

DYNAMIC PRICING UNDER INVENTORY CONSIDERATIONS AND PRICE PROTECTION

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By
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AND PRICE PROTECTION

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We certify that we have read this thesis and that in our opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In high-tech industry, customers' tendency to purchase the newest versions of products forces manufacturers to reduce the prices of older models. This puts the retailers in a vulnerable position as their own sales prices also decrease for these products. For this purpose, manufacturers and retailers compromise over different price commitment terms in their contracts. One such term is price protection. In general, a price protection term obliges a manufacturer refund a retailer a portion of the difference between the new and old wholesale prices for the inventory that the retailer have in stock and that are ordered within a time window. Sometimes, refunds may also be applied on products sold based on their sales price. We study a price protection contract over a finite horizon under stochastic demand. We have a single manufacturer and a single retailer, each endowed with a fixed amount of inventory at the beginning of the horizon. The manufacturer determines the retail price and neither manufacturing nor replenishment is allowed. The objective of the manufacturer is to set the retail price in each period given how much inventory is left at the manufacturer and the retailer. We analyze the structure of the model and provide some analytical results on the effect of different factors on optimal prices and optimal expected profits. Then, we present the results of a numerical study in which we further investigate the effect of different factors to obtain managerial insights.

Keywords: dynamic pricing, price protection, revenue management, periodic-review.

ÖZET

FİYAT KORUMASI VE ENVANTERİ GÖZ ÖNÜNDE TUTARAK DİNAMİK FİYATLANDIRMA

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Yüksek teknoloji endüstrisinde, tüketicilerin ürünlerin yeni versiyonlarını almaya yönelik eğilimleri üreticileri eski modellerin fiyatlarını düşürmeye zorlar. Bu perakendecileri bu ürünler için kendi satış fiyatları da düşeceği için korunmasız durumda bırakır. Bunu önlemek için, üreticiler ve perakendeciler belirli bir fiyat yükümlülük politikası üzerinde uzlaşırlar ve bunlardan bir tanesi fiyat korumasıdır. Genelde fiyat koruması kontratı uyarınca üretici firma perakendeci firmanın belirli bir zaman aralığında aldığı envanter için eski ve yeni toptan satış fiyatları arasındaki farkın bir kısmını geri alır. Bazı fiyat koruması kontratlarında ise satılan ürünlerin eski ve yeni satış fiyatları üzerinden ödeme yapılır. Bu çalışmada rassal talebin görüldüğü çok periyotlu ve sonlu bir ortamda fiyat koruması kontratı incelenmektedir. Satış periyodunun başında belirli miktarda envantere sahip olan bir üretici ve bir perakendeci olduğu kabul edilmektedir. İncelenen problemde, üretici perakende satış fiyatına karar vermekte, yeni üretim ve ikmale izin vermemektedir. Üreticinin amacı kendisinde ve perakendecide kalan envanter miktarlarına göre her periyodun başında kendi kârını maksimize edecek satış fiyatlarını belirlemektir. Bu problem için bir model önerilerek, modelin analitik özellikleri ve farklı faktörlerin optimal fiyat ve kâr üzerindeki etkileri analitik olarak incelenmiştir. Son olarak, yöneticilerin kararlarını desteklemek amaçlı sayısal bir çalışma yapılarak sonuçları özetlenmiştir.

Anahtar sözcükler: dinamik fiyatlama, fiyat koruması, hasılat yönetimi, periyodik gözden geçirme.

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Chapter 1

Introduction

Advances in technology, customers' inclination to buy state-of-the-art products and the disparity in financial conditions force manufacturers to reduce the prices of their soon-to-be-obsolete products and to release higher versions of these products into circulation both to avoid the financial loss and to satisfy their customers' expectations. As manufacturers organize a system to reach their direct customers, they can also sign contracts with some intermediaries which include specific terms to be obeyed by parties during the selling horizon. In addition to price terms, some other factors such as reliability and warranty durations are also involved. A manufacturer with an inventory of its own may have to decrease the wholesale price, which in turn reduces the sales price in the marketplace. This certainly hurts the retailers which already have an inventory of these items that needs to be sold at lower (sometimes negative) profit margin. Since keeping their retail partners loyal to the supply chain is of fundamental importance, manufacturers sometimes offer a compensation for price differences. Different forms of compensation used for this purpose include price protection (adjustment) and rebate.

Price protection is regarding the change in the price of a specific product in a single channel. If the wholesale price of a certain product is decreased by the manufacturer, the retailers which have purchased the product from this manufacturer at a higher price and still hold inventory want to be reimbursed to maintain

their profitability. Since the possibility of price decreases puts the retailers at risk and creates a threat against the availability of new products in the market and may eventually discourage the retailers from carrying similar products from this manufacturer, such price-protection terms are crucial for the success of the supply chain. Some manufacturers accept reimbursing their retailers the difference if they decrease the wholesale price at any time during the selling horizon. However these price-protection terms limit the flexibility of the manufacturer when pricing its own inventory. The manufacturer now needs to consider how much it needs to reimburse to its retail partners in addition to amount of inventory it owns and the future demand for this product.

Reimbursement is required for all inventory that is ordered prior to the reduction of the wholesale price. Price protection requires the reimbursed products not to have been sold. However, if an increase in price is also possible later on, manufacturers might want to reimburse only the products that are sold during which the prices are lower. This is called rebate and it can be regarded as being complementary to price protection. Rebate and price protection are pillars of the integration between manufacturers and retailers.

Manufacturers might put some restrictions over the products which are eligible for reimbursement to mitigate their own risks. One of them is a time restriction which stipulates that the price difference only for the products purchased or ordered within a time window before the price reduction. For example, if they agree on a 30-day restriction, a manufacturer is not responsible for the inventories acquired more than 30 days ago. The other possible limitation is on the percentage by which the price is reduced. If the manufacturer asks for a lower bound over this percentage, the reductions less than this lower bound will not be eligible for price protection. Another one is reimbursement rate which necessitates the manufacturer to reimburse only a portion of the difference between the current and prior prices. This reimbursement rate normally ranges between 0 and 1. If the reimbursement rate is 1, then it will be called as full-reimbursement. These restrictions can be applied alone or they can be combined.

When considering a price reduction at any time, the manufacturer should

balance between benefits of price reduction and the costs due to price protection terms. If the manufacturer has a high inventory level and has a very short time before the end of the selling season, it is expected to decrease the price if it has no commitment to its retailers. However, when reimbursement is in the picture, determining the price is a challenging task. Each unit of reduction will cause an increase in the reimbursement cost on aggregate if the new wholesale price is lower than the original wholesale price at which the retailer acquired its current inventory. Although the price reduction sometimes seems to be advantageous when the manufacturer's distressed inventory alone is considered, it may end up hurting the manufacturer at the end. Therefore, the manufacturer should take more than one factor into account when making its mind over how high to set price.

We study a supply chain which involves a manufacturer and a retailer. In this setting, the retailer has already purchased a certain number of products just at the beginning of the selling horizon and is not allowed to make any replenishment later on. Likewise, the manufacturer has already stopped the manufacturing. We fix a initial retail price at which the product has been sold until the beginning of the selling season. We also assume that the manufacturer can set a higher price than this initial retail price since these two parties might have started the selling season by signing a new price protection contract and the manufacturer might have followed a different pricing strategy before. That is, the manufacturer is expected to adapt to the changes in the contract. Throughout the selling horizon, the manufacturer determines the market price at which both the manufacturer and the retailer are committed to sell the inventories in stock. That is, the retailer has no control over the price. If the manufacturer sets a price less than the procurement cost of the retailer at the beginning of any period, then it is required to credit the price difference to the retailer for the products sold by the retailer in that period. At the end of the selling horizon, if the retailer has a positive inventory level, then the salvage value will also impose a reimbursement cost to the manufacturer. That is, we use a combination of rebate and price protection. Demand follows a Poisson process whose rate is dependent on the market price. If at an intermediate period, the manufacturer has no inventory

remaining in stock, the product is withdrawn from the market and the retailer is reimbursed based on the salvage value. Inventory holding and backlogging costs are not considered.

The rest of the thesis is organized as follows:

In Chapter 2, we review the literature on price protection and supply chain coordination.

In Chapter 3, we firstly model the single-period and the multi-period problem. Afterwards, we present our analytical results regarding the structure of the single-period model, the effect of price protection and the changes in the optimal price for the single-period problem and the optimal profit for both the single and the multi-period problem with respect to the problem parameters such as the salvage values, the retailer's procurement cost, the reimbursement rate, the manufacturer's and the retailer's inventory levels.

In Chapter 4, we present the results of our numerical study on the effect of different problem parameters exemplifying the relationships we prove for the single-period problem in the previous chapter. We also interpret the changes in the optimal price and the optimal profit verbally. We also provide a comparison between the single-period and the multi-period problem.

In Chapter 5, we summarize our work, reflect what the thesis brings to the literature and give recommendations for further research.

Chapter 2

Literature Survey

In this chapter, we present some articles written over pricing and pricing under price protection from different perspectives. Clark and Scarf [1] study a multi-period inventory problem. The retailer is allowed to make replenishment during the horizon and the lead time is assumed to be positive. The retailer determines whether to place another order or not just after the previous order is delivered. The retailer incurs an inventory holding cost and a backlogging cost. The objective is to minimize the retailer's cost. As a result, the authors show that order-up-to-level policy is optimal for the retailer.

Wang [2] models a periodic-review single-product problem in which there is a retailer that sells its inventory in multiple periods under a stochastic demand distribution. The demands in different periods are assumed to be distributed identically. The procurement cost at each period is defined as a random variable and assumed to be decreasing over time. Since the procurement cost is decreasing by the assumption, the retail price is also decreasing. The retail price is determined in two different ways. It is either the procurement cost marked up by a fixed amount or the procurement cost marked up by a fixed percentage. The author studies two types of the problem one of which is the backorder model and the other one of which is the lost sales model. The author shows that order-up-to-level is the optimal policy for the retailer.

Gavirneni [3] studies a periodic-review problem in which the retailer sells its inventory in multiple periods. The demand is assumed to be stochastic. The retailer does not determine the selling price and the price is assumed to follow a Markovian process. A probability transition matrix demonstrates how much the price declines from one period to the other one. This transition matrix is regenerative and communicative. The retailer does not incur any set-up cost, the capacity is not restricted and placed orders are delivered immediately. When the retailer cannot fully satisfy the demand in a period, the unsatisfied portion is lost and there is no back-order cost. A linear inventory holding cost is also incorporated for the products carried between consecutive periods. The author solves three types of problems that are single-period, finite horizon and discounted infinite horizon problems. For all these problems, it is shown that the base-stock policy is optimal for the retailer.

Li and Py [4] model a supply chain consisting of four echelons with the suppliers, production facilities, distributors and customers by integrating price protection and product return subsidies. Distributors are divided into two classes as independent and manufacturer-controlled. Manufacturer-controlled distributors are not eligible to utilize price protection but the independent ones are secured. They also follow a specific network depicting the relationships between the echelons like which distributors can purchase from which production facilities. The purpose is to maximize the manufacturer's profit based on its revenues arising from selling products to the independent distributors and reaching the end-customers by the manufacturer-controlled distributors. For the cost coming from the price protection, they define a random variable representing the price protection level for each of the products at each period. For the other part of the cost expression, they prefer targeted product return subsidy and for this purpose they incorporate another random variable representing the extra percentage of returned products to credit for. The other constraints are the satisfaction of the targeted profits for the independent distributors, the inventory balancing constraints for all the distributors and the production facilities, the price protection level and the extra percentage for product return subsidies ranging between 0 and 1 and the non-negativity constraints. They evaluate four different scenarios and

the base scenario is without price protection and product return subsidies. In the other three scenarios, they incorporate these costs one by one to determine their effects in the allocation of the products. After solving the non-linear model for each of the scenarios, price protection turns out to increase the manufacturer's profit.

Sourirajan, Kapuscinski and Ettle [5] study a model in which price protection is limited to the products purchased by the distributor in a limited time before the price reduction occurs. Their main purpose is to minimize the excessive stock at the distributor and to determine the optimal principle between the manufacturer and the distributor. As the base models, they utilize the centralized problem and the decentralized problem with Distributor Managed Inventory but without price protection. It is assumed that the distributor places an order at the end of each period by taking into consideration the production cost and the wholesale price of the following period together with the distribution of all the costs and the prices for the future periods which are known in advance. The ordered products are transported to the retailer just at the beginning of the following period. Another assumption is that if the distributor cannot grant its customers a certain number of products from its stock, then it is eligible to place an urgent order to the manufacturer and gets it within a negligible time. Therefore, both the manufacturer and the retailer incur a backorder cost due to the loss of goodwill, the regular production cost and the cost of accelerated production for the manufacturer and due to the wholesale price paid for the expedited products, the loss of goodwill and maybe the obligatory discount granted to its customers for the retailer. The decentralized model differs slightly from the centralized one. The differences are the wholesale price instead of the production cost and the backorder cost only for the retailer. For both of the models, they prove that the optimal policy is a non-stationary base-stock policy. Afterwards, they model the decentralized problems under price protection with Distributor Managed Inventory and Vendor Managed Inventory, respectively. In this case, the number of products purchased by the retailer within the time of a certain length before the current period is included in the state set. Again they prove the optimal policy is a base stock level and VMI contracts with risk-sharing are more suitable than DMI in the cases where there

are technological obsolescence, small-to-moderate profit margins, low inventory carrying costs and higher impact of product shortages on the manufacturer.

Zhang [6] investigates whether the price protection mechanism is effective in channel coordination or not. The problem is formulated as a two period problem with one manufacturer and one retailer. It is assumed that the retailer can purchase products from the manufacturer only before the start of the first period. The demand distribution is continuous with known mean and variance which can also be different for these two periods. The retailer incurs inventory holding and stockout costs, while the manufacturer incurs only production costs. Retail price is also assumed to be decreasing. The author first considers the integrated system where the manufacturer determines the optimal order quantity of the retailer that maximizes the expected profit of the entire system. Afterwards, the retailer is regarded as the decision maker and the objective is to maximize the retailer's expected profit in a setting that price protection is in effect. In the first case, the author assumes the system is disintegrated but there is no price protection and proves that the retailer purchases more products at optimality when the system is integrated. In the second case, the author incorporates price protection and determines an expression for the optimal profit of the retailer. This leads the author to find the optimal price protection policy with the optimal wholesale price and the reimbursement rate by benchmarking between the integrated system and the disintegrated system under price protection. In the end, this optimal policy is proven to coordinate the channel.

Liu et. al [7] evaluate the structure of the optimal solution for different contracts. For price protection, they solve a decentralized model which is composed of one manufacturer and one retailer. It is assumed that the demand is a deterministic function of the retail price and the retail price is determined by the retailer. The manufacturer does not sell any product to the end-customer so its only relationship is with the retailer. The decision mechanism for these two echelons follow a specific order. Firstly, the manufacturer determines the wholesale price by taking into consideration the retailer's optimal behavior. Then, the retailer determines the retail price and the optimal order quantity. For this purpose, at each period, they have two different but connected dynamic programming models

one of which is to maximize the profit of the manufacturer and the other one of which is to maximize the profit of the retailer. At the beginning, they attempt to find a closed-form solution by solving the single-period problem. The retailer's single period problem gives a closed-form solution for the retail price and the optimal order quantity. By inserting the expression for the optimal retail price into the manufacturer's model, this time they acquire a closed-form solution for the optimal wholesale price. Afterwards, they prove that not holding inventories between consecutive periods is the optimal solution for the retailer by showing increasing inventory level at the end of a specific period leads to a decrease in optimal price. This fact disintegrates the periods and the closed-form solutions for the single-period problem can be generalized to the multi-period problem.

Lu et. al [8] examine the effectiveness of price commitment policies. The basic ingredients of a specific policy are rebate, mid-life returns, end-life returns and price protection. Rebate is put forward to replace price protection in some situations and defined as the refund for the products sold by the retailer if the procurement cost decreases. The retailer is assumed to have a single buying opportunity or two buying opportunities. They prove that when there is a single buying opportunity, end-life returns, mid-life returns and only one of price protection or rebate terms can be used to coordinate the channel. However, they also present some conditions under which using only mid-life returns and end-life returns brings a win-win outcome. Then, using price protection, end-life returns and mid-life returns is shown to be the policy which results in a channel coordination when there are two buying opportunities if the out-of-stock costs are excluded in the first period.

Chen and Xiao [9] model a problem consisting of a manufacturer and a retailer selling a specific product over two periods. The retailer decides on how many products to order at the beginning of the selling horizon. After observing the demand in period 1, the retailer specifies the quantity of products which it will carry over to the next period. The retailer is provided with a right to send the remaining products to the manufacturer after the specification of the number of products at the beginning of period 2 and get a refund. If demand is not fully satisfied, the retailer incurs a back-order cost and for each of the products carried

over to the next period, it incurs an inventory holding cost. As a benchmark, the authors firstly solve a model with dispose-down-to policy in which the retailer returns some of the remaining products at the end of the first period but without any price contract between two parties. Afterwards, price protection terms are included and analytical expressions are derived for optimal solutions. By utilizing these expressions, they present some propositions reflecting the conditions under which using price protection, mid-life returns and end-life returns can coordinate the channel by preventing double marginalization.

Lee and Rhee [10] examine three different price commitment contracts. A two-period setting is utilized to analyze these contracts in which a manufacturer and a retailer show up. Only the retailer is allowed to sell the product and its selling horizon is limited to the first period. Retail and wholesale prices are kept fixed and the decision variables are the quantity which will be ordered by the retailer at the beginning and the quantity which will be returned to the manufacturer between period 1 and period 2. If buyback with early salvage subsidy is applied, then the retailer is reimbursed by the manufacturer for each one of the products it carries over to period 2. The retailer releases the same product in a remote market again and the demand distribution in this market is assumed to be independent of the one observed in period 1. If buyback with final salvage subsidy is applied, this time the retailer receives a refund for each one of the remaining products after period 2. When buyback with price protection is adopted, the retailer is granted a right to claim a credit for the products sold in period 2 at a lower price. The manufacturer can salvage its products and this operation is composed of two stages. First of all, it recycles the reusable products received back from the retailer after period 1 and this follows a specific distribution. That is, it is assumed that a specific portion of these products are eligible for recycling. Then, the manufacturer puts these reusable products into the market again and earns some revenue. It has another demand distribution which is independent of the first two demand distributions. As a benchmark, an integrated system in which the manufacturer is possessed of its own retailer is solved without adopting any buy-back policy. Afterwards, by including each one of the buy-back policies, the authors derive the optimal quantities which will be ordered at the beginning of

period 1 and will be returned before period 2. They continue with delivering some numerical examples regarding the change of the optimal solutions with respect to the problem parameters.

Lee [11] studies a setting which includes one manufacturer and one retailer selling a specific product in two periods and the manufacturer provides the retailer with a price protection contract. The author first assumes that the retailer will have only one buying opportunity since the lead time is long. That is, it will determine the order quantity at the beginning of the selling horizon and place the order to take the delivery just at the beginning of period 1. If any excess demand is observed, then the demand will be lost but the retailer will incur no back-order cost. The author shows that if the parameters are determined properly, price protection coordinates the channel and this will benefit both of the parties. As a benchmark, the author solves an integrated problem and prove that in the case without price protection, the retailer is inclined to order less than the optimal order quantity of the integrated system with price protection. If two buying opportunities are given to the retailer, in both of the periods, the order-up-to policy will be optimal. Whether the price protection is included or not, the optimal order-up-to level does not change in the second period but in the first period, the optimal order quantity will be higher in case of price protection. The author states that two buying opportunities will not secure the channel coordination in general and present some conditions under which the channel is coordinated.

Zhu [12] examines a two-echelon problem which is composed of a supplier and a buyer and the life cycle of the sold product is two periods. It is assumed that due to the high procurement and holding cost of the raw material, the supplier is not inclined to keep a large amount of inventory and because of the long lead time, a new order cannot be received in time. The buyer firstly decides on how many products to order at the beginning of period 1 both for period 1 and period 2. That is, the buyer is expected to place a joint order for the selling horizon. The order for period 1 is satisfied immediately but the order for period 2 stays outstanding. The supplier grants the buyer a discount for the advance order. After the realization of the demand in period 1, the buyer is allowed to place

an order for additional products or to cancel a part of the advance order. If the buyer orders additional products, the supplier stipulates the buyer to pay the normal wholesale price per product instead of the discounted price. The cost parameters are assumed to be known by both the parties at the beginning of the horizon. The retail prices set for the two periods are not changed and it is lower in period 2. Demand distributions over both of the periods are known in advance and assumed to be independent. The supplier pays an incentive per product if the buyer cancels a part of its advance order to induce the buyer to place a joint order at the beginning. The unsold products at the end of the selling horizon are returned to the supplier and the buyer is refunded a certain amount of money per product. First, the centralized problem in which the buyer is under the ownership of the supplier is solved. The optimality conditions in combination with a closed-form solution for the inventory level with which to start period 2 and the expected profit of period 2 are determined. Then, the decentralized problem is solved with two different price contracts. It is shown that price-only contract in which the wholesale prices and the canceling rebate are pre-specified can coordinate the system for the optimal inventory position determined for the centralized problem but the coordination is not guaranteed to be satisfied for the optimal inventory level for those contract parameters. To satisfy the coordination, price protection over the unit holding cost in period 1 is adopted and it is shown that under some conditions the coordination is satisfied and the win-win outcome is obtained. These conditions are presented by some propositions.

Wang [13] gives a general model for the price contracts in literature. Like most of the papers released about price contracts, a two echelon problem is dissected for a selling horizon of two periods. First, the manufacturer offers a contract to the retailer and then the retailer determines its order quantity as a joint order for both of the periods. It is assumed that the retailer is not allowed to make another replenishment until the end of the selling horizon. Demand distributions of two periods are correlated. For each one of the products carried in a period, an inventory holding cost is incurred and the unsatisfied demand is lost causing a goodwill cost per product. Leftover inventories after period 2 are salvaged

by bringing in a certain amount of revenue. There is also a production cost for the manufacturer. First, the integrated problem is solved and the optimal order quantity is determined. Second, the decentralized problem without return is solved to determine the optimal order quantity of the retailer. Afterwards, it is proven that the optimal order quantity decreases if no return is allowed. This proves the channel coordination is not satisfied. Finally, the return option is included and another parameter is defined for this setting. For the returned products, the retailer gets a refund. This time, the retailer is granted a right to return a portion of the leftover inventories after the realization of the demand in period 1. If the leftover inventory is less than the order quantity times the allowed proportion, a positive inventory level will be carried over to the next period. Otherwise, the retailer will start the second period with no products since it will return all of them to the manufacturer. An equation is presented which should be satisfied to coordinate the channel under this type of return policy. At the end, a table demonstrating the relationship between the general model and some price contracts which also include price protection.

Most of the papers in the literature focus on channel coordination, optimal replenishment policies, optimal retail and wholesale prices. The main focus of the model we study is how the manufacturer should react to sudden changes in some factors or in case of new terms in the price protection contract it has signed with the retailer. The problem definition is also different and we study the case in which the product is soon-to-be-obsolete and will be withdrawn from the market at the end of a finite selling horizon. We also have a manufacturer-controlled system in which the retailer has no control over pricing.

Chapter 3

Model

A manufacturer sells a single product at its own store as well as through a retailer. Prices at these two channels are assumed to be always the same; when the manufacturer changes the price at its own store, the retailer matches this price. The manufacturer is endowed with n_1 units of inventory. The retailer has n_2 units of inventory which it purchased when the retail price was p_0 . After the start of the selling horizon, the retailer is not allowed to make another replenishment. That is, it has single buying opportunity. In order to protect the retailer, the manufacturer is providing “price protection” in its supply contract. For any item that is sold at a price p_1 below p_0 , manufacturer reimburses $\gamma(p_0 - p_1)$, i.e., a portion of the difference between the current retail price and the initial retail price. In this definition, γ is called as “reimbursement rate”. Since the retailer cannot make another replenishment during the selling horizon, the manufacturer reimburses the retailer whenever it sets the retail price lower than the initial retail price.

The product can be sold over a selling season that consists of N periods. For simplicity, we assume that each period is one unit of time. The arrival process is Poisson whose rate ($\lambda(p)$) depends on the retail price p . The demand distributions for different periods are assumed to be independent and the parameters of the rate do not change over time. $\lambda(p)$ is a non-increasing function but there is

no limitation over its concavity as long as otherwise is not stated. $\lambda(p)$ can be linear, exponential, constant price elasticity or another demand function in the literature. It is assumed that an arriving customer buys the product from the manufacturer's store with probability α and from the retailer's store with probability $(1 - \alpha)$. Then, in each period, the demands for the manufacturer and the retailer will be Poisson random variables with means $\alpha\lambda(p)$ and $(1 - \alpha)\lambda(p)$. We denote P_1 and P_2 for the corresponding probability mass functions and F_1 and F_2 for the corresponding cumulative distribution functions for the manufacturer and the retailer. It is assumed that any inventory at the end of the selling season may have a salvage value both for the manufacturer and for the retailer. v_1 and v_2 denote the salvage values per period for the manufacturer and the retailer respectively. v_2 is also assumed to be less than p_0 for the rationality of the model since the value of a product is expected to be less at the end of the selling horizon than at the beginning and so the manufacturer tends to sell it at a higher price than v_2 .

Given the initial inventory levels, the manufacturer's problem is to determine the price over the selling season such that its expected revenue net of price protection reimbursements is maximized. Throughout the selling season, the manufacturer is committed to refunding the retailer whenever it sets a retail price which is less than the initial retail price. Price protection contract also necessitates the manufacturer to reimburse the retailer for the leftover inventories at the end of the selling horizon. We will model the multi-period problem and then show how the optimal price and the optimal profit change with respect to the problem parameters. Both the manufacturer and the retailer sell the product over the same number (N) of periods and the manufacturer decides on which price to set at the beginning of each period. We will also examine the structure of the model and the effect of the price protection.

Let $J_k^*(n_1, n_2)$ be the optimal expected profit of the manufacturer over the periods $k, k + 1, \dots, N$ given that manufacturer and the retailer have n_1 and n_2 units of inventory at the beginning of period k . The model to determine the price to charge at the beginning of period k can be written as follows:

$$\begin{aligned}
J_k^*(n_1, n_2) = \max_{p_k} \bigg\{ & \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[xp_k - \gamma(p_0 - p_k)^+ y + J_{k+1}^*(n_1 - x, n_2 - y) \right] P_1(x) P_2(y) + \\
& \sum_{x=n_1}^{\infty} \sum_{y=0}^{n_2-1} \left[n_1 p_k - \gamma(p_0 - p_k)^+ y + J_{k+1}^*(0, n_2 - y) \right] P_1(x) P_2(y) + \\
& \sum_{x=0}^{n_1-1} \sum_{y=n_2}^{\infty} \left[xp_k - \gamma(p_0 - p_k)^+ n_2 + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) + \\
& \sum_{x=n_1}^{\infty} \sum_{y=n_2}^{\infty} \left[n_1 p_k - \gamma(p_0 - p_k)^+ n_2 \right] P_1(x) P_2(y) \bigg\},
\end{aligned}$$

where

$$\begin{aligned}
J_{N+1}^*(n_1, n_2) &= n_1 v_1 - n_2 \gamma(p_0 - v_2), \\
J_k^*(0, n_2) &= -n_2 \gamma(p_0 - v_2), k = 2..N.
\end{aligned}$$

The first boundary condition states that at the end of the selling horizon, the manufacturer and the retailer earn a salvage value as we mentioned before and under the assumption of v_2 being less than p_0 , the manufacturer is assumed to be committed to reimbursing the retailer for the remaining inventories in stock. The second one states that if the manufacturer runs out of stock in any intermediate period, the product is assumed to be withdrawn from the circulation and the retailer earns a salvage value for its positive inventory by forcing the manufacturer to refund. We have this assumption for the tractability of the analysis we present. v_2 is also assumed to be constant during the selling horizon. There is also a third boundary condition for the case in which the retailer runs out of stock in an intermediate period but the manufacturer still has some products in stock. In this case, since the retailer is not in the market anymore, the value of the profit-to-go function ($J_k(n_1, 0)$) will come from the problem in which the manufacturer is the only seller and so there is no price protection. Furthermore, we should be aware that this is not equivalent to setting γ to 0 in the problem under price protection we study.

