150008

FRACTIONAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

150008

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
ÇANKAYA UNIVERSITY

BY

TANSEL AVKAR

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF

MASTER OF SCIENCE

IN

THE DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

JUNE 2004

Approval of the Graduate School of Natural and Applied Sciences.

Prof. Dr. Yurdahan Güler
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Yurdahan Güler
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Baleanu

Assoc. Prof. Dr. Dumitru Baleanu
Supervisor

Examining Committee Members

Prof. Dr. Kenan Taş

Assoc. Prof. Dr. Dumitru Baleanu — Baleanu

Prof. Dr. Hüseyin Hüseyinov

ABSTRACT

FRACTIONAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

Avkar, Tansel

M. S., Department of Mathematics and Computer Science

Supervisor: Assoc. Prof. Dr. Dumitru Baleanu

June 2004, 67 pages

The Laplace transform method for solving fractional differential equations is presented. The fractional diffusion and fractional Schrödinger equations together with their properties are investigated. The Lagrangians linear in velocities are analyzed using the fractional calculus, and the fractional Euler-Lagrange equations are derived.

Keywords: Fractional derivative, fractional integral, fractional diffusion equation, fractional Schrödinger equation, fractional Euler-Lagrange equations.

ÖZ

KESİRSEL DİFERANSİYEL DENKLEMLER VE UYGULAMALARI

Avkar, Tansel

Yüksek Lisans, Matematik ve Bilgisayar Bölümü

Tez Yöneticisi: Doç. Dr. Dumitru Baleanu

Haziran 2004, 67 sayfa

Kesirsel diferansiyel denklemleri çözmek için Laplace dönüşüm metodu sunulmaktadır. Kesirsel difüzyon ve kesirsel Schrödinger denklemleri özellikleri ile beraber araştırılmaktadır. Kesirsel kalkulus yardımıyla doğrusal hızlı Lagrangianlar analiz edilmekte ve kesirsel Euler-Lagrange denklemleri elde edilmektedir.

Anahtar Kelimeler: Kesirsel türev, kesirsel integral, kesirsel difüzyon denklemi, kesirsel Schrödinger denklemi, kesirsel Euler-Lagrange denklemleri.

To my parents, Yüksel and Şerife,

my elder brother, Göksel

and my wife, Seçil...

ACKNOWLEDGMENTS

Acknowledgment is due to Çankaya University for the facilities and support provided for the completion of this thesis for the degree of Master of Science in Mathematics and Computer Science.

I wish to thank Assoc. Prof. Dr. Dumitru BALEANU, my supervisor, for his excellent helpful guidance, and deliberate efforts to equip this thesis with innovative applications of Fractional Calculus.

I extend my thanks to Prof. Dr. Yurdahan GÜLER, the director of the Graduate School of Natural and Applied Sciences, for his understanding and contributions in reviewing our efforts in this thesis.

I would like to express my deep gratitude to my spouse Seçil Yücel AVKAR for her precious support, inestimable encouragements and endless patience.

I also extend my thanks to technical specialist Ahmet KABARCIK for his constant technical help throughout this thesis. Last but not least, I thank to my family and friends for their moral support and memorable days we shared together.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Fractional calculus is a branch of mathematics dealing with generalization of well-known differentiation and integration operations to arbitrary real or complex orders. The mathematical idea of fractional calculus, which goes back to the seventeenth century, has been investigated by many mathematicians for years [1, 2, 3]. The topic of fractional calculus has found applications in various fields of science, engineering and finance [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

One of the problems encountered in this field is what kind of fractional derivatives will be used instead of integer order derivative for a given problem [1, 2, 3, 13, 14]. Depending on the specified physical situation different authors have applied different derivatives [6, 9].

The exterior calculus has been generalized by using fractional derivatives in the definition of the standard exterior derivative [15, 16, 17].

For the past two decades, the fractional generalization of the standard diffusion equation, which is gaining a great importance in economics, has been considered in [19, 20, 21, 22].

Feynman and Hibbs used a Gaussian probability distribution in the space of all possible paths, for a quantum mechanical particle, to derive the Schrödinger equation [23]. Then, Laskin constructed space fractional quantum mechanics using Feynman's path integral approach, the difference being the use of Lévy distributions instead of Gaussian distributions for the set of possible paths [24, 25, 26].

Nonconservative Lagrangian and Hamiltonian mechanics were investigated by Riewe within fractional calculus [27, 28]. Besides, Lagrangian and Hamiltonian fractional sequential mechanics, the models with symmetric fractional derivative were studied in [29, 30]. Recently, an extension of the simplest fractional variational problem and the fractional variational problem of Lagrange was obtained by Agrawal [31, 32]. A natural and interesting generalization of Agrawal's approach is to apply the fractional calculus to the constrained systems [33, 34, 35, 36].

The plan of this thesis is as follows:

In chapter II, some commonly used approaches in fractional calculus and their noteworthy properties are presented. A natural extension of the exterior derivative to the fractional exterior derivative is considered.

In chapter III, the Laplace transform method is presented and Laplace transformation formulas are obtained. The fundamental solution of the time-fractional diffusion equation for the Cauchy problem is derived. The time-fractional Schrödinger equation is examined.

In chapter IV, as an original contribution, the concept of fractional calculus is applied to the Lagrangians with linear velocities. The corresponding fractional Euler-Lagrange equations are analyzed, and their explicit solutions are obtained.

Chapter V is devoted to conclusions, and for the sake of convenience and completeness, some essential notions of special functions in fractional calculus are presented in Appendix A.

CHAPTER II

FRACTIONAL DERIVATIVES AND INTEGRALS

Fractional derivatives and fractional integrals are the subjects of study in the field of fractional calculus. Throughout this thesis, the symbols

$$ {}_aD_t^{\alpha} f(t) \qquad \text{and} \qquad {}_aD_t^{-\alpha} f(t) \ , \tag{II.1} $$

where the order 'α' is a non-integer positive real or complex number, and 'a' and 't' are lower and upper limits, are used to denote the fractional-order derivative and fractional-order integration of the function $f(t)$, respectively. There are several definitions for the fractional derivatives and integrals.

II.1 Grünwald-Letnikov Fractional Derivatives

The notions of differentiation of integer order n and n-fold integration are usually presented separately in standard calculus. Before getting the formulas corresponding to the Grünwald-Letnikov fractional differentiation and integration, we need to describe an approach to the unification of these two notions.

Consider a continuous function $f(t)$. The first order derivative of the function $f(t)$

is defined by

$$f'(t) = \frac{df}{dt} = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h} \ . \tag{II.2}$$

By the method of mathematical induction, one can obtain

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh) \ , \tag{II.3}$$

where

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!} \tag{II.4}$$

is the binomial coefficient.

Now, consider the following expression generalizing the fractions in (II.2) and (II.3)

$$f_h^{(p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t - rh) \ , \tag{II.5}$$

where p is an arbitrary integer.

Clearly, for $p \leq n$ we have

$$\lim_{h \to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p} \ . \tag{II.6}$$

Denoting

$$\begin{bmatrix} p \\ r \end{bmatrix} = \frac{(p+r-1)!}{(p-1)!\, r!} \ , \tag{II.7}$$

we have

$$\binom{-p}{r} = (-1)^r \begin{bmatrix} p \\ r \end{bmatrix} \ , \tag{II.8}$$

and replacing p in (II.5) by $-p$ we obtain

$$f_h^{(-p)}(t) = h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t - rh) \ . \tag{II.9}$$

If n is fixed, then $f_h^{(-p)}(t)$ tends to the uninteresting limit 0 as $h \to 0$. To have a non-zero limit, take $h = \frac{t-a}{n}$ where a is a real constant and denote

$$\lim_{\substack{h \to 0 \\ nh = t-a}} f_h^{(-p)}(t) = {}_aD_t^{-p} f(t) \ . \tag{II.10}$$

By induction on p, we obtain

$${}_aD_t^{-p} f(t) = \frac{1}{(p-1)!} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau \ . \tag{II.11}$$

Now let us show that the formula (II.11) is a representation of a p-fold integral. Differentiating both sides of (II.11) with respect to t, we have

$$\frac{d}{dt} \left({}_aD_t^{-p} f(t) \right) = \frac{1}{(p-2)!} \int_a^t (t-\tau)^{p-2} f(\tau) d\tau = {}_aD_t^{-p+1} f(t) \ , \tag{II.12}$$

and then integrating from a to t

$${}_aD_t^{-p} f(t) = \int_a^t \left({}_aD_t^{-p+1} f(t) \right) dt \ . \tag{II.13}$$

Similarly,

$${}_aD_t^{-p+1} f(t) = \int_a^t \left({}_aD_t^{-p+2} f(t) \right) dt \text{ , etc.,} \tag{II.14}$$

and therefore

$$\begin{aligned} {}_aD_t^{-p} f(t) &= \int_a^t dt \int_a^t \left({}_aD_t^{-p+2} f(t) \right) dt \\ &= \int_a^t dt \int_a^t dt \int_a^t \left({}_aD_t^{-p+3} f(t) \right) dt \\ &= \underbrace{\int_a^t dt \int_a^t dt \dots \int_a^t}_{p \ times} f(t) dt \ . \end{aligned} \tag{II.15}$$

Therefore, we conclude that the derivative of an integer order n (II.3) and the p-fold integral (II.11) are particular cases of the general expression

$$ {}_aD_t^p f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) \ , \tag{II.16} $$

which shows the derivative of order m if $p = m$ and the m-fold integral if $p = -m$.

II.1.1 Integrals of Arbitrary Order

Consider the negative values of p in (II.16) where p is real number. Replacing p by $-p$, we have

$$ {}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{p} \sum_{r=0}^{n} \begin{bmatrix} p \\ \\ r \end{bmatrix} f(t-rh) \ . \tag{II.17} $$

Let us consider the following theorem to prove the existence of the limit (II.17) and to evaluate it.

Theorem 1 *Let $\{\beta_k\}_{k=1,2,\ldots}$ be a sequence of real numbers. Let $\{\alpha_{n,k}\}_{n,k=1,2,\ldots}$ be a sequence of functions. If*

$$ \lim_{k\to\infty} \quad \beta_k = 1 \ , \tag{II.18} $$

$$ \lim_{n\to\infty} \quad \alpha_{n,k} = 0 \quad \forall k \ , \tag{II.19} $$

$$ \lim_{n\to\infty} \quad \sum_{k=1}^{n} \alpha_{n,k} = \mathbf{A} \ , \tag{II.20} $$

$$ \sum_{k=1}^{n} \quad |\alpha_{n,k}| < K \quad \forall n \ , \tag{II.21} $$

then

$$ \lim_{n\to\infty} \quad \sum_{k=1}^{n} \alpha_{n,k}\beta_k = \mathbf{A} \ . \tag{II.22} $$

Proof: Let $\beta_k = 1 - \sigma_k$ where $\lim_{k\to\infty} \sigma_k = 0$. It follows from (II.19) that for every fixed r we have

$$\lim_{n\to\infty} \sum_{k=1}^{r-1} \alpha_{n,k}\beta_k = 0 \ , \tag{II.23}$$

$$\lim_{n\to\infty} \sum_{k=1}^{r-1} \alpha_{n,k} = 0 \ . \tag{II.24}$$

Now, consider

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k}\beta_k &\overset{by(II.23)}{=} \lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k}\beta_k \\
&= \lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k} - \lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k}\sigma_k \\
&\overset{by(II.24)}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k} - \lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k}\sigma_k \\
&\overset{by(II.20)}{=} \mathbf{A} - \lim_{\mathbf{n}\to\infty} \sum_{\mathbf{k}=\mathbf{r}}^{\mathbf{n}} \alpha_{\mathbf{n},\mathbf{k}}\sigma_{\mathbf{k}} \ .
\end{aligned} \tag{II.25}$$

Now, let us estimate the following

$$\begin{aligned}
\left|\mathbf{A} - \lim_{\mathbf{n}\to\infty} \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} \alpha_{\mathbf{n},\mathbf{k}}\beta_{\mathbf{k}}\right| &= \left|\lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k}\sigma_k\right| \\
&\leq \lim_{n\to\infty} \sum_{k=r}^{n} |\alpha_{n,k}|\,|\sigma_k| \\
&\leq \sigma^* \lim_{n\to\infty} \sum_{k=r}^{n} |\alpha_{n,k}| \\
&\overset{by(II.24)}{=} \sigma^* \lim_{n\to\infty} \sum_{k=1}^{n} |\alpha_{n,k}| \\
&\overset{by(II.21)}{<} \sigma^* K \ ,
\end{aligned} \tag{II.26}$$

where $\sigma^* = \max_{k\geq r} |\sigma_k|$.

It follows from $\lim_{k\to\infty} \sigma_k = 0$ that for every arbitrarily small $\varepsilon > 0$ there exists r such that $\sigma^* < \varepsilon/K$. Therefore, we obtain

$$\left|\mathbf{A} - \lim_{\mathbf{n}\to\infty} \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} \alpha_{\mathbf{n},\mathbf{k}}\beta_{\mathbf{k}}\right| < \varepsilon \ , \tag{II.27}$$

which completes the proof of the theorem.

Now, our aim is to evaluate the limit (II.17) with the help of the theorem above. To do this, write

$$
\begin{aligned}
{}_aD_t^{-p}f(t) &= \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \left[\begin{matrix} p \\ \\ r \end{matrix}\right] f(t-rh) \\
&= \lim_{\substack{h\to 0\\ nh=t-a}} \sum_{r=0}^{n} \frac{1}{r^{p-1}} \left[\begin{matrix} p \\ \\ r \end{matrix}\right] h(rh)^{p-1} f(t-rh) \\
&= \frac{1}{\Gamma(p)} \lim_{\substack{h\to 0\\ nh=t-a}} \sum_{r=0}^{n} \frac{\Gamma(p)}{r^{p-1}} \left[\begin{matrix} p \\ \\ r \end{matrix}\right] h(rh)^{p-1} f(t-rh) \\
&= \frac{1}{\Gamma(p)} \lim_{n\to\infty} \sum_{r=0}^{n} \beta_r\ \alpha_{n,r}\ , \qquad \text{(II.28)}
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_r &= \frac{\Gamma(p)}{r^{p-1}} \left[\begin{matrix} p \\ \\ r \end{matrix}\right]\ , \qquad \text{(II.29)} \\
\alpha_{n,r} &= \left(\frac{t-a}{n}\right)\left(r\frac{t-a}{n}\right)^{p-1} f\left(t-r\frac{t-a}{n}\right)\ . \qquad \text{(II.30)}
\end{aligned}
$$

Note that

$$
\lim_{n\to\infty} \alpha_{n,r} = 0 \ \ \forall r\ . \qquad \text{(II.31)}
$$

Moreover

$$
\lim_{r\to\infty} \beta_r = 1\ , \qquad \text{(II.32)}
$$

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{r=0}^{n}\alpha_{n,r} &= \lim_{n\to\infty}\sum_{r=0}^{n}\left(\frac{t-a}{n}\right)\left(r\frac{t-a}{n}\right)^{p-1} f\left(t-r\frac{t-a}{n}\right) \\
&= \lim_{h\to 0}\sum_{r=0}^{n} h(rh)^{p-1} f(t-rh) \\
&= \int_a^t (t-\tau)^{p-1} f(\tau)d\tau =: \mathbf{A}\ . \qquad \text{(II.33)}
\end{aligned}
$$

Applying Theorem 1, we conclude that

$$ {}_aD_t^{-p}f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau)d\tau \ , \tag{II.34} $$

where $\Gamma(\bullet)$ denotes the Gamma function (see A.1-4).

