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FUZZY CONTINUITY OF FUNCTIONS IN THE FUZZY TOPOLOGICAL SPACES

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ABSTRACT

This study, purpose to give fuzzy weakly continuity, fuzzy almost continuity, fuzzy quasi continuity, fuzzy somewhat continuity, fuzzy rarely continuity, fuzzy D-continuity and weakly forms of them with the relations, are four chapters.

In the first chapter are given introductory information about fuzzy sets and fuzzy topological spaces. Also some definitions and theorems and their references, in this area are given. The second, the third and the forth chapters offer the original studies of this thesis.

In the second chapter which takes from three sections, fuzzy waekly and fuzzy weakly* continuities are defined and compared with fuzzy weakly** continuity. Alsoit is shown some examples that they are independent to each other. In the second section, fuzzy almost cotinuity is defined in the sense of Stallings and studied the relations with the other fuzzy almost continuities. Also, some characterizations of the fuzzy almost continuity in the sense of Singal are given. In the last section, fuzzy quasi and fuzzy somewhat continuity are compared with weakly forms of them.

In the third chapter, some basic properties of fuzzy rarely continuity of functions are studied. Then, fuzzy rarely* continuity is defined and compared with some continuities and their weakly forms. The strongly and weakly forms of fuzzy θ-continuity of functions are defined. Also fuzzy strongly θ*-continuity is defined. Then, some comparisons are done.

In the last chapter takes from two sections. In the first section, fuzzy D-continuity and fuzzy D*-continuity of functions are defined and equivalent conditions are given. Then several properties and requirements are studied. In the second section, some properties which are preserved under the fuzzy D-continuity of functions are studied. This section is finished with the definitions of fuzzy D-Hausdorff space and fuzzy D-regular space and some properties of them.

ÖZET

Bu çalışmada, fuzzy topolojik uzaylar için fuzzy weakly süreklilik ve onun bazı varyasyonları, fuzzy almost süreklilik, fuzzy quasi süreklilik fuzzy somewhat, fuzzy rarely süreklilik ile fuzzy D-süreklilik ve bunlar arasındaki ilişkileri vermeyi amaçladık.

Birinci bölümde ön bilgiler ile bu alanda yapılan çalışmalardan bazı önemli tanım ve teoremler ve bunlara ilişkin kaynaklar verildi. İkinci, üçüncü ve dördüncü bölümlerde tezin özgün çalışmaları sunuldu.

Üç kesimden oluşan ikinci bölümde, fuzzy weakly süreklilik ve fuzzy weakly* süreklilik tanımlanarak, birbirleriyle ve fuzzy weakly** süreklilik ile karşılaştırıldı. Birbirlerinden bağımsız oldukları örneklerle gösterildi. İkinci kesimde, Stallings anlamda fuzzy almost süreklilik tanımlandı ve Singal ve Husain anlamda almost sürekliliklerle ilgili bazı teoremler verildi ve bazı sonuçlar elde edildi. Son kesimde de, fuzzy quasi süreklilik tanımı verildi ve zayıf biçimleri ile karşılaştırıldı.

Üçüncü bölümde, fuzzy rarely sürekli fonksiyonların bazı temel özellikleri çalışıldı. Daha sonra fuzzy rarely* süreklilik tanımlanarak diğer bazı sürekli fonksiyonlar ve bunların zayıf biçimleriyle karşılaştırmalar yapıldı. Ters örnekler verildi. Fuzzy θ-sürekli fonksiyonların kuvvetli ve zayıf biçimleri tanımlanarak karşılaştırmalar yapıldı. Ayrıca fuzzy strongly θ*-süreklilik tanımı verildi ve çeşitli karşılaştırmalar yapıldı.

Son bölüm iki kesimden oluştu. Birinci kesimde, fuzzy D-sürekli ve fuzzy D*-sürekli fonksiyon tanımları verildi. Denk tanımlar bir karekterizasyonla sunuldu. Daha sonra bir takım özellikleri ile bazı gerektirmeler çalışıldı. İkinci kesimde, fuzzy D-sürekli fonksiyonlar altında korunan bazı özellikler incelendi. Fuzzy D-Hausdorff uzay ve fuzzy D-regüler uzay tanımları verilerek diğer bazı ayırma aksiyomları ile karşılaştırmalar yapılarak bölüm sonlandırıldı.

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CHAPTER 1

BASIC CONCEPTS

1.1.INTRODUCTION

The fundamental concept of a fuzzy set was first introduced by Zadeh in 1965 [11]. Since then, work has been done by many authors, in several directions, which has resulted in the formation of a new mathematical field called "Fuzzy Mathematics".

Because the concept of fuzzy sets corresponds to the physical situation in which there is no precisely defined criterion for membership, fuzzy sets are having useful and increasing applications in various fields including probability theory, information theory and pattern recognition. Thus the developments in abstract mathematics using the idea of fuzzy sets posses sound footing. In accordance with this, fuzzy topological spaces, fuzzy proximity spaces, fuzzy groups, fuzzy vector spaces, fuzzy topological groups and fuzzy topological vector spaces were introduced by some mathematicians.

The theory of fuzzy topological spaces is a branch of such mathematics Chang in [3]. was the first to introduce the notion of fuzzy topology. Several others contined the work in this area. Some mathematicians consider that general point set topology can be regarded as a special case of fuzzy topology, where all membership functions are just characteristic funtions. Therefore, one would expect weaker results in the case of fuzzy topology.

Also some authors was devoted to the fuzzy topological spaces with specific attention to the weaker forms of fuzzy continuity. Weaker forms of continuity in topology have been considered by many workers using the concepts of semi open, regular open, pre open and alpha open sets. Same way, weaker forms of fuzzy continuity in fuzzy topology have been considered by many workers using the concepts of fuzzy semi open, fuzzy regular open, fuzzy pre open and fuzzy alpha open sets.

In this thesis, using a Q-neigborhood and neigborhood of a fuzzy point, we introduced several weaker forms of fuzzy continuity and studied relations between them. For example, fuzzy weakly* continuity, fuzzy rarely* continuity and fuzzy D continuity of a function

In this section, the definition of fuzzy set, some other basic concepts and general properties of fuzzy sets are given under the light of the fundamental information

Definition 1.1.1. Let X be a set. A fuzzy set in X is an element in $[0,1]^X$, i.e., a function from X into [0,1].[11].

We will denote fuzzy sets with α,β,δ, ... as Latin letters. The symbol 0 will be used to denote an empty fuzzy set ($0(x) = 0$ for all x in X). For 1, we have by definition $1(x) = 1$ for all x in X.

We will denote $\alpha(x)$ the image of x with a fuzzy set α in X. Contain, union and intersection of fuzzy sets are denoted by "$\leq$", "V" and "Λ" respectirely.

Definition 1.1.2. The complement of α denoted by α' is defined by the formula

$\alpha'(x) = 1-\alpha(x)$ for $x \in X$. [11].

Definition 1.1.3. Let α and β be two fuzzy sets in X. Then, we have the following properties for fuzzy sets α and β.

$\alpha \leq \beta \Leftrightarrow \alpha(x) \leq \beta(x)$ for all $x \in X$,

$\alpha = \beta \Leftrightarrow \alpha(x) = \beta(x)$ for all $x \in X$,

$\mu = \alpha \vee \beta \Leftrightarrow \mu(x) = \max\{\alpha(x),\beta(x)\}$ for all $x \in X$

$\delta = \alpha \wedge \beta \Leftrightarrow \delta(x) = \min\{\alpha(x),\beta(x)\}$ for all $x \in X$

$\alpha = \beta' \Leftrightarrow \alpha(x) = 1 - \beta(x)$ for all $x \in X$

More generally, for a family of fuzzy sets,

$\mu = \{\mu_i \mid i \in I\}$, and the intersection $\beta = \wedge\mu_i$ the union $\vee\mu_i = \alpha$, are defined by

$\alpha(x) = \mathrm{Sup}\,\{\mu_i(x)\}, x \in X$,

and

$\beta(x) = \mathrm{In\,f}\,\{\mu_i(x)\}, x \in X$. [3].

Definition 1.1.4. A fuzzy set in X is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α $(0 < \alpha \leq 1)$ we denote this fuzzy point by x_α, where the point x is called its support.

We can write the fuzzy point x_α, with

$$x_\alpha(y) = \begin{cases} \alpha, & y = x \\ 0, & y \neq x \end{cases}$$

and we can denote the support of x_α with supp $x_\alpha = x$ [5].

Definition 1.1.5. Let x_α, x_β be two fuzzy points in X. If supp $x_\alpha \neq$ Suppy x_β, then it is called that x_α and x_β fuzzy points are different. [5].

Definition 1.1.6. The fuzzy point x_α is said to be contained in a fuzzy set β, or belong to β, denoted by $x_\alpha \in \beta$, iff $\alpha \leq \beta(x)$. Evidently, every fuzzy set α can be expressed as the union of all the fuzzy points which belong to α. [5].

Definition 1.1.7. Two fuzzy sets α, β in X are said to be intersected iff there exists a point $x \in X$ such that $(\alpha \wedge \beta)(x) \neq 0$. For such a case, we say that α and β intesect at x. [5].

Theorem 1.1.8. Let α and β be two fuzzy sets, in X and let x_∂ be a fuzzy point. Then we have the following.

a) $x_\partial \in \alpha \wedge \beta \Leftrightarrow x_\partial \in \alpha$ and $x_\partial \in \beta$

b) $x_\partial \in \alpha \vee \beta \Leftrightarrow x_\partial \in \alpha$ or $x_\partial \in \beta$

Proof : [1].

Proposition 1.1.9. Let $\mu = \{\mu_i : i \in I\}$ be a family of fuzzy sets. For fuzzy point x_α in X,

$x_\alpha \in \vee \mu_i$ iff there is a $i_o \in I$ at least such that $x_\alpha \in \mu_{io}$. [5].

Definition 1.1.10. A fuzzy point x_α is said to be quasi-coincident with β, denoted by $x_\alpha q \beta$, iff $\alpha > \beta'(x)$, or $\alpha + \beta(x) > 1$. [5].

Definition 1.1.11. α is said to be quasi-coincident with β, denoted by $\alpha q\beta$, iff there exists a point $x \in X$ such that $\alpha(x) > \beta'(x)$, or $\alpha(x)+\beta(x)>1$. If this is true, we also say that α and β are quasi-coincident (with each other) at x. It is clear that if α and β are quasi-concident at x, both $\alpha(x)$ and $\beta(x)$ are not zero and hence α and β intersect at x. [5].

Proposition 1.1.12. Let $\mu = \{\mu_i : i \in I\}$ be a family of fuzzy sets in X. And let x_α be a fuzzy point. $x_\alpha q \vee \mu_i$, iff there exists at least $i_o \in I$ such that $x_\alpha \, q \, \mu_{i_{o}}$ [5].

1.2. Fuzzy Topological Spaces

Definition 2.2.1. A fuzzy topology is a family τ of fuzzy sets in X which satifies the following conditions:

(a) $0, 1, \in \tau$,

(b) If $\alpha, \beta \in \tau$, then $\alpha \wedge \beta \in \tau$,

(c) If $\mu_i \in \tau$ for each, $i \in I$ then $\vee_i \mu_i \in \tau$

τ is called a fuzzy topology for X, and the pair (X, τ) is a fuzzy topological space, (shortly f.t.s.) Every member of τ is called an open fuzzy set. A fuzzy set is closed fuzzy set iff its complement is open.

As in general topolgy, the indiscrete fuzzy topology contains only 0 and 1, while the discrete fuzzy topology contains all fuzzy sets. A fuzzy topology τ is said to be coaser than a fuzzy topology τ', if and only if $\tau \subset \tau'$. [5].

Definition 1.2.2. Let (X,τ) be a f.t.s. and $\alpha \in I^X$. Closure of α is denoted α or cl (α), and given by.

$$\alpha = \wedge\{ \beta : \beta \text{ is a fuzzy closed set and } \alpha \leq \beta \}$$

The interior of α is denoted by int α or α° and given by.

$$\alpha^\circ = \vee \{ \beta : \beta \text{ is fuzzy open set and } \beta \leq \alpha \} \text{ [4].}$$

Theorem 1.2.3. Let α be a fuzzy set in a f.t.s. (X, τ). Then α° is open and it is the largest open set contained in α. The fuzzy set α is open iff $\alpha = \alpha^\circ$

Proof : [3]..

Remark 1.2.4. We have the following properties for α, β fuzzy sets in a f.t.s. (X, τ),

a) $1^\circ = 1,\ 0^\circ = 0$

b) $\alpha^\circ \leq \alpha$

c) $\alpha^{\circ\circ} = \alpha^\circ$

d) $(\alpha \wedge \beta)^\circ = \alpha^\circ \wedge \beta^\circ$

e) $\vee \alpha^\circ{}_i \leq (\vee \alpha_i)^\circ$ for every $i \in I$.

Proof : [4] and [5].

Remark 1.2.5. We have the following porperties for α,β fuzzy sets in (X, τ) ft..s.

a) $1 = \overline{1},\ \overline{0} = 0$

b) $\alpha \leq \overline{\alpha}$

c) $\overline{\overline{\alpha}} = \overline{\alpha}$

d) $\overline{(\alpha \vee \beta)} = \overline{\alpha} \vee \overline{\beta}$

e) $\overline{\wedge \alpha_i} \leq (\wedge \overline{\alpha_i})$ for every $i \in I$.

Proof : [4] and [5].

Theorem 1.2.6. Let (X, τ) be a f.t.s. and $\mu \in I^X$. Then we have the following;

a) $(\mu^\circ)' = \overline{(\mu')}$

b) $(\overline{\mu})' = (\mu')^\circ$

Proof : [4] .

Definition 1.2.7. A fuzzy set α in a f.t.s. (X, τ) is called a neighborhood of fuzzy point x_∂ iff there exists a $\beta \in \tau$ such that $x_\partial \in \beta \leq \alpha$. A neighborhood α of x_∂ is said to be open iff α is fuzzy open. The family consisting of all the neighborhoods of x_∂ is called the system of neighborhoods of x_∂. [15].

We will denote the family of all the neighborhoods of x_∂ with $\mathbb{N}(x_\partial)$ and the family of all the fuzzy open neighborhood of x_∂ with $\mathbb{U}(x_\partial)$.

Definition 1.2.8. A fuzzy set α in a f.t.s., (X, τ) is called Q-neighborhood of x_∂ iff there exists a $\beta \in \tau$ such that x_∂ q $\beta \leq \alpha$. The family consisting of all the Q-neighborhoods of x_∂ is called the system of Q-neighborhoods of x_∂. [5].

We will denote the family consisting of all the Q-neighborhoods of x_∂ with $\mathbb{N}_Q(x_\partial)$ and the family consisting of all the open Q-neighborhoods of x_∂ with $\mathbb{U}_Q(x_\partial)$

Theorem 1.2.9. Let α,β be two fuzzy sets in X. Then $\alpha \leq \beta$ iff α and β' are not quasi-coincident. Particularly, $x_\partial \in \alpha$ iff x_∂ is not quasi-coincident with α'. [5].

Theorem 1.2.10. Let $\mathbb{N}_Q$ be the family of Q-neighborhoods (resp, neighborhoods) of a fuzzy point x_∂ in a f.t.s. (X, τ). Then we have the followings:

(1) If $\mu \in \mathbb{N}_Q(x_\partial)$ ($\mu \in \mathbb{N}(x_\partial)$), then x_∂ is quasi-coincident with (resp. belong to) μ.

(2) If $\alpha, \beta \in \mathbb{N}_Q(x_\partial)$ ($\alpha, \beta \in \mathbb{N}(x_\partial)$), then $\alpha\wedge\beta \in \mathbb{N}_Q(x_\partial)$, ($\alpha\wedge\beta \in \mathbb{N}(x_\partial)$)

(3) If $\alpha \in \mathbb{N}_Q(x_\partial)$ ($\alpha \in \mathbb{N}(x_\partial)$) and $\alpha \leq \beta$, then $\beta \in \mathbb{N}_Q(x_\partial)$ ($\beta \in \mathbb{N}(x_\partial)$)

(4) If $\alpha \in \mathbb{N}_Q(x_\partial)$ ($\alpha \in \mathbb{N}(x_\partial)$, then there exists a fuzzy set β such that $\beta \leq \alpha$ and for every fuzzy point x_σ which is quasi - coincident with (belong to) β.$\beta \in \mathbb{N}_Q(x_\sigma)$ ($\beta \in \mathbb{N}(x_\sigma)$) [5].

