T.R.
BOLU ABANT İZZET BAYSAL UNIVERSITY
INSTITUTE OF GRADUATE STUDIES
Department of Mathematics

ON FOURIER SERIES AND ITS EFFECTS TO CANTOR'S WORK

MASTER OF SCİENCE

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BOLU, JANUARY 2024

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ABSTRACT

ON FOURIER SERIES AND ITS EFFECTS TO CANTOR'S WORK
MSC THESIS
ABDULWAHHAB AHMAD A.WAHHAB
BOLU ABANT IZZET BAYSAL UNIVERSITY
INSTITUTE OF GRADUATE STUDIES
DEPARTMENT OF MATHEMATICS
(SUPERVISOR: PROF.DR.NİHAN UYGUN ERCAN)

BOLU, JANUARY 2024
(ix + 60)

The study explores the origin of trigonometric Fourier series (TFS), tracing back to Fourier's. Théorie analytique de la chaleur in 1822. Cantor's contributions led to the emergence of set theory. Chapter 1 introduces Fourier Analysis, focusing on trigonometric series theory. Chapter 2 delves into defining finite and infinite sets, highlighting Cantor's role and introducing the Continuum Hypothesis. Chapter 3 explores Cauchy and Dedekind's constructions of real numbers, proving no one to one correspondence with natural numbers. Chapter 4 outlines the axiomatic basis of mathematics using sets. The thesis offers a historical perspective on Cantor's early works, referencing S. R. Srivastava's paper on Cantor's discovery of Set Theory and Topology.

KEYWORDS: Trigonometric Fourier series, Set theory, Finite and infinite sets, infinite set transition, One-to-one correspondence.

ÖZET

FOURİER SERİLERİ VE CANTOR'UN ÇALIŞMALARINA OLAN ETKİSİ
YÜKSEK LİSANS TEZİ
ABDULWAHHAB AHMAD A.WAHHAB
BOLU ABANT İZZET BAYSAL ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI
(TEZ DANIŞMANI: PROF.DR.NİHAN UYGUN ERCAN)

BOLU, OCAK - 2024
(ix + 60)

Çalışma, Fourier serisine kadar uzanan trigonometrik Fourier serilerinin (TFS) kökenini araştırıyor. 1822'de Théorie analytique de la chaleur. Cantor'un katkıları küme teorisinin ortaya çıkmasına yol açtı. Bölüm 1'de trigonometrik seri teorisine odaklanan Fourier Analizi tanıtılmaktadır. 2. Bölüm, sonlu ve sonsuz kümelerin tanımlanmasını ele alıyor, Cantor'un rolünü vurguluyor ve Süreklilik Hipotezi'ni tanıtıyor. 3. Bölüm, Cauchy ve Dedekind'in doğal sayılarla birebir örtüşme olmadığını kanıtlayan gerçek sayı yapılarını araştırıyor. 4. Bölüm , kümeleri kullanan matematiğin aksiyomatik temellerini özetlemektedir. Tez, S. R. Srivastava'nın Cantor'un Küme Teorisi ve Topoloji keşfi hakkındaki makalesine atıfta bulunarak, Cantor'un ilk çalışmalarına tarihsel bir bakış açısı sunuyor

ANAHTAR KELİMELER: Trigonometrik Fourier serileri, Küme teorisi, Sonlu ve sonsuz kümeler, sonsuz küme geçişi, Bire-bir eşleşme.

TABLE OF CONTENTS

LIST OF FIGURES

Page

ACKNOWLEDGEMENTS

The author wishes to express his deepest gratitude to his supervisor Prof. Dr. Nihan Uygun Ercan for their guidance, advice, criticism, encouragements and insight throughout the research.

The author would also like to thank Prof. Dr. Zafer Ercan for his suggestions and comments

1. INTRODUCTION

In a cosmic ballet orchestrated by mathematics, the universal language of patterns and structures shapes our understanding of the world. From the celestial dance of planets to the symphony of subatomic particles, mathematics unveils the rules that govern the very fabric of the cosmos.

As the cosmos unfolded, mathematical principles became the essence of existence.Ancient civilizations, armed with basic counting systems, set the stage for the evolution of mathematics. Over time, algebraic and geometric methods emerged, transforming mathematics into an indispensable tool for unraveling the intricacies of the natural world.

” **Mathematics is the language with which God has written the universe**.”

Galileo Galilei

Within the grand tapestry of creation, mathematics emerges as the silent architect. It governs planetary movements, star orbits, and the symmetrical dance of galaxies. Precision embedded in mathematical principles shapes the cosmos, providing a systematic understanding of its fundamental laws.

Mathematics acts as a guiding light, revealing the intricacies of the natural world. From fractals in nature to the golden ratio in shells and flowers, mathematical patterns underscore the deep-rooted relationship between mathematics and nature since the dawn of time.

The evolution of mathematics mirrors human cognitive development, from basic counting to complex theories. It serves not only as a problem-solving tool but also as a mirror reflecting the growth of human thought, enabling comprehension and manipulation of the surrounding world.

Beyond theoretical abstraction, mathematics finds practical applications across scientific disciplines and technological advancements. Its synergy with physics, engineering, and computer science highlights its versatility, driving innovation and shaping the modern landscape of scientific discovery.

The study of mathematics extends beyond utility, delving into philosophical realms. Mathematical beauty and intrinsic harmony raise profound questions about the nature of reality and the order embedded in the universe. From elegant proofs to the coherence of mathematical systems, these aspects offer a glimpse into a sublime order that transcends mere calculation.

” Pure mathematics is, in its way, the poetry of logical ideas.”

Albert Einstein

The inception of set theory in the late 19th century marked a transformative moment in the landscape of mathematics, revolutionizing how mathematicians conceptualize and approach mathematical structures. Set theory has since become a foundational framework for modern mathematics, playing a pivotal role in various branches of the discipline.

Historically, set theory’s roots can be traced back to mathematicians such as Georg Cantor, Richard Dedekind, and Augustin-Louis Cauchy, who sought a more rigorous foundation for mathematical analysis, especially in dealing with the infinite and continuity.

The German mathematician Georg Cantor is recognized as the architect of set theory, a cornerstone upon which modern mathematics is built.

He revolutionized the mathematical landscape with pioneering contributions that served as the bedrock for the entire discipline. His indepth exploration of infinity and the introduction of transfinite numbers posed a formidable challenge to conventional mathematical paradigms. Cantor’s diagonal argument, coupled with the establishment of a hierarchical structure of infinities, marked pivotal advancements in the trajectory of set theory.

One of the pivotal innovations stemming from set theory is the formulation of a precise notation system. This system utilizes symbols such as ∪ (union), ∩ (intersection), and ∈ (belongs to) to elegantly express intricate relationships between sets. The elucidation of axiomatic systems, with a particular emphasis on Zermelo-Fraenkel set theory, provided a robust logical foundation for the systematic exploration of sets. Cantor's transformative work essentially demystified the concept of" philosophical infinity," rendering it a natural and manageable entity. Infinity, once perceived as a formidable monster, has been domesticated through Cantor's efforts.

This paradigm shift has underscored the realization that, while mathematics is an inherent aspect of human cognition, it requires a deliberate process of" construction."

In this context, the concept of a" set" emerged as a fundamental building block for mathematical structures. Cantor's legacy extends beyond numerical frameworks, as he fundamentally altered the philosophical underpinnings of mathematical thought, paving the way for a more nuanced and comprehensible understandingo finfinity and mathematical constructs.

Set theory addressed longstanding mathematical paradoxes, including Russell's paradox, and provided a more secure foundation for mathematical reasoning. Its influence extends across diverse mathematical disciplines, serving as the underpinning for mathematical logic, model theory, and finding applications in topology, analysis, and algebra.

The development of set theory was not without challenges and controversies. The continuum hypothesis, proposed by Cantor, led to debates about its status within set theory. Godel's incompleteness theorems and Cohen's work on the independence of the continuum hypothesis further added complexity to the field. Actually, the birth of set theory represents a watershed moment in the evolution of mathematics. From Cantor's profound insights into infinity to the development of formalized axiomatic systems, set theory has become an essential component of mathematical thinking. Its impact on various mathematical disciplines and its ability to provide a rigorous foundation for dealing with infinite structures underscore its enduring significance in the realm of mathematics. The journey from the late **19th** century to the present day reveals the richness and depth that set theory has brought to mathematical exploration and understanding.

One of the answers to the question of" What is mathematics?" is that mathematics is a sort of game. In this case, this game consists of two things, one is the players and the others are the rules. Nowadays it can be said that the players are being known "sets". The rules of mathematics came from logic. Also, the meaning of the sets is defined in a system of axioms which are some rules of the logic.

In 1638 Galileo observed the following: There is a one-to-one corresponding between N and {1, 4, 9 . . . n^2, . . . }, for example,

$$n \rightarrow n^2$$

but, those sets are not" equal". This was against human intuition and, it was the voice of the first formal notion of" infinity". Galileo had discussed this in his Concerning Two New Sciences shortly but stopped thinking any more on this subject. Galileo's observation has opened a door to infinity. In the literature, this is known as Galileo's paradox. Actually, the idea of Galileo goes back; to around 1302 Duns Scotus, comparing even numbers to whole numbers. Bolzano observed that Galileo's observation is a characteristic property of an infinite set. That is, every infinity set can be one-to-one corresponding with its proper subset, but this property had not been studied systematically up to Cantor's time.

The works of Cantor can be divided into three main parts: Number theory, analysis and set theory. He studied number theory until 1870. And then, by the suggestion of Eduard Heine started to study the representation of trigonometric series, a subject of analysis, which was to determine set A ⊂ R satisfying

$$\mathbf{S_A: \forall x \in A\ ,\ \frac{f(0)}{2} + \sum_{n=1}^{\infty}[f\ (n)\ \cos(nx)\ +\ g(n)\ \sin(nx)]\ = 0 \rightarrow f = 0, g = 0}$$

for all functions f: ω → IR and g: N → IR. Cantor proved that S_A is true for A = Q' (the irrational numbers set) but for A = Q (the rational numbers set). He questioned how although the sets Q and Q′ have many common properties their behavior on the trigonometric series was different. This has given a direction to Cantor to study the subsets of real numbers, and this concluded that there is a bijection between N and Q. In modern terminology this is denoted by

$$|N| = |Q|$$

This says that the number of the elements of the set natural numbers N (denoted by ℵ0) is as much as the set of rational numbers. In the “infinity” terms, it means that N and Q have the same type of infinity. Cantor proved also that there is no bijection between N and R. Nationally this is denoted by:

$$|N| < |IR|$$

In the set theory, this says that the "number of elements" of R (denoted by |R| or c) is bigger than” the number of elements” of N, hence

$$\aleph 0 = |N| < |IR| = c$$

We note that “the number of elements of a set” which is known as “cardinal number of a set” generalizes the notion of the number of elements of a finite set.

This is a revolutionary. This revolutionary led a theory: Set theory, in particular; Zermole (Z), Zermole-Fraenkel (ZF), and Zermole-Fraenkel with Axiom of choice (ZFC) set theory.

In the mathematics community, all agreed that the founder of the set theory was Cantor (1845-1918, a German mathematician). Cantor’s early work was on number theory which includes dissertation and Habilitation under Kummer. Then Cantor changed the direction of his work to the trigonometric series(TS) and on this subject, between 1870-1912, has published five papers. Those works, to Cantor, gave given opportunity to compare sets and then Cantor was able to give a definition of “infinity sets”. The notion of infinity sets forced mathematicians to give the definition of “sets”, to do this has taken about fifty years.

The study of trigonometric Fourier series (TFS) mainly is the problem of representing a given function by a TS, originated in problems of the theory of oscillations and the theory of heat conduction which was the subject of Fourier’s Theorie analtique de la chaleur, published in 1822. As mentioned in the previous paragraph Cantor’s works on this series have led to open a new area: set theory. Regarding this Fourier Analysis which deals with the theory of trigonometric series will be introduced in the first chapter.

Although it seems that to define “the finite set” is too “obvious”, in fact, it is not, so it is difficult as much as to give the “infinite set”. In the second chapter, it will not only be given the notion of an infinite set, but several types of infinities will be given also introduced and it will be discussed how Cantor passed to the infinity set.

In particular, we state the Continuum Hypothesis and it will be emphasized its central role in the construction of the set theory.

In chapter 3 after giving Cauchy and Dedekind constructions of real numbers it will be proven that one-to-one correspondence between the real numbers and the natural numbers does not exist.

In the last chapter 4, we will give some basic details about how mathematics is axiomatized in terms of “sets”.