II.1.2 Derivatives of Arbitrary Order

Now, our aim is to evaluate the following limit

$$ {}_aD_t^p f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(p)}(t) \ , \tag{II.35} $$

where

$$ f_h^{(p)}(t) = h^{-p}\sum_{r=0}^{n}(-1)^r \binom{p}{r} f(t-rh) \ . \tag{II.36} $$

To evaluate (II.35), first we should transform (II.36) as follows

$$ \begin{aligned} f_h^{(p)}(t) &= h^{-p}\sum_{r=0}^{n}(-1)^r\binom{p-1}{r}f(t-rh) + h^{-p}\sum_{r=1}^{n}(-1)^r\binom{p-1}{r-1}f(t-rh) \\ &= h^{-p}\sum_{r=0}^{n}(-1)^r\binom{p-1}{r}f(t-rh) \\ &\quad +h^{-p}\sum_{r=0}^{n-1}(-1)^{r+1}\binom{p-1}{r}f(t-(r+1)h) \\ &= (-1)^n\binom{p-1}{n}h^{-p}f(a) \\ &\quad +h^{-p}\sum_{r=0}^{n-1}(-1)^r\binom{p-1}{r}\left[f(t-rh)-f(t-(r+1)h)\right] \\ &= (-1)^n\binom{p-1}{n}h^{-p}f(a) + h^{-p}\sum_{r=0}^{n-1}(-1)^r\binom{p-1}{r}\Delta f(t-rh) \\ &= (-1)^n\binom{p-1}{n}h^{-p}f(a) + (-1)^{n-1}\binom{p-2}{n-1}h^{-p}\Delta f(a+h) \\ &\quad +h^{-p}\sum_{r=0}^{n-2}(-1)^r\binom{p-2}{r}\Delta^2 f(t-rh) \\ &= \ldots \end{aligned} $$

$$= \sum_{k=0}^{m}(-1)^{n-k}\binom{p-k-1}{n-k}h^{-p}\Delta^k f(a+kh)$$
$$+h^{-p}\sum_{r=0}^{n-m-1}(-1)^r\binom{p-m-1}{r}\Delta^{m+1}f(t-rh)\ . \tag{II.37}$$

Evaluation of the limit of the k-th term in the first sum in (II.37) gives

$$\lim_{\substack{h\to 0\\ nh=t-a}}(-1)^{n-k}\binom{p-k-1}{n-k}h^{-p}\Delta^k f(a+kh) = \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)}\ . \tag{II.38}$$

On the other hand, applying Theorem 1, the limit of the second sum in (II.37) is evaluated as

$$\lim_{\substack{h\to 0\\ nh=t-a}} h^{-p}\sum_{r=0}^{n-m-1}(-1)^r\binom{p-m-1}{r}\Delta^{m+1}f(t-rh)$$
$$= \frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau\ . \tag{II.39}$$

Using (II.38) and (II.39), we conclude that

$${}_aD_t^p f(t) = \sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau \tag{II.40}$$

under the assumption that the derivatives $f^{(k)}(t),\ k=1,2,\ldots,m+1$ are continuous in $[a,t]$ and that m is an integer satisfying $m > p-1$. The smallest possible value for m is determined by $m < p < m+1$.

II.1.3 Fractional Derivative of Power Function

Consider the power function

$$f(t) = (t-a)^{\nu}\ , \tag{II.41}$$

where ν is a real number.

First, we will consider the negative values of p. To do this, in (II.34), replace p by $-p$ and apply new formula to the above function. Then, we have

$$ {}_aD_t^p(t-a)^\nu = \frac{1}{\Gamma(-p)} \int_a^t (t-\tau)^{-p-1}(\tau-a)^\nu d\tau \ . \tag{II.42} $$

For the convergence of the integral, we need to suppose that $\nu > -1$. Then, making the substitution $\tau = a + \xi(t-a)$ in (II.42), we obtain

$$ \begin{aligned} {}_aD_t^p(t-a)^\nu &= \frac{1}{\Gamma(-p)}(t-a)^{\nu-p} \int_0^1 \xi^\nu (1-\xi)^{-p-1} d\xi \\ &= \frac{1}{\Gamma(-p)} B(-p, \nu+1)(t-a)^{\nu-p} \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu-p+1)}(t-a)^{\nu-p} \ , \end{aligned} \tag{II.43} $$

where $p < 0$, $\nu > -1$ and B denotes the Beta function (see A.5-6).

For the positive values of p, assume that $0 \leq m \leq p < m+1$. In this case, we will apply the formula (II.40) to the function $f(t)$ defined by (II.41). Again, for the convergence of the integral in (II.40), we need to suppose that $\nu > m$. Then, we have

$$ {}_aD_t^p(t-a)^\nu = \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} \underbrace{\frac{d^{m+1}(\tau-a)^\nu}{d\tau^{m+1}}}_{\substack{\| \\ f^{(m+1)}(\tau)}} d\tau \ . \tag{II.44} $$

Noting that

$$ f^{(m+1)}(\tau) = \nu(\nu-1)\dots(\nu-m)(\tau-a)^{\nu-m-1} = \frac{\Gamma(\nu+1)(\tau-a)^{\nu-m-1}}{\Gamma(\nu-m)} \ , \tag{II.45} $$

and making the substitution $\tau = a + \xi(t-a)$, we obtain

$$ \begin{aligned} {}_aD_t^p(t-a)^\nu &= \frac{\Gamma(\nu+1)}{\Gamma(\nu-m)\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p}(\tau-a)^{\nu-m-1} d\tau \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu-m)\Gamma(-p+m+1)} B(-p+m+1, \nu-m)(t-a)^{\nu-p} \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu-p+1)}(t-a)^{\nu-p} \ . \end{aligned} \tag{II.46} $$

From (II.43) and (II.46), we conclude that

$$ {}_aD_t^p(t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)}(t-a)^{\nu-p} \ , \tag{II.47} $$

where $(p<0,\ \nu>-1)$ or $(0\le m\le p<m+1,\ \nu>m)$.

II.1.4 Composition with Integer-order Derivatives

Let us rewrite (II.40) by writing s instead of m

$$ {}_aD_t^p f(t) = \sum_{k=0}^{s}\frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+s+1)}\int_a^t (t-\tau)^{s-p} f^{(s+1)}(\tau)d\tau \ . \tag{II.48} $$

Evaluating the derivative of integer order n of the fractional derivative of real order p in the form (II.48) under the assumption $s\ge m+n-1$, we obtain

$$ \begin{aligned} \frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) &= \sum_{k=0}^{s}\frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} \\ &\quad + \frac{1}{\Gamma(-p-n+s+1)}\int_a^t (t-\tau)^{s-p-n} f^{(s+1)}(\tau)d\tau \\ &= {}_aD_t^{p+n} f(t) \ . \end{aligned} \tag{II.49} $$

Since $s\ge m+n-1$ is arbitrary, we can take $s=m+n-1$. Thus, we have

$$ \begin{aligned} \frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) &= \sum_{k=0}^{m+n-1}\frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} \\ &\quad + \frac{1}{\Gamma(m-p)}\int_a^t (t-\tau)^{m-p-1} f^{(m+n)}(\tau)d\tau \\ &= {}_aD_t^{p+n} f(t) \ . \end{aligned} \tag{II.50} $$

On the other hand, using (II.48) we obtain

$$ \begin{aligned} {}_aD_t^p\left(\frac{d^n}{dt^n}f(t)\right) &= \sum_{k=0}^{s}\frac{f^{(n+k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\ &\quad + \frac{1}{\Gamma(-p+s+1)}\int_a^t (t-\tau)^{s-p} f^{(n+s+1)}(\tau)d\tau \ . \end{aligned} \tag{II.51} $$

Substituting $s = m - 1$, we have

$$
\begin{aligned}
{}_aD_t^p\left(\frac{d^n}{dt^n}f(t)\right) &= \sum_{k=0}^{m-1}\frac{f^{(n+k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\
&\quad + \frac{1}{\Gamma(m-p)}\int_a^t (t-\tau)^{m-p-1}f^{(m+n)}(\tau)d\tau \, . \qquad \text{(II.52)}
\end{aligned}
$$

Comparing (II.50) and (II.52), we conclude that

$$
\frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) = {}_aD_t^p\left(\frac{d^n}{dt^n}f(t)\right) + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} \, . \qquad \text{(II.53)}
$$

It follows from (II.53) that the operations $\frac{d^n}{dt^n}$ and ${}_aD_t^p$ are *commutative* only if $f^{(k)}(a) = 0, \; k = 0, 1, \ldots, n-1$.

II.1.5 Composition with Fractional Derivatives

In this section, we will consider the fractional derivative of order q of a fractional derivative of order p. As the first case, suppose $\underline{p < 0}$.

Let us take $q < 0$. Then, applying the formula (II.34) following one another to any given function $f(t)$, one can easily obtain that

$$
{}_aD_t^q\left({}_aD_t^p f(t)\right) = {}_aD_t^{p+q}f(t) \, . \qquad \text{(II.54)}
$$

Now, consider the case $0 \leq n < q < n+1$. Taking into account that $q = (n+1) + (q-n-1)$, where $q - n - 1 < 0$, and using the formulas (II.49) and (II.54) we obtain

$$
\begin{aligned}
{}_aD_t^q\left({}_aD_t^p f(t)\right) &= \frac{d^{n+1}}{dt^{n+1}}\left\{{}_aD_t^{q-n-1}\left({}_aD_t^p f(t)\right)\right\} \\
&= \frac{d^{n+1}}{dt^{n+1}}\left\{{}_aD_t^{p+q-n-1}f(t)\right\} \\
&= {}_aD_t^{p+q}f(t) \, . \qquad \text{(II.55)}
\end{aligned}
$$

From (II.54) and (II.55), we conclude that if $p < 0$, then for any real q

$$ {}_aD_t^q \left({}_aD_t^p f(t)\right) = {}_aD_t^{p+q} f(t) \ . \tag{II.56} $$

As the second case, suppose $p > 0$. Besides, assume that $0 \leq m < p < m+1$. Let us rewrite the formula (II.40)

$$ \begin{aligned} {}_aD_t^p f(t) = & \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\ & + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau) d\tau \ . \end{aligned} \tag{II.57} $$

First, take $q < 0$. Note that the functions $(t-a)^{-p+k}$ have non-integrable singularities for $k = 0, 1, \ldots, m-1$. Therefore, the derivative of real order q of ${}_aD_t^p f(t)$ exists only if

$$ f^{(k)}(a) = 0, \quad k = 0, 1, \ldots, m-1 \ . \tag{II.58} $$

The integral in the right hand side of (II.57) is equal to ${}_aD_t^{p-m-1} f^{(m+1)}(t)$. Therefore, under the conditions (II.58), we can rewrite (II.57) as follows

$$ {}_aD_t^p f(t) = \frac{f^{(m)}(a)(t-a)^{-p+m}}{\Gamma(-p+m+1)} + {}_aD_t^{p-m-1} f^{(m+1)}(t) \ . \tag{II.59} $$

It follows from (II.59) that

$$ \begin{aligned} {}_aD_t^q \left({}_aD_t^p f(t)\right) & = \frac{f^{(m)}(a)(t-a)^{-p-q+m}}{\Gamma(-p-q+m+1)} \\ & + \frac{1}{\Gamma(-p-q+m+1)} \int_a^t \frac{f^{(m+1)}(\tau)}{(t-\tau)^{p+q-m}} d\tau \ . \end{aligned} \tag{II.60} $$

Taking into account (II.57) and (II.58), we obtain

$$ {}_aD_t^q \left({}_aD_t^p f(t)\right) = {}_aD_t^{p+q} f(t) \ . \tag{II.61} $$

Now, let us consider the case $0 \leq n < q < n+1$. Assume that $f(t)$ satisfies the conditions (II.58) and take into account that $q-n-1 < 0$. Now, one can use the formula (II.61) as seen below

$$
\begin{aligned}
{}_aD_t^q\left({}_aD_t^p f(t)\right) &= \frac{d^{n+1}}{dt^{n+1}}\left\{{}_aD_t^{q-n-1}\left({}_aD_t^p f(t)\right)\right\} \\
&= \frac{d^{n+1}}{dt^{n+1}}\left\{{}_aD_t^{p+q-n-1} f(t)\right\} \\
&= {}_aD_t^{p+q} f(t)\ . \qquad \text{(II.62)}
\end{aligned}
$$

Therefore, we conclude that if $p < 0$, then (II.61) holds for arbitrary real q, and also that if $0 \leq m < p < m+1$, then (II.61) holds for arbitrary real q, under the assumption that the function $f(t)$ satisfies the conditions given in (II.58).

II.2 Riemann-Liouville Fractional Derivatives

Among several approaches, one that has been often used in the literature on fractional calculus is the one known as the Riemann-Liouville fractional integration. This approach can be interpreted as a generalization of Cauchy's repeated integration formula. For that reason, let us start with the derivation of Cauchy formula.

Suppose that $f(\tau)$ is continuous and integrable in every finite interval (a,t). Note that the function $f(t)$ may have an integrable singularity of order $r < 1$ at the point $\tau = a$, i.e.,

$$\lim_{\tau \to a}(\tau - a)^r f(t) = const(\neq 0). \qquad \text{(II.63)}$$

Then, we obtain that the integral

$$f^{(-1)}(t) = \int_a^t f(\tau)d\tau \qquad \text{(II.64)}$$

exists and has a finite value; it is equal to zero when $t \to a$. Therefore, one can consider the two fold integral

$$\begin{aligned} f^{(-2)}(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau)d\tau = \int_a^t \int_a^{\tau_1} f(\tau)d\tau d\tau_1 \\ &= \int_a^t \int_\tau^t f(\tau)d\tau_1 d\tau = \int_a^t (t-\tau)f(\tau)d\tau \ . \end{aligned} \tag{II.65}$$

Integration of (II.65) gives us

$$f^{(-3)}(t) = \frac{1}{2}\int_a^t (t-\tau)^2 f(\tau)d\tau \ , \tag{II.66}$$

and using the method of mathematical induction we obtain the *Cauchy Formula*

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau \ . \tag{II.67}$$

II.2.1 Integrals of Arbitrary Order

In the Cauchy formula (II.67), replacing n with a non-integer positive number p, the formula for the Riemann-Liouville fractional integration is obtained as follows

$${}_a\mathbf{D}_t^{-p} f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1} f(\tau)d\tau \ , \quad p > 0, \ t > a \ . \tag{II.68}$$

By the method of integration by parts, one can obtain the following expression under the assumption that $f(t)$ has continuous derivatives for $t \geq 0$

$$\lim_{p\to 0} {}_a\mathbf{D}_t^{-p} f(t) = f(t) \ . \tag{II.69}$$

If $f(t)$ is continuous for $t \geq a$, then integration of arbitrary real order defined by (II.68) has the following important property

$${}_a\mathbf{D}_t^{-p}\left({}_a\mathbf{D}_t^{-q} f(t)\right) = {}_a\mathbf{D}_t^{-p-q} f(t) \ . \tag{II.70}$$

II.2.2 Derivatives of Arbitrary Order

For the Riemann-Liouville fractional derivatives, the formula (II.68) can still be used if it is combined with the following additional step

$$ {}_a\mathbf{D}_t^p f(t) = \frac{d^k}{dt^k}\left({}_a\mathbf{D}_t^{-(k-p)} f(t)\right), \quad p > 0 , \tag{II.71}$$

where k is a positive integer and chosen so that $k-p > 0$. Thus, the Riemann-Liouville fractional integration can be applied for ${}_a\mathbf{D}_t^{-(k-p)} f(t)$. Then, $\frac{d^k}{dt^k}$ is the ordinary k-th order differential operator. Therefore, we may write

$$ {}_a\mathbf{D}_t^p f(t) = \frac{1}{\Gamma(k-p)}\left(\frac{d}{dt}\right)^k \int_a^t (t-\tau)^{k-p-1} f(\tau)d\tau \ , \quad \left\{ \begin{array}{c} p > 0, \ t > a \\ k-1 \leq p < k \end{array} \right\}. \tag{II.72}$$

The formula (II.72) is also called as the left Riemann-Liouville fractional derivative. Additionally, the right Riemann-Liouville fractional derivative is defined as

$$ {}_t\mathbf{D}_b^p f(t) = \frac{1}{\Gamma(k-p)}\left(-\frac{d}{dt}\right)^k \int_t^b (\tau-t)^{k-p-1} f(\tau)d\tau \ , \quad p > 0, \ t < b \ . \tag{II.73}$$

If p is any integer, the relations to the usual ones are as follows

$$ {}_a\mathbf{D}_t^p = \left(\frac{d}{dt}\right)^p , \quad {}_t\mathbf{D}_b^p = \left(-\frac{d}{dt}\right)^p \ . \tag{II.74}$$

Now, we will consider some properties of the Riemann-Liouville fractional derivatives. The most important property of the Riemann-Liouville fractional derivative is given by

$$ {}_a\mathbf{D}_t^p \left({}_a\mathbf{D}_t^{-p} f(t)\right) = f(t) \ , \quad p > 0 \ , \ t > a \ , \tag{II.75}$$

which means that the Riemann-Liouville fractional differentiation operator is a left inverse to the Riemann-Liouville fractional integration operator of the same order.