We can rewrite the model shown above in a different form. Firstly, we change the order of the sums since it does not affect the objective function. This operation

enables us to simplify the model by replacing the infinite sums by finite sums. For example, in the second component of the objective function, there is an infinite sum over x and the expression in the summation is not a function of x except for $P_1(x)$. Then, taking that expression out of the summation over x by leaving only $P_1(x)$, we obtain an infinite sum over the manufacturer's probability mass function. We can rewrite that infinite sum by utilizing the corresponding cumulative distribution function. The same operation can be applied in the third and last component of the objective function. After these operations and incorporating the assumptions we have made, the model is:

$$J_k^*(n_1, n_2) = \max_{p_k} \left\{ \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[xp_k - \gamma(p_0 - p_k)^+ y + J_{k+1}^*(n_1 - x, n_2 - y) \right] P_1(x) P_2(y) + \right. \\ \sum_{y=0}^{n_2-1} \left[n_1 p_k - \gamma(p_0 - p_k)^+ y - (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) (1 - F_1(n_1 - 1)) + \\ \sum_{x=0}^{n_1-1} \left[xp_k - \gamma(p_0 - p_k)^+ n_2 + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) (1 - F_2(n_2 - 1)) + \\ \left. \left[n_1 p_k - \gamma(p_0 - p_k)^+ n_2 \right] (1 - F_1(n_1 - 1)) (1 - F_2(n_2 - 1)) \right\}.$$

The objective function can be written in a more compact form. For this purpose, firstly we decompose the components of the objective function. For example, the second component can be decomposed into two sub-components. In that component, multiplying the function of y by the cumulative distribution function subtracted from 1, we obtain two separate sums. One of them is a finite sum over y and the other one is a double sum over x and y . By applying the same operation for the other components, we rewrite the objective function as follows:

$$J_k^*(n_1, n_2) = \max_{p_k} \left\{ \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[xp_k - \gamma(p_0 - p_k)^+ y + J_{k+1}^*(n_1 - x, n_2 - y) \right] P_1(x) P_2(y) - \right. \\ \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[n_1 p_k - \gamma(p_0 - p_k)^+ y - (n_2 - y) \gamma(p_0 - v_2) \right] P_1(x) P_2(y) -$$

$$\begin{aligned}
& \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[xp_k - \gamma(p_0 - p_k)^+ n_2 + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) + \\
& \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \right] P_1(x) P_2(y) + \\
& \sum_{x=0}^{n_1-1} \left[xp_k - \gamma(p_0 - p_k)^+ n_2 + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) - \\
& \sum_{x=0}^{n_1-1} \left[n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \right] P_1(x) + \\
& \sum_{y=0}^{n_2-1} \left[n_1 p_k - \gamma(p_0 - p_k)^+ y - (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) - \\
& \sum_{y=0}^{n_2-1} \left[n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \right] P_2(y) + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \Big\}.
\end{aligned}$$

Now we have some summations which can be combined. For example, we can add the functions summed over x from 0 to $n_1 - 1$ and y from 0 to $n_2 - 1$ to collect them under one summation. After the same operation for the other summations, we obtain:

$$\begin{aligned}
J_k^*(n_1, n_2) = \max_{p_k} \Big\{ & \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[xp_k - \gamma(p_0 - p_k)^+ y + J_{k+1}^*(n_1 - x, n_2 - y) - n_1 p_k + \right. \\
& \gamma(p_0 - p_k)^+ y + (n_2 - y) \gamma(p_0 - v_2) - xp_k + \gamma(p_0 - p_k)^+ n_2 - \\
& \left. J_{k+1}^*(n_1 - x, 0) + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \right] P_1(x) P_2(y) + \\
& \sum_{x=0}^{n_1-1} \left[xp_k - \gamma(p_0 - p_k)^+ n_2 + J_{k+1}^*(n_1 - x, 0) - n_1 p_k + \right. \\
& \left. n_2 \gamma(p_0 - p_k)^+ \right] P_1(x) + \\
& \sum_{y=0}^{n_2-1} \left[n_1 p_k - \gamma(p_0 - p_k)^+ y - (n_2 - y) \gamma(p_0 - v_2) - n_1 p_k + \right. \\
& \left. n_2 \gamma(p_0 - p_k)^+ \right] P_2(y) + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \Big\}.
\end{aligned}$$

$$\begin{aligned}
J_k^*(n_1, n_2) = \max_{p_k} \Big\{ & \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 - x, n_2 - y) + (n_2 - y) \gamma(p_0 - v_2) - \right. \\
& \left. J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) + \\
& \sum_{x=0}^{n_1-1} \left[(x - n_1) p_k + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) +
\end{aligned}$$

$$\sum_{y=0}^{n_2-1} \left[- (n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \Big\}.$$

Recall that we have a boundary condition necessitating us to solve another problem. The problem we should solve to determine the values of the profit-to-go function when the manufacturer has a positive inventory level but the retailer has no more products to sell is:

$$\begin{aligned} J_k^*(n_1, 0) &= \max_{p_k} \left\{ \sum_{x=0}^{n_1-1} \left[x p_k + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) + n_1 p_k (1 - F_1(n_1 - 1)) \right\} \\ &= \max_{p_k} \left\{ \sum_{x=0}^{n_1-1} \left[(x - n_1) p_k + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) + n_1 p_k \right\}. \end{aligned}$$

In the last period,

$$J_N^*(n_1, 0) = \max_{p_N} \left\{ \sum_{x=0}^{n_1-1} (v_1 - p_N)(n_1 - x) P_1(x) + n_1 p_N \right\}.$$

In the following chapter, we will evaluate how the optimal price and the optimal profit change with respect to problem parameters and the big part of the analysis will be based on the single-period problem. We know it is equivalent to the last period of the multi-period problem. Like we did for the multi-period problem, we will make some simplifications. The model at the beginning is:

$$\begin{aligned} J_N^*(n_1, n_2) &= \max_{p_N} \left\{ \sum_{x=0}^{n_1} \sum_{y=0}^{n_2} \left[x p_N + (n_1 - x) v_1 - \gamma(p_0 - p_N)^+ y - \right. \right. \\ &\quad \left. \left. (n_2 - y) \gamma(p_0 - v_2) \right] P_1(x) P_2(y) + \right. \\ &\quad \sum_{x=n_1+1}^{\infty} \sum_{y=0}^{n_2} \left[n_1 p_N - (p_0 - p_N)^+ y \gamma - (n_2 - y) \gamma(p_0 - v_2) \right] P_1(x) P_2(y) + \\ &\quad \sum_{x=0}^{n_1} \sum_{y=n_2+1}^{\infty} \left[x p_N + (n_1 - x) v_1 - \gamma(p_0 - p_N)^+ n_2 \right] P_1(x) P_2(y) + \\ &\quad \left. \sum_{x=n_1+1}^{\infty} \sum_{y=n_2+1}^{\infty} \left[n_1 p_N - n_2 \gamma(p_0 - p_N)^+ \right] P_1(x) P_2(y) \right\}. \end{aligned}$$

Here it is possible to substitute the infinite sums by finite ones, utilizing the cumulative distribution functions and the model can be rewritten as follows:

$$\begin{aligned}
J_N^*(n_1, n_2) = \max_{p_N} \Bigg\{ & \sum_{x=0}^{n_1} \sum_{y=0}^{n_2} \left[xp_N + (n_1 - x)v_1 - \gamma(p_0 - p_N)^+ y - \right. \\
& \left. (n_2 - y)\gamma(p_0 - v_2) \right] P_1(x)P_2(y) + \\
& \sum_{y=0}^{n_2} \left[n_1 p_N - (p_0 - p_N)^+ y \gamma - (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) + \\
& \sum_{x=0}^{n_1} \left[xp_N + (n_1 - x)v_1 - \gamma(p_0 - p_N)^+ n_2 \right] P_1(x) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) + \\
& \left. \left[n_1 p_N - n_2 \gamma(p_0 - p_N)^+ \right] \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right\}.
\end{aligned}$$

Now the summations can be split to obtain some similar expressions and to rewrite the model in a more compact form.

$$\begin{aligned}
J_N^*(n_1, n_2) = \max_{p_N} \Bigg\{ & \sum_{x=0}^{n_1} \sum_{y=0}^{n_2} \left[xp_N + (n_1 - x)v_1 \right] P_1(x)P_2(y) - \\
& \sum_{x=0}^{n_1} \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y)\gamma(p_0 - v_2) \right] P_1(x)P_2(y) + \\
& \sum_{y=0}^{n_2} n_1 p_N P_2(y) \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - \\
& \sum_{y=0}^{n_2} \left[(p_0 - p_N)^+ y \gamma + (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) + \\
& \sum_{x=0}^{n_1} \left[xp_N + (n_1 - x)v_1 \right] P_1(x) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) - \\
& \sum_{x=0}^{n_1} \gamma(p_0 - p_N)^+ n_2 P_1(x) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) + \\
& n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) - \\
& \left. \left[n_2 \gamma(p_0 - p_N)^+ \right] \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right\}.
\end{aligned}$$

After combining the similar expressions, we obtain the simplest model representing the one-period problem and it is literally the difference between the expected revenue and the expected loss. Therefore, we can use the following model for all the analysis we will do.

$$\begin{aligned}
J_N^*(n_1, n_2) &= \max_{p_N} \left\{ \sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - \right. \\
&\quad \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) - \\
&\quad \left. n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right\} \\
&= \max_{p_N} \left\{ \sum_{x=0}^{n_1} (n_1 - x)(v_1 - p_N) P_1(x) + \sum_{y=0}^{n_2} \left[(n_2 - y) \gamma(p_0 - p_N)^+ - \right. \right. \\
&\quad \left. \left. (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) - n_2 \gamma(p_0 - p_N)^+ + n_1 p_N \right\}.
\end{aligned}$$

When analyzing the change of the optimal price and the optimal profit in the single period problem, we will use this objective function. From this point on, this objective function is denoted by $f(p)$ for a fixed pair of inventory levels. This objective function has two constituent functions. Since the manufacturer reimburses the retailer only when it sets a retail price less than the initial retail price(p_0), the reimbursement cost per product sold at a lower retail price(p_N) than p_0 will be $\gamma(p_0 - p_N)$. The constituent function in which $(p_0 - p_N)^+ = p_0 - p_N$ is denoted by $f_1(p)$. We assume that the manufacturer can set a retail price that is higher than the initial retail price(p_0). In this case, when the manufacturer sets a retail price larger than p_0 , the reimbursement cost per product sold at that retail price is 0. The constituent function in which $(p_0 - p_N)^+ = 0$ is denoted by $f_2(p)$. Now we are ready to move on to the structural analysis.

Chapter 4

Analysis

4.1 Structural Results

We now present our results regarding both the single-period and the multi-period problem. Firstly, the structure of the model and the effect of the price protection over the optimal price and the optimal profit are examined. Afterwards, we will move on to the results over how the changes in specific factors affect the optimal price in the single-period problem and the optimal profit in the multi-period problem. All the results we derive on the change of the optimal profit is valid for all demand functions but the ones on the optimal price are special to Poisson distribution.

Throughout this chapter, we base our analysis on the fact that the optimal price will be one of the local maximums when investigating the effect of different factors over it. Since the rate of the Poisson process depends on price, some demand rate functions might lead to more than one local maximum. If the objective function has more than one local maximum, an analysis of derivative will not suffice to determine the change in the global optimum with respect to a specific problem parameter so we will have to compare the expected profits as well. We know that the objective function has two constituent functions since there is no

reimbursement if the retail price is larger than the initial retail price at which the retailer has purchased its products. The following proposition presents the necessary properties of these constituent functions for the objective function to have one local maximum.

Proposition 1: In the single period problem, if each one of $f_1(p)$ and $f_2(p)$ has only one local maximum, then the objective function($f(p)$) has also one local maximum.

Proof:

We should focus on two different parts of the graph of the objective function because we know the graph has a breaking point $p = p_0$. Therefore, there are two cases to be examined based on which price interval includes the optimal solution p^* :

Case I: ($p^* < p_0$) First, since p^* is less than p_0 and it is assumed that $f_1(p)$ has only one local maximum, we can directly state p^* to be that local maximum. This result leads us to evaluate the location of the local maximum of $f_2(p)$. For this purpose, we will take the difference between $f_1(p)$ and $f_2(p)$.

$$\begin{aligned}
f_1(p) - f_2(p) &= \left\{ \sum_{x=0}^{n_1} \left[xp + v_1(n_1 - x) \right] P_1(x) + n_1 p \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - \right. \\
&\quad \sum_{y=0}^{n_2} \left[\gamma(p_0 - p)y + (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) - \\
&\quad \left. n_2 \gamma(p_0 - p) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right\} - \\
&\quad \left\{ \sum_{x=0}^{n_1} \left[xp + v_1(n_1 - x) \right] P_1(x) + n_1 p \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - \right. \\
&\quad \left. \sum_{y=0}^{n_2} \left[(n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \right\} \\
&= - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p)y + (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) - \\
&\quad n_2 \gamma(p_0 - p) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) + \sum_{y=0}^{n_2} \left[(n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \\
&= -n_2 \gamma(p_0 - p) + \sum_{y=0}^{n_2} n_2 \gamma(p_0 - p) P_2(y) - \\
&\quad \sum_{y=0}^{n_2} \left[y\gamma(p_0 - p) + (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) +
\end{aligned}$$

$$\begin{aligned}
& \sum_{y=0}^{n_2} (n_2 - y) \gamma(p_0 - v_2) P_2(y) \\
= & -n_2 \gamma(p_0 - p) + \sum_{y=0}^{n_2} \left[(n_2 - y) \gamma(p_0 - p) + (n_2 - y) \gamma(p_0 - v_2) - \right. \\
& \left. (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
= & -n_2 \gamma(p_0 - p) + \sum_{y=0}^{n_2} (n_2 - y) \gamma(p_0 - p) P_2(y) \\
= & -\gamma(p_0 - p) \left[n_2 - \sum_{y=0}^{n_2} (n_2 - y) P_2(y) \right].
\end{aligned}$$

Since the obtained difference is positive when $p > p_0$, $f_2(p)$ will be below $f_1(p)$ in that interval. Likewise, since the difference is negative when $p < p_0$, $f_2(p)$ will be above $f_1(p)$ in the same interval. Assume the contrary and assume that $f_2(p)$ has a local maximum larger than p_0 , then it is required to start increasing again which leads it to have a local minimum which is larger than p_0 . Because the two functions should intersect at $p = p_0$. Then, there is a contradiction with the assumption.

Case II: ($p^* > p_0$) Since p^* is larger than p_0 and it is assumed that $f_2(p)$ has only one local maximum, we can directly state p^* to be that specific local maximum. Similarly to Case I, since $f_1(p)$ is below $f_2(p)$ when $p < p_0$, if it has a local maximum in that interval, it should have a local minimum smaller than p_0 . The reason is the same as in the previous case. Therefore, there is a contradiction with the assumption.

Case III: ($p^* = p_0$) For the optimal price to be equal to the initial sales price, the local maximums of these two functions do not have to be a part of the objective function. That is, the local maximum of $f_1(p)$ has to be larger than p_0 and the local maximum of $f_2(p)$ should be less than p_0 . Therefore, it is not possible for the objective function to have more than one local maximum because local maximums are not a part of it.

Proposition 2: (Price Protection Effect)

- i) In the multi-period problem, the expected optimal profit under price protection is smaller than the expected optimal profit under no price protection.
- ii) In the single period problem, the optimal price under no price protection is larger than the optimal price under price protection if the optimal price under price protection is larger than p_0 .
- iii) If the demand rate function $\lambda(p)$ is concave and the optimal price under price protection is less than p_0 , then there exists a threshold price p' such that
 - if the optimal price under price protection is larger than p' , then the optimal price under price protection is smaller than the optimal price under no price protection.
 - if the optimal price under price protection is smaller than p' , then the optimal price under price protection is larger than the optimal price under no price protection.

Proof:

Notice that excluding the price protection is equivalent to the case where the retailer has no inventory in stock because the retailer is no longer in the market and so the manufacturer does not need to take the reimbursement cost into consideration. This leads us to make a comparison with the function of the one-sided dynamic programming model.

- i) We will take the difference between the expected profit function of the problem when there is price protection and the expected profit function of the problem when there is no price protection. Let $J_N(n_1, n_2, p_N)$ be the expected profit of the single period when there are n_1 and n_2 units of inventory and the price is set at p_N .

$$\begin{aligned}
 J_N(n_1, n_2, p_N) &= J_N(n_1, 0, p_N) + \left(\sum_{x=0}^{n_1} (n_1 - x)(v_1 - p_N)P_1(x) + \sum_{y=0}^{n_2} \left[(n_2 - y)\gamma(p_0 - p_N)^+ \right. \right. \\
 &\quad \left. \left. - (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) - n_2\gamma(p_0 - p_N)^+ + n_1p_N \right) \\
 &\quad - \left(\sum_{x=0}^{n_1} (v_1 - p_N)(n_1 - x)P_1(x) + n_1p_N \right) \\
 &= J_N(n_1, 0, p_N) + \sum_{y=0}^{n_2} \left[(n_2 - y)\gamma(p_0 - p_N)^+ - (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \\
 &\quad - n_2\gamma(p_0 - p_N)^+ \\
 &= J_N(n_1, 0, p_N) - n_2\gamma(p_0 - p_N)^+ + \sum_{y=0}^{n_2} (n_2 - y)\gamma(p_0 - p_N)^+ P_2(y)
 \end{aligned}$$

$$\begin{aligned}
& - \sum_{y=0}^{n_2} (n_2 - y) \gamma(p_0 - v_2) P_2(y) \\
= & J_N(n_1, 0, p_N) + \gamma(p_0 - p_N)^+ \left(-n_2 + \sum_{y=0}^{n_2} (n_2 - y) P_2(y) \right) \\
& - \sum_{y=0}^{n_2} (n_2 - y) \gamma(p_0 - v_2) P_2(y).
\end{aligned}$$

The above remainder is strictly smaller than 0 since it is negative for any p . Let $p_N^*(n_1)$ be the optimal solution of the model under no price protection and $p_N^*(n_1, n_2)$ be the optimal solution of the model under price protection. Then we can write the following series of inequalities:

$$J_N^*(n_1, n_2) = J_N(n_1, n_2, p_N^*(n_1, n_2)) < J_N(n_1, 0, p_N^*(n_1, n_2)) < J_N(n_1, 0, p_N^*(n_1)) = J_N^*(n_1, 0).$$

Now we know that the optimal expected profit increases in the single-period problem when the price protection is excluded. For backward induction, let $J_{k+1}^*(n_1, n_2) < J_{k+1}^*(n_1, 0)$ in the period $k + 1$. By taking the difference,

$$\begin{aligned}
J_k(n_1, n_2, p_k) &= J_k(n_1, 0, p_k) + \left(\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 - x, n_2 - y) + (n_2 - y) \gamma(p_0 - v_2) \right. \right. \\
&\quad \left. \left. - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \right. \\
&\quad + \sum_{x=0}^{n_1-1} \left[(x - n_1) p_k + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \\
&\quad + \sum_{y=0}^{n_2-1} \left[- (n_2 - y) \gamma(p_0 - v_2) \right. \\
&\quad \left. + (n_2 - y) \gamma(p_0 - p_k)^+ \right] P_2(y) + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \Big) \\
&\quad - \left(\sum_{x=0}^{n_1-1} \left[(x - n_1) p_k + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) + n_1 p_k \right) \\
&= J_k(n_1, 0, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 - x, n_2 - y) \right. \\
&\quad \left. + (n_2 - y) \gamma(p_0 - v_2) - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \\
&\quad + \sum_{y=0}^{n_2-1} \left[- (n_2 - y) \gamma(p_0 - v_2) + (n_2 - y) \gamma(p_0 - p_k)^+ \right] P_2(y) \\
&\quad - n_2 \gamma(p_0 - p_k)^+
\end{aligned}$$

$$\begin{aligned}
&= J_k(n_1, 0, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 - x, n_2 - y) - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \\
&\quad + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} (n_2 - y) \gamma(p_0 - v_2) P_1(x) P_2(y) \\
&\quad - \sum_{y=0}^{n_2-1} \left[(n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
&\quad + \sum_{y=0}^{n_2-1} (n_2 - y) \gamma(p_0 - p_k)^+ P_2(y) - n_2 \gamma(p_0 - p_k)^+.
\end{aligned}$$

The above remainder is strictly smaller than 0 since it is negative for any p_k . Let $p_k^*(n_1)$ be the optimal solution of the model under no price protection and $p_k^*(n_1, n_2)$ be the optimal solution of the model under price protection. Then we can write the following series of inequalities:

$$J_k^*(n_1, n_2) = J_k(n_1, n_2, p_k^*(n_1, n_2)) < J_k(n_1, 0, p_k^*(n_1, n_2)) < J_k(n_1, 0, p_k^*(n_1)) = J_k^*(n_1, 0).$$

Since we already know the negativity of the difference for the period N , moving backwards the proof is completed. Although the optimal expected profit seems to decrease when there is a price protection contract, it does not mean this is not profitable for the manufacturer. The reason is that we exclude the revenue coming from selling products to the retailer at the beginning of the selling horizon. That is, by selecting a true wholesale price, the manufacturer can increase its profit. This part also shows the importance of the selection of the contract parameters.

ii) Let $p^*(n_1, n_2)$ be the optimal solution of the model under price protection. We have already taken the difference of two functions in part i and since we assume that $p^*(n_1, n_2)$ is larger than p_0 and so $(p_0 - p^*(n_1, n_2))^+ = 0$, we will drop some parts of that difference. Then,

$$\begin{aligned}
J(n_1, 0, p) &= J(n_1, n_2, p) + \sum_{y=0}^{n_2} (n_2 - y) \gamma(p_0 - v_2) P_2(y) \\
\frac{\partial J(n_1, 0, p)}{\partial p} &= \frac{\partial J(n_1, n_2, p)}{\partial p} + \left(\sum_{y=0}^{n_2} (n_2 - y) \gamma(p_0 - v_2) P_2(y) \right)' \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + \left(n_2 \gamma(p_0 - v_2) e^{-\lambda(p)(1-\alpha)} \right. \\
&\quad \left. + \gamma(p_0 - v_2) \sum_{y=1}^{n_2} (n_2 - y) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right)'
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial J(n_1, n_2, p)}{\partial p} - \lambda'(p)(1-\alpha)e^{-\lambda(p)(1-\alpha)}n_2\gamma(p_0 - v_2) \\
&\quad + \gamma(p_0 - v_2) \sum_{y=1}^{n_2} (n_2 - y) \left(-\lambda'(p)(1-\alpha)e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \\
&\quad \left. + \lambda'(p)(1-\alpha)e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^{y-1}}{(y-1)!} \right) \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} - \lambda'(p)(1-\alpha)n_2\gamma(p_0 - v_2)P_2(0) \\
&\quad - \gamma(p_0 - v_2)\lambda'(p)(1-\alpha) \left[(n_2 - 1)(P_2(1) - P_2(0)) \right. \\
&\quad + (n_2 - 2)(P_2(2) - P_2(1)) + (n_2 - 3)(P_2(3) - P_2(2)) \\
&\quad \left. + \dots 2(P_2(n_2 - 2) - P_2(n_2 - 3)) + (P_2(n_2 - 1) - P_2(n_2 - 2)) \right] \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} - \lambda'(p)(1-\alpha)n_2\gamma(p_0 - v_2)P_2(0) \\
&\quad + \lambda'(p)(1-\alpha)(n_2 - 1)\gamma(p_0 - v_2)P_2(0) \\
&\quad - \gamma(p_0 - v_2)\lambda'(p)(1-\alpha) \sum_{y=1}^{n_2-1} P_2(y) \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} - \gamma(p_0 - v_2)\lambda'(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) \\
\frac{\partial J(n_1, 0, p)}{\partial p} \Big|_{p=p^*(n_1, n_2)} &= \frac{\partial J(n_1, n_2, p)}{\partial p} \Big|_{p=p^*(n_1, n_2)} \\
&\quad - \gamma(p_0 - v_2)\lambda'(p^*(n_1, n_2))(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) \\
\frac{\partial J(n_1, 0, p)}{\partial p} \Big|_{p=p^*(n_1, n_2)} &= -\gamma(p_0 - v_2)\lambda'(p^*(n_1, n_2))(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) > 0.
\end{aligned}$$

Since $J(n_1, 0, p)$ is an increasing function at $p = p^*(n_1, n_2)$ which is the optimal price of the model under price protection, $p^*(n_1, 0)$ is larger than $p^*(n_1, n_2)$, which completes the proof.

iii) Let $p^*(n_1, n_2)$ be the optimal solution of the model under price protection. We will do the same analysis as in the previous part. In this case, we will use the complete expression of the difference we obtained in part i. Then,

$$\begin{aligned}
J(n_1, 0, p) &= J(n_1, n_2, p) - \gamma(p_0 - p) \left(-n_2 + \sum_{y=0}^{n_2} (n_2 - y)P_2(y) \right) \\
&\quad + \sum_{y=0}^{n_2} (n_2 - y)\gamma(p_0 - v_2)P_2(y)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial J(n_1, 0, p)}{\partial p} &= \frac{\partial J(n_1, n_2, p)}{\partial p} - \left[\gamma(p_0 - p) \left(-n_2 + \sum_{y=0}^{n_2} (n_2 - y) P_2(y) \right) \right. \\
&\quad \left. - \sum_{y=0}^{n_2} (n_2 - y) \gamma(p_0 - v_2) P_2(y) \right]' \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} - \left[-n_2 \gamma(p_0 - p) + \gamma(p_0 - p) n_2 P_2(0) \right. \\
&\quad \left. + \gamma(p_0 - p) \sum_{y=1}^{n_2} (n_2 - y) P_2(y) - n_2 \gamma(p_0 - v_2) P_2(0) \right. \\
&\quad \left. - \sum_{y=1}^{n_2} (n_2 - y) \gamma(p_0 - v_2) P_2(y) \right]' \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} - \left[n_2 \gamma - \gamma n_2 P_2(0) - \gamma \sum_{y=1}^{n_2} (n_2 - y) P_2(y) \right. \\
&\quad \left. - \lambda'(p) (1 - \alpha) \gamma(p_0 - p) n_2 P_2(0) \right. \\
&\quad \left. + \gamma(p_0 - p) \sum_{y=1}^{n_2} (n_2 - y) \left(-\lambda'(p) (1 - \alpha) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \right. \\
&\quad \left. \left. + \lambda'(p) (1 - \alpha) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^{y-1}}{(y-1)!} \right) \right. \\
&\quad \left. + \lambda'(p) (1 - \alpha) n_2 \gamma(p_0 - v_2) \right. \\
&\quad \left. - \sum_{y=1}^{n_2} (n_2 - y) \gamma(p_0 - v_2) \left(-\lambda'(p) (1 - \alpha) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \right. \\
&\quad \left. \left. + \lambda'(p) (1 - \alpha) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^{y-1}}{(y-1)!} \right) \right] \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} - \left[-\gamma \sum_{y=0}^{n_2} (n_2 - y) P_2(y) + n_2 \gamma \right. \\
&\quad \left. - \lambda'(p) (1 - \alpha) \gamma(p_0 - p) n_2 P_2(0) \right. \\
&\quad \left. - \lambda'(p) (1 - \alpha) \gamma(p_0 - p) \sum_{y=1}^{n_2} (n_2 - y) \left(P_2(y) - P_2(y-1) \right) \right. \\
&\quad \left. + \lambda'(p) (1 - \alpha) n_2 \gamma(p_0 - v_2) \right. \\
&\quad \left. + \lambda'(p) (1 - \alpha) \gamma(p_0 - v_2) \sum_{y=1}^{n_2} (n_2 - y) \left(P_2(y) - P_2(y-1) \right) \right] \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} - \left[-\gamma \sum_{y=0}^{n_2} (n_2 - y) P_2(y) + n_2 \gamma \right. \\
&\quad \left. - \lambda'(p) (1 - \alpha) \gamma(p_0 - p) n_2 P_2(0) \right. \\
&\quad \left. - \lambda'(p) (1 - \alpha) \gamma(p_0 - p) \left[(n_2 - 1) (P_2(1) - P_2(0)) \right. \right. \\
&\quad \left. \left. + (n_2 - 2) (P_2(2) - P_2(1)) + (n_2 - 3) (P_2(3) - P_2(2)) + \dots \right] \right]
\end{aligned}$$

$$\begin{aligned}
& +2(P_2(n_2 - 2) - P_2(n_2 - 3)) + (P_2(n_2 - 1) - P_2(n_2 - 2)) \Big] \\
& +\lambda'(p)(1 - \alpha)n_2\gamma(p_0 - v_2) \\
& +\lambda'(p)(1 - \alpha)\gamma(p_0 - v_2) \Big[(n_2 - 1)(P_2(1) - P_2(0)) \\
& + (n_2 - 2)(P_2(2) - P_2(1)) + (n_2 - 3)(P_2(3) - P_2(2)) + \dots \\
& +2(P_2(n_2 - 2) - P_2(n_2 - 3)) + (P_2(n_2 - 1) - P_2(n_2 - 2)) \Big] \Big] \\
= & \frac{\partial J(n_1, n_2, p)}{\partial p} - \Big[-\gamma \sum_{y=0}^{n_2} (n_2 - y)P_2(y) + n_2\gamma \\
& -\lambda'(p)(1 - \alpha)\gamma(p_0 - p)n_2P_2(0) \\
& +\lambda'(p)(1 - \alpha)\gamma(p_0 - p)(n_2 - 1)P_2(0) \\
& -\lambda'(p)(1 - \alpha)\gamma(p_0 - p) \sum_{y=1}^{n_2-1} P_2(y) \\
& +\lambda'(p)(1 - \alpha)n_2\gamma(p_0 - v_2) - \lambda'(p)(1 - \alpha)\gamma(p_0 - v_2)(n_2 - 1) \\
& +\lambda'(p)(1 - \alpha)\gamma(p_0 - v_2) \sum_{y=1}^{n_2-1} P_2(y) \Big] \\
= & \frac{\partial J(n_1, n_2, p)}{\partial p} + \gamma \sum_{y=0}^{n_2} (n_2 - y)P_2(y) - n_2\gamma \\
& +\lambda'(p)(1 - \alpha)\gamma(p_0 - p) \sum_{y=0}^{n_2-1} P_2(y) \\
& -\lambda'(p)(1 - \alpha)\gamma(p_0 - v_2) \sum_{y=0}^{n_2-1} P_2(y) \\
= & \frac{\partial J(n_1, n_2, p)}{\partial p} + \gamma \sum_{y=0}^{n_2} (n_2 - y)P_2(y) - n_2\gamma \\
& +\lambda'(p)(1 - \alpha)\gamma(v_2 - p) \sum_{y=0}^{n_2-1} P_2(y) \\
\frac{\partial J(n_1, 0, p)}{\partial p} \Big|_{p=p^*(n_1, n_2)} = & \frac{\partial J(n_1, n_2, p)}{\partial p} \Big|_{p=p^*(n_1, n_2)} + \gamma \sum_{y=0}^{n_2} (n_2 - y)P_2(y) - n_2\gamma \\
& +\lambda'(p^*(n_1, n_2))(1 - \alpha)\gamma(v_2 - p^*(n_1, n_2)) \sum_{y=0}^{n_2-1} P_2(y) \\
\frac{\partial J(n_1, 0, p)}{\partial p} \Big|_{p=p^*(n_1, n_2)} = & \gamma \sum_{y=0}^{n_2} (n_2 - y)P_2(y) - n_2\gamma \\
& +\lambda'(p^*(n_1, n_2))(1 - \alpha)\gamma(v_2 - p^*(n_1, n_2)) \sum_{y=0}^{n_2-1} P_2(y).
\end{aligned}$$

Now we will determine when the right-hand side is positive and when it is negative. When $p \leq v_2$, the right-hand side is negative, which proves in that price interval, $J(n_1, 0, p)$ is decreasing. Then, if $p^*(n_1, n_2) \leq v_2$, the optimal solution decreases when the price protection is excluded.

$$\begin{aligned}
& \left(\sum_{y=0}^{n_2} \gamma(n_2 - y)P_2(y) \right)' = \gamma \left(\sum_{y=0}^{n_2} (n_2 - y)e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right)' \\
& = \gamma \left(n_2 e^{-\lambda(p)(1-\alpha)} + \sum_{y=1}^{n_2} (n_2 - y)e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right)' \\
& = -\gamma n_2 \lambda'(p) e^{-\lambda(p)(1-\alpha)} (1-\alpha) + \gamma \sum_{y=0}^{n_2} (n_2 - y) \left(-\lambda'(p) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} (1-\alpha) \right. \\
& \quad \left. + e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^{y-1}}{(y-1)!} \lambda(p)(1-\alpha) \right) \\
& = -\gamma n_2 \lambda'(p) P_2(0)(1-\alpha) - \gamma \lambda'(p)(1-\alpha) \sum_{y=1}^{n_2} [P_2(y) - P_2(y-1)](n_2 - y) \\
& = -\gamma n_2 \lambda'(p) P_2(0)(1-\alpha) - \gamma \lambda'(p)(1-\alpha) \left[(n_2 - 1)(P_2(1) - P_2(0)) \right. \\
& \quad \left. + (n_2 - 2)(P_2(2) - P_2(1)) + (n_2 - 3)(P_2(3) - P_2(2)) + \dots \right. \\
& \quad \left. + 2(P_2(n_2 - 2) - P_2(n_2 - 3)) + (P_2(n_2 - 1) - P_2(n_2 - 2)) \right] \\
& = -\gamma n_2 \lambda'(p) P_2(0)(1-\alpha) + \gamma(n_2 - 1) \lambda'(p) P_2(0)(1-\alpha) - \lambda'(p)(1-\alpha) \gamma \sum_{y=1}^{n_2-1} P_2(y) \\
& = -\lambda'(p)(1-\alpha) \gamma \sum_{y=0}^{n_2-1} P_2(y) > 0.
\end{aligned}$$

This proves that $\gamma \sum_{y=0}^{n_2} (n_2 - y)P_2(y)$ is an increasing function and so $-\gamma \sum_{y=0}^{n_2} (n_2 - y)P_2(y) + n_2 \gamma$ is a decreasing function at every price. Now, if we can prove that $\lambda'(p)(1-\alpha) \gamma (v_2 - p) \sum_{y=0}^{n_2-1} P_2(y)$ is a continuously increasing function at the prices larger than v_2 , we will have determined the breaking point. The steps are as follows:

$$\begin{aligned}
& \left[-\gamma(p - v_2) \lambda'(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) \right]' = -\gamma(p - v_2) \lambda''(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) \\
& \quad - \gamma \lambda'(p)(1-\alpha) \left[(p - v_2) \sum_{y=0}^{n_2-1} P_2(y) \right]' \\
& = -\gamma(p - v_2) \lambda''(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) - \gamma \lambda'(p)(1-\alpha) \left[\sum_{y=0}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right]'
\end{aligned}$$

$$\begin{aligned}
& + (p - v_2) \left(\sum_{y=0}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right)' \Big] \\
= & -\gamma(p - v_2) \lambda''(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) \\
& -\gamma \lambda'(p)(1-\alpha) \left[\sum_{y=0}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} + (p - v_2) \left(e^{-\lambda(p)(1-\alpha)} \right. \right. \\
& \left. \left. + \sum_{y=1}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right)' \right] \\
= & -\gamma(p - v_2) \lambda''(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) - \gamma \lambda'(p)(1-\alpha) \left[\sum_{y=0}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \\
& + (p - v_2) \left(-\lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} + \sum_{y=1}^{n_2-1} -\lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \\
& \left. \left. + \lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} \frac{y[\lambda(p)(1-\alpha)]^{y-1}}{y!} \right) \right] \\
= & -\gamma(p - v_2) \lambda''(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) - \gamma \lambda'(p)(1-\alpha) \left[\sum_{y=0}^{n_2-1} P_2(y) \right. \\
& \left. + (p - v_2) \left(-\lambda'(p)(1-\alpha) P_2(0) + \sum_{y=1}^{n_2-1} -\lambda'(p)(1-\alpha) P_2(y) + \lambda'(p)(1-\alpha) P_2(y-1) \right) \right] \\
= & -\gamma(p - v_2) \lambda''(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) - \gamma \lambda'(p)(1-\alpha) \left[\sum_{y=0}^{n_2-1} P_2(y) \right. \\
& \left. - (p - v_2) \lambda'(p)(1-\alpha) \left(P_2(0) + \sum_{y=1}^{n_2-1} P_2(y) - P_2(y-1) \right) \right] \\
= & -\gamma(p - v_2) \lambda''(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) - \gamma \lambda'(p)(1-\alpha) \left[\sum_{y=0}^{n_2-1} P_2(y) \right. \\
& \left. - (p - v_2) \lambda'(p)(1-\alpha) \left(P_2(0) + P_2(n_2-1) - P_2(0) \right) \right] \\
= & -\gamma(p - v_2) \lambda''(p)(1-\alpha) \sum_{y=0}^{n_2-1} P_2(y) - \gamma \lambda'(p)(1-\alpha) \left[\sum_{y=0}^{n_2-1} P_2(y) \right. \\
& \left. - (p - v_2) \lambda'(p)(1-\alpha) P_2(n_2-1) \right] > 0.
\end{aligned}$$

This proves that starting from zero, this function always increases as p increases above v_2 . Then, at some point p' which is the unique solution of the equation $-\sum_{y=0}^{n_2} (n_2 - y) P_2(y) + n_2 = \lambda'(p)(1-\alpha)(v_2 - p) \sum_{y=0}^{n_2-1} P_2(y)$, the right-hand side will be positive, which means the optimal solution increases if the price protection is excluded. In this case, if $p^*(n_1, n_2)$ is less than p' , $p^*(n_1, 0)$ is less than

$p^*(n_1, n_2)$ and if $p^*(n_1, n_2)$ is larger than p' , $p^*(n_1, 0)$ is larger than $p^*(n_1, n_2)$. At equality, the optimal prices are the same.

