probability 0.5 and nothing with probability 0.5. The second option promises eighty units of a prize with probability 0.5 and ten units of a prize with probability 0.5. Since the expected gain of the first option (50 units) is greater than the expected gain of the second option (45 units), a risk-neutral decision maker naturally prefers the first option. However, depending on his objective function and his degree of risk-aversion, a risk-averse decision maker may prefer the second option, since it promises a better gain in the worst case.

There have been many ways of representing risk-aversion. In the first part of chapter 3, we use a financial constraint protecting the decision maker against some possible undesired profit levels but, in return, preventing him from some very favorable outcomes. In the second part of chapter 3, we set an objective such that the decision maker needs to achieve a target profit level, applies a policy to avoid falling behind this target and does not care about exceeding the target. In chapters 4 and 5, we employ an exponential utility function to express risk-aversion.

## 1.3 The Newsvendor Problem

The newsvendor problem is a famous single-period single-item inventory management problem. Consider a newspaper seller who should decide how many newspapers to order at the beginning of the day, in order to sell during the day, when the demand for newspapers is random. The key here is that the newspapers can only be sold on the day they are published for. At the end of the day, unsold newspapers become worthless or may be salvaged for a value less then their costs. On the other hand, if the demand for a newspaper exceeds the number of available newspapers, the seller loses some sales and misses the opportunity of making some more profit. The classical newsvendor problem aims to offer solutions to this situation to maximize the expected profit obtained by selling newspapers. Of course, the newsvendor provides solutions not only for the case of newspaper sellers, but also for many varying analogies like plant capacity, overbooking, and target production levels for economies [10].

In the first part of chapter 3, we work on a modified version of the classical newsvendor problem. Apart from deciding on order quantity, in our model, the seller also sets the price. Since demand is a random function of the given price, the profit function of the seller includes two decision variables and becomes very interesting. Relating to the main motivation of our study (effect of financial considerations on inventory management), we insert a financial constraint to the problem. The seller is obligated to make a payment at the end of the selling season and he needs to assure that the probability of gaining enough cash to cover the payment is more than a predetermined risk level. While maximizing his expected profit, the seller also needs to satisfy the payment constraint. This constraint somehow may prevent the seller to order in the amount of the expected profit maximizer order quantity and to set the expected profit maximizer price freely. This case gives us a good chance to observe how binding that kind of a payment constraint is on newsvendor's decisions and how it affects these decisions.

### *1.4 The Dynamic Inventory Management Problem*

The dynamic inventory management problem deals with managing inventory level in a multi-period horizon. In the classical dynamic inventory management problem, there is a seller selling single item with or without initial inventory. The selling season may consist of discrete or continuous periods. At the beginning of each period, he decides how many items to order, in a stochastic demand environment, when price is not a decision. Since there is a holding cost for items in inventory, the seller aims to keep as less items as possible in inventory while satisfying the demand as much as possible. The trade off is between holding and lost sale costs. If the seller orders more items than the real demand, he pays the cost of keeping the unsold items for the next period. If the demand occurs to be more than available inventory, he loses excess demand and misses the opportunity for some additional profit. There may or may not be a fixed ordering cost charged once per each order. Solutions for the traditional dynamic inventory management problems to maximize the expected profit were presented in a very distant past by Scarf [20]. When there is a fixed ordering cost, Scarf proved that an $(s, S)$ policy is optimal. An $(s, S)$ policy suggests ordering up to the inventory level of $S$, if the inventory level is less than $s$, and ordering nothing, otherwise. He also showed that if there is no fixed ordering cost, a special case of $(s, S)$ policy, a case when $s = S$, which is known as 'base-stock policy', is optimal.

In chapter 4, we examine dynamic inventory management problem with random supply. In our problem, the seller has an initial inventory and decides whether to order or not, when there is no fixed ordering cost. Since each order includes one item, order size is not a decision. Unlike the classical case, receiving the ordered item is not guaranteed and it arrives with a probability less than 1. We analyze this problem for the expected profit maximization and for the expected utility maximization objectives and derive some properties of optimal solution for both objectives. Since it reflects risk-aversion, the expected utility maximization objective conducts this problem to the general framework of this thesis.

## *1.5 The Dynamic Pricing Problem*

Dynamic pricing application has a wide range of usage from airline companies to hotel reservation systems. It concerns the determination of selling prices in a multi-period problem. Having a number of initial inventory, a seller decides on price at the beginning of each period. No replenishment is allowed during the selling season and the unsold items are disposed in the end. So the decision maker needs to balance the risk of having unsold items at the end of the selling season and the risk of selling an item for a low price when there is a chance of selling it for a relatively higher price. Demand is a random function of the given price. The classical dynamic pricing problem solves the optimization problem for each period and for each inventory level and it was first studied by Kincaid and Darling [16]. They formulated a continuous time stochastic dynamic program and derived structural properties of the value function. About thirty years later, Gallego and Ryzin [11] presented some structural properties of the optimal decision and stated a heuristic for solving the problem.

In the second part of the chapter 3, we consider a dynamic pricing problem with the expected shortfall minimization objective. Differently from the classical dynamic pricing problem, the seller wants to minimize the expected shortfall between a targeted level and the realized profit and this objective reflects his risk-averse attitude. We run a numerical analysis comparing the results of the classical model and ours.

In the first part of the chapter 5, we summarize the background information and well-known structural properties the classical dynamic pricing problem. In the second part, we investigate the problem from a risk-averse decision maker's perspective. We propose the risk-aversion in a utility maximization formulation. The seller aims to maximize the expected exponential utility obtained by selling his inventories. We show that the structural properties of the classical problem are valid also for utility maximization problem.

## *1.6 Outline*

The organization of the thesis is as follows: Chapter 2 presents a literature review on many variations of newsvendor, inventory, and pricing problems. Chapter 3 investigates two different financially constrained inventory management and pricing problems with numerical analysis on them. Chapter 4 examines a dynamic inventory management problem with random supply. Chapter 5 analyzes a dynamic pricing problem with utility maximization objective. Chapter 6 gives concluding remarks with possible future extensions.

Chapter 2

# LITERATURE REVIEW

We survey the related literature under five main groups according to the problems studied.

## *2.1 Newsvendor Models*

Gotoh and Takano [13] consider the minimization of the conditional value-at-risk (CVaR), a most preferable risk measure in financial risk management, in the context of the well-known single period newsvendor problem, which is formulated as the maximization of the expected profit or the minimization of the expected cost. In this paper, they show that downside risk measures including the CVaR are tractable in the problem due to their convexity, and consequently, under mild assumptions on the probability distribution of the demand of the product. They provide analytical solutions or linear programming formulation of the minimization of the CVaR measures defined with two different loss functions. They also give numerical examples to clarify the difference among the models analyzed in the paper. These studies also demonstrate the efficiency of the LP solutions.

In his paper, Lau [15] considers the newsvendor problem under two new objectives, rather than maximizing the expected profit. These two new objectives are: maximizing the expected utility and maximizing the probability of achieving a budgeted profit. The relevance of the objective of maximizing the expected utility is well established in decision theory. In the paper, he develops formulas for handling the partial moments arising from adopting this objective. These formulas are useful to many other management models involving partial moments. The second objective, maximizing the probability of achieving a budgeted profit, is adopted by managers but largely ignored in the literature.

Eeckhoudt, Gollier, and Schlesinger [10] examine the effects of risk and risk-aversion in the single period inventory problem, known as the newsvendor problem. Comparative-static effects of changes in the various price and cost parameters are determined and related to the newsvendor's risk-aversion. The addition of a random background wealth and of an increase in the riskiness of newspaper demand are also examined. Although many of the comparative effects generally are ambigious, some simple restrictions on preferences are shown to lead to qualitatively deterministic results.

In our study, in the first part of the chapter 3, we insert pricing decision into the newsvendor model. Adding a financial constraint, we reflect the risk-averse character of the seller.

## 2.2 Dynamic Inventory Management Models

The paper written by Buzacott and Zhang [6] is the first attempt which incorporates asset-based financing into production decisions. They model the available cash in each period as a function of assets and liabilities that may be periodically updated according to the dynamics of the production activities, instead of setting a known, exogenously determined budgetary constraint as most existing models suggest. Their models allow different interest rates on cash balance and outstanding loans, which is an enhancement over most traditional models in that inventory financed by a loan can be more expensive than that by out-of-pocket cash. They show the importance of joint consideration of production and financing decisions in a start-up setting in which the ability to grow the firm is mainly constrained by its limited capital and dependence of bank financing. They then explain the motivation for asset-based financing by examining the decision making at a bank and a set of retailers in a newsvendor setting.

Chao, Chen and Wang [8] propose a general framework for incorporating financial states of an organization in multi-period inventory models considering lost sales. This approach differs from the traditional ones in that operational decisions are correlated with and constrained by financial flows of the firm. They study the borrowing and lending actions of the firm with or without loan limit. They characterize the optimal inventory management decisions and its dependence structure on the financial decisions.

Bouakiz and Sobel [5] show that optimal ordering policy is given by a sequence of critical numbers if the ordering costs are linear and the penalty and the holding costs are convex. These critical numbers depend on the time horizon, the discount rate and the parameter, $\mu$. the infinite horizon policy is ultimately stationary and approaches the risk-neutral policy as $n$ gets large.

Archibald, Thomas, Betts, and Johnston [2] examine the implications for companies operating decisions by considering an abstraction of the inventory problem faced by a start-up manufacturing company. They model this problem under two criteria as a Markov decision process. They also compare the characteristics of the optimal policies under the two criteria. They show that although the start-up company should be more conservative in its component purchasing strategy than if it were a well-established company, it should not be too conservative. This strategy is shown as monotone in the amount of capital that is available in the company. They extend these models to allow for interest on investment and inflation.

Chen, Sim, Simchi-Levi, and Sun [9] propose a framework for incorporating risk-aversion in multi-period inventory models as well as multi-period models that coordinate inventory and pricing strategies. They show that the structure of the optimal policy for a decision maker with an exponential utility function is almost identical to the structure of the optimal risk-neutral inventory policies. They extend these results to models in which the decision maker has access to a complete financial market and can hedge its operational risk through trading financial securities.

In the fourth chapter of this study, we analyze a simplified version of dynamic pricing problem with random supply. We draw some properties of the optimal solution for both risk-neutral (profit maximization) and risk-averse (utility maximization) cases.

## 2.3 Dynamic Pricing Models

Monalhan, Petruzzi, and Zhao [17] approach the dynamic pricing problem in a novel way by formulating a dynamic optimization model in which demand fluctuation is isoelastic but the random demand process is general. They develop structural properties that define an optimal pricing strategy over a finite horizon and investigate how that policy impacts a newsvendor's optimal procurement policy and optimal expected profit. They also establish a practical and efficient algorithm for computing the optimal prices. And finally, they examine how market parameters affect the optimal solution through a series of numerical experiments that utilize the algorithm.

Barz and Waldman [3] investigate dynamic capacity control problem with risk-averse approach and show that structural properties of risk-neutral problem also hold for risk-averse problem.

Zhao and Zheng [21] consider a dynamic pricing model for selling a given stock of a perishable product over a finite time horizon. They assume that the customers arrive according to a nonhomogeneous Poisson process. Reservation price distribution of these customers changes over time. They show that at any given time, the optimal price decreases with inventory. They also identify a sufficient condition under which the optimal price decreases over time for a given inventory level. This sufficient condition requires that the willingness of a customer to pay a premium for the product does not increase over time. They conduct numerical studies to show the profit impact of dynamic price policies. Price changes are set to compensate for statistical fluctuations of demand and to respond to shifts of the reservation price. The examples which they give in the paper show that using optimal dynamic policies achieves 2.4-7.3% profit improvement over the optimal single price policy.

Ziya, Ayhan, and Foley [23] discuss the relationships among three assumptions appear frequently in the pricing and profit management literature. These assumptions are decreasing marginal profit with respect to demand, decreasing marginal profit with respect to price, and increasing price elasticity of demand. They provide proofs and examples to show that none of these conditions implies any other. But they can be ordered from strongest to weakest over restricted regions, and the ordering depends upon the region.

In the second part of chapter 3, we examine the dynamic pricing problem with shortfall minimization objective. In chapter 5 we work again on the dynamic pricing problem but with utility maximization objective. For both cases, we present a comparison on risk-neutral and risk-averse versions of the studied problems.

## 2.4 Combined Models

Agrawal and Seshadri [1] consider a single period inventory model in which a risk averse retailer faces uncertain customer demand and makes a purchasing order quantity and selling price decisions with the objective of maximizing the expected utility. They consider two different ways in which price affects the distribution of demand. In the first model, they assume that a change in price affects the scale of the distribution. In the second model, a change in price only affects the location of the distribution. They present a methodology by which this problem with two decision variables can be simplified by reducing it to a problem in a single variable. They show that comparative to a risk-neutral retailer, a risk averse retailer in the first model will charge higher price and order less; whereas in the second model, a risk averse retailer will charge a lower price.

Gayon, Talay-Değirmenci, Karaesmen, and Örmeci [12] investigate a dynamic pricing and replenishment problem with Markov-Modulated demand where actual demand is depends on given price. They draw some structural properties of optimal pricing and replenishment decisions. They also present a numerical study and show the effect of dynamic pricing in compare to static pricing.

Petruzzi and Dada [18] insert pricing into newsvendor problem. They show how endogenous demand affect quantity decision by examining two demand-price models; additive and multiplicative.

## 2.5 Economic Order Quantity Models

Goyal [14] derives mathematical models to obtain the economic order quantity for an item for which the supplier permits a fixed delay in settling the amount owed to him. To illustrate the method he solves a numerical example. Then, Chand and Ward [7] investigated Goyal's model within the framework of the classical EOQ model.

Rachamadugu [19] addresses the problem of determining the economic order quantity when the vendor permits delay in payment. He first shows that contrary to the claims made by some earlier authors, the optimal order quantity increases as payment delay-time increases. He also shows that the Chand and Ward [7] approach yields an order quantity which is an upper bound on the true optimum. Together with the fact that the net present value function is convex, it is helpful in determining the optimum order quantity. The hybrid procedure suggested by Chand and Ward [7] is robust. The analysis given in Rachamadugu [19] has some research and pedagogical implications. Even if the average cost analysis is computationally adequate for most practical purposes, it can lead to conceptually anomalous conclusions.

# Chapter 3

# SOME STUDIES ON FINANCIALLY CONSTRAINED INVENTORY MODELS

## *3.1 Financially Constrained Newsvendor with Pricing*

In the classical newsvendor problem, a seller needs to decide how many items to order of a single-type perishable product, for one period. In this study, a modified version of the newsvendor problem is examined. Seller decides on price parameter, as well as quantity parameter. Also there exists a payment constraint. Seller needs to achieve enough cash position to make the payment at the end of the period. The aim of this section is to examine and present solution methodologies to satisfy payment constraint by sacrificing as little profit as possible.

### *3.1.1 Problem Definition*

There is a single seller with single type of product. At the beginning of each period, the seller needs to decide on both how many items to order and how much price to set. Since it is a kind of newsvendor problem, there is only one period. $Q$ is the quantity ordered and $w$ is the unit purchasing cost. Because pricing has an effect on consumer behavior, the demand $D$ is dependent on price $p$. The demand-price relationship is assumed to be linear. At the end of the selling period, the seller needs to have enough amount of cash to make the payment on time. $K$ is the amount of payment. As a constraint, the seller wants to keep the risk (probability) of not having enough cash to make the payment at the end of the selling season at less than or equal to a predetermined value $\alpha$. In other words, the seller needs to be sure that probability of making the payment on time is equal to or greater than $1 - \alpha$. Thus, we regard $\alpha$ as the risk tolerance level. The probability of making the payment on time can be manipulated by changing the price, indirectly influencing the demand. The probability also can be manipulated by the order quantity. Obviously, ordering less means spending less on purchase and keeping more cash at hand.