Theorem 1.2.11. Let μ be a fuzzy set in a f.t.s. (X, τ). Then μ is fuzzy open if and only if for each fuzzy point x_α which is quasi-coincident with μ, $\mu \in \mathbb{N}_Q(x_\alpha)$

(μ is fuzzy open $\Leftrightarrow [(\forall\, x_\alpha, x_\alpha q \mu) \Rightarrow \mu \in \mathbb{N}_Q(x_\alpha)]$). [5].

Proposition 1.2.12. Let x_α be a fuzzy point in (X, τ). Then, $\mu \in \mathbb{N}_Q(x_\alpha) \Rightarrow \mu^\circ \in \mathbb{U}_Q(x_\alpha)$. [5].

Theorem 1.2.13. A fuzzy point $x_\partial \in \overline{\alpha}$ iff each Q-neighborhood of x_∂ is quasi-coincident with α. [5].

Definition 1.2.14. A fuzzy point x_∂ is called an adherence point of a fuzzy set α, if every Q-neighborhood of x_∂ is quasi-coincident with α. [5].

Corollary 1.2.15. $\overline{\alpha}$ is the union of all the adherence points of α. [5].

Proposition 1.2.16. Let (X, τ) be a f.t.s. and $0 \neq \mu \in I^X$ $\bar{\mu} = 1$ if and only if for each $\alpha \in \tau$, $\alpha q \mu$. [5].

Definition 1.2.17. Let $\mu \in I^X$ and μ^b (or $\partial\mu$) is defined to be the fuzzy boundary of μ which is the infimum of of all fuzzy closed sets σ with the property $\sigma(x) \geq \bar{\mu}(x)$ for all $x \in X$ for which we have $[\bar{\mu} \wedge (1-\bar{\mu})](x) > 0$. Clearly $\partial\mu$ is fuzzy closed and $\partial\mu \leq \bar{\mu}$. [6].

Definition 1.2.18. Let (X, τ) be a f.t.s. We define the following notions.

i) A fuzzy set μ is said to be fuzzy boundary if and only if $\overline{1-\mu} = 1$

ii) If $\alpha \leq \mu$ and μ is fuzzy boundary then α is fuzzy boundary too.

iii) If μ is fuzzy boundary then $\partial\mu \geq \bar{\mu} - \mu^{\circ}$.

iv) If μ is a fuzzy set then $\bar{\mu} = \partial\mu \vee \mu$. [6].

Proposition 1.2.19. Let (X, τ) be a f.t.s. Then, we have the followings:

i) $\mu \in I^X$ is fuzzy boundary if and only if $\mu^{\circ} = 0$

ii) $\mu \in I^X$ is fuzzy boundary if and only if $\mu \leq \partial\mu$.

Proof: [6].

Proposition 1.2.20. $\bar{\mu}$ is fuzzy boundary if and only if $\bar{\mu} \leq \overline{1-\bar{\mu}}$ if and only if $\mu \leq \overline{1-\bar{\mu}}$

Proof: [6].

Theorem 1.2.21. Fuzzy boundary sets are stable under fuzzy continuous surjections .

Proof: [6].

Definition 1.2.22 A fuzzy point e is called a boundary point of a fuzzy set β iff $e \in \bar{\beta} \wedge \bar{\beta}'$. The union of all the boundary points of β is called a boundary of β, denoted by $\partial\beta$. It is clear that $\partial\beta = \bar{\beta} \wedge (1-\beta^{\circ})$. [5].

1.3. Fuzzy Continuous Functions

Definition 1.3.1. Let X and Y be two f.t.s. and let f be a function from X to Y. Also let β be a fuzzy set in Y. Then the inverse of β, written as $f^{-1}(\beta)$, is a fuzzy set in X which is defined by

$f^{-1}(\beta)(x) = \beta(f(x))$ for all x in X.

Conversely, let α be a fuzzy set in X. The image of α, written as $f(\alpha)$, is fuzzy set in Y which is defined by

$$\begin{cases} \sup\limits_{z \in f^{-1}(y)} \{\alpha(z)\}, & \text{if } f^{-1}(y) \text{ is non empty} \\ 0 & \text{Other wise} \end{cases}$$

for all y in Y, where $f^{-1}(y) = \{x \mid f(x) = y\}$. [11].

Note that: If x_α is a fuzzy point in X, then $f(x_\alpha)$ is a fuzzy point in Y. If supp $x_\alpha = x$, then supp $(f(x_\alpha))=f(x)$. If $y_\alpha$is a fuzzy point in Y. Then $f^{-1}(y_\alpha)$ doesn't need a fuzzy point in X. If f is one-to-one and y_α is a fuzzy point in f(X), then $f^{-1}(y_\alpha)$ is a fuzzy point in X. In this case, If supp y_α=y, supp $f^{-1}(y_\alpha) = f^{-1}(y)$.

Theorem 1.3.2. Let $f : X \to Y$ be a function from X to Y then we have the following properties:

1) $f^{-1}(\beta') = \{ f^{-1}(\beta)\}'$ for any fuzzy set β in Y.
2) $f^{-1}(\alpha') \geq \{ f^{-1}(\alpha)\}'$ for any fuzzy set α in X.
3) $\beta_1 \leq \beta_2 \Rightarrow f^{-1}(\beta_1) \leq f^{-1}(\beta_2)$, where β_1 and β_2 are fuzzy sets in Y.
4) $\alpha_1 \leq \alpha_2 \Rightarrow f(\alpha_1) \leq f(\alpha_2)$, where α_1 and α_2 are fuzzy sets in X.
5) $f(f^{-1}(\beta)) \leq \beta$, for any fuzzy set β in Y.
6) $\alpha \leq f(f^{-1}(\alpha))$, for any fuzzy set α in X.
7) if $\alpha_i \in I^X$, $f(\vee\alpha_i) = \vee f(\alpha_i)$ For each $i \in I$
8) if $\beta_i \in I^Y$, $f(\vee\beta_i) = \vee f^{-1}(\beta_i)$ For each $i \in I$
9) If f is one-to-one, for $\alpha \in I^X$, $f^{-1}(f(\alpha)) = \alpha$
10) If f is surjection, for $\beta \in I^Y$, $f(f^{-1}(\beta)) = \beta$
11) If f is one-to-one and surjection, for $\alpha \in I^X$, $\{f(\alpha)\}' = f(\alpha')$
12) For $\alpha, \delta \in I^X$, $f(\alpha \wedge \delta) \leq f(\alpha) \wedge f(\delta)$
13) For each $i \in I$ if, $\beta \in I^Y$, $f^{-1}(\wedge\beta_i) = \wedge f^{-1}(\beta_i)$ For each $i \in I$

14) Let g be a function from Y to Z Then;

a) For $\beta\in I^Z$, $(gof)^{-1}(\beta) = f^{-1}(g^{-1}(\beta))$

b) For $\alpha\in I^X$, $(gof)(\alpha) = g(f(\alpha))$

where gof is the composition of g and f. [2], [4] and [7].

Proposition 1.3.3. Let f be a function from X to Y and x_α be a fuzzy point in X. Then;

1) For $\beta\in I^Y$, if $f(x_\alpha)q\beta$, $x_\alpha\, q\, f^{-1}(\beta)$

2) For $\alpha\in I^Y$, if $f\, x_\alpha\, q\, \delta$, $f(x_\alpha)\, q\, f^{-1}(\delta)$ [7].

Definition 1.3.4. A function f from a f.t.s. (X, τ) to a f.t.s. (Y, τ') is fuzzy continuous iff the inverse of each τ'-open fuzzy set is τ-open. (That is; if $\beta\in\tau'$ then $f^{-1}(\beta)\in\tau$). [7].

Clearly if f is a fuzzy continuous function from X to Y and g is a fuzzy continuous function from Y to Z, then the composition gof is a fuzzy continuous function from X to Z, for $(gof)^{-1}(\beta) = f^{-1}(g^{-1}(\beta))$ for each fuzzy set β in Z, and using the fuzzy continuity of g and f it follows that if β is open so is $(gof)^{-1}(\beta)$.

Definition 1.3.5. Let (X, τ) and (Y, τ') be two f.t.s., $f: X\to Y$ be a function and x_α be a fuzzy point in X. For each μ Q-neighborhood of $f(x_\alpha)$. If there is a Q-neighborhood δ of x_α, such that $f(\delta)\leq\mu$. Then it is called that f is fuzzy continuous at x_α, [7].

Definition 1.3.6 : Let D be a non - void set. Let "$\geq$" be a semi-order on D.The pair $(D, \geq)$ is called a directed set, directed by "$\geq$" iff for every pair $m, n\in D$, there exists a $p\in D$ such that $p\geq m$ and $p\geq n$. [5].

Definition 1.3.7 : Let $(D,\geq)$ be a directed set. Let X be an ordinary set and let g be the collection of all the fuzzy points in X. The function $S: D\to g$ is called a fuzzy net in X. In other words a fuzzy set is a pair $(S, \geq)$ such that S is a function from D to g and "$\geq$" directs the domain of S. For $n\in D$, S(n) is often denoted by S_n and hence a net S is often denoted by $\{S_n : n\in D\}$. [5].

Definition 1.3.8. Let S= $\{S_n:n\in D\}$ be a fuzzy net in X. S is said to be quasi-coincident with α iff for each $n\in D$, S_n is quasi-coincident with α.

S is said to be eventvally quasi-coincident with α iff there is an element m of D such that if $n\in D$ and $n \geq m$ then S_n is quasi-coincident with α.

S is said to be frequently quasi coincident with α iff for each m in D, there is an n in D such that $n \geq m$ and S_n is quasi-coincident with α. S is said to be in α, iff for each $n\in D$, $S_n \in\alpha$. [5].

Definition 1.3.9. A net S in a f.t.s. (X, τ) is said to be converge to a fuzzy point x_α in X relative to τ iff S is eventually quasi - coincident with each Q - neighborhood of x_α. [5].

Theorem 1.3.10. In a f.t.s. (X, τ), a fuzzy point $x_\alpha\in\mu$ iff there is a fuzzy net S in μ such that S converges to x_α. [5].

Theorem 1.3.11 : Let $f : (X, \tau) \rightarrow (Y, \tau')$ be a function; then followings are equivalent:

(1) f is fuzzy continuous

(2) For every τ'-closed β, $f^{1}(\beta)$ is τ closed

(3) For each fuzzy open set β in Y $f^{1}(\beta)$ is fuzzy open set in X.

(4) For each fuzzy point x_α in X and each neighborhood β of $f(x_\alpha)$, there exists a neighbborhood μ of x_α such that $f(\mu) \leq\beta$.

(5) For each fuzzy point x_α in X and each Q - neigborhood β of $f(x_\alpha)$. there exists a Q-neighborhood μ of x_α such that $f(\mu) \leq \beta$

(6) For each fuzzy net S= $\{S_n : n \in D\}$, if S converges to x_α , then foS= $\{f(S_n): n \in D\}$ is a fuzzy set in Y and converges to $f(x_\alpha)$.

(7) For any fuzzy set α in X , $f(\bar{\alpha}) \leq \overline{f(\alpha)}$

(8) For any fuzzy set β in Y , $\overline{f^{1}(\beta)} \leq f^{1}(\bar{\beta})$ [7].

Definition 1.3.12. Let (X, τ) be a f.t.s and $\mu \in I^X$. If $\overset{\circ}{\overline{\mu}}=\mu$, it is called that μ is a fuzzy regular open set (f.r.o.) in X. If $\overline{\overset{\circ}{\mu}} = \mu$, it is called that μ is a fuzzy regular closed set (f.r.c.). in X [4].

Theorem 1.3.13.

a) The closure of a fuzzy open set is a fuzzy regular closed set

b) The interior of a fuzzy closed set is a fuzzy regular open set.

But the inverse isn't true. [4].

Definition 1.3.14. A f.t.s. X is called a fuzzy regular space if for each fuzzy point x_α in X, $0<\alpha< 1$, and if for every open fuzzy set μ properly $x_\alpha\, q\, \mu$, there exists a fuzzy open set η in X such that $x_\alpha\, q\, \eta \leq \overline{\eta} \leq \mu$. [1].

Definition 1.3.15. A f.t.s. X is called a fuzzy almost regular space if for each f.r.o. set μ in X and for each fuzzy point x_α properly $x_\alpha\, q\, \mu$, there exists a fuzzy open set β in X such that $x_\alpha\, q\, \beta \leq \overline{\beta} \leq \mu$ [1].

Definition 1.3.16. A f.t.s. X called a fuzzy semi - regular space if for each fuzzy open set μ in X and for every fuzzy point x_α with $x_\alpha\, q\, \mu$, there exists a fuzzy open set β in X such that $x_\alpha\, q\, \beta \leq \overset{\circ}{\overline{\beta}} \leq \mu$. [1].

It can be seen from above definitions that a fuzzy regular space is a fuzzy semi regular and a fuzzy almost regular space. But the inverse isn't true.

Theorem 1.3.17. Let (X, τ) be a f.t.s.Then, (X, τ) is a fuzzy almost regular space and a fuzzy semi - regular space if and only if (X, τ) is a fuzzy regular space. [1].

Definition 1. 3.18. Let (X, τ) be a f.t.s. if for each $\mu \in \tau$, $\overline{\mu} \in \tau$, then, it is called that (X, τ) is a fuzzy extremally disconnected space. [1].

CHAPTER 2

WEAKLY AND ALMOST FORMS OF SOME CONTINUITIES

In this section, firstly, fuzzy weakly and fuzzy weakly* continuities are defined. Then, some comparisons are given about these continuities. Finally, fuzzy quasi and fuzzy somewhat continuities with weakly forms are studied and some comparisons are given about them.

2.1. Fuzzy Weakly Continuity, Fuzzy Weakly* Continuity and Fuzzy Weakly** Continuity

Using two neigborhoods of a fuzzy point, two different definitions are given. Also, with the boundary of a fuzzy set a different fuzzy weakly continuity is obtained and they are compared to each other.

Definition 2.1.1: A mapping f from a fuzzy topological space X to another fuzzy topologiol space Y is called fuzzy weakly continuous (f.w.c.) if for each fuzzy open set μ in Y $f^{-1}(\mu) \leq [f^{-1}(\bar{\mu})]^\circ$

Theorem 2.1.2: A function $f:X \to Y$ is f.w.c. if and only if for each fuzzy open set β with $f(x_\alpha) \in \beta$ there exists a fuzzy open set μ in X with $x_\alpha \in \mu$ such that $f(\mu) \leq \bar{\beta}$

Proof: **($\Rightarrow$):** Let $f(x_\alpha) \in \beta$ and β is fuzzy open in Y. Then, since f is f.w.c. from the definition 2.1.1. $f^{-1}(\beta) \leq [f^{-1}(\bar{\beta})]^\circ$, $f(x_\alpha) \in \beta \Rightarrow x_\alpha \in f^{-1}(\beta) \Rightarrow x_\alpha \in [f^{-1}(\bar{\beta})]^\circ = \mu$ Thus $f(\mu) \leq \bar{\beta}$

($\Leftarrow$): Let β be a fuzzy open set in Y. Suppose that $x_\alpha \in f^{-1}(\beta)$. Then we will show that $x_\alpha \in [f^{-1}(\bar{\beta})]^\circ$. $x_\alpha \in f^{-1}(\beta) \Rightarrow f(x_\alpha) \in \beta$ by the hypothesis, there exists a fuzzy open set μ in X with $x_\alpha \in \mu$ such that $f(\mu) \leq \bar{\beta}$. Hence $f(x_\alpha) \in f(\mu)$ and $f(x_\alpha) \in \bar{\beta} \Rightarrow x_\alpha \in \mu \leq f^{-1}(\bar{\beta}) \Rightarrow x_\alpha \in [f^{-1}(\bar{\beta})]^\circ$ Thus, $f^{-1}(\beta) \leq [f^{-1}(\bar{\beta})]^\circ$.

Definition 2.1.3: A mapping $f:X \to Y$ is called fuzzy weakly* continuous (f.w*.c.) if for each fuzzy open set β in Y, $f^{-1}(\partial\beta)$ is closed in X.

Theorem 2.1.4: Let f: $X \rightarrow Y$ be a function.Then f is f.c. if f is f.w.c. and f is f.w*.c.

Proof: It is clear from the definition 2.1.1 and 2.1.3.

Definition 2.1.5: A mapping f: $X \rightarrow Y$ is called fuzzy weakly** continuous (f.w**.c.) if for each fuzzy open set β in Y, with $f(x_\alpha)\ q\ \beta$ there exists a fuzzy open set μ in X with $x_\alpha\ q\ \mu$ such that $f(\mu) \leq \bar{\beta}$.[4].

Proposition 2.1.6: If f: $X \rightarrow Y$ is f.c. then f is f.w** c.