We note that this thesis mainly contains the historical perspective of Cantor’s early works and its relation to “the birth of the set theory”. So most theorems are stated without proof or given with a sketch proof, not in the modern form. Most of those theorem can be generalized and proofs of them can be found in any classical books on that topic.

We also note that this thesis mainly is followed by the paper “How did Cantor Discover Set Theory and Topology?” by S. R. Srivastava [21].

2. TRIGONOMETRIC FOURIER SERIES

One of Cauchy'sresults, which is given in his book *Cours d'analyse* published in 1821, was that the sum of series function of a sequence of continuous functions is also continuous. In 1826, Abel gave a counterexample to show that Cauchy's result was incorrect. That counterexample was a series of sin x and cos$x$ functions which is now known Fourier series (FS). Although these series were first considered briefly by Daniel Bernoulli in the middle of the eighteenth century, in detail, they had been worked on by Joseph Fourier in his work on heat conduction in the early nineteenth century.

The TFS represents a periodic function(PF) of a sum of sine and cosine functions. This representation, which was introduced by Daniel Bernoulli in the 18^{th} century and extensively developed by Joseph Fourier in the 19^{th} century, plays a crucial role in the analysis of periodic phenomena [12].

The series is denoted by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{r} + b_n sin\frac{n\pi x}{r}\right),$$

where the coefficients a_n and b_n are calculated through integrals involving the function *f(x)*. It converges to *f(x)* under certain conditions and provides a powerful tool for approximating periodic functions in various fields [13].

Recall that the form of a partial differential equation

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}_1^2} + \cdots + \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}_\mathrm{n}^2},$$

where $u: U \times I \rightarrow \mathrm{IR}$ with $U \subset \mathrm{IR}^n$, where $(x_1, \cdots, x_n, t)$ denotes the general point of the domain $U \times I$, is known as a **heat equation**.

In this chapter, we introduce the notion of the FS with some basic results.

2.1 An introduction to Trigonometric Fourier Series.

Definition 2.1. *Let* $r > 0$ *and* f*:* $(a, b) \to \mathbb{R}$ *be a function such that for each the following integrals*

$$\int_a^b f(x) cos\frac{n\pi x}{r} dx \text{ and } \int_a^b f(x) sin\frac{n\pi x}{r} dx$$

exist for all $n \in \omega$*, where* $r = \frac{b-a}{2}$*. Then the series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{r} + b_n sin\frac{n\pi x}{r}\right),$$

where, for each $n \in \omega$

$$a_n = \frac{1}{r}\int_{-r}^{r} f(x)\, cos\frac{n\pi x}{r} dx, \quad b_n = \frac{1}{r}\int_{-r}^{r} f(x)\, sin\frac{n\pi x}{r} dx,$$

is known as the **trigonometric Fourier series(TFS) of** f *on the interval* $[a, b]$.

Although in the definition the domain is given in the form $[a, b]$, to understand the notion of the TFS, without loss of any generality, it is enough to study the interval $[-r, r]$ for $r > 0$ or $[0, r]$.

By the definition, the TFS of any Riemann integral function (in particular continuous function) exists.

Examples 2.2. *We give some examples (which are taken from* [17]*) of the TFS of functions as follows:*

i. *The TFS of the* f*:* $\mathbb{R} \to \mathbb{R}$*, given by* $f(x) = |x|$*, for each* $x \in [-\pi, \pi]$ *is*

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x.$$

Prove: *The general form of a TFS for a function f(x) for each* $x \in [-L, L]$ *is as follows:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nx}{2L}\right) + b_n \sin\left(\frac{2\pi nx}{2L}\right)\right]$$

where the coefficients of f(x), that is, a_0, a_n, b_n are obtained by:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{2n\pi x}{2L}\right)dx\ ,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{2n\pi x}{2L}\right)dx\ .$$

In this case, f(x) = |x| and L = π (as given in the problem). Let's calculate the coefficients:

Since f(x) = |x| is an even that symmetrically extends about the y-axis, b_n coefficients will be zero.

Now, let's find a_0 and a_n coefficients:

Since |x| is an even, so the integral over the symmetric interval [−π, π] simplifies to twice the integral over [0, π]:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx.$$

Now, let's solve this integral:

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

Now, let's calculate a_n

$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$ *Applying integral by parts:*

Let x = u then dx = du,

and $\cos nx\,.\,dx = dv \Rightarrow v = \frac{\sin nx}{n}$

$$a_n = \frac{2}{\pi}[u.v - \int v\,.\,du]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[x\frac{\sin nx}{n} - \int_0^{\pi} \frac{\sin nx}{n}.\,dx\right]$$

$$a_n = \frac{2}{\pi}\left[x.\frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin nx}{n}.\,dx = 0 + \frac{2}{\pi}.\frac{1}{n}\left[\frac{1}{n}\cos n\,x\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$a_n = \begin{cases} 0 \text{ , if n is even} \\ \frac{-4}{\pi(n)^2} \text{ , if n is odd} \end{cases}.$$

So we can write:

$$a_0 = \pi, \ a_n = \frac{-4}{\pi(2n-1)^2} \ , \ b_n = 0$$

$$\Rightarrow |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum \frac{\cos(2n-1)x}{(2n-1)^2}$$

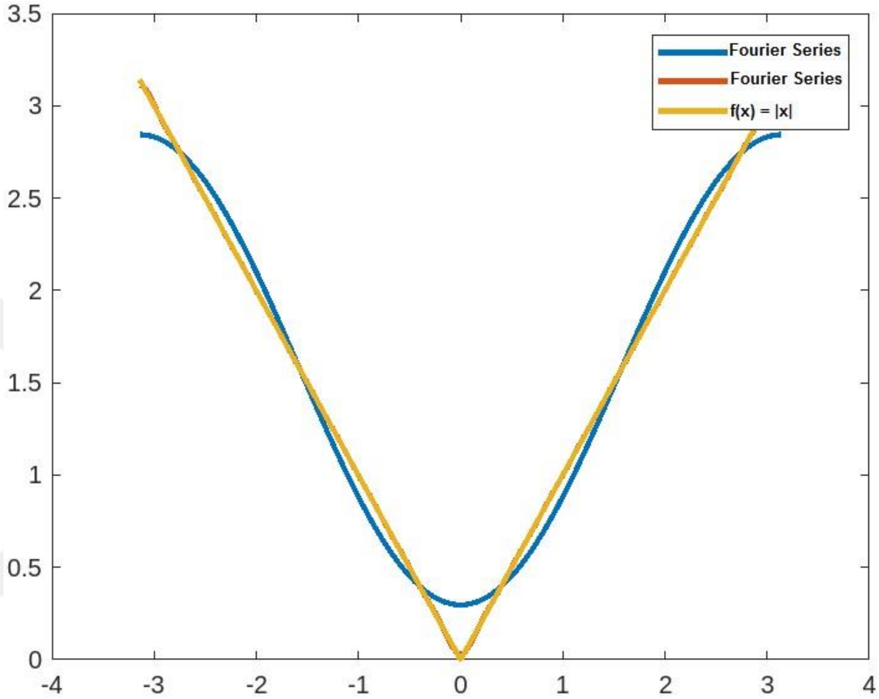

Figure 2.1. Fourier Representation for *f*(*x*) = |*x*|

ii. *The TFS of the function f:* R→R, *f*(*x*) = *x on the interval* [−4,4] *is*

$$\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{4}$$

Assume that f(-x) = -f(x) then f(x) is an odd. Then, a_n = 0, n = 0,1,2... and we can calculate

$$b_n = \frac{1}{r}\int_{-r}^{r} f(x) \sin\frac{n\pi x}{r} \, dx$$

$$\Rightarrow \mathrm{b_n} = \frac{2}{4} x \sin\frac{\mathrm{n}\pi x}{4}\,\mathrm{d}x$$

Applying integration by parts, let

$$x = \mathrm{u} \Rightarrow \mathrm{d}x = \mathrm{du}\,, \sin\frac{\mathrm{n}\pi x}{4}\mathrm{d}x = \mathrm{dv}$$

Then

$$\mathrm{v} = \frac{-4}{\pi\mathrm{n}}\cos\frac{\mathrm{n}\pi x}{4}$$

$$\Rightarrow \mathrm{b_n} = \frac{1}{2}\left[\mathrm{u.v} - \int \mathrm{v\,du}\right]$$

$$= \frac{1}{2}\left[\frac{-4}{\mathrm{n}\pi} x\cos\ \frac{\mathrm{n}\pi x}{4} + \frac{4}{\mathrm{n}\pi}\int \cos\ \frac{\mathrm{n}\pi x}{4}\mathrm{d}\,x\right]_0^4$$

$$= \frac{1}{2}\left[\frac{-4}{\mathrm{n}\pi}\mathrm{x}\cos\ \frac{\mathrm{n\pi x}}{4}\right]_0^4 + \frac{1}{2}\left[\frac{4}{\mathrm{n}\pi}\int\cos\ \frac{\mathrm{n\pi x}}{4}\mathrm{d\,x}\right]_0^4$$

$$\mathrm{b_n} = \frac{-8(-1)^{\mathrm{n}}}{\pi\mathrm{n}} + \frac{1}{2}\left[\frac{4}{\pi\mathrm{n}}.\frac{4}{\pi\mathrm{n}}\sin\ \frac{\mathrm{n\pi x}}{4}\right]_0^4$$

$$\mathrm{b_n} = \frac{-8(-1)^{\mathrm{n}}}{\pi\mathrm{n}} + \frac{8}{\pi^2\mathrm{n}^2}(0)$$

$$\mathrm{b_n} = \frac{8(-1)^{\mathrm{n}+1}}{\mathrm{n}\pi}$$

By substitution in the main formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nx}{2L}\right) + b_n\sin\left(\frac{2\pi nx}{2L}\right)\right]$$

$$\mathrm{f(x)} = \frac{8}{\pi}\sum_{\mathrm{n}=1}^{\infty}\frac{(-1)^{\mathrm{n}+1}}{\mathrm{n}}\sin\ \frac{\mathrm{n\pi x}}{4}.$$

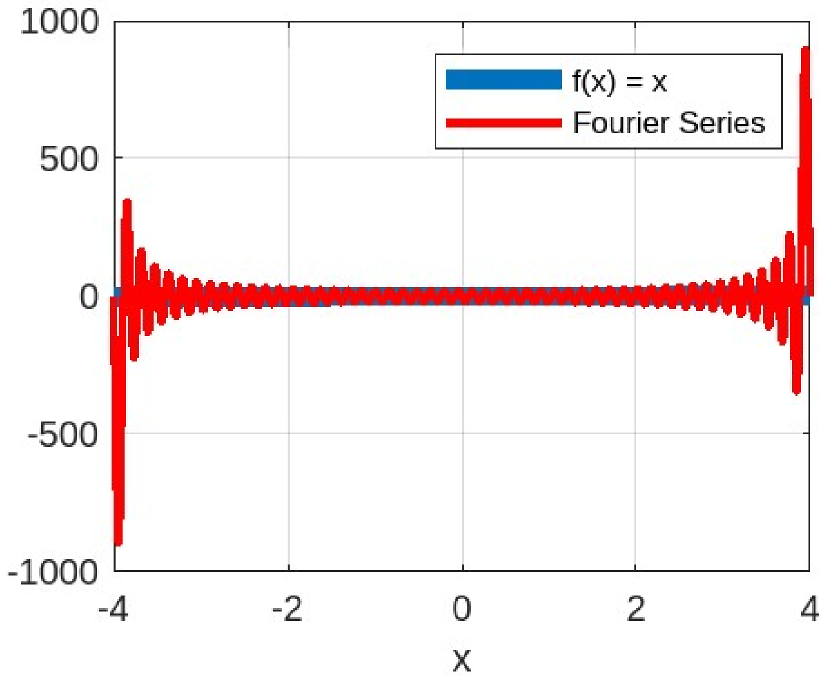

Figure 2.2. Fourier Representation for *f*(*x*) = *x*

iii. *Let* f: $[-\pi, \pi] \to \mathbb{R}$ *be given by* $f(x) = \pi\chi_{[-\pi,0]} + x\chi_{[0,\pi]}$. *The TFS of f on the interval* $[-\pi, \pi]$ *is*

$$\frac{3\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos n\,x - \frac{1}{n}\sin n\,x\right].$$

Prove:

$$f(x) = \begin{cases} \pi, & \text{if } x \in [-\pi, 0] \\ x, & \text{if } x \in [0, \pi] \end{cases}$$

Let's calculate Fourier Coefficients;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi d\,x + \int_{0}^{\pi} x\,dx\right]$$
$$= \frac{1}{\pi}[\pi x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left[\pi^2 + \frac{\pi^2}{2}\right] = \frac{3\pi}{2}\ .$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n\,.\,cdx$$
$$= \frac{1}{\pi}\int_{-\pi}^{0} \pi\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