The property (II.75) is a particular case of a more general property

$$ {}_a\mathbf{D}_t^p \left({}_a\mathbf{D}_t^{-q} f(t) \right) = {}_a\mathbf{D}_t^{p-q} f(t) \tag{II.76} $$

under the assumptions that $f(t)$ is continuous, and that if $p \geq q \geq 0$, then ${}_a\mathbf{D}_t^{p-q} f(t)$ exists.

As with usual integer-order differentiation and integration, fractional differentiation and integration do not commute.

If the fractional derivative ${}_a\mathbf{D}_t^p f(t)$, $(k-1 \leq p < k)$ of the function $f(t)$ is integrable, then

$$ {}_a\mathbf{D}_t^{-p} \left({}_a\mathbf{D}_t^p f(t) \right) = f(t) - \sum_{j=1}^{k} \left[{}_a\mathbf{D}_t^{p-j} f(t) \right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)} \ . \tag{II.77} $$

The property (II.77) is also a particular case of a more general property

$$ {}_a\mathbf{D}_t^{-p} \left({}_a\mathbf{D}_t^q f(t) \right) = {}_a\mathbf{D}_t^{q-p} f(t) - \sum_{j=1}^{k} \left[{}_a\mathbf{D}_t^{q-j} f(t) \right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)} \ , \tag{II.78} $$

where $0 \leq k-1 \leq q < k$.

II.2.3 Fractional Derivative of Power Function

Similarly to the Grünwald-Letnikov approach, one can conclude that the Riemann-Liouville fractional derivative of the power function $(t-a)^\nu$ is given by the formula

$$ {}_a\mathbf{D}_t^p (t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)} (t-a)^{\nu-p} \ , \tag{II.79} $$

which is the same as the relationship (II.47). In this case, the only restriction is $\nu > -1$.

II.2.4 Composition with Integer-order Derivatives

From the formula (II.72), one can easily obtain

$$\frac{d^n}{dt^n}\left({}_a\mathbf{D}_t^p f(t)\right) = {}_a\mathbf{D}_t^{n+p} f(t)\ . \tag{II.80}$$

On the other hand, using (II.77) and (II.79) we have

$$\begin{aligned}
{}_a\mathbf{D}_t^p\left(\frac{d^n}{dt^n} f(t)\right) &= {}_a\mathbf{D}_t^{p+n}\left({}_a\mathbf{D}_t^{-n} f^{(n)}(t)\right) \\
&= {}_a\mathbf{D}_t^{p+n}\left(\frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f^{(n)}(\tau)d\tau\right) \\
&\overset{by(II.77)}{=} {}_a\mathbf{D}_t^{p+n}\left(f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)}\right) \\
&\overset{by(II.79)}{=} {}_a\mathbf{D}_t^{p+n} f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^{-p-n+j}}{\Gamma(-p-n+j+1)}\ ,
\end{aligned} \tag{II.81}$$

which is the same as the relationship (II.53).

It follows from (II.81) that the operations $\frac{d^n}{dt^n}$ and ${}_a\mathbf{D}_t^p$ are *commutative* only if $f^{(j)}(a) = 0,\ j = 0, 1, \ldots, n-1$.

II.2.5 Composition with Fractional Derivatives

Let $m - 1 \leq p < m$ and $n - 1 \leq q < n$. It follows from the relationships (II.71), (II.78) and (II.80) that

$${}_a\mathbf{D}_t^p\left({}_a\mathbf{D}_t^q f(t)\right) = {}_a\mathbf{D}_t^{p+q} f(t) - \sum_{j=1}^{n}\left[{}_a\mathbf{D}_t^{q-j} f(t)\right]_{t=a}\frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}\ . \tag{II.82}$$

Let us interchange p and q (and of course m and n) in (II.82):

$${}_a\mathbf{D}_t^q\left({}_a\mathbf{D}_t^p f(t)\right) = {}_a\mathbf{D}_t^{p+q} f(t) - \sum_{j=1}^{m}\left[{}_a\mathbf{D}_t^{p-j} f(t)\right]_{t=a}\frac{(t-a)^{-q-j}}{\Gamma(1-q-j)}\ . \tag{II.83}$$

If the relationships (II.82) and (II.83) are compared, then one can conclude that Riemann-Liouville fractional derivative operators ${}_a\mathbf{D}_t^p$ and ${}_a\mathbf{D}_t^q$ *commute* only if

$$\left[{}_a\mathbf{D}_t^{p-j} f(t)\right]_{t=a} = 0\,, \qquad j = 1, \ldots, m \tag{II.84}$$

and

$$\left[{}_a\mathbf{D}_t^{q-j} f(t)\right]_{t=a} = 0\,, \qquad j = 1, \ldots, n\,. \tag{II.85}$$

II.3 Caputo's Fractional Derivatives

We now observe that an alternative definition of the fractional derivative, originally introduced by Caputo in [12], is

$${}_a^C D_t^p f(t) = \frac{1}{\Gamma(n-p)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{p+1-n}}\,, \quad (n-1 < p < n)\,. \tag{II.86}$$

Now, assume that $0 \leq n-1 < p < n$ and that the function $f(t)$ has $n+1$ continuous bounded derivatives in every finite interval $[a,t]$. Then, we have

$$\begin{aligned} \lim_{p\to n} {}_a^C D_t^p f(t) &= \lim_{p\to n} \left\{ \frac{f^{(n)}(a)(t-a)^{n-p}}{\Gamma(n-p+1)} + \frac{1}{\Gamma(n-p+1)} \int_a^t \frac{f^{(n+1)}(\tau)d\tau}{(t-\tau)^{p-n}} \right\} \\ &= f^{(n)}(a) + \int_a^t f^{(n+1)}(\tau)d\tau = f^{(n)}(t)\,. \end{aligned} \tag{II.87}$$

Therefore, we conclude that for $p \to n$ the Caputo derivative becomes usual $n-$th order derivative of the function $f(t)$ under the above assumptions.

The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations.

To see the difference in the form of the initial conditions which must accompany fractional differential equations in terms of the Riemann-Liouville and Caputo derivatives, one should write the corresponding Laplace transform formulas (see section III.1.1).

Another difference between the Riemann-Liouville definition and the Caputo definition is that the Caputo derivative of a constant is 0, whereas in the cases of a finite value of the lower terminal 'a' the Riemann-Liouville fractional derivative of a constant C is not 0. Please, see the relationship (II.89).

II.4 Properties of Fractional Derivatives

Similarly to their integer-order counterparts, fractional operators are linear operators. That is,

$$D^p\left(\lambda f(t)+\mu g(t)\right)=\lambda\ D^p f(t)+\mu\ D^p g(t)\ , \tag{II.88}$$

where D^p denotes any notation of the fractional operator. Proof of the property (II.88) follows directly from the corresponding definition.

One of the distinctions between conventional and fractional derivatives is that of the *nonlocal* property of the fractional derivatives. As can be seen from (II.72), this means that not only do fractional derivatives of a function evaluated at a point 't' depend on the point 't' and its neighboring points, but also, they do depend on values of the function between 't' and the lower limit point 'a'. However, this is not the case for the conventional derivative.

Another interesting property of fractional derivatives can be observed when one evaluates the p-th order Riemann-Liouville fractional derivative of a constant function $f(t) = C$. According to (II.79), p-th order derivative of $f(t) = C$ (with the lower limit $a = 0$) can be obtained as

$$
{}_0\mathbf{D}_t^p C = \frac{C\Gamma(1)}{\Gamma(1-p)} t^{-p} = \frac{Ct^{-p}}{\Gamma(1-p)} \ , \quad t > 0 \ . \tag{II.89}
$$

The *Leibniz rule* for the fractional differentiation is given by

$$
{}_aD_t^p(\varphi(t)f(t)) = \sum_{k=0}^{\infty} \binom{p}{k} \varphi^{(k)}(t)\ {}_aD_t^{p-k} f(t) \ , \tag{II.90}
$$

where $\varphi(t)$ and $f(t)$ along with all its derivatives are continuous in $[a, t]$.

Remember that the well known rule for the differentiation of an integral depending on a parameter is given by

$$
\frac{d}{dt} \int_0^t F(t,\tau) d\tau = \int_0^t \frac{\partial F(t,\tau)}{\partial t} d\tau + F(t,t) \ . \tag{II.91}
$$

The relationship (II.91) has the following analogue for fractional-order differentiation

$$
{}_aD_t^\alpha \int_0^t K(t,\tau) d\tau = \int_0^t {}_\tau D_t^\alpha K(t,\tau) d\tau + \lim_{\tau \to t} {}_\tau D_t^{\alpha-1} K(t,\tau) \ . \tag{II.92}
$$

II.5 The Fractional Differential Forms

Fractionalized exterior calculus is considered by allowing the partial derivatives in the exterior derivative to assume fractional orders. First of all, let us give a brief review of exterior calculus. It is by no means complete, but is sufficient for the purposes of this section. A clear introduction to the field, with emphasis on applications, is given in [15].

II.5.1 The Standard Exterior Calculus

A differential k-form is a tensor of tensor rank 'k' which is antisymmetric under exchange of any pair of indices. The number of algebraically independent components in E^n (n-dimensional Euclidean Space) is given by the binomial coefficient $\binom{n}{k}$. In particular, a one-form 'α', two-form 'β' and k-form 'ω' are the quantities of the following type

$$\alpha = \sum_{i=1}^{n} a_i dx_i \ , \tag{II.93}$$

$$\beta = \sum_{i,j=1}^{n} b_{ij} dx_i \wedge dx_j \ , \tag{II.94}$$

$$\omega = \sum_{|I|=k} w_I dx_{i_1} \wedge \ldots \wedge dx_{i_k} \ , \tag{II.95}$$

where the $\{x_i\}$ are the cartesian coordinates of E^n and I ranges over all increasing subsets of k elements from $\{1, \ldots, n\}$ with the constraint that

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \ . \tag{II.96}$$

The functions a_i, b_{ij}, etc. may be real or complex depending on the application.

If α and β are differential forms of degrees p and q, respectively, then

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \ . \tag{II.97}$$

Note that the result is zero if $p + q > n$. The exterior product is not (in general) commutative, but it is associative, distributive and anti-symmetric. Now, denote the vector space of k-forms over $P \in E^n$ by $F(k,k,n)$. We need the apparently redundant 'k' in the above notation for the fractional form case.

The exterior derivative of a function f is the one-form

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \ . \tag{II.98}$$

Thinking of a function as a zero-form, the exterior derivative extends linearly to all differential forms using the formulas

$$d(\alpha + \beta) = d\alpha + d\beta \ , \tag{II.99}$$

$$d(\alpha \wedge \gamma) = d\alpha \wedge \gamma + (-1)^p \alpha \wedge d\gamma \ , \tag{II.100}$$

$$d(d\alpha) = 0 \quad \text{(Poincaré Lemma)} \ , \tag{II.101}$$

where α and β are p-forms, and γ is a q-form. Note also that d maps k-forms into $k+1$-forms. If $d\alpha = 0$, then α is called a *closed form*. If $\alpha = d\beta$, then α is called an *exact form*, so any exact form is also closed. An example of a closed form which is not exact is $d\theta$ on the circle. Since θ is a function defined up to a constant multiple of 2π, $d\theta$ is a well defined one-form, but there is no function for which it is the exterior derivative.

II.5.2 The Fractional Form Spaces

If the partial derivatives in (II.98) are replaced by fractional order partial derivatives, fractional exterior derivative of a function f is obtained as

$$d^\mu f = \sum_{i=1}^{n} \frac{\partial^\mu f}{\left(\partial \left(x_i - a_i\right)\right)^\mu} dx_i^\mu \ , \tag{II.102}$$

where 'μ' denotes the order of the fractional exterior derivative, and a_i's are the components of the initial point of the derivative. As an example, in 2-dimension, the fractional exterior derivative of order μ of x^p, with the initial point zero, is obtained as follows

$$d^\mu x^p = x^{p-\mu}\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}dx^\mu + \frac{x^p}{y^\mu\Gamma(1-\mu)}dy^\mu \ . \tag{II.103}$$

We know that, by the analogy with standard exterior calculus, vector spaces can be constructed using the dx_i^μ . So, denote vector space at $P \in E^n$ by $F(\mu, m, n)$ where 'μ' denotes the sum of the fractional differential orders of the basis elements, 'm' denotes the number of coordinate differentials appearing in the basis elements and 'n' is the number of coordinates. The set $\{dx_1^\mu, dx_2^\mu, \ldots, dx_n^\mu\}$ is a basis set for $F(\mu, 1, n)$ and an arbitrary element of $F(\mu, 1, n)$ may be written as

$$\alpha = \sum_{i=1}^{n} \alpha_i dx_i^\mu \ . \tag{II.104}$$

If μ is fixed, then the resulting vector space is an n-dimensional vector space. Note that different values of μ yield different vector spaces, and that one forms from the conventional exterior calculus are recovered if $\mu = 1$. Now suppose that the basis elements are made up of two coordinate differentials, $F(\mu, 2, n)$. In this case, the basis set is defined as

$$\left\{ dx_i^{\sigma_1} \wedge dx_j^{\sigma_2} \ \middle| \ i, j \in \{1, \ldots, n\} \ , \quad \sigma_1, \sigma_2 \geq 0 \ , \quad \sigma_1 + \sigma_2 = \mu \right\} \ . \tag{II.105}$$

An arbitrary element of $F(\mu, 2, n)$ can be expressed as

$$\alpha = \sum_{i,j=1}^{n} \int_0^\mu \left(\alpha_{ij}(\mu_i, \mu - \mu_i) dx_i^{\mu_i} \wedge dx_j^{\mu-\mu_i} \right) d\mu_i \ , \tag{II.106}$$

while any element of $F(\mu, 3, n)$ would be expressed as a sum of the form

$$\sum_{i,j,k=1}^{n} \int_0^\mu \int_0^{\mu-\mu_i} \left(\alpha_{ijk}(\mu_i, \mu - \mu_j, \mu - \mu_i - \mu_j) dx_i^{\mu_i} \wedge dx_j^{\mu_j} \wedge dx_k^{\mu-\mu_i-\mu_j} \right) d\mu_j d\mu_i. \tag{II.107}$$

As can be seen above, with each step up on the middle index of $F(\mu, m, n)$ another integral and summation is included [16]. A basis for this vector space would be

$$\left\{ dx_{i_1}^{\sigma_{i_1 1}} \wedge dx_{i_2}^{\sigma_{i_2 1}} \wedge \ldots \wedge dx_{i_m}^{\sigma_{i_m 1}}, \ldots\ldots \left| \sum_{k=1}^{m} \sigma_{i_k j} = \mu \right. \right\} . \tag{II.108}$$

For $m > 1$, the vector space $F(\mu, m, n)$ is clearly uncountably infinite dimensional for any value of μ.