Proof: [4].

Fuzzy weakly continuity and fuzzy weakly* continuity are independent shown by the following examples.

Example 2.1.7: Let $X=\{a,b\}$, $\tau=\{0,1,\mu\}$ and $Y=\{c,d\}$, $\tau'=\{0,1,\beta\}$ where $\mu(a)=\mu(b)=0.8$ and $\beta(c)=0.2$ $\beta(d)=0.6$. Consider the mapping $f:(X,\tau)\rightarrow(Y,\tau')$ defined by $f(a)=c$, $f(b)=c$. Hence f is f.w.c. but not f.w*.c.

Proof : Clearly, f isn't f.w*.c. Because, for $\beta\in\tau'$, $\partial\beta = \bar{\beta} \wedge (\beta^\circ)' = 1 \wedge (1-\beta)$ and

$f^{-1}(\partial\beta) = 1 - f^{-1}(\beta) = \mu$. Thus $f^{-1}(\partial\beta)$ isn't fuzzy closed in X. f is f.w.c. Because for $\beta\in\tau', f^{-1}(\beta) \leq \mu$ and $[f^{-1}(\bar{\beta})]^\circ = 1 \Rightarrow f^{-1}(\beta) \leq [f^{-1}(\bar{\beta})]^\circ$.

Example 2.1.8: Let $X = \{a, b\}$, $\tau = \{0.1,\mu\}$ and let $Y=\{c,d\}$, $\tau'=\{0,1,\beta_1,\beta_2\}$ where $\beta_1(d)=\beta_1(c)=\frac{1}{3}$, $\beta_2(d)=\beta_2(c)=\frac{2}{3}$ and $\mu(a)=\mu(b)=\frac{2}{3}$. Then, we corsider $f:(X,\tau) \rightarrow (Y,\tau')$ defined by $f(a)=f(b)=c$. Then f is f.w*.c. but not f.w.c.

Proof : Clearly f is f.w*.c. Because, for $\beta_1, \beta_2 \in \tau'$, $\partial\beta_1 = \partial\beta_2 = (1-\beta_1) \wedge (1-\beta_2) = 1-\beta_2$ and $f^{-1}(\partial\beta_1) = f^{-1}(\partial\beta_2) = 1 - f^{-1}(\beta_2) = 1 - \mu$. Thus $f^{-1}(\partial\beta_1)$ and $f^{-1}(\partial\beta_2)$ are fuzzy closed in X. But $f^{-1}(\bar{\beta}_1) = f^{-1}(1-\beta_2) = 1 - f^{-1}(\beta_2) = 1 - \mu$ and $[f^{-1}(\bar{\beta}_1)]^\circ = 0$. Thus $f^{-1}(\beta_1) \nleq [f^{-1}(\bar{\beta}_1)]^\circ$ and f is not f.w.c.

Corollary 2.1.9: Let $f: X \to Y$ be function . If f is f.w.c. (f.w*.c) then f isn't f.c. But if f is f.c. then f is f.w.c. (f.w*.c.)

Proof: Obvious.

Fuzzy weakly continuity and fuzzy weakly** continuity are independent concepts shown by the following examples.

Example 2.1.10: Let $X = \{a, b\}$, $\tau = \{0.1, \mu\}$ and let $Y = \{c,d\}$, $\tau' = \{0,1,\beta_1,\beta_2\}$ where $\beta_1(d)=\beta_1(c)=\frac{1}{3}$, $\beta_2(d)=\beta_2(c)=\frac{2}{3}$ and $\mu(a)=\mu(b)=\frac{2}{3}$. Then, we corsider $f:(X,\tau) \to (Y,\tau')$. defined by f(a)=f(b)=c. Then f is f.w**.c. but not f.w.c at the fuzzy point $x_{1/3}$.

Proof: For $f(x_{1/3})$ q 1, $1 \in \tau'$ and $x_{1/3}$. q 1, $1 \in \tau$ and $f(1) \leq \overline{1} = 1$. Thus f is f.w**.c. at $x_{1/3}$. But for $f(x_{1/3}) \in \beta_1$, $x_{1/3} \in \mu$ and $f(\mu) \nleq \overline{\beta}_1 = 1-\beta_2$. Hence f is not f.w.c. at $x_{1/3}$.

Example 2.1.11: Let $X=\{a,b\}$, $\tau=\{0,1,\mu\}$ and $Y=\{c,d\}$, $\tau'=\{0,1,\beta\}$ where $\mu(a)=\mu(b)=0.8$ and $\beta(c) = \beta(d)=1/2$. Consider the mapping $f:(X,\tau)\to(Y,\tau')$ defined by f(a)=c, f(b)=c. Hence f is f.w.c. but not f.w**.c at the fuzzy point $x_{102/200}$.

Proof: For $f(x_{102/200})$ q β, $x_{102/200}$ q μ and $f(\mu) \nleq \overline{\beta} = \beta$. Thus f is not f.w**.c. at $x_{102/200}$. But f is f.w.c. at $x_{102/200}$. Because for $x_{102/200}$. $\in 1$, $1 \in \tau'$ $x_{102/200}$. $\in \mu$ and $f(\mu) \leq \overline{1} = 1$.

Fuzzy weakly* continuity and fuzzy weakly** continuity are independent concepts shown by the following examples.

Example 2.1.12: Let $X=\{a,b\}$, $\tau=\{0,1,\mu\}$ and $Y=\{c,d\}$, $\tau'=\{0,1,\beta\}$ where $\mu(a)=\mu(b)=0.8$ and $\beta(c)=0.2$, $\beta(d)=0.6$. Consider the mapping $f:(X,\tau)\to(Y,\tau')$ defined by f(a)=c, f(b)=c. Hence f is f.w**.c. but not f.w*.c at the fuzzy point $x_{1/2}$.

Proof: From the example 2.1.7, f is not f.w*.c . But for $f(x_{1/2})$ q β, $x_{1/2}$ q μ and $f(\mu) \leq \overline{\beta}=1$. Thus f is f.w**.c . at $x_{1/2}$.

Example 2.1.13: Let $X = \{a, b\}$, $\tau = \{0.1, \mu\}$ and let $Y = \{c,d\}$, $\tau' = \{0,1,\beta_1,\beta_2\}$ where $\beta_1(d)=\beta_1(c)=\frac{1}{3}$, $\beta_2(d)=\beta_2(c)=\frac{2}{3}$ and $\mu(a)=\mu(b)=\frac{2}{3}$. Then, we corsider $f:(X,\tau) \to (Y,\tau')$ defined by $f(a)=f(b)=c$. Then f is $f.w^*.c.$ but not $f.w^{**}.c.$ at the fuzzy point $x_{201/300}$.

Proof: From the example 2.1.8, f is $f.w^*.c$. But for $f(x_{201/300})\, q\, \beta_1$ $x_{201/300}\, q\, \mu$ and $f(\mu) \nleq \overline{\beta}_1 = \beta_1 = 1-\beta_2$. Thus f is not $f.w^{**}.c$. at $x_{201/300}$.

Now on we will concept that fuzzy weakly continuity is in the sense of f.w**.c.

2.2.Fuzzy Almost Continuity and Its Properties:

In today's math literature, there exist three kind of definitions of almost continuity. These definitions are defined by Singal [12], Husain [13] and Stallings [14], respectively.

They are investigated to the fuzzy continuity and relations between them are studied. Now,lets give the definitions of the fuzzy almost continuity.

Definition 2.2.1: A mapping $f: X \to Y$ from a fuzzy space X to another fuzzy space Y is called a fuzzy almost continuous (f.a.c.) function in the sense of Singal if for each fuzzy open set β of Y with $f(x_\alpha)\, q\, \beta$ there exists a fuzzy open set μ in X with $x_\alpha\, q\, \mu$ such that $f(\mu) \leq \overset{\circ}{\overline{\beta}}$.

Definition 2.2.2: A mapping $f: X \to Y$ from a fuzzy space X to another fuzzy space Y is called a fuzzy almost continuous (f.a.c.) function in the sense of Husain if for each fuzzy open set β in Y with $f(x_\alpha)\, q\, \beta$, $\overline{f^{-1}(\beta)}$ is a Q- neighborhood of x_α.

Definition 2.2.3: A mapping $f: X \to Y$ from a fuzzy space X to another fuzzy space Y is called a fuzzy almost continuous (f.a.c.) function in the sense of Stallings if for each W set of XxY with $G(f) \subset W$ where G(f) is the graph of f, there exists a function $g : X \to Y$ such that $G(g) \subset W$ and g is fuzzy continuous.

It is obvious that fuzzy continuity implies fuzzy almost continuity in sense of Singal and f.a.continuity in the sense of Husain and f.a.continuity in the sense of Stallings respectively. However the converse is not true. Also fuzzy almost continuities in sense of Singal and Husain, are different as seen the following examples.

Example 2.2.4: Let X be a non empty set. Let $\tau = \{0,1,\mu\}$ and $\tau' = \{0,1,\beta\}$, where for each x in X, $\mu(x)=2/3$, and $\beta(x)=1/2$. Then, the identity mapping $i=f:(X,\tau) \to (X,\tau')$ is f.a.c. in the sense of Husain but not f.a.c. in the sense of Singal at the fuzzy point $x_{101/200}$.

Proof: For $f(x_{101/200})\ q\ \beta$, $x_{101/200}\ q\ \mu$ and $f(\mu) \nleq \overset{\circ}{\overline{\beta}} = \beta$. Thus f is not f.a.c. in the sense of singal. But for $f(x_{101/200})\ q\ \beta$, $\overline{f^{-1}(\beta)}=1$ and 1 is Q - nbd of $x_{101/200}$. Thus f is f.a.c. in the sense of Husain.

Example 2.2.5: Let X be a non empty set. Let $\tau = \{0,1,\mu\}$ and $\tau' = \{0,1,\beta\}$, where for each x in X, $\mu(x)=1/3$, and $\beta(x)=2/3$. Then, the identity mapping $i=f:(X,\tau) \to (X,\tau')$ is f.a.c. in the sense of Singal but not f.a.c. in the sense of Husain at the fuzzy point $x_{101/200}$.

Proof: For $f(x_{101/200})\ q\ \beta$, $x_{101/200}\ q\ 1$ and $f(1) \leq \overset{\circ}{\overline{\beta}} = 1 = 1$. Thus f is f.a.c. in the sense of singal. But for $f(x_{101/200})\ q\ \beta$, $\overline{f^{-1}(\beta)} = 1-\mu$ and $\overline{f^{-1}(\beta)}$ is Q - nbd of $x_{101/200}$. Thus f is not f.a.c. in the sense of Husain at $x_{101/200}$.

Theorem 2.2.6: Let $f : X \to Y$ be a mapping . Then the following are equivalent.

i) f is f.a.c. (Singal) at each fuzzy point x_α in X, $(0<\alpha<1)$.

ii) For each fuzzy regular open set β with $f(x_\alpha)\ q\ \beta$, there exists a fuzzy open set μ with $x_\alpha\ q\ \mu$ such that $f(\mu) \leq \beta$.

iii) For each fuzzy net $S=\{S_n : n \in D)$ in X which S converges to x_α, if β is fuzzy reguler open Q - neighborhood of $f(x_\alpha)$ then, $\{f(S_n): n \in D\}$ is eventually quasi-coincident with β.

Proof: (i) $\Rightarrow$ (ii): Let β be a fuzzy reqular open set in Y with $f(x_\alpha)\ q\ \beta$. Then, β is fuzzy open. From (i), there exists a fuzzy open set μ in X with $x_\alpha\ q\ \mu$ such that $f(\mu) \leq \beta$.

(ii) ⇒ (i): Let β be a fuzzy open set in Y with $f(x_\alpha)$ q β. Then, $\beta = \overset{\circ}{\overline{\beta}}$ and β is fuzzy regular open in Y. From (ii), there exists a fuzzy open set μ in X with x_α q μ such that $f(\mu) \leq \overset{\circ}{\overline{\beta}}$. Thus f is f.a.c. (Singal) at x_α

(ii) ⇒ (iii): Let $S = \{S_n ; n \in D\}$ be a fuzzy net converges to x_α. Let β be a fuzzy regular open set in Y and let β be Q-neighborhood of $f(x_\alpha)$. Then, by the condition (ii), there exists a fuzzy open set μ with x_α q μ such that $f(\mu) \leq \beta$. Then, $\{f(S_n): n \in D\}$ is eventually quasi-coincident with $f(\mu)$. Since f $\mu) \leq \beta$, $\{f(S_n): n \in D\} = f(S)$ is eventually quasi coincident with β.

(iii) ⇒ (i): Suppose, if it is possible, f is not f.a.c. in the sense of singal at x_α. Then, there exists a fuzzy reguler open set β such that $f(x_\alpha)$ q β and for every fuzzy open set μ properly quasi-coincident with x_α, $f(\mu) \nleq \beta$.

Now, let $\mathcal{U}$ be the collection of all fuzzy open sets in X properly quasi-coincident with x_α. Then, for every μ_i in $\mathcal{U}$, there exists a point x^i in X such that $f(\mu_i)(f(x^i)) \nleq \beta(f(x^i))$. Choose a positive real number α_i such that $f(\mu_i)(f(x^i)) > \alpha_i' > \beta f(x^i))$, where $\alpha_i' = 1 - \alpha_i$. Now, $(\mathcal{U}, \geq)$, where " $\geq$ " is the inverse relation of " fuzzy set inclusion " is a directed set as reflexivity and transitivity are obvious; and for every μ_i, $\mu_j \in \mathcal{U}$, $\mu_i \vee \mu_j \in \mathcal{U}$. But $\mu_i \vee \mu_j \geq \mu_i$ and $\mu_i \vee \mu_j \geq \mu_j$. Now let us write $S\mu_i = x_{\alpha i}^{i}$. Then $S = \{S\mu_i\}\mu_i$ is a fuzzy net in X. We show that this fuzzy net converges to x_α. Let δ be a fuzzy open Q-neighborhood of x_α. Then x_α qδ and consequently $\delta \in \mathcal{U}$. Let $S_\delta \in S$ and S_δ q δ at x^p in X. Then for every $\mu_j \geq \delta$ in $\mathcal{U}$ $S\mu j$ quasi-coincident with δ at x^p. Thus the fuzzy net S is eventually quasi-coincident with δ. Hence S converges to x_α. But, for every μ_i

$$f(S\mu_i)(f(x^i)) + \beta(f(x^i)) < \alpha'_i + \alpha_i = 1.$$

Thus no member of f(S) quasi-coincident with β, which is contradicts (iii).

Teorem 2.2.7: Let f: X → Y be a surjection mapping from a fuzzy topological space X to fuzzy topological space Y. Then, followings are equivalent:

(i) f is f.a.c. in the sense of Singal

(ii) For each fuzzy regular open set β in Y, $f^{-1}(\beta)$ is a fuzzy open set in X

(iii) For each fuzzy regular closed set η in Y, $f^{-1}(\eta)$ is a fuzzy closed set in X

(iv) For every fuzzy regular open set β in Y with $f(x_\alpha)\, q\beta$, there exist a fuzzy open set β with $x_\alpha\, q\, \mu$ such that $f(\mu) \le \beta$

(v) For every fuzzy open set β in Y, $f^{-1}(\beta) \le [\, f^{-1}(\overset{\circ}{\bar{\beta}})]^\circ$

(vi) For every fuzzy closed set η in Y , $[f^{-1}(\bar{\overset{\circ}{\eta}})]^- \le f^{-1}(\eta)$

(vii) For each fuzzy net $S=\{S_n:n \in D\}$ such that S converges to x_α , if β is fuzzy regular open Q-neighborhood of x_α then, $f(S)=\{f(S_n):n \in D\}$ is eventually quasi-coincident with β.

Proof: (i) ⇒ (ii): Let β be a fuzzy regular open set in Y and let $x_\alpha\, q\, f^{-1}(\beta)$. Then, $f(x_\alpha)\, q\beta$ By condition (i), there exists a fuzzy open set μ with $x_\alpha\, q\, \mu$. Such that $f(\mu) \le \beta$. Then $x_\alpha q\mu \le f^{-1}(f(\mu)) \le f^{-1}(\beta)$ .Then $f^{-1}(\beta)$ is fuzzy open in X.

(ii) ⇒ (iii): Let η be a fuzzy regular closed set in Y. Then $1\text{-}\eta$ is fuzzy regular open. From (ii), $f^{-1}(1\text{-}\eta) = 1\text{-}\, f^{-1}(\eta)$ is fuzzy open in X. Then $f^{-1}(\eta)$ is fuzzy closed in X.