Assume:

$$I_1 = \int_{-\pi}^{0} \pi \cos n\, x dx\, ,\ I_2 = \int_{0}^{\pi} x \cos n\, x dx.$$

$$\mathrm{I}_1 = \left[\frac{\pi}{n}\sin nx\right]_{-\pi}^{0} = 0$$

Using the method of integral by parts on I$_2$:

Let x = u then dx = du , and $\cos n\, xdx = dv \Rightarrow v = \sin\frac{nx}{n}$

$$\mathrm{I}_2 = \left[x \cdot \frac{\sin n\, x}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{\sin n\, x}{n} d\, x$$

$$= 0 - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_0^{\pi}.$$

By substitution in a$_n$

$$a_n = \frac{1}{\pi}\left[0 + \frac{(-1)^n - 1}{n^2}\right].$$

$$a_n = \frac{(-1)^n - 1}{\pi n^2}.$$

Now let's calculate b$_n$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin n\, x\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \pi\, .\sin n\, .\, xdx + \frac{1}{\pi}\int_{0}^{\pi} x\, .\sin n\, axdx$$

$$\frac{1}{\pi} = \left\{\left[\frac{-\pi}{n}\cos n\, x\right]_{-\pi}^{0} + \mathrm{I}_3\right\}$$

$$b_n = \left[\frac{(-1)^n - 1}{n} + \frac{1}{\pi}\mathrm{I}_3\right]$$

Since $\mathrm{I}_3 = \int_0^{\pi} x \sin n\, xdx$; *Applying Integration by parts then,*

Let x = u then dx = du and $\sin n\, x.\, dx = dv \Rightarrow v = \frac{-\cos(nx)}{n}$

$$\mathrm{I}_3 = \left[\frac{-x.\cos n\, x}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos n\, xd\, x = \frac{\pi(-1)^{n+1}}{n} + 0$$

$$\Rightarrow b_n = \frac{(-1)^{n+1}}{n} + \frac{(-1)^n - 1}{n} = \frac{-1}{n}$$

Now let's substitute in the main formula:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\, x + b_n \sin n\, x)$$

$$\Rightarrow f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos n\, x - \frac{1}{n}\sin n\, x\right]$$

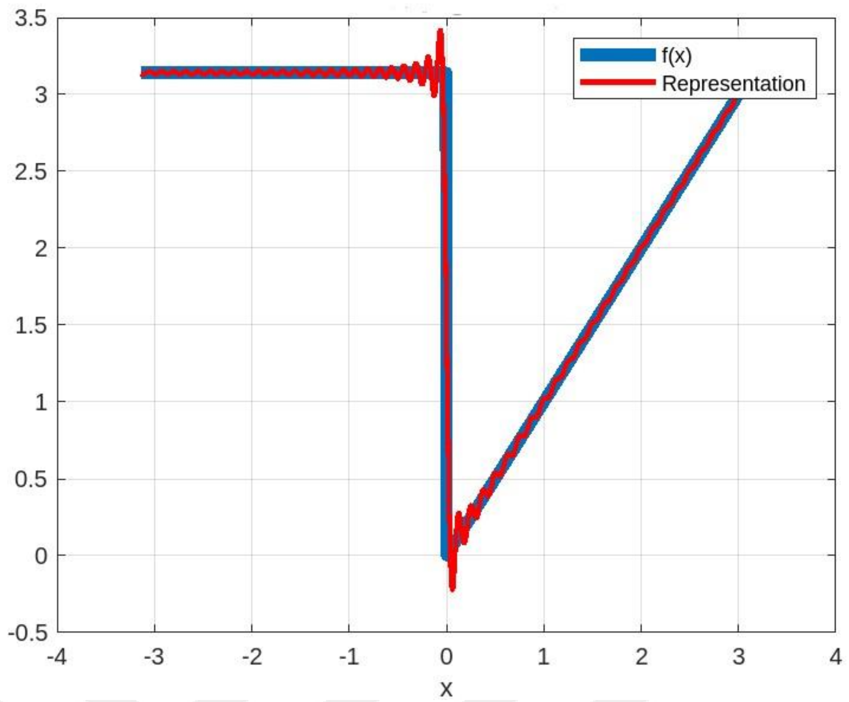

Figure 2.3. Fourier Representation for $f(x) = \pi\chi_{[-\pi,0]} + x\chi_{[0,\pi]}$

A function defined f: $D \subset$R→R is said to be:

i **Even** if $f(x) = f(-x)$.

ii **Odd** if $f(x) = -f(-x)$.

For all $x \in D$ with $-x \in D$. The proof of the following theorem can be given very easily.

Theorem 2.3. *Let f: $[-r, r] \to$R be a function for which its TFS exits. Then:*

i. If f is even then its TFS is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{2L}\right) + b_n \sin\left(\frac{2\pi n x}{2L}\right)\right]$$

$$\frac{1}{2}\mathrm{a}_0 + \sum_{\mathrm{n}=1}^{\infty} \mathrm{a}_\mathrm{n} \cos\frac{\mathrm{n}\pi\mathrm{x}}{\mathrm{r}},$$

where

$$a_n = \frac{2}{r}\int_0^r f(x)\cos\frac{n\pi x}{r}\,d\,x.$$

ii. If f is odd then its TFS is

$$\sum_{n=1}^{\infty} b_n \sin\ \frac{n\pi x}{r},$$

where

$$b_n = \frac{2}{r}\int_0^r f(x)\sin\frac{n\pi x}{r}\,d\,x.$$

2.2 Convergence Problem.

Two fundamental problems of the TFS of a function can be stated as follows:

i **The Convergence Problem:** When does the TFS of a function converge at given a point?

ii **The Convergence Problem:** What is the relation between the convergent point of the TFS and the value of the function?

That is, let f: $[-r, r] \to \mathrm{R}$ be a function for which its TFS exists. Let

A = $\{x \in [-r, r]$: the TFS converges at $a\}$,

and

B = $\{x \in A$: the TFS converges at to $f(x)\}$.

The above-given problems now are:

i. When $A = [-r, r]$?

ii. When $B = [-r, r]$?

iii. When $A = B$?

iv. Is it possible that $A = \emptyset$ or $B = \emptyset$?

A partial answer to the question can be given for some special functions, in particular piece-wise smooth functions. Recall that function f: $[a, b] \to \mathrm{R}$ is known a piece-wise smooth function if there exists

$$x_0 = a < x_1 < \cdots < x_n = b$$

Such that

i. For each $x \in (x_{i-1}, x_i)$ $(i = 1, \cdots n)$ the derivative $f'(x)$ of f at the point x exists.

ii. f' is continuous on the interval (x_{i-1}, x_i) for each i.

iii. $\lim_{t \to x_i, t < x_i} f(x), \lim_{t \to x_i, t > x_i} f(x), \lim_{t \to x_i, t < x_i} f'(x)$ and $\lim_{t \to x_i, t > x_i} f'(x)$, exist for each possible x_i.

The right and left limit of a function f at a point x is, respectively, denoted by $f(x^+)$ and $f(x^-)$. Now, we can state the following theorem.

Theorem 2.4. *Assume that f: R→R be a periodic function with period r and also suppose that it is a piece-wise smooth on the interval $[-r, r]$. Then, for each $x \in [-r, r]$, the TFS of f converges to*

$$\frac{f(x^+) + f(x^-)}{2}.$$

As a corollary of this theorem, we have the following.

Corollary 2.5. *Let f be a continuous function from R into R which is also a piece-wise smooth function with period r, then at each point of the interval $[-r, r]$, the TFS of f converges to $f(x)$.*

As an application of the previous theorem, the values of π and π^2 can be evaluated as in following theorem.

Theorem 2.6. $\pi^2 = 8\sum_{n=1}^{\infty} \frac{1}{(2n-1)!}$.

***Proof*.** Consider periodic function f: R→R with period 2π such that

$$f(x) = \pi\chi[-\pi, 0) + x\chi\,[0, \pi] \text{ on } [-\pi, \pi].$$

It is easy to show that f satisfies the condition of the previous Theorem. In particular, the TFS of f on $[-\pi, \pi]$ is

$$S(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos n\,x - \frac{1}{n}\sin n\,x\right].$$

From the above theorem for each $x \in [-\pi, \pi]$,

$$S(x) = \frac{f(x^+) + f(x^-)}{2}.$$

In this equation letting $x = \pi$ we get the required result.

The following theorem deals with the divergence of the TFS of a function.

Theorem 2.7. *Let $D \subset [0,2\pi]$ be a countable subset. Then there exists a function which is continuous and periodic with period 2π such that its trigonometric Fourier form divergence at each point of D.*

A proof of this theorem can be seen in [1], Example 35.11. The proof uses the Principle of Condensation of Singularities.

We mention that the TFS is an important special class of **FS** in the following sense: Let (φ_n) *orthonormal system with respect to weight function* $r: [a, b] \to \mathbb{R}$ if for each $n \neq m$,

$$\int_a^b \varphi_n(x)\varphi_m(x)r(x)dx = 0$$

and

$$\int_a^b \varphi_n^2(x)r(x)dx = 1.$$

Definition 2.8. *Let (φ_n) be an orthonormal system with respect to weight function r. Then the series*

$$\sum_{n=1}^{\infty} c_n\, \varphi_n(x),$$

where

$$\int_a^b \varphi_n^2(x)r(x)dx = 1.$$

is known the FS of f relative to the system (φ_n).

We note that, letting,

$$\varphi_1(x) = \frac{1}{\sqrt{2L}},\ \varphi_{2n}(x) = \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L} \text{ and } \varphi_{2n+1}(x) = \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}$$

and $r(x) = 1$ on $[-L, L]$ then the FS of a function f to the system (φ_n) with respect to r is TFS of f.

Finally, we note that the FS is related to the Sturm-Liouville problem which is out of the scope of the thesis.

2.3 Another Representation of the trigonometric Fourier series.

Let $a_0,\cdots,a_m$ and $b_0,\cdots,b_m$ be complex numbers. Let

$$c_0 = \frac{1}{2}$$

and for each $1 \leqslant n \leqslant m$ defines

$$c_0 = \frac{1}{2}a_0,\quad c_n = \frac{1}{2}(a_n - ib_n) \quad and\ c_{-n} = \frac{1}{2}(a_n + ib_n)\ .$$

Then one can see that

$$\frac{1}{2}a_0 + \sum_{n=1}^{m}(a_n \cos nx + b_n\, sin\, nx) = \sum_{n=-m}^{m} c_n\, e^{inx}.$$

Conversely, for each $-m \leqslant n \leqslant m$, c_n be a complex number. Let $a_0 = 2c_0$ and for

$1 \leqslant n \leqslant m$ let

$$a_n = c_n + c_{-n} \text{ and } b_n = i\,(c_n - c_{-n})$$

Then one can easily to show that

$$\frac{1}{2}a_0 + \sum_{n=1}^{m}(a_n \cos nx + b_n\, sin\, nx) = \sum_{n=-m}^{m} c_n\, e^{inx}.$$

Hence, we get the following theorem.

Theorem 2.9. *Every TFS can be represented in this form:*

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

The converse of this is also true.

We also note that, as it is mentioned after Definition 1.1, the TFS of a function can be defined for any interval. For example, if f: $[0,2\pi]$ →R for each *n*,

$$a_n = \int_0^{2\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n = \int_0^{2\pi} f(x)\sin nx\,dx$$

exist for all *n*, then the TFS of the f*(x) becomes*

$$\frac{1}{2\pi} + \sum_{n=1}^{\infty} (a_n \cos nx\,\mathrm{dx} + b_n \sin nx).$$

Despite its versatility, the TFS has limitations, as demonstrated by Abel in 1826 with functions that do not satisfy certain conditions. This led to the development of more general forms of FS, including complex FS, enhancing its applicability and addressing theoretical challenges [23].

3. CANTOR'S CONTRUBUTION ON TRIGNOMETRIC FOURIER SERIES

For given sequences a: $\omega \rightarrow$R and b: N→R, r > 0 and x ∈R, the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{r} + b_n sin\frac{n\pi x}{r}\right)$$

will be denoted by *TFS* (*a, b, r, x*). Now, for given *D* ⊂ R we define a "function"

TFS (*a, b, r*) from *D* into the collection

{TFS (a, b, r, x): x ∈D}.