In the following proposition, we will examine how an increase in the manufacturer's salvage value affects the optimal expected profit in the multi-period problem and the optimal price in the single-period problem.

Proposition 3:

- i) The optimal expected profit is increasing in the manufacturer's salvage value.
- ii) The optimal price for the single period problem is increasing in manufacturer's salvage value.

Proof:

- i) Let $J_N^{v_1}(n_1, n_2, p_N)$ be the expected profit when the manufacturer's salvage value is v_1 . Assume that we are increasing v_1 by a very small amount of a . Then, if we take the difference between the expected profit when the salvage value is $v_1 + a$ and the expected profit when the salvage value is v_1 , we obtain:

$$\begin{aligned}
J_N^{v_1+a}(n_1, n_2, p_N) &= J_N^{v_1}(n_1, n_2, p_N) + \left(\sum_{x=0}^{n_1} \left[xp_N + (v_1 + a)(n_1 - x) \right] P_1(x) \right. \\
&\quad + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
&\quad - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
&\quad - \left(\sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right. \\
&\quad - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
&\quad \left. \left. - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right) \right) \\
&= J_N^{v_1}(n_1, n_2, p_N) + \left(\sum_{x=0}^{n_1} a(n_1 - x) P_1(x) + \sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) \right. \\
&\quad + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
&\quad - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
&\quad \left. - \left(\sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right) \right)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
& - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
& = J_N^{v_1}(n_1, n_2, p_N) + \sum_{x=0}^{n_1} a(n_1 - x) P_1(x).
\end{aligned}$$

The above remainder is strictly positive since it is larger than 0 for any p_N . This implies that at the same price, if v_1 increases, the expected profit also increases. Let $p_N^{v_1,*}(n_1, n_2)$ be the optimal price when the manufacturer's salvage value is v_1 . Since for all prices, the difference of the expected profits is positive; it turns out that at $p_N = p_N^{v_1,*}(n_1, n_2)$, the expected profit ($J_N^{v_1+a}(n_1, n_2, p_N)$) will be larger. Furthermore, whether the optimal price increases or not, the optimal profit for the problem when the salvage value is $v_1 + a$ will be even larger than the expected profit ($J_N^{v_1+a}(n_1, n_2, p_N)$) at $p_N = p_N^{v_1,*}(n_1, n_2)$. This relationship can be written as the following series of inequalities:

$$J_N^{v_1+a,*}(n_1, n_2) > J_N^{v_1+a}(n_1, n_2, p_N^{v_1,*}(n_1, n_2)) > J_N^{v_1}(n_1, n_2, p_N^{v_1,*}(n_1, n_2)) = J_N^{v_1,*}(n_1, n_2)$$

For backward induction, assume that for the period $k + 1$, the optimal profit increases as the manufacturer's salvage value increases. Then, for the period k , we can write:

$$\begin{aligned}
J_k^{v_1+a}(n_1, n_2, p_k) &= J_k^{v_1}(n_1, n_2, p_k) + \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1, v_1+a}^*(n_1 - x, n_2 - y) \right. \right. \\
&\quad \left. \left. + (n_2 - y) \gamma(p_0 - v_2) - J_{k+1, v_1+a}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \right. \\
&\quad \left. + \sum_{x=0}^{n_1-1} \left[(x - n_1) p_k + J_{k+1, v_1+a}^*(n_1 - x, 0) \right] P_1(x) \right. \\
&\quad \left. + \sum_{y=0}^{n_2-1} \left[- (n_2 - y) \gamma(p_0 - v_2) + (n_2 - y) \gamma(p_0 - p_k)^+ \right] P_2(y) \right. \\
&\quad \left. + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \right] \\
&\quad - \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1, v_1}^*(n_1 - x, n_2 - y) + (n_2 - y) \gamma(p_0 - v_2) \right. \right. \\
&\quad \left. \left. - J_{k+1, v_1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1,v_1}^*(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} \left[- (n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \\
& + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \Big] \\
= & J_k^{v_1}(n_1, n_2, p_k) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1,v_1+a}^*(n_1 - x, n_2 - y) - J_{k+1,v_1}^*(n_1 - x, n_2 - y) \right. \\
& + J_{k+1,v_1}^*(n_1 - x, 0) - J_{k+1,v_1+a}^*(n_1 - x, 0) + (n_2 - y)\gamma(p_0 - v_2) \\
& \left. - (n_2 - y)\gamma(p_0 - v_2) \right] P_1(x) P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k - (x - n_1)p_k + J_{k+1,v_1+a}^*(n_1 - x, 0) \right. \\
& \left. - J_{k+1,v_1}^*(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} \left[- (n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \\
& - \sum_{y=0}^{n_2-1} \left[- (n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \\
& + n_1 p_k - n_1 p_k - n_2 \gamma(p_0 - p_k)^+ + n_2 \gamma(p_0 - p_k)^+ \\
= & J_k^{v_1}(n_1, n_2, p_k) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1,v_1+a}^*(n_1 - x, n_2 - y) - J_{k+1,v_1}^*(n_1 - x, n_2 - y) \right. \\
& + J_{k+1,v_1}^*(n_1 - x, 0) - J_{k+1,v_1+a}^*(n_1 - x, 0) \Big] P_1(x) P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[J_{k+1,v_1+a}^*(n_1 - x, 0) - J_{k+1,v_1}^*(n_1 - x, 0) \right] P_1(x) \\
= & J_k^{v_1}(n_1, n_2, p_k) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1,v_1+a}^*(n_1 - x, n_2 - y) - J_{k+1,v_1}^*(n_1 - x, n_2 - y) \right] P_1(x) P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[J_{k+1,v_1+a}^*(n_1 - x, 0) - J_{k+1,v_1}^*(n_1 - x, 0) \right] P_1(x) \\
& - \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1,v_1+a}^*(n_1 - x, 0) - J_{k+1,v_1}^*(n_1 - x, 0) \right] P_1(x) P_2(y).
\end{aligned}$$

The above remainder is strictly larger than 0 since it is positive for any p_k . It

is obvious that if only the manufacturer is active in the market and its salvage value increases, the optimal expected profit increases for each period. Since we also have the assumption we made at the beginning of the proof, the above expression turns out to be positive. Let $p_k^{v_1,*}(n_1, n_2)$ be the optimal solution for the manufacturer's salvage value of v_1 and $p_k^{v_1+a,*}(n_1, n_2)$ be the optimal solution for the manufacturer's salvage value of $v_1 + a$. Then, we can write the following series of inequalities:

$$\begin{aligned} J_k^{v_1+a,*}(n_1, n_2) &= J_k^{v_1+a}(n_1, n_2, p_k^{v_1+a,*}(n_1, n_2)) > J_k^{v_1+a}(n_1, n_2, p_k^{v_1,*}(n_1, n_2)) \\ &> J_k^{v_1}(n_1, n_2, p_k^{v_1,*}(n_1, n_2)) = J_k^{v_1,*}(n_1, n_2). \end{aligned}$$

Since we know the positivity of the difference for the period N , we can generalize the proposition to all the other periods by moving backwards.

ii) Let $p^{v_1,*}(n_1, n_2)$ be the optimal price in the single period problem when the manufacturer's salvage value is v_1 . For the optimal price to increase, at $p = p^{v_1,*}(n_1, n_2)$ $J^{v_1+a}(n_1, n_2, p)$ should be increasing.

We know

$$J^{v_1+a}(n_1, n_2, p) = J^{v_1}(n_1, n_2, p) + \sum_{x=0}^{n_1} a(n_1 - x)P_1(x)$$

We should take the derivative of the above remainder to check whether it is an increasing function or not

$$\begin{aligned} \left(\sum_{x=0}^{n_1} a(n_1 - x)P_1(x) \right)' &= a \left(\sum_{x=0}^{n_1} (n_1 - x) e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^x}{x!} \right)' \\ &= a \left(n_1 e^{-\lambda(p)\alpha} + \sum_{x=1}^{n_1} (n_1 - x) e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^x}{x!} \right)' \\ &= -an_1 \lambda'(p) e^{-\lambda(p)\alpha} \alpha \\ &\quad + a \sum_{x=0}^{n_1} (n_1 - x) \left(-\lambda'(p) e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^x}{x!} \alpha \right. \\ &\quad \left. + e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^{x-1}}{(x-1)!} \lambda(p)\alpha \right) \\ &= -an_1 \lambda'(p) P_1(0)\alpha - a \lambda'(p)\alpha \sum_{x=1}^{n_1} [P_1(x) - P_1(x-1)](n_1 - x) \\ &= -an_1 \lambda'(p) P_1(0)\alpha - a \lambda'(p)\alpha \left[(n_1 - 1)(P_1(1) - P_1(0)) \right. \\ &\quad \left. + (n_1 - 2)(P_1(2) - P_1(1)) + (n_1 - 3)(P_1(3) - P_1(2)) + \dots \right. \\ &\quad \left. + 2(P_1(n_1 - 2) - P_1(n_1 - 3)) + (P_1(n_1 - 1) - P_1(n_1 - 2)) \right] \end{aligned}$$

$$\begin{aligned}
&= -an_1\lambda'(p)P_1(0)\alpha + a(n_1 - 1)\lambda'(p)P_1(0)\alpha - \lambda'(p)\alpha a \sum_{x=1}^{n_1-1} P_1(x) \\
&= -\lambda'(p)\alpha a \sum_{x=0}^{n_1-1} P_1(x) > 0
\end{aligned}$$

Now we have

$$\begin{aligned}
\frac{\partial J^{v_1+a}(n_1, n_2, p)}{\partial p} &= \frac{\partial J^{v_1}(n_1, n_2, p)}{\partial p} - \lambda'(p)\alpha a \sum_{x=0}^{n_1-1} P_1(x) \\
\left. \frac{\partial J^{v_1+a}(n_1, n_2, p)}{\partial p} \right|_{p=p^{v_1,*}(n_1, n_2)} &= \left. \frac{\partial J^{v_1}(n_1, n_2, p)}{\partial p} \right|_{p=p^{v_1,*}(n_1, n_2)} \\
&\quad - \lambda'(p^{v_1,*}(n_1, n_2))\alpha a \sum_{x=0}^{n_1-1} P_1(x) \\
\left. \frac{\partial J^{v_1+a}(n_1, n_2, p)}{\partial p} \right|_{p=p^{v_1,*}(n_1, n_2)} &= -\lambda'(p^{v_1,*}(n_1, n_2))\alpha a \sum_{x=0}^{n_1-1} P_1(x) > 0.
\end{aligned}$$

Since we have proven $J^{v_1+a}(n_1, n_2, p)$ is an increasing function at $p = p^{v_1,*}(n_1, n_2)$ that is the optimal price when the salvage value is v_1 , this completes the proof.

In the following proposition, we will examine how an increase in the retailer's salvage value affects the optimal expected profit in the multi-period problem and the optimal price in the single-period problem.

Proposition 4:

- i) The optimal expected profit is increasing in the retailer's salvage value.
- ii) In the single period problem, the optimal price is increasing in the retailer's salvage value.

Proof: i) Let $J_N^{v_2}(n_1, n_2, p_N)$ be the expected profit in the last period when the retailer's salvage value is v_2 and the price is (p_N) . Assume that we are increasing v_2 by a very small amount of a . Then, if we take the difference between the expected profit under the salvage value of $v_2 + a$ and the expected profit under the salvage value of v_2 , we obtain:

$$\begin{aligned}
J_N^{v_2+a}(n_1, n_2, p_N) &= J_N^{v_2}(n_1, n_2, p_N) + \left(\sum_{x=0}^{n_1} \left[xp_N + (v_1 + a)(n_1 - x) \right] P_1(x) \right. \\
&\quad \left. + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right. \\
&\quad \left. - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - (v_2 + a)) \right] P_2(y) \right)
\end{aligned}$$

$$\begin{aligned}
& - \left(\sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right. \\
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
& \left. - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right) \\
= & J_N^{v_2}(n_1, n_2, p_N) + \left(\sum_{y=0}^{n_2} a \gamma(n_2 - y) P_2(y) + \sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) \right. \\
& + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
& - \left(\sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right. \\
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
& \left. - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right) \\
= & J_N^{v_2}(n_1, n_2, p_N) + \sum_{y=0}^{n_2} a \gamma(n_2 - y) P_2(y).
\end{aligned}$$

The above remainder is strictly larger than 0 since it is positive for any p_N . This implies that at the same price, if v_2 increases, the expected profit also increases. Let $p_N^{v_2,*}(n_1, n_2)$ be the optimal price when the retailer's salvage value is v_2 . Since for all prices, the difference of the expected profits is positive, it turns out that the expected profit ($J_N^{v_2+a}(n_1, n_2, p_N)$) will be larger at $p = p_N^{v_2,*}(n_1, n_2)$. Furthermore, whether the optimal price increases or not, the optimal expected profit for the problem under the salvage value of $v_2 + a$ will be even larger than the expected profit ($J_N^{v_2+a}(n_1, n_2, p_N)$) at $p = p_N^{v_2,*}(n_1, n_2)$. This relationship can be written as the following series of inequalities:

$$J_N^{v_2+a,*}(n_1, n_2) > J_N^{v_2+a}(n_1, n_2, p_N^{v_2,*}(n_1, n_2)) > J_N^{v_2}(n_1, n_2, p_N^{v_2,*}(n_1, n_2)) = J_N^{v_2,*}(n_1, n_2).$$

For backward induction, assume that for the period $k + 1$, the optimal profit increases as the retailer's salvage value increases. Then, for the period k , we can write:

$$\begin{aligned}
J_k^{v_2+a}(n_1, n_2, p_k) &= J_k^{v_2}(n_1, n_2, p_k) + \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{v_2+a,*}(n_1-x, n_2-y) \right. \right. \\
&\quad \left. \left. + (n_2-y)\gamma(p_0-v_2-a) - J_{k+1}^{v_2+a,*}(n_1-x, 0) \right] P_1(x) P_2(y) \right. \\
&\quad \left. + \sum_{x=0}^{n_1-1} \left[(x-n_1)p_k + J_{k+1}^{v_2+a,*}(n_1-x, 0) \right] P_1(x) \right. \\
&\quad \left. + \sum_{y=0}^{n_2-1} \left[-(n_2-y)\gamma(p_0-v_2-a) + (n_2-y)\gamma(p_0-p_k)^+ \right] P_2(y) \right. \\
&\quad \left. + n_1 p_k - n_2 \gamma(p_0-p_k)^+ \right] \\
&\quad - \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{v_2,*}(n_1-x, n_2-y) + (n_2-y)\gamma(p_0-v_2) \right. \right. \\
&\quad \left. \left. - J_{k+1}^{v_2,*}(n_1-x, 0) \right] P_1(x) P_2(y) \right. \\
&\quad \left. + \sum_{x=0}^{n_1-1} \left[(x-n_1)p_k + J_{k+1}^{v_2,*}(n_1-x, 0) \right] P_1(x) \right. \\
&\quad \left. + \sum_{y=0}^{n_2-1} \left[-(n_2-y)\gamma(p_0-v_2) + (n_2-y)\gamma(p_0-p_k)^+ \right] P_2(y) \right. \\
&\quad \left. + n_1 p_k - n_2 \gamma(p_0-p_k)^+ \right] \\
&= J_k^{v_2}(n_1, n_2, p_k) \\
&\quad + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{v_2+a,*}(n_1-x, n_2-y) - J_{k+1}^{v_2,*}(n_1-x, n_2-y) \right. \\
&\quad \left. + (n_2-y)\gamma(p_0-v_2-a) - (n_2-y)\gamma(p_0-v_2) \right. \\
&\quad \left. - J_{k+1}^{v_2+a,*}(n_1-x, 0) + J_{k+1}^{v_2,*}(n_1-x, 0) \right] P_1(x) P_2(y) \\
&\quad + \sum_{x=0}^{n_1-1} \left[(x-n_1)p_k - (x-n_1)p_k + J_{k+1}^{v_2+a,*}(n_1-x, 0) \right. \\
&\quad \left. - J_{k+1}^{v_2,*}(n_1-x, 0) \right] P_1(x) \\
&\quad + \sum_{y=0}^{n_2-1} \left[(n_2-y)\gamma(p_0-p_k)^+ - (n_2-y)\gamma(p_0-p_k)^+ \right. \\
&\quad \left. + (n_2-y)\gamma(-p_0+v_2+a+p_0-v_2) \right] P_2(y) + n_1 p_k - n_1 p_k \\
&\quad - n_2 \gamma(p_0-p_k)^+ + n_2 \gamma(p_0-p_k)^+ \\
&= J_k^{v_2}(n_1, n_2, p_k)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{v_2+a,*}(n_1-x, n_2-y) - J_{k+1}^{v_2,*}(n_1-x, n_2-y) \right. \\
& - a(n_2-y)\gamma - J_{k+1}^{v_2+a,*}(n_1-x, 0) \\
& \left. + J_{k+1}^{v_2,*}(n_1-x, 0) \right] P_1(x) P_2(y) + \sum_{y=0}^{n_2-1} \left[(n_2-y)\gamma a \right] P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[J_{k+1}^{v_2+a,*}(n_1-x, 0) - J_{k+1}^{v_2,*}(n_1-x, 0) \right] P_1(x).
\end{aligned}$$

When the retailer has no inventory in stock, we know that the retailer is not active in the market anymore and the manufacturer is not supposed to take the reimbursement cost into consideration. Then, v_2 is negligible in the process of determining the optimal prices and in each period, the optimal profit is independent of v_2 . This implies $J_{k+1}^{v_2,*}(n_1-x, 0) = J_{k+1}^{v_2+a,*}(n_1-x, 0)$. This enables us to simplify the above expression and it turns into:

$$\begin{aligned}
J_k^{v_2+a}(n_1, n_2, p_k) &= J_k^{v_2}(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{v_2+a,*}(n_1-x, n_2-y) \right. \\
& \left. - J_{k+1}^{v_2,*}(n_1-x, n_2-y) - a(n_2-y)\gamma \right] P_1(x) P_2(y) \\
& + \sum_{y=0}^{n_2-1} (n_2-y)\gamma a P_2(y) \\
&= J_k^{v_2}(n_1, n_2, p_k) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{v_2+a,*}(n_1-x, n_2-y) \right. \\
& \left. - J_{k+1}^{v_2,*}(n_1-x, n_2-y) \right] P_1(x) P_2(y) + \sum_{y=0}^{n_2-1} (n_2-y)\gamma a P_2(y) \\
& - \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} a(n_2-y)\gamma P_1(x) P_2(y) > 0.
\end{aligned}$$

Since we have the assumption we made at the beginning of the proof, the above remainder turns out to be strictly positive. Let $p_k^{v_2,*}(n_1, n_2)$ be the optimal price under the retailer's salvage value of v_2 and $p_k^{v_2+a,*}(n_1, n_2)$ be the optimal solution for the retailer's salvage value of $v_2 + a$. Then, we can write the following series of inequalities:

$$J_k^{v_2+a,*}(n_1, n_2) = J_k^{v_2+a}(n_1, n_2, p_k^{v_2+a,*}(n_1, n_2)) > J_k^{v_2+a}(n_1, n_2, p_k^{v_2,*}(n_1, n_2))$$

$$> J_k^{v_2}(n_1, n_2, p_k^{v_2,*}(n_1, n_2)) = J_k^{v_2,*}(n_1, n_2).$$

Since we know the expected profit is increasing in the retailer's salvage value for the period N , we can generalize it to all the other periods by moving backwards.

ii) Let $p^{v_2,*}(n_1, n_2)$ be the optimal price in the single-period problem when the retailer's salvage value is v_2 . For the optimal price to increase, $J^{v_2+a}(n_1, n_2, p)$ should be increasing at $p = p^{v_2,*}(n_1, n_2)$.

We know

$$J^{v_2+a}(n_1, n_2, p) = J^{v_2}(n_1, n_2, p) + \sum_{y=0}^{n_2} a\gamma(n_2 - y)P_2(y).$$

We should take the derivative of the above remainder to check whether it is an increasing function or not.

$$\begin{aligned} \left(\sum_{y=0}^{n_2} a\gamma(n_2 - y)P_2(y) \right)' &= a\gamma \left(\sum_{y=0}^{n_2} (n_2 - y) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right)' \\ &= a\gamma \left(n_2 e^{-\lambda(p)(1-\alpha)} \right. \\ &\quad \left. + \sum_{y=1}^{n_2} (n_2 - y) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right)' \\ &= a\gamma \sum_{y=0}^{n_2} (n_2 - y) \left(-\lambda'(p) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} (1-\alpha) \right. \\ &\quad \left. + e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^{y-1}}{(y-1)!} \lambda(p)(1-\alpha) \right) \\ &\quad - a\gamma n_2 \lambda'(p) e^{-\lambda(p)(1-\alpha)} (1-\alpha) \\ &= -a\gamma n_2 \lambda'(p) P_2(0) (1-\alpha) \\ &\quad - a\gamma \lambda'(p) (1-\alpha) \sum_{y=1}^{n_2} [P_2(y) - P_2(y-1)] (n_2 - y) \\ &= -a\gamma n_2 \lambda'(p) P_2(0) (1-\alpha) \\ &\quad - a\gamma \lambda'(p) (1-\alpha) \left[(n_2 - 1)(P_2(1) - P_2(0)) \right. \\ &\quad \left. + (n_2 - 2)(P_2(2) - P_2(1)) + (n_2 - 3)(P_2(3) - P_2(2)) + \dots \right. \\ &\quad \left. + 2(P_2(n_2 - 2) - P_2(n_2 - 3)) + (P_2(n_2 - 1) - P_2(n_2 - 2)) \right] \\ &= -a\gamma n_2 \lambda'(p) P_2(0) (1-\alpha) + a\gamma (n_2 - 1) \lambda'(p) P_2(0) (1-\alpha) \\ &\quad - \lambda'(p) (1-\alpha) a\gamma \sum_{y=1}^{n_2-1} P_2(y) \\ &= -\lambda'(p) (1-\alpha) a\gamma \sum_{y=0}^{n_2-1} P_2(y) > 0. \end{aligned}$$

Now we have

$$\begin{aligned}
\frac{\partial J_{v_2+a}(n_1, n_2, p)}{\partial p} &= \frac{\partial J_{v_2}(n_1, n_2, p)}{\partial p} - \lambda'(p)(1-\alpha)a\gamma \sum_{y=0}^{n_2-1} P_2(y), \\
\left. \frac{\partial J^{v_2+a}(n_1, n_2, p)}{\partial p} \right|_{p=p^{v_2,*}(n_1, n_2)} &= \left. \frac{\partial J^{v_2}(n_1, n_2, p)}{\partial p} \right|_{p=p^{v_2,*}(n_1, n_2)} \\
&\quad - \lambda'(p^{v_2,*}(n_1, n_2))(1-\alpha)a\gamma \sum_{y=0}^{n_2-1} P_2(y), \\
\left. \frac{\partial J^{v_2+a}(n_1, n_2, p)}{\partial p} \right|_{p=p^{v_2,*}(n_1, n_2)} &= -\lambda'(p^{v_2,*}(n_1, n_2))(1-\alpha)a\gamma \sum_{y=0}^{n_2-1} P_2(y) > 0.
\end{aligned}$$

Since we have proven $J^{v_2+a}(n_1, n_2, p)$ is an increasing function at $p = p^{v_2,*}(n_1, n_2)$ that is the optimal price when the salvage value is v_2 , this completes the proof.

In the following proposition, we will examine how an increase in the initial sales price affects the optimal expected profit in the multi-period problem and the optimal price in the single-period problem.

Proposition 5:

- i) The optimal expected profit is decreasing in the initial sales price p_0 .
- ii) In the single period problem, the optimal price is decreasing in the initial sales price p_0 until it hits p_0 and stays constant once the optimal price is less than p_0 .

Proof:

- i) Since the model requires no reimbursement when the price set at the beginning of a specific period is above the initial sales price (p_0) and due to the prices in the interval between p_0 and $p_0 + a$, we should check whether the optimal profit decreases or not in three cases. By doing this, we can take the non-negativity $((p_0 - p)^+)$ into consideration without any complication.

Case I: ($p_N \geq p_0 + a$) In this case, p_N is larger than p_0 , there will be no reimbursement and $(p_0 - p_N)^+$ will be equal to 0. Then, by taking the difference of the expected profits for $p_0 + a$ and p_0 , we have:

$$\begin{aligned}
J_N^{p_0}(n_1, n_2, p_N) &= J_N^{p_0+a}(n_1, n_2, p_N) + \left(\sum_{x=0}^{n_1} [xp_N + v_1(n_1 - x)] P_1(x) \right. \\
&\quad \left. + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - n_2 \gamma (p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
& - \left(\sum_{x=0}^{n_1} \left[x p_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right. \\
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 + a - p_N)^+ y + (n_2 - y) \gamma(p_0 + a - v_2) \right] P_2(y) \\
& \left. - n_2 \gamma(p_0 + a - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right) \\
= & J_N^{p_0+a}(n_1, n_2, p_N) - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
& + n_2 \gamma(p_0 + a - p_N)^+ + \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y \right] P_2(y) + \sum_{y=0}^{n_2} \left[\gamma(p_0 + a - p_N)^+ y \right] P_2(y) \\
& - \sum_{y=0}^{n_2} \left[(n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) + \sum_{y=0}^{n_2} \left[(n_2 - y) \gamma(p_0 + a - v_2) \right] P_2(y) \\
= & J_N^{p_0+a}(n_1, n_2, p_N) + \sum_{y=0}^{n_2} a \gamma(n_2 - y) P_2(y).
\end{aligned}$$

The above remainder is strictly positive. Then, we can say that when $p_N \geq p_0 + a$, the expected profit ($J^{p_0}(n_1, n_2, p_N)$) decreases as p_0 increases. Then, at optimality this result can be generalized since we know that the optimal expected profit is larger than the expected profit at any other possible price value.

Case II: ($p_N < p_0$) This time since p_N is less than p_0 , there will be reimbursement and $(p_0 - p)^+$ will be equal to $p_0 - p$. Then again by taking the difference of the expected profit under the initial sales price p_0 and the expected profit under the initial sales price $p_0 + a$, we have:

$$\begin{aligned}
J_N^{p_0}(n_1, n_2, p_N) &= J_N^{p_0+a}(n_1, n_2, p_N) + \left(\sum_{x=0}^{n_1} \left[x p_N + (v_1)(n_1 - x) \right] P_1(x) \right. \\
& + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - (v_2)) \right] P_2(y) \\
& \left. - \left(\sum_{x=0}^{n_1} \left[x p_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right) \right)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 + a - p_N)^+ y + (n_2 - y) \gamma(p_0 + a - v_2) \right] P_2(y) \\
& - n_2 \gamma(p_0 + a - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
= & J_N^{p_0+a}(n_1, n_2, p_N) - n_2 \gamma(p_0 - p_N) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
& + n_2 \gamma(p_0 + a - p_N) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
& - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N) y \right] P_2(y) + \sum_{y=0}^{n_2} \left[\gamma(p_0 + a - p_N) y \right] P_2(y) \\
& - \sum_{y=0}^{n_2} \left[(n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) + \sum_{y=0}^{n_2} \left[(n_2 - y) \gamma(p_0 + a - v_2) \right] P_2(y) \\
= & J_N^{p_0+a}(n_1, n_2, p_N) + a n_2 \gamma \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) + \sum_{y=0}^{n_2} a \gamma y P_2(y) \\
& + \sum_{y=0}^{n_2} a (n_2 - y) \gamma P_2(y).
\end{aligned}$$

The above remainder is strictly positive. This implies that the same argument as in Case I holds when $p_N < p_0$.