(iii) ⇒ (iv): Let β be a fuzzy regular open set in Y with $f(x_\alpha)\, q\, \beta$. Then $1\text{-}\beta$ fuzzy reqular closed. By the condition (iii), $f^{-1}(1\text{-}\beta) = 1\text{-}f^{-1}(\beta)$ is fuzzy closed set in X. Thus $f^{-1}(\beta)$ is fuzzy open and $x_\alpha\, q\, f^{-1}(\beta)$. Let $f^{-1}(\beta)= \mu$. Then $f(\mu) \le \beta$.

(iv) ⇒ (ii): Let β be a fuzzy reqular open set in Y with $f(x_\alpha)\, q\, \beta$. From (iv), there exists a fuzzy open set μ with $x_\alpha\, q\, \mu$ such that $f(\mu) \le \beta$. Thus , $x_\alpha\, q\, \mu$ and $\mu \le f^{-1}(f(\beta)) \le f^{-1}(\beta)$. So, $f^{-1}(\beta)$ is fuzzy open set in X.

(iv) ⇒ (v): Let β be a fuzzy open set in Y. Then $\overset{\circ}{\bar{\beta}}$ is fuzzy reqular open and $f^{-1}(\beta) \le f^{-1}(\overset{\circ}{\bar{\beta}})$. From (iv) ⇒ (ii), $f^{-1}(\overset{\circ}{\bar{\beta}})$ is fuzzy open. Thus $f^{-1}(\beta) \le [\, f^{-1}(\overset{\circ}{\bar{\beta}})]^\circ$.

(v) ⇒(vi): Let η be a fuzzy closed set in Y. Then, $1\text{-}\eta$ is fuzzy open set in Y. From (v),

$$f^{-1}(1\text{-}\eta) \le [f^{-1}(\overset{\circ}{\overline{1\text{-}\eta}})]^\circ$$

$$\Rightarrow 1\text{-}f^{-1}(\eta) \le [f^{-1}(1\text{-}\,\overset{\circ}{\bar{\eta}})]^\circ$$

$$\Rightarrow 1\text{-}f^{-1}(\eta) \le [1 - f^{-1}(\overset{\circ}{\bar{\eta}})]^\circ = 1\text{-}\overline{[f^{-1}(\overset{\circ}{\bar{\eta}})]}$$

$\Rightarrow \overline{[f^{-1}(\mathring{\bar{\eta}})]} \leq f^{-1}(\eta)$.

(vi) $\Rightarrow$ (vii): Let a fuzzy net $S=\{S_n : n \in D\}$ converges to x_α and let β be a fuzzy open Q-neighborhood of $f(x_\alpha)$. Then $x_\alpha \, q \, f^{-1}(\beta)$ and from (vi),

$[f^{-1}((1-\beta)^{\circ -})]^{-} \leq f^{-1}(1-\beta)$

Then, $[f^{-1}(1-\beta)]^{-} \leq f^{-1}(1-\beta)$

$\Rightarrow 1-[f^{-1}(\beta)]^{\circ} \leq 1-f^{-1}(\beta)$

$\Rightarrow f^{-1}(\beta) \leq [f^{-1}(\beta)]^{\circ}$

$\Rightarrow f^{-1}(\beta)$ is fuzzy open set in X. Also , $f^{-1}(\beta)$ is a Q-neighborhood of x_α :

Since the net $S=\{S_n ; n \in D\}$ converges to x_α S is eventually quasi-coincident with $f^{-1}(\beta)$. Hence, the net $f(S_n) = f(S)$ is eventually quasi-coincident with $f(f^{-1}(\beta))$. Then, $f(f^{-1}(\beta) \leq \beta \Rightarrow$ f(S) is eventually quasi-coincident with β .

(vii) $\Rightarrow$ (i): From theorem 2.2.6 it is obvious.

Theorem 2.2.8: Let $f : X \to Y$ be a f.a.c. (Singal) and let β be a fuzzy open set in Y. Then if $x_\alpha \in \overline{f^{-1}(\beta)}$, $f(x_\alpha) \in \bar{\beta}$.

Proof: As β be a fuzzy open set in Y, $\bar{\beta}$ is a fuzzzy reqular closed set. By the theorem 2.2.7, $f^{-1}(\bar{\beta})$ is fuzzy closed in X. Thus $\overline{f^{-1}(\beta)} = f^{-1}(\bar{\beta})$. Hence we have,

$(\bar{\beta})(f(x)) = f^{-1}(\bar{\beta})(x) = \overline{f^{-1}(\bar{\beta})}(x) \geq \overline{f^{-1}(\beta)}(x)$. But $x_\alpha \in \overline{f^{-1}(\beta)}$, therefore

$\alpha \leq \overline{f^{-1}(\beta)}(x) \leq (\bar{\beta})(f(x))$ and consequently $f(x_\alpha) \in \bar{\beta}$.

Proposition 2.2.9: Let $f : X \to Y$ be a function if f is f.a.c. Then , for each fuzzy open set β in Y, $\overline{f^{-1}(\beta)} \leq f^{-1}(\bar{\beta})$.

Proof: By the tehorem 2.2.8, $f(\overline{f^{-1}(\beta)}) \leq (\bar{\beta})$, for any fuzzy open set β in Y. Also since $\overline{f^{-1}(\beta)} \leq f^{-1}(f(\overline{f^{-1}(\beta)}))$, we have $\overline{f^{-1}(\beta)} \leq f^{-1}(\bar{\beta})$.

Theorem 2.2.10 : Let $f : X \to Y$ be a fuzzy open mapping from a f.t.s. X onto a f.t.s. Y. Then for every fuzzy set β in Y, $f^{-1}(\overline{\beta}) \leq \overline{f^{-1}(\beta)}$.

Proof: Let $x_\alpha \notin \overline{f^{-1}(\beta)}$ be a fuzzy point in X. Then there exists a fuzzy open Q-nbd μ of x_α such that $\mu \not q f^{-1}(\beta)$, which implies that $f(\mu) \leq f(1 - f^{-1}(\beta)) = 1 - f(f^{-1}(\beta)) = 1 - \beta$. Since f is fuzzy open $f(\mu)$ is also fuzzy open. Again $x_\alpha \, q \, \mu$ implies that $f(x_\alpha) \, q \, f(\mu)$. Thus $f(\mu)$ is a Q-nbd of $f(x_\alpha)$. Therefore we have $f(x_\alpha) \notin \overline{\beta}$, because $\mu \not q f^{1}(\beta) \Rightarrow f(\mu) \not q \beta$, $\beta = f(f^{-1}(\beta))$. So $x_\alpha \notin f^{-1}(\overline{\beta})$. This completes the proof.

Corollary 2.2.11: Let $f : X \to Y$ be a fuzzy open function from a f.t.s. X onto a f.t.s. Y. Then for every fuzzy open set β in Y, $f^{-1}(\overline{\beta}) = \overline{f^{-1}(\beta)}$.

Proof: From the theorem 2.2.10. and the corollary 2.2.9, the proof is obvious.

Definition 2.2.12: A function $f : X \to Y$ is called fuzzy almost open iff for every fuzzy reqular open set β of Y, $f^{-1}(\overline{\beta}) \leq \overline{f^{-1}(\beta)}$.

Corollary 2.2.13: Let $f : X \to Y$ be fuzzy almost open function and f.a.c. (Singal). Then for every fuzzy reqular open set β is in Y, then $f^{-1}(\overline{\beta}) = \overline{f^{-1}(\beta)}$

Proof: It is obvious from the definition 2.2.12 an the corollary 2.2.9.

Theorem 2.2.14: Let $f : X \to Y$ be fuzzy continuous mappings with $f(X) = Y$ and f is fuzzy open and let $g: Y \to Z$ be a function. Then, the composition $gof: X \to Z$ is f.a.c. (Singal) if and only if g is f.a.c. (Singal).

Proof: ($\Rightarrow$) **:** Let β be a fuzzy reqular open set in Z. Then $(gof)^{-1}(\beta) = f^{-1}(g^{-1}(\beta))$ is fuzzy open in X. Since f is fuzzy open $f(f^{-1}(g^{-1}(\beta))$ is fuzzy open in Y. Since $f(X) = Y$, $f(f^{-1}(g^{-1}(\beta)) = g^{-1}(\beta)$ and $g^{-1}(\beta)$ is fuzzy open. Thus g is .f.a.c. (Singal)

($\Leftarrow$): Let β be a fuzzy reqular open set in Z. Then, since g is f.a.c. (Singal) $g^{-1}(\beta)$ is fuzzy open in Y. Also since f is f.c., $f^{-1}(g^{-1}(\beta))$ is fuzzy open in X. Thus $(gof)^{-1}(\beta)$ is fuzzy open and so gof is f.a.c. (Singal).

Theorem 2.2.15: If $f: X \to Y$ be a fuzzy open function. If f is fuzzy open and f.w.c., then f is f.a.c. (Singal).

Proof: Let β be a fuzzy open set in Y with $f(x_\alpha)\ q\ \beta$. Since f is f.w.c., there exists a fuzzy open set μ in X with $x_\alpha\ q\ \mu$ such that $f(\mu) \leq \bar{\beta}$. Also , since f is fuzzy open, $f(\mu)$ is fuzzy open. Thus, $f(\mu) \leq \mathring{\bar{\beta}}$.

Corallary 2.2.16: A fuzzy open function f is f.a.c. (Singal) if and only if f is f.w.c.

Proof: It is clear.

Theorem 2.2.17 : Let $f: X \to Y$ is f.a.c. (Singal) and fuzzy almost open, then f is f.a.c.(Husain).

Proof: Let β be a fuzzy open set in Y with $f(x_\alpha)\ q\ \beta$. Then, since f is fuzzy almost continuous (Singal), $\overline{f^{-1}(\beta)} \leq f^{-1}(\bar{\beta})$ and $f^{-1}(\mathring{\bar{\beta}})$ is fuzzy open . Then,

$$f^{-1}(\mathring{\bar{\beta}}) \leq \overline{f^{-1}(\beta)} \leq f^{-1}(\bar{\beta}) .$$

We know that,

$$f(x_\alpha)\ q\ \beta \Rightarrow x_\alpha\ q\ f^{-1}(\beta),\ x_\alpha\ q\ f^{-1}(\mathring{\bar{\beta}}) \leq \overline{f^{-1}(\beta)} .$$

Hence $\overline{f^{-1}(\beta)}$ is a Q- neighborhood of x_α . Consequently f is f.a.c. (Husain) at x_α.

Theorem 2.2.18 : Let $f: X \to Y$ be a function as surjection, fuzzy open and f.a.c. (Husain) . Then, f is f.a.c.(Singal) if and only if for every fuzzy open set β , $f^{-1}(\bar{\beta}) = \overline{f^{-1}(\beta)}$.

Proof:($\Rightarrow$): Necessity follows in view of Corollary 2.2.11.

($\Leftarrow$): To prove sufficiency , let β be a fuzzy open set in Y with $f(x_\alpha)\ q\ \beta$. Since f is f.a.c. (Husain), there exists a fuzzy open set μ in X such that $x_\alpha\ q\ \mu$ and $\mu \leq \overline{f^{-1}(\beta)}$. Since f is fuzzy open, $\overline{f^{-1}(\beta)} \leq f^{-1}(\bar{\beta})$. Thus $f(\mu) \leq \bar{\beta}$ and f is f.w.c. at x_α . From the theorem 2.2.14, f is f.a.c. (Singal).

2.3. Variants of Almost and Weakly of Some Fuzzy Continuities.

In recent years, the almost and weakly variants of the quasi and somewhat continuities are defined by W. Balaz. He has also studied that in which case these continuities imply each other. We have applied his work to fuzzy continuity.

Definition 2.3.1: A mapping f from a fuzzy topological space X to another fuzzy topological space Y is called a fuzzy quasi continuous if for each fuzzy open set μ and for each open set β with x_α q μ and $f(x_\alpha)$ q β, there exists a fuzzy open set σ with x_α q σ and $\sigma \neq 0$ such that $f(\mu) \leq \beta$ and $\sigma \leq \mu$.

Definition 2.3.2: A mapping f from a fuzzy topological space X to another fuzzy topological space Y is called a fuzzy somewhat continuous mapping if for each fuzzy open set β in Y properly $f^{-1}(\beta) \neq 0$, $[f^{-1}(\beta)]^o \neq 0$, [2].

Definition 2.3.3: A mapping f from a fuzzy topological space to another fuzzy topological space Y is called a fuzzy almost quasi continuous mapping if for each fuzzy open set β with $f(x_\alpha)$ q β and for each fuzzy open μ with x_α q μ there exists a fuzzy open set σ with x_α q σ and $\sigma \leq \beta$, $f(\sigma) \leq \overset{o}{\overline{\beta}}$.

Definition 2.3.4: A mapping f from a fuzzy topological space X to another fuzzy topological space Y is called a fuzzy almost somewhat continuous mapping if for each fuzzy open set β in Y properly $f^1(\beta) \neq 0$, $[f^1(\overset{o}{\overline{\beta}})]^o \neq 0$, [2].

Definition 2.3.5: A mapping f from a fuzzy topological space X to another fuzzy topological space Y is called a fuzzy weakly quasi continuous mapping if for each fuzzy open set β with $f(x_\alpha)$ q β and for each fuzzy open μ with x_α q μ there exists a fuzzy open set σ with x_α. q σ and $\sigma \leq \mu$, $f(\sigma) \leq \overline{\beta}$.

Definition 2.3.6: A mapping f from a fuzzy topological space X to another fuzzy topological space Y is called a fuzzy weakly somewhat continuous mapping if $[f^{-1}(\overline{\beta})]^o \neq 0$, for each fuzzy open set β in Y properly $f^{-1}(\beta) \neq 0$, $[f^{-1}(\overline{\beta})]^o$ [2].

It can be seen from the definitions that, almost variences of fuzzy continuties imply weakly variences of fuzzy continuities. However, the inverse is not always true. The results shown below, are related to be the same meaning of almost varience of continuities with originals and weakly variences of continuities with almost variences of continuities. Now on, the fuzzy almost continuity is in the sense of Singal.

Theorem 2.3.7: Let $f:X\rightarrow Y$ be a function from a fuzzy topological space X to a fuzzy semi-regular space Y. Then we obtain the followings:

i) f is a fuzzy continuous mapping iff f is fuzzy almost continuous.

ii) f is a fuzzy quasi continuous mapping iff f is a fuzzy almost quasi continuous mapping.

iii) f is a fuzzy somewhat continuous mapping iff f is fuzzy almost somewhat continuous mapping.

Proof: (i) ($\Rightarrow$): It is obvious using the definitions.

($\Leftarrow$): Let β be a fuzzy open set in Y with $f(x_\alpha)\ q\ \beta$. Since Y is fuzzy semi-regular space, there exists a fuzzy open set δ such that $f(x_\alpha)\ q\ \delta \leq \overset{o}{\overline{\delta}} \leq \beta$.Also, since f is f.a.c., there exists a fuzzy open Q-neighborhood μ of x_α such that $f(\mu) \leq \overset{o}{\overline{\delta}} \leq \beta$. Hence $f(\mu)\leq \beta$.

(ii) ($\Rightarrow$): Obvious.

($\Leftarrow$): Let β be a fuzzy open Q-neighborhood of $f(x_\alpha)$ in Y and let μ be a fuzzy open set in X with $x_\alpha\ q\ \mu$. Then, since Y is fuzzy semi-regular space there exists a fuzzy open set δ such that $f(x_\alpha)\ q\ \delta \leq \overset{o}{\overline{\delta}} \leq \beta$. Since f is fuzzy almost quasi continuous, there exists a fuzzy open set σ with $x_\alpha\ q\ \sigma$ such that $\sigma \leq \mu$ and $f(\sigma) \leq \overset{o}{\overline{\delta}}$. Thus, $f(\sigma) \leq \beta$.

(iii) ($\Rightarrow$): Obvious.

($\Leftarrow$): Let β be a fuzzy open Q-neighborhood of $f(x_\alpha)$ and $f^{-1}(\beta) \neq 0$. Since Y is a fuzzy semi- regular space, there exists a fuzzy open set δ such that $f(x_\alpha)\ q\ \delta \leq \overset{o}{\overline{\delta}} \leq \beta$. Also, since f is fuzzy almost somewhat continuous $[f^{1}(\overset{o}{\overline{\delta}})]^{o} \neq 0$. Thus $[f^{1}(\beta)]^{o} \neq 0$.