As

TFS (a, b, r) (x) = TFS (a, b, r, x),

here the equality symbol "=" is used in sense of "denoted". If the series *TFS* (*a, b, r, x*):

i Is convergent then its convergence point will be denoted by *TFS* (*a, b, r, c*(*x*)).

ii Is divergent to ∞ then we write *TFS* (*a, b, r, x,* ∞).

iii Is divergent to −∞ we write *TFS* (*a, b, r, x,* −∞).

iv Is divergent we write *TFS* (*a, b, x, div*).

Let *f*: [*r, r*] → be a function and suppose that its TFS exists. Then, for each *x* ∈ [−*r, r*] we denote the *TFS* (*a, b, r, x*) by *TFS* (*f, x*), of course, *a, b*: $\omega \rightarrow$R defined by

$$\mathrm{a(n)} = \frac{1}{r}\int_{-r}^{r} f(x)\cos\frac{n\pi x}{r}dx \qquad and \qquad b(n) = \frac{1}{r}\int_{-r}^{r} f(x)\, sin\frac{n\pi x}{r}dx,$$

One can define two different functions *f, g* : [−*r, r*] →R such that

$$\int_{-r}^{r} f(x)\, cos\frac{n\pi x}{r}\, dx \quad = \int_{-r}^{r} g(x)\, cos\frac{n\pi x}{r}\, dx,$$

and

$$\int_{-r}^{r} f(x)\, sin\frac{n\pi x}{r}\, dx \quad = \int_{-r}^{r} g(x)\, sin\frac{n\pi x}{r}\, dx$$

From this, we observe that although *TFS* (*f*) = *TFS* (*g*) (in the sense that *TFS*(*f*, *x*) = *TFS*(*g*, *x*) for all *x*), *f* and *g* may not be equal.

If the series *TFS*(*f*) of a function *f:* [*a*, *b*] →R is convergent then its convergence value is denoted by *TFS*(*f*, *c*(*x*)). In particular, ”*TFS* (*f*, *c*(*x*)) exits” means that *TFS*(*f*) is convergent at the point *x*.

3.1 The representation Problem.

This problem can be stated as follows: Let f: [a, b] → R be a function with the trigonometric series T(f). If T (f, c(x)) exits then what is the relation between T (f, c(x)) and f(x)? For example, TFS (f, c(x)) = f(x). We give an example to show that, in general, TFS (f, c(x)) ≠ f(x).

Example 3.1. Let *f*: [0,2*π*] →R be defined by,

$$f(x) = \chi\ [0,\ \pi) \cup \{2\pi\}\ (x).$$

That is, *f is* the characteristic function of the set $\chi_{[0,\pi]\cup\{2\pi\}}$.

For each $n \in \omega$,

i. $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,d\,x = \frac{1}{\pi}\int_0^{2\pi}\ dx = 1.$

Prove:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} d\,x = \frac{1}{\pi}\int_0^{\pi} d\,x + 0 = \frac{\pi}{\pi} = 1.$$

ii. $\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos n\,x\,dx = \frac{1}{\pi}\int_0^{\pi}\cos n\,x\,dx = 0.$

Prove:

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos n\,x\,dx \frac{1}{\pi} = \left[\frac{\sin n\,x}{n}\right]_0^{\pi} = 0.$$

iii. $b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin n\,x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin n\,x\,dx = \frac{1-(-1)^n}{n\pi}.$

Prove:

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin n\,x\,dx = \frac{-1}{\pi n}[\cos n\,x]_0^{\pi} = \frac{-1}{\pi n}[(-1)^n - 1] = \frac{2}{\pi(2n-1)}$$

This shows that for each $x \in [0,2\pi]$,

$$TFS(f,x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin\big((2n-1)x\big)\,dx.$$

In particular,

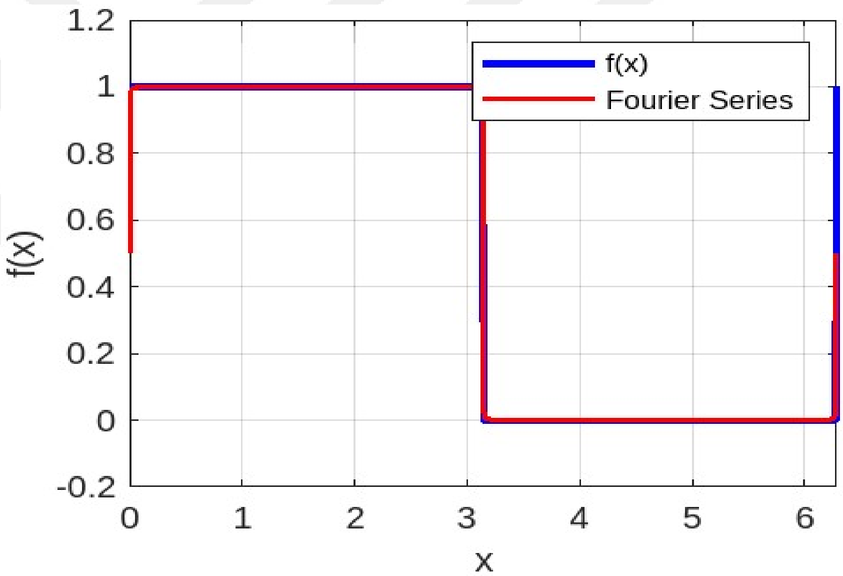

Figure 3.1 Fourier Representation For $f(x) = \chi_{[0,\pi)\cup\{2\pi\}}(x)$

$$TFS\big(f, c(x)\big) = f(x)\chi_{(0,\pi)\cup(\pi,2\pi)} + \frac{1}{2}\chi_{0,\pi,2\pi},$$

and of course

$TFS\ (f, c\ (0)) \neq f\ (0)$, $TFS\ (f, c\ (\pi)) \neq f\ (\pi)$ and $TFS\ (f, c\ (2\pi)) \neq f(2\pi)$.

3.2 The Background in Cantor's Work on Fourier Analysis.

In this section, we present some fundamental results on the TFS, which were backgrounds of Cantor's work on this topic.

For given a function f: $(a, b) \to \mathrm{R}$ for each $x \in (a, b)$ and $h \in \mathrm{R}$ with $x-h$, $x+h \in (a, b)$ we write

$\Delta^2 f(x, h) = f(x + h) + f(x - h) - 2f(x)$. The following theorem is due to Schwart.

Theorem 3.2. *Let* $f : (a, b) \to \mathrm{R}$ *be a continuous function and*

$$D^2 f(x) = \lim_{h \to 0} \frac{\Delta^2 f(x, h)}{h^2}$$

exits for all $x \in (a, b)$. *Then the following are held.*

i. *If* $D^2 f \geqslant 0$ *then* f *is convex. That is, for each* $x, y \in (a, b)$ *and* $0 \leqslant \lambda \leqslant 1$, $f(\lambda x + (1 - \lambda y) \leqslant \lambda f(x) + (1 - \lambda) f(y)$.

ii. *If* $D^2 f = 0$ *then* f *is linear on* (a, b).

We note that in the above theorem if f is continuously two times differentiable function then $D^2 f = f''$, where f'' represents the second derivative of f. Thus, the above result is a generalization of a standard result on functions with second derivative.

Riemann has stated two fundamental results on the convergence of a TFS of a function. Those results are given in terms of $D^2 f$.

For given a bounded sequence $c : \mathrm{Z} \to \mathbb{R}$, the function $f : \mathbb{R} \to \mathbb{R}$ where in the next two lemmas defined by

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \frac{c_0}{2} x^2 - \sum_{n \neq 0} \frac{c(n)}{n^2} e^{inx}.$$

We note that f *is* continuous and $D^2 f(x)$ exits.

Theorem 3.3 (Riemann's first lemma)**.** *Let* c: $\mathrm{Z} \to \mathrm{R}$ *be a bounded sequence. If the series*

$$\sum_{n=-\infty}^{\infty} c(n) e^{inx}$$

is convergent at the point then its convergent point is equal to $D^2 f(x)$.

Theorem 3.4 (Riemann's second lemma)**.** *Let c*: Z→R *be a sequence with* $c_n \to 0$

Then

$$\lim_{h\to 0} \frac{\Delta^2 f(x,h)}{h} = 0$$

for all $x \in \mathbb{R}$.

3.3 Suggested problem to Cantor.

An important result is that the Fourier coefficients of the TFS of a function converge to zero. That is:

Theorem 3.5. *Let* $f : [a, b] \to \mathbb{R}$ *be an integrable function with its FS* $\sum_{n=-\infty}^{\infty} c_n e^{inx}$. *Then,*

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} c_{-n} = 0.$$

A proof of this theorem can be found in [1], Theorem 35.2.

The following question (which is related to theorem 2.5) was suggested to Cantor by one of his colleagues Edward Heine: **Heine's question to Cantor** Suppose the trigonometric series,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\,x + \sin n\,x)$$

converges to the zero for all $x \in$R. Must then for each $n \in \omega$,

$$a_n = 0 \text{ and } b_n = 0?$$

At that time this problem was tackled by many well-known mathematicians, including Peter Dirichlet, Eduard Heine, Rudolph Lipschitz and Bernhard Riemann but no full solution was given until Cantor. Cantor proved the following theorem.

Theorem 3.6 (Cantor, 1970)**.** *Every continiuous function defined on a closed interval can have at most one trigonometric Fourier representation.*

This is the full answer to Heine's question. To prove this theorem Cantor first proved that if a series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\, x + \sin n\, x)$$

converge to zero then

$$\lim_{n\to\infty} \text{ an} = 0 \text{ and } \lim_{n\to\infty} \text{bn} = 0.$$

This is equivalent to this: If c: Z→R is a function providing

$$\sum_n c(n)e^{i\pi nx} = 0 \text{ for all n,}$$

then $\lim\limits_{n\to\infty}$ c(n) = 0. This can be sated as follows and a proof of this result can be found in [18].

Theorem 3.7. *If* $\sum n\, e^{inx}$ *is a trigonometric series of a function from* R *into* R *on a closed interval* [*a*, *b*] *and* f^2 *is integrable then*

$$\sum_{n=1}^{\infty}|c_n|^2 \le \int_a^b |f(x)|^2\, dx.$$

From this theorem, it implies that $\lim\limits_{n\to\infty} c_n = 0$. More generally the above theorem can be stated also as follows:

Theorem 3.8 (Bessel's inequality). If $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \sin n\, x + b_n \cos n\, x)$. *is the trigono-metric FS of the function f:* R→R *with* $f^{\,2}$ *is Lebesgue integrable function then*

$$\frac{|a_0|}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) \le \int_a^b [|f(x)|^2]^{\frac{1}{2}}\, dx.$$

A proof of this theorem can be found in [19].

Following the above theorem, we have

$$\lim_{n\to\infty} a_n = 0 \text{ and } \lim_{n\to\infty} b_n = 0.$$

Now the Cantor theorem can be proved as follows: Let $(c(n))_{n\in Z}$ be Fourier coefficient of TFS of a continuous function and suppose that the series is convergent to the value of the function at each point. Then, from the above theorem (c_n) is bounded. Let f: R→R be defined by

$$\mathrm{f(x)} = \frac{\mathrm{c}(0)}{2}\mathrm{x}^2 - \sum_{\mathrm{n}\neq 0}\frac{\mathrm{c(n)}}{\mathrm{n}^2}\mathrm{e}^{\mathrm{inx}}$$

is a continuous function. It can easily be shown that $D^2f(x) = 0$ for all x ∈R. By Schwarz's theorem, f *is* linear. So there exists m, l ∈R such that

$$\mathrm{f(x)} = \tfrac{\mathrm{c}(0)}{2}\mathrm{x}^2 - \textstyle\sum_{\mathrm{n}\neq 0}\tfrac{\mathrm{c(n)}}{\mathrm{n}^2}\mathrm{e}^{\mathrm{inx}} = \mathrm{mx} + \mathrm{l} \quad \text{for all } x \in \mathrm{R}.$$

Letting $x = \pi$ and $x = -\pi$ one can get $m = 0$. Next, by letting $x = 0$ and $x = 2\pi$ we get $c_0 = 0$. This implies that for all x ∈R,

$$\sum_{\mathrm{n}\neq 0}\frac{\mathrm{c(n)}}{\mathrm{n}^2}\mathrm{e}^{\mathrm{inx}} = -\mathrm{l}.$$

For each k ∈Z we have

$$\sum_{\mathrm{n}\neq 0}\frac{\mathrm{c(n)}}{\mathrm{n}^2}\mathrm{e}^{\mathrm{i(n-k)x}} = -\mathrm{l}\mathrm{e}^{-\mathrm{i(n-k)x}}.$$

Since f is uniformly continuous, we have

$$2\pi\frac{\mathrm{c(k)}}{\mathrm{k}^2} = \sum_{\mathrm{n}\neq 0}^{2\pi}\int_0^{\mathrm{c(n)}}\mathrm{e}^{\mathrm{i(n-k)x}}\,\mathrm{dx} = -\int_0^{2}\pi\mathrm{l}\,\mathrm{e}^{\mathrm{i(n-k)x}} = 0.$$

So, $c(k) = 0$. Since k is arbitrary $c = 0$.