### *3.1.2 Mathematical Model*

Demand randomness makes the profit a random function of the price and the order quantity. If the demand occurs to be more than the quantity ordered, the seller sells all items in inventory and loses excess demand. If the demand occurs to be less than the quantity ordered, unsold items are salvaged at a value $s$. So $min(Q, D)$ gives the sales amount and $(Q - D)^+$ gives the amount of salvaged items, where $(Q - D)^+$ means $max((Q - D), 0)$. The profit function consists of summation of these two type of incomes subtracted by purchasing cost. So the following equation is the random profit function of seller;

$$\Pi(p, Q) = p\, min(Q, D) + s\, (Q - D)^+ - w\, Q \tag{3.1}$$

In order to determine optimal price and quantity, seller needs to maximize the expectation of the profit function, as follows;

$$\begin{aligned} \max_{p,Q} E[\Pi(p, Q)] &= pE[min(Q, D)] + sE[(Q - D)^+] - wQ \\ \text{s. t. } & P(\Pi(p, Q) < K) \leq \alpha \end{aligned} \tag{3.2}$$

As mentioned before, demand is a price-dependent function with a random disturbance $e$. Since we assume the relationship between demand and price to be linear, the demand is decreasing from a certain ceiling value $a$ according to the price level, where decrease on demand per unit increase on price is $b$. So the demand equation is given as follows;

$$D = a - bp + e \tag{3.3}$$

We assume $e$ to be uniformly distributed between $-\overline{e}$ and $\overline{e}$. Doing the necessary calculations, we obtain the expected profit as follows:

$$\begin{aligned} E[\Pi(p,Q)] = p\left(\frac{(Q-a+bp+\overline{e})(Q+a-bp-\overline{e})}{4\overline{e}} + \frac{Q(a-bp+\overline{e}-Q)}{2\overline{e}}\right) \\ +s\left(\frac{Q^2-(2Q(a-bp-\overline{e})-(a-bp-\overline{e})^2)}{4\overline{e}}\right) - wQ \end{aligned} \quad (3.4)$$

### *3.1.3 Numerical Results*

The problem has two decision variables: number of items to order and price to charge. Price is not constrained to a limited number of potential prices. Instead, the seller can choose all non-negative values as price. For the existence of an optimal solution, the expected profit function should be jointly concave according to the price and quantity variables; $p$ and $Q$. Unfortunately, the expected profit function is jointly concave only under some conditions. As a counterexample, figure 3.1 shows the distribution of the expected profit for parameters $w = 3$, $s = 2$, $a = 100$, $b = 1$, and $\overline{e} = 10$.

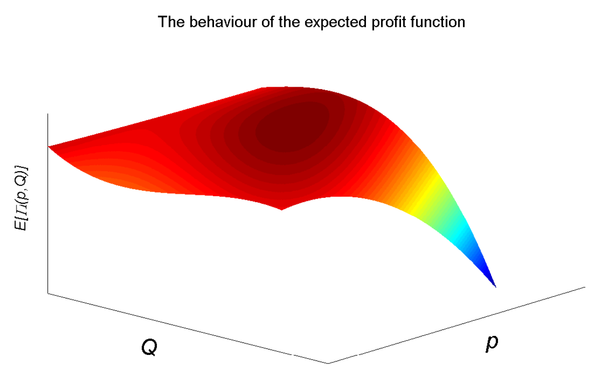


Figure 3.1: The behavior of the expected profit function

In figure 3.1, we observe that $Q$ curve for low $p$ values (lower-left edge) has a convex-looking pattern, while for high $p$ values (upper-right edge) it has concavelooking pattern. This visual example indicates that the function does not hold jointly concavity property.

This situation disables us from drawing a general optimal solution for price and quantity. Therefore, we solve the optimization problem numerically using GAMS for different $K$ and $\alpha$ levels. For sample problem, we set $w = 10$, $s = 2$, $a = 100$, $b = 1$, and $\bar{e} = 10$. For these parameters; the optimal expected profit when there is no financial constraint is 1957.08 when $p^* = 54.88$ and $Q^* = 52.08$. Table 3.1 gives the results. $\alpha$ levels are shown in first row and $K$ levels are shown in the first column. $E[\Pi]$ symbolizes the optimal expected profit and $\Delta(E[\Pi])$ symbolizes the deviation from the unconstrained optimal (which is 1957.08 for this sample case) in percentage.

$$\Delta(E[\Pi]) = \frac{100(1957.08 - E[\Pi])}{1957.08} \tag{3.5}$$

| $K$ | $\alpha$ | 0.01 | 0.05 | 0.1 |
|---|---|---|---|---|
| 1400 | $p^*$ | 54.88 | 54.88 | 54.88 |
| | $Q^*$ | 52.08 | 52.08 | 52.08 |
| | $E[\Pi]$ | 1957.08 | 1957.08 | 1957.08 |
| | $\Delta(E[\Pi])$ | 0 | 0 | 0 |
| 1500 | $p^*$ | 52.12 | 54.42 | 54.88 |
| | $Q^*$ | 51.06 | 52.18 | 52.08 |
| | $E[\Pi]$ | 1933.08 | 1956.72 | 1957.08 |
| | $\Delta(E[\Pi])$ | 1.22 | 0.02 | 0 |
| 1600 | $p^*$ | 50.18 | 50.97 | 52.25 |
| | $Q^*$ | 41.01 | 45.02 | 49.67 |
| | $E[\Pi]$ | 1646.46 | 1800.90 | 1920.18 |
| | $\Delta(E[\Pi])$ | 15.87 | 7.98 | 1.88 |
| 1650 | $p^*$ | | | 51.34 |
| | $Q^*$ | | | 44.51 |
| | $E[\Pi]$ | Infeas. | Infeas. | 1798.11 |
| | $\Delta(E[\Pi])$ | | | 8.12 |

Table 3.1: Analysis of the effect of financial constraint

### *3.1.4 Conclusions*

Table 3.1 shows that the optimal decisions are not affected by the constraint, when $K = 1400$, for all $\alpha$ levels. Naturally, as payment amount $K$ increases, the financial constraint becomes more binding. Similarly, as risk tolerance level $\alpha$ increases, the financial constraint becomes less binding. For the effect of financial constraint on optimal decisions, we cannot draw a general conclusion, like "*the seller responses to the financial constraint by changing the order quantity more than the price*" or vice versa. For example, in "$\alpha = 0.01$" column of the table 3.1, the decrease in the optimum price is more than the decrease in the optimum order quantity, as $K$ goes from 1400 to 1500. On the other hand, the comparison is quite opposite, as $K$ goes from 1500 to 1600. In "$\alpha = 0.05$" column, the optimum order quantity slightly increases while the optimum price decreases, as $K$ goes from 1400 to 1500. The latter example can be explained by that the objective function is not jointly concave in $p$ and $Q$.

## 3.2 Dynamic Pricing Problem with the Expected Shortfall Minimization

In this part, we approach the dynamic pricing problem from a risk-averse perspective. A detailed definition of the classical dynamic pricing problem was already given in section 1.5. The sample problem is examined according to two different objective functions and results are compared numerically. First objective is the expected profit maximization and the other is the expected shortfall minimization.

### *3.2.1 Problem Definition*

In this sample problem, there is a single seller selling single product. At the beginning of the selling season, he has some predetermined number of items in inventory. No replenishment is assumed during the selling season, which consists $N$ periods. In each period $n$, one customer arrives or not. Each arriving customer purchases exactly one product. Arriving probability of customers $\lambda(p)$ depends on the given price $p$. At the beginning of each period, the seller has to pick a price from a discrete price set. Price decision is basically affected by two factors: number of items remained in the inventory and the number of periods remained in the selling season. If the seller runs out of inventory at the beginning of any period $n$, the problem ends. At the end of the selling season, the unsold items are disposed without a salvage value.

### *3.2.2 Mathematical Model*

Since it is a multi-period problem, we set up a dynamic programming formula to solve this sample problem.

#### *3.2.2.1 Profit Maximization*

Let $J_n(I)$ be the maximum expected total profit from the current period $n$ to the end, when the current inventory level is $I$. In each period, seller chooses the price which maximizes $J_n(I)$. The value function $J_n(I)$ iterates recursively. The seller sets price $p$. With probability $\lambda(p)$, a customer arrives and purchases one item. The seller gets the instant cash $p$ and goes to the next period $n+1$ with $I-1$ items in inventory. With probability $(1-\lambda(p))$, no customer arrives, the seller does not get any instant cash and goes to the next period $n+1$ with still $I$ items in inventory. Corresponding dynamic recursion equation is as follows;

$$J_n(I) = \begin{cases} \max\limits_{p} \lambda(p)p, & for \;\; n = N \\ \max\limits_{p} \lambda(p)(p + J_{n+1}(I-1)) + (1-\lambda(p))J_{n+1}(I), & for \;\; n < N \end{cases} \tag{3.6}$$

*3.2.2.2 Shortfall Minimization*

For this case, the objective of the seller is to minimize the expected positive gap between the total profit and a predetermined target value (shortfall). In other words, the seller wants to minimize the expected difference between the target value and total profit but only cares about cases in which the total profit is less than the target value. So he needs to keep track of not only the current inventory level but also the current cash level $X$. The mathematical representation of shortfall is as follows;

$$Shortfall = (T - X_N)^+ \tag{3.7}$$

where $T$ is the target profit and $X_N$ is the total profit, and $(T - X_N)^+$ means $max((T - X_N), 0)$.
So the aim of the seller is;

$$\min_{p} E[(T - X_N)^+] \tag{3.8}$$

Let $J_n^{sh}(X, I)$ be the minimum expected amount of shortfall at the end of the selling season, when the current period is $n$, the current cash level is $X$, and the current inventory level is $I$. Similar to the expected profit maximization case, the seller sets price $p$. With probability $\lambda(p)$, a customer arrives and purchases one item.

Differently from the expected profit maximization case, instant cash obtained by the sale is not added to the value function. Instead, it is added the current cash level $X$ and the seller goes to the next period $n+1$ with $X+p$ amount of cash at hand and with $I-1$ items in inventory. With probability $(1-\lambda)$, no customer arrives, cash level of the seller does not change and the seller goes to the next period $n+1$ with still $X$ amount of cash at hand and with $I$ items in inventory. Corresponding dynamic recursion equation is as follows;

$$J_n^{sh}(X,I) = \begin{cases} \min\limits_p \lambda(p)(T-X+p)^+ + (1-\lambda(p))(T-X)^+, & for\ \ n=N \\ \min\limits_p \lambda(p)(J_{n+1}^{sh}(X+p,I-1)) + (1-\lambda(p))J_{n+1}^{sh}(X,I), & for\ \ n<N \end{cases} \tag{3.9}$$

### *3.2.3 Numerical Results*

We assume that the seller picks the prices from a discrete set. We make this assumption instead of letting the seller determine the optimal prices continuously because for the shortfall minimization objective case, the seller must keep track of the current cash level. The theory behind the dynamic programming algorithm requires to provide the optimal solutions for any possible state. If we let the seller choose the prices continuously, there would be infinitely many possible cash positions and it would be theoretically impossible to obtain the optimal prices for any possible state. Although this situation does not cause any problem for the expected profit maximization objective case since the seller does not need to keep track of the current cash level, for a meaningful comparison of two objective cases, we use the same price-demand set for both cases. Price set and corresponding customer arrival probabilities are given in table 3.2.

For the sample problem, the seller has an initial inventory of 6 items. The selling season consists of 10 periods. Since these two factors, number of periods and beginning inventory, have less effect on deviation of price decision of two objectives, we

| Price ($p$) | 3 | 5 | 7 |
|---|---|---|---|
| Arrival Probability ($\lambda(p)$) | 0.8 | 0.5 | 0.2 |

Table 3.2: Price-demand set

only changed target profit and compared the expected total profit ($ETP$) and the expected shortfall ratio ($ESR$) of two objectives. By the expected shortfall ratio we mean in what percentage the seller falls behind the target values. The mathematical formulation of $ESR$ is as follows;

$$ESR = \frac{100\ E[(T - X_N)^+]}{T} \tag{3.10}$$

Table 3.3 gives the computational results in terms of objective values. First column shows the various target levels. To simplify the comparison, the last two columns list the difference in the expected total profits ($DiETP$) and the difference in the expected shortfall ratios ($DiESR$).

| Target ($T$) | Objective | $ETP$ | $ESR$ | $DiETP$ | $DiESR$ |
|---|---|---|---|---|---|
| 10 | Prof. Max. | 24.03 | 0.59 | 13.37 | 0.49 |
| | Sh.fall Min. | 10.66 | 0.10 | | |
| 15 | Prof. Max. | 24.03 | 2.21 | 6.88 | 1.13 |
| | Sh.fall Min. | 17.15 | 1.08 | | |
| 18 | Prof. Max. | 24.03 | 4.72 | 6.56 | 1.73 |
| | Sh.fall Min. | 17.48 | 2.99 | | |
| 20 | Prof. Max. | 24.03 | 5.97 | 5.00 | 0.99 |
| | Sh.fall Min. | 19.03 | 4.98 | | |
| 22 | Prof. Max. | 24.03 | 8.94 | 3.64 | 1.50 |
| | Sh.fall Min. | 20.39 | 7.44 | | |
| 25 | Prof. Max. | 24.03 | 12.53 | 1.52 | 0.50 |
| | Sh.fall Min. | 22.51 | 12.03 | | |
| 30 | Prof. Max. | 24.03 | 21.73 | 0.22 | 1.09 |
| | Sh.fall Min. | 23.81 | 20.64 | | |
| 35 | Prof. Max. | 24.03 | 31.38 | 0.01 | 0.01 |
| | Sh.fall Min. | 24.02 | 31.37 | | |

Table 3.3: Comparison of two objective functions with respect to the expected total profit and the expected shortfall ratio

The figures 3.2 and 3.3 illustrates the results in table 3.3.

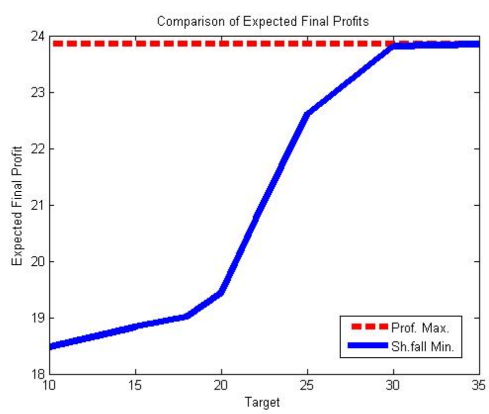


Figure 3.2: Comparison of the expected total profits

The tables 3.4 and 3.5 gives the computational results in term of optimal prices. Table 3.4 lists the optimal prices with respect to the changes in inventory and period,

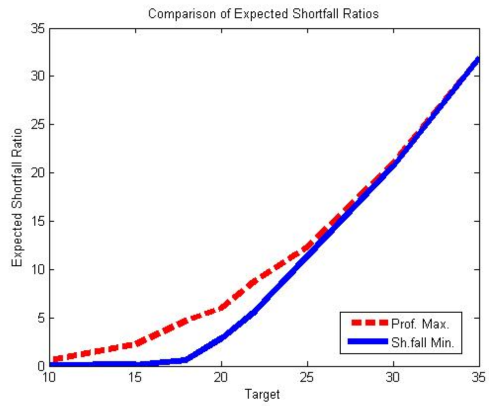


Figure 3.3: Comparison of the expected shortfall ratios

while keeping the cash level fixed at 10. Each column contains two subcolumns separated by a dashed line. The left one lists the optimal prices for the expected profit maximization objective (Prof.) and the right one is for the expected shortfall minimization objective (Sh.fall). Similarly, table 3.4 lists the optimal prices with respect to the changes in inventory and cash level, while keeping the period fixed at 5. Notice that the optimal prices do not change with respect to the cash level for the expected profit maximization objective.

| inventory \ periods | 1 Prof. | 1 Sh.fall | 2 Prof. | 2 Sh.fall | 3 Prof. | 3 Sh.fall | 4 Prof. | 4 Sh.fall | 5 Prof. | 5 Sh.fall | 6 Prof. | 6 Sh.fall | 7 Prof. | 7 Sh.fall |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 5 | 7 | 5 | 7 | 5 | 7 | 5 | 7 | 5 | 7 | 5 | 5 | 5 | 5 | 5 |
| 4 | 7 | 3 | 7 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 |
| 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 |
| 2 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 |
| 1 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 |

Table 3.4: Comparison of the optimal decisions of two objectives with respect to inventory and period when $X = 10$

| inventory \ cash level | 4 | | 6 | | 8 | | 10 | | 12 | | 14 | | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Prof. | Sh.fall | Prof. | Sh.fall | Prof. | Sh.fall | Prof. | Sh.fall | Prof. | Sh.fall | Prof. | Sh.fall | Prof. | Sh.fall |
| 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 5 | 7 | 3 |
| 5 | 7 | 7 | 7 | 5 | 7 | 5 | 7 | 5 | 7 | 3 | 7 | 3 | 7 | 3 |
| 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 |
| 3 | 5 | 5 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 |
| 2 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 |
| 1 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 |

Table 3.5: Comparison of the optimal decisions of two objectives with respect to inventory and cash level when $n = 5$

### *3.2.4 Conclusions*

Obviously the expected profit maximization objective always generates the same amount of the expected total profit. As the target value increases, the expected shortfall ratio of the expected profit maximization problem also increases. The expected shortfall minimization objective generates an expected profit less than the expected profit maximization objective's and, naturally, the expected shortfall ratio of the expected shortfall minimization objective is less than the expected profit maximization objective's. As the target value increases and goes beyond the maximum possible amount of profit, both algorithms converge to the same policy and they generate same amount of expected total profit and same expected shortfall ratio. In general, we can conclude that the expected shortfall minimization objective generates profit from a more stable (with lower variance) distribution. In terms of optimal prices, we can see from the tables 3.4 and 3.5 that the expected shortfall minimization objective never sets higher prices than the expected profit maximization objective's. This is not a surprising observation since the expected shortfall minimization objective represents a risk-averse approach and the expected profit maximization objective stands for a risk-neutral approach.