Theorem 2.3. 8 : Let $f : X\rightarrow Y$ be a function from a fuzzy topological space X to a fuzzy almost reagular space Y.Then we obtain the followings:

(i) f is a fuzzy almost continuous iff f is fuzzy weakly continuous

(ii) f is fuzzy almost quasi continuous iff is fuzzy weakly quasi continuous

(iii) f is fuzzy almost somewhat continuous iff f is fuzzy weakly somewhat continuous

Proof: (i) ($\Rightarrow$): Obvious.

(i) ($\Leftarrow$): Let β be fuzzy open Q-neighborhood of $f(x_\alpha)$. Since Y is fuzzy almost regular, for fuzzy open set β with $f(x_\alpha)\ q\ \beta$, there exists a fuzzy regular open set δ such that $f(x_\alpha)\ q\ \delta \leq \overline{\delta} \leq \overset{\circ}{\overline{\beta}}$. Since f is fuzzy weakly continuous,there exists a fuzzy open set η with $x_\alpha\ q\ \eta$ such that $f(\eta)\leq \overline{\delta}$. Then,$f(\eta)\leq \overset{\circ}{\overline{\beta}}$.

(ii) ($\Rightarrow$): Obvious.

(ii) ($\Leftarrow$): Let β be fuzzy open Q-neighborhood of $f(x_\alpha)$ and let μ be a fuzzy open Q-nbd of x_α.Then, since Y is fuzzy almost regular space, there exists a fuzzy regular open set δ such that $f(x_\alpha)\ q\ \delta \leq \overline{\delta} \leq \overset{\circ}{\overline{\beta}}$. Since f is fuzzy weakly quasi continuous,there exists a fuzzy open set η with $x_\alpha\ q\ \eta$ such that $\eta \leq \delta$ and $f(\eta)\leq \overline{\delta}$. Then,$f(\eta)\leq \overset{\circ}{\overline{\beta}}$.

(iii) ($\Rightarrow$): Obvious.

(iii) ($\Leftarrow$): Let β be a fuzzy open Q-neighborhood of $f(x_\alpha)$ and $f^{-1}(\beta) \neq 0$. Since Y is a fuzzy almost regular space, there exists a fuzzy regular open set δ such that $f(x_\alpha)\ q\ \delta\ \ \delta \leq \overline{\delta} \leq \overset{\circ}{\overline{\beta}}$.Also, since f is fuzzy weakly somewhat continuous $[f^{1}(\overline{\delta})]^{o}\neq 0$.Thus $[f^{1}(\overset{\circ}{\overline{\beta}})]^{o}\neq 0$.

Now we arrive the following corollary.

Corollary 2.3.9. Let $f:X\rightarrow Y$ be a function from a fuzzy topological space X to a fuzzy regular space Y. Then we obtain the followings:

(i) f is fuzzy continuous iff f is fuzzy weakly continuous

(ii) f is fuzzy quasi continuous iff f is fuzzy weakly quasi continuous

(iii) f is fuzzy somewhat continuous iff f is fuzzy weakly somewhat continuous

Proof: We know that a fuzzy regular space is a fuzzy almost regular space and a fuzzy semi-regular space. So The proof follows from the theorems 2.3.7. and 2.3.8.

If f:X→Y is a function, then we have the followings requirements:

f.c.	⇒ f.a.c. (Singal)
f.c.	⇒ f.a.c. (Husain)
f.c.	⇒ f.a.c. (Stallings)
f.c.	⇒ f.w.c.
f.c.	⇒ f.w*.c.
f.c.	⇒ f.w**.c.
f.c.	⇒ f.somewhat c.
f.c.	⇒ f. quasi c.
f.c.	⇒ f.a.somewhat c.
f.c.	⇒ f.a.quasi c.
f.c.	⇒ f.w. somewhat c.
f.c.	⇒ f.w.quasi c.
f.a.c.(Singal)	⇒ f.w**.c.
f.a.c.(Singal) and f.a. open	⇒ f.a.c.(Husain)
f.w**.c. and fuzzy open	⇒ f.a.c.(Singal)
f.a.somewhat c.	⇒ f.w.somewhat c.
f.a.quasi c.	⇒ f.w.quasi c.

If Y is fuzzy semi-regular space

f is f.c.	⇔ f.a.c. (Singal)
f is f.quasi c.	⇔ f is f.a.quasi c.
f is f.somewhat c.	⇔ f is f.a.somewhat c.

If Y is a fuzzy almost regular space

f is f.a.c. (Singal)	⇔ f is f.w**.c.
f is f.a.quasi c.	⇔ f is f.w.quasi c.
f is f.a.somewhat c.	⇔ f is f.w.somewhat c.

If Y is a fuzzy regular space

f is f.c.	⇔ f is f.w**.c.
f is f.quasi c.	⇔ f is f.w.quasi c.
f is f.somewhat c.	⇔ f is f.w.somewhat c.

CHAPTER 3

FUZZY RARELY CONTINUITY AND FUZZY STRONGLY θ- CONTINUITY

In this chapter, some properties of fuzzy rarely continuous and fuzzy strongly θ-continuous functions are studied. Also fuzzy rarely* continuity and fuzzy strongly θ*-continuity are defined and some relations with several continuities are given.

3.1. Fuzzy Rarely Continiuity And Fuzzz Rarely* Continiuity

Fuzzy rarely continuity is defined and weaker forms are studied by TAN [2]. In this section some properties of fuzzy rarely continuity are given. Fuzzy rarely* continuity is defined and weaker forms are studied and some comparisons are given.

Definition 3.1.1: If (X,τ) be a f.t.s. and μ is a fuzzy set in X with $\mu^o=0$, then it is called that μ is a fuzzy rare set in X.

Definition 3.1.2: Let f: $X\to Y$ be a function. If for each fuzzy open set β in Y with $f(x_\alpha)q\,\beta$, there exists a fuzzy rare set R_β with $\overline{R_\beta}\not{q}\,\beta$ and there exists a fuzzy apen set μ with $x_\alpha\, q\, \mu$ such that.$f(\mu)\leq\beta\vee R_\beta$.Then it is called that f is fuzzy rarely continuous (f. rarely c.) at a fuzzy point x_α (Briefly: $\forall_\beta\in U_Q\,(f(x_\alpha))$, $(\exists\ R_\beta\in I^Y, R^o{}_\beta=0, \overline{R_\beta}\not{q}\,\beta)$ $(\exists\ \mu\in U_Q\,(x_\alpha)\ni(f(\mu)\leq\beta\vee R_\beta))$ [2].

Example 3.1.3: Let X= [0, 1] and $a\in X$ be a constant element of X. Now we define α, β, σ, μ fuzzy sets in X with for every $x\in X$,

$$\alpha(x)=\begin{cases}5/6, & x=a\\ 0, & x\neq a\end{cases},\qquad \beta(x)=\begin{cases}7/8, & x=a\\ 0, & x\neq a\end{cases}$$

$$\sigma(X)=\begin{cases}5/6, & x=a\\ 0, & x\neq a\end{cases},\qquad \mu(x)=\begin{cases}1/9, & x=a\\ 0, & x\neq a\end{cases}$$

If we take two topologies τ and τ' in X with $\tau = \{1, 0, \alpha\}$, $\tau' = \{1, 0, \beta, \sigma\}$, the identity function $f = I: (X, \tau) \to (X, \tau')$ is fuzzy rarely continuous.

Proof: For $\beta \in \tau'$, if we take $R_\beta = \mu$, then $\overset{\circ}{R_\beta} = 0, \overline{R}_\beta = 1 - \beta, \overline{R}_\beta \not q \beta$ and $\beta V R_\beta = \beta$ So for each $x_{\lambda,}$ $\lambda > 1/6$, $x_\lambda q \alpha$ and $f(\alpha) \leq \beta \vee R_\beta = \beta$. For $\sigma \in \tau'$, if we take $R_\sigma = \mu$, we arrire that for $x_{\lambda,}$ $\lambda > 1/6$, $x_\lambda q \alpha$ and $f(\alpha) \leq \sigma \vee R_\sigma = \sigma$. So f is fuzzy rarely continuous.

Corollary 3.1.4: Let f: X→Y be a function. If f is fuzzy continuous, then f is fuzzy rarely continuous. But inverse isn't true.[2].

Theorem 3.1.5: If f: $(X, \tau) \to (Y, \tau')$ is fuzzy continuous and g: $(Y, \tau') \to (Z, \tau'')$ is fuzzy rarely c, then the function gof is f. rarely c. [2].

Definition 3.1.6: A function f: X→Y is said to be fuzzy weakly continuous at the fuzzy point x_α if for each fuzzy open set β in Y with $f(x_\alpha)\, q\, \beta$ there exists a fuzzy open set μ in X with $x_\alpha\, q\, \mu$ such that $f(\mu) \leq \overline{\beta}$

Fuzzy weakly continuous function and fuzzy rarely continuous function are indeperdent to each other as seen the following examples.

Example 3.1.7: Let $f:(X, \tau) \to (X, \tau')$ be a function as defined f (a) = a, f(b)= b f(c) = c where X= {a,b,c}, $\tau = \{0,1,\alpha\}$ and $\tau' = \{0,1,\beta\}$, with $\alpha(a) = \alpha(b) = \alpha(c) = 2/3$ and $\beta(a) = \beta(b) = \beta(c) = 1/3$. Then f is f.w.c but f is not f.rarely continuous at $a_{202/200}$

Proof : For $f(a_{202/200})\, q\, \beta$, $a_{202/200} q\, \alpha$ and $f(\alpha) \leq \overline{\beta}$. Thus, f is f.w.c. at $a_{202/200}$. But, for every f.rare set $R_{\beta\,.}$ $f(\alpha) \not\leq R_\beta \vee \beta$ Because If $\overset{\circ}{R_\beta} = 0$ then. $R_\beta(a) < 1/3$, $R_\beta(b) < 1/3$ or $R_\beta(c) < 1/3$ .Suppose $R_\beta(a) < 1/3$, so $f(\alpha)(a) > (R_\beta \vee \beta)(a)$. Thus f is not f rarely c. at $a_{202/200}$

Example 3.1.8. Let f: $(X,\tau) \to (X,\tau')$ be a function as defined f(a)=a f(b)=b, f(c)=c where X= {a, b, c}, $\tau = \{0, 1, \alpha\}$, $\tau' = \{0, 1, \beta_1, \beta_2\}$ and $\alpha(a) = \alpha(b) = \frac{2}{3}$, $\alpha(c) = \frac{1}{2}$, $\beta_1(a) = \beta_1(b) = \frac{1}{3}$, $\beta_1(c) = \frac{1}{2}$ and $\beta_2(a) = \beta_2(b) = \frac{2}{3}$, $\beta_2(c) = \frac{1}{2}$. Then f is f. rarely c. at $\frac{c_{102}}{200}$ and but f is not f w.c. at $\frac{c_{102}}{200}$.

Proof: For $f(\frac{c_{102}}{200})$ q β_1, $\frac{c_{102}}{200}$ qα and $f(\alpha) \leq \overline{\beta}_1 = 1-\beta_2$ .Thus, f is not f.w.c. at $\frac{c_{102}}{200}$.
However, for $f(\frac{c_{102}}{200})$qβ_1, if we take $R_{\beta 1}(a) = R_{\beta 1}(b) = \frac{2}{3}$ and $R_{\beta 1}(c) = \frac{1}{100}$. Then $\mathring{R}_{\beta 1}=0$ and $\overline{R}_{\beta 1} \not{q}\, \beta_1$. Finally, $f(\alpha) \leq R_{\beta 1} \vee \beta_1$ with $\frac{c_{102}}{200}$qα. Also for $f(\frac{c_{102}}{200})$qβ_2, if we take $R_{\beta 2}(a) = R_{\beta 2}(b) = R_{\beta 2}(c) = 0$, then $\mathring{R}_{\beta 2}=0$ and $\overline{R}_{\beta 2} \not{q} \beta_2$ and for αq$\frac{c_{102}}{200}$, $f(\alpha) \leq R_{\beta 2} \vee \beta_2$. Thus, f is f. rarely c. at $\frac{c_{102}}{200}$.

Fuzzy almost continuous function and fuzzy rarely continuous function are independent to each other as seen the following examples.

Example 3.1.9. Let $f:(X,\tau) \to (X,\tau')$ be a function as defined f(a) = a,f(b) = b, f(c) = c where $X=\{a,b,c\}$, $\tau=\{0,1,\alpha\}$, $\tau'=\{0,1,\beta_1,\beta_2\}$ and $\alpha(a) = \alpha(b) = \frac{3}{4}$, $\alpha(c) = \frac{1}{2}$, $\beta_1(a) = \beta_1(b) = \frac{1}{4}$, $\beta_1(c) = \frac{1}{2}$ and $\beta_2(a) = \beta_2(b) = \frac{3}{4}$ and $\beta_2(c) = \frac{1}{2}$. Then f is f. rarely c. but f is not f.a.c. (in the sense of Singal) at the fuzzy point $\frac{c_{103}}{200}$.

Proof: For $f(\frac{c_{103}}{200})$ qβ_1, $\frac{c_{103}}{200}$ q α and $f(\alpha) \not\leq \mathring{\overline{\beta}}_1 = 1-\beta_2 = \beta_1$ Thus, f is not f.a.c. at this point.
Hovewer, for $f(\frac{c_{103}}{200})$ qβ_1, if we choose $R_{\beta 1}(a) = R_{\beta 1}(b) = 3/4$, $R_{\beta 1}(c) = 1/100$, then $\mathring{R}_{\beta 1} = 0$, $\overline{R}_{\beta 1} \not{q}\, \beta_1$ and for α q $\frac{c_{103}}{200}$, $f(\alpha) \leq R_{\beta 1} \vee \beta_1$. For $f(\frac{c_{103}}{200})$q β_2, if we choose $\mathring{R}_{\beta 2}=0$, then, for α q $\frac{c_{103}}{200}$, $f(\alpha) \leq R_{\beta 2} \vee \beta_2$.Thus f is f. rarely c. at this point.

Example.3.1.10: Let $f:(X,\tau) \to (X,\tau')$ be a function as defined f(a) = a, f(b) = b, f(c) = c where $X=\{a,b,c\}$, $\tau=\{0,1,\alpha\}$, $\tau'=\{0,1,\beta_1,\beta_2\}$ with $\alpha(a) = \alpha(b) = \alpha(c) = \frac{1}{2}$, $\beta_1(a) = \beta_1(b) = \beta_1(c) = \frac{1}{3}$ and $\beta_2(a) = \beta_2(b) = \beta_2(c) = \frac{1}{2}$. Then f is f.a.c. but f is not f. rarely c. at $\frac{a_{202}}{300}$.

Proof: For $f(\frac{a_{202}}{300})\ q\ \beta_1$, for every fuzzy rare set $R_{\beta 1}$, $f(\alpha) \le R_{\beta 1} \vee \beta_1$ with $\overline{R_{\beta 1}} \not q\ \beta_1$, because, if $\overset{o}{R}_{\beta 1} = 0$, $R_{\beta 1}$ (a) < 1/3, $R_{\beta 1}$ (b) < 1/3 or $R_{\beta 1}$ (c) < 1/3. Suppose $R_{\beta 1}$ (a) < 1/3 then, $f(\alpha)$ (a) > $(R_{\beta 1} \vee \beta_1)$(a) . Thus f is not f rarely c. at $\frac{a_{202}}{300}$.For $f(\frac{a_{202}}{300})\ q\ \beta_1$, $\frac{a_{202}}{300}\, q\ \alpha$ and $f(\alpha) \le \overset{o}{\beta}_1$ and for $f(\frac{a_{202}}{300})\ q\ \beta_2$, $f(\alpha) \le \overset{o}{\beta}_2$.Thus, f is f.a.c. at this point.

Theorem 3.1.11: If f: $X \to Y$ is f rarely c. and $A \subset X$, then $f|_A: A \to Y$ is f rarely c.

Proof: Let x_α be a fuzzy point in A and let β be a fuzzy open set in Y with $f(x_\alpha)\ q\beta$. Then since f is f. rarely c., there is a fuzzy open set μ in X with $x_\alpha\ q\mu$ and there is a fuzzy rare set R_β in Y such that $\overline{R_\beta} \not q\ \beta$ and $f(\mu) \le R_\beta \vee \beta$. We know that $x_\alpha\ q\ \mu \Leftrightarrow \alpha+\mu(x) > 1$, so $\mu_A = \mu|_A$ is a fuzzy open set in A such that $x_\alpha\ q\ \mu A$. Since $\mu|_A \le \mu$, $f(\mu|_A) \le f(\mu) \le R_\beta \vee \beta$. So $f|_A$ is f rarely c. at x_α

Lemma 3.1.12: If g ; $X \to Y$ is f.c. and one - to - one, then g preserves fuzzy rare sets.