Let $\varphi \in F(\mu, m, n)$ and $\psi \in F(\sigma, k, n)$ be defined at the point $P \in E^n$. Then, antisymmetry property (II.97) is still recovered as follows

$$\varphi \wedge \psi = (-1)^{km} \psi \wedge \varphi \quad \in \; F(\mu + \sigma, k + m, n) . \tag{II.109}$$

But, in this case, the result need not be zero if $k + m > n$. Equation (II.99) is also maintained due to the linearity of the fractional derivative. But, because of (II.90), equation (II.100) is not maintained. Note also that d^μ maps $F(\sigma, k, n)$ into $F(\sigma + \mu, k + 1, n)$.

Using the analogy with the standard exterior calculus, the definitions of being closed and exact may be extended to the fractional differential forms. $\alpha \in F(\sigma, k, n)$ is called as *μ-closed* if $d^\mu \alpha = 0$, and *μ-exact* if there exists $\beta \in F(\sigma - \mu, k - 1, n)$ such that $\alpha = d^\mu \beta$.

To examine the notion of exactness (integrability conditions) in the fractional form case, the kernel is needed for the fractional derivative operator. Thus, we need to solve

$$\frac{\partial^\mu f}{\partial x_i^\mu} = 0 , \tag{II.110}$$

where the initial point of the derivative is taken to be the origin. To solve equation

(II.110), the formula (II.72) should be written in terms of partial derivatives

$$\frac{\partial^\mu f(x)}{(\partial\,(x-a))^\mu} = \frac{1}{\Gamma(k-\mu)}\left(\frac{\partial}{\partial x}\right)^k \int_a^x (x-\tau)^{k-\mu-1} f(\tau)d\tau \ , \ \mu > 0, \ x > a. \quad \text{(II.111)}$$

Let n be an integer such that $n-1 \leq \mu < n$. Then, using (II.77) we have

$$f = (x_i)^{\mu-n} \sum_{m=0}^{n-1} c_m (x_i)^m \ , \quad \text{(II.112)}$$

where c_m's are functions. The kernel for the operator d^μ is similarly constructed as follows

$$d^\mu f = \sum_{i=1}^{n} \frac{\partial^\mu f}{\partial x_i^\mu} dx_i^\mu = 0 \quad \text{(II.113)}$$

$$\Rightarrow \quad f = \left(\prod_{i=1}^{n} x_i\right)^{\mu-m} \left(\sum_{k_1,\ldots,k_n=0}^{m-1} C_{k_1\ldots k_n} (x_1)^{k_1} \ldots (x_n)^{k_n}\right) \ , \quad \text{(II.114)}$$

where $C_{k_1\ldots k_n}$ are constants and 'm' is the integer which fulfills $m-1 \leq \mu < m$. Now, we will derive the fractional integrability conditions. Let $\alpha \in F(\mu, 1, n)$ such that

$$\alpha = \sum_{i=1}^{n} \alpha_i dx_i^\mu \ . \quad \text{(II.115)}$$

Our aim is to solve the equation $d^\mu f = \alpha$ for the function f. If such an f exists, then using (II.112) it may be written as

$$f = \partial_i^{-\mu}(\alpha_i) + (x_i)^{\mu-m} \sum_{k=0}^{m-1} c_k (x_i)^k \ , \quad \text{(II.116)}$$

where the operator $\partial_i^{-\mu}$ denotes fractional integration of the order μ. Since $\partial_j^\mu f = \alpha_j$, where ∂_j^μ denotes $\frac{\partial^\mu}{\partial x_j^\mu}$, then we have

$$\partial_j^\mu \left(\partial_i^{-\mu}(\alpha_i) + (x_i)^{\mu-m} \sum_{k=0}^{m-1} c_k (x_i)^k\right) = \alpha_j \ . \quad \text{(II.117)}$$

Equation (II.117) may be written as

$$\sum_{k=0}^{m-1} \left(\partial_j^\mu c_k\right) (x_i)^k = \frac{\alpha_j - \partial_j^\mu \left(\partial_i^{-\mu}(\alpha_i)\right)}{(x_i)^{\mu-m}} \ . \quad \text{(II.118)}$$

As can be seen above, the left hand side of (II.118) is a polynomial of order $m-1$. Thus, the only condition for it to be solved for c_k is that

$$\frac{\partial^m}{\partial x_i^m}\left(\frac{\alpha_j - \partial_j^{\mu}\left(\partial_i^{-\mu}(\alpha_i)\right)}{(x_i)^{\mu-m}}\right) = 0 \ . \tag{II.119}$$

The above equation is called as *integrability condition* for fractional differential forms of type $F(\mu, 1, n)$. Note that the standard integrability conditions from exterior calculus are recovered if $\mu = m = 1$.

Now, we will examine what it takes to be closed for fractional forms. Consider the fractional exterior derivative of order σ of μ-form α given by (II.115),

$$d^{\sigma}\alpha = \sum_{i=1}^{n} d^{\sigma}\left(\alpha_i dx_i^{\mu}\right) \ . \tag{II.120}$$

Using the formula (II.90), one can easily obtain

$$d^{\sigma}\alpha = \sum_{i=1}^{n}\sum_{j=1}^{n} dx_j^{\sigma} \wedge \sum_{k=0}^{\infty} \binom{\sigma}{k} \left(\frac{\partial^{\sigma-k}}{\partial x_j^{\sigma-k}}\alpha_i\right)\left(\frac{\partial^k}{\partial x_j^k} dx_i^{\mu}\right) \ . \tag{II.121}$$

It is clear that

$$\frac{\partial^k}{\partial x_j^k}(dx_i^{\mu}) = 0, \quad \forall k \geq 1 \ . \tag{II.122}$$

This yields that

$$d^{\sigma}\alpha = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^{\sigma}}{\partial x_j^{\sigma}}\alpha_i\right) dx_j^{\sigma} \wedge dx_i^{\mu} \ . \tag{II.123}$$

Therefore, under the assumption that $\sigma \neq \mu$ or $i \neq j$, we concluded $d^{\sigma}\alpha = 0$ if and only if

$$\frac{\partial^{\sigma}}{\partial x_j^{\sigma}}\alpha_i = 0 \ , \tag{II.124}$$

i.e. $\alpha_i \in Ker(\partial_j^{\sigma})$. Note that if $\sigma = \mu = 1$ in (II.123) then the usual result

$$\frac{\partial}{\partial x_i}\alpha_j - \frac{\partial}{\partial x_j}\alpha_i = 0 \tag{II.125}$$

is recovered.

Now, we will consider the coordinate transformations for the fractional differentials. Let $\{x_i\}$ and $\{y_i\}$ be two coordinate systems. And, assume that there exists a one to one map between them. We will take $\{x_i\}$ and $\{y_i\}$ to be cartesian and curvilinear coordinates, respectively. Assume that $\{x_i\}$ can be written in terms of the $\{y_i\}$, i.e.

$$x_i = x_i(y) \ . \tag{II.126}$$

Taking the exterior derivative of both sides of (II.126), one can easily obtain

$$dx_i = \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k} dy_k \ . \tag{II.127}$$

The above equation can be generalized for the fractional case. In the two coordinate systems the fractional exterior derivative takes the following forms

$$d^\mu = \sum_{i=1}^{n} \frac{\partial^\mu}{(\partial (x_i - a_i))^\mu} dx_i^\mu \tag{II.128}$$

and

$$d^\mu = \sum_{i=1}^{n} \frac{\partial^\mu}{(\partial (y_i - b_i))^\mu} dy_i^\mu \ , \tag{II.129}$$

where the components a_i's and b_i's represent the same initial point, but in different coordinate systems.

Consider the following function

$$\alpha_k = \frac{1}{\Gamma(\mu+1)} \left(\prod_{\substack{i=1 \\ i \neq k}}^{n} (x_i - a_i) \right)^{\mu - m} (x_k - a_k)^\mu \ . \tag{II.130}$$

The above function was chosen so that it is in the kernel of $\frac{\partial^\mu}{(\partial(x_i - a_i))^\mu}$ for $i \neq k$, and for $i = k$

$$\frac{\partial^\mu \ \alpha_k}{(\partial (x_k - a_k))^\mu} = 1 \ . \tag{II.131}$$

Applying (II.128) and (II.129) to α_k and taking (II.131) into account, we obtain the following coordinate transformation rule

$$dx_k^{\mu} = \frac{1}{\Gamma(\mu+1)} \sum_{i=1}^{n} \frac{\partial^{\mu}}{\left(\partial\left(y_i - b_i\right)\right)^{\mu}} \left(\left(\prod_{\substack{i=1 \\ i\neq k}}^{n} (x_i - a_i) \right)^{\mu - m} (x_k - a_k)^{\mu} \right) dy_i^{\mu}. \quad \text{(II.132)}$$

To illustrate our approach, let us consider the coordinate transformation for two dimensional cartesian coordinates to polar coordinates with $a_i = b_i = 0$. Let

$$x_1 = r\cos(\theta) \ , \quad x_2 = r\sin(\theta) \ . \quad \text{(II.133)}$$

Using the formula (II.132) and remembering the equation (II.79), we obtain

$$dx_1^{\mu} = \frac{1}{\Gamma(\mu+1)} \left[dr^{\mu} \frac{\partial^{\mu}}{\partial r^{\mu}} \left(x_2^{\mu-m} x_1^{\mu} \right) + d\theta^{\mu} \frac{\partial^{\mu}}{\partial \theta^{\mu}} \left(x_2^{\mu-m} x_1^{\mu} \right) \right] \ , \quad \text{(II.134)}$$

$$dx_2^{\mu} = \frac{1}{\Gamma(\mu+1)} \left[dr^{\mu} \frac{\partial^{\mu}}{\partial r^{\mu}} \left(x_1^{\mu-m} x_2^{\mu} \right) + d\theta^{\mu} \frac{\partial^{\mu}}{\partial \theta^{\mu}} \left(x_1^{\mu-m} x_2^{\mu} \right) \right] \ . \quad \text{(II.135)}$$

$$dx_1^{\mu} = \frac{1}{\Gamma(\mu+1)} \left[dr^{\mu} \frac{\cos^{\mu}(\theta)}{\sin^{m-\mu}(\theta)} \frac{\partial^{\mu}}{\partial r^{\mu}} \left(r^{2\mu-m} \right) + d\theta^{\mu} r^{2\mu-m} \frac{\partial^{\mu}}{\partial \theta^{\mu}} \left(\frac{\cos^{\mu}(\theta)}{\sin^{m-\mu}(\theta)} \right) \right] \ , \quad \text{(II.136)}$$

$$dx_2^{\mu} = \frac{1}{\Gamma(\mu+1)} \left[dr^{\mu} \frac{\sin^{\mu}(\theta)}{\cos^{m-\mu}(\theta)} \frac{\partial^{\mu}}{\partial r^{\mu}} \left(r^{2\mu-m} \right) + d\theta^{\mu} r^{2\mu-m} \frac{\partial^{\mu}}{\partial \theta^{\mu}} \left(\frac{\sin^{\mu}(\theta)}{\cos^{m-\mu}(\theta)} \right) \right] \ . \quad \text{(II.137)}$$

$$\begin{aligned} dx_1^{\mu} &= \frac{\cos^{\mu}(\theta)}{\sin^{m-\mu}(\theta)} \frac{\Gamma(2\mu-m+1)}{\Gamma(\mu+1)\Gamma(\mu-m+1)} r^{\mu-m} dr^{\mu} \\ &\quad + \frac{r^{2\mu-m}}{\Gamma(\mu+1)} \frac{\partial^{\mu}}{\partial \theta^{\mu}} \left(\frac{\cos^{\mu}(\theta)}{\sin^{m-\mu}(\theta)} \right) d\theta^{\mu} \ , \quad \text{(II.138)} \\ dx_2^{\mu} &= \frac{\sin^{\mu}(\theta)}{\cos^{m-\mu}(\theta)} \frac{\Gamma(2\mu-m+1)}{\Gamma(\mu+1)\Gamma(\mu-m+1)} r^{\mu-m} dr^{\mu} \\ &\quad + \frac{r^{2\mu-m}}{\Gamma(\mu+1)} \frac{\partial^{\mu}}{\partial \theta^{\mu}} \left(\frac{\sin^{\mu}(\theta)}{\cos^{m-\mu}(\theta)} \right) d\theta^{\mu} \ . \quad \text{(II.139)} \end{aligned}$$

The standard transformation equations

$$dx_1 = \cos(\theta) dr - r\sin(\theta) d\theta \ , \quad \text{(II.140)}$$

$$dx_2 = \sin(\theta) dr + r\cos(\theta) d\theta \quad \text{(II.141)}$$

are recovered if $\mu = m = 1$.

CHAPTER III

THE FRACTIONAL DIFFERENTIAL EQUATIONS

A fractional differential equation is an equation which contains fractional derivatives. Differential equations of that type appear in certain applied problems and in theoretical researches. To begin with, let us consider the Laplace transform method.

III.1 The Laplace Transform Method for Solving Fractional Differential Equations

Recall some basic concepts of the theory of Laplace transformation.

The function $F(s)$ of the complex variable s defined by

$$F(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt \tag{III.1}$$

is called the Laplace transform of the function $f(t)$.

A function $f(t)$ is said to be of exponential order if there exist three constants α, $T > 0$ and $M > 0$ such that

$$e^{-\alpha t}|f(t)| \leq M \ , \quad \forall t > T \ . \tag{III.2}$$

We claim that every bounded function is of exponential order. To see this, let f

be bounded in $[0,\infty]$. Then, $\exists M > 0$ s.t. $|f(t)| \leq M \quad \forall t \in [0,\infty]$. Thus, choosing $\alpha = 0$ implies f is of exponential order.

Hence, we conclude that $f(t)$ must be of exponential order for the existence of integral in (III.1).

Convolution product of the functions $f(t)$ and $g(t)$ is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau \ . \tag{III.3}$$

Let $f(t)$ and $g(t)$ be of exponential order such that they are identically zero for $t < 0$. Then,

$$L\{f(t) * g(t); s\} = F(s)\ G(s)\ . \tag{III.4}$$

Finally, note that the following very useful and well known formula

$$L\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0)\ , \tag{III.5}$$

where $f^{(n)}(t)$ denotes n-th order derivative, can be obtained from the definition (III.1) by integrating by parts.