Chapter 4

# DYNAMIC INVENTORY REPLENISHMENT WITH RANDOM SUPPLY

In this chapter, a dynamic inventory management problem is examined. Two types of objective function are analyzed; the expected profit maximization and the expected utility maximization, where the first one implies a risk-neutral perspective and the latter implies a risk-averse perspective. Since it is a multi-period problem, we set up a dynamic programming formula to solve the problem. We derive some properties of the optimal solution and exemplify them with a numerical illustration.

## 4.1 Problem Definition

A detailed definition of the dynamic inventory management problem is already given in section 1.4 and our problem is a modified version of it. Price is not a decision but at the beginning of each period, seller checks the inventory level and decides whether to order or not. Since each order includes only one item, order size is not a decision. If he orders, the item will be delivered at the beginning of the next period with probability of $q$. With probability of $1 - q$, the order will not be delivered. Order decision is basically affected by two factors: number of items remained in the inventory and the number of periods remained in the selling season.

## 4.2 Profit Maximization Objective

The aim of the seller is to maximize the expected total profit. Let $J_n(I)$ be the maximum expected total profit from the current period $n$ to the end, when the current inventory level is $I$. The value function $J_n(I)$ iterates recursively. Price $p$ is fixed, so is demand $\lambda(p)$. At each period, independently of order decision, the seller pays the holding cost. The cost of holding one item for one period is $h$. The holding cost is a linear function of the inventory level at the beginning of the period. If the seller does not order an item, there exist two possible cases. With probability $\lambda(p)$, a customer arrives and purchases one item, the seller gets the instant cash $p$ and goes to the next period $n+1$ with $I-1$ items in inventory. With probability $(1-\lambda(p))$, no customer arrives, the seller does not get any instant cash and goes to the next period $n+1$ with still $I$ items in inventory. If the seller orders an item, there exist four possible cases. With probability $\lambda(p)q$, a customer arrives, purchases one item, the ordered item arrives, the seller gets the instant cash $p$, pays the cost of purchasing $w$, and goes to the next period $n+1$ with $I$ items in inventory. With probability $\lambda(p)(1-q)$, a customer arrives, purchases one item but the ordered item does not arrive, the seller gets the instant cash $p$ but does not pay any purchasing cost, and goes to the next period $n+1$ with $I-1$ items in inventory. With probability $(1-\lambda(p))q$, no customer arrives but the ordered item arrives, the seller does not get any instant cash but pays the cost of purchasing $w$, and goes to the next period $n+1$ with $I+1$ items in inventory. With probability $(1-\lambda(p))(1-q)$, no customer arrives and the ordered item does not arrive. The seller does not get any instant cash, does not pay any purchasing cost, and goes to the next period $n+1$ with still $I$ items in inventory. Corresponding dynamic recursion equation is as follows;

$$
\begin{aligned}
J_n(I) = -hI + \max \Big[ & \lambda(J_{n+1}(I-1)+p) + (1-\lambda)J_{n+1}(I), \\
& \lambda(q(J_{n+1}(I)-w) + (1-q)J_{n+1}(I-1)+p) + \\
& (1-\lambda)(q(J_{n+1}(I+1)-w) + (1-q)J_{n+1}(I))\Big]
\end{aligned} \tag{4.1}
$$

The seller will order if the expected future profit of ordering is greater than the expected future profit of not ordering. In other words, the seller orders if the inequality below holds;

$$\lambda(J_{n+1}(I) - J_{n+1}(I-1)) + (1-\lambda)(J_{n+1}(I+1) - J_{n+1}(I)) \geq w \tag{4.2}$$

The comparison above is simply derived from the recursion equation. The left-hand-side of the inequality represents the marginal value of an extra item in inventory and it is a convex combination of two functions.

If we verify that both of the functions are decreasing in $I$, we can conclude that a threshold policy is optimal. Next, some structural results, including concavity on inventory, are given.

**Proposition 1** *$J_n(I)$ is concave in $I$, for all $n$.*

$$J_n(I+1) - J_n(I) \leq J_n(I) - J_n(I-1) \tag{4.3}$$

The proposition 1 states that the marginal value of a single item in inventory decreases as inventory level increases.

**Corollary 1** *A threshold policy on inventory is optimal.*

Corollary 1 implies that it is optimal to order if $I_n \leq I_n^*$ where $I_n$ is the current inventory level and $I_n^*$ is the threshold level for period $n$.

**Proposition 2** *Supermodularity in period and inventory holds when $h = 0$.*

$$J_n(I+1) - J_n(I) \leq J_{n-1}(I+1) - J_{n-1}(I) \tag{4.4}$$

The proposition 2 states that the marginal value of a single item in inventory decreases as the period increases because probability of selling that additional item and getting some additional profit decreases as the season goes to the end.

**Corollary 2** *A threshold policy on period is optimal when $h = 0$.*

$$I_n^* \leq I_{n-1}^* \tag{4.5}$$

Corollary 2 implies that if it is optimal to order in period $n$ with $I$ inventory, then it is also optimal to order in any period earlier than $n$, too.

Detailed proofs are given in the appendix.

### 4.3 Utility Maximization Objective

In this section, we use an exponential utility function as objective. The reason that we choose exponential utility will be explained just after a brief definition.

The formulation of the expected utility maximization problem is quite similar to the expected profit maximization's. Differently from the expected profit maximization case, the aim of the seller is to maximize the expected exponential utility gathered by the total profit.

Let $\Pi_n$ be the profit obtained in period $n$ and $\sum_{n=1}^{N} \Pi_n$ be the total profit obtained in the whole selling season. The mathematical formulation of the exponential utility is as follows;

$$1 - e^{-\gamma\left(\sum_{n=1}^{N} \Pi_n\right)} \tag{4.6}$$

where $\gamma$ is the degree of risk-aversion. So, the objective of the seller is;

$$\max E[1 - e^{-\gamma\left(\sum_{n=1}^{N} \Pi_n\right)}] \tag{4.7}$$

Utility function of the seller decreases exponentially from 1 and converges to 0. We may name the subtracted part of the utility function ($e^{-\gamma\left(\sum_{n=1}^{N} \Pi_n\right)}$) as 'disutility' and simplify the objective function as 'disutility minimization'. So the objective of the seller may be defined as follows;

$$\min E[e^{-\gamma\left(\sum_{n=1}^{N} \Pi_n\right)}] \tag{4.8}$$

For simplicity, the latter will be used along this section and we will express the objective as 'utility minimization' by referring the above function.

We use exponential utility because its concave character is highly successful at representing the risk-aversion. This property makes exponential utility function very accordant with the scope of this thesis. The other reason that we use the exponential utility function is the ease it provides in multiplicative recursion. In fact, the profit level of the seller iterates additively. He sells an item, adds the price obtained by selling this item to his value function, and goes to the next period. Let $\Pi_{n+1}(I-1)$ be the expected total profit obtained by applying utility minimizing price policy from the period $n+1$ to the end. Then the value function of the seller when he makes a sale in period $n$ becomes $(p+\Pi_{n+1}(I-1))$. The corresponding exponential utility of his *value* is $e^{-\gamma(p+\Pi_{n+1}(I-1))}$ and we can rewrite this exponential utility as $(e^{-\gamma p}e^{-\gamma(\Pi_{n+1}(I-1))})$. Since the second multiplier $(e^{-\gamma(\Pi_{n+1}(I-1))})$ is the exponential utility of the value function of the seller when he is in period $n+1$ with $I-1$ items in inventory. So we may write the exponential utility of the value function of the seller when he makes a sale in period $n$ as multiplication of the exponential utility of the instant cash $p$ and the exponential utility of the value function of the seller when he is in period $n+1$ with $I-1$ items in inventory. This property makes us be able to iterate the recursion equation multiplicatively.

We assume that $\gamma \geq 0$ because risk-averse decision makers have concave utility functions. Higher $\gamma$ values imply more risk-averse decision makers. As $\gamma$ increases, the seller gets better utility from the same amount of profit. In other words, to achieve some level of utility, a risk-averse decision maker needs less amount of profit than one with a lower degree of risk-aversion. Figure 4.1 illustrates the relation between $\gamma$ and the utility obtained from some amount of profit.

Let's get back to our current dynamic inventory management problem and its mathematical formulation. Assume $G_n(I)$ be the minimum expected total utility (in other words, $G_n(I)$ is the exponential utility of the expected total profit obtained by applying utility minimizing price policy) from the current period $n$ to the end, when the current inventory level is $I$. The value function $G_n(I)$ iterates recursively. The problem definition is the same with the expected profit maximization objective's.

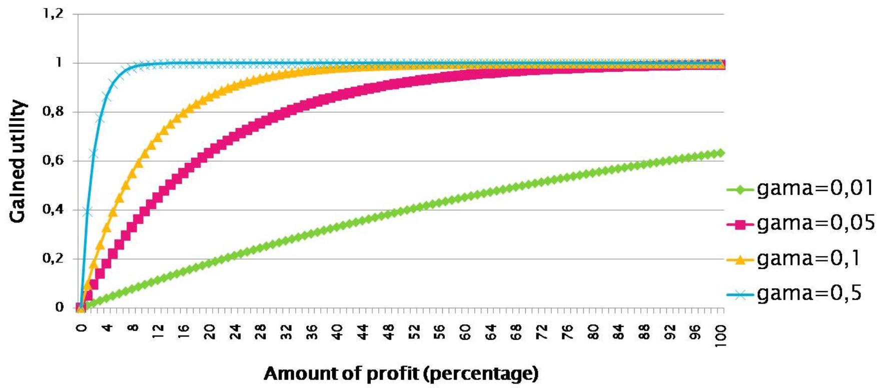


Figure 4.1: Gamma Effect on Utility

The holding cost is represented in the value function as the utility lost by paying the holding cost. The seller makes the decision (ordering or not) minimizing his expected utility. Possible outcomes of these two decisions are the same with the expected profit maximization case. Since iteration mechanism of the recursion equation is explained there in detail, we skip this part to avoid repetition. So the recursive dynamic equation is as follows;

$$\begin{aligned} G_n(I) = e^{\gamma h I} \min \Big[ & \lambda G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)G_{n+1}(I), \\ & (\lambda(qG_{n+1}(I)e^{\gamma w} + (1-q)G_{n+1}(I-1))e^{-\gamma p}) + \\ & (1-\lambda)(qG_{n+1}(I+1)e^{\gamma w} + (1-q)G_{n+1}(I)) \Big] \end{aligned} \tag{4.9}$$

The seller orders an item if the expected utility of ordering is better than the expected utility of not ordering. Similar to the expected profit maximization case, the seller orders an item if the inequality below holds;

$$\frac{\lambda G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)G_{n+1}(I)}{\lambda G_{n+1}(I)e^{-\gamma p} + (1-\lambda)G_{n+1}(I+1)} \geq e^{\gamma w} \tag{4.10}$$

The comparison above is simply derived from recursion equation. The left-hand-side of the inequality is ratio of $G_{n-1}(I)$ to $G_{n-1}(I+1)$ when there is no replenishment and it represents marginal utility value of an extra item in inventory.

If we verify that function $G_{n-1}(I)$ is log-concave in $I$, we can conclude that a threshold policy is optimal because if $G_{n-1}(I)$ is log-concave in $I$, needless to say, it is also log-concave in no replenishment case (log-concavity in no replenishment case is sufficient for a threshold policy, see equation 4.10 above). Following, similar structural properties to the expected profit maximization problems', including log-concavity on inventory, are given.

**Proposition 3** *$G_n(I)$ is log-concave in $I$, for all $n$.*

$$\frac{G_n(I+1)}{G_n(I)} \geq \frac{G_n(I)}{G_n(I-1)} \tag{4.11}$$

The proposition 3 states that the marginal utility value of a single item in inventory decreases as inventory level increases.

**Corollary 3** *A threshold policy on inventory is optimal.*

Corollary 3 implies that it is optimal to order if $I_n \leq I_n^*$ where $I_n$ is the current inventory level and $I_n^*$ is the threshold level for period $n$.

**Proposition 4** *Supermodularity in period and inventory holds when $h = 0$.*

$$\frac{G_n(I)}{G_n(I-1)} \geq \frac{G_{n-1}(I)}{G_{n-1}(I-1)} \tag{4.12}$$

The proposition 4 states that the marginal value of a single item in inventory decreases as the period increases because probability of selling that additional item and getting some additional profit to improve the utility decreases as the season goes to the end.

**Corollary 4** *A threshold policy on period is optimal when $h = 0$.*

$$I_n^* \leq I_{n-1}^* \tag{4.13}$$

Corollary 4 implies that if it is optimal to order in period $n$ with $I$ inventory, then it is also optimal to order in any period earlier than $n$, too.

Detailed proofs are given in the appendix.

### *4.4 Numerical Illustration*

In this section, we present a numerical study to compare the optimal solutions for the expected profit and the expected utility problems. For the sample problem, we assume that the selling season consist of 20 periods. We set the other parameters as follows: $p = 20, \lambda(p) = 0.7, q = 0.5, w = 3, h = 0.5$, and $\gamma = 0.1$. Following figure 4.2 shows the results.

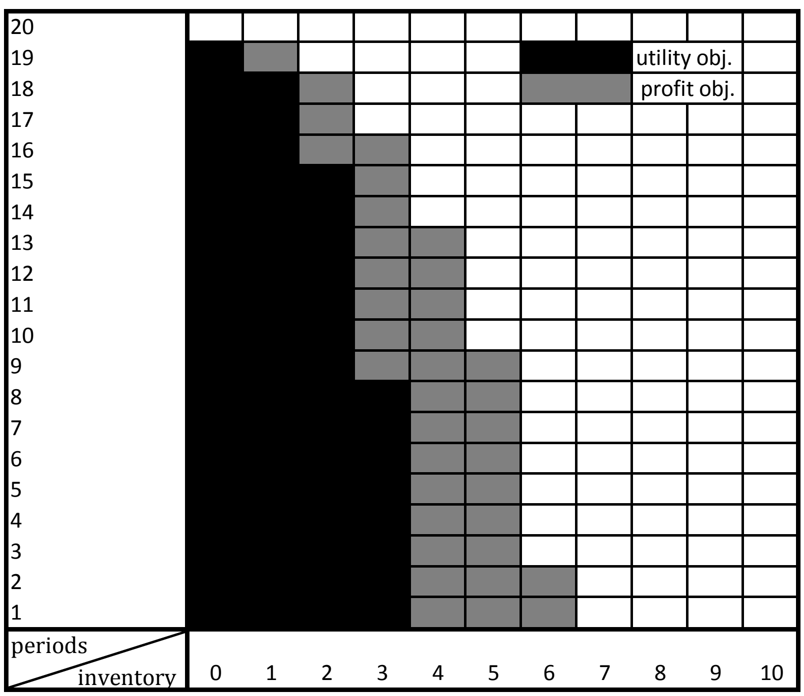


Figure 4.2: Comparison of the optimal decisions of two objectives

In figure 4.2, the black blocks represent the states that it is optimal to order for both the expected utility (risk-averse decision maker) and the expected profit (risk-neutral decision maker) problems. The grey blocks represent the states that it is optimal to order only for the expected profit problem. The white blocks are the states that it is optimal not to order for both problems. The borders between the black and grey regions and between the grey and white regions represent the threshold levels for the risk-averse decision maker and for the risk-neutral decision maker, respectively.

## 4.5 Conclusions

In this chapter, we analyze the dynamic inventory management problem for single item with random supply. We approach to the problem from two perspectives in terms of the objective function. First, we set the expected profit maximization as the objective, which reflects the risk-neutral retailer's point of view. Second, we use the expected utility maximization as the objective, which reflects the risk-averse retailer's point of view. We derive some structural properties of the value function and of the optimal solution for the expected profit maximization case. Then, we show that these structural properties are valid also for the expected utility case. With a numerical study, we compare the optimal decisions of the risk-neutral and risk-averse decision makers and we observe that the threshold level of a risk-averse decision maker is always lower than a risk-neutral one's, which is quite natural because of the idea behind risk-aversion.