Case III: ($p_0 \leq p_N < p_0 + a$) Although we are increasing the retailer's initial sales price by a very small amount of a , we cannot neglect the prices ranging between the old initial sales price and the new one since we are trying to prove that the graph will completely shift down. The fact behind this requirement is the change of the function. Then, we can show the steps as follows:

$$\begin{aligned}
J_N^{p_0}(n_1, n_2, p_N) &= J_N^{p_0+a}(n_1, n_2, p_N) + \left[\sum_{x=0}^{n_1} \left[x p_N + v_1(n_1 - x) \right] P_1(x) \right. \\
& \quad \left. + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - \sum_{y=0}^{n_2} (n_2 - y) \gamma(p_0 - v_2) P_2(y) \right] \\
& \quad - \left[\sum_{x=0}^{n_1} \left[x p_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right. \\
& \quad \left. - \sum_{y=0}^{n_2} \left[\gamma(p_0 + a - p_N) y + (n_2 - y) \gamma(p_0 + a - v_2) \right] P_2(y) \right]
\end{aligned}$$

$$\begin{aligned}
& -n_2\gamma(p_0 + a - p_N)\left(1 - \sum_{y=0}^{n_2} P_2(y)\right) \Big] \\
= & J_N^{p_0+a}(n_1, n_2, p_N) + \sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) \\
& - \sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) \\
& + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x)\right) - n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x)\right) \\
& + \sum_{y=0}^{n_2} (n_2 - y)\gamma(-p_0 + v_2)P_2(y) + n_2\gamma(p_0 + a - p_N)\left(1 - \sum_{y=0}^{n_2} P_2(y)\right) \\
& + \sum_{y=0}^{n_2} \left[\gamma(p_0 + a - p_N)y + (n_2 - y)\gamma(p_0 + a - v_2) \right] P_2(y) \\
= & J_N^{p_0+a}(n_1, n_2, p_N) + \sum_{y=0}^{n_2} \left[\gamma(p_0 + a - p_N)y + (n_2 - y)\gamma a \right] P_2(y) \\
& + n_2\gamma(p_0 + a - p_N)\left(1 - \sum_{y=0}^{n_2} P_2(y)\right).
\end{aligned}$$

The above remainder is strictly positive. Then in general, we proved that as p_0 increases, the expected profit ($J_N^{p_0}(n_1, n_2, p_N)$) decreases. Let $p_N^{p_0,*}(n_1, n_2)$ be the optimal solution for the retailer's initial sales price of p_0 and $p_N^{p_0+a,*}(n_1, n_2)$ be the optimal solution for the retailer's initial sales price of $p_0 + a$. Then, we can write the following series of inequalities:

$$\begin{aligned}
J_N^{p_0+a,*}(n_1, n_2) &= J_N^{p_0+a}(n_1, n_2, p_N^{p_0+a,*}(n_1, n_2)) < J_N^{p_0}(n_1, n_2, p_N^{p_0+a,*}(n_1, n_2)) \\
&< J_N^{p_0}(n_1, n_2, p_N^{p_0,*}(n_1, n_2)) = J_N^{p_0,*}(n_1, n_2).
\end{aligned}$$

For backward induction, assume $J_{k+1}^{p_0+a,*}(n_1, n_2) < J_{k+1}^{p_0,*}(n_1, n_2)$. We will evaluate the model in three cases as we did for the last period.

Case I: ($p_k \geq p_0 + a$) In this case, p_k is larger than or equal to p_0 , there will be no reimbursement and $(p_0 - p_k)^+$ will be equal to 0. Then, by taking the difference of the expected profit under the initial sales price of $p_0 + a$ and the expected profit under the initial sales of p_0 , we have:

$$\begin{aligned}
J_k^{p_0}(n_1, n_2, p_k) &= J_k^{p_0+a}(n_1, n_2, p_k) + \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) \right. \right. \\
&\quad \left. \left. + (n_2 - y)\gamma(p_0 - v_2) - J_{k+1}^{p_0,*}(n_1 - x, 0) \right] P_1(x)P_2(y) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1}^{p_0,*}(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} \left[- (n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \\
& + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \Big] \\
& - \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) + (n_2 - y)\gamma(p_0 + a - v_2) \right. \right. \\
& \left. \left. - J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) P_2(y) + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k \right. \right. \\
& \left. \left. + J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) \right. \\
& \left. + \sum_{y=0}^{n_2-1} \left[- (n_2 - y)\gamma(p_0 + a - v_2) + (n_2 - y)\gamma(p_0 + a - p_k)^+ \right] P_2(y) \right. \\
& \left. + n_1 p_k - n_2 \gamma(p_0 + a - p_k)^+ \right] \\
= & J_k^{p_0+a}(n_1, n_2, p_k) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) - J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) \right. \\
& + (n_2 - y)\gamma(p_0 - v_2 - p_0 - a + v_2) \\
& \left. - J_{k+1}^{p_0,*}(n_1 - x, 0) + J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k - (x - n_1)p_k + J_{k+1}^{p_0,*}(n_1 - x, 0) \right. \\
& \left. - J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) + n_1 p_k - n_1 p_k \\
& + \sum_{y=0}^{n_2-1} (n_2 - y)\gamma(-p_0 + v_2 + a + p_0 - v_2) P_2(y) \\
= & J_k^{p_0+a}(n_1, n_2, p_k) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) - J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) \right. \\
& \left. - a(n_2 - y)\gamma - J_{k+1}^{p_0,*}(n_1 - x, 0) + J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[J_{k+1}^{p_0,*}(n_1 - x, 0) - J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} (n_2 - y)\gamma a P_2(y)
\end{aligned}$$

When the retailer has no inventory on hand, we know that the retailer is not active in the market anymore and the manufacturer is not supposed to take the reimbursement cost into consideration. Then, p_0 is negligible in the process of determining the optimal prices and in each period, the optimal profit is independent of p_0 . This implies $J_{k+1}^{p_0+a,*}(n_1 - x, 0) = J_{k+1}^{p_0,*}(n_1 - x, 0)$. This enables us to simplify the above expression and it turns into:

$$\begin{aligned}
J_k^{p_0}(n_1, n_2, p_k) &= J_k^{p_0+a}(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) \right. \\
&\quad \left. - J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) - a(n_2 - y)\gamma \right] P_1(x)P_2(y) \\
&\quad + \sum_{y=0}^{n_2-1} (n_2 - y)\gamma a P_2(y) \\
&= J_k^{p_0+a}(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) \right. \\
&\quad \left. - J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) \right] P_1(x)P_2(y) + \sum_{y=0}^{n_2-1} (n_2 - y)\gamma a P_2(y) \\
&\quad - \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} a(n_2 - y)\gamma P_1(x)P_2(y) \\
&= J_k^{p_0+a}(n_1, n_2, p_k) \\
&\quad + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) - J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) \right] P_1(x)P_2(y) \\
&\quad + \left(1 - \sum_{x=0}^{n_1-1} P_1(x) \right) \sum_{y=0}^{n_2-1} a(n_2 - y)\gamma P_2(y).
\end{aligned}$$

Since we have the assumption we have made at the beginning of the proof, the above remainder turns out to be positive. Then, the expected profit decreases in the initial retail price when $p_k \geq p_0 + a$.

Case II: ($p_k < p_0$) In this case, p_k is less than p_0 , the reimbursement will be included and $(p_0 - p_k)^+$ will be larger than 0. Then, by taking the difference of the expected profit under the initial sales price of $p_0 + a$ and the expected profit under the initial sales price of p_0 , we have:

$$J_k^{p_0}(n_1, n_2, p_k) = J_k^{p_0+a}(n_1, n_2, p_k) + \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) \right. \right.$$

$$\begin{aligned}
& +(n_2 - y)\gamma(p_0 - v_2) - J_{k+1}^{p_0,*}(n_1 - x, 0) \Big] P_1(x)P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1}^{p_0,*}(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} \left[-(n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \\
& + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \Big] \\
& - \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) + (n_2 - y)\gamma(p_0 + a - v_2) \right. \right. \\
& \left. \left. - J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x)P_2(y) + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k \right. \right. \\
& \left. \left. + J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) \right. \\
& \left. + \sum_{y=0}^{n_2-1} \left[-(n_2 - y)\gamma(p_0 + a - v_2) + (n_2 - y)\gamma(p_0 + a - p_k)^+ \right] P_2(y) \right. \\
& \left. + n_1 p_k - n_2 \gamma(p_0 + a - p_k)^+ \right] \\
= & J_k^{p_0+a}(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) - J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) \right. \\
& \left. + (n_2 - y)\gamma(p_0 - v_2 - p_0 - a + v_2) \right. \\
& \left. - J_{k+1}^{p_0,*}(n_1 - x, 0) + J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x)P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k - (x - n_1)p_k + J_{k+1}^{p_0,*}(n_1 - x, 0) \right. \\
& \left. - J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} \left[(n_2 - y)\gamma(-p_0 + v_2 + a + p_0 - v_2) \right. \\
& \left. + (n_2 - y)\gamma(p_0 - p_k - p_0 - a + p_k) \right] P_2(y) + n_1 p_k - n_1 p_k \\
& + n_2 \gamma(p_0 + a - p_k - p_0 + p_k) \\
= & J_k^{p_0+a}(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) - J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) \right. \\
& \left. - a(n_2 - y)\gamma - J_{k+1}^{p_0,*}(n_1 - x, 0) \right. \\
& \left. + J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x)P_2(y)
\end{aligned}$$

$$+ \sum_{x=0}^{n_1-1} \left[J_{k+1}^{p_0,*}(n_1-x, 0) - J_{k+1}^{p_0+a,*}(n_1-x, 0) \right] P_1(x) + an_2\gamma.$$

When the retailer has no inventory on hand, we know that the retailer is not active in the market anymore and the manufacturer is not supposed to take the reimbursement cost into consideration. Then, p_0 is negligible in the process of determining the optimal prices and in each period, the optimal profit is independent of p_0 . This implies $J_{k+1}^{p_0+a,*}(n_1-x, 0) = J_{k+1}^{p_0,*}(n_1-x, 0)$. This enables us to simplify the above expression and it turns into:

$$\begin{aligned} J_k^{p_0}(n_1, n_2, p_k) &= J_k^{p_0+a}(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1-x, n_2-y) \right. \\ &\quad \left. - J_{k+1}^{p_0+a,*}(n_1-x, n_2-y) - a(n_2-y)\gamma \right] P_1(x)P_2(y) + an_2\gamma \\ &= J_k^{p_0+a}(n_1, n_2, p_k) \\ &\quad + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1-x, n_2-y) - J_{k+1}^{p_0+a,*}(n_1-x, n_2-y) \right] P_1(x)P_2(y) \\ &\quad + an_2\gamma - \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} a(n_2-y)\gamma P_1(x)P_2(y) \\ &= J_k^{p_0+a}(n_1, n_2, p_k) \\ &\quad + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1-x, n_2-y) - J_{k+1}^{p_0+a,*}(n_1-x, n_2-y) \right] P_1(x)P_2(y) \\ &\quad + an_2\gamma \left(1 - \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} P_1(x)P_2(y) \right) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} ay\gamma P_1(x)P_2(y). \end{aligned}$$

Since we have the assumption we made at the beginning of the proof, the above remainder turns out to be positive.

Case III: ($p_0 \leq p_k < p_0 + a$) Like in part i, we will check the positivity for the prices ranging between p_0 and $p_0 + a$. Then,

$$\begin{aligned} J_k^{p_0}(n_1, n_2, p_k) &= J_k^{p_0+a}(n_1, n_2, p_k) + \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1-x, n_2-y) \right. \right. \\ &\quad \left. \left. + (n_2-y)\gamma(p_0 - v_2) - J_{k+1}^{p_0,*}(n_1-x, 0) \right] P_1(x)P_2(y) \right. \\ &\quad \left. + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1}^{p_0,*}(n_1-x, 0) \right] P_1(x) \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{y=0}^{n_2-1} -(n_2 - y)\gamma(p_0 - v_2)P_2(y) + n_1 p_k \Big] \\
& - \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) + (n_2 - y)\gamma(p_0 + a - v_2) \right. \right. \\
& \quad \left. \left. - J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) P_2(y) \right. \\
& + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} \left[-(n_2 - y)\gamma(p_0 + a - v_2) + (n_2 - y)\gamma(p_0 + a - p_k) \right] P_2(y) \\
& \quad \left. + n_1 p_k - n_2 \gamma(p_0 + a - p_k) \right] \\
= & J_k^{p_0+a}(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) - J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) \right. \\
& + (n_2 - y)\gamma(p_0 - v_2 - p_0 + v_2 - a) \\
& \quad \left. - J_{k+1}^{p_0,*}(n_1 - x, 0) + J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k - (x - n_1)p_k - J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right. \\
& \quad \left. + J_{k+1}^{p_0,*}(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} (n_2 - y)\gamma(-p_0 + v_2 + p_0 - v_2 + a)P_2(y) \\
& - \sum_{y=0}^{n_2-1} (n_2 - y)\gamma(p_0 + a - p_k)P_2(y) + n_1 p_k - n_1 p_k + n_2 \gamma(p_0 + a - p_k) \\
= & J_k^{p_0+a}(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1 - x, n_2 - y) - J_{k+1}^{p_0+a,*}(n_1 - x, n_2 - y) \right. \\
& \quad \left. - J_{k+1}^{p_0,*}(n_1 - x, 0) + J_{k+1}^{p_0+a,*}(n_1 - x, 0) \right] P_1(x) P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[-J_{k+1}^{p_0+a,*}(n_1 - x, 0) + J_{k+1}^{p_0,*}(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} (n_2 - y)\gamma a P_2(y) - \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} (n_2 - y)\gamma a P_1(x) P_2(y) \\
& - \sum_{y=0}^{n_2-1} (n_2 - y)\gamma(p_0 + a - p_k)P_2(y) + n_2 \gamma(p_0 + a - p_k) \\
= & J_k^{p_0+a}(n_1, n_2, p_k)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{p_0,*}(n_1-x, n_2-y) - J_{k+1}^{p_0+a,*}(n_1-x, n_2-y) \right] P_1(x)P_2(y) \\
& + \sum_{y=0}^{n_2-1} (n_2-y)\gamma a P_2(y) - \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} (n_2-y)\gamma a P_1(x)P_2(y) \\
& - \sum_{y=0}^{n_2-1} (n_2-y)\gamma(p_0+a-p_k)P_2(y) + n_2\gamma(p_0+a-p_k).
\end{aligned}$$

Since we have the assumption we made at the beginning of the proof, the above remainder turns out to be positive. Then, we proved that the graph of the function of the expected profit shifts down when p_0 increases. Let $p_k^{p_0,*}(n_1, n_2)$ be the optimal solution for the retailer's initial sales price of p_0 and $p_k^{p_0+a,*}(n_1, n_2)$ be the optimal solution for the retailer's initial sales price of $p_0 + a$. In this case, we can write the following series of inequalities:

$$\begin{aligned}
J_k^{p_0+a,*}(n_1, n_2) &= J_k^{p_0+a}(n_1, n_2, p_k^{p_0+a,*}(n_1, n_2)) < J_k^{p_0}(n_1, n_2, p_k^{p_0+a,*}(n_1, n_2)) \\
&< J_k^{p_0}(n_1, n_2, p_k^{p_0,*}(n_1, n_2)) = J_k^{p_0,*}(n_1, n_2).
\end{aligned}$$

Since we know the optimal expected profit is decreasing in the initial sales price in the last period, we can generalize the result to all the other periods by moving backwards.

ii) For this part, we will examine the model in two cases.

Case I: ($p^* > p_0$) Let $p^{p_0,*}(n_1, n_2)$ be the optimal price when the initial sales price is p_0 . For the optimal price to decrease as p_0 increases, $J^{p_0+a}(n_1, n_2, p)$ should be decreasing at $p = p^{p_0,*}(n_1, n_2)$. From part i, we know:

$$J^{p_0+a}(n_1, n_2, p) = J^{p_0}(n_1, n_2, p) - \sum_{y=0}^{n_2} a\gamma(n_2-y)P_2(y).$$

To check whether it is decreasing or not, we should take the derivative of the above equation. Then, the derivative gives:

$$\frac{\partial J^{p_0+a}(n_1, n_2, p)}{\partial p} = \frac{\partial J^{p_0}(n_1, n_2, p)}{\partial p} - \left(\sum_{y=0}^{n_2} a\gamma(n_2-y)P_2(y) \right)'.$$

We have already proven $\left(\sum_{y=0}^{n_2} a\gamma(n_2-y)P_2(y) \right)' > 0$ for any price in the proof of Proposition 4. Then we have

$$\frac{\partial J^{p_0+a}(n_1, n_2, p)}{\partial p} = \frac{\partial J^{p_0}(n_1, n_2, p)}{\partial p} - \left(\sum_{y=0}^{n_2} a\gamma(n_2-y)P_2(y) \right)',$$

$$\begin{aligned}
\left. \frac{\partial J^{p_0+a}(n_1, n_2, p)}{\partial p} \right|_{p=p^{p_0,*}(n_1, n_2)} &= \left. \frac{\partial J^{p_0}(n_1, n_2, p)}{\partial p} \right|_{p=p^{p_0,*}(n_1, n_2)} \\
&\quad - \left(\sum_{y=0}^{n_2} a\gamma(n_2 - y)P_2(y) \right)' \Big|_{p=p^{p_0,*}(n_1, n_2)}, \\
\left. \frac{\partial J^{p_0+a}(n_1, n_2, p)}{\partial p} \right|_{p=p^{p_0,*}(n_1, n_2)} &= - \left(\sum_{y=0}^{n_2} a\gamma(n_2 - y)P_2(y) \right)' \Big|_{p=p^{p_0,*}(n_1, n_2)} < 0.
\end{aligned}$$

We have proved that $J^{p_0+a}(n_1, n_2, p)$ is a decreasing function at $p = p^{p_0,*}(n_1, n_2)$ which is the optimal price when the initial sales price is p_0 . This implies that the optimal price is decreasing in the initial sales price when it is larger than p_0 .

Case II: ($p < p_0$) We know from Case II of Part i:

$$\begin{aligned}
J^{p_0+a}(n_1, n_2, p) &= J^{p_0}(n_1, n_2, p) - \left[an_2\gamma \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) + \sum_{y=0}^{n_2} a\gamma y P_2(y) \right. \\
&\quad \left. + \sum_{y=0}^{n_2} a(n_2 - y)\gamma P_2(y) \right] = J^{p_0}(n_1, n_2, p) - an_2\gamma.
\end{aligned}$$

If we take the derivative of the above equation, we obtain

$$\begin{aligned}
\frac{\partial J_{p_0+a}(n_1, n_2, p)}{\partial p} &= \frac{\partial J_{p_0}(n_1, n_2, p)}{\partial p}, \\
\left. \frac{\partial J_{p_0+a}(n_1, n_2, p)}{\partial p} \right|_{p=p^{p_0,*}(n_1, n_2)} &= \left. \frac{\partial J_{p_0}(n_1, n_2, p)}{\partial p} \right|_{p=p^{p_0,*}(n_1, n_2)} = 0.
\end{aligned}$$

Then, we can say that the optimality of $p^{p_0,*}(n_1, n_2)$ is preserved when the initial retail price increases if $p^{p_0,*}(n_1, n_2) < p_0$. In the following proposition, we will examine how an increase in the reimbursement rate affects the optimal expected profit in the multi-period problem and the optimal price in the single-period problem.

Proposition 6:

- i) The optimal expected profit is decreasing in the reimbursement rate.
- ii) In the single period problem,
 - When the optimal price is larger than the initial sales price p_0 , the optimal price is decreasing in the reimbursement rate.
 - When the optimal price is smaller than the initial price p_0 , assuming the demand rate function $(\lambda(p))$ is concave, there exists a threshold price p' such that the optimal price is increasing in the reimbursement rate until this threshold price, decreasing after it and constant at that price.

Proof:

i) We will subtract the expected profit under the reimbursement rate of γ from the expected profit under the reimbursement rate of $\gamma - a$ at any price p_N . This time it does not necessitate evaluating in two cases. That is, we can keep $(p_0 - p)^+$ in the picture. After the subtraction, we obtain:

$$\begin{aligned}
J_N^{\gamma-a}(n_1, n_2, p_N) &= J_N^\gamma(n_1, n_2, p_N) + \left(\sum_{x=0}^{n_1} \left[xp_N + (v_1)(n_1 - x) \right] P_1(x) \right. \\
&\quad \left. + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right. \\
&\quad \left. - \sum_{y=0}^{n_2} \left[(\gamma - a)(p_0 - p_N)^+ y + (n_2 - y)(\gamma - a)(p_0 - v_2) \right] P_2(y) \right. \\
&\quad \left. - n_2 (\gamma - a)(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right) \\
&\quad - \left(\sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right. \\
&\quad \left. - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \right. \\
&\quad \left. - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right) \\
&= J_N^\gamma(n_1, n_2, p_N) + \sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) \\
&\quad - \sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) \\
&\quad + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) - n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \\
&\quad + (a - \gamma)(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) + \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
&\quad + \sum_{y=0}^{n_2} \left[(a - \gamma)(p_0 - p_N)^+ y + \gamma(p_0 - p_N)^+ y \right] P_2(y) \\
&\quad + \sum_{y=0}^{n_2} \left[(n_2 - y)(a - \gamma)(p_0 - v_2) + (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \\
&= J_N^\gamma(n_1, n_2, p_N) + \sum_{y=0}^{n_2} \left[a(p_0 - p_N)^+ y + (n_2 - y)a(p_0 - v_2) \right] P_2(y) \\
&\quad + n_2 a(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right).
\end{aligned}$$

The above remainder is strictly positive. Then, by the same logic as in the previous propositions, $J_N^{\gamma-a}(n_1, n_2, p_N) > J_N^\gamma(n_1, n_2, p_N)$ at $p = p_N^{\gamma,*}(n_1, n_2)$ and the optimal expected profit of the problem under the reimbursement rate of $\gamma - a$ being larger than $J_N^{\gamma-a}(n_1, n_2, p_N^{\gamma,*}(n_1, n_2))$ implies the optimal expected profit ($J_N^{\gamma,*}(n_1, n_2)$) increases as γ decreases. We can express the same argument with the following series of inequalities:

$$\begin{aligned} J_N^{\gamma,*}(n_1, n_2) &= J_N^\gamma(n_1, n_2, p_N^{\gamma,*}(n_1, n_2)) < J_N^{\gamma-a}(n_1, n_2, p_N^{\gamma,*}(n_1, n_2)) \\ &< J_N^{\gamma-a}(n_1, n_2, p_N^{\gamma-a,*}(n_1, n_2)) = J_N^{\gamma-a,*}(n_1, n_2). \end{aligned}$$

For backward induction, assume that for the period $k+1$, the optimal profit increases as the reimbursement rate decreases. Then, for the period k , we can write:

$$\begin{aligned} J_k^{\gamma-a}(n_1, n_2, p_k) &= J_k^\gamma(n_1, n_2, p_k) + \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1,\gamma-a}^*(n_1-x, n_2-y) \right. \right. \\ &\quad \left. \left. + (n_2-y)(\gamma-a)(p_0-v_2) - J_{k+1,\gamma-a}^*(n_1-x, 0) \right] P_1(x) P_2(y) \right. \\ &\quad \left. + \sum_{x=0}^{n_1-1} \left[(x-n_1)p_k + J_{k+1,\gamma-a}^*(n_1-x, 0) \right] P_1(x) \right. \\ &\quad \left. + \sum_{y=0}^{n_2-1} \left[-(n_2-y)(\gamma-a)(p_0-v_2) + (n_2-y)(\gamma-a)(p_0-p_k)^+ \right] P_2(y) \right. \\ &\quad \left. + n_1 p_k - n_2(\gamma-a)(p_0-p_k)^+ \right] \\ &\quad - \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{\gamma,*}(n_1-x, n_2-y) + (n_2-y)\gamma(p_0-v_2) \right. \right. \\ &\quad \left. \left. - J_{k+1}^{\gamma,*}(n_1-x, 0) \right] P_1(x) P_2(y) \right. \\ &\quad \left. + \sum_{x=0}^{n_1-1} \left[(x-n_1)p_k + J_{k+1}^{\gamma,*}(n_1-x, 0) \right] P_1(x) \right. \\ &\quad \left. + \sum_{y=0}^{n_2-1} \left[-(n_2-y)\gamma(p_0-v_2) + (n_2-y)\gamma(p_0-p_k)^+ \right] P_2(y) \right. \\ &\quad \left. + n_1 p_k - n_2 \gamma(p_0-p_k)^+ \right] \\ &= J_k^\gamma(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1,\gamma-a}^*(n_1-x, n_2-y) - J_{k+1}^{\gamma,*}(n_1-x, n_2-y) \right. \\ &\quad \left. + (n_2-y)(\gamma-a-\gamma)(p_0-v_2) - J_{k+1,\gamma-a}^*(n_1-x, 0) \right] \end{aligned}$$

$$\begin{aligned}
& + J_{k+1}^{\gamma,*}(n_1 - x, 0) \Big] P_1(x) P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[(x - n_1) p_k - (x - n_1) p_k + J_{k+1, \gamma-a}^*(n_1 - x, 0) \right. \\
& \left. - J_{k+1}^{\gamma,*}(n_1 - x, 0) \right] P_1(x) + \sum_{y=0}^{n_2-1} \left[(n_2 - y)(-\gamma + a + \gamma)(p_0 - v_2) \right. \\
& \left. + (n_2 - y)(\gamma - a - \gamma)(p_0 - p_k)^+ \right] P_2(y) + n_1 p_k - n_1 p_k \\
& + n_2(-\gamma + a + \gamma)(p_0 - p_k)^+ \\
= & J_k^\gamma(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1, \gamma-a}^*(n_1 - x, n_2 - y) - J_{k+1}^{\gamma,*}(n_1 - x, n_2 - y) \right. \\
& \left. - (n_2 - y)a(p_0 - v_2) - J_{k+1, \gamma-a}^*(n_1 - x, 0) \right. \\
& \left. + J_{k+1}^{\gamma,*}(n_1 - x, 0) \right] P_1(x) P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[J_{k+1, \gamma-a}^*(n_1 - x, 0) - J_{k+1}^{\gamma,*}(n_1 - x, 0) \right] P_1(x) \\
& + \sum_{y=0}^{n_2-1} \left[(n_2 - y)a(p_0 - v_2) - (n_2 - y)a(p_0 - p_k)^+ \right] P_2(y) \\
& + n_2 a(p_0 - p_k)^+.
\end{aligned}$$

When the retailer has no inventory on hand, we know that the retailer is not active in the market anymore and the manufacturer is not supposed to take the reimbursement cost into consideration. Then, γ is negligible in the process of determining the optimal prices and in each period, the optimal profit is independent of γ . This implies $J_{k+1}^{\gamma,*}(n_1 - x, 0) = J_{k+1}^{\gamma-a,*}(n_1 - x, 0)$. This enables us to simplify the above expression and it turns into:

$$\begin{aligned}
J_k^{\gamma-a}(n_1, n_2, p_k) &= J_k^\gamma(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{\gamma-a,*}(n_1 - x, n_2 - y) \right. \\
& \left. - J_{k+1}^{\gamma,*}(n_1 - x, n_2 - y) - (n_2 - y)a(p_0 - v_2) \right] P_1(x) P_2(y) \\
& + \sum_{y=0}^{n_2-1} \left[(n_2 - y)a(p_0 - v_2) - (n_2 - y)a(p_0 - p_k)^+ \right] P_2(y) \\
& + n_2 a(p_0 - p_k)^+ \\
= & J_k^\gamma(n_1, n_2, p_k) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{\gamma-a,*}(n_1 - x, n_2 - y) - J_{k+1}^{\gamma,*}(n_1 - x, n_2 - y) \right] P_1(x) P_2(y)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{y=0}^{n_2-1} \left[(n_2 - y)a(p_0 - v_2) \right] P_2(y) \\
& - \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} (n_2 - y)a(p_0 - v_2) P_1(x) P_2(y) \\
& + n_2 a(p_0 - p_k)^+ - \sum_{y=0}^{n_2-1} (n_2 - y)a(p_0 - p_k)^+ P_2(y) \\
= & J_k^\gamma(n_1, n_2, p_k) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^{\gamma-a,*}(n_1 - x, n_2 - y) - J_{k+1}^{\gamma,*}(n_1 - x, n_2 - y) \right] P_1(x) P_2(y) \\
& + \left(1 - \sum_{x=0}^{n_1-1} P_1(x) \right) \sum_{y=0}^{n_2-1} (n_2 - y)a(p_0 - v_2) P_2(y) \\
& + a(p_0 - p_k)^+ \left(n_2 - \sum_{y=0}^{n_2-1} (n_2 - y) P_2(y) \right).
\end{aligned}$$

Since we have the assumption we made at the beginning of the proof, the above remainder turns out to be positive. Let $p_k^{\gamma,*}(n_1, n_2)$ be the optimal solution under the reimbursement rate of γ and $p_k^{\gamma-a,*}(n_1, n_2)$ be the optimal solution under the reimbursement rate of $\gamma - a$. Then, we can write the following series of inequalities:

$$\begin{aligned}
J_k^{\gamma,*}(n_1, n_2) &= J_k^\gamma(n_1, n_2, p_k^{\gamma,*}(n_1, n_2)) < J_k^{\gamma-a}(n_1, n_2, p_k^{\gamma,*}(n_1, n_2)) \\
&< J_k^{\gamma-a}(n_1, n_2, p_k^{\gamma-a,*}(n_1, n_2)) = J_k^{\gamma-a,*}(n_1, n_2).
\end{aligned}$$

Since we know the optimal expected profit increases in the last period as the reimbursement rate decreases, we can generalize the result to all the other periods by moving backwards.

ii) Since we know the difference of the expected profit under the reimbursement rate of $\gamma - a$ and the expected profit under the reimbursement rate of γ respectively, our job is very easy for this part. $p_0 - p < 0$ will drop the expression $(p_0 - p)^+$ from the result of the subtraction in part i and we obtain:

$$J^{\gamma-a}(n_1, n_2, p) = J^\gamma(n_1, n_2, p) + \sum_{y=0}^{n_2} (n_2 - y)a(p_0 - v_2) P_2(y).$$

After taking the derivative of the above equation with respect to p and assuming $p^{\gamma,*}(n_1, n_2)$ is the optimal solution of the problem under the reimbursement rate of γ , we have:

$$\left. \frac{\partial J^{\gamma-a}(n_1, n_2, p)}{\partial p} \right|_{p=p^{\gamma,*}(n_1, n_2)} = \left. \frac{\partial J^\gamma(n_1, n_2, p)}{\partial p} \right|_{p=p^{\gamma,*}(n_1, n_2)}$$

$$\frac{\partial J^{\gamma-a}(n_1, n_2, p)}{\partial p} \Big|_{p=p^{\gamma,*}(n_1, n_2)} = \sum_{y=0}^{n_2} (n_2 - y)a(p_0 - v_2)P_2(y)' \Big|_{p=p^{\gamma,*}(n_1, n_2)} > 0.$$

We had already proven the positivity of the function we have obtained for all possible prices in the proof of Proposition 4. Since $J^{\gamma-a}(n_1, n_2, p)$ is an increasing function at $p = p^{\gamma,*}(n_1, n_2)$ which is the optimal price under the reimbursement rate of γ , the optimal price increases as the reimbursement rate decreases when $p^{\gamma,*}(n_1, n_2) > p_0$.