III.1.1 Laplace Transformation Formulas for the Fractional Derivatives

First, let us consider the RL fractional integral $\left(i.e.\ {}_a\mathbf{D}_t^{-p} f(t), \quad p > 0\right)$. Assume $a = 0$. Then, we have

$${}_0\mathbf{D}_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau = \frac{t^{p-1}}{\Gamma(p)} * f(t)\ . \tag{III.6}$$

Noting the following rule for the Laplace transform of the function t^{p-1}

$$G(s) = L\{t^{p-1}; s\} = \Gamma(p)\ s^{-p}\ , \tag{III.7}$$

and also using (III.4), we obtain

$$L\{{}_0\mathbf{D}_t^{-p} f(t)\} = s^{-p}\ F(s)\ . \tag{III.8}$$

Now, we will try to evaluate that the Laplace transform of the RL fractional derivative, in other words that $L\{{}_0\mathbf{D}_t^{p} f(t)\}$ where $p > 0$.

First, write ${}_0\mathbf{D}_t^{p} f(t) = g^{(n)}(t)$ where

$$g(t) = {}_0\mathbf{D}_t^{-(n-p)} f(t) = \frac{1}{\Gamma(n-p)} \int_0^t (t-\tau)^{n-p-1} f(\tau) d\tau, \quad n-1 \leq p < n. \tag{III.9}$$

By (III.8), we have

$$G(s) = L\{g(t); s\} = s^{-(n-p)}\ F(s)\ . \tag{III.10}$$

And, we also obtain from (III.5) that

$$L\{{}_0\mathbf{D}_t^{p} f(t)\} = s^n G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0)\ . \tag{III.11}$$

Now, let us evaluate the following

$$g^{(n-k-1)}(t) = \frac{d^{n-k-1}}{dt^{n-k-1}}\ {}_0\mathbf{D}_t^{-(n-p)} f(t) = {}_0\mathbf{D}_t^{p-k-1} f(t)\ . \tag{III.12}$$

Finally, substituting (III.10) and (III.12) into (III.11) we conclude that

$$L\{{}_0\mathbf{D}_t^{p} f(t)\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0\mathbf{D}_t^{p-k-1} f(t)\right]_{t=0}\ ,\ n-1 \leq p < n\ . \tag{III.13}$$

To obtain the formula for the Laplace transform of the Caputo's fractional derivative, one should first write that ${}_0^C D_t^p f(t) = {}_0\mathbf{D}_t^{-(n-p)} h(t)$ where $h(t) = f^{(n)}(t)$ and $n-1 < p \leq n$. Then, from the formula (III.8), we have

$$L\{{}_0^C D_t^p f(t)\} = s^{-(n-p)} H(s)\ . \tag{III.14}$$

Thus, taking into account the formula (III.5), the Laplace transform formula for the Caputo fractional derivative may be written as

$$L\{{}_0^C D_t^p f(t)\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0) \ , \tag{III.15}$$

where $\quad n-1 < p \leq n$.

In [13], Gorenflo and Mainardi have pointed out the utility of the Caputo fractional derivative in the treatment of differential equations of fractional order for physical applications. In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of bounded values assumed by the field variables and its derivatives of integer order, despite the fact that the governing evolution equation may be a fractional differential equation.

III.1.2 Examples

Now, we will consider some examples of the solution of ordinary linear fractional differential equations.

Example 1. Solve the fractional differential equation given by

$${}_0\mathbf{D}_t^{1/2} f(t) + a\ f(t) = 0 \ , \quad t > 0 \ , \tag{III.16}$$

with the initial condition

$$\left[{}_0\mathbf{D}_t^{-1/2} f(t)\right]_{t=0} = C \ . \tag{III.17}$$

Solution: We apply the formula for the Laplace transform of the Riemann-

Liouville fractional derivative, that is

$$L\left\{ {}_0\mathbf{D}_t^{1/2} f(t) \right\} + a\ F(s) = 0\ . \tag{III.18}$$

Then, using the formula (III.13) we obtain

$$s^{1/2} F(s) - \left[{}_0\mathbf{D}_t^{-1/2} f(t) \right]_{t=0} + a\ F(s) = 0\ . \tag{III.19}$$

$$\Rightarrow \qquad (s^{1/2} + a) F(s) = C\ , \tag{III.20}$$

$$\Rightarrow \qquad F(s) = \frac{C}{s^{1/2} + a}\ . \tag{III.21}$$

Now, application of the formula (A.17) yields

$$f(t) = C\ t^{-1/2}\ E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t})\ , \tag{III.22}$$

where E denotes the two-parameter Mittag-Leffler function (see A.7-19).

Example 2. Solve the following problem for the function $f(t)$.

$${}_0\mathbf{D}_t^Q f(t) + {}_0\mathbf{D}_t^q f(t) = h(t)\ , \quad 0 < q < Q < 1\ , \tag{III.23}$$

$$\left[{}_0\mathbf{D}_t^{q-1} f(t) + {}_0\mathbf{D}_t^{Q-1} f(t) \right]_{t=0} = C\ . \tag{III.24}$$

Solution: Applying the Laplace transform formula (III.13) for the RL fractional derivative, we obtain

$$F(s) = \frac{C + H(s)}{s^Q + s^q}\ . \tag{III.25}$$

The equation above may be written as

$$F(s) = (C + H(s)) \frac{s^{-q}}{s^{Q-q} + 1}\ . \tag{III.26}$$

Then, by using the formula (A.16) and taking into account that

$$L^{-1}\{H(s)\ G(s)\} = h(t) * g(t)\ , \tag{III.27}$$

one can obtain the result

$$f(t) = C\ g(t) + \int_0^t h(t-\tau)\ g(\tau)\ d\tau\ , \tag{III.28}$$

where

$$g(t) = t^{Q-1}\ E_{Q-q,Q}(-t^{Q-q})\ . \tag{III.29}$$

Although the initial value problems with initial conditions, which are described in examples 1 and 2, can be successfully solved, their solutions are practically useless, because there is no known physical meaning for such types of initial conditions. In other words, the Laplace transform of the Riemann-Liouville fractional derivative allows utilization of initial conditions of the type

$$\left[{}_0\mathbf{D}_t^{p-k-1} f(t)\right]_{t=0} = C\ , \tag{III.30}$$

where $k = 0, \ldots, n-1$, which can cause problems with their physical interpretations.

On the contrary, the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with known physical interpretations. Now, let us consider the following initial value problem for a non-homogenous differential equation, which contains a Caputo fractional derivative of order $p > 0$.

Example 3. Solve the following initial value problem

$${}_0^C D_t^p f(t) + f(t) = h(t)\ , \quad t > 0\ ; \quad f^{(j)}(0) = c_j\ ,\ j = 0, \ldots, n-1\ , \tag{III.31}$$

where n is a positive integer uniquely determined by $n - 1 < p \leq n$.

Solution: Application of the Laplace transform formula (III.15) for the Caputo fractional derivative yields

$$F(s) = \sum_{j=0}^{n-1} c_j \frac{s^{p-j-1}}{s^p+1} + \frac{H(s)}{s^p+1} \ . \tag{III.32}$$

Using the formula (A.16), one can conclude that

$$L^{-1}\left\{\frac{s^{p-1}}{s^p+1}\right\} = E_{p,1}(-t^p) \ , \tag{III.33}$$

and taking into account (III.8), that

$$L^{-1}\left\{\frac{s^{p-j-1}}{s^p+1}\right\} = {}_0D_t^{-j}(E_{p,1}(-t^p)) \ , \tag{III.34}$$

where ${}_0D_t^{-j}$ denotes the j-th fold repeated integration.

On the other hand, again by using the formula (A.16), we have

$$L^{-1}\left\{\frac{1}{s^p+1}\right\} = t^{p-1}E_{p,p}(-t^p) \ . \tag{III.35}$$

Finally, considering also the relationship (III.27), we obtain the result

$$f(t) = \sum_{j=0}^{n-1} c_j \ {}_0D_t^{-j}(E_{p,1}(-t^p)) + \int_0^t h(t-\tau)\ \tau^{p-1}\ E_{p,p}(-\tau^p)\ d\tau \ . \tag{III.36}$$

In particular, the formula (III.36) encompasses the solutions of the problems, *fractional relaxation* and *fractional oscillation*, for $0 < p < 1$ and $1 < p < 2$, respectively.

III.2 The Fractional Diffusion Equation

When a linear one-dimensional diffusion equation is fractionalized, we can replace either the time derivative or the space derivative by a suitable fractional order derivative. In this section, first, we recall the basic results for the standard diffusion equation concerning the fundamental solution of the Cauchy problem, and then we examine the time-fractional diffusion equation by replacing the time derivative in usual case by Caputo fractional derivative.

III.2.1 The Standard Diffusion Equation

By the standard diffusion equation, we mean

$$\frac{\partial}{\partial t}u(x,t) = c\,\frac{\partial^2}{\partial x^2}u(x,t)\ , \tag{III.37}$$

where $u = u(x,t)$ denotes the concentration of the diffusing material, 'c' is the diffusion coefficient which is positive and whose magnitude helps determine the speed at which the diffusion takes place, and, 't' and 'x' are the time and space variables, respectively.

One of the basic problem for the equation (III.37) is the Cauchy problem, which concerns the space-time domain $-\infty < x < +\infty,\ t > 0$. Letting $g(x)$ be a sufficiently well-behaved function, this problem is formulated as

$$u(x,0^+) = g(x)\ ,\ -\infty < x < +\infty\ ;\quad u(\mp\infty,t) = 0\ ,\ t > 0\ . \tag{III.38}$$

Now, we will derive some classical results for the Cauchy problem. It is well-known that this initial value problem can be easily solved by the Fourier transform and its fundamental solution can be found in terms of Green function. Remember that the Fourier transform of a continuous and absolutely integrable function $f(x)$ is defined by

$$\widehat{f}(w) = F\{f(x);w\} = \int_{-\infty}^{+\infty} e^{iwx}f(x)dx\ ,\quad w \in R\ . \tag{III.39}$$

For the inverse Fourier transform, we have the following relationship

$$f(x) = F^{-1}\{\widehat{f}(w);x\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iwx}\widehat{f}(w)dw\ . \tag{III.40}$$

The above transformation rule may be denoted by $f(x) \div \widehat{f}(w)$. Application of the method of Fourier transformation to the equation (III.37) with the conditions (III.38) yields

$$\left(\frac{d}{dt} + w^2 c\right) \widehat{u}(w,t) = 0 \ , \quad \widehat{u}(w,0^+) = \widehat{g}(w) \ , \tag{III.41}$$

and the transformed solution $\widehat{u}(w,t)$ turns out to be

$$\widehat{u}(w,t) = \widehat{g}(w) e^{-w^2 ct} \ . \tag{III.42}$$

Denoting

$$G_c^d(x,t) \div \widehat{G_c^d}(w,t) = e^{-w^2 ct} \ , \tag{III.43}$$

where d refers to standard diffusion equation, one can express the solution, obtained by inversion of (III.42), in terms of

$$u(x,t) = \int_{-\infty}^{\infty} G_c^d(\tau,t) g(x-\tau) d\tau \ , \tag{III.44}$$

where

$$G_c^d(x,t) = \frac{1}{\sqrt{4\pi ct}} \, e^{-\frac{x^2}{4ct}} \ . \tag{III.45}$$

It is well-known that the function $G_c^d(x,t)$ is called as *Green* function for the Cauchy problem, since it corresponds to $g(x) = \delta(x)$. $G_c^d(x,t)$ is an even and normalized function in the variable x, in other words

$$G_c^d(x,t) = G_c^d(|x|,t) \ , \tag{III.46}$$

and

$$\int_{-\infty}^{+\infty} G_c^d(x,t) dx = 1 \ . \tag{III.47}$$

Note that the relationship

$$|x|G_c^d(|x|,t) = \frac{\xi}{2}M^d(\xi) \ , \tag{III.48}$$

where $\xi = |x|/(\sqrt{ct})$ is the *similarity* variable and

$$M^d(\xi) = \frac{1}{\sqrt{\pi}}e^{-\frac{\xi^2}{4}} \ . \tag{III.49}$$

Also, note that $M^d(\xi)$ satisfies the condition

$$\int_0^\infty M^d(\xi)d\xi = 1 \ . \tag{III.50}$$

The interpretation (III.45) of the *Green* function is very easy to obtain in probability since

$$G_c^d(x,t) = p_G(x;\sigma) := \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}} \ , \quad \sigma^2 = 2ct \ , \tag{III.51}$$

where $p_G(x;\sigma)$ denotes the *Gauss* probability density function, whose moment of the second order, the *variance*, is σ^2. The associated cumulative distribution function is defined by

$$P_G(x;\sigma) := \int_{-\infty}^{x} p_G(\tau;\sigma)d\tau = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)\right] \ , \tag{III.52}$$

where $\operatorname{erf}(z)$ denotes the *error function* defined by (A.11). Moreover, the moments of even order of the *Gauss* probability density function turn out to be

$$\int_{-\infty}^{+\infty} x^{2n}p_G(x;\sigma)dx = \sigma^{2n}(2n-1)!! \ , \tag{III.53}$$

and, thus

$$\int_{-\infty}^{+\infty} x^{2n}G_c^d(x,t)dx = (2ct)^n(2n-1)!! \ , \quad n = 1,2,\dots \ . \tag{III.54}$$

III.2.2 The Time-Fractional Diffusion Equation

We will use the following interpretation for the time-fractional diffusion equation

$$ {}_0^C D_t^p u(x,t) = c\,\frac{\partial^2}{\partial x^2} u(x,t)\ , \tag{III.55} $$

where ${}_0^C D_t^p$ denotes the Caputo fractional derivative of order p with $0 < p < 2$. Let us recall the definition of the Caputo fractional derivative

$$ {}_0^C D_t^p f(t) = \frac{1}{\Gamma(n-p)} \int_0^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{p+1-n}}\ ,\quad (n-1 < p < n) \tag{III.56} $$

with the lower limit $a = 0$. According to the above formula, we should distinguish the cases $0 < p < 1$ and $1 < p < 2$. Let us consider the function $\Phi_v(t)$ defined by

$$ \Phi_v(t) = \begin{cases} \frac{t^{v-1}}{\Gamma(v)} & if \quad t > 0 \\ 0 & if \quad t \le 0 \end{cases}\ ,\qquad v > 0\ . \tag{III.57} $$

Using the formula (III.3), we obtain

$$ \Phi_{1-p}(t) * \frac{\partial u}{\partial t} = \frac{1}{\Gamma(1-p)} \int_0^t (t-\tau)^{-p}\,\frac{\partial u}{\partial \tau}\,d\tau = c\,\frac{\partial^2 u}{\partial x^2},\ 0 < p < 1, \tag{III.58} $$

$$ \Phi_{2-p}(t) * \frac{\partial^2 u}{\partial t^2} = \frac{1}{\Gamma(2-p)} \int_0^t (t-\tau)^{1-p}\,\frac{\partial^2 u}{\partial \tau^2}\,d\tau = c\,\frac{\partial^2 u}{\partial x^2},\ 1 < p < 2. \tag{III.59} $$

The Cauchy problem for the integro-differential equations (III.58) and (III.59) is formulated as in (III.38)

$$ u(x, 0^+) = g(x)\ ,\ -\infty < x < +\infty\ ;\quad u(\mp\infty, t) = 0\ ,\ t > 0\ . \tag{III.60} $$

The time-fractional diffusion equation (III.55) with the conditions (III.60) is equivalent to the following integro-differential equation

$$ u(x,t) = g(x) + \frac{c}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} \frac{\partial^2 u}{\partial \tau^2}\,d\tau\ ,\ 0 < p < 2\ . \tag{III.61} $$

Such integro-differential equations were investigated in [18, 19, 20, 21].