# Chapter 5

# DYNAMIC PRICING MODEL WITH EXPONENTIAL UTILITY MAXIMIZATION

In this chapter, we analyze the dynamic pricing problem. The objective of the seller is exponential utility maximization, which reflects the risk-averse behavior of the seller. In the section 5.1, some background information about dynamic pricing problem is given and structural properties of dynamic pricing problem with the expected profit maximization objective (risk-neutral) which are done by Gallego and Ryzin [11] are summarized. In section 5.2, our model is defined and structural properties of the optimal solution are investigated. In section 5.4, a numerical example is presented and what is done along the study is briefly summarized.

## 5.1 Background

A detailed definition of the dynamic pricing problem is already given in section 1.5. In the classical dynamic pricing problem, there is a single seller with single product. At the beginning, he has some predetermined number of items in inventory. No replenishment is assumed during the selling season, which consists of $N$ periods. In each period $n$, one customer arrives or not. Each arriving customer purchases exactly one product. Arriving probability of customers $\lambda(p)$ depends on the given price $p$ is a function of the given price. At the beginning of each period, the seller chooses the price maximizing the expected total profit. We assume that there exists a price that makes the first derivative of the recursive equation equal to zero. Price decision is basically affected by two factors: number of items remained in the inventory and the number of periods remained in the selling season. If the seller runs out of inventory at the beginning of any period $n$, the problem ends. At the end of the selling season, the unsold items are disposed without a salvage value. Gallego and Ryzin [11] showed some structural properties of the dynamic pricing problem with profit maximizing objective. In that problem, the seller wants to maximize overall profit gained along $N$ periods. The maximum expected total profit to be gained from period $n$ to the end when the seller has $I$ amount of current inventory (recursive dynamic equation) is as follows;

Since it is a multi-period problem, a dynamic programming algorithm is applied to solve the dynamic pricing problem. Let $R_n(I,p)$ be the maximum expected total profit from the current period $n$ to the end if the seller sets the price $p$, when the current inventory level is $I$. And let $J_n(I)$ be the value of $R_n(I,p)$ when the optimal price for period $n$ is set. In each period, seller chooses the price which maximizes $J_n(I)$. The value function $J_n(I)$ iterates recursively. The seller sets price $p$. With probability $\lambda(p)$, a customer arrives and purchases one item. The seller gets the instant cash $p$ and goes to the next period $n+1$ with $I-1$ items in inventory. With probability $(1-\lambda(p))$, no customer arrives, the seller does not get any instant cash and goes to the next period $n+1$ with still $I$ items in inventory. Below is the dynamic recursion equation for $R_n(I,p)$:

$$R_n(I,p) = \begin{cases} \lambda(p)p, & for \;\; n = N \\ \lambda(p)(p + J_{n+1}(I-1)) + (1-\lambda(p))J_{n+1}(I), & for \;\; n < N \end{cases} \tag{5.1}$$

By definition;

$$J_n(I) = \max_p R_n(I,p) \tag{5.2}$$

The optimal price for period $n$ is maximizer of the $R_n(I,p)$ function. So we take the first derivative of $R_n(I,p)$;

$$\frac{R_n(I,p)}{dp} = \lambda'(p)(p + J_{n+1}(I-1)) + \lambda(p) - \lambda'(p)J_{n+1}(I) \tag{5.3}$$

and

$$\frac{dR_n(I,p)}{dp} = 0 \qquad at \;\; p = p^* \tag{5.4}$$

So the optimal price $p^*$ should hold the following optimality condition;

$$p^* = \frac{-\lambda(p^*)}{\lambda'(p^*)} + (J_{n+1}(I) - J_{n+1}(I-1)) \tag{5.5}$$

Following are the some structural properties of the expected profit maximization model derived by Gallego and Ryzin [11]. For completeness, we present these properties in our notation. Corresponding properties of the expected utility maximization model will be given in section 5.2.

**Property 1** *$J_n(I)$ is concave in $I$, for all $n$.*

$$J_n(I+1) - J_n(I) \leq J_n(I) - J_n(I-1) \tag{5.6}$$

The property 1 states that the marginal value of a single item in inventory decreases as inventory level increases.

**Property 2** *The optimal price is monotonic in inventory.*

$$p_1 = argmax(J_n(I+1)) \tag{5.7}$$

$$p_2 = argmax(J_n(I)) \tag{5.8}$$

$$\Rightarrow p_1 \leq p_2 \tag{5.9}$$

The property 2 implies that the seller wants to charge lower prices if the inventory level is higher.

**Property 3** *Supermodularity in period and inventory holds.*

$$J_n(I+1) - J_n(I) \leq J_{n-1}(I+1) - J_{n-1}(I) \tag{5.10}$$

The property 3 states marginal value of a single item in inventory decreases as the period increases because probability of selling that additional item and getting some additional profit decreases as the season goes to the end.

**Property 4** *The optimal price is monotonic in period.*

$$p_1 = argmax(J_{n+1}(I)) \tag{5.11}$$

$$p_2 = argmax(J_n(I)) \tag{5.12}$$

$$\Rightarrow p_1 \leq p_2 \tag{5.13}$$

The property 4 implies that the optimal price for a fixed level of inventory decreases as the period increases. This is because the seller has less time to sell the items inventory and wants to speed up the sales.

Detailed proofs are given in the appendix.

## 5.2 Exponential Utility Maximization

In this section, the dynamic pricing problem with exponential utility maximization objective is examined. The problem definition is the same with previous section. Differently, the objective of the seller in this case is to maximize the expected exponential utility obtained by selling the items in his inventory. For the properties of the exponential utility function and the reasons that we use it, please refer to the explanations in section 4.3. The exponential utility function of the seller is;

$$\max_{\omega} E[(1 - e^{-\gamma \Pi(\omega)})] \tag{5.14}$$

where $\Pi(\omega)$ denotes the expected total profit if the price policy $\omega$ is applied.

Utility function of the seller decreases exponentially from 1 and converges to 0. We may name the subtracted part of the utility function ($e^{-\gamma \Pi(\omega)}$) as 'disutility' and simplify the objective function as 'disutility minimization'. So the objective of the seller may be defined as follows;

$$\min_{\omega} E[(e^{-\gamma\Pi(\omega)})] \tag{5.15}$$

For simplicity, the latter will be used along this section and we will express the objective as 'utility minimization' by referring the above function.

Let $U_n(I,p)$ be the minimum expected total utility if the seller sets the price $p$ for the current period $n$ and follows the utility minimizing price policy from the next period $n+1$ to the end, when the current inventory level is $I$. And let $G_n(I)$ be the value of $U_n(I,p)$ when the optimal price for period $n$ is set. The value function $G_n(I)$ iterates recursively. The iteration mechanism of the problem is same with the expected profit maximization problem's. The recursive dynamic equation is as follows;

$$G_n(I) = \min_{p} U_n(I,p) \tag{5.16}$$

Below equation shows how the function $U_n(I,p)$ iterates;

$$U_n(I,p) = \begin{cases} \lambda(p)e^{-\gamma p} + (1-\lambda(p)), & for \;\; n = N \\ \lambda(p)(e^{-\gamma p}G_{n+1}(I-1)) + (1-\lambda(p))G_{n+1}(I), & for \;\; n < N \end{cases} \tag{5.17}$$

The optimal price is the minimizer of the above function. So we take the first derivative of $U_n(I,p)$;

$$\frac{dU_n(I,p)}{dp} = \lambda'(p)(e^{-\gamma p}G_{n+1}(I-1) - G_{n+1}(I)) - \gamma\lambda(p)G_{n+1}(I-1)e^{-\gamma p} \tag{5.18}$$

and

$$\frac{dU_n(I,p)}{dp} = 0 \qquad at \;\; p = p^* \tag{5.19}$$

$$\Rightarrow \lambda'(p^*)(e^{-\gamma p^*}G_{n+1}(I-1) - G_{n+1}(I)) - \gamma\lambda(p^*)G_{n+1}(I-1)e^{-\gamma p^*} = 0 \qquad (5.20)$$

So the optimal solution $p^*$ should satisfy the following optimality condition;

$$\Rightarrow e^{-\gamma p^*}\left(1 - \frac{\gamma\lambda(p^*)}{\lambda'(p^*)}\right) = \frac{G_{n+1}(I)}{G_{n+1}(I-1)} \qquad (5.21)$$

### *5.2.1 Structural Properties*

The structural properties similar to the expected profit maximization problems' can be drawn for this problem, too.

**Proposition 5** *$G_n(I)$ is log-concave in $I$, for all $n$.*

$$\frac{G_n(I+1)}{G_n(I)} \geq \frac{G_n(I)}{G_n(I-1)} \qquad (5.22)$$

The proposition 5 states that the marginal utility value of a single item in inventory decreases as inventory level increases.

**Corollary 5** *The optimal price is monotonic in inventory.*

$$p_1 = argmin(G_n(I+1)) \qquad (5.23)$$

$$p_2 = argmin(G_n(I)) \qquad (5.24)$$

$$\Rightarrow p_1 \leq p_2 \qquad (5.25)$$

The corollary 5 implies that the seller wants to charge lower prices if the inventory level is higher.

**Proposition 6** *If the hazard rate function of $p$ (also known as failure rate function, $h(p) = \frac{f(p)}{1-F(p)}$, where $f(p)$ and $F(p)$ are the probability and cumulative density* *functions of p, respectively) is non-decreasing in p; supermodularity in period and inventory holds.*

$$\frac{G_n(I+1)}{G_n(I)} \leq \frac{G_{n+1}(I+1)}{G_{n+1}(I)} \tag{5.26}$$

The proposition 6 states that the marginal utility value of a single item in the inventory decreases as the period increases because probability of selling that additional item and getting some additional profit to improve the utility decreases as the season goes to the end.

**Corollary 6** *If $h(p)$ is non-decreasing in $p$, the optimal price is monotonic in period.*

$$p_1 = argmin(G_{n+1}(I)) \tag{5.27}$$

$$p_2 = argmin(G_n(I)) \tag{5.28}$$

$$\Rightarrow p_1 \leq p_2 \tag{5.29}$$

The corollary 6 implies that the optimal price for a fixed level of inventory decreases as the period increases. This is because the seller has less time to sell the items in the inventory and wants to speed up the sales.

**Proposition 7** *An increase in $\gamma$ improves the optimal expected utility when other variables are the same.*

$${G_n}^{\gamma_1}(I) \geq {G_n}^{\gamma_2}(I) \tag{5.30}$$

*when $\gamma_1 \leq \gamma_2$.*

Under equal conditions, a more risk averse decision maker gains a better optimal expected utility.

This argument can be proven by a similar argument with Zhao and Zheng's [21]. Detailed proofs are given in the appendix.

## 5.3 Exponential Demand Case

In this section, we apply the structural properties presented in sections 5.1 and 5.2 to the exponential demand case. For this case, customer arrival probability is an exponential function of price:

$$\lambda(p) = e^{-bp} \tag{5.31}$$

where $b$ reflects the demand-price relation.

The dynamic recursion equations for the expected profit and the expected utility maximization problems are as follows:

$$J_n(I) = \max_p (e^{-bp}(p + J_{n+1}(I-1)) + (1 - e^{-bp})J_{n+1}(I)) \tag{5.32}$$

and

$$G_n(I) = \min_p (e^{-bp}(e^{-\gamma p} G_{n+1}(I-1)) + (1 - e^{-bp})G_{n+1}(I)) \tag{5.33}$$

By taking the first derivatives of both equations, it is easy to see that optimal prices are:

for the expected profit maximization problem:

$$p^* = \begin{cases} \frac{1}{b}, & for \;\; n = N \\ \frac{1}{b} + (J_{n+1}(I) - J_{n+1}(I-1)), & for \;\; n < N \end{cases} \tag{5.34}$$

and for the expected utility maximization problem:

$$p^* = \begin{cases} \frac{-ln\left(\frac{b}{b+\gamma}\right)}{\gamma}, & for \;\; n = N \\ \frac{-ln\left(\frac{b}{(b+\gamma)}\right)}{\gamma} + \frac{-ln\left(\frac{(G_{n+1}(I))}{(G_{n+1}(I-1))}\right)}{\gamma}, & for \;\; n < N \end{cases} \tag{5.35}$$

## 5.4 Numerical Illustration

For the sample problem, we use the properties of the exponential demand case derived in section 5.3 and set $b = 0.05$. The demand-price relation is given in figure 5.1.

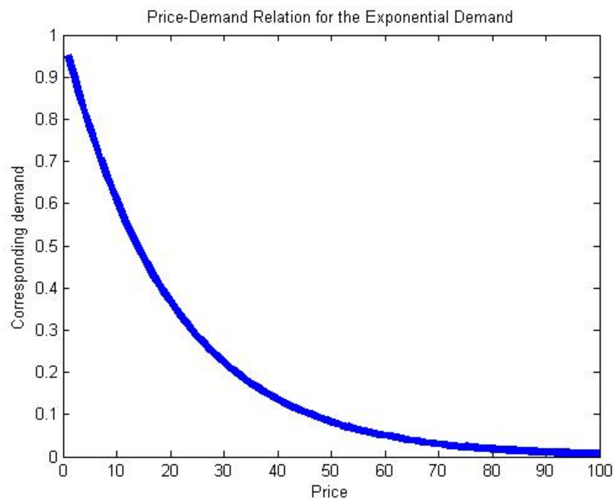


Figure 5.1: Price-demand curve for the exponential demand function

### *5.4.1 Comparison of Two Objectives With Respect to Their Performances*

In this part, we present a numerical example comparing the performances of the expected profit and the expected utility objectives. While we keep number of periods fixed ($N = 30$), we try 4 different initial inventory level as 5, 10, 20, and 30. For the degree of risk-aversion, we try 5 different levels as 0.01, 0.05, 0.1, 0.5, and 1. In order to compare the performance of two objectives in the long-run, we coded the sample problem and ran 10000 replications. In each replication, we used the same demand realization for both objectives. Following table 5.1 shows performance of two objectives in terms of the average total profit (AP) and the variance of total profits (Var).

| | $I$ | 5 | | 10 | | 20 | | 30 | |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma$ | Objective | AP | Var | AP | Var | AP | Var | AP | Var |
| 0.01 | Prof. Max. | 158.16 | 38.40 | 208.17 | 49.23 | 220.63 | 52.84 | 220.92 | 52.99 |
| | Util. Max. | 157.04 | 33.60 | 207.44 | 43.84 | 219.81 | 49.01 | 219.86 | 48.98 |
| 0.05 | Prof. Max. | 158.16 | 38.40 | 208.17 | 49.23 | 220.63 | 52.84 | 220.92 | 52.99 |
| | Util. Max. | 146.64 | 21.68 | 193.19 | 29.46 | 207.65 | 37.85 | 207.97 | 38.03 |
| 0.1 | Prof. Max. | 158.16 | 38.40 | 208.17 | 49.23 | 220.63 | 52.84 | 220.92 | 52.99 |
| | Util. Max. | 132.19 | 14.63 | 172.60 | 20.06 | 189.67 | 29.01 | 190.26 | 29.64 |
| 0.5 | Prof. Max. | 158.16 | 38.40 | 208.17 | 49.23 | 220.63 | 52.84 | 220.92 | 52.99 |
| | Util. Max. | 76.87 | 3.96 | 94.83 | 5.40 | 108.76 | 7.62 | 113.24 | 10.76 |
| 1 | Prof. Max. | 158.16 | 38.40 | 208.17 | 49.23 | 220.63 | 52.84 | 220.92 | 52.99 |
| | Util. Max. | 53.70 | 2.04 | 64.28 | 2.73 | 73.58 | 3.72 | 78.47 | 5.84 |

Table 5.1: The average and the variance of profit comparison of two objectives

The figures 5.2 and 5.3 illustrate the results in the table 5.1 for initial inventory $I = 5$.