Now we will show how the optimal price changes if it is less than the initial sales price. In this part; since $p_0 - p^* > 0$, $(p_0 - p)^+$ will be kept in the objective function. However, we will follow the same procedure. At the beginning, we have:

$$\begin{aligned} J^{\gamma-a}(n_1, n_2, p) &= J^{\gamma}(n_1, n_2, p) + \sum_{y=0}^{n_2} \left[a(p_0 - p)^+ y + (n_2 - y)a(p_0 - v_2) \right] P_2(y) \\ &\quad + n_2 a(p_0 - p)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\ &= J^{\gamma}(n_1, n_2, p) + \sum_{y=0}^{n_2} \left[a(p_0 - p)y + (n_2 - y)a(p_0 - v_2) \right] P_2(y) \\ &\quad + \left(n_2 a(p_0 - p) - \sum_{y=0}^{n_2} n_2 a(p_0 - p) P_2(y) \right) \\ &= J^{\gamma}(n_1, n_2, p) + \sum_{y=0}^{n_2} \left[a(p_0 - p)y - n_2 a(p_0 - p) + (n_2 - y)a(p_0 - v_2) \right] P_2(y) \\ &\quad + n_2 a(p_0 - p) \\ &= J^{\gamma}(n_1, n_2, p) + \sum_{y=0}^{n_2} \left[a(p_0 - p)(y - n_2) + (n_2 - y)a(p_0 - v_2) \right] P_2(y) \\ &\quad + n_2 a(p_0 - p) \\ &= J^{\gamma}(n_1, n_2, p) + \sum_{y=0}^{n_2} (n_2 - y)a(p - p_0 + p_0 - v_2)P_2(y) + n_2 a(p_0 - p) \\ &= J^{\gamma}(n_1, n_2, p) + \sum_{y=0}^{n_2} (n_2 - y)a(p - v_2)P_2(y) + n_2 a(p_0 - p) \\ &= J^{\gamma}(n_1, n_2, p) + a(p - v_2) \sum_{y=0}^{n_2} (n_2 - y)P_2(y) + n_2 a(p_0 - p) \end{aligned}$$

Now, by taking the derivative of the above equation, we will obtain the relationship between the change rate of the function with the reimbursement rate $\gamma - a$

and the change rate of the function with the reimbursement rate γ . The way to this relationship is as follows:

$$\begin{aligned}
J^{\gamma-a}(n_1, n_2, p) &= J^\gamma(n_1, n_2, p) + a(p - v_2) \sum_{y=0}^{n_2} (n_2 - y) P_2(y) + n_2 a(p_0 - p) \\
&= J^\gamma(n_1, n_2, p) + a(p - v_2) n_2 e^{-\lambda(p)(1-\alpha)} \\
&\quad + a(p - v_2) \sum_{y=1}^{n_2} (n_2 - y) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \\
&\quad + n_2 a(p_0 - p) \\
\frac{\partial J^{\gamma-a}(n_1, n_2, p)}{\partial p} &= \frac{\partial J^\gamma(n_1, n_2, p)}{\partial p} - n_2 a + n_2 a e^{-\lambda(p)(1-\alpha)} \\
&\quad + a(p - v_2) n_2 \left(-\lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} \right) \\
&\quad + \left[a(p - v_2) \sum_{y=1}^{n_2} (n_2 - y) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right]' \\
&= \frac{\partial J^\gamma(n_1, n_2, p)}{\partial p} - n_2 a + n_2 a e^{-\lambda(p)(1-\alpha)} \\
&\quad + a(p - v_2) n_2 \left(-\lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} \right) \\
&\quad + \sum_{y=1}^{n_2} a(n_2 - y) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \\
&\quad + a(p - v_2) \sum_{y=1}^{n_2} (n_2 - y) \left[-\lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \\
&\quad \left. + \lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} \frac{y[\lambda(p)(1-\alpha)]^{y-1}}{y!} \right] \\
&= \frac{J^\gamma(n_1, n_2, p)}{\partial p} - n_2 a + n_2 a P_2(0) \\
&\quad + a(p - v_2) n_2 \left(-\lambda'(p)(1-\alpha) P_2(0) \right) + \sum_{y=1}^{n_2} a(n_2 - y) P_2(y) \\
&\quad - a(p - v_2) \lambda'(p)(1-\alpha) \sum_{y=1}^{n_2} (n_2 - y) [P_2(y) - P_2(y-1)] \\
&= \frac{\partial J^\gamma(n_1, n_2, p)}{\partial p} - n_2 a + \sum_{y=0}^{n_2} a(n_2 - y) P_2(y) \\
&\quad + a(p - v_2) n_2 \left(-\lambda'(p)(1-\alpha) P_2(0) \right) \\
&\quad - a(p - v_2) \lambda'(p)(1-\alpha) \sum_{y=1}^{n_2} (n_2 - y) [P_2(y) - P_2(y-1)]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial J^\gamma(n_1, n_2, p)}{\partial p} - n_2 a + \sum_{y=0}^{n_2} a(n_2 - y)P_2(y) \\
&\quad + a(p - v_2)n_2 \left(-\lambda'(p)(1 - \alpha)P_2(0) \right) \\
&\quad - a(p - v_2)\lambda'(p)(1 - \alpha) \left[(n_2 - 1)(P_2(1) - P_2(0)) \right. \\
&\quad \left. + (n_2 - 2)(P_2(2) - P_2(1)) + (n_2 - 3)(P_2(3) - P_2(2)) + \dots \right. \\
&\quad \left. + 2(P_2(n_2 - 2) - P_2(n_2 - 3)) + (P_2(n_2 - 1) - P_2(n_2 - 2)) \right] \\
&= \frac{\partial J^\gamma(n_1, n_2, p)}{\partial p} - n_2 a + \sum_{y=0}^{n_2} a(n_2 - y)P_2(y) \\
&\quad - a(p - v_2)n_2\lambda'(p)(1 - \alpha)P_2(0) \\
&\quad + a(p - v_2)(n_2 - 1)\lambda'(p)(1 - \alpha)P_2(0) - a(p - v_2)\lambda'(p)(1 - \alpha) \sum_{y=1}^{n_2-1} P_2(y) \\
&= \frac{\partial J^\gamma(n_1, n_2, p)}{\partial p} - n_2 a + \sum_{y=0}^{n_2} a(n_2 - y)P_2(y) \\
&\quad - a(p - v_2)\lambda'(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y)
\end{aligned}$$

Assuming $p^{\gamma,*}(n_1, n_2)$ is the optimal solution of the problem under the reimbursement rate of γ and so with the known equation $\left. \frac{\partial J_\gamma(n_1, n_2, p)}{\partial p} \right|_{p=p^{\gamma,*}(n_1, n_2)} = 0$, we can infer that if $p^{\gamma,*}(n_1, n_2) \leq v_2$, then the remainder will be negative and as γ decreases, the optimal solution ($p^{\gamma,*}(n_1, n_2)$) decreases as well. The reason for the remainder to be negative is so obvious. $\sum_{y=0}^{n_2} a(n_2 - y)P_2(y)$ is the expected number of unsold products multiplied by a very small positive number a , which is always less than an_2 . Therefore, $-an_2 + \sum_{y=0}^{n_2} a(n_2 - y)P_2(y)$ will be negative and because of the assumption saying the demand rate function ($\lambda(p)$) is decreasing which implies $\lambda'(p) < 0$, $a(p - v_2)\lambda'(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y)$ will be positive. After subtraction, it makes the remainder negative by leading us to reach the relationship $\left. \frac{\partial J^{\gamma-a}(n_1, n_2, p)}{\partial p} \right|_{p=p^{\gamma,*}(n_1, n_2)} < 0$. At $p^{\gamma,*}(n_1, n_2) = v_2$, the negativity of the remainder still holds. For this reason, we have to characterize the structure of the optimal solution for the prices larger than v_2 and less than p_0 . In purpose for acquiring this structure, we will do a rate of change analysis. We already know that at $p = v_2$, $-a(p - v_2)\lambda'(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) < an_2 - \sum_{y=0}^{n_2} a(n_2 - y)P_2(y)$. All we have to do is to determine where the value of the function on the left-hand side will start being larger than the value of the function on the right-hand side and whether that relationship continues incessantly as p increases. In the course

of analysis, we will benefit from the result we obtained in the proof of Proposition 4 which says $\sum_{y=0}^{n_2} a(n_2 - y)P_2(y)$ is an increasing function with respect to p . As a result of that relationship, we can directly say $an_2 - \sum_{y=0}^{n_2} a(n_2 - y)P_2(y)$ is a decreasing function and converges to zero. Then, our job is to show whether $-a(p - v_2)\lambda'(p)(1 - \alpha)\sum_{y=0}^{n_2-1} P_2(y)$ is increasing or not and if so, to show until which point it is decreasing. The steps of the analysis are as follows:

$$\begin{aligned}
& \left[-a(p - v_2)\lambda'(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) \right]' = -a(p - v_2)\lambda''(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) \\
& \quad - a\lambda'(p)(1 - \alpha) \left[(p - v_2) \sum_{y=0}^{n_2-1} P_2(y) \right]' \\
& = -a(p - v_2)\lambda''(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) - a\lambda'(p)(1 - \alpha) \left[\sum_{y=0}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \\
& \quad \left. + (p - v_2) \left(\sum_{y=0}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right)' \right] \\
& = -a(p - v_2)\lambda''(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) - a\lambda'(p)(1 - \alpha) \left[\sum_{y=0}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \\
& \quad \left. + (p - v_2) \left(e^{-\lambda(p)(1-\alpha)} + \sum_{y=1}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right)' \right] \\
& = -a(p - v_2)\lambda''(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) - a\lambda'(p)(1 - \alpha) \left[\sum_{y=0}^{n_2-1} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \\
& \quad \left. + (p - v_2) \left(-\lambda'(p)(1 - \alpha)e^{-\lambda(p)(1-\alpha)} + \sum_{y=1}^{n_2-1} -\lambda'(p)(1 - \alpha)e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \right. \\
& \quad \left. \left. + \lambda'(p)(1 - \alpha)e^{-\lambda(p)(1-\alpha)} \frac{y[\lambda(p)(1-\alpha)]^{y-1}}{y!} \right) \right] \\
& = -a(p - v_2)\lambda''(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) - a\lambda'(p)(1 - \alpha) \left[\sum_{y=0}^{n_2-1} P_2(y) \right. \\
& \quad \left. + (p - v_2) \left(-\lambda'(p)(1 - \alpha)P_2(0) + \sum_{y=1}^{n_2-1} -\lambda'(p)(1 - \alpha)P_2(y) + \lambda'(p)(1 - \alpha)P_2(y - 1) \right) \right] \\
& = -a(p - v_2)\lambda''(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) - a\lambda'(p)(1 - \alpha) \left[\sum_{y=0}^{n_2-1} P_2(y) \right. \\
& \quad \left. - (p - v_2)\lambda'(p)(1 - \alpha) \left(P_2(0) + \sum_{y=1}^{n_2-1} P_2(y) - P_2(y - 1) \right) \right]
\end{aligned}$$

$$\begin{aligned}
&= -a(p - v_2)\lambda''(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) - a\lambda'(p)(1 - \alpha) \left[\sum_{y=0}^{n_2-1} P_2(y) \right. \\
&\quad \left. - (p - v_2)\lambda'(p)(1 - \alpha) \left(P_2(0) + P_2(n_2 - 1) - P_2(0) \right) \right] \\
&= -a(p - v_2)\lambda''(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) - a\lambda'(p)(1 - \alpha) \left[\sum_{y=0}^{n_2-1} P_2(y) \right. \\
&\quad \left. - (p - v_2)\lambda'(p)(1 - \alpha) P_2(n_2 - 1) \right] > 0
\end{aligned}$$

The positivity of the final expression arises from the assumption in the definition of the proposition stating that the demand function $\lambda(p)$ is also a concave function apart from being a decreasing function. Now we can put forward that at each price p above the retailer's salvage value (v_2), $-a(p - v_2)\lambda'(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y)$ is an increasing function starting from 0 at $p = v_2$. Then, $-a(p - v_2)\lambda'(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y)$ and $an_2 - \sum_{y=0}^{n_2} a(n_2 - y)P_2(y)$ intersects at a point p' until which the first function is less than the second one. Since the first function continues to increase without any exception above $p = v_2$, after p passes p' , the first function will always be larger than the second one. Then, if $p^{\gamma,*}(n_1, n_2) < p'$, the optimal solution $(p^{\gamma,*}(n_1, n_2))$ decreases as the reimbursement rate (γ) decreases since $-a(p - v_2)\lambda'(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) - an_2 + \sum_{y=0}^{n_2} a(n_2 - y)P_2(y) < 0$ and if $p^{\gamma,*}(n_1, n_2) > p'$, the optimal solution $(p^{\gamma,*}(n_1, n_2))$ increases as the reimbursement rate (γ) decreases since $-a(p - v_2)\lambda'(p)(1 - \alpha) \sum_{y=0}^{n_2-1} P_2(y) - an_2 + \sum_{y=0}^{n_2} a(n_2 - y)P_2(y) > 0$. At equality, no matter how much the reimbursement rate decreases, the optimal solution will not change.

The results we have obtained when investigating the impact of price protection on the optimal price are very similar to the ones in Proposition 6. To comment on how the optimal price changes as the reimbursement rate decreases, although we are required to solve the equation we have determined, it is apparently not that necessary. If we know the corresponding optimal prices for the problem under no price protection where the manufacturer is responsible only for its revenue, we can directly infer how the optimal price changes in the problem under price protection. The following corollary gives the conditions.

Corollary 1: Let $p^{\gamma,*}(n_1, 0)$ be the optimal price in the single-period problem under no price protection and $p^{\gamma,*}(n_1, n_2)$ be the optimal price under price protection and the reimbursement rate of γ ,

- i) If $p^{\gamma,*}(n_1, 0) < p^{\gamma,*}(n_1, n_2)$ then $p^{\gamma-a,*}(n_1, n_2) < p^{\gamma,*}(n_1, n_2)$.
- ii) If $p^{\gamma,*}(n_1, 0) > p^{\gamma,*}(n_1, n_2)$ then $p^{\gamma-a,*}(n_1, n_2) > p^{\gamma,*}(n_1, n_2)$.
- iii) If $p^{\gamma,*}(n_1, 0) = p^{\gamma,*}(n_1, n_2)$ then $p^{\gamma-a,*}(n_1, n_2) = p^{\gamma,*}(n_1, n_2)$.

For decreasing reimbursement rate, the optimal prices under no price protection give bounds on the optimal prices under price protection. The following corollary states which one is a lower bound and which one is an upper bound. This enables us to narrow the feasible region to get even closer to the optimal price since we will have an idea about in which interval the optimal price will be.

Corollary 2: Let $p_{f_1}^*$ and $p_{f_2}^*$ be the optimal prices for the constituent functions $f_1(p)$ and $f_2(p)$ we have defined, respectively and p_{wr}^* be the optimal price of the model under no price protection. Then, p_{wr}^* is a bound for $p_{f_1}^*$ and $p_{f_2}^*$ and it is always an upper bound for $p_{f_2}^*$ since $p_{f_2}^*$ always increases as γ decreases.

In Proposition 6, we have assumed that the equation to determine the threshold price has only one root if the demand rate function is concave. However, for the other kinds of demand rate functions, we can encounter more than one threshold price. The following corollary states what the case is when the demand rate function is not concave.

Corollary 3: How the optimal price changes as γ decreases can be determined for all demand rate functions if we know the optimal prices of the problem under no price protection as presented in Corollary 1.

In the following proposition, we will examine how an increase in the manufacturer's inventory level affects the optimal expected profit in the multi-period problem.

Proposition 7:

The optimal expected profit is increasing in the level of manufacturer's inventory.

Proof:

We will apply the same procedure as in the previous proofs and show that at each possible price, the expected profit increases as the manufacturer's inventory

level increases. In this case, since the inventory level cannot be a decimal-point number, we will evaluate the difference of the expected profits increasing the inventory level by one. The steps are as follows:

$$\begin{aligned}
J_N(n_1 + 1, n_2, p_N) &= J_N(n_1, n_2, p_N) + \left(\sum_{x=0}^{n_1+1} [xp_N + v_1(n_1 + 1 - x)] P_1(x) \right. \\
&\quad + (n_1 + 1)p_N \left(1 - \sum_{x=0}^{n_1+1} P_1(x) \right) - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
&\quad - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
&\quad - \left(\sum_{x=0}^{n_1} [xp_N + v_1(n_1 - x)] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \right. \\
&\quad - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
&\quad \left. - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \right) \\
&= J_N(n_1, n_2, p_N) - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y) \gamma(p_0 - v_2) \right] P_2(y) \\
&\quad - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
&\quad + \sum_{y=0}^{n_2} \left[(\gamma)(p_0 - p_N)^+ y + (n_2 - y) (\gamma)(p_0 - v_2) \right] P_2(y) \\
&\quad + n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) \\
&\quad + (n_1 + 1)p_N P_1(n_1 + 1) + \sum_{x=0}^{n_1} [xp_N + v_1(n_1 + 1 - x)] P_1(x) \\
&\quad - \sum_{x=0}^{n_1} [xp_N + (v_1)(n_1 - x)] P_1(x) \\
&\quad + (n_1 + 1)p_N - n_1 p_N - (n_1 + 1)p_N P_1(n_1 + 1) \\
&\quad - (n_1 + 1)p_N \sum_{x=0}^{n_1} P_1(x) + n_1 p_N \sum_{x=0}^{n_1} P_1(x) \\
&= J_N(n_1, n_2, p_N) + (n_1 + 1 - n_1)p_N + (-n_1 - 1 + n_1)p_N \sum_{x=0}^{n_1} P_1(x) \\
&\quad + \sum_{x=0}^{n_1} [xp_N + (v_1)(n_1 + 1 - x - n_1 + x) - xp_N] P_1(x)
\end{aligned}$$

$$\begin{aligned}
&= J_N(n_1, n_2, p_N) + p_N + \sum_{x=0}^{n_1} v_1 P_1(x) - p_N \sum_{x=0}^{n_1} P_1(x) \\
&= J_N(n_1, n_2, p_N) + \sum_{x=0}^{n_1} v_1 P_1(x) + p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right).
\end{aligned}$$

Since the above remainder is positive for all possible price values, $J_N(n_1 + 1, n_2, p_N^*(n_1, n_2)) > J_N(n_1, n_2, p_N^*(n_1, n_2))$. Even if we do not know whether $p_N^*(n_1, n_2)$ is the optimal solution of the model under the inventory level of $n_1 + 1$ or not, assuming $p_N^*(n_1 + 1, n_2)$ is the optimal price under the inventory level of $n_1 + 1$, we can say that $J_N(n_1 + 1, n_2, p_N^*(n_1 + 1, n_2)) > J_N(n_1 + 1, n_2, p_N^*(n_1, n_2))$. Then, combining these two inequalities, it is obvious that $J_N(n_1 + 1, n_2, p_N^*(n_1 + 1, n_2)) > J_N(n_1, n_2, p_N^*(n_1, n_2))$ which completes the proof for period N .

For backward induction, assume that for the period $k + 1$, the optimal profit increases as the manufacturer's inventory level increases. Then, for the period k , we can write:

$$\begin{aligned}
J_k(n_1 + 1, n_2, p_k) &= J_k(n_1, n_2, p_k) + \left[\sum_{x=0}^{n_1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 + 1 - x, n_2 - y) \right. \right. \\
&\quad \left. \left. + (n_2 - y)\gamma(p_0 - v_2) - J_{k+1}^*(n_1 + 1 - x, 0) \right] P_1(x) P_2(y) \right. \\
&\quad \left. + \sum_{x=0}^{n_1} \left[(x - n_1 - 1)p_k + J_{k+1}^*(n_1 + 1 - x, 0) \right] P_1(x) \right. \\
&\quad \left. + \sum_{y=0}^{n_2-1} \left[- (n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \right. \\
&\quad \left. + (n_1 + 1)p_k - n_2\gamma(p_0 - p_k)^+ \right] \\
&\quad - \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 - x, n_2 - y) + (n_2 - y)\gamma(p_0 - v_2) \right. \right. \\
&\quad \left. \left. - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \right. \\
&\quad \left. + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \right. \\
&\quad \left. + \sum_{y=0}^{n_2-1} \left[- (n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \right. \\
&\quad \left. + n_1 p_k - n_2\gamma(p_0 - p_k)^+ \right] \\
&= J_k(n_1, n_2, p_k)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(1, n_2 - y) - J_{k+1}^*(1, 0) + (n_2 - y)\gamma(p_0 - v_2) \right] P_1(n_1)P_2(y) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 + 1 - x, n_2 - y) - J_{k+1}^*(n_1 - x, n_2 - y) \right. \\
& + (n_2 - y)\gamma(p_0 - v_2) - (n_2 - y)\gamma(p_0 - v_2) \\
& \left. - J_{k+1}^*(n_1 + 1 - x, 0) + J_{k+1}^*(n_1 - x, 0) \right] P_1(x)P_2(y) \\
& + \left[-p_k + J_{k+1}^*(1, 0) \right] P_1(n_1) + \sum_{x=0}^{n_1-1} \left[(x - n_1 - 1)p_k - (x - n_1)p_k \right. \\
& \left. + J_{k+1}^*(n_1 + 1 - x, 0) - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \\
& + (n_1 + 1)p_k - n_1p_k - n_2\gamma(p_0 - p_k)^+ + n_2\gamma(p_0 - p_k)^+ \\
= & J_k(n_1, n_2, p_k) \\
& + \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(1, n_2 - y) - J_{k+1}^*(1, 0) + (n_2 - y)\gamma(p_0 - v_2) \right] P_1(n_1)P_2(y) \\
& + \sum_{x=0}^{n_1-1} \left[-p_k + J_{k+1}^*(n_1 + 1 - x, 0) - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \\
& - p_k P_1(n_1) + J_{k+1}^*(1, 0) P_1(n_1) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 + 1 - x, n_2 - y) - J_{k+1}^*(n_1 - x, n_2 - y) \right. \\
& \left. - J_{k+1}^*(n_1 + 1 - x, 0) + J_{k+1}^*(n_1 - x, 0) \right] P_1(x)P_2(y) + p_k \\
= & J_k(n_1, n_2, p_k) + \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(1, n_2 - y) - J_{k+1}^*(0, n_2 - y) \right] P_1(n_1)P_2(y) \\
& + \left(1 - \sum_{y=0}^{n_2-1} P_2(y) \right) \sum_{x=0}^{n_1-1} \left[J_{k+1}^*(n_1 + 1 - x, 0) - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \\
& + J_{k+1}^*(1, 0) P_1(n_1) \left(1 - \sum_{y=0}^{n_2-1} P_2(y) \right) + p_k \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 + 1 - x, n_2 - y) \right. \\
& \left. - J_{k+1}^*(n_1 - x, n_2 - y) \right] P_1(x)P_2(y).
\end{aligned}$$

When the manufacturer has some inventories in stock but the retailer does not, we know that the retailer is not active in the market anymore and the manufacturer is not supposed to take the reimbursement cost into consideration. In this case, when the manufacturer's inventory level increases, the expected profit

increases since the manufacturer does not encounter any cost and has a chance to sell more. In the first part of the remainder, we face the assumption that when the manufacturer has no product remaining in stock but the retailer has still positive inventory level at the end of any period, the product is withdrawn from the market and the manufacturer reimburses the retailer $\gamma(p_0 - v_2)$ per product. The positivity of that part can be shown as follows:

We will take the difference between the function of the expected profit ($J_N(n_1, n_2, p_N)$) and the profit-to-go function ($J_N(0, n_2) = -n_2\gamma(p_0 - v_2)$). Then, we will check what happens if the price is set to the retailer's salvage value v_2 .

$$\begin{aligned}
J_N(n_1, n_2, p_N) &= J_N(0, n_2) + \sum_{x=0}^{n_1} \left[xp_N + v_1(n_1 - x) \right] P_1(x) + n_1 p_N \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \\
&\quad - \sum_{y=0}^{n_2} \left[\gamma(p_0 - p_N)^+ y + (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \\
&\quad - n_2 \gamma(p_0 - p_N)^+ \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) + n_2 \gamma(p_0 - v_2), \\
J_N(n_1, n_2, v_2) &= J_N(0, n_2) + \sum_{x=0}^{n_1} \left[xv_2 + v_1(n_1 - x) \right] P_1(x) + n_1 v_2 \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \\
&\quad - \sum_{y=0}^{n_2} \left[\gamma(p_0 - v_2) y + (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \\
&\quad - n_2 \gamma(p_0 - v_2) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) + n_2 \gamma(p_0 - v_2) \\
&= J_N(0, n_2) + \sum_{x=0}^{n_1} \left[xv_2 + v_1(n_1 - x) \right] P_1(x) + n_1 v_2 \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \\
&\quad - \sum_{y=0}^{n_2} n_2 \gamma(p_0 - v_2) P_2(y) + \sum_{y=0}^{n_2} n_2 \gamma(p_0 - v_2) P_2(y) \\
&\quad - n_2 \gamma(p_0 - v_2) + n_2 \gamma(p_0 - v_2) \\
&= J_N(0, n_2) + \sum_{x=0}^{n_1} \left[xv_2 + v_1(n_1 - x) \right] P_1(x) + n_1 v_2 \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) > 0
\end{aligned}$$

The above remainder is positive at $p = v_2$. This implies

$$J_N^*(n_1, n_2) - J_N(0, n_2) > J_N(n_1, n_2, v_2) - J_N(0, n_2) > 0$$

For backward induction, assume that the difference is positive for the period $k + 1$. Then,

$$\begin{aligned}
J_k(n_1, n_2, p_k) &= J_k(0, n_2) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 - x, n_2 - y) + (n_2 - y)\gamma(p_0 - v_2) \right. \\
&\quad \left. - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \\
&\quad + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \\
&\quad + \sum_{y=0}^{n_2-1} \left[-(n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \\
&\quad + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ + n_2 \gamma(p_0 - v_2) \\
J_k(n_1, n_2, v_2) &= J_k(0, n_2) + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 - x, n_2 - y) + (n_2 - y)\gamma(p_0 - v_2) \right. \\
&\quad \left. - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \\
&\quad + \sum_{x=0}^{n_1-1} \left[(x - n_1)v_2 + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \\
&\quad + \sum_{y=0}^{n_2-1} \left[-(n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - v_2) \right] P_2(y) \\
&\quad + n_1 v_2 - n_2 \gamma(p_0 - v_2) + n_2 \gamma(p_0 - v_2) \\
&= J_k(0, n_2) + n_1 v_2 - \sum_{x=0}^{n_1-1} (n_1 - x)v_2 P_1(x) + \sum_{x=0}^{n_1-1} J_{k+1}^*(n_1 - x, 0) P_1(x) \\
&\quad - \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} J_{k+1}^*(n_1 - x, 0) P_1(x) P_2(y) \\
&\quad + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 - x, n_2 - y) - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) > 0
\end{aligned}$$

The above remainder is positive at $p = v_2$. This implies

$$J_k^*(n_1, n_2) - J_k(0, n_2) > J_k(n_1, n_2, v_2) - J_k(0, n_2) > 0$$

Since we know the positivity holds for the period N , we can generalize the result for all the other periods by moving backwards.

Then, we can benefit from $J_{k+1}^*(n_1, n_2) - J_{k+1}(0, n_2)$ to show the positivity of the remainder. Since we also have the assumption we made at the beginning of the proof, the remainder we have obtained by taking the difference of the expected profit under the inventory level of $n_1 + 1$ and the expected profit under the inventory level of n_1 turns out to be positive. Let $p_k^*(n_1, n_2)$ be the optimal solution

for the manufacturer's inventory level of n_1 and $p_k^*(n_1 + 1, n_2)$ be the optimal solution for the manufacturer's inventory level of $n_1 + 1$. Then, we can write the following series of inequalities:

$$\begin{aligned} J_k^*(n_1, n_2) &= J_k(n_1, n_2, p_k^*(n_1, n_2)) < J_k(n_1 + 1, n_2, p_k^*(n_1, n_2)) \\ &< J_k(n_1 + 1, n_2, p_k^*(n_1 + 1, n_2)) = J_k^*(n_1 + 1, n_2) \end{aligned}$$

Since we know the optimal profit is increasing in the manufacturer's inventory level in the last period, we can generalize the result to all the other periods by moving backwards.

In the following proposition, we will examine how an increase in the manufacturer's inventory level affects the optimal price in the single-period problem under the linear demand rate function.

Proposition 8: Assume that the demand rate function is linear ($\lambda(p) = a - bp$). Then, there exists a threshold price p' such that

- The optimal price is increasing in the manufacturer's inventory level if it is less than p' .
- The optimal price is decreasing if it is larger than p' .