III.2.3 The Cauchy Problem for the Time-Fractional Diffusion Equation

In equation (III.55), replacing the order p by 2μ where $0 < \mu < 1$, we obtain

$$ {}_0^C D_t^{2\mu} u(x,t) = c \, \frac{\partial^2}{\partial x^2} u(x,t) \ . \tag{III.62} $$

Application of the method of Fourier transformation with respect to spatial variable 'x' to the equation (III.62) with the conditions (III.60) yields

$$ {}_0^C D_t^{2\mu} \widehat{u}(w,t) + w^2 c \, \widehat{u}(w,t) = 0 \ , \tag{III.63} $$

$$ \widehat{u}(w,0^+) = \widehat{g}(w) \ , \tag{III.64} $$

where 'w' is the Fourier transform parameter. Applying the Laplace transform formula (III.15) to the equation (III.63) with respect to time t and using the initial condition (III.64), we obtain

$$ \widehat{U}(w,s) = \frac{s^{2\mu-1}}{s^{2\mu} + w^2 c} \, \widehat{g}(w) \ . \tag{III.65} $$

The inverse Laplace transform of (III.65) using (A.16) gives

$$ \widehat{u}(w,t) = \widehat{g}(w) E_{2\mu,1}(-w^2 c t^{2\mu}) \ , \tag{III.66} $$

and consequently for the *Green* function we have

$$ G_c(x,t) = G_c(|x|,t) \div \widehat{G_c}(w,t) = E_{2\mu,1}(-w^2 c t^{2\mu}) \ . \tag{III.67} $$

Using the formula (A.12), we have

$$ \widehat{G_c}(w,t) = E_{2\mu,1}(-w^2 c t^{2\mu}) = \left[E_{\mu,1}(+iw\sqrt{c}\, t^{\mu}) + E_{\mu,1}(-iw\sqrt{c}\, t^{\mu}) \right]/2 \ . \tag{III.68} $$

If $\mu \neq 1/2$, then the inversion of the Fourier transform in (III.68) can not be obtained by using a standard table of Fourier transform pairs. Such an inversion can be obtained by using the Laplace transform pair (A.24) with $r = |x|$ and $s = \pm iw$. Specifically, we obtain from (III.68) and (A.24) that

$$G_c(x,t) = G_c(|x|,t) = \frac{1}{2\sqrt{c}\, t^{\mu}} M\left(\frac{|x|}{\sqrt{c}\, t^{\mu}};\mu\right) \ , \tag{III.69}$$

where $M(\xi;\mu)$ is a special function, defined by (A.21), and

$$\xi = |x|/(\sqrt{c}\, t^{\mu}) \tag{III.70}$$

is the *similarity* variable. The solution of the Cauchy problem for the time-fractional diffusion equation can be expressed in terms of a space convolution

$$u(x,t) = \int_{-\infty}^{\infty} G_c(\tau,t) g(x-\tau) d\tau \ . \tag{III.71}$$

We also note the relationship

$$|x| G_c(|x|,t) = \frac{\xi}{2} M(\xi;\mu) \ , \tag{III.72}$$

which generalizes the identity (III.48) to the time-fractional diffusion equation. From (A.25), we have

$$\int_{0}^{\infty} M(\xi;\mu) d\xi = 1 \ , \tag{III.73}$$

and, thus $M(\xi;\mu)$ is the *normalized auxiliary* function for the time-fractional diffusion equation. Using (III.70), (III.72) and (A.22), we obtain

$$G_c(x,t) \sim \widetilde{a}(t) |x|^{\frac{\mu-\frac{1}{2}}{1-\mu}} \exp\left[-\widetilde{b}(t)|x|^{\frac{1}{1-\mu}}\right] \ , \qquad |x| \to +\infty \ , \tag{III.74}$$

where $\widetilde{a}(t)$ and $\widetilde{b}(t)$ are certain functions of time. Moreover, the exponential decay in x provided by (III.74) shows that all the absolute moments of positive order of $G_c(x,t)$ are finite. Particularly, taking into account the identity (III.72) and the relationship (A.25), for moments of even order, we obtain the formula

$$\int_{-\infty}^{+\infty} x^{2n} G_c(x,t) dx = \frac{\Gamma(2n+1)}{\Gamma(2\mu n+1)} (c\, t^{2\mu})^n \ , \quad n = 0,1,2,\dots \ , \tag{III.75}$$

which yields a generalization for the formula (III.54).

III.3 The Fractional Schrödinger Equation

The Schrödinger equation has the mathematical appearance of a diffusion equation and can be derived by considering probability distributions. In this section, the Schrödinger equation and its properties will be considered with the first order time derivative changed to a Caputo fractional derivative.

III.3.1 The Time-Fractional Schrödinger Equation

To fractionalize the time derivative of the Schrödinger equation, it is necessary to preserve the units of the wave function. The Plank units are given in [37] as

$$L_p = \sqrt{\frac{Gh}{c^3}} \ , \quad T_p = \sqrt{\frac{Gh}{c^5}} \ , \quad M_p = \sqrt{\frac{hc}{G}} \ , \quad E_p = M_p c^2 \ , \tag{III.76}$$

where L_p, T_p, M_p and E_p denote Plank length, time, mass and energy, respectively. The standard Schrödinger equation in one space and one time dimension is defined in [38] as

$$ih\partial_t\psi = -\frac{h^2}{2m}\partial_x^2\psi + V\psi \ . \tag{III.77}$$

The equation above may be written in terms of Plank units

$$iT_p\partial_t\psi = -\frac{L_p^2 M_p}{2m}\partial_x^2\psi + \frac{V}{E_p}\psi \ . \tag{III.78}$$

Denoting $N_m = m/M_p$ as the number of Plank masses in m and $N_V = V/E_p$ as the number of Plank energies in V, we obtain

$$iT_p\partial_t\psi = -\frac{L_p^2}{2N_m}\partial_x^2\psi + N_V\psi \ , \tag{III.79}$$

where the operators $T_p\partial_t$ and $L_p^2\partial_x^2$ are dimensionless. To fractionalize the time derivative, we have two possibilities, i.e.

$$(iT_p)^\mu \, {}_0^C D_t^\mu \psi = -\frac{L_p^2}{2N_m}\partial_x^2\psi + N_V\psi \, , \tag{III.80}$$

or

$$i\,(T_p)^\mu \, {}_0^C D_t^\mu \psi = -\frac{L_p^2}{2N_m}\partial_x^2\psi + N_V\psi \, . \tag{III.81}$$

Due to the simpler physical behavior of (III.80) and the role of 'i' in a Wick rotation, equation (III.80) is more convenient for a time fractional Schrödinger equation. Now, for $0 < \mu < 1$, we will note the following identity for Caputo fractional derivatives

$${}_0^C D_t^{1-\mu}\left({}_0^C D_t^\mu f(t)\right) = \frac{d}{dt}f(t) - \frac{\left[{}_0^C D_t^\mu f(t)\right]_{t=0}}{t^{1-\mu}\Gamma(\mu)} \, . \tag{III.82}$$

Letting

$$\alpha = \frac{N_V}{T_p^\mu} \, , \tag{III.83}$$

and

$$\beta = \frac{L_p^2}{2N_m T_p^\mu} \, , \tag{III.84}$$

the equation (III.80) may be written in the following form

$${}_0^C D_t^\mu \psi = -\frac{\beta}{i^\mu}\partial_x^2\psi + \frac{\alpha}{i^\mu}\psi \, . \tag{III.85}$$

Using the identity (III.82), the equation (III.85) turns out to be

$$\partial_t\psi = -\frac{\beta}{i^\mu}\partial_x^2\left({}_0^C D_t^{1-\mu}\psi\right) + \frac{\alpha}{i^\mu}\left({}_0^C D_t^{1-\mu}\psi\right) + \frac{\left[{}_0^C D_t^\mu \psi(t)\right]_{t=0}}{t^{1-\mu}\Gamma(\mu)} \, . \tag{III.86}$$

For a possible interpretation for the non-local object ${}_0^C D_t^{1-\mu}\psi$ in the equation (III.86), recall the interpretation of the first order time derivative in standard quantum mechanics [38],

$$\frac{\partial}{\partial t} = \frac{E}{ih} \, , \tag{III.87}$$

where E is the energy operator. Then,

$$\int_{-\infty}^{+\infty} \psi^*(t,x)\ {}_0^C D_t^{1-\mu}(\psi(t,x))dx \tag{III.88}$$

can be interpreted as being proportional to the weighted time average of the energy of the wave function. Hereafter, we will use the notation $\widetilde{\psi}$ instead of ${}_0^C D_t^{1-\mu}\psi$. Thus, the equation (III.86) can be written as

$$\partial_t \psi = -\frac{\beta}{i^\mu}\partial_x^2 \widetilde{\psi} + \frac{\alpha}{i^\mu}\widetilde{\psi} + \frac{\left[{}_0^C D_t^\mu \psi(t)\right]_{t=0}}{t^{1-\mu}\Gamma(\mu)} \ . \tag{III.89}$$

The probability current equation for a free particle can be constructed as follows

$$P = \psi\psi^* \ , \tag{III.90}$$

$$\partial_t P = \partial_t \psi \psi^* + \psi \partial_t \psi^* \ , \tag{III.91}$$

$$\begin{aligned} \partial_t P \quad &= \left(-\frac{\beta}{i^\mu}\partial_x^2 \widetilde{\psi} + \frac{\left[{}_0^C D_t^\mu \psi(t,x)\right]_{t=0}}{t^{1-\mu}\Gamma(\mu)} \right)\psi^* \\ &\quad + \psi \left(-\frac{\beta}{(-i)^\mu}\partial_x^2 \widetilde{\psi^*} + \frac{\left[{}_0^C D_t^\mu \psi^*(t,x)\right]_{t=0}}{t^{1-\mu}\Gamma(\mu)} \right) \ . \end{aligned} \tag{III.92}$$

The last equation may be written in the following form

$$\begin{aligned} \partial_t P + \beta \partial_x \left(\frac{\partial_x \widetilde{\psi}\psi^*}{i^\mu} + \frac{\partial_x \widetilde{\psi^*}\psi}{(-i)^\mu} \right) &= \beta \left(\frac{\partial_x \widetilde{\psi}\partial_x \psi^*}{i^\mu} + \frac{\partial_x \widetilde{\psi^*}\partial_x \psi}{(-i)^\mu} \right) \\ &+ \frac{\psi^* \left[{}_0^C D_t^\mu \psi(t,x)\right]_{t=0} + \psi \left[{}_0^C D_t^\mu \psi^*(t,x)\right]_{t=0}}{t^{1-\mu}\Gamma(\mu)} \ . \end{aligned} \tag{III.93}$$

If $\mu \to 0$, then right hand side of (III.93) would be zero. The probability current equation can be defined as

$$J = \frac{\beta}{i^\mu}\left(\partial_x \widetilde{\psi}\right)\psi^* + \frac{\beta}{(-i)^\mu}\psi\left(\partial_x \widetilde{\psi^*}\right) \ . \tag{III.94}$$

The right hand side of equation (III.93) can be viewed as a source in the probability current equation. The non-zero source term in (III.93) confirms that probability will not be preserved for solutions of the time fractional Schrödinger equation. Then, denoting

$$S(x,t) = \frac{\beta}{i^{\mu}}\partial_x\widetilde{\psi}\partial_x\psi^* + \frac{\beta}{(-i)^{\mu}}\partial_x\widetilde{\psi^*}\partial_x\psi + \frac{\psi^*\left[{}_0^C D_t^{\mu}\psi(t,x)\right]_{t=0} + \psi\left[{}_0^C D_t^{\mu}\psi^*(t,x)\right]_{t=0}}{t^{1-\mu}\Gamma(\mu)} \ , \tag{III.95}$$

we obtain

$$\partial_t P + \partial_x J = S \ . \tag{III.96}$$

Integrating (III.96) over all space and letting the wave function and its first derivative go to zero at spatial infinity yields

$$\partial_t \int_{-\infty}^{+\infty} P \ dx = \int_{-\infty}^{+\infty} S \ dx \ . \tag{III.97}$$

III.3.2 Free Particle and Potential Well Solution

The time fractional Schrödinger equation for a free particle is given by

$$(iT_p)^{\mu}\, {}_0^C D_t^{\mu}\psi = -\frac{L_p^2}{2N_m}\partial_x^2\psi \ . \tag{III.98}$$

Application of the method of Fourier transformation with $F[\psi(x,t)] = \Psi(w,t)$ yields

$$(iT_p)^{\mu}\, {}_0^C D_t^{\mu}\Psi = \frac{(L_p w)^2}{2N_m}\Psi \ . \tag{III.99}$$

Taking into account the formulas (A.16) and (A.18), the solution of (III.99) can be found by Laplace transform as follows

$${}_0^C D_t^{\mu}\Psi = \frac{(L_p w)^2}{2N_m T_p^{\mu} i^{\mu}}\Psi \ , \tag{III.100}$$

$$\Psi = \Psi_0 E_{\mu,1}\left(\lambda(-i)^{\mu} t^{\mu}\right) \ , \tag{III.101}$$

$$\Psi = \frac{\Psi_0}{\mu}\left(e^{-i\lambda^{1/\mu}t} - \mu\Phi_\mu\left(\lambda(-i)^\mu, t\right)\right) \ , \tag{III.102}$$

where

$$\lambda = \frac{(L_p w)^2}{2N_m T_p^\mu} \ . \tag{III.103}$$

In equation (III.102), the first term is oscillatory, and the second one is decay in the time variable. Inverting the Fourier transformation, we obtain

$$\psi(x,t) = F^{-1}[\Psi(w,t)] = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{iwx}\frac{\Psi_0}{\mu}\left(e^{-i\lambda^{1/\mu}t} - \mu\Phi_\mu\left(\lambda(-i)^\mu, t\right)\right) dw. \tag{III.104}$$

This result may be separated into two part, a Schrödinger like piece divided by μ, and a decay term that goes to zero as time goes to infinity. Then, we have

$$\psi(x,t) = \psi_s(x,t) + \psi_d(x,t) \ , \tag{III.105}$$

where

$$\psi_s(x,t) = \frac{1}{2\pi\mu}\int\limits_{-\infty}^{+\infty} e^{iwx}\Psi_0 e^{-i\lambda^{1/\mu}t} dw \ , \tag{III.106}$$

and

$$\psi_d(x,t) = -\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{iwx}\Psi_0\Phi_\mu\left(\lambda(-i)^\mu, t\right) dw \ . \tag{III.107}$$

Note that if μ goes to one, $\psi_d(x,t)$ goes to zero and $\psi_s(x,t)$ becomes the standard Schrödinger term. Choose Ψ_0 so that the initial probability is one, i.e.

$$\int\limits_{-\infty}^{+\infty} \psi(x,0)\psi^*(x,0)dx = 1 \ . \tag{III.108}$$

Now, let us consider what happens to the total probability as time goes to infinity, in other words evaluate the following limit

$$\lim_{t\to\infty}\int\limits_{-\infty}^{+\infty} \psi(x,t)\psi^*(x,t)dx \ . \tag{III.109}$$

$$= \lim_{t\to\infty} \int_{-\infty}^{+\infty} F^{-1}\left\{\frac{\Psi_0}{\mu}\left(e^{-i\lambda^{1/\mu}t} - \mu\Phi_\mu\left(\lambda(-i)^\mu, t\right)\right)\right\} \tag{III.110}$$

$$F^{-1}\left\{\frac{\Psi_0}{\mu}\left(e^{-i\lambda^{1/\mu}t} - \mu\Phi_\mu\left(\lambda(-i)^\mu, t\right)\right)\right\}^* dx \tag{III.111}$$

$$= \frac{2\pi}{\mu^2} \lim_{t\to\infty} \int_{-\infty}^{+\infty} \Psi_0 \left(e^{-i\lambda^{1/\mu}t} - \mu\Phi_\mu\left(\lambda(-i)^\mu, t\right)\right)$$

$$\left(\Psi_0\left(e^{-i\lambda^{1/\mu}t} - \mu\Phi_\mu\left(\lambda(-i)^\mu, t\right)\right)\right)^* dw \tag{III.112}$$

$$= \frac{2\pi}{\mu^2} \lim_{t\to\infty} \int_{-\infty}^{+\infty} \Psi_0 \; e^{-i\lambda^{1/\mu}t} \; \Psi_0^* \; e^{i\lambda^{1/\mu}t} dw \tag{III.113}$$

$$= \frac{2\pi}{\mu^2} \lim_{t\to\infty} \int_{-\infty}^{+\infty} \Psi_0 \; \Psi_0^* \; dw \tag{III.114}$$

$$= \frac{1}{\mu^2} \lim_{t\to\infty} \int_{-\infty}^{+\infty} \psi_0 \; \psi_0^* \; dx \tag{III.115}$$

$$= \frac{1}{\mu^2} \; . \tag{III.116}$$

Therefore, we concluded that, since μ is less than one, the total probability increases as time goes to infinity.