Proof : Suppose R is a fuzzy rare set such that g(R) is not fuzzy rare set in Y.Then there exists a fuzzy point y_α in g(R) such that $y_\alpha \in [g(R)]^o$. That is; there is a fuzzy open set β in Y such that $y_\alpha q\ \beta$ and $\beta \le g(R)$ Now we consider the fuzzy point x_α in X such that $f(x_\alpha) = y_\alpha$ and let μ be any fuzzy open set with $x_\alpha q\ \mu$. Since R is fuzzy rare set in X, the fuzzy set μ must contain at least one fuzzy point $z_{\alpha'}$ in X such that $z_{\alpha'} \notin R$. Thus $g(z_{\alpha'}) \notin g(R)$ so that $g(\mu) \le \beta$ which shows that g is not fuzzy continuous at x_α. We conclude that if R is fuzzy rare set in X, then g(R) is fuzzy rare set in Y.

Theorem 3.1.13: If f: $X \to Y$ is f. rarely c. and g: $Y \to Z$ is f.c., one-to-one, onto Z. Then gof:$X \to Z$ is f. rarely c.

Proof: Let x_α be a fuzzy point in X and let β be any fuzzy open set in Z with $(gof)(x_\alpha)q\beta$. Since g is f.c., $g^{-1}(\beta)$ is fuzzy open in Y where $g^{-1}(z_\alpha)=f(x_\alpha)$. Also, since f is f. rarely c., there exists a fuzzy rare set R_β and a fuzzy open set μ with $x_\alpha q\ \mu$ such that $\overline{R_\beta} \not q\ g^{-1}(\beta)$ and $f(\mu) \le R_\beta \vee g^{-1}(\beta)$ Thus, $g(f(\mu)) \le g(R_\beta \vee g^{-1}(\beta)) \Rightarrow (gof)(\mu) \le g(R_\beta) \vee g(g^{-1}(\beta)) \le g(R_\beta) \vee \beta$. By lemma 3.1.12., $g(R_\beta)$ is a fuzzy rare set and in Z and since $\overline{R_\beta} \not q\, g^{-1}(\beta)$, then $g(\overline{R_\beta}) \not q \beta$. Hence, gof is f. rarely c. at x_α.

Theorem 3.1.14: Let f: X→Y be a fuzzy open surjection and 1-1 function and let g: Y→Z be a function such that gof: X→Z is f. rarely c. Then g is f. rarely c.

Proof: Let $f(x_\alpha)=y_\alpha$ and let σ be a fuzzy open set with $(gof)(x_\alpha)=z_\alpha$, $z_\alpha\, q\, \sigma$. Then there exists a fuzzy rare set R_σ and there is a fuzzy open set μ with $x_\alpha q \mu$ such that $\overline{R_\sigma} \not q \sigma$ and $(gof)(\mu) \leq R_\sigma \vee \sigma$. But since f is fuzzy open, $f(\mu)$ is fuzzy open set in Y with $y_\alpha q\, f(\mu)$ having the property $(gof)(\mu) \leq R_\sigma \vee \sigma$. Consequently g is f. rarely c. at y_α.

Now lets define fuzzy rarely* continuity of a function using the neighborhood of a fuzzy point.

Definition 3.1.15: Let f: X→Y be a function and let let β be a fuzzy open set in Y with $f(x_\alpha) \in \beta$. Then if there exists a fuzzy open set μ with $x_\alpha \in \mu$ in X and there exists a fuzzy rare set R_β such that $f(\mu) \leq R_\beta \vee \beta$ and $\overline{R_\beta} \wedge \beta = 0$. Then it is called that f is fuzzy rarely* continuous (f. rarely* c.) at x_α.

Fuzzy rarely contin›ity and fuzzy rarely* continuity are independent to each other as seen the following examples.

Example 3.1.16: Let f:(X=[0,1], τ)→(X, τ') be a function as defined f(x)=x for all x∈X and $\tau=\{0,1,\alpha\}$, $\tau'=\{0,1,\beta_1,\beta_2\}$ where

$\alpha(x) = \begin{cases} 5/6, & x=a \\ 0, & x\neq a \end{cases}$, $\beta_1(x)=\begin{cases} 7/8, & x=a \\ 0, & x\neq a \end{cases}$, $\beta_2(x)=\begin{cases} 4/5, & x=a \\ 0, & x\neq a \end{cases}$ with a is a contast point in X. Then f is f. rarely c. but f isn't f. rarely* c. at $\frac{a_{106}}{600}$.

Proof: For, $f(\frac{a_{106}}{600})q\, \beta_1$, $\frac{a_{106}}{600} q\alpha$ and if we choose $R_{\beta 1}$, $R_{\beta 1}(x)=\begin{cases} 0, & x=a \\ 1, & x\neq a \end{cases}$, $then\, R^o_{\beta 1}=0$

and $\overline{R_{\beta 1}}= 1-\beta_1 \not q\, \beta_1$. Thus $f(\alpha) \leq R_{\beta 1} \vee \beta_1$. So f is f. rarely c. at $\frac{a_{106}}{600}$. But for $f(\frac{a_{106}}{600}) \in \beta_2$,

$\frac{a_{106}}{600} \in\alpha$ and for every fuzzy rare set $R_{\beta2}$, $f(\alpha) \nleq R_{\beta2} \vee \beta_2$. Because, if $R^{o}_{\beta2} = 0$, suppose $R_{\beta2}(a) < 4/5$, then $f(\alpha) \leq R_{\beta2} \vee \beta_2$. Hence, f is not f. rarely* c. at $\frac{a_{106}}{600}$.

Example 3.1.17: Let f: $(X, \tau)\rightarrow(X, \tau')$ be a function as defined $f(x)=x$ for all $x\in X$ and $\tau=\{0,1,\alpha\}$ $\tau'=\{0,1,\beta\}$ where $\alpha(x)=1/2$, $\beta(x)=2/3$ for all $x\in X$. Then f is f. rarely* c. at $x_{1/2}$ but f is not f. rarely c. at this point.

Proof: For $f(x_{1/2})\in\beta$, $x_{1/2}\in\alpha$, and for $R_\beta= 0$, $\overline{R}_\beta \wedge \beta = 0$. So $f(\alpha)\leq R_\beta\vee\beta$. Thus f is f.rarely*c. at $x_{1/2}$ But for $f(x_{1/2})q\beta$, $x_{1/2}q1$, if $R^{o}_{\beta}=0$ then $R_\beta(x)<1/3$. Thus $f(1) \nleq R_\beta \vee \beta$. So f is not f. rarely c. at $x_{1/2}$.

Corollary 3.1.18: Let f: $X\rightarrow Y$ be a function. If f is f.c., then f is f. rarely* c.

Proof: Obvious.

The inverse isn't true as seen the following example.

Example 3.1.19: Let $f:(X,\tau)\rightarrow(X,\tau')$ be a function as defined $f(x)=x$ for all $x\in X$ and $\tau=\{0,1,\alpha\}$, $\tau' =\{0,1,\beta\}$ where $X=\{a,b,c\}$ and $\alpha(a)=2/3$, $\alpha(b)= 4/5$, $\alpha(c)=0$, $\beta(a)=0$, $\beta(b)=1$, $\beta(c)=0$ Then f is f rarely* c. but f is not f.c. at $b_{1/4}$.

Proof : For $f(b_{1/4})$ q β, $b_{1/4}$ q α but $f(\alpha)\nleq\beta$, so f is not f.c. at $b_{1/4}$. Also for $f(b_{1/4}) \in \beta$, $b_{1/4} \in \alpha$ and if we choose R_β with $R_\beta(a)=2/3=R_\beta(c)$ and $R_\beta(b)=0$. Then $\mathring{R}_\beta=0$ and $\overline{R}_\beta=1-\beta$ $\overline{R}_\beta \wedge\beta=0$.Thus we arrive that $f(\alpha) \leq R_\beta \vee\beta$. So f is f. rarely* c. at $b_{1/4}$.

Finally, If we can use the definition of f. rarely* c. functions, we see that it satisfies all the thorems that are proved for f. rarely c. Now, lets define the weakly froms of f.rarely continous and fuzzy rarely* continuous of functions.

Definition 3.1.20: Let f: $(X,\tau)\rightarrow(Y,\tau')$ be a function. If for each fuzzy open set β in Y with $f(x_\alpha)q\ \beta$ ($f(x_\alpha) \in \beta$), there exists a fuzzy rare set R_β with $\overline{R}_\beta \not{q}\ \beta$ ($\overline{R}_\beta \wedge \beta =0$) and there exists a fuzzy open set μ with x_α q μ ($x_\alpha\in\mu$) such that $f(\mu) \leq \mathring{\overline{\beta}}\vee R_\beta$. Then, it is called that f is fuzzy rarely almost continuous (fuzzy almost rarely* continuous) at x_α.[2].

Corollary 3.1.21: If f: $X \to Y$ is a f . rarely c. Then f is f. a. rarely c. The converse isn't true [2].

Corollary 3.1.22: If f: $X \to Y$ is f. a.c. (Singal). Then f is f.a.rarely c.

Proof: Let $x_\alpha q\ \beta$ and β be a fuzzy open set in Y. Since f is f.a.c. there is a fuzzy open set μ with $x_\alpha q\ \mu$ such that $f(\mu) \leq \overset{o}{\overline{\beta}}$. If we choose $R_\beta = 0$, then $\overline{R}_\beta \not q\ \beta$ and $f(\mu) \leq \beta \vee R_\beta$. Thus, f is fuzzy almost rarely continuous at x_α.

Corollary 3.1.23: If f: $X \to Y$ is f. rarely* c.. Then f is f. a. rarely* c. The converse is not true.

Proof: It is clear from the definitions 3.1.15 and 3.1.20.

Definition 3.1.24: Let $f:(X,\tau) \to (Y,\tau')$ be a function If for each fuzzy open set β in Y with $f(x_\alpha) q\ \beta$ ($f(x_\alpha) \in \beta$), there exists a fuzzy rare set R_β with $\overline{R}_\beta \not q\ \beta$ ($\overline{R}_\beta \wedge \beta = 0$) and there exists a fuzzy apen set μ with $x_\alpha\ q\ \mu$ ($x_\alpha \in \mu$) such that $f(\mu) \leq \overline{\beta} \vee R_\beta$. Then It is called that f is fuzzy weakly rarely continuous (fuzzy weakly rarely* continuous) at x_α. [2].

The proofs of the following corollaries are clear from the their definitions.

Corollary 3.1.25: If f: $X \to Y$ is f. rarely c. Then f is f.w.rarely c.The converse isn't true[2].

Corollary 3.1.26: If f: $X \to Y$ is f.a.rarely.c.Then f is f.w. rarely c. [2].

Corollary 3.1.27: If f: $X \to Y$ is f. rarely* c.Then f is f.w.rarely* c.

Proof: Obvious.

Corollary 3.1.28: If f: $X \to Y$ is f.a. rarely* c.Then f is f.w.rarely* c.

Proof: Obvious.

Corollary 3.1.29: If f: $X \to Y$ is f.w**.c. Then,f is f.w.rarely c.

Proof: Let β be a fuzzy open set in Y with $f(x_\alpha) q \beta$. Since f is f.w**.c., there is a fuzzy open set μ with $x_\alpha\ q\ \mu$ such that $f(\mu) \leq \overline{\beta}$. If we choose a fuzzy rare set R_β as $R_\beta = 0$, then $\overline{R}_\beta \not{q} \beta$ and $f(\mu) \leq \overline{\beta} \vee R_\beta$. Thus f is f.w.rarely c. at x_α.

Corollary 3.1.30: If f: X→Y is f.w.c.Then f is f.w.rarely* c.

Proof: Let β be a fuzzy open set in Y with $f(x_\alpha) \in \beta$. Since f is f.w.c., there is a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\mu) \leq \overline{\beta}$. If we choose a fuzzy rare set R_β as $R_\beta = 0$, then $\overline{R}_\beta \wedge \beta = 0$ and $f(\mu) \leq \overline{\beta} \vee R_\beta$. Thus f is f.w.rarely* c. at x_α.

3.2. Fuzzy Strongly θ - Continuous Functions

In this section fuzzy strongly θ- continuity and fuzzy strongly θ*- continuity of functions are defined and some comparisons are given.

Definition 3.2.1: Let f: X → Y be a function. If for each fuzzy open set β in Y with $f(x_\alpha)\ q\beta$, exists a fuzzy open set μ in X such that $x_\alpha\ q\mu$ and $f(\overline{\mu}) \leq \beta$. Then it is called that f is fuzzy strongly θ-continuous (Brietly, f . st . θ. c) at the fuzzy point. x_α.

If f is f. st. θ.c., then f is f.c. But inverse isn't true in general as seen the following example.

Example 3.2.2: Let f: $(X, \tau) \to (X, \tau')$ be the identity function with $\tau = \{0,1, \alpha\}$ and $\tau' = \{0,1, \beta\}$ be two topologies on $X = \{a, b\}$ where $\alpha(a) = \alpha(b) = 1/3$ and $\beta(a) = \beta(b) = 1003/3000$. Then f is f.c. but f isn't f. st. θ.c. at the fuzzy point $a_{2003/3000}$.

Proof: Let $a_{2003/3000}$ be a fuzzy point in X. Then for $f(a_{2003/3000})\ q\ \beta$, $a_{2003/3000}\ q\ \alpha$ and $f(\overline{\alpha}) \nleq \beta$. Thus f is not f. st. θ.c. at $a_{2003/3000}$. But, since $f(\alpha) \leq \beta$, f is f.c. at $a_{2003/3000}$.

Definition 3.2.3: Let f: X→Y be a function. . If for each fuzzy open set β in Y with $f(x_\alpha)\ q\beta$, there exists a fuzzy open set μ with $x_\alpha\ q\ \mu$ such that $f(\overset{o}{\overline{\mu}}) \leq \beta$. Then, it is called that f is fuzzy almost strongly θ-continuous (f.a.st. θ.c.) at the fuzzy point x_α.

Let f: $X \to Y$ be a function and let μ and β be two fuzzy open sets with x_α q μ and f (x_α) qβ. Then, since $\mu \leq \overset{\circ}{\overline{\mu}} \leq \overline{\mu}$, $f(\mu) \leq f(\overset{\circ}{\overline{\mu}}) \leq f(\overline{\mu})$. So, if f is f.st. θ.c., then f. is f. a. st. θ.c. and f. is f.c. That is,

f. st. θ.c. $\Rightarrow$ f. a st. θ.c. $\Rightarrow$ f.c.

Theorem 3.2.4: Let f: $X \to Y$ be a function from a fuzzy almost regular space X to a fuzzy topological space Y. Then f .is f. st. θ.c. if and only if f is f. a. st. θ.c.

Proof: ($\Rightarrow$): [8]

($\Leftarrow$) : Suppose that f is f. a. st. θ.c. and (X, τ) is fuzzy almost regular space. Let x_α be a fuzzy point in X and let β be a fuzzy point in Y with f (x_α) q β. Then, since f is f a.st. θ.c. , there exists a fuzzy open set μ in X with x_α q μ such that $f(\overset{\circ}{\overline{\mu}}) \leq \beta$. We know that $\overset{\circ}{\overline{\mu}}$ is a fuzzy regular open set in X. So since, X is fuzzy almost regular space, there is a fuzzy open set σ such that x_α q $\sigma \leq \overline{\sigma} \leq \overset{\circ}{\overline{\mu}}$. Thus $f(\overline{\sigma}) \leq f(\overset{\circ}{\overline{\mu}}) \leq \beta$ and f is st. θ .c. at x_α

Theorem 3.2.5: Let f: $X \to Y$ be a function from a fuzzy semi-regular space X to a fuzzy topological space Y. Then f is f. a. st. θ.c. if and only if f is f.c.

Proof: **($\Rightarrow$) :** It is clear.

($\Leftarrow$): Let β be a fuzzy open set in Y with f (x_α) q β. Since f is f.c., there exists a fuzzy open set μ in X with x_α q μ such that $f(\mu) \leq \beta$. Also since X is fuzzy semi-regular space, there is a fuzzy open set σ in X such that x_α q σ and $\sigma \leq \overset{\circ}{\overline{\sigma}} \leq \mu$. Hence $f(\overset{\circ}{\overline{\sigma}}) \leq f(\mu) \leq \beta$. Thus f is f. a. st. θ.c. at x_α.

Corollary 3.2.6 : Let f: $X \to Y$ be a function from a fuzzy regular space X to a fuzzy topological space Y. Then f is f.st. θ.c. if and only if f is f.c.

Proof: Since a fuzzy regular space is fuzzy almost regular space and fuzzy semi- regular space, the proof is clear.