Cantor's theorem can be generalized to the Lebesgue integrable functions as follows:

Theorem 3.9. *Every Lebesque integrable function defined on a closed interval with zero TFS is almost everywhere zero.*

A proof of this theorem can be found in [1] Theorem 35.3 or in [19] Theorem 8.10.5.

3.4 Sets of Uniqueness of Fourier Series.

Theorem 3.2 is knowna sort of the uniqueness problem of the TFS. This Theorem has been improved by Cantor and this has opened some notion of the topological spaces and led to the development of the theory of ordinal numbers.

Definition 3.10. *A subset D ⊂R is knowna* **set of uniqueness** *if every function a, b: ω →R is zero providing*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a(n)\cos n\,x + b(n)\sin n\,x) = 0$$

for all x ∈R \ D.

In [2] he asserted that an empty set is a set of uniqueness. For the proof, he addressed Weierstrass's theorem: a continuous fonction on a finite closed interval assumes its maximum value.

Cantor's theorem can be re-read in terms of uniqueness as follows.

Theorem 3.11. *The empty set is a set uniqueness.*

Because of this Cantor has also observed that:

Theorem 3.12. *Let a, b*: *ω →*R *be two functions. If the*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\,x + b_n\sin n\,x)$$

series is zero on an open interval then

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$

Cantor has improved his early results as follows.

Theorem 3.13 (Cantor, 1871)**.** *Let f*: Z→R *be a strictly increasing function. If* inf *f* (Z) = ∞ *and* sup*f* (Z) = ∞ *then f*(Z) *is a set of uniqueness.*

A proof of this theorem can be found in [21].

We note that in 1871 [3], he also proved that every finite subset of real numbers is a set of uniqueness.

4. HOW DID CANTOR PASS TO INFINITY?

” How did Cantor pass to Infinity?” delves into Cantor’s transformation contributions to the concept of infinity.

The notion of infinity, for about a thousand years, had been discussed on the philosophical level without arriving at any certain conclusion.

According to Ferreiros (2007) [11], Cantor revolutionized the discourse on infinity in 1872 by introducing cardinality and transfinite numbers. This marked the beginning of the unification of these notions in Zermelo-Fraenkel set theory (ZFC), challenging traditional philosophical discussions and paving the way for a more rigorous mathematical treatment.

As detailed by Dauben (1979) [10], Cantor’s shift from studying trigonometric series to focusing on the” size” of subsets of real numbers between 1872-1880 was a crucial period. He meticulously classified sets into” finite,” ”countable infinite,” and” continuum,” laying the groundwork for a formal understanding of infinity. Cantor’s unique perspective on infinity, driven by the need for a more formal classification, culminated in the development of transfinite numbers.

Cantor’s creation of” symbols in infinity” using derived sets, described by Moore (1982) [16], represented a pivotal step toward defining transfinite numbers. The first transfinite number, surpassing all natural numbers, played a foundational role in modern definitions. Cantor’s interpretation, highlighted by Grattan-Guinness (2000), provided a meaningful representation within the context of derived sets [14].

The definition, which is given here, is not a formal definition but we note that the modern definition of transfinite numbers based on this approach.

Everybody knows where the infinity is, it is at the end of the following three dots:

1, 2, 3...

That is, if" infinity" is denoted by" ∞" we have

1, 2, 3...∞.

In this picture the meaning of the numbers 1, 2, 3, ... more or less were clear but not ∞. Cantor, in 1880, had looked at his 1872 paper and then he was able to give a meaning to ∞ it in terms of a derived set.

In this chapter, we deal with infinity and transfinite sets.

4.1 Derived sets.

We understand a" set" to be any collection M of certain distinct objects of our thought or intuition (the" elements" of M) into a whole. (Cantor, 1895) [6].

A point $x \in$ R is known **an accumulation point** of a subset $A \subset$ R if for each $r > 0$, the set

$$(-r, r) \cap (a \setminus \{x\})$$

is nonempty, which, was introduced by Heine. The **derived set** of A is the set of all its accumulation points, which was introduced by Cantor.

The set of derived points of a set A will be denoted by A'.

There is no accumulation point and finite set. A countable set may have but only one; for example zero is an accumulation point of the set $\{\frac{1}{n} : n \in \mathbb{N}\}$. On the other hand, if $D \subset$R is a countable set and let f: Z→D be a strictly increasing fonction with inf $D = -\infty$ and sup $D = \infty$. Then, it is obvious that there is no accumulation point of D. Cantor has improved these to the following theorem.

Theorem 4.1. *Let $P \subset$R be given. If $P'' = \emptyset$ then P is a set of uniqueness.*

Proof. (Sketch) Since $P'' = \emptyset$, P' must be a subset $\{x_n : n \in$Z$\}$ such that $(x_n)_{n\in Z}$ is increasing with

$$\lim_{n\to-\infty} = -\infty \text{ and } \lim_{n\to\infty} = \infty.$$

Let $i \in \mathrm{Z}$ be fixed. Choose $m < M$ with $[m, M] \subset (x_i, x_{i+1})$. One can show that, since $[m, M]$ is compact it contains no accumulation point of P. By the Bolzano-Weierstrass

Theorem, $[m, M]$ contains at most finitely many points from P, so it can be written as

$$m = \leqslant a1 \leqslant \cdots \leqslant ak \leqslant M.$$

Define f: R→R be defined by

$$f(x) = \frac{c_0}{2}x^2 - \sum_{n \neq 0} \frac{c(n)}{n^2} e^{i\,n\,x},$$

where c : Z→R is a fonction defined by $c(n) = x_n$. Some efforts show that f is linear on the interval (x_i, x_{i+1}). An application of Riemann's second lemma shows that f is linear. From this observation, the required result is followed.

Indeed, Cantor proved that his arguments can be easily extended to more general sets. For any set P of real numbers, Cantor inductively defined inductively,

$$P^0 = P,\ P^{n+l} = (P^n)',$$

where n is a natural number.

Definition 4.2. *A subset* $P \subset \mathrm{R}$ *is called a* set of first species *if*

$$P^n = \emptyset$$

for some $n \in \mathrm{N}$.

Theorem 4.3. Every first species set is a set of uniqueness.

4.2 What is Infinity.

Let $P \subset \mathrm{R}$ be given. Let

$$P^{\omega} = \cup_n P^n.$$

Then, for each $n \in \mathbb{N}$, when $P^{\omega+n}$ is defined, let

$$P^{\omega+(n+1)} = (P^{\omega+n})'.$$

Cantor has proved the following important theorem.

Theorem 4.4. *For each $n \in \omega$ there exists a subset $P \subset \mathbb{R}$ such that*

$$P^{\omega+n} \neq \phi \text{ and } P^{\omega+(n+1)} = \phi.$$

Following the above theorem one can define a function f from $\{1,2,\cdots,\}\cup\{\infty\}$ to $\{P^1,P^2,\cdots,P^{\omega}\}$ by

$$f(n) = P^n \text{ and } f(\infty) = P^{\omega}.$$

Hence, in some sense, Cantor is defined ∞ as a set.

Definition 4.5. $\infty = P^{\omega}$.

Definition 4.6. *For each $n \in \omega$,*

$$\infty + n = P^{\omega+n}.$$

The reader should be careful that here, the "definition" of $\infty + n$ depends on the given set P, so in some sense, it is not "well defined". But, it is a just discussion of infinity.

Similar to the above definition following can be given. For each $n \in \omega$,

$$P^{\omega+\omega} = \bigcap_{n\in\omega} P^{\omega+n},$$

$$P^{\omega+\omega+(n+1)} = \left(\bigcap_{n\in\omega} P^{\omega+n}\right)'.$$

By following the above observations, Cantor let:

Definition 4.7. *For each $n \in N$,*

$$\infty + \infty + \mathrm{n} = \mathrm{P}^{\omega+\omega+\mathrm{n}}.$$

where

$$\infty + \infty = \infty + \infty + 0 = \mathrm{P}^{\omega+\omega}.$$

It should be noted that nowadays in the literature it is shown "ω" instead of the symbol "∞" Transfinite numbers in modern literature are known ordinals which is a generalization of natural numbers. Arithmetical operations on ordinals can be well-defined.

One should think about:

$$\infty + \frac{1}{2},\ \infty + \sqrt{2},\ \text{or}\ \sqrt{\infty}.$$

4.3 Size of a set.

For each $n \in \mathrm{N}$ the number of the elements of the following set

$$A = \{0,1,\cdots,n-1\}$$

is defined by n. If X is a set and there exists a bijection between A and X then the number of elements of X is said to be also n [9]. But, the meaning of a number of elements of an "infinite set" may not be so clear. For example, without definition and without any discussion we suppose that the cardinal number of the set of natural numbers N is ∞, but denoting this by $|N| = \infty$.

It includes some risk because the meaning of ∞ is not clear. Here, it should not be confused with ∞ what was introduced in the previous section. For every set X a unique symbol can be assigned that designates of the number of elements of X, which is knowna cardinal number of X and usually denoted $|X|$ or *card*(X).

Meaningful the "equality" of the cardinal numbers of two different sets also could be defined. To understand the meaning of infinity as the size of a set we introduce the following definition.

Definition4.8 (Cantor)**.** *Two sets are said to be equivalent if there exists a bijection between them.*

If the sets X and Y are equivalent, we say they have the same cardinal number and, in this case, it may be denoted by

$$|X| = |Y| \text{ or } card(X) = card(Y).$$

If there exists a one-to-one map from X to Y, then it is written

$$|X| \leqslant |Y|.$$

In the case of $|X| \leqslant |Y|$ and not being $|X| = |Y|$ we write $|X| < |Y|$. If $f: X \to Y$ is an injective map, then $|X| = |f(X)|$. The following theorem can be proved in *ZFC* but not in *ZF*.

Theorem 4.9. *Let X and Y be non-empty sets.* $|X| \leqslant |Y|$ *if and only if there exists a surjective map from Y onto X.*

A set A is known a finite if

$$|A| = \{0,1,\cdots,n-1\}$$

for some $n \in$ N. By induction, it can be proved that union, intersection, set difference, and product of given two finite sets are finite. It is "obvious" (actually not obvious) that N is not finite, that is, there is no $n \in$N with

$$|\{0, 1,\cdots,n-1\}| = \text{N}.$$

A set, which is equivalent to a subset of the natural numbers, is known countable. Therefore, every finite set is countable but not converse. A set X is said to be countable infinite if

$$|X| = |\mathrm{N}|.$$

In this case, it is written

$$|X| = \aleph_0.$$

Is there a set that is “bigger” than infinity? This is dealt with in the next section.

5. COUNTABILITY AND UNCOUNTABILITY

Is there a set that is "bigger" than infinity? Of course, for example $\wp$(N) is bigger than $\aleph_0$, that is,

$$\aleph_0 = |\mathrm{N}| < |\wp\ (\mathrm{N})|.$$

However, the word "infinity" of the question is not well defined, although in the previous chapter, it has been tried to define it, but not fully succeeded. Questions that are more correct might be as follows: Is there a set X without any set Y satisfying:

$$|\mathrm{N}| < |Y| < |X|.$$

To answer this set best candidate is R. However, what is R?

5.1 Foundation of the real numbers by Cantor.

Rational numbers are not as natural as natural numbers and, real numbers are not as natural as irrational numbers! Indeed, for example which one is more natural

$$2,\ \frac{1}{2}\ \text{or}\ \sqrt{2}\ ?$$

Until Cantor's time, no one had needed to give a systematic foundation to real numbers. Cantor has introduced Cauchy sequences for better understanding the uniqueness problem of the trigonometric series, in particular, for understanding the first species and to describe accumulation points. Dedekind looked at the Cauchy sequence from a different view and pointed out to Cantor that the real numbers can be described by equivalence classes of Cauchy sequences of rational numbers. This gave an opportunity to Cantor to start the systematic foundation of real numbers and he also defined irrational numbers in terms of rational numbers in 1872, [4]. In the same year, Dedekind defined real numbers in the terms of" Dedekind cuts" giving reference to Cantor's 1872 paper. The work [4] also led to general topology. The technical terms closed, dense,

dense in itself, perfect, interior point and open sets originate from his works. Why the construction of real numbers needs to be discussed in more detail in [16].

In this section, we introduce real numbers which enable us to be safe.