Now, let us consider a particle in a potential well,

$$V(x) = \begin{cases} 0 & \text{if} \quad 0 < x < a \\ \infty & \text{if} \quad \text{elsewhere} \end{cases} \; , \tag{III.117}$$

$$(iT_p)^\mu \, {}_0^C D_t^\mu \psi = -\frac{L_p^2}{2N_m} \partial_x^2 \psi \; , \tag{III.118}$$

$$\psi(0,t) = 0 \; , \qquad \psi(a,t) = 0 \; . \tag{III.119}$$

The above problem can be solved by separation of variables. So, letting $\psi = A(t)B(x)$, we have

$$(iT_p)^\mu \frac{{}_0^C D_t^\mu A}{A} = -\frac{L_p^2}{2N_m} \frac{\partial_x^2 B}{B} = \kappa \; . \tag{III.120}$$

First, solve the spatial component

$$B'' + \kappa \frac{2N_m}{L_p^2} B = 0 \; . \tag{III.121}$$

Using the boundary conditions, we have

$$B_n = c_n \sin\left(\frac{n\pi x}{a}\right) \; , \quad n = 1, 2, \dots \; , \tag{III.122}$$

$$\kappa_n = \left(\frac{n\pi L_p}{a}\right)^2 \frac{1}{2N_m} \; , \quad n = 1, 2, \dots \; . \tag{III.123}$$

For the time equation, we have

$${}_0^C D_t^\mu A = \frac{\kappa_n}{(iT_p)^\mu} A \; . \tag{III.124}$$

The solution of equation (III.124) with $A(0) = 1$ can be found by the method of Laplace transform as

$$A_n = E_{\mu,1}(\lambda_n (-i)^\mu t^\mu) \tag{III.125}$$

$$= \frac{1}{\mu}\left(e^{-i\lambda_n^{1/\mu} t} - \mu \Phi_\mu\left(\lambda_n (-i)^\mu, t\right)\right) \; , \quad n = 1, 2, \dots \; , \tag{III.126}$$

where

$$\lambda_n = \frac{\kappa_n}{T_p^\mu} \; , \quad n = 1, 2, \dots \; . \tag{III.127}$$

We also note that

$$\lim_{t\to\infty} |A_n(t)| = \frac{1}{\mu} \; . \tag{III.128}$$

As the normalized spatial eigenfunctions, we have

$$\psi_n(x, 0) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \; , \quad n = 1, 2, \dots \; , \tag{III.129}$$

$$\int_0^a \psi_n(x, 0) \psi_n(x, 0) dx = 1 \; , \quad n = 1, 2, \dots \; . \tag{III.130}$$

As a result, we obtain the solution

$$\psi(x,t) = \sum_{n=1}^{\infty} \psi_n(x,t) \ , \tag{III.131}$$

where

$$\psi_n(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \frac{1}{\mu} \left(e^{-i\lambda_n^{1/\mu} t} - \mu \Phi_\mu \left(\lambda_n (-i)^\mu, t\right)\right) \ . \tag{III.132}$$

The energy spectrum for the potential well can be computed as follows

$$E_n(t) = \int_0^a \psi^* \ ih \ \partial_t \psi \ dx \tag{III.133}$$

$$= \frac{2ih}{a\mu^2} \left(e^{i\lambda_n^{1/\mu} t} - \mu \Phi_\mu \left(\lambda_n i^\mu, t\right)\right) \partial_t \left(e^{-i\lambda_n^{1/\mu} t} - \mu \Phi_\mu \left(\lambda_n (-i)^\mu, t\right)\right) \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx \ . \tag{III.134}$$

Taking into account that

$$\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a}{2} \ , \quad n = 1, 2, \dots \ , \tag{III.135}$$

we obtain

$$E_n(t) = \frac{ih}{\mu^2} \left(e^{i\lambda_n^{1/\mu} t} - \mu \Phi_\mu \left(\lambda_n i^\mu, t\right)\right) \partial_t \left(e^{-i\lambda_n^{1/\mu} t} - \mu \Phi_\mu \left(\lambda_n (-i)^\mu, t\right)\right) \ . \tag{III.136}$$

Letting time t go to infinity in (III.136), we obtain the same energy spectrum

$$\lim_{t\to\infty} E_n(t) = \frac{h\kappa_n^{1/\mu}}{\mu^2 T_p} = \frac{h(n\pi L_p)^{2/\mu}}{\mu^2 T_p (2a^2 N_m)^{1/\mu}} \ , \tag{III.137}$$

which was found for the standard Schrödinger equation except for the factor of $1/\mu^2$.

Now, suppose that $1 < \mu \leq 2$. Consider the problem for a free particle

$$(iT_p)^\mu \, {}_0^C D_t^\mu \psi = -\frac{L_p^2}{2N_m} \partial_x^2 \psi \ , \tag{III.138}$$

where

$$\psi(x,0) = \psi_0 \ , \qquad \partial_t(\psi(x,t))\Big|_{t=0} = \psi_1 \ . \tag{III.139}$$

Denote the Fourier transform as

$$\Psi(w,t) = F[\psi(x,t)] , \tag{III.140}$$

$$\Psi_0 = F(\psi_0) , \tag{III.141}$$

$$\Psi_1 = F(\psi_1) . \tag{III.142}$$

In Fourier space, the solution for (III.138) and (III.139) can be written in terms of

$$\Psi = \Psi_0 \left(\frac{e^{-i\lambda^{1/\mu} t}}{\mu} + \Phi_\mu \left(\lambda(-i)^\mu, t\right) \right) + \Psi_1 \left(\frac{e^{-i\lambda^{1/\mu} t}}{\lambda^{1/\mu}\mu} - \Phi_{\mu-1} \left(\lambda(-i)^\mu, t\right) \right) , \tag{III.143}$$

where

$$\lambda = \frac{(L_p w)^2}{2 N_m T_p^\mu} . \tag{III.144}$$

CHAPTER IV

APPLICATION

This chapter has three objectives. The first is to present the formulation of the Euler-Lagrange equations for fractional variational problems. The second objective is to describe the basis of the procedure at the construction of the fractional generalization of a Lagrangian with linear velocity. The third objective is to illustrate our approach.

IV.1 The Fractional Euler Lagrange Equations for Lagrangians with Linear Velocities

Let $J[q^1,\ldots,q^n]$ be a functional of the form

$$\int_a^b L\left(t, q^1,\ldots,q^n, {}_a\mathbf{D}_t^\alpha q^1,\ldots,{}_a\mathbf{D}_t^\alpha q^n, {}_t\mathbf{D}_b^\beta q^1,\ldots,{}_t\mathbf{D}_b^\beta q^n\right) dt \;, \qquad \text{(IV.1)}$$

defined on the set of functions $q^i(t),\ i = 1,\ldots,n$ which have continuous left RL fractional derivative of order α and right RL fractional derivative of order β in $[a,b]$ and satisfy the boundary conditions $q^i(a) = q^i_a$ and $q^i(b) = q^i_b$. In [31], a necessary condition for $J[q^1,\ldots,q^n]$ to have an extremum for given functions $q^i(t),\ i = 1,\ldots,n$ was found as Euler-Lagrange equations

$$\frac{\partial L}{\partial q^j} + {}_t\mathbf{D}_b^\alpha \frac{\partial L}{\partial_a \mathbf{D}_t^\alpha q^j} + {}_a\mathbf{D}_t^\beta \frac{\partial L}{\partial_t \mathbf{D}_b^\beta q^j} = 0 \;, \qquad j = 1,\ldots,n \;. \qquad \text{(IV.2)}$$

Let us consider the following Lagrangian

$$L = a_j\left(q^i\right)\dot{q}^j - V\left(q^i\right) \ , \tag{IV.3}$$

where $a_j\left(q^i\right)$ and $V\left(q^i\right)$ are functions of their arguments.

The first step is to construct the corresponding fractional generalization of (IV.3). The fractional Lagrangian is not unique. In other words, there are several possibilities to replace the time derivative with fractional derivatives. The requirement is to obtain the same Lagrangian expression if the order 'α' is 1. Having in mind the above considerations, for $0 < \alpha \leq 1$, we propose two fractional Lagrangians. The first one is as follows

$$L' = a_j\left(q^i\right) {}_a\mathbf{D}_t^\alpha q^j - V\left(q^i\right) \ . \tag{IV.4}$$

Using (IV.2) and (IV.4), the corresponding Euler-Lagrange equations emerge as

$$\frac{\partial L'}{\partial q^k} + {}_t\mathbf{D}_b^\alpha a_k\left(q^i\right) = 0 \ , \tag{IV.5}$$

or, explicitly,

$$\frac{\partial a_j\left(q^i\right)}{\partial q^k} {}_a\mathbf{D}_t^\alpha q^j - \frac{\partial V\left(q^i\right)}{\partial q^k} + {}_t\mathbf{D}_b^\alpha a_k\left(q^i\right) = 0 \ . \tag{IV.6}$$

The second Lagrangian is given by

$$L' = -a_j\left(q^i\right) {}_t\mathbf{D}_b^\alpha q^j - V\left(q^i\right) \ . \tag{IV.7}$$

Using (IV.2) and (IV.7), the corresponding Euler-Lagrange equations become

$$\frac{\partial a_j\left(q^i\right)}{\partial q^k} {}_t\mathbf{D}_b^\alpha q^j + \frac{\partial V\left(q^i\right)}{\partial q^k} + {}_a\mathbf{D}_t^\alpha a_k\left(q^i\right) = 0 \ . \tag{IV.8}$$

IV.2 Examples

Example 1. Consider the following Lagrangian

$$L = \dot{q}^1 q^2 - \dot{q}^2 q^1 - (q^1 - q^2)q^3 \ , \tag{IV.9}$$

which is a gauge invariant. We propose the corresponding fractional Lagrangian [39] as

$$L' = \left({}_a\mathbf{D}_t^\alpha q^1\right) q^2 - \left({}_a\mathbf{D}_t^\alpha q^2\right) q^1 - (q^1 - q^2)q^3 \ . \tag{IV.10}$$

Using (IV.6), the Euler-Lagrange equations corresponding to (IV.10) have the form

$$\begin{aligned} \frac{\partial L'}{\partial q^3} &= 0 \ , && \text{(IV.11)} \\ \frac{\partial L'}{\partial q^1} + {}_t\mathbf{D}_b^\alpha \frac{\partial L'}{\partial \left({}_a\mathbf{D}_t^\alpha q^1\right)} &= 0 \ , && \text{(IV.12)} \\ \frac{\partial L'}{\partial q^2} + {}_t\mathbf{D}_b^\alpha \frac{\partial L'}{\partial \left({}_a\mathbf{D}_t^\alpha q^2\right)} &= 0 \ . && \text{(IV.13)} \end{aligned}$$

After calculations, the above system becomes

$$q^1 = q^2 \ , \quad -{}_a\mathbf{D}_t^\alpha q^2 - q^3 + {}_t\mathbf{D}_b^\alpha q^2 = 0 \ , \quad {}_a\mathbf{D}_t^\alpha q^1 + q^3 - {}_t\mathbf{D}_b^\alpha q^1 = 0 \ . \tag{IV.14}$$

The solution of (IV.14) is given by

$$\begin{aligned} q^2 &= q^1 \ , && \text{(IV.15)} \\ q^3 &= (-{}_a\mathbf{D}_t^\alpha + {}_t\mathbf{D}_b^\alpha)\, q^1. && \text{(IV.16)} \end{aligned}$$

The classical solutions are recovered if $\alpha \to 1$.

Example 2. Let us consider the second Lagrangian, which is given by

$$L = \dot{q}^1 q^2 + \dot{q}^3 q^4 - V(q^2, q^3, q^4), \tag{IV.17}$$

where $V(q^2, q^3, q^4) = \frac{-1}{2}\left[(q^4)^2 - 2q^2q^3\right]$. We observe that (IV.17) corresponds to a second class constrained system in Dirac's classification [33, 34, 35].

We propose the fractional generalization of (IV.17) to be as follows [39]

$$L' = -\left[({}_t\mathbf{D}_b^\alpha q^1)q^2 + ({}_t\mathbf{D}_b^\alpha q^3)q^4 + V(q^2,q^3,q^4)\right] . \tag{IV.18}$$

Using (IV.8), the Euler-Lagrange equations of (IV.18) are given by

$${}_a\mathbf{D}_t^\alpha q^2 = 0 , \tag{IV.19}$$

$${}_t\mathbf{D}_b^\alpha q^1 + q^3 = 0 , \tag{IV.20}$$

$${}_a\mathbf{D}_t^\alpha q^4 + q^2 = 0 , \tag{IV.21}$$

$${}_t\mathbf{D}_b^\alpha q^3 - q^4 = 0 . \tag{IV.22}$$

From (IV.19), the solution for q^2 is given by

$$q^2(t) = C_1(t-a)^{\alpha-1} . \tag{IV.23}$$

Substituting (IV.23) into (IV.21), we obtain

$${}_a\mathbf{D}_t^\alpha q^4 = -C_1(t-a)^{\alpha-1} \tag{IV.24}$$

having the solution as

$$q^4(t) = C_2(t-a)^{\alpha-1} - \frac{C_1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}(\tau-a)^{\alpha-1}d\tau . \tag{IV.25}$$

Using (IV.22) and (IV.25), the solution of q^3 has the form

$$\begin{aligned} q^3(t) &= C_3(b-t)^{\alpha-1} + \frac{C_2}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}(\tau-a)^{\alpha-1}d\tau \\ &\quad - \frac{C_1}{(\Gamma(\alpha))^2}\int_t^b (\tau-t)^{\alpha-1}\int_a^\tau (\tau-\eta)^{\alpha-1}(\eta-a)^{\alpha-1}d\eta d\tau . \end{aligned} \tag{IV.26}$$

Finally, the equation (IV.20) together with (IV.26) give the solution for q^1 as follows

$$q^1(t) = C_4(b-t)^{\alpha-1} - \frac{C_3}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}(b-\tau)^{\alpha-1}d\tau$$

$$
-\frac{C_2}{(\Gamma(\alpha))^2}\int_t^b(\tau-t)^{\alpha-1}\int_\tau^b(\eta-\tau)^{\alpha-1}(\eta-a)^{\alpha-1}d\eta d\tau
$$
$$
+\frac{C_1}{(\Gamma(\alpha))^3}\int_t^b(\sigma-t)^{\alpha-1}\int_\sigma^b(\tau-\sigma)^{\alpha-1}\int_a^\tau(\tau-\eta)^{\alpha-1}(\eta-a)^{\alpha-1}d\eta d\tau d\sigma\ , \tag{IV.27}
$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. Letting $\alpha \to 1$, $a \to 0$, $b \to 1$, and also redefining the constants in (IV.23), (IV.25), (IV.26) and (IV.27) as

$$
C_1 = C_4^{'},\quad C_2 = C_3^{'},\quad C_3 = C_2^{'} + \frac{C_4^{'}}{2} - C_3^{'},\quad C_4 = C_1^{'} + C_2^{'} - \frac{C_3^{'}}{2} + \frac{C_4^{'}}{6}\ , \tag{IV.28}
$$

the standard solutions

$$
q^1 = \frac{C_4^{'}t^3}{6} - \frac{C_3^{'}t^2}{2} + C_2^{'}t + C_1^{'}\ , \tag{IV.29}
$$
$$
q^2 = C_4^{'}\ , \tag{IV.30}
$$
$$
q^3 = \frac{C_4^{'}}{2}t^2 - C_3^{'}t + C_2^{'}\ , \tag{IV.31}
$$
$$
q^4 = -C_4^{'}t + C_3^{'} \tag{IV.32}
$$

are recovered.