Proof:

Now, we will assume that $p^*(n_1, n_2)$ is the optimal price for the problem in which the manufacturer starts the last selling period with the inventory level of n_1 . Then,

$$\begin{aligned} J(n_1 + 1, n_2, p) &= J(n_1, n_2, p) + \sum_{x=0}^{n_1} v_1 P_1(x) + p \left(1 - \sum_{x=0}^{n_1} P_1(x) \right) \\ &= J(n_1, n_2, p) + p(1 - P_1(0)) - p \sum_{x=1}^{n_1} P_1(x) + v_1 P_1(0) \\ &\quad + \sum_{x=1}^{n_1} v_1 P_1(x) \\ &= J(n_1, n_2, p) + p + (v_1 - p)P_1(0) + \sum_{x=1}^{n_1} (v_1 - p)P_1(x), \\ \frac{\partial J(n_1 + 1, n_2, p)}{\partial p} &= \frac{\partial J(n_1, n_2, p)}{\partial p} + \left[p + (v_1 - p)P_1(0) + \sum_{x=1}^{n_1} (v_1 - p)P_1(x) \right]' \\ &= \frac{\partial J(n_1, n_2, p)}{\partial p} \\ &\quad + \left[p + (v_1 - p)e^{-\lambda(p)\alpha} + \sum_{x=1}^{n_1} (v_1 - p)e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^x}{x!} \right]' \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + 1 - e^{-\lambda(p)\alpha} + \lambda'(p)\alpha e^{-\lambda(p)\alpha}(p - v_1) \\
&\quad - \sum_{x=1}^{n_1} e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^x}{x!} \\
&\quad + (p - v_1)\lambda'(p)\alpha \sum_{x=1}^{n_1} \left[e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^x}{x!} - e^{-\lambda(p)\alpha} \frac{x[\lambda(p)\alpha]^{x-1}}{x!} \right] \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + 1 - P_1(0) + \lambda'(p)\alpha P_1(0)(p - v_1) \\
&\quad - \sum_{x=1}^{n_1} P_1(x) + (p - v_1)\lambda'(p)\alpha \sum_{x=1}^{n_1} [P_1(x) - P_1(x-1)] \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + 1 - \sum_{x=0}^{n_1} P_1(x) + \lambda'(p)\alpha P_1(0)(p - v_1) \\
&\quad + (p - v_1)\lambda'(p)\alpha \left[P_1(n_1) - P_1(n_1 - 1) + P_1(n_1 - 1) \right. \\
&\quad \left. - P_1(n_1 - 2) + \dots + P_1(2) - P_1(1) + P_1(1) - P_1(0) \right] \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + 1 - \sum_{x=0}^{n_1} P_1(x) + \lambda'(p)\alpha P_1(0)(p - v_1) \\
&\quad + (p - v_1)\lambda'(p)\alpha P_1(n_1) - (p - v_1)\lambda'(p)\alpha P_1(0) \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + 1 - \sum_{x=0}^{n_1} P_1(x) + (p - v_1)\lambda'(p)\alpha P_1(n_1), \\
\frac{\partial J(n_1 + 1, n_2, p)}{\partial p} \Big|_{p=p^*(n_1, n_2)} &= 1 - \sum_{x=0}^{n_1} P_1(x) + (p - v_1)\lambda'(p)\alpha P_1(n_1) \Big|_{p=p^*(n_1, n_2)}
\end{aligned}$$

We can directly say that if $p^*(n_1, n_2) \leq v_1$, the remainder we found will be positive. Therefore, as the manufacturer's inventory level(n_1) increases to $n_1 + 1$, the optimal price increases as well. ($p^*(n_1 + 1, n_2) > p^*(n_1, n_2)$) Now we should determine how the optimal price changes when $p^*(n_1, n_2) > v_1$. At $p = v_1$, $(p - v_1)\lambda'(p)\alpha P_1(n_1) = 0 > \sum_{x=0}^{n_1} P_1(x) - 1$. We will check whether the sign will change or not as p increases over v_1 .

$$\begin{aligned}
\left(\sum_{x=0}^{n_1} P_1(x) - 1 \right)' &= \left(\sum_{x=0}^{n_1} e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^x}{x!} - 1 \right)' \\
&= \left(e^{-\lambda(p)\alpha} + \sum_{x=1}^{n_1} e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^x}{x!} - 1 \right)' \\
&= -\lambda'(p)\alpha e^{-\lambda(p)\alpha} - \lambda'(p)\alpha \sum_{x=1}^{n_1} e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^x}{x!}
\end{aligned}$$

$$\begin{aligned}
& -e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^{x-1}}{x-1!} \\
& = -\lambda'(p)\alpha P_1(0) - \lambda'(p)\alpha \sum_{x=1}^{n_1} P_1(x) - P_1(x-1) \\
& = -\lambda'(p)\alpha P_1(0) + \lambda'(p)\alpha P_1(0) - \lambda'(p)\alpha P_1(n_1) \\
& = -\lambda'(p)\alpha P_1(n_1) = b\alpha P_1(n_1) > 0
\end{aligned}$$

This means that $\sum_{x=0}^{n_1} P_1(x) - 1$ is an increasing function with respect to p and it approaches 0 as p goes to infinity.

$$\begin{aligned}
\left[(p - v_1)\lambda'(p)\alpha P_1(n_1) \right]' & = \alpha P_1(n_1)[\lambda'(p) + \lambda''(p)(p - v_1)] \\
& + (p - v_1)\lambda'(p)\alpha[\lambda'(p)\alpha(P_1(n_1 - 1) - P_1(n_1))] < 0 \\
& \text{if } P_1(n_1 - 1) - P_1(n_1) \leq 0 \\
P_1(n_1 - 1) - P_1(n_1) \leq 0 & \Rightarrow P_1(n_1 - 1) \leq P_1(n_1) \\
& \Rightarrow e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^{n_1-1}}{(n_1 - 1)!} \leq e^{-\lambda(p)\alpha} \frac{[\lambda(p)\alpha]^{n_1}}{n_1!} \Rightarrow n_1 \leq \lambda(p)\alpha
\end{aligned}$$

In this case, we can reveal that for the p values such that $p > v_1$ and $n_1 \leq \lambda(p)\alpha$, $(p - v_1)\lambda'(p)\alpha P_1(n_1)$ is a decreasing function and starts to decrease from 0. However, we need some further analysis to characterize the same function when $n_1 > \lambda(p)\alpha$.

$$\begin{aligned}
& \Rightarrow \alpha P_1(n_1)[\lambda'(p) + \lambda''(p)(p - v_1)] + (p - v_1)\lambda'(p)\alpha[\lambda'(p)\alpha(P_1(n_1 - 1) - P_1(n_1))] \\
& = \alpha P_1(n_1)\lambda'(p) + \alpha P_1(n_1)\lambda''(p)(p - v_1) + (p - v_1)\lambda'(p)\alpha[\lambda'(p)\alpha P_1(n_1 - 1) - \lambda'(p)\alpha P_1(n_1)] \\
& = \alpha P_1(n_1)\lambda'(p) + \alpha P_1(n_1)\lambda''(p)(p - v_1) + (p - v_1)(\lambda'(p)\alpha)^2[P_1(n_1 - 1) - P_1(n_1)] \\
& = \alpha P_1(n_1)\lambda'(p) + \alpha P_1(n_1)\lambda''(p)(p - v_1) + (p - v_1)(\lambda'(p)\alpha)^2 \left[\frac{n_1 P_1(n_1)}{\lambda(p)\alpha} - P_1(n_1) \right] \\
& = \alpha P_1(n_1)\lambda'(p) + \alpha P_1(n_1)\lambda''(p)(p - v_1) + (p - v_1)(\lambda'(p)\alpha)^2 \left[\left(\frac{n_1}{\lambda(p)\alpha} - 1 \right) P_1(n_1) \right] \\
& = P_1(n_1) \left[\alpha\lambda'(p) + \alpha\lambda''(p)(p - v_1) + (p - v_1)(\lambda'(p)\alpha)^2 \left[\left(\frac{n_1}{\lambda(p)\alpha} - 1 \right) \right] \right] \\
& = P_1(n_1) \left[-\alpha b + (p - v_1)(\lambda'(p)\alpha)^2 \left[\left(\frac{n_1}{\lambda(p)\alpha} - 1 \right) \right] \right]
\end{aligned}$$

We know that for the price values which are larger than v_1 and make $\lambda(p)\alpha$ larger than or equal to n_1 , $(p - v_1)\lambda'(p)\alpha P_1(n_1)$ is decreasing since $\alpha P_1(n_1)[\lambda'(p) + \lambda''(p)(p - v_1)] + (p - v_1)\lambda'(p)\alpha[\lambda'(p)\alpha(P_1(n_1 - 1) - P_1(n_1))] < 0$. Now we continue

increasing the price above p'' that results in $n_1 = \lambda(p'')\alpha$. We will determine until which point the function is decreasing and whether its rate of increase is higher or lower than the rate of increase of the other function which is $b\alpha P_1(n_1)$. The remaining part of the analysis is as follows:

$$\begin{aligned} \left[(p - v_1)(\lambda'(p)\alpha)^2 \left[\left(\frac{n_1}{\lambda(p)\alpha} - 1 \right) \right] \right]' &= \left[(\lambda'(p)\alpha)^2 + 2\lambda''(p)(\alpha)^2 \lambda'(p)(p - v_1) \right] \left(\frac{n_1}{\lambda(p)\alpha} - 1 \right) \\ &\quad + \left(\frac{-\lambda'(p)\alpha n_1}{(\lambda(p)\alpha)^2} \right) (p - v_1)(\lambda'(p)\alpha)^2 > 0 \\ &\text{since } n_1 > \lambda(p)\alpha \end{aligned}$$

Since it is an increasing function for all the price values larger than p'' , at a specific price value, $(p - v_1)(\lambda'(p)\alpha)^2 \left[\left(\frac{n_1}{\lambda(p)\alpha} - 1 \right) \right]$ will be equal to αb . This means, until that price, $(p - v_1)\lambda'(p)\alpha P_1(n_1)$ is decreasing and after passing that price, it will increase forever. We can highlight the rate of increase of the function $\sum_{x=0}^{n_1} P_1(x) - 1$ is $\alpha b P_1(n_1)$ and until $(p - v_1)(\lambda'(p)\alpha)^2 \left[\left(\frac{n_1}{\lambda(p)\alpha} - 1 \right) \right]$ reaches $2\alpha b$, its rate of increase will be larger than the rate of increase of the latter. Then, if we set p''' to the price making $(p - v_1)(\lambda'(p)\alpha)^2 \left[\left(\frac{n_1}{\lambda(p)\alpha} - 1 \right) \right]$ equal to $2\alpha b$, we can say that if these two functions do not intersect until p''' , they can never intersect in the future. The reason for this result is that $(p - v_1)\lambda'(p)\alpha P_1(n_1)$ will increase faster than $\sum_{x=0}^{n_1} P_1(x) - 1$. However this is impossible since these two functions will intersect at the largest possible price a/b . Then, there should be a threshold price p' which is less than a/b such that if the optimal price is less than this threshold price, it is increasing in the manufacturer's inventory level and if the optimal price is larger than this threshold price, it is decreasing. Since a/b is the largest possible price to set for the linear demand rate function, we do not need to pay attention to the prices larger than it.

In the following proposition, we will examine how an increase in the retailer's salvage value affects the optimal expected profit in the multi-period problem and the optimal price in the single-period problem when the optimal price is larger than the initial sales price p_0 .

Proposition 9:

- i) The optimal expected profit is decreasing in the retailer's inventory.
- ii) In the single period problem, the optimal price is decreasing in the retailer's inventory when it is larger than p_0 .

Proof:

- i) We will take the difference of the expected profit under the inventory level of n_2 and the expected profit under the inventory level of $n_2 + 1$. The steps are as follows:

$$\begin{aligned}
J_N(n_1, n_2 + 1, p_N) &= J_N(n_1, n_2, p_N) - \gamma(p_0 - p_N)^+(n_2 + 1) \\
&\quad + \sum_{y=0}^{n_2+1} \gamma(p_0 - p_N)^+(n_2 + 1)P_2(y) \\
&\quad - \sum_{y=0}^{n_2+1} \left(\gamma y(p_0 - p_N)^+ + (n_2 + 1 - y)\gamma(p_0 - v_2) \right) P_2(y) \\
&\quad + \gamma(p_0 - p_N)^+ n_2 - \sum_{y=0}^{n_2} \gamma(p_0 - p_N)^+(n_2)P_2(y) \\
&\quad + \sum_{y=0}^{n_2} \left(\gamma y(p_0 - p_N)^+ + (n_2 - y)\gamma(p_0 - v_2) \right) P_2(y) \\
&= J_N(n_1, n_2, p_N) + \gamma(p_0 - p_N)^+(-n_2 - 1 + n_2) \\
&\quad + \gamma(p_0 - p_N)^+(n_2 + 1)P_2(n_2 + 1) \\
&\quad + \sum_{y=0}^{n_2} \gamma(p_0 - p_N)^+(n_2 + 1 - n_2)P_2(y) \\
&\quad - \gamma(p_0 - p_N)^+(n_2 + 1)P_2(n_2 + 1) \\
&\quad + \sum_{y=0}^{n_2} (n_2 - y - n_2 - 1 + y)\gamma(p_0 - v_2)P_2(y) \\
&= J_N(n_1, n_2, p_N) - \gamma(p_0 - p_N)^+ + \sum_{y=0}^{n_2} \gamma(p_0 - p_N)^+ P_2(y) \\
&\quad - \sum_{y=0}^{n_2} \gamma(p_0 - v_2)P_2(y) \\
&= J_N(n_1, n_2, p_N) - \gamma(p_0 - p_N)^+ \left[1 - \sum_{y=0}^{n_2} P_2(y) \right] - \sum_{y=0}^{n_2} \gamma(p_0 - v_2)P_2(y).
\end{aligned}$$

The above remainder is negative for any price p_N . This implies that at the same price, if n_2 decreases, the expected profit increases. Let $p_N^*(n_1, n_2 + 1)$ be the optimal price when the retailer's inventory level is $n_2 + 1$ and $p_N^*(n_1, n_2)$ be the

optimal price when the retailer's inventory level is n_2 . Then,

$$\begin{aligned} J_N^*(n_1, n_2) &= J_N(n_1, n_2, p_N^*(n_1, n_2)) > J_N(n_1, n_2, p_N^*(n_1, n_2 + 1)) \\ &> J_N(n_1, n_2 + 1, p_N^*(n_1, n_2 + 1)) = J_N^*(n_1, n_2 + 1). \end{aligned}$$

For backward induction, assume that for the period $k + 1$, the optimal profit increases as the retailer's inventory level decreases. Then, for the period k , we can write:

$$\begin{aligned} J_k(n_1, n_2 + 1, p_k) &= J_k(n_1, n_2, p_k) + \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2} \left[J_{k+1}^*(n_1 - x, n_2 + 1 - y) \right. \right. \\ &\quad \left. \left. + (n_2 + 1 - y)\gamma(p_0 - v_2) - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \right. \\ &\quad \left. + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \right. \\ &\quad \left. + \sum_{y=0}^{n_2} \left[- (n_2 + 1 - y)\gamma(p_0 - v_2) + (n_2 + 1 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \right. \\ &\quad \left. + n_1 p_k - (n_2 + 1)\gamma(p_0 - p_k)^+ \right] \\ &\quad - \left[\sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1 - x, n_2 - y) + (n_2 - y)\gamma(p_0 - v_2) \right. \right. \\ &\quad \left. \left. - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) P_2(y) \right. \\ &\quad \left. + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \right. \\ &\quad \left. + \sum_{y=0}^{n_2-1} \left[- (n_2 - y)\gamma(p_0 - v_2) + (n_2 - y)\gamma(p_0 - p_k)^+ \right] P_2(y) \right. \\ &\quad \left. + n_1 p_k - n_2 \gamma(p_0 - p_k)^+ \right] \\ &= J_k(n_1, n_2, p_k) \\ &\quad + \sum_{x=0}^{n_1-1} \left[J_{k+1}^*(n_1 - x, 1) - J_{k+1}^*(n_1 - x, 0) + \gamma(p_0 - v_2) \right] P_1(x) P_2(n_2) \\ &\quad + \sum_{x=0}^{n_1-1} \left[(x - n_1)p_k + J_{k+1}^*(n_1 - x, 0) - (x - n_1)p_k \right. \\ &\quad \left. - J_{k+1}^*(n_1 - x, 0) \right] P_1(x) \end{aligned}$$

$$\begin{aligned}
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1-x, n_2+1-y) - J_{k+1}^*(n_1-x, n_2-y) \right. \\
& + (n_2+1-y)\gamma(p_0-v_2) - (n_2-y)\gamma(p_0-v_2) \\
& \left. - J_{k+1}^*(n_1-x, 0) + J_{k+1}^*(n_1-x, 0) \right] P_1(x)P_2(y) \\
& + \left[-\gamma(p_0-v_2) + \gamma(p_0-p_k)^+ \right] P_2(n_2) \\
& + \sum_{y=0}^{n_2-1} \left[(y-n_2-1)\gamma(p_0-v_2) + (n_2-y)\gamma(p_0-v_2) \right. \\
& + (n_2+1-y)\gamma(p_0-p_k)^+ + (y-n_2)\gamma(p_0-p_k)^+ \left. \right] P_2(y) \\
& + n_1 p_k - n_1 p_k + (-n_2-1)\gamma(p_0-p_k)^+ + n_2 \gamma(p_0-p_k)^+ \\
= & J_k(n_1, n_2, p_k) \\
& + \sum_{x=0}^{n_1-1} \left[J_{k+1}^*(n_1-x, 1) - J_{k+1}^*(n_1-x, 0) + \gamma(p_0-v_2) \right] P_1(x)P_2(n_2) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1-x, n_2+1-y) - J_{k+1}^*(n_1-x, n_2-y) \right. \\
& \left. + \gamma(p_0-v_2) \right] P_1(x)P_2(y) \\
& + \left[-\gamma(p_0-v_2) + \gamma(p_0-p_k)^+ \right] P_2(n_2) \\
& + \sum_{y=0}^{n_2-1} \left[-\gamma(p_0-v_2) + \gamma(p_0-p_k)^+ \right] P_2(y) - \gamma(p_0-p_k)^+ \\
= & J_k(n_1, n_2, p_k) + \sum_{x=0}^{n_1-1} \left[J_{k+1}^*(n_1-x, 1) - J_{k+1}^*(n_1-x, 0) \right] P_1(x)P_2(n_2) \\
& + \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} \left[J_{k+1}^*(n_1-x, n_2+1-y) - J_{k+1}^*(n_1-x, n_2-y) \right] P_1(x)P_2(y) \\
& + \left(\sum_{x=0}^{n_1-1} P_1(x) - 1 \right) \gamma(p_0-v_2) \sum_{y=0}^{n_2-1} P_2(y) \\
& + \left(\sum_{x=0}^{n_1-1} P_1(x) - 1 \right) \gamma(p_0-v_2) P_2(n_2) \\
& + \left(\sum_{y=0}^{n_2} P_2(y) - 1 \right) \gamma(p_0-p_k)^+.
\end{aligned}$$

In Proposition 2, we have proven that the optimal expected profit is always larger than it is under price protection. We also know that this implies the case in which the retailer has no inventory on hand is equivalent to the exclusion of the price protection. Then, the first summation of the above equation turns out

to be negative. The assumption we made at the beginning of the proof leads the second summation to be negative, as well. The negativity of the remainder comes from an easy probability concept. As a result, the above remainder is negative. Let $p_k^*(n_1, n_2)$ be the optimal solution for the retailer's inventory level of n_2 and $p_k^*(n_1, n_2 + 1)$ be the optimal solution for the retailer's inventory level of $n_2 + 1$. Then, we can write the following series of inequalities:

$$\begin{aligned} J_k^*(n_1, n_2) &= J_k(n_1, n_2, p_k^*(n_1, n_2)) > J_k(n_1, n_2, p_k^*(n_1, n_2 + 1)) \\ &> J_k(n_1, n_2 + 1, p_k^*(n_1, n_2 + 1)) = J_k^*(n_1, n_2 + 1). \end{aligned}$$

Since we know the optimal profit increases as the retailer's inventory level decreases in the last period, we can generalize the result to all the other periods by moving backwards.

ii) Since p^* is assumed to be larger than p_0 , we will equalize the expression $(p_0 - p)^+$ to 0. Then, the difference will be:

$$J(n_1, n_2 + 1, p) - J(n_1, n_2, p) = - \sum_{y=0}^{n_2} \gamma(p_0 - v_2) P_2(y)$$

Now, we will take the derivative of the above expression and assume that $p^*(n_1, n_2)$ is the optimal solution of the problem in which the retailer's starting inventory level at the beginning of the last selling period is n_2 . Then,

$$\begin{aligned} J(n_1, n_2 + 1, p) &= J(n_1, n_2, p) - \sum_{y=0}^{n_2} \gamma(p_0 - v_2) P_2(y), \\ \frac{\partial J(n_1, n_2 + 1, p)}{\partial p} &= \frac{\partial J(n_1, n_2, p)}{\partial p} - \left[\gamma(p_0 - v_2) \sum_{y=0}^{n_2} P_2(y) \right]' \\ &= \frac{\partial J(n_1, n_2, p)}{\partial p} - \left[\gamma(p_0 - v_2) \left(P_2(0) + \sum_{y=1}^{n_2} P_2(y) \right) \right]' \\ &= \frac{\partial J(n_1, n_2, p)}{\partial p} - \left[\gamma(p_0 - v_2) \left(e^{-\lambda(p)(1-\alpha)} \right. \right. \\ &\quad \left. \left. + \sum_{y=1}^{n_2} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right) \right]' \\ &= \frac{\partial J(n_1, n_2, p)}{\partial p} - \gamma(p_0 - v_2) \left(-\lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} \right. \\ &\quad \left. + \sum_{y=1}^{n_2} -\lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right) \end{aligned}$$

$$\begin{aligned}
& +\lambda'(p)(1-\alpha)e^{-\lambda(p)(1-\alpha)}\frac{y[\lambda(p)(1-\alpha)]^{y-1}}{y!}\Bigg) \\
= & \frac{\partial J(n_1, n_2, p)}{\partial p} - \gamma(p_0 - v_2)(-\lambda'(p)(1-\alpha))\left(P_2(0) + \sum_{y=1}^{n_2} P_2(y) \right. \\
& \left. - P_2(y-1)\right) \\
= & \frac{\partial J(n_1, n_2, p)}{\partial p} - \gamma(p_0 - v_2)(-\lambda'(p)(1-\alpha))(P_2(0) + P_2(n_2) \\
& - P_2(n_2 - 1) + P_2(n_2 - 1) - P_2(n_2 - 2) + \dots + P_2(2) - P_2(1) \\
& + P_2(1) - P_2(0)) \\
= & \frac{\partial J(n_1, n_2, p)}{\partial p} + \gamma(p_0 - v_2)\lambda'(p)(1-\alpha)P_2(n_2), \\
\frac{\partial J(n_1, n_2 + 1, p)}{\partial p}\Bigg|_{p=p^*(n_1, n_2)} = & \frac{\partial J(n_1, n_2, p)}{\partial p}\Bigg|_{p=p^*(n_1, n_2)} \\
& + \gamma(p_0 - v_2)\lambda'(p)(1-\alpha)P_2(n_2)\Bigg|_{p=p^*(n_1, n_2)}, \\
\frac{\partial J(n_1, n_2 + 1, p)}{\partial p}\Bigg|_{p=p^*(n_1, n_2)} = & \gamma(p_0 - v_2)\lambda'(p)(1-\alpha)P_2(n_2)\Bigg|_{p=p^*(n_1, n_2)} < 0.
\end{aligned}$$

Then, we proved the optimal price decreases as the retailer's inventory level increases since $J(n_1, n_2 + 1, p)$ is decreasing at $p = p^*(n_1, n_2)$ which is the optimal solution of the problem with n_2 .

In the following proposition, we will examine how an increase in the retailer's inventory level affects the optimal price in the single-period problem under the linear demand rate function when the optimal price is less than the initial sales price p_0 .

Proposition 10: Assume that the demand rate function is linear ($\lambda(p) = a - bp$).

Then, there exists a threshold price p' such that

- The optimal price is increasing in the retailer's inventory level if it is less than p' .
- The optimal price is decreasing in the retailer's inventory level if it is larger than p' .

Proof:

Now, we will assume that $p^*(n_1, n_2)$ is the optimal price for the problem in which the retailer starts the last selling period with the inventory level of n_2 . We know that this assumption implies $\frac{\partial J(n_1, n_2, p^*(n_1, n_2))}{\partial p} = 0$. We defined the function for

the problem in which the retailer has $n_2 + 1$ products at the beginning and can use it to determine the relationship between the rates of change as follows:

$$\begin{aligned}
J(n_1, n_2 + 1, p) &= J(n_1, n_2, p) - \gamma(p_0 - p) \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) - \sum_{y=0}^{n_2} \gamma(p_0 - v_2) P_2(y) \\
&= J(n_1, n_2, p) - \gamma(p_0 - p) + \gamma(p_0 - p) \sum_{y=0}^{n_2} P_2(y) - \sum_{y=0}^{n_2} \gamma(p_0 - v_2) P_2(y) \\
&= J(n_1, n_2, p) - \gamma(p_0 - p) + \sum_{y=0}^{n_2} \gamma(p_0 - p - p_0 + v_2) P_2(y) \\
&= J(n_1, n_2, p) - \gamma(p_0 - p) + \sum_{y=0}^{n_2} \gamma(v_2 - p) P_2(y), \\
\frac{\partial J(n_1, n_2 + 1, p)}{\partial p} &= \frac{\partial J(n_1, n_2, p)}{\partial p} + \left[-\gamma(p_0 - p) + \gamma(v_2 - p) P_2(0) \right. \\
&\quad \left. + \sum_{y=1}^{n_2} \gamma(v_2 - p) P_2(y) \right]' \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + \left[-\gamma(p_0 - p) + \gamma(v_2 - p) e^{-\lambda(p)(1-\alpha)} \right. \\
&\quad \left. + \sum_{y=1}^{n_2} \gamma(v_2 - p) e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right]' \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + \gamma - \gamma e^{-\lambda(p)(1-\alpha)} \\
&\quad + \lambda'(p)(1-\alpha) e^{-\lambda(p)(1-\alpha)} \gamma(p - v_2) \\
&\quad - \gamma \sum_{y=1}^{n_2} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \\
&\quad + \gamma(p - v_2) \lambda'(p)(1-\alpha) \sum_{y=1}^{n_2} \left[e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \right. \\
&\quad \left. - e^{-\lambda(p)(1-\alpha)} \frac{y[\lambda(p)(1-\alpha)]^{y-1}}{y!} \right] \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + \gamma - \gamma P_2(0) + \lambda'(p)(1-\alpha) P_2(0) \gamma(p - v_2) \\
&\quad - \gamma \sum_{y=1}^{n_2} P_2(y) + \gamma(p - v_2) \lambda'(p)(1-\alpha) \sum_{y=1}^{n_2} \left[P_2(y) - P_2(y-1) \right] \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + \gamma - \gamma \sum_{y=0}^{n_2} P_2(y) + \lambda'(p)(1-\alpha) P_2(0) \gamma(p - v_2) \\
&\quad + \gamma(p - v_2) \lambda'(p)(1-\alpha) \left[P_2(n_2) - P_2(n_2-1) + P_2(n_2-1) \right. \\
&\quad \left. - P_2(n_2-2) + \dots + P_2(2) - P_2(1) + P_2(1) - P_2(0) \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + \gamma - \gamma \sum_{y=0}^{n_2} P_2(y) + \lambda'(p)(1-\alpha)P_2(0)\gamma(p-v_2) \\
&\quad + \gamma(p-v_2)\lambda'(p)(1-\alpha)P_2(n_2) - \gamma(p-v_2)\lambda'(p)(1-\alpha)P_2(0) \Big] \\
&= \frac{\partial J(n_1, n_2, p)}{\partial p} + \gamma \left(1 - \sum_{y=0}^{n_2} P_2(y) \right) - \gamma(v_2-p)\lambda'(p)(1-\alpha)P_2(n_2).
\end{aligned}$$

We can directly say that if $p^*(n_1, n_2) \leq v_2$, the remainder we found will be positive. Therefore, as the retailer's inventory level (n_2) increases to $n_2 + 1$, the optimal price increases as well. ($p^*(n_1, n_2 + 1) > p^*(n_2)$) Now we should determine how the optimal price changes when $p^*(n_1, n_2) > v_2$. At $p = v_2$, $\gamma(p-v_2)\lambda'(p)(1-\alpha)P_2(n_2) = 0 > \gamma \left(\sum_{y=0}^{n_2} P_2(x) - 1 \right)$. We will check whether the sign will change or not as p increases over v_2 .

$$\begin{aligned}
\left(\sum_{y=0}^{n_2} P_2(y) - 1 \right)' &= \left(\sum_{y=0}^{n_2} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} - 1 \right)' \\
&= \left(e^{-\lambda(p)(1-\alpha)} + \sum_{y=1}^{n_2} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} - 1 \right)' \\
&= -\lambda'(p)(1-\alpha)e^{-\lambda(p)(1-\alpha)} - \lambda'(p)(1-\alpha) \sum_{y=1}^{n_2} e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^y}{y!} \\
&\quad - e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^{y-1}}{y-1!} \\
&= -\lambda'(p)(1-\alpha)P_2(0) - \lambda'(p)(1-\alpha) \sum_{y=1}^{n_2} P_2(y) - P_2(y-1) \\
&= -\lambda'(p)(1-\alpha)P_2(0) + \lambda'(p)(1-\alpha)P_2(0) - \lambda'(p)(1-\alpha)P_2(n_2) \\
&= -\lambda'(p)(1-\alpha)P_2(n_2) > 0
\end{aligned}$$

This means that $\sum_{y=0}^{n_2} P_2(y) - 1$ is an increasing function with respect to p and it approaches 0 as p goes to infinity.

$$\begin{aligned}
\left[(p-v_2)\lambda'(p)(1-\alpha)P_2(n_2) \right]' &= (1-\alpha)P_2(n_2)[\lambda'(p) + \lambda''(p)(p-v_2)] \\
&\quad + (p-v_2)\lambda'(p)(1-\alpha)[\lambda'(p)(1-\alpha)(P_2(n_2-1) - P_2(n_2))] < 0 \\
&\quad \left(\text{if } P_2(n_2-1) - P_2(n_2) \leq 0 \right) \\
&\Rightarrow P_2(n_2-1) - P_2(n_2) \leq 0 \Rightarrow P_2(n_2-1) \leq P_2(n_2) \\
&\Rightarrow e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^{n_2-1}}{(n_2-1)!} \leq e^{-\lambda(p)(1-\alpha)} \frac{[\lambda(p)(1-\alpha)]^{n_2}}{n_2!} \\
&\Rightarrow n_2 \leq \lambda(p)(1-\alpha)
\end{aligned}$$

In this case, we can reveal that for the p values such that $p > v_2$ and $n_2 \leq \lambda(p)(1 - \alpha)$, $(p - v_2)\lambda'(p)(1 - \alpha)P_2(n_2)$ is a decreasing function and starts to decrease from 0. However, we need some further analysis to characterize the same function when $n_2 > \lambda(p)(1 - \alpha)$. After this point, we will assume that the demand rate function ($\lambda(p)$) is linear and $\lambda(p) = a - bp$.

$$\begin{aligned}
&\Rightarrow (1 - \alpha)P_2(n_2)[\lambda'(p) + \lambda''(p)(p - v_2)] \\
&\quad + (p - v_2)\lambda'(p)(1 - \alpha)[\lambda'(p)(1 - \alpha)(P_2(n_2 - 1) - P_2(n_2))] \\
&= (1 - \alpha)P_2(n_2)\lambda'(p) + (1 - \alpha)P_2(n_2)\lambda''(p)(p - v_2) \\
&\quad + (p - v_2)\lambda'(p)(1 - \alpha)[\lambda'(p)(1 - \alpha)P_2(n_2 - 1) - \lambda'(p)(1 - \alpha)P_2(n_2)] \\
&= (1 - \alpha)P_2(n_2)\lambda'(p) + (1 - \alpha)P_2(n_2)\lambda''(p)(p - v_2) \\
&\quad + (p - v_2)(\lambda'(p)(1 - \alpha))^2[P_2(n_2 - 1) - P_2(n_2)] \\
&= (1 - \alpha)P_2(n_2)\lambda'(p) + (1 - \alpha)P_2(n_2)\lambda''(p)(p - v_2) \\
&\quad + (p - v_2)(\lambda'(p)(1 - \alpha))^2\left[\frac{n_2 P_2(n_2)}{\lambda(p)(1 - \alpha)} - P_2(n_2)\right] \\
&= (1 - \alpha)P_2(n_2)\lambda'(p) + (1 - \alpha)P_2(n_2)\lambda''(p)(p - v_2) \\
&\quad + (p - v_2)(\lambda'(p)(1 - \alpha))^2\left[\left(\frac{n_2}{\lambda(p)(1 - \alpha)} - 1\right)P_2(n_2)\right] \\
&= P_2(n_2)\left[(1 - \alpha)\lambda'(p) + (1 - \alpha)\lambda''(p)(p - v_2)\right. \\
&\quad \left.+ (p - v_2)(\lambda'(p)(1 - \alpha))^2\left[\left(\frac{n_2}{\lambda(p)(1 - \alpha)} - 1\right)\right]\right] \\
&= P_2(n_2)\left[-(1 - \alpha)b + (p - v_2)(\lambda'(p)(1 - \alpha))^2\left[\left(\frac{n_2}{\lambda(p)(1 - \alpha)} - 1\right)\right]\right]
\end{aligned}$$

We know that for the price values which are larger than v_2 and making $\lambda(p)(1 - \alpha)$ larger than or equal to n_2 , $(p - v_2)\lambda'(p)(1 - \alpha)P_2(n_2)$ is decreasing since $(1 - \alpha)P_2(n_2)[\lambda'(p) + \lambda''(p)(p - v_2)] + (p - v_2)\lambda'(p)(1 - \alpha)[\lambda'(p)(1 - \alpha)(P_2(n_2 - 1) - P_2(n_2))] < 0$. Now we continue increasing the price above p'' such that $n_2 = \lambda(p'')(1 - \alpha)$. The remaining part of the analysis is as follows:

$$\begin{aligned}
&\Rightarrow \left[(p - v_2)(\lambda'(p)(1 - \alpha))^2\left[\left(\frac{n_2}{\lambda(p)(1 - \alpha)} - 1\right)\right] \right]' \\
&= \left[(\lambda'(p)(1 - \alpha))^2 + 2\lambda''(p)(1 - \alpha)^2\lambda'(p)(p - v_2) \right] \left(\frac{n_2}{\lambda(p)(1 - \alpha)} - 1 \right) \\
&+ \left(\frac{-\lambda'(p)(1 - \alpha)n_2}{(\lambda(p)(1 - \alpha))^2} \right) (p - v_2)(\lambda'(p)(1 - \alpha))^2 > 0 \text{ since } n_2 > \lambda(p)(1 - \alpha)
\end{aligned}$$

Since it is an increasing function for all the price values larger than p' , at a specific price value, $(p - v_2)(\lambda'(p)(1 - \alpha))^2 \left[\left(\frac{n_2}{\lambda(p)(1 - \alpha)} - 1 \right) \right]$ will be equal to $(1 - \alpha)b$. This means, until that price, $(p - v_2)\lambda'(p)(1 - \alpha)P_2(n_2)$ is decreasing and after passing that price, it will increase forever. We can highlight the rate of increase of the function $\sum_{y=0}^{n_2} P_2(y) - 1$ is $(1 - \alpha)bP_2(n_2)$ and until $(p - v_2)(\lambda'(p)(1 - \alpha))^2 \left[\left(\frac{n_2}{\lambda(p)(1 - \alpha)} - 1 \right) \right]$ reaches $2(1 - \alpha)b$, its rate of increase will be larger than the rate of increase of the latter. Then, if we set p''' to the price making $(p - v_2)(\lambda'(p)(1 - \alpha))^2 \left[\left(\frac{n_2}{\lambda(p)(1 - \alpha)} - 1 \right) \right]$ equal to $2(1 - \alpha)b$, we can say that if these two functions do not intersect until p''' , they can never intersect in the future. The reason for this result is that $(p - v_2)\lambda'(p)(1 - \alpha)P_2(n_2)$ will increase faster than $\sum_{y=0}^{n_2} P_2(y) - 1$. However, it is impossible since they will intersect at the largest price a/b . Then, there should be a threshold price p' such that if the optimal price is less than this threshold price, it is increasing in the retailer's inventory level and if the optimal price is larger than this threshold price, it is decreasing.