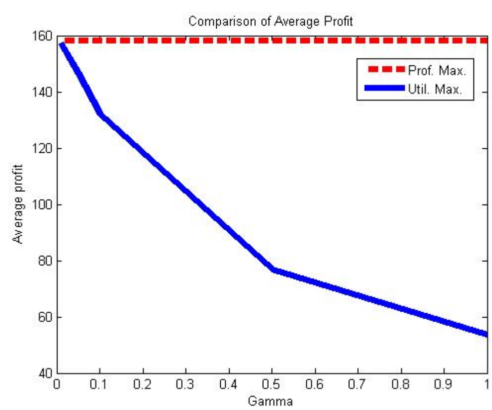


Figure 5.2: Comparison of the average profit with respect to $\gamma$ change when $I = 5$

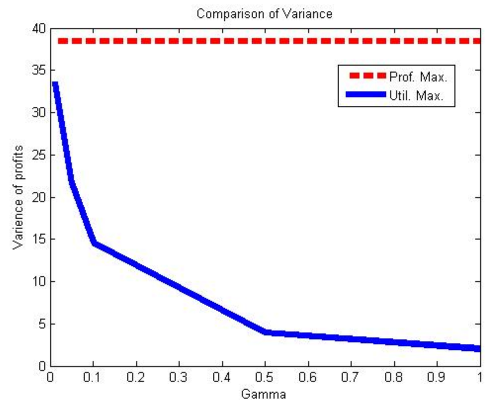


Figure 5.3: Comparison of variance of profits with respect to $\gamma$ change when $I = 5$

And the figures 5.4 and 5.5 illustrate the results in the table 5.1 for the degree of risk-aversion $\gamma = 0.1$.

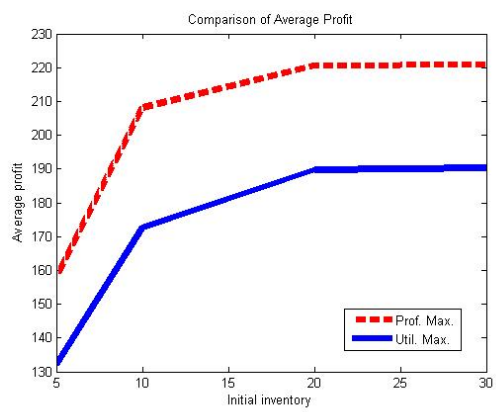


Figure 5.4: Comparison of the average profit with respect to $I$ change when $\gamma = 0.1$

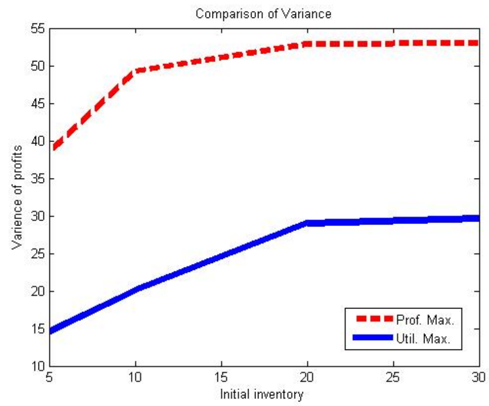


Figure 5.5: Comparison of variance of profits with respect to $I$ change when $\gamma = 0.1$

The figure 5.6 shows the histogram of the realized profit levels for both objectives.

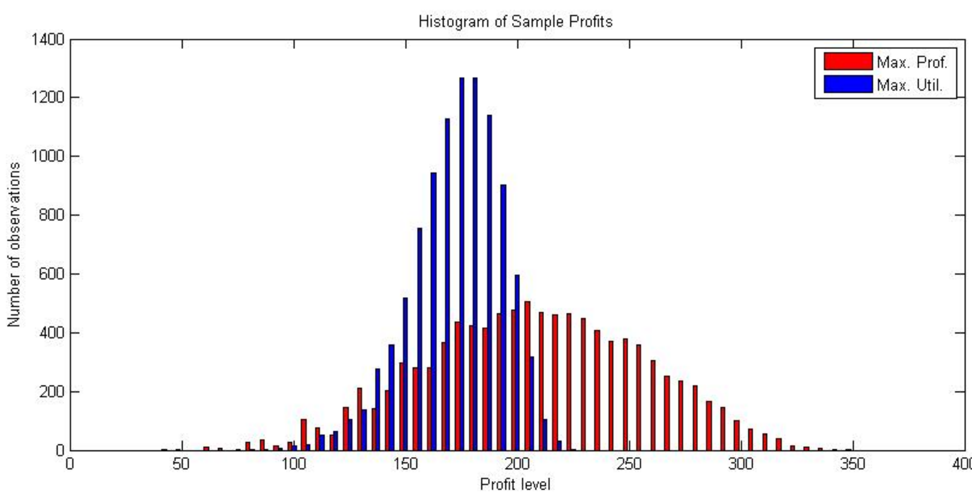


Figure 5.6: Histogram comparing the realized profits of the expected profit and the expected utility objectives

In general, the expected utility maximization objective sacrifices some amount of profit but leads to a more stable profit distribution. Intuitively, the expected profit maximization objective generates better profits with higher variance than the expected utility maximization objective's.

### 5.4.2 Comparison of Two Objectives With Respect to Their Optimal Decisions

In this part, we compare the optimal decisions of the expected profit maximization (risk-neutral) and the expected utility maximization (risk-averse) problems. For comparison, we keep $N = 30$, the initial inventory level $I = 10$, and the degree of risk-aversion $\gamma = 0.1$. Figure 5.7 shows the optimal price decisions for the expected profit and the expected utility objectives.

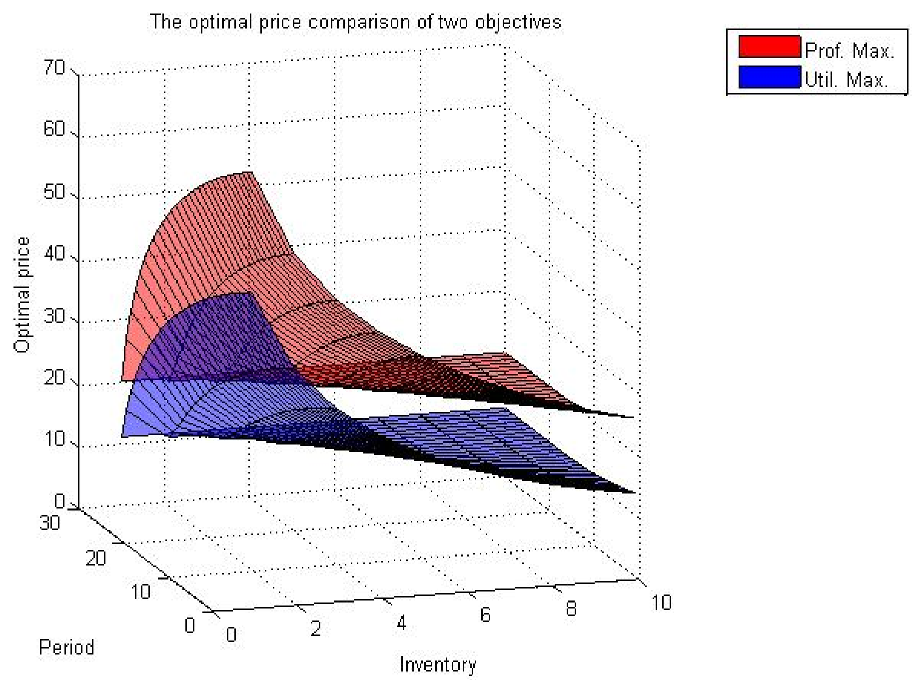


Figure 5.7: Comparison of the optimal prices of two objectives

Since the expected profit and the expected utility maximization objectives imply risk-neutral and risk-averse decision makers, respectively, it is not surprising that a risk-averse decision maker always sets lower prices than a risk-neutral one's. By figure 5.7, we also can observe that the optimal prices decrease with inventory level and periods.

### 5.4.3 Observed Results on the Degree of risk-aversion Effect

In this part, we analyze the effect of a change in the degree of risk-aversion on the optimal price decisions and on the marginal value of an additional item in the inventory. Needless to say, this part is for the exponential utility objective.

#### 5.4.3.1 On Inventory

A change in the degree of risk-aversion has an effect on the marginal value of an additional item in the inventory. To express the change in the marginal value mathematically, we define $\frac{G_n(I+1)}{G_n(I)}$ as the inventory ratio in $(n, I)$. The inventory ratio implies the marginal value of the $(I+1)^{th}$ item in period $n$. Since $G_n(I)$ is the minimum expected total utility from period $n$ to the end when the inventory level is $I$, the inventory ratio is always less than or equal to 1. So the lower values of the inventory ratio means higher difference between $G_n(I+1)$ and $G_n(I)$ and consequently means higher marginal value for the $(I+1)^{th}$ item in the inventory.

Theoretically, it is not possible to argue that the degree of risk-aversion has a monotonic effect on inventory ratio but a numerical analysis shows that the following inequality holds for the most cases:

$$\frac{{G_n}^{\gamma_1}(I+1)}{{G_n}^{\gamma_1}(I)} \geq \frac{{G_n}^{\gamma_2}(I+1)}{{G_n}^{\gamma_2}(I)} \tag{5.36}$$

when $\gamma_1 \leq \gamma_2$.

The inequality above implies that an additional item in the inventory is more valuable for a relatively more risk-averse decision maker. This is quite intuitive because selling this additional item brings a better utility for a more risk-averse seller (See proposition 7).

The figure 5.8 illustrates the effect of the degree of risk-aversion on the inventory ratio when $I = 5$. In other words, the figure shows the change in the marginal value of $6^{th}$ item for some periods with respect to the change in the degree of risk-aversion.

In the figure 5.8, lines represent periods. The most below line represents first

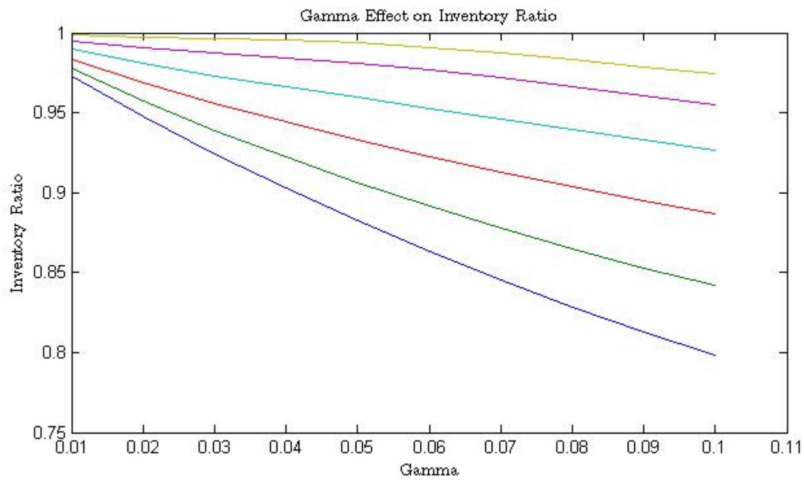


Figure 5.8: Gamma Effect on the Inventory Ratio for $I = 5$

period, the second below represents the second period, and so forth. Axis 'x' is the gamma (the degree of risk-aversion) values and axis 'y' is $\frac{G_n(6)}{G_n(5)}$. So we can conclude that the marginal value of an additional item in the inventory increases as the degree of risk-aversion increases. This practical result implies the effect of the degree of risk-averion on the optimal price, which is given in the following part.

#### *5.4.3.2 On Price*

Another important effect of the degree of risk-aversion is on the optimal price. Since an increase in the degree of risk-aversion means more risk-averse decisions, it is intuitive to say that as the degree of risk-aversion increases, the optimal price decreases under some general circumstances.

$$\arg\min {G_n}^{\gamma_1}(I) \geq \arg\min {G_n}^{\gamma_2}(I) \tag{5.37}$$

when $\gamma_1 \leq \gamma_2$.

The figure 5.9 shows how the optimal price is affected by a change in the degree of risk-aversion.

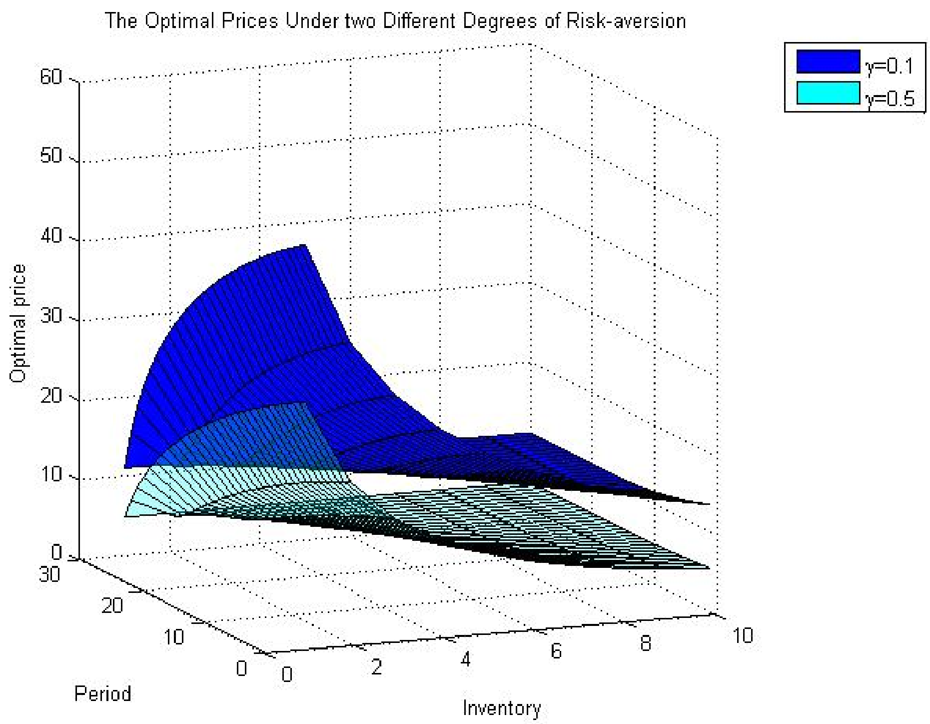


Figure 5.9: Optimal prices under two different degrees of risk-aversion

## 5.5 Conclusions

In this chapter, we study the dynamic pricing problem with the expected exponential utility maximization objective. Setting utility maximization objective implies the risk-averse character of the decision maker. For this problem, we show that the structural properties of the classical problem (the expected profit maximization - risk-neutral approach) are valid also for the expected utility maximization (risk-averse approach) problem. With a numerical study, we compare the performances and the optimal decisions of the classical problem and ours. We also draw some observed results on the effect of the degree of risk-aversion on the optimal decisions.

Chapter 6

# CONCLUSION

In this thesis, we intend to interpret some problems of inventory and pricing management in relation with financial considerations. We take financial considerations into account by inserting a financially binding constraint or setting a risk-averse objectives. There are many ways of reflecting risk-averse behavior and we employ an exponential utility function, since its concavity successfully represents the risk-aversion.

In the first chapter of the thesis, we first explore a financially constrained newsvendor problem with a payment constraint. Since there are two decision variables, we need to check the joint concavity of the objective function but we see that the objective function does not have joint concavity property which disables us from drawing a general solution. Instead, we perform a numerical study to analyze the effect of a financial constraint on the objective value. The decision maker has two instruments to satisfy the financial constraint: price and order quantity. We observe that none of them is dominating the other in being used in order to manipulate the constraint. In the next part of the first chapter, we analyze the dynamic pricing problem with a shortfall minimization objective. The objective of the seller is to minimize the expected shortfall with respect to a target profit. He pays no attention to achieving higher profit levels than the target. This attitude reflects the risk-averse character of the seller and relates the problem to the scope of this thesis. We provide a numerical analysis and compare the results with classical (profit maximization) dynamic pricing problems'. Since the expected shortfall minimization objective represents a risk-averse approach, it always sets prices less than or equal to those of the expected profit maximization objective's.

In the next chapter, we investigate a dynamic inventory management problem with single item with random supply. A seller decides on either to order or not at each period and any ordered item arrives in the next period with a probability less than 1. We analyze the problem with respect to two objective functions: the expected profit maximization and the expected utility maximization, while the first objective implies a risk-neutral decision maker and the second objective implies a risk-averse one. We derive the structural properties of the optimal decision and show that a threshold policy on inventory and period is optimal for both, risk-neutral and risk-averse, problems. We also present a numerical study comparing the performances and behavior of two problems. With this numerical study, we observe that the risk-averse (the expected utility maximizer) decision maker has always lower threshold level than the risk-neutral (the expected profit maximizer) one.