CHAPTER V

CONCLUSION

Grünwald-Letnikov, Riemann-Liouville and Caputo fractional derivatives together with their noteworthy properties were presented. A brief review of exterior calculus was given to fix notation and to provide a convenient reference. Fractional form spaces were defined based on the fractional exterior derivative. For the new vector spaces, basis sets were given and the notions of closed and exact forms were examined. The coordinate transformation rules were also computed.

The Laplace transform method was presented and Laplace transformation formulas for Riemann-Liouville and Caputo fractional derivatives were derived. To illustrate the approach, three examples were investigated in details. The difference in the form of the initial conditions which must accompany fractional differential equations in terms of the Riemann-Liouville and Caputo derivatives was given.

The basic results for the standard diffusion equation concerning the fundamental solution of the Cauchy problem were presented. In particular, we provided the derivation of the solution by the Fourier transform and its interpretation in terms of Gauss probability density function. The fundamental solution of the time-fractional diffusion equation for the Cauchy problem was shown to provide probability density function evolving in time which are related to certain stable distributions. This property is a noteworthy generalization of what happens in the case of the standard diffusion equation and can be relevant in treating financial and economical problems where stable probability distributions are known to play a key role.

The time-fractional Schrödinger equation was examined. It was solved for a free particle and for a potential well. Probability and the resulting energy levels were found to increase over time to a limiting value depending on the order of the fractional time derivative.

The Lagrangians with linear velocities were investigated using left and right Riemann Liouville fractional derivatives. The corresponding fractional Lagrangians were proposed and the fractional Euler-Lagrange equations were obtained. Although the fractional Lagrangians contain only left or right derivatives, both derivatives are involved in Euler-Lagrange equations and both played an important role in finding the solutions. The exact solutions of the fractional Euler-Lagrange equations were obtained for two examples corresponding to first and second-class constrained systems. A "gauge invariance" was reported for the first example. The recovery of the usual results was also discussed.

REFERENCES

[1] K. B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, New York, (1974).

[2] K. S. Miller, B. Ross, *An introduction to the Fractional Integrals and Derivatives-Theory and Applications*, Gordon and Breach, Longhorne, PA, (1993).

[3] I. Podlubny, *Fractional Differential Equations*, Academic Press, London, (1999).

[4] F. Mainardi, *Fractional calculus: Some basic problems in continuum and statistical mechanics*, in A. Carpinteri and F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien, (1997).

[5] F. Mainardi, G. Pagnini, *The Wright functions as solutions of the time-fractional diffusion equation*, Appl. Math. and Comp., **vol. 141**, **no.1**, 51-62, (2003).

[6] R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*, Word Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, (2000).

[7] S. Z. Rida, El-Sayed AMA, *Fractional calculus and generalized Rodrigues formula*, Appl. Math. and Comp., **147**, **no.1**, 29-43, (2004).

[8] H. C. Rosu, L. Madueno, J. Socorro, *Transform of Riccati equation of constant coefficients through fractional procedure*, J. Phys. A, **36**, 1087-1094, (2003).

[9] O. P. Agrawal, *Solution for a Fractional Diffusion-Wave Equation Defined in a Bounded Domain*, Nonlinear Dynamics, **29**, 145-155, (2002).

[10] N. Engheta, *On fractional paradigm and intermediate zones in electromagnetism: I - Planar observation*, Microw. Opt. Techn. Let., **vol. 22**, **no.4**, 236-241, (1999).

[11] V. V. Anh, C. N. Nguyen, *Semimartingale representation of fractional Riesz-Bassel motion*, Finance Stoch., **5**, **no.1**, 83-101, (2001).

[12] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent*, Part II., Geophys. J. R. Astr. Soc., **13**, 529-539, (1967).

[13] R. Gorenflo, F. Mainardi, *Fractional calculus: Integral and differential equations of fractional order*, in A. Carpinteri and F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien, (1997).

[14] R. Hilfer, *Experimental evidence for fractional time evolution in glass forming materials*, Chem. Phys., **284**, **no.(1-2)**, 399-408, (2002).

[15] H. Flanders, *Differential Forms with Applications to the Physical Sciences*, Dover, (1989).

[16] C. Shepherd, M. Naber, *Fractional Differential Forms*, J. Math. Phys., **42**, **no.5**, (2003).

[17] C. Shepherd, M. Naber, *Fractional Differential Forms II*, arXiv:math-ph/0301016, (2003).

[18] H. Engler, *Similarity solutions for a class of hyperbolic integrodifferential equations*, **10**, **no.5**, 815-840, (1997).

[19] Y. Fujita, *Integrodifferential equation which interpolates the heat equation and the wave equation*, Osaka J. Math., **27**, 309-321, (1990).

[20] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, (1993).

[21] W. R. Schneider, W. Wyss, *Fractional diffusion and wave equations*, J. Math. Phys., **30**, 134-144, (1989).

[22] F. Mainardi, M. Tomirotti, *On a special function arising in the time fractional diffusion-wave equation*, in P. Rusev, I. Dimovski and V. Kiryakova (Eds.), Transform Methods and Special Functions, Sofia, (1994), Science Culture Technology, Singapore, (1995).

[23] R. P. Feynman, A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGrawHill, Inc., (1965).

[24] N. Laskin, *Fractional quantum mechanics*, Phys. Rev. E, **62**, **no.3**, 3135-3145, (2000).

[25] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A, **268**, 298-305, (2000).

[26] N. Laskin, *Fractional Schrödinger equation*, Phys. Rev. E, **66**, art. no. 056108, (2002).

[27] F. Riewe, *Nonconservative Lagrangian and Hamiltonian mechanics*, Phys. Rev. E, **53**, 1890-1899, (1996).

[28] F. Riewe, *Mechanics with fractional derivatives*, Phys. Rev. E, **55**, 3582-3592, (1997).

[29] M. Klimek, *Lagrangian and Hamiltonian fractional sequential mechanics*, Czech. Journ. Phys., **52**, **no.11**, 1247-1253, (2002).

[30] M. Klimek, *Fractional sequaential mechanics - models with symmetric fractional derivative*, Czech. Journ. Phys., **51**, **no.12**, 1348-1354, (2001).

[31] O. P. Agrawal, *Formulation of Euler-Lagrange equations for fractional variational problems*, J. Math. Anal. Appl., **272**, 368-379, (2002).

[32] O. P. Agrawal, *Lagrangian and Lagrange equation of motion for fractionally damped systems*, Transactions of the ASME, J. Appl. Mech., **68**, **no.2**, 339-341, (2001).

[33] A. Hanson, T. Regge, C. Teitelboim, *Constrained Hamiltonian Systems*, Academia Nationale dei Lincei, Rome, (1976).

[34] K. Sundermeyer, *Constrained Dynamics*, Lecture Notes in Physics, **vol.169**, Springer-Verlag, New-York, (1982).

[35] M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, (1992).

[36] D. Baleanu, Y. Güler, *A general treatment of singular Lagrangians with linear velocities*, Nuovo Cimento B, **115**, **no.3**, 319-324, (2000).

[37] B. Hatfield,*Quantum Field Theory of Point Particles and Strings*, Addison-Wesley, (1992).

[38] S. Gasiorowicz, *Quantum Physics*, John Wiley & Sons, (1974).

[39] D. Baleanu, T. Avkar, *Lagrangians with linear velocities within Riemann-Liouville fractional derivatives*, accepted for publication in Il Nuovo Cimento B, arXiv:math-ph/0405012, **v1**, (2004).

APPENDIX A

SPECIAL FUNCTIONS IN FRACTIONAL CALCULUS

Hereafter, a review of definitions and some noteworthy properties of the Gamma, Beta, Mittag-Leffler and Wright functions is given. They play very important role in the theory of fractional calculus.

A.1 Gamma Function

Euler's Gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \ , \quad Re(z) > 0 \ . \tag{A.1}$$

The integral in (A.1) converges absolutely and uniformly in any strip of the kind $\{z : 0 < a \leq Re(z) \leq A < \infty\}$ and hence it is an analytic function in $\{z : Re(z) > 0\}$. Using the technique of integration by parts, it follows from (A.1) that

$$\Gamma(z+1) = z\Gamma(z) \ , \qquad \forall z \in C \quad s.t. \quad Re(z) > 0 \ . \tag{A.2}$$

Note also that the application of the method of mathematical induction yields

$$\Gamma(n+1) = n! \ , \qquad n = 0, 1, 2, \dots \ , \tag{A.3}$$

and limit representation of the Gamma function is given by

$$\Gamma(z) = \lim_{n\to 0} \frac{n! n^z}{z(z+1)\dots(z+n)} \ , \quad Re(z) > 0. \tag{A.4}$$

A.2 Beta Function

The beta function is defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \ , \qquad Re(a) > 0 \ , \quad Re(b) > 0 \ . \tag{A.5}$$

A noteworthy property of the Beta function is based on the following formula

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \ , \quad \forall a, b \in \mathbf{C} \ . \tag{A.6}$$

A.3 Mittag-Leffler Function

The two-parameter Mittag-Leffler function, named after the great Swedish mathematician, is defined by the following series, valid in the whole complex plane,

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)} \ , \qquad \alpha > 0, \ \beta > 0 \ . \tag{A.7}$$

Some elementary functions can be easily derived from the formula (A.7) as follows

$$E_{2,1}(+z^2) = \sum_{m=0}^{\infty} \frac{z^{2m}}{\Gamma(2m+1)} = \sum_{m=0}^{\infty} \frac{z^{2m}}{(2m)!} = \cosh(z) \ , \tag{A.8}$$

$$E_{2,1}(-z^2) = \sum_{m=0}^{\infty} \frac{(-z^2)^m}{\Gamma(2m+1)} = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{(2m)!} = \cos(z) \ , \tag{A.9}$$

and that

$$E_{\frac{1}{2},1}(\pm z) = e^{z^2}\left[1 + \mathrm{erf}(\pm z)\right] = e^{z^2}\mathrm{erfc}(\mp z) \ , \tag{A.10}$$

where

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \ , \qquad \mathrm{erfc}(z) := 1 - \mathrm{erf}(z) \ , \quad z \in \mathbf{C} \ . \tag{A.11}$$

One of the most important property of the Mittag-Leffler function is the following one, called as *duplication formula*

$$2E_{\alpha,\beta}(z^2) = E_{\frac{\alpha}{2},\beta}(z) + E_{\frac{\alpha}{2},\beta}(-z) \ . \tag{A.12}$$

The Mittag-Leffler function has also the following property

$$\int_0^\infty e^{-t} \ t^{\beta-1} \ E_{\alpha,\beta}(zt^\alpha)dt = \frac{1}{1-z} \ , \qquad \alpha > 0 \tag{A.13}$$

and we obtain from (A.13) that the Laplace transform of the function

$$t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm z t^\alpha) \ , \tag{A.14}$$

where

$$E^{(k)}_{\alpha,\beta}(y) \equiv \frac{d^k}{dy^k} E_{\alpha,\beta}(y) \tag{A.15}$$

is given by

$$\int_0^\infty e^{-st} \ t^{\alpha k+\beta-1} \ E^{(k)}_{\alpha,\beta}(\pm a t^\alpha)dt = \frac{k! \ s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}} \ , \qquad (Re(s) > |a|^{1/\alpha}) \ . \tag{A.16}$$

The particular case of (A.16) for $\alpha = \beta = \frac{1}{2}$ is that

$$\int_0^\infty e^{-st} \ t^{\frac{k-1}{2}} \ E^{(k)}_{\frac{1}{2},\frac{1}{2}}(\pm a\sqrt{t})dt = \frac{k!}{(\sqrt{s} \mp a)^{k+1}} \ , \qquad (Re(s) > a^2) \ . \tag{A.17}$$

Additionally, the "Euler identity" for the Mittag-Leffler function is given by

$$E_{\alpha,1}(\lambda i^\alpha t^\alpha) = \frac{e^{i\lambda^{1/\alpha} t}}{\alpha} - \Phi_\alpha(\lambda i^\alpha, t) \ , \tag{A.18}$$

where

$$\Phi_\alpha(\zeta, t) = \frac{\zeta \sin(\alpha\pi)}{\pi} \int_0^\infty \frac{e^{-rt} r^{\alpha-1} dr}{r^{2\alpha} - 2\zeta \cos(\alpha\pi) r^\alpha + \zeta^2} \ . \tag{A.19}$$

A.4 Wright Function

The Wright function, named after the British mathematician, is defined by the following series, valid in the whole complex plane,

$$W_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{m! \, \Gamma(\alpha m + \beta)} \ , \qquad \alpha > -1, \ \beta > 0 \ . \tag{A.20}$$

Using the Wright function, Mainardi's function $M(z;\mu)$ is defined by

$$M(z;\mu) := W_{-\mu,1-\mu}(-z) \ , \quad 0 < \mu < 1 \ . \tag{A.21}$$

When the argument z is real and positive, i.e. $z = r > 0$, the existence of the Laplace transform of $M(r;\mu)$ is ensured by the asymptotic behavior [22]

$$M(r;\mu) \sim a(\mu) r^{\frac{\mu - \frac{1}{2}}{1-\mu}} \exp\left[-b(\mu) r^{\frac{1}{1-\mu}}\right] \ , \qquad r \to +\infty \ , \tag{A.22}$$

where

$$a(\mu) = \frac{1}{\sqrt{2\pi(1-\mu)}} \ , \quad b(\mu) = \frac{1-\mu}{\mu} \ . \tag{A.23}$$

It should be noted that we have the following Laplace transform pair

$$M(r;\mu) \div E_{\mu,1}(-s) \ , \quad 0 < \mu < 1 \ . \tag{A.24}$$

Note also that the above relationship allows us to compute the moments of any real order δ of $M(r;\mu)$ in the positive real axis as follows

$$\int_0^{+\infty} r^\delta M(r;\mu) dr = \frac{\Gamma(\delta + 1)}{\Gamma(\mu\delta + 1)} \ , \quad \delta \geq 0 \ . \tag{A.25}$$