In this chapter, we have analyzed the effect of the changes in some factors on the optimal price and the optimal profit in the single-period and multi-period problem. Firstly, we have presented the condition which should be satisfied for the objective function of the single-period problem to have only one local maximum to facilitate our job. Secondly, we have evaluated what price protection changes for the manufacturer when setting the retail price and whether a contract of this sort is profitable or not. Afterwards, we have started to analyze the effect of problem parameters. We have showed that the optimal profit is increasing in salvage values and manufacturers inventory level, decreasing in initial retail price, reimbursement rate, retailers inventory level and price sensitivity of demand. Moreover, we have showed that the optimal price is increasing in salvage values and decreasing in initial retail price, reimbursement rate, retailers inventory level when the optimal price is larger than initial retail price and decreasing in price sensitivity of demand in any case. The other result is that the optimal price does not change in initial retail price when it is larger than initial retail price. The most interesting result of the analysis is that the manufacturer encounters a trade-off on how the optimal price changes in reimbursement rate and retailers inventory level when it is less than initial retail price and in manufacturers inventory level

in any case.

In the following section, we analyze how the optimal price changes when it is equal to the initial retail price at which the objective function of the single-period problem is non-differentiable.

4.2 Optimality at the Non-differentiable Point

p_0

When evaluating how the optimal price changes with respect to the problem parameters, we took into consideration the relationship between the derivatives of the functions having only the changing parameter differently. However, since the actual function consists of two different complementary functions whose derivatives take different values at p_0 , it is non-differentiable at that point and we cannot say the derivative takes the value of 0 if p_0 is the optimal solution. Furthermore, for p_0 to be the optimal price of a specific problem setting, there are some different cases on the contrary to the case for the other prices. Throughout the analysis, let $p_{f_1}^*$ and $p_{f_2}^*$ be the optimal prices for the functions $f_1(p)$ and $f_2(p)$ we have defined before, respectively. In this subsection, we will assess the behavior of the optimal price for the case in which $p_{f_1}^* \geq p_0$ and $p_{f_2}^* \leq p_0$ if it is currently p_0 . However, there is one combination which is not likely to appear because of the structure of the model. The equality cannot occur since in Proposition 1-ii we have proven that if $p_{f_2}^*$ is less than or equal to p_0 , at that point $f_1(p)$ should be increasing.

The first two parameters we will consider are the manufacturer's and the retailer's salvage values. Since the change of the optimal price with respect to these two parameters are the same, we can evaluate them together. We have already proven that the local maximum points of $f_1(p)$ and $f_2(p)$ shift right as the salvage values increase. Then, $p_{f_1}^*$ will always stay above p_0 and $p_{f_2}^*$ will shift right towards p_0 . Then, whenever $p_{f_2}^*$ passes the threshold p_0 , it will start determining the optimal

price since the local maximum point takes a position in the active zone for the actual function. If $p_{f_2}^*$ is equal to p_0 at the beginning, then the optimal price will increase.

The next parameter to be examined is the retailer's initial sales price (p_0). We know that $p_{f_1}^*$ does not change but $p_{f_2}^*$ increases as p_0 decreases. Then, like in the case for the salvage values, $p_{f_1}^*$ will stay larger than or equal to p_0 and $p_{f_2}^*$ will shift right towards p_0 . Until $p_{f_2}^*$ and p_0 meet at a specific point, the optimal price will decrease since it is equal to p_0 . Again, if $p_{f_2}^*$ is equal to p_0 at the beginning, then the optimal price will not stay equal to p_0 .

The other parameter is the reimbursement rate (γ). In this case, we will base our analysis on Corollary 1 and Corollary 2. We know that the relationship between the optimal price for the model with reimbursement and the optimal price for the model without reimbursement determines how the optimal price changes by Corollary 1. Afterwards, we have generalized this argument to both of the constituent functions and put forward that the optimal price for the model without reimbursement is an upper or lower bound for $p_{f_1}^*$ and $p_{f_2}^*$. Let p_{wr}^* denote the optimal price of the model without reimbursement. p_{wr}^* cannot be less than $p_{f_2}^*$ since we expect $p_{f_2}^*$ to increase as γ decreases, which necessitates p_{wr}^* being larger than $p_{f_2}^*$ by Corollary 2. Then, there are two possible locations for this optimal price. It can be either between $p_{f_1}^*$ and $p_{f_2}^*$ or larger than both of them. If p_{wr}^* is both between $p_{f_1}^*$ and $p_{f_2}^*$ and less than p_0 , we can say that $p_{f_2}^*$ can never pass the threshold p_0 since its upper bound is below that level. Then, $p_{f_1}^*$ will determine the optimal solution if it can get below p_0 since it always decreases approaching a price larger than or equal to p_{wr}^* . However, until it hits p_0 , the optimal price will keep being p_0 if it is not equal to p_0 at the beginning. In the other case, both of $p_{f_1}^*$ and $p_{f_2}^*$ will increase and p_{wr}^* will be an upper bound for both of the functions. This time, everything is like in the situation where the salvage values increase. If p_{wr}^* is equal to p_0 , it is certain that the optimal price will never be different from p_0 . However, notice that p_{wr}^* cannot be equal to p_0 if $p_{f_2}^*$ is equal to p_0 . If p_{wr}^* and $p_{f_1}^*$ are equal to p_0 , then $p_{f_1}^*$ will not change and since p_{wr}^* is an upper bound for $p_{f_2}^*$, the optimal solution will never be different from p_0 .

For the manufacturer's and the retailer's inventory level, the situation is more

complicated but we can still comment on the behavior of the optimal price. However, since p' values are dependent on the current inventory level, we should keep track of the changes of the local maximum points.

In the following chapter, we present a numerical experiment.

Chapter 5

Numerical Experiments

In this section, we report the results of a numerical study that investigates the effect of various parameters on the optimal prices set by the manufacturer and the corresponding profits. These results attempt to explain various trade-offs that the manufacturer faces when making pricing decisions and there is a price-protection contract in place between the manufacturer and the retailer. We present our results in two parts. In the first part, we present the results for the single period case, where most of the relationships are already shown analytically in Chapter 4. In the second part, we present the results for the multiple period case.

5.1 Single-period Problem

We first study the effect of manufacturer's salvage value on manufacturer's pricing decision and the resulting expected profit. We have already proven in Proposition 3 that the manufacturer tends to set a higher price under a higher salvage value. Since the value of the unsold products at the end of the selling horizon is higher, the manufacturer avoids setting lower prices and sells less during the regular season. Likewise, the optimal expected profit is shown to increase in the manufacturer's salvage value. In Table 5.1 and Table 5.2, we provide results for the

Table 5.1: The effect of manufacturer's salvage value for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

v_1	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
0	2.67	-3.49	2.70	6.18	3.26	-6.85
0.25	2.75	-2.54	2.79	7.15	3.29	-6.75
0.50	2.84	-1.55	2.88	8.15	3.33	-6.65
0.75	2.93	-0.54	2.98	9.18	3.36	-6.55
1.00	3.01	0.49	3.07	10.24	3.40	-6.44
1.25	3.10	1.57	3.16	11.33	3.44	-6.32
1.50	3.19	2.67	3.25	12.45	3.49	-6.20
1.75	3.28	3.80	3.34	13.61	3.50	-6.07
2.00	3.38	4.97	3.43	14.79	3.50	-5.95
2.50	3.50	7.39	3.50	17.23	3.50	-5.70
3.00	3.57	9.85	3.66	19.72	3.52	-5.45
3.50	3.82	12.44	3.88	22.36	3.74	-5.16
4.00	4.07	15.20	4.11	25.16	3.98	-4.79

linear demand case ($\lambda(p) = 10 - 2p$) and exponential demand case ($\lambda(p) = e^{2.5-p}$) respectively.

The rates of change in the optimal prices and the resulting optimal profits vary by the other problem parameters. Whether the retailer's salvage value is high or low, the increases both in the optimal price and in the optimal profit are almost the same when the manufacturer's salvage value increases. Likewise, the initial sales price is not very effective. The reimbursement rate does not have a significant impact on the change of the optimal profits but a higher reimbursement rate lead to a smaller amount of increase in the optimal price. Low price sensitivity of demand triggers the increase in the optimal price but is not effective on the optimal profit. The change in the optimal price increases till the manufacturer's inventory level reaches a certain number but afterwards it gets balanced. On the contrary, the change in the optimal profit is much higher for the manufacturer's higher inventory levels.

The second parameter we examine is the retailer's salvage value. We have

Table 5.2: The effect of manufacturer's salvage value for the exponential demand rate function ($\lambda(p) = e^{2.5-p}$), $p_0=3.5$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

v_1	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
0	1.27	-10.74	1.26	-0.77	1.69	-11.50
0.25	1.45	-9.54	1.45	0.43	1.79	-11.30
0.50	1.63	-8.25	1.64	1.73	1.90	-11.09
0.75	1.81	-6.88	1.83	3.10	2.02	-10.85
1.00	2.00	-5.45	2.02	4.54	2.15	-10.58
1.25	2.19	-3.96	2.20	6.03	2.29	-10.29
1.50	2.37	-2.43	2.38	7.56	2.44	-9.98
1.75	2.56	-0.86	2.57	9.13	2.61	-9.64
2.00	2.74	0.74	2.75	10.73	2.78	-9.28
2.50	3.11	4.00	3.12	14.00	3.13	-8.50
3.00	3.49	7.35	3.49	17.35	3.49	-7.65
3.50	3.64	10.72	3.65	20.72	3.64	-6.78
4.00	4.14	14.13	4.14	24.13	4.14	-5.87

already proven in Proposition 4 that the optimal price and the optimal profit increase in the retailer's salvage value. When the retailer's salvage value increases, the manufacturer is less worried about the reimbursement cost arising from the retailer's unsold products at the end of the selling horizon. In this case, it does not refrain from setting higher prices to earn more. Likewise, the optimal profit increases since at any price, the reimbursement cost at the end of the horizon is less. In Table 5.3 and Table 5.4, we provide results for the linear demand rate case ($\lambda(p) = 10 - 2p$) and exponential demand rate case ($\lambda(p) = e^{2.5-p}$), respectively.

The level of the manufacturer's salvage value has no important effect over the change in the optimal price and the optimal profit when the retailer's salvage value increases. Although the initial sales price seems not to have a profound impact on the optimal price and the optimal profit, only if the manufacturer's inventory level is very low, the change in the optimal price is much higher for low values of the initial sales price. A high reimbursement rate impose a high amount of change in the optimal price and this is even more overt when the

Table 5.3: The effect of retailer's salvage value for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_1=1$, $\gamma=0.8$, $\alpha=0.7$

v_2	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
0	2.89	-4.12	2.98	9.46	3.23	-11.23
0.25	2.92	-2.97	3.00	9.66	3.28	-10.04
0.50	2.95	-1.82	3.02	9.85	3.32	-8.85
0.75	2.98	-0.66	3.05	10.05	3.36	-7.65
1.00	3.01	0.50	3.07	10.24	3.40	-6.44
1.25	3.05	1.66	3.09	10.44	3.44	-5.23
1.50	3.08	2.83	3.11	10.64	3.48	-4.01
1.75	3.11	4.00	3.13	10.85	3.50	-2.80
2.00	3.14	5.17	3.16	11.05	3.50	-1.58
2.50	3.20	7.54	3.20	11.47	3.50	0.86
3.00	3.26	9.91	3.25	11.89	3.50	3.30
3.50	3.33	12.30	3.30	12.32	3.57	5.75

Table 5.4: The effect of retailer's salvage value for the exponential demand rate function ($\lambda(p) = e^{2.5-p}$), $p_0=3.5$, $v_1=1$, $\gamma=0.8$, $\alpha=0.7$

v_2	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
0	1.75	-10.60	1.79	3.36	1.69	-15.79
0.25	1.81	-9.32	1.85	3.64	1.79	-14.50
0.50	1.87	-8.04	1.90	3.94	1.90	-13.20
0.75	1.94	-6.75	1.96	4.23	2.02	-11.90
1.00	2.00	-5.45	2.02	4.54	2.15	-10.58
1.25	2.06	-4.15	2.07	4.85	2.29	-9.27
1.50	2.13	-2.84	2.13	5.16	2.44	-7.94
1.75	2.19	-1.52	2.19	5.48	2.61	-6.62
2.00	2.26	-0.20	2.25	5.80	2.78	-5.29
2.50	2.38	2.46	2.38	6.46	3.13	-2.61
3.00	2.51	5.13	2.50	7.13	3.49	0.08
3.50	2.64	7.82	2.63	7.82	3.64	2.78

manufacturer's inventory level is low. Likewise, the retailer's salvage value affects the optimal profit more dramatically when the reimbursement rate is high. The price sensitivity of demand does not cause very big differences in the change of the optimal price and the optimal profit but only if the manufacturer's inventory level is low, lower price sensitivity shows greater impacts. When the manufacturer's inventory level increases, the difference in the optimal prices for the retailer's different salvage values falls. The rate of increase in the optimal price increases until the retailer's inventory level reaches a specific number and afterwards it gets balanced. The retailer's higher inventory level leads to a higher rate of increase in the optimal profit.

The third parameter we study is the initial sales price. We have already proven in Proposition 5 that the optimal price does not change in the initial sales price as long as it is under the initial sales price. Once the optimal price is larger than the initial sales price, it is decreasing in the initial sales price. In the same proposition, the optimal profit is shown to decrease in the initial sales price. The change in the optimal profit is very obvious since the reimbursement costs fall at any price. For the prices higher than the initial sales price, the manufacturer is committed to refunding the retailer only for the unsold products at the end of the horizon. Therefore, the increase in the initial sales price leads the manufacturer to prefer lower prices purposefully for the retailer to sell more products. For the prices less than the initial sales price, the change in the reimbursement cost per product is the same both throughout the horizon and at the end. Therefore, the optimality of the current price is maintained. In Table 5.5 and Table 5.6, we provide results for the linear demand rate case ($\lambda(p) = 10 - 2p$) and exponential demand rate case ($\lambda(p) = e^{2.5-p}$), respectively.

Table 5.5: The effect of the initial sales price for the linear demand rate function ($\lambda(p)=10-2p$), $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

p_0	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
5.00	3.01	-7.90	3.07	7.84	3.40	-14.84
4.75	3.01	-6.50	3.07	8.24	3.40	-13.44
4.50	3.01	-5.10	3.07	8.64	3.40	-12.04
4.25	3.01	-3.70	3.07	9.04	3.40	-10.64
4.00	3.01	-2.30	3.07	9.44	3.40	-9.24
3.75	3.01	-0.90	3.07	9.84	3.40	-7.84
3.50	3.01	0.50	3.07	10.24	3.40	-6.44
3.25	3.01	3.70	3.07	10.64	3.25	-5.07
3.00	3.00	3.90	3.00	11.04	3.07	-3.86
2.75	2.75	4.56	2.84	11.27	3.15	-2.69
2.50	2.77	5.69	2.86	11.45	3.23	-1.50
2.25	2.81	6.83	2.89	11.63	3.29	-0.31
2.00	2.85	7.97	2.91	11.82	3.36	0.89

When the manufacturer's salvage value is low, the change in the optimal price is high but it has no significant impact over the optimal profit. Only if the manufacturer's inventory level is low, the change in the optimal price is high when the retailer's salvage value is low. The change in the optimal profit is very balanced for the retailer's salvage value. The big values of the reimbursement rate have a drastic effect over the change in the optimal price and the optimal profit. There is a breaking point for the effect of the price sensitivity of demand. Whereas the change in the optimal price is much higher under high price sensitivity of demand when the manufacturer's inventory level is low, the lower values of the price sensitivity of demand changes the optimal price more dramatically when the manufacturer's inventory level is high. Although it can be stated that the change in the optimal profit is higher for higher price sensitivity of demand, it is usually balanced. As the manufacturer's inventory level increases, the rate of change firstly decreases but then gets balanced. We can make this inference for the case in which the optimal price is larger than the initial sales price because otherwise, the optimal price does not change. The change in the optimal profit

Table 5.6: The effect of the initial sales price for the exponential demand rate function ($\lambda(p) = e^{2.5-p}$), $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

p_0	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
5.00	2.00	-13.85	2.02	2.14	2.15	-18.98
4.75	2.00	-12.45	2.02	2.54	2.15	-17.58
4.50	2.00	-11.05	2.02	2.94	2.15	-16.18
4.25	2.00	-9.65	2.02	3.34	2.15	-14.78
4.00	2.00	-8.25	2.02	3.74	2.15	-13.38
3.75	2.00	-6.85	2.02	4.14	2.15	-11.98
3.50	2.00	-5.45	2.02	4.54	2.15	-10.58
3.25	2.00	-4.05	2.02	4.94	2.15	-9.18
3.00	2.00	-2.65	2.02	5.34	2.15	-7.78
2.75	2.00	-1.25	2.02	5.74	2.15	-6.38
2.50	2.00	0.15	2.02	6.14	2.15	-4.98
2.25	2.00	1.55	2.02	6.54	2.15	-3.58
2.00	2.00	2.95	2.00	6.94	2.00	-2.20

is high when the manufacturer's inventory level is low. As the rate of change in the optimal profit is higher for the retailer's higher inventory levels, the rate of change in the optimal price increases until the inventory level reaches a certain number and then gets balanced.

The fourth parameter we discuss is the reimbursement rate. We have already proven in Proposition 6 that the optimal profit decreases in the reimbursement rate. Since the reimbursement cost per product increases, this result can be expected. However, the case is much more complicated for the optimal price. We have shown in the same proposition, when the optimal price is less than the initial sales price, the manufacturer encounters a trade-off. Although setting a higher price will compensate for the increase in the reimbursement cost per product within the period since the retailer sells less, at the end of the period it causes higher number of unsold products for the retailer which means a higher reimbursement cost. This trade-off is reflected by the equation from which we obtain the threshold price. The case in which the optimal price is larger than the

Table 5.7: The effect of the reimbursement rate for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\alpha=0.7$

γ	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
1.00	3.011	-2.52	3.080	9.66	3.37	-9.48
0.90	3.014	-1.01	3.077	9.95	3.39	-7.96
0.80	3.018	0.50	3.073	10.24	3.40	-6.44
0.70	3.021	2.01	3.066	10.53	3.42	-4.92
0.60	3.022	3.52	3.062	10.83	3.44	-3.40
0.50	3.024	5.03	3.058	11.12	3.45	-1.88
0.40	3.025	6.54	3.054	11.41	3.47	-0.35
0.30	3.027	8.05	3.044	11.70	3.50	1.17
0.20	3.028	9.56	3.036	11.99	3.50	2.69
0.10	3.029	11.07	3.032	12.29	3.51	4.22
0	3.029	12.58	3.029	12.58	3.57	5.75

initial sales price is very clear since the manufacturer is not committed to refund the retailer in the corresponding period. In Table 5.7 and Table 5.8, we provide results for the linear demand case ($\lambda(p) = 10 - 2p$) and exponential demand case ($\lambda(p) = e^{2.5-p}$), respectively.

In Table 5.7, when evaluating how the optimal price changes in the reimbursement rate, we compare the current optimal price to the threshold price we obtain from the equation presented in Proposition 6. As we have proven, the threshold price does not depend on the reimbursement rate but the retailer's inventory level. For example, the threshold price is 3 for 7 products in stock and it is 3.219 for 2 products in stock. As we can see in Table 5.7, when the optimal price is larger than the threshold price, it increases when the reimbursement rate decreases. For the exponential demand function, the case could be different but in Table 5.8, we still have one threshold price. In this table, the threshold price is 2 for 7 products in stock and it is 2.056 for 2 products in stock. This time we have an example showing that the optimal price does not change in the reimbursement rate when it is equal to the threshold price. However, even if we had more than one threshold price, we could still define how the optimal price changes in different intervals

Table 5.8: The effect of the reimbursement rate for the exponential demand rate function ($\lambda(p) = e^{2.5-p}$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\alpha=0.7$

γ	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
1.00	2.00	-8.85	2.017	3.63	2.139	-13.99
0.90	2.00	-7.15	2.016	4.09	2.143	-12.28
0.80	2.00	-5.45	2.015	4.54	2.148	-10.58
0.70	2.00	-3.75	2.013	4.99	2.153	-8.88
0.60	2.00	-2.05	2.012	5.44	2.158	-7.18
0.50	2.00	-0.35	2.010	5.89	2.163	-5.48
0.40	2.00	1.35	2.008	6.35	2.169	-3.78
0.30	2.00	3.05	2.007	6.80	2.176	-2.08
0.20	2.00	4.75	2.005	7.25	2.183	-0.38
0.10	2.00	6.45	2.002	7.70	2.190	1.32
0	2.00	8.15	2.000	8.15	2.198	3.03

since the sign changes at each one of the threshold prices. We have one more way to infer how the optimal price changes in the reimbursement rate as we have stated in Corollary 1. It is evident that if we know the optimal prices of the problem under no price protection which is simpler to solve compared to the problem under price protection since only the manufacturer is involved in this case, we do not need to calculate the threshold prices. As we can see, when the optimal price of the problem under price protection is less than the corresponding optimal price of the problem under no price protection, it increases as the reimbursement rate decreases. We also know from Corollary 2 that the optimal price of the problem under no price protection is an upper bound for the optimal price of the problem under price protection. That is, if we keep decreasing the reimbursement rate from 0.1, the optimal price will converge to a value less than or equal to the optimal price of the problem under no price protection. For example, in Table 5.7, when each one of the manufacturer and the retailer have 7 products in stock, the optimal price converges to 3.029 which is the optimal price under no price protection, in other words, when the reimbursement rate is 0.

When the manufacturer's inventory level is low, the manufacturer's high salvage values increases the effect of the reimbursement rate over the optimal price but mostly the effect is higher for lower values. However, it has no significant impact over the optimal profit. When the manufacturer's inventory level is low, although the retailer's low salvage values leads to high rates of change in the optimal price, this change is balanced for the manufacturer's high inventory levels. The change in the optimal profit is much higher for the retailer's smaller salvage values. Both the optimal price and the optimal profit change more drastically when the initial sales price is high than they do when it is low. A low price sensitivity of demand imposes a high rate of change in the optimal price but although a high price sensitivity of demand seems to increase the rate of change in the optimal profit, it is almost balanced. As the manufacturer's inventory level increases, the change in the optimal price and the optimal profit decreases. Although the retailer's inventory level does not have a profound effect over the change in the optimal price, when it is high, the optimal profit changes more dramatically.

The fifth parameter is the retailer's inventory level. In Proposition 9, we have proven that the optimal profit and the optimal price being larger than the initial sales price decrease in the retailer's inventory level. The change in the optimal price is very obvious when it is larger than the initial sales price because the manufacturer is committed to refunding the retailer only for the unsold products at the end of the period. The manufacturer tends to set a lower price in purpose for enabling the retailer to sell more which decreases the reimbursement cost. However, the case is more complicated when the optimal price is less than the initial sales price. Like in the case for the reimbursement case, the manufacturer faces a trade-off. This time, setting a lower price will increase the reimbursement cost within the period. This trade-off is reflected by the equation in Proposition 10 from which we obtain the threshold price. Table 5.9 and 5.10 show the examples for the linear and the exponential distributions respectively.

As we have shown in Proposition 10, the optimal price being less than the initial sales price requires calculating the threshold price to determine how the optimal price changes in the retailer's inventory level. We also know that the threshold price should be updated for each inventory level. For example, in

Table 5.9: The effect of retailer's inventory level for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_1=1$, $v_2 = 1$, $\gamma=0.8$, $\alpha=0.7$

n_2	$n_1 = 1$		$n_1 = 2$		$n_1 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
1	3.5000	2.38	3.5000	4.93	3.1000	11.71
2	3.5000	0.84	3.4700	3.39	3.0700	10.24
3	3.5000	-1.04	3.4200	1.52	3.0400	8.43
4	3.4910	-3.01	3.4100	-0.45	3.0280	6.48
5	3.4897	-5.01	3.4090	-2.44	3.0250	4.49
6	3.4893	-7.00	3.4080	-4.44	3.0210	2.50
7	3.4884	-9.01	3.4070	-6.44	3.0190	0.50
8	3.4873	-11.01	3.4060	-8.44	3.0170	-1.50
9	3.4862	-13.01	3.4050	-10.44	3.0140	-3.50
10	3.4850	-15.01	3.4040	-12.44	3.0110	-5.50
15	3.4831	-25.01	3.4010	-22.44	3.0107	-15.50
20	3.4809	-35.01	3.4005	-32.44	3.0102	-25.50
25	3.4802	-45.01	3.4002	-42.44	3.0100	-35.50

Table 5.10: The effect of retailer's inventory level for the exponential demand rate function ($\lambda(p) = e^{2.5-p}$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

n_2	$n_1 = 1$		$n_1 = 2$		$n_1 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
1	2.39800	0.07	2.20400	1.35	2.04900	6.47
2	2.33300	-1.88	2.16200	-0.59	2.01500	4.54
3	2.31900	-3.87	2.15000	-2.58	2.00300	2.55
4	2.31700	-5.87	2.14880	-4.58	2.00100	0.55
5	2.31680	-7.87	2.14860	-6.58	2.00080	-1.45
6	2.31670	-9.87	2.14850	-8.58	2.00060	-3.45
7	2.31650	-11.87	2.14840	-10.58	2.00050	-5.45
8	2.31630	-13.87	2.14830	-12.58	2.00030	-7.45
9	2.31620	-15.87	2.14820	-14.58	2.00020	-9.45
10	2.31610	-17.87	2.14810	-16.58	2.00010	-11.45
15	2.31606	-27.87	2.14805	-26.58	2.00008	-21.45
20	2.31602	-37.87	2.14802	-36.58	2.00005	-31.45
25	2.31600	-47.87	2.14800	-46.58	2.00002	-41.45

Table 5.9 in which we present the results for the linear demand rate function, the threshold price for 5 products in stock is 1.74 and since the optimal prices are larger than 1.74, the optimal price decreases in the retailer's inventory level. The threshold price for 7 products in stock is 1.55 but again the optimal prices decrease in the retailer's inventory level. That is, if we want to determine how the optimal price changes if the retailer has more products, we should proceed by increasing the inventory level by one at each step. The reason is that we can encounter some ups and downs in the optimal price. We can reach the same result for the exponential demand rate function but this time we can encounter more than one threshold price. For example, in Table 5.10 in which we present the results for the exponential demand rate function, the threshold prices for 5 products in stock is 1.19 and 34.02. As we can see, we have two threshold prices and the optimal prices are between these values. However, we can still comment on the change of the optimal price since we know that till the first threshold price, the optimal price is increasing in the retailer's price and the sign will change at each one of the threshold prices. As it seems in Table 5.10, the optimal price is decreasing in the retailer's inventory level when the retailer has 5 products.

The sixth parameter is the manufacturer's inventory level. In Proposition 7, we have proven that the optimal profit is increasing in the manufacturer's inventory level. This is very obvious because it will increase the expected profit at any price. In Proposition 8, the manufacturer is proven to come across a trade-off on pricing. This time even when the optimal price is larger than the initial sales price, the case does not change because the manufacturer is inclined to set a higher price for a lower inventory level to earn more but this also increases the number of unsold products for the retailer causing the reimbursement rate to increase, as well. When proving how the optimal price changes in Proposition 8, since we assumed that the demand rate function is linear, for the exponential distribution, we can expect more than two threshold values. For the same reason as in the case of decreasing γ , we can still reach a general statement for the exponential distribution similarly. Table 5.11 and 5.12 are the examples for the linear and exponential distributions.

In Table 5.11, the threshold price for 7 products in stock is 1.76 and since

Table 5.11: The effect of the manufacturer's inventory level for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

n_1	$n_2 = 1$		$n_2 = 2$		$n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
1	3.50000	2.38	3.50000	0.84	3.48000	-9.01
2	3.50000	4.93	3.47000	3.38	3.40000	-6.44
3	3.38000	6.83	3.32000	5.32	3.27000	-4.48
4	3.25000	8.33	3.21000	6.84	3.16000	-2.94
5	3.17000	9.57	3.13000	8.09	3.08000	-1.67
6	3.12000	10.67	3.09000	9.20	3.04000	-0.55
7	3.10000	11.71	3.07000	10.24	3.01000	0.50
8	3.09000	12.72	3.06000	11.26	3.00800	1.51
9	3.08700	13.73	3.05600	12.26	3.00500	2.52
10	3.08500	14.73	3.05200	13.26	3.00300	3.52
15	3.08080	19.73	3.05110	18.26	3.00060	8.52
20	3.08020	24.73	3.05040	23.26	3.00008	13.52
25	2.08006	29.73	3.05008	28.26	3.00001	18.52

Table 5.12: The effect of the manufacturer's inventory level for the exponential demand rate function ($\lambda(p) = e^{2.5-p}$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

n_1	$n_2 = 1$		$n_2 = 2$		$n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
1	2.39800	0.07	2.33300	-1.88	2.31600	-11.87
2	2.20400	1.35	2.16200	-0.59	2.14800	-10.58
3	2.10500	2.44	2.06900	0.50	2.05500	-9.49
4	2.06500	3.46	2.03100	1.53	2.01700	-8.46
5	2.05300	4.47	2.01800	2.54	2.00400	-7.45
6	2.05000	5.47	2.01500	3.54	2.00180	-6.45
7	2.04950	6.47	2.01470	4.54	2.00160	-5.45
8	2.04920	7.47	2.01450	5.54	2.00130	-4.45
9	2.04916	8.47	2.01420	6.54	2.00110	-3.45
10	2.04913	9.47	2.01409	7.54	2.00090	-2.45
15	2.04906	14.47	2.01406	12.54	2.00030	2.55
20	2.04902	19.47	2.01402	17.54	2.00005	7.55
25	2.04900	24.47	2.01400	22.54	2.00001	12.55

the optimal prices are larger than this value, the optimal prices are decreasing in the manufacturer's inventory level regardless of the retailer's inventory level. As in the case of the change of the optimal price in the retailer's inventory level, the threshold price depends on the manufacturer's inventory level. For example, the threshold price for 5 products in stock 2.047 and the optimal price is still decreasing in the manufacturer's inventory level. However, the fact that we can encounter some ups and downs in the optimal price necessitates increasing the inventory level by one and updating the threshold price at each step if the difference between inventory levels is larger than one. For the exponential demand rate function, we can have more than one threshold price. For example, if the manufacturer has 5 products in stock, the threshold prices are 1.25 and 31.57. Since we know that the optimal price is increasing in the manufacturer's inventory level till the first threshold price and the sign will change at each one of the threshold prices, the optimal price is decreasing when the manufacturer's inventory level increases from 5 to 6.

The last parameter which we study is the price sensitivity of demand. We have no proof over how it changes the optimal price and the optimal profit but we can make an inference from the tables shown below. As the price sensitivity increases, the optimal price and the optimal profit decrease. Table 5.13 and 5.14 are the examples for the linear and exponential distributions.