In the last chapter, we study the dynamic pricing problem with the expected utility maximization objective, which means a risk-averse decision maker. We compare some structural results to those of the the classical, the expected profit maximization (risk-neutral), model. In this context, we show that basic structural properties of classical model, concavity on inventory, monotonicity on optimal price, supermodularity on optimal price and inventory, and optimal price monotonicity in period, are valid also in the expected utility maximization model. We also show that a decision maker with relatively higher degree of risk-aversion obtains a better utility than a decision maker with lower degree of risk-aversion, when the other state variables (period and inventory level) are the same. We provide a numerical illustration comparing the risk-neutral and the risk-averse cases and observe that the risk-averse price policy generates sample profits from distribution with lower variance and lower expected utility than the risk-neutral policy's. In other words, a risk-averse decision maker sacrifices some level of profit in order to have a more stable distribution of outcomes. By numerical example, we also show the risk-averse decision maker sets relatively lower prices than a risk-neutral one does. These observations are quite accordant with the theory of risk-aversion. Afterwards, we numerically examine the effect of the degree of risk-aversion on price and on inventory and observe that an increase in the degree of risk-aversion leads a decrease in the optimal price, while increasing the marginal value of an additional item in the inventory.

For a future work, the scope of this thesis can be extended in several directions. For the dynamic inventory management problem with random supply, inserting the price decision may be interesting and challenging. Similarly, for the dynamic pricing problem with the expected utility maximization objective, one may add inventory decisions. The seller may decide on the order quantity at each period or he may decide on the initial inventory level at the beginning of the selling season. Along this thesis, we generally study on the multi-period problems so considering the time value of money can also be another good way of future extension. The seller may aim to maximize the net present value of the expected utility. Lastly, lending/borrowing options may be included in the inventory management problem with random supply.

# VITA

Abdullah Bilgin was born in Bursa, Turkey in 1987. He received his B.Sc. degree in Industrial Engineering from Koç University, İstanbul, in 2009. From August 2009 to August 2011, he worked as a teaching and research assistant in Koç University.

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# Appendix A

## PROOF OF PROPOSITION 1

Since the holding cost has the same effect on both sides of inequality, it is ignored during the proof.

Assume

$$J_{n+1}(I+1) - J_{n+1}(I) \leq J_{n+1}(I) - J_{n+1}(I-1) \qquad for\ \ any\ \ I \geq 1 \tag{A.1}$$

There are four possible cases for optimal decisions:

| Inventory Position | | I+1 | I | I-1 |
|---|---|---|---|---|
| Decision | Case 1 | N | N | N |
| | Case 2 | N | N | O |
| | Case 3 | N | O | O |
| | Case 4 | O | O | O |

where "$N$" is "not order" and "$O$" is "order". Other cases are impossible. See appendix B for details.

Case 1:

$$\begin{aligned} &\lambda(J_{n+1}(I) - J_{n+1}(I-1)) + (1-\lambda)(J_{n+1}(I+1) - J_{n+1}(I)) \leq \\ &\lambda(J_{n+1}(I-1) - J_{n+1}(I-1)) + (1-\lambda)(J_{n+1}(I) - J_{n+1}(I-1)) \end{aligned} \tag{A.2}$$

$$\lambda(J_{n+1}(I) - J_{n+1}(I-1)) + (1-\lambda)(J_{n+1}(I+1) - J_{n+1}(I)) \leq$$
$$J_{n+1}(I) - J_{n+1}(I-1) \leq \lambda(J_{n+1}(I-1) - J_{n+1}(I-2)) + (1-\lambda)(J_{n+1}(I) - J_{n+1}(I-1)) \tag{A.3}$$

The above conditions is true by induction assumption.

Case 2 and 3:

Let's first check the condition below:

$$\underbrace{J_n(I+1)}_{N} - \underbrace{J_n(I)}_{N} \leq \underbrace{J_n(I)}_{O} - \underbrace{J_n(I-1)}_{O} \tag{A.4}$$

$$\begin{aligned} &\lambda(J_{n+1}(I) - J_{n+1}(I-1)) + (1-\lambda)(J_{n+1}(I+1) - J_{n+1}(I)) \leq \\ &\lambda(q(J_{n+1}(I) - J_{n+1}(I-1)) + (1-q)(J_{n+1}(I-1) - J_{n+1}(I-2)))+ \\ &(1-\lambda)(q(J_{n+1}(I+1) - J_{n+1}(I)) + (1-q)(J_{n+1}(I) - J_{n+1}(I-1))) \end{aligned} \tag{A.5}$$

The above condition is true by induction assumption.

The above condition holds in both Case 2 and Case 3. For Case 2 (if not ordering in $J_n(I)$ is optimal), then;

$$\underbrace{J_n(I)}_{O} \leq \underbrace{J_n(I)}_{N} \tag{A.6}$$

and inserting $\underbrace{J_n(I)}_{N}$ instead of $\underbrace{J_n(I)}_{O}$ on the right hand-side of the inequality A.4 will be valid since inequality A.4 holds and that implies that above condition covers Case 2.

For Case 3 (if ordering in $J_n(I)$ is optimal), then;

$$\underbrace{J_n(I)}_{N} \leq \underbrace{J_n(I)}_{O} \tag{A.7}$$

and inserting $\underbrace{J_n(I)}_{O}$ instead of $\underbrace{J_n(I)}_{N}$ on the left hand-side of the inequality A.4 will be valid since inequality A.4 holds and that implies that above condition covers Case 3, too.

Case 4:

$$\begin{aligned}
\lambda(q(J_{n+1}(I+1) - J_{n+1}(I)) + (1-q)(J_{n+1}(I) - J_{n+1}(I-1)))+ \\
(1-\lambda)(q(J_{n+1}(I+2) - J_{n+1}(I+1)) + (1-q)(J_{n+1}(I+1) - J_{n+1}(I))) \leq \\
\lambda(q(J_{n+1}(I) - J_{n+1}(I-1)) + (1-q)(J_{n+1}(I-1) - J_{n+1}(I-2)))+ \\
(1-\lambda)(q(J_{n+1}(I+1) - J_{n+1}(I)) + (1-q)(J_{n+1}(I) - J_{n+1}(I-1)))
\end{aligned} \tag{A.8}$$

By assumption, each term on the left hand side is less then or equal to its corresponding term on the right hand side. This completes the proof.

# Appendix B

## PROOF OF COROLLARY 1

Functions

$$J_n(I) - J_n(I-1) \tag{B.1}$$

and

$$J_n(I+1) - J_n(I) \tag{B.2}$$

are both decreasing in $I$ (see appendix A). Since convex combination of two decreasing functions is also decreasing in $I$:

$$\lambda(J_n(I) - J_n(I-1)) + (1-\lambda)(J_n(I+1) - J_n(I)) \geq w \tag{B.3}$$

The above is the order condition. Seller makes an order if marginal value of ordering is greater than or equal to ordering cost. Since the order condition is decreasing in $I$, left side of the inequality will become less than $w$ after some $I$ level. So we can conclude that a threshold policy is optimal.

# Appendix C

## PROOF OF PROPOSITION 2

Assume

$$J_{n+1}(I) - J_{n+1}(I-1) \leq J_n(I) - J_n(I-1) \qquad for\ any\ I \geq 1 \tag{C.1}$$

There are 6 possible cases for optimal decisions:

<table>
<tr><td colspan="2">Period</td><td colspan="2">n</td><td colspan="2">n-1</td></tr>
<tr><td colspan="2">Inventory Position</td><td>I</td><td>I-1</td><td>I</td><td>I-1</td></tr>
<tr><td rowspan="6">Decision</td><td>Case 1</td><td>N</td><td>N</td><td>N</td><td>N</td></tr>
<tr><td>Case 2</td><td>N</td><td>N</td><td>N</td><td>O</td></tr>
<tr><td>Case 3</td><td>N</td><td>N</td><td>O</td><td>O</td></tr>
<tr><td>Case 4</td><td>N</td><td>O</td><td>O</td><td>O</td></tr>
<tr><td>Case 5</td><td>N</td><td>O</td><td>N</td><td>O</td></tr>
<tr><td>Case 6</td><td>O</td><td>O</td><td>O</td><td>O</td></tr>
</table>

where "$N$" is "not order" and "$O$" is "order". Other cases are impossible. See proofs B and D for details.

Case 1:

$$\begin{aligned}\lambda(J_{n+1}(I-1) - J_{n+1}(I-2)) + (1-\lambda)(J_{n+1}(I) - J_{n+1}(I-1)) \leq \\ \lambda(J_n(I-1) - J_n(I-2)) + (1-\lambda)(J_n(I) - J_n(I-1))\end{aligned} \tag{C.2}$$

The above condition is true by assumption.

Case 5:

To prove that;

$$J_n(I) - J_n(I-1) \leq J_{n-1}(I) - J_{n-1}(I-1) \tag{C.3}$$

we may use that;

$$\underbrace{J_{n-1}(I-1)}_{O} - \underbrace{J_n(I-1)}_{O} \leq \underbrace{J_{n-1}(I)}_{N} - \underbrace{J_n(I)}_{N} \tag{C.4}$$

$$\begin{aligned} \lambda(q(J_n(I-1) - J_{n+1}(I-1)) + (1-q)(J_n(I-2) - J_{n+1}(I-2)))+ \\ (1-\lambda)(q(J_n(I) - J_{n+1}(I)) + (1-q)(J_n(I-1) - J_{n+1}(I-1))) \leq \\ \lambda(J_n(I-1) - J_{n+1}(I-1)) + (1-\lambda)(J_n(I) - J_{n+1}(I)) \end{aligned} \tag{C.5}$$

The above condition is true by assumption.

The above condition holds in all Case 2, Case 3, and Case 4.

For Case 2 (if not ordering in $J_n(I-1)$ is optimal), then;

$$\underbrace{J_n(I-1)}_{O} \leq \underbrace{J_n(I-1)}_{N} \tag{C.6}$$

and inserting $\underbrace{J_n(I-1)}_{O}$ instead of $\underbrace{J_n(I-1)}_{N}$ on the left hand-side of the inequality C.4 will be valid since inequality C.4 holds and that implies that above condition covers Case 2.

For Case 3 (if not ordering in $J_n(I-1)$ is optimal and if ordering in $J_{n-1}(I)$ is optimal), then;

$$\underbrace{J_n(I-1)}_{O} \leq \underbrace{J_n(I-1)}_{N} \tag{C.7}$$

and

$$\underbrace{J_{n-1}(I)}_{N} \leq \underbrace{J_{n-1}(I)}_{O} \tag{C.8}$$

and inserting $\underbrace{J_n(I-1)}_{N}$ instead of $\underbrace{J_n(I-1)}_{O}$ on the left hand-side of the inequality C.4 and inserting $\underbrace{J_{n-1}(I)}_{N}$ instead of $\underbrace{J_{n-1}(I)}_{O}$ on the right hand-side of the inequality C.4 will be valid since inequality C.4 holds and that implies that above condition covers Case 3.

For Case 4 (if ordering in $J_{n-1}(I)$ is optimal), then;

$$\underbrace{J_{n-1}(I)}_{N} \leq \underbrace{J_{n-1}(I)}_{O} \tag{C.9}$$

and inserting $\underbrace{J_{n-1}(I)}_{O}$ instead of $\underbrace{J_{n-1}(I)}_{N}$ on the right hand-side of the inequality C.4 will be valid since inequality C.4 holds and that implies that above condition covers Case 4.

Case 6:

$$\begin{aligned} \lambda(q(J_{n+1}(I) - J_{n+1}(I-1)) + (1-q)(J_{n+1}(I-1) - J_{n+1}(I-2)))+ \\ (1-\lambda)(q(J_{n+1}(I+1) - J_{n+1}(I)) + (1-q)(J_{n+1}(I) - J_{n+1}(I-1))) \leq \\ \lambda(q(J_n(I) - J_n(I-1)) + (1-q)(J_n(I-1) - J_n(I-2)))+ \\ (1-\lambda)(q(J_n(I+1) - J_n(I)) + (1-q)(J_n(I) - J_n(I-1))) \end{aligned} \tag{C.10}$$

The above condition is true by assumption.

This completes the proof.

# Appendix D

## PROOF OF COROLLARY 2

Assume to order in period $n$ with inventory $I$. Then following ordering condition holds for period $n$ with inventory $I$;

$$\lambda(J_{n+1}(I) - J_{n+1}(I-1)) + (1-\lambda)(J_{n+1}(I+1) - J_{n+1}(I)) \geq w \tag{D.1}$$

By appendix C;

$$\begin{aligned} \lambda(J_{n+1}(I) - J_{n+1}(I-1)) + (1-\lambda)(J_{n+1}(I+1) - J_{n+1}(I)) \leq \\ \lambda(J_n(I) - J_n(I-1)) + (1-\lambda)(J_n(I+1) - J_n(I)) \end{aligned} \tag{D.2}$$

So;

$$\lambda(J_n(I) - J_n(I-1)) + (1-\lambda)(J_n(I+1) - J_n(I)) \geq w \tag{D.3}$$

ordering condition for period $n-1$ with inventory $I$ also holds. This is valid for any period earlier than $n$.

This completes the proof.

# Appendix E

## PROOF OF PROPOSITION 3

Since the holding cost has the same effect on both sides of inequality, it is ignored during the proof.

Assume

$$\frac{G_{n+1}(I+1)}{G_{n+1}(I)} \geq \frac{G_{n+1}(I)}{G_{n+1}(I-1)} \qquad for\ \ any\ \ I \geq 1 \tag{E.1}$$

There are four possible cases for optimal decisions:

| Inventory Position | | I+1 | I | I-1 |
|---|---|---|---|---|
| Decision | Case 1 | N | N | N |
| | Case 2 | N | N | O |
| | Case 3 | N | O | O |
| | Case 4 | O | O | O |

where "$N$" is "not order" and "$O$" is "order". Other cases are impossible. See appendix F for details.

Case 1:

$$\frac{\lambda(G_{n+1}(I)e^{-\gamma p} + (1-\lambda)(G_{n+1}(I+1)}{\lambda(G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)(G_{n+1}(I)} \geq \frac{\lambda(G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)(G_{n+1}(I)}{\lambda(G_{n+1}(I-2)e^{-\gamma p} + (1-\lambda)(G_{n+1}(I-1)} \tag{E.2}$$

By cross multiplication, it is easy to see the above conditions is true by induction assumption.

Case 2 and 3:

Let's first check the condition below:

$$\frac{\overbrace{G_n(I+1)}^{N}}{\underbrace{G_n(I)}_{N}} \geq \frac{\overbrace{G_n(I)}^{O}}{\underbrace{G_n(I-1)}_{O}} \tag{E.3}$$

$$\frac{\lambda(G_{n+1}(I)e^{-\gamma p} + (1-\lambda)(G_{n+1}(I+1)}{\lambda(G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)(G_{n+1}(I)} \geq \frac{\lambda(q(G_{n+1}(I)e^{\gamma w} + (1-q)(G_{n+1}(I-1))e^{-\gamma p} + (1-\lambda)(q(G_{n+1}(I+1)e^{\gamma w} + (1-q)(G_{n+1}(I)))}{\lambda(q(G_{n+1}(I-1)e^{\gamma w} + (1-q)(G_{n+1}(I-2))e^{-\gamma p} + (1-\lambda)(q(G_{n+1}(I)e^{\gamma w} + (1-q)(G_{n+1}(I-1)))} \tag{E.4}$$

By cross multiplication, it is easy to see the above conditions is true by induction assumption.

The above condition holds in both Case 2 and Case 3. For Case 2 (if not ordering in $G_n(I)$ is optimal), then;

$$\underbrace{G_n(I)}_{O} \geq \underbrace{G_n(I)}_{N} \tag{E.5}$$

and inserting $\underbrace{G_n(I)}_{N}$ instead of $\underbrace{G_n(I)}_{O}$ on the right hand-side of the inequality E.3 will be valid since inequality E.3 holds and that implies that above condition covers Case 2.

For Case 3 (if ordering in $G_n(I)$ is optimal), then;

$$\underbrace{G_n(I)}_{N} \geq \underbrace{G_n(I)}_{O} \tag{E.6}$$

and inserting $\underbrace{G_n(I)}_{O}$ instead of $\underbrace{G_n(I)}_{N}$ on the left hand-side of the inequality E.3 will be valid since inequality E.3 holds and that implies that above condition covers Case 3, too.

Case 4:

$$\frac{\lambda(q(G_{n+1}(I+1)e^{\gamma w}+(1-q)(G_{n+1}(I))e^{-\gamma p}+(1-\lambda)(q(G_{n+1}(I+2)e^{\gamma w}+(1-q)(G_{n+1}(I+1)))}{\lambda(q(G_{n+1}(I)e^{\gamma w}+(1-q)(G_{n+1}(I-1))e^{-\gamma p}+(1-\lambda)(q(G_{n+1}(I+1)e^{\gamma w}+(1-q)(G_{n+1}(I)))} \geq \frac{\lambda(q(G_{n+1}(I)e^{\gamma w}+(1-q)(G_{n+1}(I-1))e^{-\gamma p}+(1-\lambda)(q(G_{n+1}(I+1)e^{\gamma w}+(1-q)(G_{n+1}(I)))}{\lambda(q(G_{n+1}(I-1)e^{\gamma w}+(1-q)(G_{n+1}(I-2))e^{-\gamma p}+(1-\lambda)(q(G_{n+1}(I)e^{\gamma w}+(1-q)(G_{n+1}(I-1)))} \tag{E.7}$$

By cross multiplication, it is easy to see the above conditions is true by induction assumption. This completes the proof.