Corollary 3.2.7 : Let f: X→Y be a function from a fuzzy regular space X to a fuzzy topological regular space X to a fuzzy regular space Y. Then, f is f. st. θ.c. if and only if f is f.w**.c.

Proof: (⇒) : A f. st. θ.c. function is f.c. and a f.c. function is f.w**.c.

(⇐) : Suppose f is f.w**.c. Then from [8] theorem 2.1.14., since Y is fuzzy regular space, f is. f.c. Also, since X is fuzzy regular space, from the corolllary 3.2.6 f is f. st. θ.c.

Corollary 3.2.8 : Let f: X→Y be a function if f is f. st. θ.c., then f is f. rarely c.

Proof: Since f is f. st. θ.c. ⇒ f.c., then f is f. rarely. c.

Corollary 3.2.9 : Let f: X→Y be a function from a fuzzy regular space X to a fuzzy regular space Y. Then f is. f.w**.c. if and only, if f is f a.st. θ.c.

Proof: If X in fuzzy regular space, then f is f.a.st. θ.c. ⇔ f is f.c. Also since Y is fuzzy regular space, f is f.c. ⇔ f.w**.c.

Corolory 3.2.10: Let f: X→Y be a function from a fuzzy semi regular space X to a fuzzy semi regular space Y. f is. f.a.c. (Singal), if and only if, f is f st. θ.c.

Proof: Since X is a fuzzy semi-regular space, f is f.st. θ.c. ⇔ f.c. and also since Y is a fuzzy semi regular space, f is f.c. ⇔ f is f.a.c. Thus f is f.a.c. ⇔ f is f. st. θ.c.

Definition 3.2.11: Let f: X→Y be a function. . If for each fuzzy open set β in Y with $f(x_\alpha) \in \beta$, there exists a fuzzy open set μ in X with $x_\alpha \in \mu$ such that $f(\overline{\mu}) \leq \beta$. Then, it is called that f is fuzzy strongly θ^* continuous (Briefly f.st. θ^*. c.) at the fuzzy point x_α.

If f is f. st. θ^*.c, then f is f.c. But invesse isn't true in general as seen the following example.

Example 3.2.12: Let f $(X, \tau) \to (X, \tau')$ be the identity function. Also let $\tau = \{0,1, \alpha\}$ and $\tau' = \{0,1, \beta\}$ be two topologies on $X = \{a,b\}$ where $\alpha(a) = \alpha(b) = 1/3$ and $\beta(a) = \beta(b) = 11/30$. Then f is f.c. but f is not f. st. θ^*c. at $a_{1/3}$.

Proof: : Let β be a fuzzy open set in Y with $f(a_{1/3}) \in \beta$. Then $a_{1/3} \in \alpha$ and $f(\alpha) \leq \beta$. So f is f.c. at $a_{1/3}$. But $\overline{\alpha} = 1 - \alpha$ and $f(\overline{\alpha}) \nleq \beta$. Thus f is not f. st. θ^*.c. at $a_{1/3}$.

Fuzzy strongly θ- continuity and Fuzzy strogly θ^* continuity are independent to each other as seen the following examples.

Example 3.2.13: Let f $(X, \tau) \to (X, \tau')$ be the identity function with $X = \{a,b\}$ and $\tau = \{0,1, \alpha\}$, $\tau' = \{0,1, \beta\}$, $\alpha(a) = \alpha(b) = 12/30$ and $\beta(a) = \beta(b) = 2/3$. Then f is f. st. θ^*.c. at $a_{11/30}$. But f is not f.st. θ.c. at this point.

Proof: : Let β be a fuzzy open set in Y with $f(a_{11/30})\, q\, \beta$. Then $a_{11/30}\, q\, 1$ and $f(\overline{1}) \nleq \beta$. Thus f is not f. st. θ.c. at $a_{11/30}$. But for $f(a_{11/30}) \in \beta$, $a_{11/30} \in \alpha$ and $f(\overline{\alpha}) = f(1-\alpha) \leq \beta$. So f is f. st. θ^*.c. at $a_{11/30}$.

Example 3.2.14: Let f $(X, \tau) \to (X, \tau')$ be the identity function with $X = \{a,b\}$ and $\tau = \{0,1, \alpha\}$, $\tau' = \{0,1, \beta\}$ where $\alpha(a) = \alpha(b) = 1/3$ and $\beta(a) = \beta(b) = 11/30$. Then f is f. st. θ.c. at $a_{1/3}$ but f is not f.st. θ^*.c. at this point.

Proof: : It is clear from the example 3.2.12.

Definition 3.2.15: Let f: $X \to Y$ be a function. if for each fuzzy open set β in Y with $f(x_\alpha) \in \beta$, there exists a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\overset{\circ}{\overline{\mu}}) \leq \beta$. Then, it is called that f is fuzzy almost strongly θ^*. continuous at the fuzzy point x_α (Briefly f.a.st. θ^*. c.) .

If f: $X \to Y$ be a function and μ and β are two fuzzy open sets in X and in Y respectively with $x_\alpha \in \mu$ and $f(x_\alpha) \in \beta$. Then, since $\mu \leq \overset{\circ}{\overline{\mu}} \leq \overline{\mu}$, $f(\mu) \leq f(\overset{\circ}{\overline{\mu}}) \leq f(\overline{\mu})$. So, if f is f.st. θ^*.c., then f is f. a st. θ^*c. and f is f.c. That is,

f.st. θ^*c. $\Rightarrow$ f a. st. θ^*.c. $\Rightarrow$ f.c.

Theorem 3.2.16: If f is f is f.st. θ^*.c., then f is f. rarely*. c. and f. rarely. c.

Proof: : Since if f is f .c., then f is f. rarely*.c. and f.rarely.c. and since if f is f.st. θ^*.c., then f.is f.c. Thus, the proof is clear.

Corollary 3.2.17 : If f: $X \to Y$ is f is f. a. st. θ^*.c., then f is f.c.

Proof: : Let β be a fuzzy open set in Y with $f(x_\alpha) \in \beta$ then, there is a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\overset{o}{\overline{\mu}}) \leq \beta$. Since $\mu \leq \overline{\mu}$, $f(\mu) \leq \beta$. Thus f is. f.c.

Corollary 3.2.18 : If $f: X \rightarrow Y$ is f. a. st. θ^*.c., then f is f. a.c. (Singal), f.w.c., f.w*.c. and f. w**.c.

Proof: : From the corollary 3.2.17., the proof is clear.

Corollary 3.2.19: If $f: X \rightarrow Y$ is f. a. st. θ^*.c. Then f is f. a.rarely*.c. and f is f w. rarely*.c.

Proof: : From corollary 3.1.23, the proof is clear.

Let f: $(X, \tau) \to (X, \tau')$ be a function. Then we have the followings requirements:

f is. f.c.	$\Rightarrow$ f is f. rarely . c.
f is. f.c.	$\Rightarrow$ f is f. rarely* . c.
f is f. rarely .c.	$\Rightarrow$ f is f. a. rarely . c.
f is f.a.c. (Singal)	$\Rightarrow$ f is f. a. rarely . c.
f is f. rarely* .c.	$\Rightarrow$ f is f. a. rarely *. c.
f is f. rarely .c.	$\Rightarrow$ f is f. w. rarely. c.
f is f. a. rarely .c.	$\Rightarrow$ f is f. w. rarely. c.
f is f. rarely* .c.	$\Rightarrow$ f is f. w. rarely*.c.
f is f. a. rarely* .c.	$\Rightarrow$ f is f. w. rarely*.c.
f is f. w** .c.	$\Rightarrow$ f is f. w. rarely .c.
f is f. w .c.	$\Rightarrow$ f is f. w. rarely* .c.
f is f. st. θ .c.	$\Rightarrow$ f is f.c.
f is f. st. θ .c.	$\Rightarrow$ f is f.a.st. θ.c.
f is f. a.st. θ .c.	$\Rightarrow$ f is f..c.
f is f. st. θ .c.	$\Rightarrow$ f is f. rarely. c.
f is f. st. θ* .c.	$\Rightarrow$ f is f.c.
f is f.st. θ*.c.	$\Rightarrow$ f is f.a.st. θ*.c.
f is f. a.st. θ* .c.	$\Rightarrow$ f is f. c.
f is f. st. θ*.c.	$\Rightarrow$ f is f.rarely*. c.
f is f. st. θ*.c.	$\Rightarrow$ f is f.rarely. c.
f is f. st. θ*.c.	$\Rightarrow$ f a.c. (Singal)
f is f. a. st. θ*.c.	$\Rightarrow$ f is f. w. c. and f.w*.c. and f.w**.c.
f is f. a. st. θ*.c.	$\Rightarrow$ f is f.rarely* c.,f.a.rarely*.c.,f.w. rarely. c.,f.w. rarely*.c.

If X is fuzzy almost regular space.

f is f. st. θ.c.	$\Leftrightarrow$ f is f. a. st. θ.c.

If X is fuzzy semi regular space

f is f. a. st. θ.c. $\Leftrightarrow$ f is f. c

If Y is fuzzy regular Space;

f is f. w.c. $\Leftrightarrow$ f is f.c.

If X is fuzzy regular space;

f is f.st. θ.c. $\Leftrightarrow$ f is f.

If Y is fuzzy semi regular space;

f is f. a.c. (Singal) $\Leftrightarrow$ f.is .f.c.

If X is fuzzy regular space and Y is fuzzy regular space;

F is f. st. θ.c. $\Leftrightarrow$ f is f. w.c.

f is f. w.c. $\Leftrightarrow$ f is f. a.st. θ.c.

If X is fuzzy semi regular space and Y is fuzzy semi regular space

f is f. a.c. (Singal) $\Leftrightarrow$ f is f. st. θ.c.

CHAPTER 4

FUZZY D-CONTINUITY, FUZZY D-REGULAR SPACES AND FUZZY D-HAUSDORFF SPACES

In this Chapter, fuzzy D- continuous function fuzzy D*- continuous function are defined. Also weakly and almost forms of these continuities are studied and some relations are given betwen them. Also, basic properties of fuzzy D- continuous functions are summarized; and sufficient conditions on domain and/or range implying fuzzy continuity of fuzzy D- continuous functions are given. Fuzzy D- regular space and fuzzy D-Hausdorff space are defined and some topological propeties are proved.

4.1 Fuzzy D-Continuous Functions:

In this section, fuzzy D-conitnuity and fuzzy D*- continuity of functions are defined and basic properties of fuzzy D- continuous functions are summarized; and sufficient conditions on domain and/or range implying fuzzy continuity of fuzzy D-continuous functions are given.

Definition 4.1.1: A fuzzy set in X is a fuzzy $G_{\delta-}$set iff it is a countable intersection of fuzzy open sets.

Definition 4.1. 2: A fuzzy set in X in as fuzzy F_{σ}-set iff it is a countable union of fuzzy closed sets.

The complement of a fuzzy $G_{\delta-}$ set is a fuzzy $F_{\sigma-}$ set and vice versa.

Lemma 4.1. 3: A fuzzy F_{σ}-set can be written as the union of an increasing sequence $\sigma_1 \leq \sigma_2 \leq \dots$ of fuzzy closed sets (Hence, a fuzzy $G_{\delta-}$set can be written as a decreasing intersection) .

Proof: It is clear from the definitions 4.1.1. and 4.1.2.

Definition 4.1.4: A function $f:X \to Y$ is said to be fuzzy D- continuous (f.D.c.) if for each fuzzy open $F_{\sigma-}$ set β in Y with $f(x_\alpha)\, q\beta$ there is a fuzzy open set μ with $x_\alpha\, q\mu$ such that $f(\mu) \leq \beta$.

Theorem 4.1. 5: Let $f: X \to Y$ be a function. If f is fuzzy continuous then f is f.D.c.

Proof: Since each fuzzy open $F_{\sigma-}$ set is fuzzy open, the proof is clear.

Example 4.1. 6: Let X be a non-empty set and τ be the indiscrete fuzzy topology (that is $\tau=\{0,1\}$ on X). Let $\tau' = \{0,1,\beta_n\}$ where $\beta_n(x) = \frac{1}{n}$, $n \in$ IN and α (x)=11/30. Then the identity mapping $f: (X,\tau) \to (X, \tau')$ is f.D.c. but f is not f.c. at $x_{21/30}$

Proof: For $\beta_3 \in \tau'$ with $f(x_{21/30})q\, \beta_3$, $x_{21/30}\, q\, \alpha$ and $f(\alpha) \leq \beta_3$ so is f is not f.c. at $x_{21/30}$. But, if we show the family of fuzzy open F_σ-sets with A in (X,τ') then we arrive that $A=\{1,\beta_2\}$. Thus, for $f(x_{21/30})\, q\, 1$, $x_{21/30}q\, 1$ and $f(1) \leq 1$ and for $f(x_{21/30})q\, \beta_2$, $x_{21/30}\, q\, \alpha$ and $f(\alpha) \leq \beta_2$ so f is f.D.c. at $x_{21/30}$.

Theorem 4.1.7: Let $f: X\to Y$ be a functions. Then the following statements are equivalent.

(a) f is f.D.c.

(b) If β is a fuzzy open $F_{\sigma-}$ set in Y, then $f^{1}(\beta)$ is a fuzzy open set in X.
(c) If σ is a fuzzy closed $G_{\delta-}$ set in Y, then $f^{-1}(\sigma)$ is fuzzy closed in X.
Proof: (a) $\Rightarrow$ (b) : I f β is a fuzzy open $F_{\sigma-}$ set in Y, then for each fuzzy point x_α in X with $x_\alpha q\, f^{1}(\beta)$, $f(x_\alpha)\, q\, \beta$. From (a), there is a fuzzy open set μ in X with $x_\alpha q\, \mu$ such that $f(\mu) \leq \beta$ Thus $x_\alpha q\, \mu$ and $\mu \leq f^{-1}(\beta)$ so $f^{-1}(\beta)$ is fuzzy Q-nbd of x_α. By the definition 1.2.12. in chapter 1, $f^{-1}(\beta)$ is fuzzy open set in X.

(b) $\Rightarrow$ (a): Let β be a fuzzy open F_σ - set in Y with. $f(x_\alpha)\, q\beta$. From (b), $f^{-1}(\beta)$ is a fuzzy open set in Y with $x_\alpha q\, f^{-1}(\beta)$. Thus, with $f^{-1}(\beta)= \mu$, $f(\mu) \leq \beta$.

(b) $\Rightarrow$ (c): Let σ be a fuzzy closed $G_{\delta-}$ set in Y, then $1- \sigma$ is a fuzzy open $F_{\sigma-}$ set and so from (b), $f^{-1}(1-\sigma) = 1 - f^{1}(\sigma)$ is fuzzy open. Thus $f^{-1}(\sigma)$ is fuzzy closed in X.

(c) $\Rightarrow$ (b): Let β be a fuzzy open F_{σ}- set . Then $1-\beta$ is fuzzy closed G_{δ}-set and so $f^{-1}(1-\beta)=1-f^{-1}(\beta)$ is fuzzy closed. Thus $f^{-1}(\beta)$ is fuzzy open set in X.

Proposition 4.1. 8: Let f: X→ Y be a function. If f is f.D.c., then for each fuzzy point x_{α} in X and each fuzzy net $\{S_n : n\in D\}$ which converges to x_{α}, the fuzzy net $\{f(S_n): n\in D\}$ is eventually quasi-coincident with each fuzzy open F_{σ}- set β with $f(x_{\alpha})\, q\beta$.

Proof: By the theorem 4.1.7, f is f.D.c. $\Leftrightarrow$ if β is a fuzzy open F_{σ}- set in Y, then $f^{-1}(\beta)$ is a fuzzy open set in X. Now, let $\{S_n: n\in D\}$ be a fuzzy net in X which converges to x_{α} and let β be a fuzzy open F_{σ}-set in Y with $f(x_{\alpha})\ q\ \beta$. Then $f^{-1}(\beta)$ is a fuzzy open set with $x_{\alpha} q\ f^{-1}(\beta)$. Thus $\{S_n: n\in D\}$ is eventually quasi-coincident with $f^{-1}(\beta)$. Hence, $\{f(S_n): n\in D\}$ is eventually quasi-coincident with β.

Definition 4.1. 9: Let f: → be any function. Then the function g: X→XxY, defined by g(x) = (x, f(x)) is called the graph function with respect to f.

Theorem 4.1.10: Let f: X→ Y be a function such that the graph function is f.D.c. Then f is f.D.c.