Definition 5.1 (Cantor). *A sequence* $f\colon \mathrm{N} \to \mathrm{Q}$ *is called a* **fundamental sequence** *if for each rational number* $\varepsilon > 0$ *there exists a natural number* $k \in \mathrm{N}$ *such that*

$$|f(n) - f(m)| < \varepsilon$$
$$\text{for all } k > n, m$$

It means that f is known as Cauchy if

$$\lim_{n,m\to\infty} |f(n) - f(m)| = 0.$$

We note that the fundamental sequence is named as **Cauchy sequence** nowadays. Let *Cauchy*(Q) be the set of all Cauchy sequences on Q. If Q^{N} is considered an ordered ring with point-wise algebraic operations and point-wise order, then *Cauchy* (Q) becomes a sub-ordered ring of that.

Two Cauchy sequences f and g are said to be **equivalent** if

$$\lim_{n\to\infty} |f(n) - g(n)| = 0$$

For each Cauchy sequence f, the equivalence class of f is defined and usually denoted by

$$[f] = \{g \in \mathit{Cauchy}\,(\mathrm{Q}) : f \text{ and } g \text{ are equivalent}\}.$$

Definition 5.2. *The equivalence class of any Cauchy sequence on rational is known as a* real number.

The set of real numbers is denoted by R, that is

$$\mathrm{R} = \{[f] : f \text{ is a Cauchy sequence}\}.$$

In R addition operation and multiplication operation can be defined by

$$[f + g] = [f] + [g] \text{ and } [fg] = [f][g].$$

It is easy to show that R is a commutative ring with identity with respect to these algebraic operations. Of course, the zero of R is [0], where 0 is the zero sequence. We write 0 = [0]. Similarly, the identity of R is [1], where 1 is the constant one fonction.

If $0 \neq x \in \mathrm{R}$, then there exists $f \in Cauchy$ (Q) and $0 < r \in \mathrm{Q}$ such that $x = [f]$ and

$$r < |f(n)|$$

for all n. In this case $g = \frac{1}{f} \in Cauchy$ (Q) and $[f][g] = 1$. Hence, R is a field.

If $[f]$, $[g] \in \mathrm{R}$ and there exists a rational number r and natural number n_0 satisfying

$$f(n) < g(n) + r$$

for all $n \geqslant n_0$ we write

$$[f] < [g].$$

If

$$[f] = [g] \text{ or } [f] < [g]$$

we write

$$[f] \leqslant [g].$$

The relation $\leqslant$ is called **order of real numbers**. This order is linear, that is, for each f, $g \in Cauchy(\mathrm{Q})$ only one of the following holds:

$$[f] < [g],\ [g] < [f] \text{ or } [f] = [g].$$

Moreover, if $x \leqslant y$ in R then for each $z \in$R, $x + z \leqslant y + z$ is satisfied. If $0 \leqslant z$ then $xz \leqslant yz$. Hence, R is an ordered field.

Let $x_0 \in$R and $A \subset$R be given. x_0 is called an **upper bound** of A if $x \leqslant x_0$ for all $x \in A$. If such x_0 exits, then A is called **bounded from above**. If x_0 is an upper bound and it is smaller than all other upper bounds of A then x_0 is called **least upper bound** of A, and in it is denoted by writing

$$x_0 = \inf A.$$

In particular, if $x_0 \in A$ it is written by

$$x_0 = \max A,$$

and it said **maximum** of A. Similarly, **lower bound**, **bounded from below**, **greatest lower bound** and **minimum** of A is defined and standard notations are used.

Real numbers has the **Archimedes property**, that is, for each positive real number x,

$$\lim_n \frac{1}{n} x = 0.$$

Another important property of real numbers is the **supermom property** that is every nonempty subset that is bounded from above has a unique least upper bound. This is known as **completeness property**. In modern terminology, we can state the following theorem.

Theorem 5.3. *Real numbers are a completely ordered field.*

The constructed real numbers as in above, in general, are known **Cauchy real numbers**. On the other hand, all complete ordered fields are ring and order isomorphic.

Hence, we can consider that R is a unique complete ordered field.

A sequence f: N→R is known a **Cauchy sequence** if

$$\lim_{n} |f(n) - f(m)| = 0.$$

One of the important theorem on real numbers can be stated below. We state it without proof.

Theorem 5.4. *Real numbers is complete, that is, a sequence in real numbers is convergent if and only if it is Cauchy.*

Let $x \in$R be given. **Entire part** of x is defined by

$$[x] = \max \{Z \cap (-\infty, x]\}.$$

It is obvious that, for each $n \in$N

$$x - n^{-1} < n^{-1}[nx] \leqslant x,$$

so we have

$$x = \lim_{n\to\infty} \frac{[nx]}{n} = x\,.$$

Now the following theorem can be proved.

Theorem 5.5. *Every real number is a limit point of rational numbers.*

In topological terminology, Q is dense in R. This theorem also can be proved by its construction.

5.2 Dedekind Cuts.

Another well known constructions of real numbers is given by Dedekind. A nonempty set $\alpha \subset Q$ is known a **Dedekind cut** if for given any $x \in \alpha$, $y \in \alpha$ whenever $y < x$ and there exists $z \in \alpha$ satisfying $x < z$. Let RD be the set of all Dedekind cuts. Let

$$0_D = \{x \in Q: x < 0\}.$$

It is obvious that 0_D is a cut. It is easy to show that for each cut α,

$$0_D = \alpha O_D = 0_D\alpha.$$

In RD define $\alpha < \beta$ whenever $\alpha \subset \beta$ and $\alpha \neq \beta$. For each α the one of the following holds.

$$0_\Delta = \mathrm{a},\ 0_\Delta < \mathrm{a},\ \mathrm{a} < 0_\mathrm{D}.$$

For each $\alpha, \beta \in$ RD, $\alpha + \beta$ and

$$\alpha\beta = \begin{cases} [(-\alpha)(-\beta)] & : \alpha, \beta < 0_D \\ -[(-\alpha)\beta] & : \alpha < 0_D < \beta \\ \alpha\beta & : \alpha, \beta > 0_D\ . \end{cases}$$

are cuts. It can been shown that under the given order <, addition operation $\alpha + \beta$ and multiplication $\alpha\beta$, RD is an complete ordered field.

In the survey paper [22] many other different constructions of real numbers can be found.

5.3 Countability of rational and algebraic numbers.

Probably in the mind of Cantor, there was to find a non-countable infinite subset of real number, and so he had studied on the cardinal numbers and the set of algebraic numbers.

We note that if A and B are nonempty and one of them is countably infinite then $A \times B$ is countably infinite. One of the main result of the Cantor is:

Theorem 5.6 (Cantor [7]). *The set of rational numbers is countable infinite.*

Proof. Let $X = (\mathrm{Z}\times\{0\}) \cup (\mathrm{Z}\times\{0\})$. X is countable. Define a bijection from the set of rational numbers Q into $\wp^{<\omega}(X)$ by

$$\frac{\mathrm{m}}{\mathrm{n}} \to \{(\mathrm{m}, 0), (\mathrm{n}, 1)\}.$$

Obviously, it is an injective map. Since $\wp^{<\omega}(X)$ is countable infinite and Q is infinite set, it is also countable infinite.

Slightly a different proof also can be given as follow: Let

$$X = \{(m, n) : m, n \in \mathrm{N} \text{ and relatively prime}\}.$$

Then $|\mathrm{Q}^+| = |X|$. In particular, the map $(n, m) \to 2^n (2m - 1)$ from X into N is injective. Since X is infinite, it is countable infinity, so Q^+ is countable infinite. Hence, $\mathrm{Q} = \mathrm{Q}^+ \cup (-\mathrm{Q}^+) \cup \{0\}$ is countable infinite.

One another proof ([20]) of the countable infinity of rational numbers as follows: Let $m, n \in \mathrm{N}$ relatively prime numbers. Write

$$\mathrm{n} = \mathrm{p}_1^{\mathrm{m}_1}\mathrm{p}_2^{\mathrm{m}_2} \cdots \mathrm{p}_\mathrm{k}^{\mathrm{m}_\mathrm{k}},$$

where $n_i, m_i \in \mathrm{N}$ and p_i , q_i are prime numbers. The map defined by

$$\frac{m}{n} \to \frac{m^2 n^2}{p_1 p_2 .. p_k} \ .$$

From rational to natural numbers is bijective.

An algebraic number, which is a root of a polynomial fonction with rational coefficients on complex number. That is, a complex number x is knownan algebraic number if there exist rational numbers a_0, a_1 ,... a_n such that

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0.$$

We note that the set of polynomials with rational coefficients on R, denote it by Q[x] , is countable. Indeed, a map from Q[x] into $\wp^{<\infty}$(Q×N) defined by

$$p \to \{(a_0,0), (a_n, n)\},$$

where $p(x) = a_0 + a_1x + ...a_nx^n$ is an injective map. Thus, Q[x] is countable. The Fundamental Theorem of Algebra says that on complex numbers the set of root of any nonzero polynomial is finite. So, if p ∈Q[x] then

$$A_p = \{x \in R : p(x) = 0\}.$$

Hence, the set of the algebraic number is

$$A = \bigcup_{p \in \mathbb{Q}[x]} \{x \in \ \mathbb{R}: p(x) = 0\}.$$

We note that every rational number is an algebraic number. Now the proof of the following theorem is more or less clear.

Theorem 5.7 (Cantor [5])**.** *The set of algebraic numbers is countable infinite.*

5.4 Uncountability of real numbers.

Most of the set theorists agree that the set theory was borne with the following theorem, in late 1873.

Theorem 5.8 (Cantor [5]). *The set of the real numbers is uncountable.*

Wenner. It is obvious that R is not finite. Suppose that R is countable. Then there exists a bijection map $f\colon \omega \to$R. For each $k \in \omega$ we can consider that $f(k)$ is an equivalence class of a Cauchy sequence. Let us write $f(k) = f_k$. We can choose a natural number N_0 such that

$$|\mathrm{f}_0(\mathrm{n}) - \mathrm{f}_0(\mathrm{m})| < \frac{1}{4}$$

for all $n, m \geqslant N_0$. Choose $b_0 \in$Q satisfying

$$|\mathrm{b}_0 - \mathrm{f}_0(\mathrm{N}_0)| \geq \frac{1}{2}.$$

Now suppose that N_{k-1} and b_{k-1} are defined. We can find a natural number $N_k > N_{k-1}$ such that

$$|\mathrm{f}_\mathrm{k}(\mathrm{m}) - \mathrm{f}_\mathrm{k}(\mathrm{n})| < \frac{1}{3^{\mathrm{k}+2}}$$

for all $n, m \geqslant N_k$. Let b_k be defined as follows: If $f_k(N_k) \leqslant b_k - 1$ let,

$$\mathrm{b}_\mathrm{k} = \mathrm{b}_{\mathrm{k}-1} + \frac{1}{3^{\mathrm{K}+1}},$$

and otherwise let,

$$\mathrm{b}_\mathrm{k} = \mathrm{b}_{\mathrm{k}-1} - \frac{1}{3^{\mathrm{k}+1}}.$$

It is obvious that

$$|b_k - f_k(N_k)| \geqslant \frac{1}{2^{3k+1}}$$

Hence, we have a function $g\colon \omega \to$Q defined by

$$g(n) = b_n.$$

g is a Cauchy sequence. Indeed, let $\varepsilon > 0$ be given. Choose a natural number N with

$\frac{1}{3^N} < \varepsilon$. Then for $n \geqslant m \geqslant N$ we have

$$|g(n) - g(m)| \leq \sum_{i=m+1}^{n} |g(i-1) - g(i)| \leq \sum_{i=m+1}^{n} \frac{1}{3^i} < \frac{1}{3^N} < \varepsilon.$$

Now, we show that $[g] \neq f(\omega)$. Let $k \in \omega$ be given. It can show that for each

$n \geqslant N_k$,

$|g(n) - f_k(n)| = |g(n) + g(k) - g(k) + f_k(N_k) - f_k(N_k) - f_k(n)|$

$\geqslant |(g(k) - f_k(N_k))| - |g(n) - g(k)| - |f_k(N_k) - f_k(n)|$

$$> \frac{1}{2^{3k+1}} - \sum_{i=1}^{n} \frac{1}{3^i}$$

$$> \frac{1}{2^{3k+1}} - \sum_{i=1}^{\infty} \frac{1}{3^i}$$

$$= \frac{1}{2^{3k+1}} - \frac{1}{7}\frac{1}{2^{3k}}$$

$$= \frac{3}{28}\frac{1}{3^k}$$

This shows that $[g] \neq f_k$. Since k is arbitrary the proof is completed.