# Appendix F

## PROOF OF COROLLARY 3

Since the holding cost has the same effect on both sides of inequality, it is ignored during the proof.

Assume it is optimal to order in $G_n(I)$. So the order condition must hold for $G_n(I)$;

$$\frac{\lambda G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)G_{n+1}(I)}{\lambda G_{n+1}(I)e^{-\gamma p} + (1-\lambda)G_{n+1}(I+1)} \geq e^{\gamma w} \tag{F.1}$$

The above is the order condition. Seller makes an order if marginal utility value of ordering is better than or equal to marginal ordering "disutility". Since the order condition is decreasing in $I$ (see appendix E), left side of the inequality will become less than $e^{\gamma w}$ after some $I$ level. So we can conclude that a threshold policy is optimal.

# Appendix G

## PROOF OF PROPOSITION 4

Assume

$$\frac{G_{n+1}(I)}{G_{n+1}(I-1)} \geq \frac{G_n(I)}{G_n(I-1)} \qquad \textit{for any } I \geq 1 \tag{G.1}$$

There are 6 possible cases for optimal decisions:

| Period | | n | | n-1 | |
|---|---|---|---|---|---|
| Inventory Position | | I | I-1 | I | I-1 |
| Decision | Case 1 | N | N | N | N |
| | Case 2 | N | N | N | O |
| | Case 3 | N | N | O | O |
| | Case 4 | N | O | O | O |
| | Case 5 | N | O | N | O |
| | Case 6 | O | O | O | O |

where "$N$" is "not order" and "$O$" is "order". Other cases are impossible. See proofs F and H for details.

Case 1:

$$\frac{\lambda G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)G_{n+1}(I)}{\lambda G_{n+1}(I-2)e^{-\gamma p} + (1-\lambda)G_{n+1}(I-1)} \geq \frac{\lambda G_n(I-1)e^{-\gamma p} + (1-\lambda)G_n(I)}{\lambda G_n(I-2)e^{-\gamma p} + (1-\lambda)G_n(I-1)} \tag{G.2}$$

By cross multiplication, it is easy to see the above conditions is true by induction assumption.

Case 3:

Let's first check the condition below:

$$\frac{\overbrace{G_n(I)}^{N}}{\underbrace{G_n(I-1)}_{N}} \geq \frac{\overbrace{G_{n-1}(I)}^{O}}{\underbrace{G_{n-1}(I-1)}_{O}} \tag{G.3}$$

$$\frac{\lambda(G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)(G_{n+1}(I))}{\lambda(G_{n+1}(I-2)e^{-\gamma p} + (1-\lambda)(G_{n+1}(I-1)} \geq \frac{\lambda(q(G_n(I)e^{\gamma w} + (1-q)(G_{n+1}(I-1))e^{-\gamma p} + (1-\lambda)(q(G_{n+1}(I+1)e^{\gamma w} + (1-q)(G_{n+1}(I)))}{\lambda(q(G_{n+1}(I-1)e^{\gamma w} + (1-q)(G_{n+1}(I-2))e^{-\gamma p} + (1-\lambda)(q(G_{n+1}(I)e^{\gamma w} + (1-q)(G_{n+1}(I-1)))} \tag{G.4}$$

By cross multiplication, it is easy to see the above conditions is true by induction assumption.

The above condition holds in all Case 2, Case 3, and Case 4.

For Case 2 (if not ordering in $G_{n-1}(I)$ is optimal), then;

$$\underbrace{G_{n-1}(I)}_{O} \geq \underbrace{G_{n-1}(I)}_{N} \tag{G.5}$$

and inserting $\underbrace{G_{n-1}(I)}_{N}$ instead of $\underbrace{G_{n-1}(I)}_{O}$ on the right hand-side of the inequality G.3 will be valid since inequality G.3 holds and that implies that above condition covers Case 2.

For Case 4 (if ordering in $G_n(I-1)$ is optimal), then;

$$\underbrace{G_n(I-1)}_{N} \geq \underbrace{G_n(I-1)}_{O} \tag{G.6}$$

and inserting $\underbrace{G_n(I-1)}_{O}$ instead of $\underbrace{G_n(I-1)}_{N}$ on the left hand-side of the inequality G.3 will be valid since inequality G.3 holds and that implies that above condition covers Case 4.

For Case 5 (if ordering in $G_n(I-1)$ is optimal and if not ordering in $G_{n-1}(I)$ is optimal), then;

$$\underbrace{G_n(I-1)}_{N} \geq \underbrace{G_n(I-1)}_{O} \tag{G.7}$$

and

$$\underbrace{G_{n-1}(I)}_{O} \geq \underbrace{G_{n-1}(I)}_{N} \tag{G.8}$$

and inserting $\underbrace{G_n(I-1)}_{N}$ instead of $\underbrace{G_n(I-1)}_{O}$ on the left hand-side of the inequality G.3 and inserting $\underbrace{G_{n-1}(I)}_{N}$ instead of $\underbrace{G_{n-1}(I)}_{O}$ on the right hand-side of the inequality G.3 will be valid since inequality G.3 holds and that implies that above condition covers Case 5.

Case 6:

$$\frac{\lambda(q(G_{n+1}(I)e^{\gamma w}+(1-q)(G_{n+1}(I-1))e^{-\gamma p}+(1-\lambda)(q(G_{n+1}(I+1)e^{\gamma w}+(1-q)(G_{n+1}(I)))}{\lambda(q(G_{n+1}(I-1)e^{\gamma w}+(1-q)(G_{n+1}(I-2))e^{-\gamma p}+(1-\lambda)(q(G_{n+1}(I)e^{\gamma w}+(1-q)(G_{n+1}(I-1)))} \geq \frac{\lambda(q(G_n(I)e^{\gamma w}+(1-q)(G_n(I-1))e^{-\gamma p}+(1-\lambda)(q(G_n(I+1)e^{\gamma w}+(1-q)(G_n(I)))}{\lambda(q(G_n(I-1)e^{\gamma w}+(1-q)(G_n(I-2))e^{-\gamma p}+(1-\lambda)(q(G_n(I)e^{\gamma w}+(1-q)(G_n(I-1)))} \tag{G.9}$$

By cross multiplication, it is easy to see the above conditions is true by induction assumption.

This completes the proof.

# Appendix H

## PROOF OF COROLLARY 4

Assume it is optimal to order in $G_n(I)$. So the order condition must hold for $G_n(I)$;

$$\frac{\lambda G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)G_{n+1}(I)}{\lambda G_{n+1}(I)e^{-\gamma p} + (1-\lambda)G_{n+1}(I+1)} \geq e^{\gamma w} \tag{H.1}$$

The above is the order condition. Seller makes an order if marginal utility value of ordering is better than or equal to marginal ordering "disutility". Since the order condition is decreasing in $n$ (see appendix G), left side of the inequality will become greater in period $n$. By appendix G;

$$\frac{\lambda G_n(I-1)e^{-\gamma p} + (1-\lambda)G_n(I)}{\lambda G_n(I)e^{-\gamma p} + (1-\lambda)G_n(I+1)} \geq \frac{\lambda G_{n+1}(I-1)e^{-\gamma p} + (1-\lambda)G_{n+1}(I)}{\lambda G_{n+1}(I)e^{-\gamma p} + (1-\lambda)G_{n+1}(I+1)} \tag{H.2}$$

So;

$$\frac{\lambda G_n(I-1)e^{-\gamma p} + (1-\lambda)G_n(I)}{\lambda G_n(I)e^{-\gamma p} + (1-\lambda)G_n(I+1)} \geq e^{\gamma w} \tag{H.3}$$

ordering condition for period $n-1$ with inventory $I$ also holds, if it holds for period $n$. This is valid for any period earlier than $n$.

This completes the proof.

# Appendix I

## PROOF OF PROPERTY 1

$$J_n(I) = \max_p R_n(I,p) \tag{I.1}$$

where $R_n(I,p)$ denotes the expected total profit when the current inventory is $I$ and given price is $p$:

$$R_n(I,p) = \lambda(p)(p + J_{n+1}(I-1)) + (1-\lambda(p))J_{n+1}(I) \tag{I.2}$$

Assume

$$J_{n+1}(I+1) - J_{n+1}(I) \leq J_{n+1}(I) - J_{n+1}(I-1) \qquad for\ \ any\ \ I \geq 1 \tag{I.3}$$

Assume that optimum prices are $p_1$, $p_2$, and $p_3$ for $J_n(I+1)$, $J_n(I)$, and $J_n(I-1)$, respectively.

So;

$$R_n(I;p_1) \leq R_n(I;p_2) = J_n(I) \tag{I.4}$$

and

$$R_n(I;p_3) \leq R_n(I;p_2) = J_n(I) \tag{I.5}$$

$$J_n(I+1) = R_n(I+1,p_1) = \lambda(p_1)(p_1 + J_{n+1}(I)) + (1-\lambda(p_1))J_{n+1}(I+1) \tag{I.6}$$

$$J_n(I) = R_n(I, p_2) = \lambda(p_2)(p_2 + J_{n+1}(I-1)) + (1-\lambda(p_2))J_{n+1}(I) \tag{I.7}$$

$$J_n(I-1) = R_n(I-1, p_3) = \lambda(p_3)(p_3 + J_{n+1}(I-2)) + (1-\lambda(p_3))J_{n+1}(I-1) \tag{I.8}$$

To prove the following;

$$J_n(I+1) - J_n(I) \leq J_n(I) - J_n(I-1) \tag{I.9}$$

we first analyze the left part of the inequality;

$$\begin{aligned} J_n(I+1) - J_n(I) \leq R_n(I+1; p_1) - R_n(I; p_1) = \\ \lambda(p_1)(J_{n+1}(I) - J_{n+1}(I-1)) + (1-\lambda(p_1))(J_{n+1}(I+1) - J_{n+1}(I)) \leq \\ J_{n+1}(I) - J_{n+1}(I-1) \end{aligned} \tag{I.10}$$

$$\Rightarrow J_n(I+1) - J_n(I) \leq J_{n+1}(I) - J_{n+1}(I-1) \tag{I.11}$$

then the right part of the inequality;

$$\begin{aligned} J_n(I) - J_n(I-1) \geq R_n(I; p_3) - R_n(I-1; p_3) = \\ \lambda(p_3)(J_{n+1}(I-1) - J_{n+1}(I-2)) + (1-\lambda(p_3))(J_{n+1}(I) - J_{n+1}(I-1) \geq \\ J_{n+1}(I) - J_{n+1}(I-1) \end{aligned} \tag{I.12}$$

$$\Rightarrow J_{n+1}(I) - J_{n+1}(I-1) \leq J_n(I) - J_n(I-1) \tag{I.13}$$

Therefore;

$$J_n(I+1) - J_n(I) \leq J_n(I) - J_n(I-1) \tag{I.14}$$

This completes the proof.

# Appendix J

## PROOF OF PROPERTY 2

$$J_n(I) = \max_p R_n(I, p) \tag{J.1}$$

where $R_n(I, p)$ denotes the expected total profit when the current inventory is $I$ and given price is $p$:

$$R_n(I, p) = \lambda(p)(p + J_{n+1}(I - 1)) + (1 - \lambda(p))J_{n+1}(I) \tag{J.2}$$

Since $p_1$ is optimal for $J_n(I + 1)$;

$$R_n(I + 1; p_1) \geq R_n(I + 1; p_2) \tag{J.3}$$

$$\lambda(p_1)(p_1+J_{n+1}(I))+(1-\lambda(p_1))J_{n+1}(I+1) \geq \lambda(p_2)(p_2+J_{n+1}(I))+(1-\lambda(p_2))J_{n+1}(I+1) \tag{J.4}$$

Similarly, since $p_2$ is optimal for $J_n(I)$;

$$R_n(I; p_2) \geq R_n(I; p_1) \tag{J.5}$$

$$\lambda(p_2)(p_2+J_{n+1}(I-1))+(1-\lambda(p_2))J_{n+1}(I) \geq \lambda(p_1)(p_1+J_{n+1}(I-1))+(1-\lambda(p_1))J_{n+1}(I) \tag{J.6}$$

By adding two inequalities;

$$\begin{aligned}&\lambda(p_1)(p_1 + J_{n+1}(I)) + (1 - \lambda(p_1))J_{n+1}(I + 1) + \lambda(p_2)(p_2 + J_{n+1}(I - 1)) + (1 - \lambda(p_2))J_{n+1}(I)\\ &\geq \lambda(p_2)(p_2 + J_{n+1}(I)) + (1 - \lambda(p_2))J_{n+1}(I + 1) + \lambda(p_1)(p_1 + J_{n+1}(I - 1)) + (1 - \lambda(p_1))J_{n+1}(I)\end{aligned} \tag{J.7}$$

$$\begin{aligned}&\Rightarrow \lambda(p_1)(J_{n+1}(I) - J_{n+1}(I + 1)) + \lambda(p_2)(J_{n+1}(I - 1) - J_{n+1}(I))\\ &\geq \lambda(p_2)(J_{n+1}(I) - J_{n+1}(I + 1)) + \lambda(p_1)(J_{n+1}(I - 1) - J_{n+1}(I))\end{aligned} \tag{J.8}$$

$$\Rightarrow (\lambda(p_1)-\lambda(p_2))(J_{n+1}(I)-J_{n+1}(I+1)) \geq (\lambda(p_1)-\lambda(p_2))(J_{n+1}(I-1)-J_{n+1}(I)) \tag{J.9}$$

$$\Rightarrow (\lambda(p_1) - \lambda(p_2))(2J_{n+1}(I) - J_{n+1}(I + 1) - J_{n+1}(I - 1)) \geq 0 \tag{J.10}$$

By appendix J, the second term is non-negative. So the first term also has to be non-negative;

$$\Rightarrow \lambda(p_1) \geq \lambda(p_2)) \tag{J.11}$$

$$\Rightarrow p_1 \leq p_2 \tag{J.12}$$

This completes the proof.

# Appendix K

## PROOF OF PROPERTY 3

$$J_n(I) = \max_p R_n(I,p) \tag{K.1}$$

where $R_n(I,p)$ denotes the expected total profit when the current inventory is $I$ and given price is $p$:

$$R_n(I,p) = \lambda(p)(p + J_{n+1}(I-1)) + (1-\lambda(p))J_{n+1}(I) \tag{K.2}$$

Assume

$$J_{n+1}(I+1) - J_{n+1}(I) \leq J_n(I+1) - J_n(I) \qquad for\ \ any\ \ I \geq 1 \tag{K.3}$$

Equivalently;

$$J_{n+1}(I+1) - J_n(I+1) \leq J_{n+1}(I) - J_n(I) \tag{K.4}$$

To prove that;

$$J_n(I+1) - J_n(I) \leq J_{n-1}(I+1) - J_{n-1}(I) \tag{K.5}$$

we may use that;

$$J_n(I+1) - J_{n-1}(I+1) \leq J_n(I) - J_{n-1}(I) \tag{K.6}$$

Assume that optimum prices are $p_1$, $p_2$, $p_3$, and $p_4$ for $J_n(I+1)$, $J_{n-1}(I+1)$, $J_n(I)$, and $J_{n-1}(I)$, respectively.

$$\begin{aligned} R_n(I+1,p_1) - R_{n-1}(I+1,p_2) \leq R_n(I+1,p_1) - R_{n-1}(I+1,p_1) \\ \leq R_n(I,p_4) - R_{n-1}(I,p_4) \leq R_n(I,p_3) - R_{n-1}(I,p_4) \end{aligned} \tag{K.7}$$

$$R_n(I+1,p_1) - R_{n-1}(I+1,p_1) \leq R_n(I,p_4) - R_{n-1}(I,p_4) \tag{K.8}$$

$$\begin{aligned} \lambda(p_1)(J_{n+1}(I) - J_n(I)) + (1-\lambda(p_1))(J_{n+1}(I+1) - J_n(I+1)) \leq \\ \lambda(p_4)(J_{n+1}(I-1) - J_n(I-1)) + (1-\lambda(p_4))(J_{n+1}(I) - J_n(I)) \end{aligned} \tag{K.9}$$

By assumption;

$$\begin{aligned} \lambda(p_1)(J_{n+1}(I) - J_n(I)) + (1-\lambda(p_1))(J_{n+1}(I+1) - J_n(I+1)) \leq J_{n+1}(I) - J_n(I) \leq \\ \lambda(p_4)(J_{n+1}(I-1) - J_n(I-1)) + (1-\lambda(p_4))(J_{n+1}(I) - J_n(I)) \end{aligned} \tag{K.10}$$

This completes the proof.