Table 5.13: The effect of the price sensitivity for the linear demand rate function ($\lambda(p) = 10 - bp$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

b	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
1.00	5.17	9.85	5.33	19.36	6.36	-1.00
1.25	4.14	6.39	4.31	15.90	5.00	-3.09
1.50	3.50	4.08	3.62	13.60	4.07	-4.47
1.75	3.38	2.11	3.44	11.79	3.50	-5.46
2.00	3.01	0.50	3.07	10.24	3.40	-6.44
2.25	2.73	-0.74	2.78	9.05	3.06	-7.31
2.50	2.51	-1.72	2.54	8.11	2.79	-8.00
2.75	2.32	-2.52	2.35	7.35	2.56	-8.57
3.00	2.17	-3.17	2.20	6.73	2.38	-9.04
3.25	2.04	-3.71	2.06	6.20	2.22	-9.43
3.50	1.93	-4.17	1.95	5.76	2.08	-9.76
3.75	1.84	-4.56	1.85	5.39	1.97	-10.05
4.00	1.75	-4.89	1.76	5.07	1.87	-10.30

Table 5.14: The effect of the price sensitivity for the exponential demand rate function ($\lambda(p) = e^{2.5-cp}$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

c	$n_1 = 7, n_2 = 7$		$n_1 = 7, n_2 = 2$		$n_1 = 2, n_2 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
1.00	2.00	-5.45	2.01	4.54	2.15	-10.58
1.25	1.80	-6.03	1.81	3.96	1.89	-11.09
1.50	1.67	-6.37	1.67	3.63	1.72	-11.40
1.75	1.57	-6.58	1.57	3.42	1.60	-11.59
2.00	1.50	-6.71	1.50	3.28	1.52	-11.72
2.25	1.44	-6.80	1.45	3.20	1.45	-11.80
2.50	1.40	-6.86	1.40	3.14	1.41	-11.86
2.75	1.36	-6.90	1.36	3.10	1.37	-11.90
3.00	1.33	-6.93	1.33	3.07	1.34	-11.93
3.25	1.31	-6.95	1.31	3.05	1.31	-11.95
3.50	1.29	-6.96	1.29	3.04	1.29	-11.96
3.75	1.27	-6.97	1.27	3.03	1.27	-11.97
4.00	1.25	-6.98	1.25	3.02	1.25	-11.98

In the following section, we evaluate the multi-period problem and present the results over how the optimal price and the optimal profit change with respect to different problem parameters. We also show the correlation between the single-period problem and the multi-period problem.

5.2 Multi-period Problem

In this section, we will present some results on how the optimal price and the optimal profit change with respect to some problem parameters in the multi-period problem.

The first parameter we will examine is the manufacturer's salvage value. The relationship from Proposition 3 that the optimal price is increasing in the manufacturer's salvage value in the single-period problem as long as the optimal price is not equal to p_0 still holds in the multi-period problem. This turns out to be very intuitive since the manufacturer is less worried about the unsold products at the end of the horizon and so it tends to keep the price high. Furthermore, we have already proven that the optimal expected profit is increasing in the manufacturer's salvage value at any period of the selling horizon. As in the single-period problem, at a specific price value, the manufacturer will earn more since the value of the unsold products at the end of the selling horizon is higher and this guarantees that the graph of the expected profit function will shift up. Then, regardless of how the optimal price changes, we can state the optimal expected profit is increasing in the manufacturer's salvage value.

Table 5.15 presents results for the linear demand function. In this table, we also have examples for the case in which the optimal price is equal to p_0 . The objective function for each one of the periods in the multi-period problem has two constituent functions as the objective function of the single-period problem does. We already know that this case occurs when the local maximums of these two functions are not included in the objective function. Therefore, the optimal price is expected to stay at the initial sales price till the local maximum of the

constituent function which is equivalent to the objective function for the prices larger than p_0 gets larger than it.

The second parameter we will examine is the retailer's salvage value. The relationship from Proposition 4 that the optimal price is increasing in the retailer's salvage value in the single-period problem as long as the optimal price is not equal to p_0 still holds in the multi-period problem. This turns out to be intuitive since the reimbursement cost per product at the end of the horizon is less and so the retailer's unsold products are less effective. Furthermore, we have already proven that the optimal expected profit is increasing in the retailer's salvage value at any period of the selling horizon. As in the single-period problem, at a specific price value, the manufacturer's expected profit will be higher since the reimbursement cost per product in the period is less and the total expected profit till the end of the horizon is higher. This guarantees that the graph of the expected profit function will shift up. Then, regardless of how the optimal price changes, we can state the optimal expected profit is increasing in the retailer's salvage value. Table 5.16 presents results for the linear demand function. In this table, we also have examples for the case in which the optimal price is equal to p_0 . The optimal price is expected to stay at the initial sales price till the local maximum of the constituent function which is equivalent to the objective function for the prices larger than p_0 gets larger than it.

The manufacturer and the retailer are expected to compromise on the terms of the price protection contract at the beginning and the only parameter to be specified is the reimbursement rate. In negotiations, the manufacturer should be stricter about the reimbursement rate if the retailer's salvage value is very low and it starts the horizon with a relatively high inventory level. Especially, if the retailer starts with a higher inventory level than the manufacturer's, it is inevitable that the manufacturer will be exposed to a large financial loss. In this case, for the rationality of the price protection contract, the manufacturer will be inclined to increase p_0 to increase the revenue. However, this will put the manufacturer in an even more vulnerable position since it increases the reimbursement cost per product at the end.

Table 5.15: The effect of manufacturer's salvage value in the multi-period problem for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

v_1	$n_1 = 5, n_2 = 10$		$n_1 = 10, n_2 = 10$		$n_1 = 10, n_2 = 5$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
	$t = 1$					
0	3.95	2.78	3.33	15.74	3.37	25.01
0.25	3.96	2.83	3.35	16.02	3.40	25.32
0.50	3.97	2.88	3.38	16.33	3.42	25.65
0.75	3.98	2.94	3.40	16.66	3.45	26.01
1.00	3.99	3.00	3.44	17.03	3.49	26.41
1.25	4.00	3.06	3.47	17.42	3.50	26.84
1.50	4.01	3.13	3.50	17.85	3.50	27.30
1.75	4.03	3.21	3.50	18.32	3.50	27.80
2.00	4.04	3.29	3.50	18.82	3.50	28.33
2.50	4.08	3.48	3.54	19.91	3.60	29.49
3.00	4.12	3.70	3.68	21.13	3.74	30.84
3.50	4.17	3.98	3.86	22.71	3.90	32.56
4.00	4.25	4.39	4.08	24.82	4.10	34.77
	$t = 2$					
0	3.64	0.72	3.02	9.42	3.06	19.00
0.25	3.65	0.81	3.06	9.92	3.10	19.53
0.50	3.67	0.91	3.11	10.46	3.15	20.10
0.75	3.69	1.01	3.15	11.04	3.20	20.71
1.00	3.71	1.13	3.20	11.67	3.25	21.36
1.25	3.73	1.24	3.26	12.35	3.30	22.07
1.50	3.76	1.37	3.32	13.07	3.36	22.83
1.75	3.78	1.51	3.38	13.86	3.42	23.64
2.00	3.81	1.66	3.45	14.71	3.49	24.52
2.50	3.87	1.99	3.50	16.58	3.50	26.41
3.00	3.94	2.36	3.60	18.52	3.64	28.41
3.50	4.03	2.84	3.83	20.83	3.85	30.78
4.00	4.16	3.50	4.07	23.62	4.08	33.61

Table 5.16: The effect of retailer's salvage value in the multi-period problem for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_1=1$, $\gamma=0.8$, $\alpha=0.7$

v_2	$n_1 = 5, n_2 = 10$		$n_1 = 10, n_2 = 10$		$n_1 = 10, n_2 = 5$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
	$t = 1$					
0	3.965	-3.060	3.380	12.130	3.450	25.160
0.25	3.974	-1.550	3.400	13.340	3.460	25.470
0.50	3.979	-0.040	3.410	14.560	3.470	25.780
0.75	3.985	1.470	3.420	15.790	3.480	26.090
1.00	3.991	3.000	3.440	17.030	3.490	26.410
1.25	3.996	4.520	3.450	18.270	3.500	26.730
1.50	4.002	6.060	3.460	19.520	3.500	27.050
1.75	4.008	7.590	3.480	20.770	3.500	27.380
2.00	4.013	9.130	3.490	22.030	3.500	27.710
2.50	4.025	12.210	3.500	24.580	3.500	28.380
3.00	4.036	15.310	3.500	27.140	3.500	29.060
3.50	4.048	18.420	3.500	29.730	3.500	29.760
	$t = 2$					
0	3.670	-5.040	3.120	6.340	3.190	19.820
0.25	3.680	-3.500	3.140	7.660	3.200	20.200
0.50	3.690	-1.960	3.160	8.990	3.220	20.580
0.75	3.700	-0.420	3.180	10.330	3.230	20.970
1.00	3.710	1.130	3.200	11.670	3.250	21.360
1.25	3.720	2.680	3.220	13.020	3.260	21.760
1.50	3.730	4.230	3.250	14.390	3.280	22.160
1.75	3.740	5.800	3.270	15.750	3.300	22.570
2.00	3.750	7.360	3.290	17.130	3.310	22.980
2.50	3.780	10.490	3.340	19.910	3.350	23.820
3.00	3.800	13.640	3.380	22.720	3.390	24.690
3.50	3.820	16.800	3.430	25.560	3.430	25.580

The third parameter we will examine is the initial sales price (p_0). The relationship from Proposition 5 that the optimal price is decreasing in the initial sales price in the single-period problem if it is larger than p_0 and the optimal price does not change if it is less than p_0 still holds in the multi-period problem. Furthermore, we have already proven that the optimal expected profit is larger for a smaller initial sales price at any period of the selling horizon. As in the single-period problem, at a specific price value, the manufacturer's expected profit will be higher since the reimbursement cost per product in the period is less and the total expected profit till the end of the horizon is higher. This guarantees that the graph of the expected profit function will shift up. Then, regardless of how the optimal price changes, we can state the optimal expected profit increases if the manufacturer decreases p_0 . Table 5.17 presents results for the linear demand function. In this table, we also have examples for the case in which the optimal price is equal to p_0 . The optimal price is expected to decrease and stay equal to p_0 till the local maximum of the constituent function which is equivalent to the objective function for the prices larger than p_0 becomes larger than it.

The fourth parameter we will examine is the reimbursement rate (γ). Like in the single-period problem, when the optimal price is less than p_0 , it behaves like there exists a threshold price p' such that the optimal price is decreasing in the reimbursement rate if it is larger than p' , it is increasing if it is less than p' and it does not change if it is equal to p' . However, we do not have an equation to obtain this threshold price but Corollary 1 seems to hold for the multi-period problem, as well. By that corollary, if we know the optimal prices of the problem under no price protection, we have a chance to comment on how the optimal price changes without needing the threshold price. As we can see, if the optimal price of the problem under no price protection is larger than the optimal price of the problem under price protection for a specific pair of inventory levels, the optimal price increases if the manufacturer decreases the reimbursement rate. We can also state that it is an upper bound for the optimal price and no matter how small the reimbursement rate is, the optimal price cannot be larger than it. Furthermore, just the opposite is true if the optimal price of the problem under no price protection is less than the optimal price under price protection.

Table 5.17: The effect of the initial sales price in the multi-period problem for the linear demand rate function ($\lambda(p)=10-2p$), $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

p_0	$n_1 = 5, n_2 = 10$		$n_1 = 10, n_2 = 10$		$n_1 = 10, n_2 = 5$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
	$t = 1$					
5.00	4.030	-8.390	3.420	5.080	3.470	20.470
4.75	4.030	-6.390	3.420	7.080	3.470	21.470
4.50	4.030	-4.390	3.420	9.080	3.470	22.470
4.25	4.030	-2.390	3.420	11.080	3.470	23.470
4.00	4.000	-0.420	3.420	13.080	3.470	24.460
3.75	3.970	1.340	3.420	15.070	3.470	25.450
3.50	3.990	3.000	3.420	17.030	3.470	26.410
3.25	4.011	4.610	3.270	18.780	3.330	27.100
3.00	4.015	6.170	3.310	20.170	3.370	27.500
2.50	4.022	9.190	3.350	22.580	3.380	28.090
2.00	4.029	12.240	3.380	25.000	3.400	28.690
	$t = 2$					
5.00	3.810	-10.610	3.200	-0.320	3.240	15.370
4.75	3.810	-8.610	3.200	1.680	3.240	16.370
4.50	3.810	-6.610	3.200	3.680	3.240	17.370
4.25	3.810	-4.610	3.200	5.680	3.240	18.370
4.00	3.810	-2.610	3.200	7.680	3.240	19.370
3.75	3.750	-0.640	3.200	9.670	3.240	20.360
3.50	3.710	1.130	3.200	11.670	3.240	21.360
3.25	3.740	2.810	3.200	13.650	3.240	22.340
3.00	3.750	4.400	3.010	15.410	3.070	23.030
2.50	3.770	7.460	3.070	18.060	3.110	23.780
2.00	3.790	10.550	3.120	20.680	3.150	24.510

Furthermore, we have already proven that the optimal expected profit is larger for a smaller reimbursement rate at any period of the selling horizon. As in the single-period problem, at a specific price value, the manufacturer's expected profit will be higher since the reimbursement cost per product in the period is less and the total expected profit till the end of the horizon is higher. This guarantees that the graph of the expected profit function will shift up. Then, regardless of how the optimal price changes, we can state the optimal expected profit increases if the manufacturer decreases γ . Table 5.18 presents results for the linear demand function.

The fifth parameter we will examine is the manufacturer's inventory level (n_1). In Proposition 8, we have proven that how the optimal price changes depends on the threshold prices we obtain from the presented equation. That is, whether the optimal price is smaller or larger than p_0 does not matter in this case and there is no definite argument on the change of the optimal price. The same thing holds for the multi-period problem, as well but mostly the optimal price is expected to be smaller for larger inventory levels. The reason is that the manufacturer tends to keep the price low to increase the demand and to sell more. However, if the optimal price is less than p_0 , we know that a lower price will mean a higher reimbursement cost at the end of each period since the number of the products sold by the retailer will increase and the reimbursement cost per product is higher. The case is not the same when the optimal price is larger than p_0 . This time, the manufacturer is committed to refunding the retailer only when it has no remaining products in stock at the end of the period. In this case, a lower price implies a larger probability of the manufacturer's inventory level hitting 0 and a smaller number of unsold products in the retailer's stock which can compensate for each other. Therefore, we expect the manufacturer to be more flexible when the optimal price is larger than p_0 . The flexibility can be said to be even more when the retailer's inventory level is high. When the manufacturer's inventory level increases, the change in the optimal price decreases and the optimal price is almost the same for the manufacturer's high inventory levels.

Furthermore, we have already proven in Proposition 7 that the optimal expected profit is larger for the manufacturer's larger inventory levels at any period

Table 5.18: The effect of the reimbursement rate in the multi-period problem for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\alpha=0.7$

γ	$n_1 = 5, n_2 = 10$		$n_1 = 10, n_2 = 10$		$n_1 = 10, n_2 = 5$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
	$t = 1$					
1.0	3.9780	-0.8200	3.4250	13.8100	3.4970	25.5300
0.9	3.9870	1.0900	3.4290	15.4200	3.4940	25.9700
0.8	3.9950	3.0000	3.4360	17.0300	3.4910	26.4100
0.7	4.0040	4.9100	3.4390	18.6400	3.4870	26.8500
0.6	4.0080	6.8300	3.4420	20.2500	3.4830	27.2900
0.5	4.0130	8.7500	3.4441	21.8600	3.4770	27.7300
0.4	4.0180	10.6800	3.4443	23.4800	3.4710	28.1700
0.3	4.0240	12.6100	3.4446	25.0900	3.4660	28.6100
0.2	4.0320	14.5400	3.4447	26.7100	3.4620	29.0600
0.1	4.0380	16.4800	3.4448	28.3300	3.4530	29.5000
0	4.0400	18.4300	3.4449	29.9500	3.4449	29.9500
	$t = 2$					
1.0	3.6850	-2.7700	3.2010	8.0800	3.2590	20.1900
0.9	3.6930	-0.8200	3.2040	9.8800	3.2550	20.7800
0.8	3.7140	1.1300	3.2080	11.6700	3.2520	21.3600
0.7	3.7270	3.0800	3.2130	13.4700	3.2490	21.9400
0.6	3.7360	5.0300	3.2150	15.2600	3.2460	22.5300
0.5	3.7480	6.9900	3.2160	17.0600	3.2440	23.1100
0.4	3.7650	8.9500	3.2190	18.8500	3.2410	23.7000
0.3	3.7780	10.9100	3.2220	20.6500	3.2380	24.2800
0.2	3.8030	12.8800	3.2240	22.4400	3.2350	24.8600
0.1	3.8140	14.8500	3.2260	24.2400	3.2300	25.4500
0	3.8270	16.8300	3.2280	26.0400	3.2280	26.0400

of the selling horizon. As in the single-period problem, at a specific price value, the manufacturer's expected profit will be higher since the manufacturer has more products to sell which increase the expected revenue and the total expected profit till the end of the horizon is higher. This guarantees that the graph of the expected profit function will shift up. Then, regardless of how the optimal price changes, we can state the optimal expected profit is increasing in the manufacturer's inventory level. Table 5.19 presents the results for the linear demand function. In this table, we also have examples for the case in which the optimal price is equal to p_0 . The optimal price is expected to stay equal to p_0 till the local maximum of one of the constituent functions gets included in the objective function. However, since the number of products is discrete, the change from p_0 to another price is more immense.

The sixth parameter we will examine is the retailer's inventory level (n_2). In Proposition 10, we have proven that how the optimal price changes depends on the threshold prices we obtain from the presented equation if the optimal price is less than p_0 . However, it is proven in Proposition 9, the optimal price is decreasing in the retailer's inventory level if it is larger than p_0 . That is, there is no definite argument on the change of the optimal price which is less than the initial sales price but there is on the change of the optimal price which is larger than the initial sales price. However, the case is more complicated for the multi-period problem. In this case, we can encounter increasing price even when the optimal price is larger than the initial sales price. The reasons are that if the manufacturer decreases the price at any period, then the probability of its inventory level hitting 0 will be higher and although the demand observed by the retailer increases, the retailer's higher inventory level might increase the expected number of unsold products at the end of the period. This condition increases the reimbursement cost arising from the assumption that the product is withdrawn from the market as soon as the manufacturer has no more products in stock at the end of any period. In Table 5.20, we can see the examples of this different case from the single-period problem.

Furthermore, we have already proven in Proposition 9 that the optimal expected profit is larger for the retailer's smaller inventory levels at any period of

Table 5.19: The effect of the manufacturer's inventory level in the multi-period problem for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

n_1	$n_2 = 1$		$n_2 = 5$		$n_2 = 10$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
	$t = 1$					
1	4.56	3.20	4.64	-4.36	4.64	-14.36
2	4.50	7.57	4.53	0.55	4.52	-9.43
3	4.34	11.48	4.34	5.03	4.34	-4.92
4	4.19	14.99	4.17	9.13	4.16	-0.79
5	4.04	18.16	4.00	12.85	3.99	3.00
6	3.90	21.01	3.84	16.23	3.82	6.45
7	3.77	23.59	3.69	19.27	3.66	9.59
8	3.65	25.91	3.55	21.98	3.50	12.41
9	3.53	27.99	3.50	24.36	3.50	14.88
10	3.50	29.86	3.49	26.41	3.44	17.03
15	3.26	36.78	3.19	33.66	3.10	24.67
20	3.19	42.04	3.13	38.92	3.01	30.04
25	3.18	47.06	3.13	43.93	3.01	35.05
	$t = 2$					
1	4.42	3.09	4.47	-4.42	4.47	-14.41
2	4.35	7.17	4.35	0.18	4.34	-9.81
3	4.17	10.72	4.13	4.25	4.12	-5.72
4	3.99	13.83	3.92	7.85	3.91	-2.09
5	3.82	16.54	3.73	11.03	3.71	1.13
6	3.66	18.92	3.54	13.81	3.52	3.95
7	3.52	21.01	3.50	16.18	3.50	6.36
8	3.50	22.82	3.43	18.17	3.40	8.39
9	3.43	24.38	3.33	19.87	3.29	10.14
10	3.36	25.77	3.25	21.36	3.20	11.67
15	3.19	31.39	3.08	27.11	3.02	17.52
20	3.17	36.41	3.07	32.14	3.00	22.56
25	3.17	41.41	3.07	37.14	3.00	27.56

the selling horizon. As in the single-period problem, at a specific price value, the manufacturer's expected profit will be higher since the retailer has less products to sell which decrease the reimbursement cost when the optimal price is less than the initial sales price, the retailer has less products remaining in stock when the manufacturer's inventory level hits 0 which decrease the reimbursement cost arising from the withdrawal of the product from the market and the total expected profit till the end of the horizon is higher. This guarantees that the graph of the expected profit function will shift up. Then, regardless of how the optimal price changes, we can state the optimal expected profit is larger for the retailer's smaller inventory levels. Table 5.20 presents the results for the linear demand function. In this table, we also have examples for the case in which the optimal price is equal to p_0 . The optimal price is expected to stay equal to p_0 till the local maximum of one of the constituent functions gets included in the objective function. However, since the number of products is discrete, the change from p_0 to another price is more immense.

The other perspective from which we can approach the multi-period problem is how the optimal price and the optimal profit evolve over time. By the nature of the prices of high-tech products, we can expect the optimal price to decrease as the selling season progresses to the end. The existence of a price protection contract does not change the behavior of the price over time. Likewise, the optimal profit is expected to be higher at the early part of the selling season for a specific pair of inventory levels. The reason is that the manufacturer has more time to sell its products and if the manufacturer and the retailer cannot sell a portion of the inventory in stock, then this leads to a decrease in the revenue due to a low salvage value and an increase in the reimbursement cost due to the retailer's leftover inventories. In Table 5.21, we can see the results for ten periods.

In this chapter, we have presented some numerical experiments over the change of the optimal price and the optimal profit in the multi-period problem. We numerically analyzed the model with respect to some problem parameters such as salvage values, initial retail price, reimbursement rate, inventory levels, price sensitivity of demand and time. Furthermore, we have taken an insight into how the rates of change in the optimal price and the optimal profit with respect to a

Table 5.20: The effect of retailer's inventory level in the multi-period problem for the linear demand rate function ($\lambda(p) = 10 - 2p$), $p_0=3.5$, $v_1=1$, $v_2 = 1$, $\gamma=0.8$, $\alpha=0.7$

n_2	$n_1 = 1$		$n_1 = 2$		$n_1 = 7$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
	$t = 1$					
1	4.56	3.20	4.04	18.16	3.50	29.86
2	4.56	1.49	4.04	17.42	3.50	29.55
3	4.61	-0.41	4.03	16.21	3.50	28.88
4	4.63	-2.37	4.01	14.64	3.50	27.82
5	4.64	-4.36	4.00	12.85	3.49	26.41
6	4.64	-6.36	4.00	10.94	3.46	24.74
7	4.64	-8.36	3.99	8.98	3.45	22.91
8	4.64	-10.36	3.99	6.99	3.44	20.98
9	4.64	-12.36	3.99	5.00	3.44	19.01
10	4.64	-14.36	3.99	3.00	3.44	17.03
15	4.64	-24.36	3.99	-7.00	3.43	7.03
20	4.64	-34.36	3.99	-17.00	3.43	-2.97
25	4.64	-44.36	3.99	-27.00	3.43	-12.97
	$t = 2$					
1	4.42	3.09	3.82	16.54	3.36	25.77
2	4.37	1.41	3.80	15.76	3.36	25.25
3	4.43	-0.47	3.77	14.49	3.33	24.31
4	4.47	-2.43	3.74	12.86	3.28	22.99
5	4.47	-4.42	3.73	11.03	3.25	21.36
6	4.47	-6.41	3.72	9.09	3.22	19.54
7	4.47	-8.41	3.71	7.12	3.21	17.63
8	4.47	-10.41	3.71	5.12	3.21	15.66
9	4.47	-12.41	3.71	3.13	3.20	13.67
10	4.47	-14.41	3.71	1.13	3.20	11.67
15	4.47	-24.41	3.71	-8.87	3.20	1.67
20	4.47	-34.41	3.71	-18.87	3.20	-8.33
25	4.47	-44.41	3.71	-28.87	3.20	-18.33

Table 5.21: The evolution of the optimal price and the optimal profit over time in the multi-period problem for the linear demand rate function ($\lambda(p) = 10 - bp$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

t	$n_1 = 5, n_2 = 10$		$n_1 = 10, n_2 = 10$		$n_1 = 10, n_2 = 5$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
	$t = 1$					
1	4.57	6.72	4.20	29.39	4.22	37.97
2	4.53	6.43	4.12	28.40	4.14	37.01
3	4.47	6.08	4.02	27.19	4.04	35.85
4	4.4	5.62	3.89	25.65	3.92	34.39
5	4.3	5.02	3.72	23.65	3.77	32.21
6	4.17	4.20	3.50	20.93	3.56	29.99
7	3.99	3.00	3.44	17.03	3.49	26.41
8	3.71	1.13	3.20	11.67	3.25	21.36
9	3.48	-2.10	3.04	4.95	3.06	14.88
10	3.08	-7.67	3.00	-2.48	3.00	7.52

specific problem parameter are affected by the other factors. Consequently, we have observed that the behaviors of the optimal price and the optimal profit do not change in different periods except for the behavior of the optimal price with respect to the retailer's inventory level. The other result we have obtained is that the manufacturer should determine how many units of inventory to sell to the retailer at the beginning of the selling season to secure its profitability.

The last parameter we will examine is the price sensitivity of demand. Like in the single-period problem, the optimal price and the optimal profit are decreasing in the sensitivity. Table 5.22 presents the results for the linear demand function.

Table 5.22: The effect of the price sensitivity in the multi-period problem for the linear demand rate function ($\lambda(p) = 10 - bp$), $p_0=3.5$, $v_1=1$, $v_2=1$, $\gamma=0.8$, $\alpha=0.7$

b	$n_1 = 5, n_2 = 10$		$n_1 = 10, n_2 = 10$		$n_1 = 10, n_2 = 5$	
	Optimal Price	Expected Profit	Optimal Price	Expected Profit	Optimal Price	Expected Profit
	$t = 1$					
1.00	8.00	21.19	6.60	46.43	6.66	55.52
1.25	6.40	13.97	5.28	34.99	5.34	44.08
1.50	5.34	9.17	4.40	27.37	4.45	36.46
1.75	4.58	5.72	3.75	21.83	3.82	30.97
2.00	3.99	3.00	3.44	17.03	3.49	26.41
2.25	3.51	0.74	3.06	12.80	3.10	22.30
2.50	3.24	-1.42	2.77	9.39	2.81	18.99
2.75	2.95	-3.24	2.54	6.62	2.57	16.29
3.00	2.71	-4.75	2.35	4.32	2.38	14.05
3.25	2.51	-6.03	2.19	2.39	2.21	12.17
3.50	2.33	-7.12	2.05	0.75	2.07	10.57
3.75	2.18	-8.06	1.93	-0.66	1.95	9.20
4.00	2.05	-8.88	1.83	-1.87	1.84	8.01
	$t = 2$					
1.00	7.49	17.60	5.98	36.73	6.04	46.15
1.25	5.99	11.10	4.78	27.21	4.85	36.64
1.50	4.99	6.78	3.98	20.87	4.05	30.30
1.75	4.27	3.66	3.50	16.08	3.50	25.63
2.00	3.71	1.13	3.20	11.67	3.25	21.36
2.25	3.40	-1.12	2.88	8.17	2.92	17.92
2.50	3.07	-3.12	2.62	5.38	2.65	15.19
2.75	2.80	-4.75	2.41	3.12	2.44	12.97
3.00	2.57	-6.11	2.24	1.25	2.26	11.13
3.25	2.39	-7.25	2.10	-0.31	2.11	9.60
3.50	2.23	-8.23	1.97	-1.63	1.99	8.30
3.75	2.09	-9.06	1.87	-2.76	1.88	7.19
4.00	1.97	-9.79	1.78	-3.73	1.79	6.23

Chapter 6

Conclusion

We study a problem in which there are a manufacturer and a retailer that sell the same product over multiple periods. The manufacturer determines the retail price in the market and the wholesale price in a specific period is determined by multiplying the retail price of that period by the discount factor compromised on at the beginning of the selling horizon. The manufacturer sells a specific number of products to the retailer at the beginning of the horizon at a wholesale price γp_0 which is the retail price (p_0) just before the selling horizon multiplied by a discount rate (γ). The manufacturer does not have new production and the retailer is not allowed to make any replenishment till the end of the horizon. A different version of price protection is incorporated to the model which necessitates refunding the retailer for the products sold at a retail price less than p_0 . The manufacturer is committed to refunding the retailer only a portion of the difference between the retail price and the initial sales price per product if the manufacturer sets a price less than the initial sales price. This portion is determined by the reimbursement rate (γ) which is a contract parameter these two parties should agree on at the beginning of the selling horizon. This means the problem can be defined in two different ways. If price protection is based on the wholesale price, γ stands for the discount rate and the manufacturer fully reimburses the retailer the difference between the current wholesale price and the initial wholesale price. However, the wholesale price is not to be considered since the retailer is not allowed to make any

replenishment during the selling horizon. Then, if it is based on the retail price, γ stands for the reimbursement rate like it does. These two parties observe a Poisson demand distribution whose rate is dependent on the retail price. If there is any unsold product at the end of the horizon, each one of the products has a salvage value both for the manufacturer and for the retailer. If the manufacturer's inventory is sold out in an intermediate period, this product is not sold in the market anymore and the retailer is reimbursed for the unsold products since the salvage value is assumed to be less than the initial sales price. Salvage values do not change over time.

First, we analytically analyze the model and derive a number of structural results. We show that as salvage values increase, the optimal price increases irrespective of whether the optimal price is less or more than the starting retail price. The decrease in the starting retail price is proven to lead to an increase in the optimal price if the optimal price is larger than it and a non-changing optimal price if the optimal price is less than it. If the optimal price is equal to the starting retail price, they will go down together until the optimal solution of the model that is equivalent to the model we study for the prices larger than the initial retail price becomes larger than the initial retail price. A decrease in the reimbursement rate creates a trade-off and a threshold is determined above which the optimal price increases and below which it decreases continuously. We also show that if we know the optimal prices for the problem without price protection, we can comment on the change of the optimal price by comparing the current optimal price to it. If the optimal price is equal to the starting retail price, then the same rule as for the previous parameters is proven to hold but this time there is a possibility that the optimal price never changes. For increasing inventory level of the manufacturer, we analyze the model for the linear demand rate function and prove that there are two thresholds one of which is the maximum price. Therefore, the change in the optimal price is proven to have a break at a specific point below which the optimal price increases and above which it decreases regardless of the location of the optimal price. For increasing inventory level of the retailer, the optimal price decreases when it is above the starting retail price and we again prove that there is a threshold below which it increases and above which it decreases when the demand rate function is linear and the optimal price is less than the starting

retail price. The optimal profit is shown to increase in all periods of the selling horizon when the salvage values and the manufacturer's inventory level increase, the starting retail price, the reimbursement rate and the retailer's inventory level decrease.

In an extensive numerical study, we provide examples for the behavior we show analytically for the single period problem. We also investigate the effect of various factors on optimal prices and profits for the multiple period problem.

A number of extensions are possible for this problem. One can incorporate the back-order cost at the manufacturer and the retailer to account for the loss of customer goodwill. The assumption saying that the product is not sold anymore if the manufacturer runs out of stock in an intermediate period might be relaxed by permitting the retailer to go on selling and incorporating a rule over the retail price after that period. One can also consider allowing the retailer to make replenishment during the selling horizon. However, an inventory replenishment policy for the retailer should be specified in advance. Furthermore, the retailer should be reimbursed whenever the wholesale price is reduced by the manufacturer instead of the retail price in this case. Then, the initial wholesale price will replace the initial retail price in this new problem but the relationship between them can be preserved for convenience.

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