One another proof: Of course R is not finite. Suppose that it is countable. Let f: N→R be a bijection. Let g, h: N→R be defined by

$$-g(0) = f(0)$$

$$-h(0) = f(\min\{k: g(0) < f(k)\}),$$

whenever $g(n)$ and $h(n)$ are defined let,

$$-\ g(n+1) = f(\min\{k: g(n) < f(k) < h(n)\}),$$

$$-\ h(n+1) = f(\min\{k: g(n+1) < f(k) < h(n)\}).$$

It is obvious that g is strictly increasing, h strictly decreasing and $g(n) < h(m)$ for all $n, m \in \omega$. Let

$$a = \sup g(\omega).$$

It can be shown that $a \neq f(\omega)$.

A survey paper about the unaccountability of real numbers is [22] where some others different proofs can be found in it.

The cardinality of a set which is equivalent to real numbers is known **continuum** and is denoted by c, that is

$$|\mathrm{R}| = \mathrm{c}.$$

Now we have

$$0 < 1 < \cdots < \aleph_0 < \mathrm{c}.$$

A real number which is not rational is known irrational. Obviously, the set of irrational numbers is uncountable and infinite. An irrational number that is not algebraic is known **transcendental number**. In 1851, Liouville proved that a transcendental number exists, but, in fact, there are many transcendental numbers that exist. Moreover:

Theorem 5.9 (Cantor)**.** *The set of transcendental numbers is uncountable.*

5.5 Cantor's Theorem.

For each set X there exists at least one injective map from X into $\wp(\mathrm{X})$, for example, $\mathrm{x} \to \{\mathrm{x}\}$ defines an infective map. Hence, $|X| \leqslant |\wp(X)|$. But there is no surjective map from X into $\wp(\mathrm{X})$. Indeed, if f: $\mathrm{X} \to \wp(\mathrm{X})$ is a map then there would be no $\mathrm{x} \in \mathrm{X}$ satisfying

$$f(x) = \{x \in X : x \notin f(x)\}.$$

Hence, we have the following theorem which is known fundamental theorem of the set theory.

Theorem 5.10 (Cantor [8])**.** *For each set X,* $|X| < |\wp(X)|$.

As a result of Cantor's theorem, one can say that each infinity set is smaller than an infinity set.

The set of all fonctions from X into $\{0,1\}$ is denoted by 2^X. The function from $\wp(X)$ into 2^X is defined by

$$A \to \chi_A$$

is a bijective map.

Theorem 5.11. *For each set X,* $|\wp(X)| = |2^X|$.

An application of this theorem shows that $\wp(\mathrm{N})$ is an uncountable set.

As a result of this theorem is $|X| < |2^X|$. An alternative proof of this can be given as follows: Suppose that $F: X \to 2^X$ is surjective. Define $f: X \to \{0,1\}$ by

$$f(x) = 1 - F(x)(x).$$

There is no $x \in X$ with $F(x) = f$.

5.6 Cantor-Bernstein Theorem.

If for each two sets there are injective maps from each to other then the cardinalities of them are equal. This is as much important and useful as legs and arms for a body.

Let X be a nonempty set. A function $\pi\colon \wp(X) \to \wp(X)$ is called increasing if $\pi(A) \subset \pi(B)$ whenever $A \subset B$. $F \subset X$ is called a fixed point if $\pi(F) = F$.

Theorem 5.12. *Every increasing function from the power set of a set into itself has a fixed point.*

Proof. Let X be a nonempty set and $\pi : \wp(X) \to \wp(X)$ be an increasing fonction. Let

$$C = \{A : A \subset f(A)\}$$

It is obvious that $\emptyset \in C$, so C is nonempty. Let $F = {}^{S}C$. Then $F \in C$. Therefore, $F \subset f(F)$, which implies that $f(F) \subset f(f(F))$. Hence $f(F) \in C$. Thus, $f(F) \subset F$.

This completes the proof.

Theorem 5.13. *Let X and Y be two sets. If* $|X| \leqslant |Y|$ *and* $|Y| \leqslant |X|$ *then* $|X| = |Y|$.

Proof. Let $f: X \to Y$ and $g: Y \to X$ be one-to-one functions. Let $\varphi: \wp(X) \to \wp(Y)$ be defined by

$$\varphi(A) = g(Y \setminus f(X \setminus A)).$$

It is obvious that φ is increasing so by from the previous theorem it has a fixed point, say F. Define $h: X \to Y$ by

$$h(x) = \begin{cases} f(x) & : x \in F \\ g^{-1}(x) & : x \notin F. \end{cases}$$

It is obvious that h is a bijective map. And the proof is completed.

Although this theorem is known Cantor-Bernstein it is also given some different names. Extensive details of this work are given in [15].

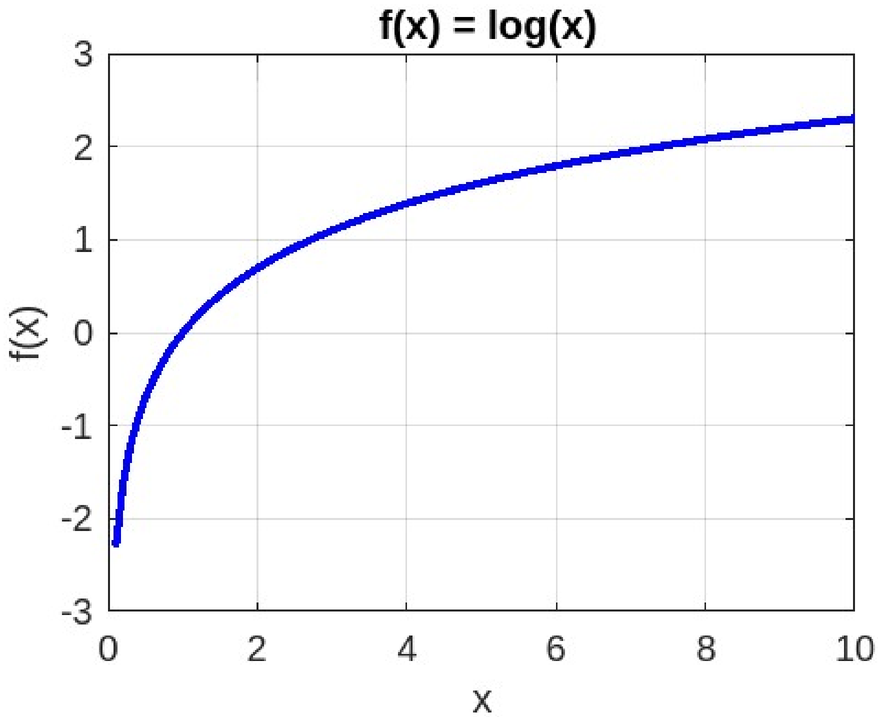

Figure 5.1 cardinalities of sets 1

6. SOME EXAMPLES

In this section, we give some examples of the cardinalities of sets.

Example 6.1. a. $|(0,\infty)| = |R|$: $f : (0,\infty) \to R$, $f(x) = \log x$.

Prove: Let us prove that the cardinality of the set of positive real numbers $(0, \infty)$ is equal to the cardinality of the set of real numbers R.

We know that the fonction $f(x) = log\ x$, is a one to one correspondence between $(0, \infty)$ and R. This means that for every value in the set $(0, \infty)$, there is a unique corresponding value in the set of real numbers R, and vice versa.

In other words, if $f(x_1) = f(x_2)$, then $x_1 = x_2$, indicating the injective nature of the function.

Since the f(x) = log x is a bijection, the number of elements in each set is equal. Thus, we can say that $|(0,\infty)| = |R|$.

Therefore, the desired equality has been proven.

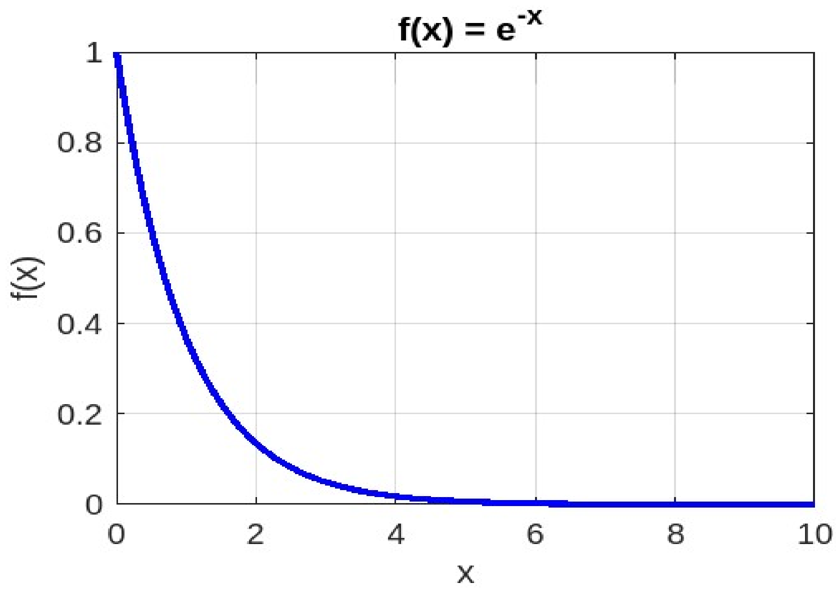

Figure 6.1 cardinalities of sets 2

b. $|(0,\infty)| = |(0,1)|$: $f : (0,\infty) \to (0,1)$, $f(x) = e^{-x}$.

Prove: Let's prove that the cardinality of the set of positive real numbers $(0, \infty)$ is equal to the cardinality of the set $(0,1)$.

We know that the $f(x) = e^{-x}$ establishes a one to one correspondence between these two sets. It can be stated that if $f(x_1) = f(x_2)$, then $x_1 = x_2$, demonstrating the injective nature of the function.

In other words, for every value in the set $(0, \infty)$, there exists a unique corresponding value in the set $(0,1)$, and vice versa.

Thus, we can assert that $|(0,\infty)| = |(0,1)|$, and consequently, the equality has been established.

c. $|(0,\infty)| = |[0,\infty)|$: $f(x) = x\chi_{R\setminus N}(x) + (x-1)\chi_N(x)$.

Prove: Let's prove that the cardinality of the set of positive real numbers $(0, \infty)$ is equal to the cardinality of the set $[0, \infty)$. We define the function:

$$f(x) = x\chi_{R\setminus N}(x) + (x-1)\,\chi_N(x)$$

Where: - $\chi_{R\setminus N}(x)$ is the characterstic function for the set of real numbers outside the natural numbers. - $\chi_N(x)$ is the characteristic function for the set of natural numbers.

Let's examine the analytical properties of this function:

1. The $f(x)$ takes distinct values for natural numbers and non-natural numbers.
2. For every value in $(0, \infty)$, there exists a unique corresponding value in $[0, \infty)$, and vice versa.

Based on these properties, we can conclude that $|(0,\infty)| = |[0,\infty)|$, confirming the desired equality.

d. $|[0,1)| = |[0,\infty)| = |(0,\infty)| = |(0,1)|$.

Prove: Let's prove that the cardinality of the sets $[0,1)$, $[0, \infty)$, $(0, \infty)$, and $(0,1)$ are all equal.

1. To show the equality between [0,1) and [0, ∞): We can use $f(x) = \frac{1}{1-x}$ to establish a one to one correspondence between these two sets. The following properties hold: - $f(x)$ is a one to one function between [0,1) and [0, ∞). - $f(x)$ takes distinct values in each set.

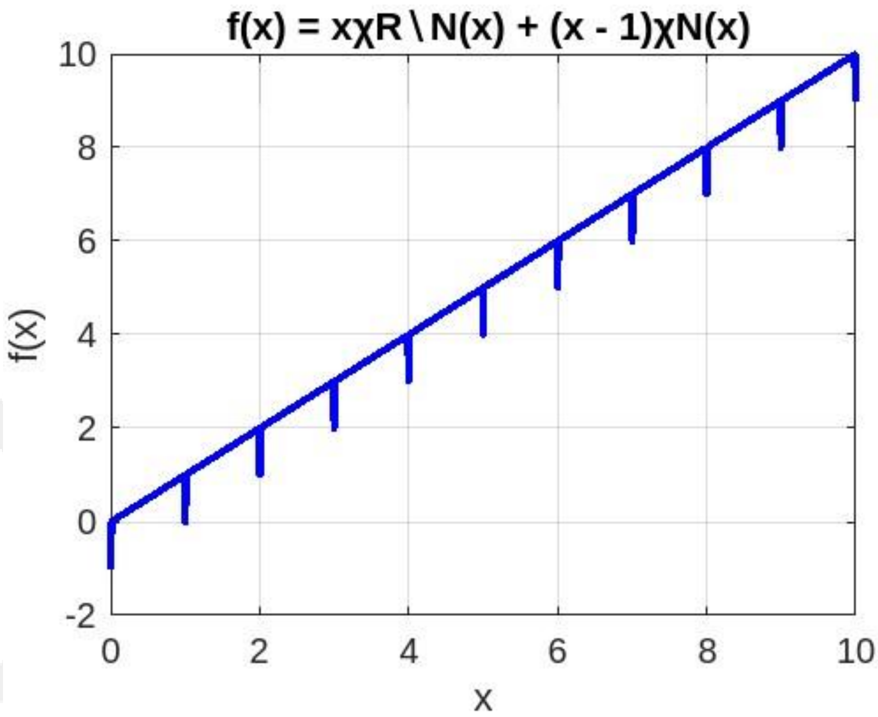

Figure 6.2 cardinalities of sets 3

For every value in [0,1), there is a unique corresponding value in [0, ∞), and vice versa. Based on these properties, we can say that |[0,1)| = |[0,∞)|.