# Appendix L

## PROOF OF PROPERTY 4

$$J_n(I) = \max_p R_n(I, p) \tag{L.1}$$

where $R_n(I, p)$ denotes the expected total profit when the current inventory is $I$ and given price is $p$:

$$R_n(I, p) = \lambda(p)(p + J_{n+1}(I - 1)) + (1 - \lambda(p))J_{n+1}(I) \tag{L.2}$$

Assume that optimum prices are $p_1$ and $p_2$ for $J_{n+1}(I)$ and $J_n(I)$, respectively. So;

$$J_{n+1}(I) = R_{n+1}(I; p_1) \geq R_{n+1}(I; p_2) \tag{L.3}$$

$$\lambda(p_1)(p_1+J_{n+2}(I-1))+(1-\lambda(p_1))J_{n+2}(I) \geq \lambda(p_2)(p_2+J_{n+2}(I-1))+(1-\lambda(p_2))J_{n+2}(I) \tag{L.4}$$

and

$$J_n(I) = R_n(I; p_2) \geq R_n(I; p_1) \tag{L.5}$$

$$\lambda(p_2)(p_2+J_{n+1}(I-1))+(1-\lambda(p_2))J_{n+1}(I) \geq \lambda(p_1)(p_1+J_{n+1}(I-1))+(1-\lambda(p_1))J_{n+1}(I) \tag{L.6}$$

By adding two inequalities;

$$\lambda(p_1)(p_1 + J_{n+2}(I-1)) + (1-\lambda(p_1))J_{n+2}(I) + \lambda(p_2)(p_2 + J_{n+1}(I-1)) + (1-\lambda(p_2))J_{n+1}(I)$$
$$\geq \lambda(p_2)(p_2 + J_{n+2}(I-1)) + (1-\lambda(p_2))J_{n+2}(I) + \lambda(p_1)(p_1 + J_{n+1}(I-1)) + (1-\lambda(p_1))J_{n+1}(I) \tag{L.7}$$

$$\begin{aligned} &\Rightarrow \lambda(p_1)(J_{n+2}(I-1) - J_{n+1}(I)) + \lambda(p_2)(J_{n+1}(I-1) - J_{n+1}(I)) \\ &\geq \lambda(p_2)(J_{n+2}(I-1) - J_{n+2}(I)) + \lambda(p_1)(J_{n+1}(I-1) - J_{n+1}(I)) \end{aligned} \tag{L.8}$$

$$\Rightarrow (\lambda(p_1) - \lambda(p_2))(J_{n+1}(I) - J_{n+1}(I-1)) + (\lambda(p_1) - \lambda(p_2))(J_{n+2}(I-1) - J_{n+2}(I)) \geq 0 \tag{L.9}$$

$$\Rightarrow (\lambda(p_1) - \lambda(p_2))((J_{n+1}(I) - J_{n+1}(I-1)) - (J_{n+2}(I) - J_{n+2}(I-1))) \geq 0 \tag{L.10}$$

By appendix K, the second term is non-negative. So the first term also has to be non-negative;

$$\Rightarrow \lambda(p_1) \geq \lambda(p_2)) \tag{L.11}$$

$$\Rightarrow p_1 \leq p_2 \tag{L.12}$$

This completes the proof.

# Appendix M

## PROOF OF PROPOSITION 5

Assume

$$\frac{G_{n+1}(I+1)}{G_{n+1}(I)} \geq \frac{G_{n+1}(I)}{G_{n+1}(I-1)} \qquad for\ \ any\ \ I \geq 1 \tag{M.1}$$

Assume that optimum prices are $p_1$, $p_2$, and $p_3$ for $G_n(I+1)$, $G_n(I)$, and $G_n(I-1)$ respectively ($p_1 \leq p_2 \leq p_3$, see the second proof).

Since

$$U_n(I;p_1) \geq U_n(I;p_2) = G_n(I) \tag{M.2}$$

and

$$U_n(I;p_3) \geq U_n(I;p_2) = G_n(I) \tag{M.3}$$

$$\Rightarrow \frac{G_n(I+1)}{G_n(I)} \geq \frac{G_n(I+1)}{U_n(I;p_1)} \geq \frac{U_n(I;p_3)}{G_n(I-1)} \geq \frac{G_n(I)}{G_n(I-1)} \tag{M.4}$$

So, proving the following will be sufficient:

$$\frac{G_n(I+1)}{U_n(I;p_1)} \geq \frac{U_n(I;p_3)}{G_n(I-1)} \tag{M.5}$$

$$\Rightarrow \frac{\lambda(p_1)(e^{-\gamma(p_1)}G_{n+1}(I)) + (1-\lambda(p_1))G_{n+1}(I+1)}{\lambda(p_1)(e^{-\gamma(p_1)}G_{n+1}(I-1)) + (1-\lambda(p_1))G_{n+1}(I)} \geq \frac{\lambda(p_3)(e^{-\gamma(p_3)}G_{n+1}(I-1)) + (1-\lambda(p_3))G_{n+1}(I)}{\lambda(p_3)(e^{-\gamma(p_3)}G_{n+1}(I-2)) + (1-\lambda(p_3))G_{n+1}(I-1)} \tag{M.6}$$

By cross multiplication, it is easy to see that the inequality holds. This completes the proof.

# Appendix N

## PROOF OF COROLLARY 5

Since $p_1$ is optimal for $G_n(I+1)$;

$$G_n(I+1) = U_n(I+1; p_1) \leq U_n(I+1; p_2) \tag{N.1}$$

$$\lambda(p_1)(e^{-\gamma(p_1)}G_{n+1}(I)) + (1-\lambda(p_1))G_{n+1}(I+1) \leq \lambda(p_2)(e^{-\gamma(p_2)}G_{n+1}(I)) + (1-\lambda(p_2))G_{n+1}(I+1) \tag{N.2}$$

$$\Rightarrow \lambda(p_1)(e^{-\gamma(p_1)}G_{n+1}(I) - G_{n+1}(I+1)) \leq \lambda(p_2)(e^{-\gamma(p_2)}G_{n+1}(I) - G_{n+1}(I+1)) \tag{N.3}$$

Similarly, since $p_2$ is optimal for $G_n(I)$;

$$G_n(I) = U_n(I; p_2) \leq G_n(I; p_1) \tag{N.4}$$

$$\lambda(p_2)(e^{-\gamma(p_2)}G_{n+1}(I-1)) + (1-\lambda(p_2))G_{n+1}(I) \leq \lambda(p_1)(e^{-\gamma(p_1)}G_{n+1}(I-1)) + (1-\lambda(p_1))G_{n+1}(I) \tag{N.5}$$

$$\Rightarrow \lambda(p_2)(e^{-\gamma(p_2)}G_{n+1}(I-1) - G_{n+1}(I)) \leq \lambda(p_1)(e^{-\gamma(p_1)}G_{n+1}(I-1) - G_{n+1}(I)) \tag{N.6}$$

We multiply inequalities N.3 N.6. Since all terms are non-positive, multiplying two inequalities will change the direction of the inequality;

$$\begin{aligned} -(G_{n+1}(I))^2 e^{-\gamma(p_2)} - G_{n+1}(I-1)G_{n+1}(I+1)e^{-\gamma(p_1)} \leq \\ -(G_{n+1}(I))^2 e^{-\gamma(p_1)} - G_{n+1}(I-1)G_{n+1}(I+1)e^{-\gamma(p_2)} \end{aligned} \tag{N.7}$$

$$(e^{-\gamma(p_1)} - e^{-\gamma(p_2)})(G_{n+1}(I))^2 - G_{n+1}(I-1)G_{n+1}(I+1)) \leq 0 \tag{N.8}$$

Since the second term is non-positive (see proof 5.2.1), first term has to be non-negative. So;

$$\Rightarrow e^{-\gamma(p_1)} \geq e^{-\gamma(p_2)} \tag{N.9}$$

$$\Rightarrow p_1 \leq p_2 \tag{N.10}$$

proof completed.

# Appendix O

## PROOF OF PROPOSITION 6

$$G_n(I) = \min_p U_n(I, p) \tag{O.1}$$

$$U_n(I, p) = \lambda(p)(e^{-\gamma p} G_{n+1}(I-1)) + (1-\lambda(p))G_{n+1}(I) \tag{O.2}$$

$$\Rightarrow U_n(I, p) = \lambda(p)(e^{-\gamma p} G_{n+1}(I-1) - G_{n+1}(I)) + G_{n+1}(I) \tag{O.3}$$

$$\frac{dU_n(I, p)}{dp} = \lambda'(p)(e^{-\gamma p} G_{n+1}(I-1) - G_{n+1}(I)) - \gamma\lambda(p)G_{n+1}(I-1)e^{-\gamma p} \tag{O.4}$$

$p^*$ is the minimizer for $U_n(I, p)$ so;

$$G_n(I) = U_n(I, p^*) \tag{O.5}$$

and

$$\frac{dU_n(I, p)}{dp} = 0 \qquad at \;\; p = p^* \tag{O.6}$$

$$\Rightarrow \lambda'(p^*)(e^{-\gamma p^*} G_{n+1}(I-1) - G_{n+1}(I)) - \gamma\lambda(p^*)G_{n+1}(I-1)e^{-\gamma p^*} = 0 \tag{O.7}$$

$$\Rightarrow e^{-\gamma p^*}(1-\frac{\gamma\lambda(p^*)}{\lambda'(p^*)}) = \frac{G_{n+1}(I)}{G_{n+1}(I-1)} \tag{O.8}$$

Optimum prices hold the equation above.

$$\Rightarrow e^{-\gamma p^*}G_{n+1}(I-1) = \frac{G_{n+1}(I)\lambda'(p^*)}{\lambda'(p^*)-\gamma\lambda(p^*)} \tag{O.9}$$

Assume $p_1$ is optimum for $G_n(I+1)$ and $p_2$ is optimum for $G_n(I)$;

by appendix N;

$$p_1 \leq p_2 \tag{O.10}$$

and

$$\lambda(p_1) \geq \lambda(p_2) \tag{O.11}$$

$$G_n(I+1) = \lambda(p_1)(e^{-\gamma p_1}G_{n+1}(I) - G_{n+1}(I+1)) + G_{n+1}(I+1) \tag{O.12}$$

$$G_n(I+1) = \lambda(p_1)(\frac{G_{n+1}(I+1)\lambda'(p_1)}{\lambda'(p_1)-\gamma\lambda(p_1)} - G_{n+1}(I+1)) + G_{n+1}(I+1) \tag{O.13}$$

$$G_n(I+1) = G_{n+1}(I+1)(1-\frac{\gamma(\lambda(p_1))^2}{\gamma\lambda(p_1)-\lambda'(p_1)}) \tag{O.14}$$

$$\frac{G_n(I+1)}{G_{n+1}(I+1)} = (1-\frac{\gamma(\lambda(p_1))^2}{\gamma\lambda(p_1)-\lambda'(p_1)}) \tag{O.15}$$

Similarly;

$$\frac{G_n(I)}{G_{n+1}(I)} = (1-\frac{\gamma(\lambda(p_2))^2}{\gamma\lambda(p_2)-\lambda'(p_2)}) \tag{O.16}$$

To prove the following is true;

$$\frac{G_n(I+1)}{G_n(I)} \leq \frac{G_{n+1}(I+1)}{G_{n+1}(I)} \tag{O.17}$$

or simply the following;

$$\Rightarrow \frac{G_n(I+1)}{G_{n+1}(I+1)} \leq \frac{G_n(I)}{G_{n+1}(I)} \tag{O.18}$$

we can use their equivalents;

$$1 - \frac{\gamma(\lambda(p_1))^2}{\gamma\lambda(p_1) - \lambda'(p_1)}) \leq 1 - \frac{\gamma(\lambda(p_2))^2}{\gamma\lambda(p_2) - \lambda'(p_2)} \tag{O.19}$$

$$\Rightarrow \frac{\gamma(\lambda(p_2))^2}{\gamma\lambda(p_2) - \lambda'(p_2)} \leq \frac{\gamma(\lambda(p_1))^2}{\gamma\lambda(p_1) - \lambda'(p_1)}) \tag{O.20}$$

$$\gamma\lambda(p_1)\lambda(p_2)(\lambda(p_1) - \lambda(p_2)) + ((\lambda(p_2))^2\lambda'(p_1) - (\lambda(p_1))^2\lambda'(p_2)) \geq 0 \tag{O.21}$$

Since $p_1 \leq p_2$; and $\lambda(p_1) \geq \lambda(p_2)$, first term is non-negative.

Since $p$ has an non-decreasing Hazard Rate Function, $\frac{(\lambda(p))^2}{\lambda'(p)}$ is also non-decreasing in p;

$\Rightarrow \frac{(\lambda(p_2))^2}{\lambda'(p_2)} \geq \frac{(\lambda(p_1))^2}{\lambda'(p_1)}$ so second term is non-negative, too.

This completes the proof.

# Appendix P

## PROOF OF COROLLARY 6

Assume $p_1$ and $p_2$ are optimal for $G_{n+1}(I)$ and $G_n(I)$, respectively.

So $p_1$ and $p_2$ satisfies the optimality condition for $G_{n+1}(I)$ and $G_n(I)$, respectively.

$$\Rightarrow e^{-\gamma p_1}(1 - \frac{\gamma\lambda(p_1)}{\lambda'(p_1)}) = \frac{G_{n+2}(I)}{G_{n+2}(I-1)} \tag{P.1}$$

$$\Rightarrow e^{-\gamma p_2}(1 - \frac{\gamma\lambda(p_2)}{\lambda'(p_2)}) = \frac{G_{n+1}(I)}{G_{n+1}(I-1)} \tag{P.2}$$

By appendix K;

$$\frac{G_{n+1}(I)}{G_{n+1}(I-1)} \leq \frac{G_{n+2}(I)}{G_{n+2}(I-1)} \tag{P.3}$$

So;

$$e^{-\gamma p_1}(1 - \frac{\gamma\lambda(p_1)}{\lambda'(p_1)}) \geq e^{-\gamma p_2}(1 - \frac{\gamma\lambda(p_2)}{\lambda'(p_2)}) \tag{P.4}$$

All terms are positive. Since both $e^{-\gamma p}$ and $(1 - \frac{\gamma\lambda(p)}{\lambda'(p)})$ are non-increasing in $p$, we can conclude that;

$$p_1 \leq p_2 \tag{P.5}$$

This completes the proof.

Appendix Q

# PROOF OF PROPOSITION 7

A similar method with Zhao and Zheng's [21] can be used to prove this.

Assume $\omega_1$ is the optimal (utility minimizer) price policy for a decision maker with the degree of risk-aversion $\gamma_1$ and $\omega_2$ is optimal (utility minimizer) price policy for another decision maker with the degree of risk-aversion $\gamma_2$ when $\gamma_1 \leq \gamma_2$.

If both of them applies the price policy $\omega_1$, their expected total profit will be equal from any period $n$ on.

$$R_n{}^{\gamma_1}(I;\omega_1) = R_n{}^{\gamma_2}(I;\omega_1) \tag{Q.1}$$

However, since their risk-aversion degrees are different, their utilities gained from same amount of profit will also be different.

$$e^{-\gamma_1(R_n{}^{\gamma_1}(I;\omega_1))} \geq e^{-\gamma_2(R_n{}^{\gamma_2}(I;\omega_1))} \tag{Q.2}$$

Since second decision maker has a higher risk-aversion degree, intuitively he will get better utility from the same amount of profit. Besides, $\omega_1$ is a suboptimal policy for second decision maker. He will get much better utility if he applies his own optimal policy $\omega_2$.

$$e^{-\gamma_2(R_n{}^{\gamma_2}(I;\omega_1))} \geq e^{-\gamma_2(R_n{}^{\gamma_2}(I;\omega_2))} \tag{Q.3}$$

So we can say that;

$$e^{-\gamma_1(R_n{}^{\gamma_1}(I;\omega_1))} \geq e^{-\gamma_2(R_n{}^{\gamma_2}(I;\omega_2))} \tag{Q.4}$$

$$\Rightarrow {G_n}^{\gamma_1}(I) \geq {G_n}^{\gamma_2}(I) \tag{Q.5}$$

This completes the proof.