1. Now, to prove the equality between [0, ∞) and (0, ∞): We can use $g(x) = x-1$ to establish a one to one correspondence between these two sets. The following properties hold: - $g(x)$ is a one to one between [0, ∞) and (0, ∞). - $g(x)$ takes distinct values in each set. For every value in [0, ∞), there is a unique corresponding value in (0, ∞), and vice versa.

Therefore, |[0,∞)| = |(0,∞)|.

2. Finally, to demonstrate the equality between (0, ∞) and (0,1): We can use the function $h(x) = \frac{x}{x+1}$ to establish a one to one correspondence between these two sets. The following properties hold: - $h(x)$ is a one to one between (0, ∞) and

(0,1). - $h(x)$ takes distinct values in each set. For every value in (0, ∞), there is a unique corresponding value in (0,1), and vice versa.

Consequently, |(0,∞)| = |(0,1)|.

In conclusion, the equality between all these sets has been established.

e. $|[0,1)| = \left|\left[0,\frac{1}{2}\right] \cup \left(\frac{1}{2}\right) = \left[0,\frac{1}{2}\right] \cup \left(\frac{1}{2},1\right)\right| = |[0,1)| = |(0,1)|$

Prove: Let's prove that the $[0,1)$, $\left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$, $\left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, sets and (0,1) all have the same cardinality.

1. Between [0,1) and $\left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$:

We can use $f(x) = \frac{x}{2}$ to establish a one to one correspondence between these two sets. The following properties hold:

- $f(x)$ is a one to one function between [0,1) and

-f(x) takes distinct values in each set. $\left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$.

- For every value in [0,1), there is a unique corresponding value in $\left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$, and vice versa.

Based on these properties, we can say that $[0,1) = \left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$.

2. Between [0,1) and $[0,1) = \left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$

We use the same function $f(x) = \frac{x}{2}$ to establish a one to one correspondence between these two sets. The properties are the same as in the previous point.

3. Between [0,1) and (0,1):

We use the function $g(x) = x$ to establish a one to one correspondence between these two sets. The following properties hold:

- $g(x)$ is a one to one function between [0, 1) and (0,1).

- $g(x)$ takes distinct values in each set.

- For every value in [0, 1), there is a unique corresponding value in (0, 1), and vice versa.

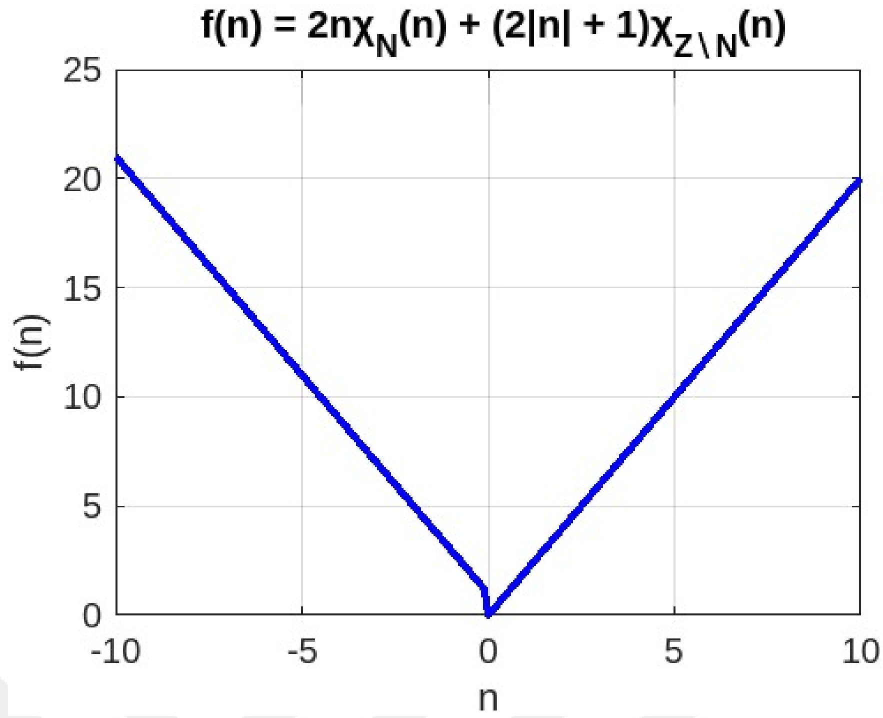

Figure 6.3 cardinalities of sets 4

Therefore, [0, 1) = (0, 1).

In conclusion,we have proven that $|[0,1)| = \left|\left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)\right| = \left|\left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)\right| = |(0,1)|$.

$|Z| = |N|$: f: $Z \rightarrow N$, $f(n) = 2n\chi_N(n) + (2|n| + 1)\chi_{R\backslash N}(n)$.

Prove: Let us prove that the cardinality of the set of integers Z is equal to the cardinality of the set of natural numbers N. We will use the following function to demonstrate this:

$f(n) = 2n\chi_N(n) + (2|n| + 1)\chi_{Z\backslash N}(n)$

where: - $\chi_N(n)$ is the characteristic function for the set of natural numbers. - $\chi_{Z\backslash N}(n)$ is the characterstic function for the difference between the set of integers and the set of natural numbers.

We will examine the following properties of this function:

1. The function $f(n)$ takes distinct values for natural numbers and non-natural numbers.

2. For every integer, there is a unique corresponding value in the set of natural numbers, and vice versa.

Based on these properties, we can conclude that $|Z| = |N|$, and thus, the desired equality has been proven.

f. $|Q \times Q| = |N|$: $f : Q \times Q \to N$, $f((n,m)) = 2^n(2m - 1)$.

Prove: Let's prove that the cardinality of the Cartesian product of rational numbers Q×Q is equal to the cardinality of the set of natural numbers N. We will use the following function to demonstrate this:

$$f((n,m)) = 2n\,(2m - 1)$$

Let's examine the following properties:

1. The function $f((n, m))$ takes distinct values for all pairs (n,m) in Q×Q.
2. For every pair (n, m) in Q × Q, there is a unique corresponding value in N, and vice versa.

Focusing on these properties:

-The first property ensures that the function uniquely maps each pair of rational numbers, indicating that it operates uniquely on each pair.

-The second property means that each natural number has a unique corresponding pair in Q × Q, and vice versa.

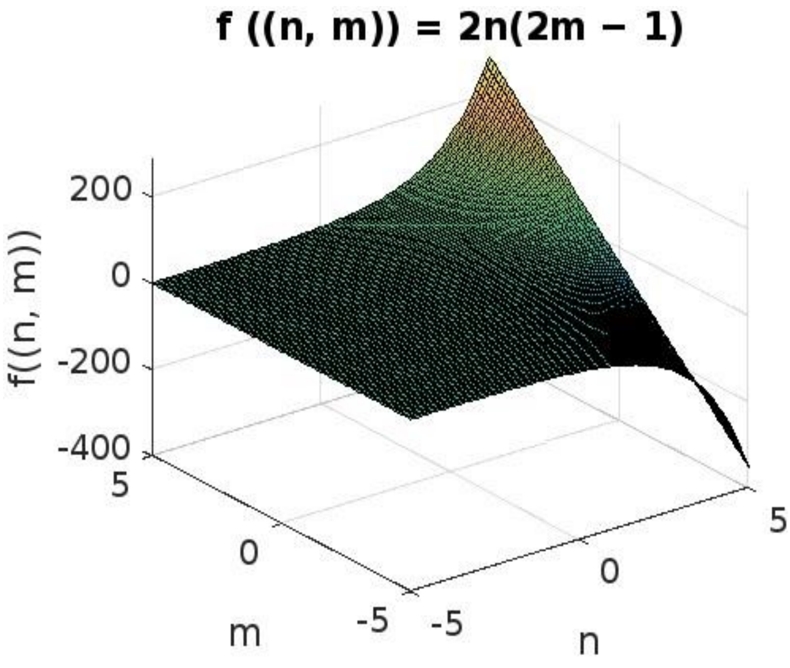

Figure 6.4 cardinalities of sets 5

Since both properties are satisfied, we can conclude that $|Q \times Q| = |N|$, indicating that the cardinality of the Cartesian product of rational numbers is equal to the cardinality of the set of natural numbers.

Theorem 6.2. *Let X be a set with $|X| \leqslant |N|$. Then X is finite or $|X| = |N|$.*

***Proof*.** Suppose that X is not finite. Without any restriction, we can suppose that

$X \subset N$. Define $f : X \to N$ by, $f(1) = \min X$ and

$f(n + 1) = \min\{f(1), f(2), \cdots, f(n)\}$.

Clearly, f is one to one .

Theorem 5.3. $|\wp(\mathbb{N})| = |\mathbb{R}|$

***Proof*.** $\wp(\mathbb{N}) \to [0,1]$ Be defined by

$$f(A) = \sum_{n=1}^{\infty} \chi_A(n) 3^{-n}$$

is a one to one, so

$$|\wp(\mathbb{N}| \leq ||0,1||.$$

Also,$g: \wp(\mathbb{N}) \to [0,1]$ defined by

$$f(A) = \sum_{n=1}^{\infty} \chi_A(n) 2^{-n}$$

is an onto. This implies

$$|[0,1]| \leq |\wp(\mathbb{N}|.$$

Combining those, we have the required result.

Theorem 6.4. $|R| = |R \times R|$

Proof. We $|\mathbb{R}| = |\wp(\mathbb{N})| = |\wp(\mathbb{Z} \setminus \mathbb{N})|$. have this implies that

$$|\mathbb{R} \times \mathbb{R}| = |\wp(\mathbb{N}) \times \wp(\mathbb{Z} \setminus \mathbb{N})| = |\wp(\mathbb{Z})| = |\wp(\mathbb{N}| = |\wp(\mathbb{R}|.$$

Finally, we give the following two theorems without proof.

Theorem 6.5. *Let* $I \subset \mathrm{R}$ *be an interval and* $C(I)$ *be the set of all continuous functions from* I *into* R. *Then* $|C(I)| = |\mathrm{R}|$.

Theorem 6.6. *Let* X *be a nonempty set with* $|X| \leqslant |\mathrm{R}|$, *and* X^{N} *be the set of all functions from* N *into* X. *Then,*

$|X^{\mathrm{N}}| = |\mathrm{R}|$.

The above results are taken from *A second course on real functions* written by A. C. M. Van Rooij and W. H. Schikhof [24].

7. RESULTS AND RECOMENDATIONS

7.1 Results.

(1) Discovery of Set Theory:

- Georg Cantor, a brilliant mathematician, played a pivotal role in developing set theory in the late nineteenth century.
- He introduced the concept of the infinite and non-countable sets.
- Cantor demonstrated the existence of multiple levels of infinity.

(2) Contributions to Topology:

- Cantor laid the foundations for topology by understanding spatial arrangementsand sets.
- He introduced ideas about dimensions and advanced concepts in the studyof space.

(3) Impact on Mathematics:

- Cantor's ideas brought a radical change to the way we understand mathematicsand infinity.
- His thoughts formed the basis for continued evolution in various fields of modern mathematics.

7.2 Recommendations.

(1) Study More of George Cantor's Works:

- Complete an in-depth study of his original works to understand the nuances of his set theory and topology.

(2) Examine Cultural and Social Influences:

- Investigate how the cultural and social context of Cantor's time influenced his discoveries and mathematical understanding.

(3) Explore Applications of Set Theory:

- Examine how set theory can be applied in areas outside of mathematics.

(4) Investigate Modern Topology:

- Explore how Cantor’s concepts in topology have evolved and how they have been influenced by recent research in the field.

(5) Examine the Impact on Educational Systems:

- Investigate how schools of mathematics have been influenced by Georg Cantor's ideas and how they have been incorporated into curricula.

8. REFERENCES

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