Proof: Let x_{α} be a fuzzy point in X and let β be a fuzzy open F_{σ}-set with $f(x_{\alpha})\, q\beta$. Since $1-\beta$ is a fuzzy closed G_{δ} - set, so $1 \times (1-\beta) = (1 \times 1) - p_2^{-1}(\beta)$ is a fuzzy closed G_{δ}- set. Thus $p_2^{-1}(\beta)$ is a fuzzy open F_{σ} - set of (1×1). Since g is f.D.c., there is a fuzzy open set μ with $x_{\alpha}\ q\ \mu$ such that $g(\mu) \leq p_2^{-1}(\beta)$. It follows that $p_2(g(\mu)) = f(\mu) \leq \beta$, and so f is f.D.c.

Theorem 4.1. 11: Let f: X→ Y be any function. If f is f.D.c., then for $A\subset X$, $f_{/A}: A \to Y$ is f.D.c.

Proof: Let β be a fuzzy open F_{σ}-set in Y with $f(x_{\alpha})\, q\beta$. Since f is f.D.c., there exists a fuzzy open set μ with $x_{\alpha}\, q\mu$ such that $f(\mu) \leq \beta$. We know that if μ is fuzzy open set in X, then $\mu_{|A}=\mu_A$ is fuzzy open in A and $\mu_A \leq \mu$.Thus $f(\mu_A) \leq f(\mu) \leq \beta$ and so $f_{/A}$ is f.D.c.

Theorem 4.1.12: If f: X→Y is f.c. and g: Y→Z is f.D.c., then gof is f.D.c.

Proof: Let σ be a fuzzy closed G_δ - set in Z. Then $g^{-1}(\sigma)$ is fuzzy closed in Y and since f is f.c., $(go\ f)^{-1}(\sigma) = f^{-1}(g^{-1}(\sigma))$ is fuzzy closed in X. Thus gof is f.D.c.

Theorem 4.1.13: Let f: X→Y be either a fuzzy open or fuzzy closed surjection and let g: Y→Z be any function such that gof is f.D.c. Then g is f.D.c.

Proof: Suppose f is fuzzy open (respectively, fuzzy closed), and let β be a fuzzy open F_σ - set in Z (respectively,β be a fuzzy closed G_δ-set). Since gof is f.D.c.,$(gof)^{-1}(\beta)= f^{-1}(g^{-1}(\beta))$ is fuzzy open (respectively, fuzzy closed) and since f is surjection and fuzzy open $f(f^{-1}(g^{-1}(\beta))) = g^{-1}(\beta)$ is fuzzy open (respectively, fuzzy closed) and consequently. g is f.D.c.

Definition 4.1.14: Let (X,τ) be a fuzzy topological space. Let R be an equivalence relation on X. Let X/R be the quotient set, and let P: X → X/R be the projection (quotient map). Let υ be the family of fuzzy sets in X defined by,$\upsilon =\{ \beta \mid p^{-1}(\beta) \in\tau\}$.Then υ is obviously a fuzzy topology for X/R, called the quotient fuzzy topology and (X/R, υ) is called the quotient fuzzy space of (X,τ) (relative to the quotient map). Here P is f.c.

Theorem 4.1.15: Let f: X→ Y be a quotient map. Then a function g: Y→ Z is f.D.c. if and only if gof is f.D.c.

Proof:($\Rightarrow$): It is immediate from the theorem 4.1.12.

($\Leftarrow$): Let β. be a fuzzy open F_σ - set in Z. Then, $(go\ f)^{-1}(\beta) = f^{-1}(g^{-1}(\beta))$ is fuzzy open in X. Since f is a quotient map $g^{-1}(\beta)$ is fuzzy open in Y and so g is f.D.c.

Theorem 4.1.16: For each $\alpha\in$ I, let $f_\alpha : X_\alpha\to Y_\alpha$ be a function, and let f: $\Pi X_\alpha\to\Pi Y_\alpha$ be a function defined by $f(x_\alpha) = (f_\alpha(x_\alpha))$ for each fuzzy point (x_α) in ΠX_α. If f is f.D.c., then each f_α is f.D.c.

Proof: Let σ_i be a fuzzy closed G_δ - set in Y_i . Then $\sigma_i \times \Pi 1_\alpha$ ($\alpha \neq i$) is a fuzzy closed G_δ - set in ΠY_α. Since f is f.D.c., by the theorem 4.1.7, $f^{-1}(\sigma_i \times (\Pi 1_\alpha)) = f_i^{-1}(\sigma_i) \times (\Pi 1_\alpha)$ is fuzzy closed in ΠX_α where $\alpha \neq i$. Consequently, $f_i^{-1}(\sigma_i)$ is fuzzy closed in X_i and so f_i is f.D.c.

Theorem 4.1. 17: Let f: $X \to \Pi X_\alpha$ be a function into a fuzzy product space. If f is f.D.c., then for each $P_\alpha : \Pi X_\alpha \to X_\alpha$, $P_\alpha o f$ is f.D.c.

Proof: Let σ_i be a fuzzy closed G_δ-set in Y_i. Then , $(P_i o f)^{-1}(\sigma_i) = f^{-1}(P_i^{-1}(\sigma_i)) = f^{-1}(\sigma_i x \Pi 1_\alpha)$ where $\alpha \neq i$. Since f is f.D.c. and since $\sigma_i \; x \, \Pi 1_\alpha$ is a fuzzy closed G_δ -set, $f^{-1}(\sigma_i \; x \, \Pi 1_\alpha)$

is fuzzy closed in X . Thus by the theorem 4.1. 7, P_α of is f.D.c.

Definition 4.1.18: A function f: $X \to Y$ is said to be fuzzy almost D-continuous (f.a.D.c.) if for each fuzzy open F_σ-set β with $f(x_\alpha) q \beta$, there exists a fuzzy open set μ , with $x_\alpha q \mu$ such that $f(\mu) \leq \overset{\circ}{\overline{\beta}}$.

Theorem 4.1.19: Let f: $X \to Y$ be a function. If f is f.D.c., then f is f.a.D.c.

Proof: For a fuzzy open set β, since $\beta \leq \overset{\circ}{\overline{\beta}}$, the proof is clear.

Theorem 4.1.20: Let f: $X \to Y$ be a function. Then the following statements are equivalent.

(a) f is f.a.D.c.

(b) For each fuzzy regular open F_σ-set β in Y, $f^{-1}(\beta)$ is fuzzy open in X.

(c) For each fuzy regular closed G_δ - set β in Y, $f^{-1}(\beta)$ is fuzzy closed in X.

(d) For each fuzzy regular open F_σ-set β in Y with $f(x_\alpha) q \beta$, there is a fuzzy open set μ with $x_\alpha q \mu$ such that $f(\mu) \leq \beta$.

(e) For every fuzzy open F_σ-set β in Y, $f^{-1}(\beta) \leq [f^{-1}(\overset{\circ}{\overline{\beta}})]^\circ$.

(f) For every fuzy closed G_δ - set β in Y, $\overline{[f^{-1}(\overline{\overset{\circ}{\beta}})]} \leq f^{-1}(\beta)$.

Proof: (a)$\Rightarrow$(b): If β is a fuzzy regular open F_σ-set in Y, then for each fuzzy point x_α in X with $x_\alpha q f^{-1}(\beta)$, $f(x_\alpha) q \beta$. From (a), there is a fuzzy open set μ with $x_\alpha q \mu$ such that $f(\mu) \leq \overset{\circ}{\overline{\beta}}$. Since $\beta = \overset{\circ}{\overline{\beta}}$, $f(\mu) \leq \beta$. Thus $x_\alpha q \mu$ and $\mu \leq f^{-1}(\beta)$. So $f^{-1}(\beta)$ is Q-nbd of x_α and $f^{-1}(\beta)$ is fuzzy open in X.

(b) $\Rightarrow$(c): Let β be a fuzy regular closed G_δ - set in Y, then $1-\beta$ is a fuzzy regular open F_σ-set in Y. So $f^{-1}(1-\beta)=1-f^{-1}(\beta)$ is fuzzy open in X from (b). Thus $f^{-1}(\beta)$ is fuzzy closed in X.

(c) $\Rightarrow$(d): Let β be a fuzzy regular open F_σ-set in Y with $f(x_\alpha)q\,\beta$. From (c), $1-\beta$ is fuzzy regular closed G_δ - set in Y and $f^{-1}(1-\beta)=1-f^{-1}(\beta)$ is fuzzy closed in X. Thus $f^{-1}(\beta)$ is fuzzy open in X and $x_\alpha q\, f^{-1}(\beta)$. Let $\mu=f^{-1}(\beta)$, then $f(\mu)\le\beta$.

(d) $\Rightarrow$(e): Let β be a fuzzy open F_σ-set in Y. Then $\overset{\circ}{\overline{\beta}}$ is fuzzy regular open F_σ-set and $f^{-1}(\beta)\le f^{-1}(\overset{\circ}{\overline{\beta}})$. From (d)$\Rightarrow$(b), $f^{-1}(\overset{\circ}{\overline{\beta}})$ is fuzzy open. Thus $f^{-1}(\beta)\le[f^{-1}(\overset{\circ}{\overline{\beta}})]^\circ$.

(d) $\Rightarrow$(b): Let β be a fuzzy regular open F_σ-set in Y and let $x_\alpha q\, f^{-1}(\beta)$. Then $f(x_\alpha)q\,\beta$ and from (d), there is a fuzzy open set μ such that $x_\alpha q\,\mu$ and $f(\mu)\le\beta$.Thus $x_\alpha q\,\mu$ and $\mu\le f^{-1}(\beta)$. So $f^{-1}(\beta)$ is fuzzy open set in X.

(e) $\Rightarrow$(f): Let β be a fuzzy closed G_δ - set in Y. Then $1-\beta$ is a fuzzy open F_σ-set in Y. From (e), $f^{-1}(1-\beta)=1-f^{-1}(\beta)\le[f^{-1}(\overset{\circ}{\overline{1-\beta}})]^\circ=[1-f^{-1}(\overline{\overset{\circ}{\beta}})]^\circ=[1-\overline{f^{-1}(\overline{\overset{\circ}{\beta}})}]$. Thus$\overline{(f^{-1}(\overline{\overset{\circ}{\beta}}))}\le f^{-1}(\beta)$.

(f) $\Rightarrow$(a): Let β be a fuzzy open F_σ-set in Y with $f(x_\alpha)q\,\beta$. Then, $1-\beta$ is fuzzy closed G_δ - set in Y. From (f), $[\overline{f^{-1}(1-\overline{\overset{\circ}{\beta}})}]\le 1-f^{-1}(\beta)$

$\Rightarrow[\overline{f^{-1}(1-\overset{\circ}{\overline{\beta}})}]\le 1-f^{-1}(\beta)\Rightarrow\overline{1-f^{-1}(\overset{\circ}{\overline{\beta}})}\le 1-f^{-1}(\beta)$

$\Rightarrow 1-[f^{-1}(\overset{\circ}{\overline{\beta}})]^\circ\le 1-f^{-1}(\beta)\Rightarrow f^{-1}(\beta)\le[f^{-1}(\overset{\circ}{\overline{\beta}})]^\circ$. Let $\mu=[f^{-1}(\overset{\circ}{\overline{\beta}})]^\circ$. Then $x_\alpha q\,\mu$ and $f(\mu)\le\overset{\circ}{\overline{\beta}}$.

Thus f is f.a.D.c. at x_α.

Proposition 4.1.21: Let f: X$\rightarrow$Y be a function. If f is f.a.D.c.,then for each fuzzy point x_α

in X and for each fuzzy net $S=\{S_n:n\in D\}$ which converges to x_α, the fuzzy net $f(S)=\{f(S_n):n\in D\}$ is eventually quasi-coincident with each fuzzy regular open F_σ-set β with $f(x_\alpha)q\,\beta$.

Proof: Let $S=\{S_n:n\in D\}$ be a fuzzy net in X which converges to x_α and let β be a fuzzy regular open F_σ-set in Y with $f(x_\alpha)q\,\beta$. Then, $f^{-1}(\beta)$ is a fuzzy open set in X with $x_\alpha q\, f^{-1}(\beta)$, from the theorem 4.1.20. Thus since S is converges to $x_{\alpha.}$ S is eventually quasi-coincident with $f^{1}(\beta)$. Hence $f(S)=\{f(S_n):n\in D\}$ is eventually quasi-coincident with β.

Theorem 4.1.22: Let f: $X\rightarrow Y$ be a f.c. mapping and if g: $Y\rightarrow Z$ is f.a.D.c., then gof is f.a.D.c.

Proof: Let β be a fuzzy regular closed G_δ - set in Z. Then, by the theorem 4.1.20., $g^{-1}(\beta)$ is fuzzy closed in Y. Since f is f.c., $f^{1}(g^{-1}(\beta))=(gof)^{-1}(\beta)$ is fuzzy closed in X. Thus, gof is f.a.D.c.

Corollary 4.1.23: If f: $X\rightarrow Y$ is f.c. Then f is f.a.D.c.

Proof: f is f.c. $\Rightarrow$ f is f.D.c. $\Rightarrow$f is f.a.D.c.

Theorem 4.1.24: If f: $X\rightarrow Y$ is f.a.c. (Singal). Then f is f.a.D.c.

Proof: Since a fuzzy regular open F_{σ}-set is a fuzzy regular open set, the proof is clear.

Definiton 4.1.25: A function f: $X\rightarrow Y$ is said to be fuzzy weakly D-continuous (f.w.D.c.) iff for each fuzzy open F_{σ}-set β with $f(x_\alpha)q\,\beta$, there exists a fuzzy open set μ with $x_\alpha q\,\mu$ such that $f(\mu)\leq\overline{\beta}$.

Theorem 4.1.26: If f: $X\rightarrow Y$ is f.D.c. (Singal). Then f is f.w.D.c.

Proof: For a fuzzy set β, since $\beta\leq\overline{\beta}$, the proof is clear.

Crollary 4.1.27: If f: $X\rightarrow Y$ is f.c. Then f is f.w.D.c.

Proof: f is f.c. . $\Rightarrow$ f is f.D.c. $\Rightarrow$f is f.w.D.c.

Crollary 4.1.28: If f: $X\rightarrow Y$ is f.a.D.c. Then f is f.w.D.c.

Proof: For a fuzzy set β, since $\beta\leq\overset{\circ}{\overline{\beta}}$, the proof is clear.

Corollary 4.1.29: If f: $X\rightarrow Y$ is f.a.c.(Singal). Then f is f.w.D.c.

Proof: f is f.a.c. . $\Rightarrow$ f is f.a.D.c. $\Rightarrow$f is f.w.D.c.

Theorem 4.1.30: If f: $X\rightarrow Y$ is f.w**.c. Then f is f.w.D.c.

Proof: Since a fuzzy regular open F_{σ}-set is a fuzzy open set, the proof is clear.

Definiton 4.1.31: A function f: $X \rightarrow Y$ is said to be fuzzy D*-continuous (f.D*.c.) if for each fuzzy open F_{σ}-set β with $f(x_{\alpha}) \in \beta$, there exists a fuzzy open set μ with $x_{\alpha} \in \mu$ such that $f(\mu) \leq \beta$.

Theorem 4.1.32: If f: $X \rightarrow Y$ is f.c. Then f is f.D*.c.

Proof: Since a fuzzy open F_{σ}-set is a fuzzy open set, the proof is clear.

Example 4.1.33: Let X be a non-empty set and τ be the indiscrete fuzzy topology (that is $\tau=\{0,1\}$) on X. Let $\tau' = \{0,1,\beta_n\}$ where $\beta_n(x) = \frac{1}{n}$, $n \in$ IN and $\alpha(x)=11/30$. Then the identity mapping f: $(X,\tau) \rightarrow (X, \tau')$ is f.D*.c. but f is not f.c. at $x_{21/30}$

Proof: f is not f.c. from the example 4.1.6. For $f(x_{21/30}) \in 1$, $x_{21/30} \in 1$ and $f(1) \leq 1$. Thus f is f.D*.c. at $x_{21/30}$.

Definition 4.1.34: A function f: $X \rightarrow Y$ is said to be fuzzy almost D*-continuous (f.a.D*.c.) if for each fuzzy open F_{σ}-set β with $f(x_{\alpha}) \in \beta$, there exists a fuzzy open set μ with $x_{\alpha} \in \mu$ such that $f(\mu) \leq \overset{\circ}{\overline{\beta}}$.

Theorem 4.1.35: Let $f: X \rightarrow Y$ is a function.If f is f.D*.c., then f is f.a.D*.c.

Proof: For a fuzzy set β, since $\beta \leq \overset{\circ}{\overline{\beta}}$, the proof is clear.

Corollary 4.1.36: If f: $X \rightarrow Y$ is f.c. Then f is f.a.D*.c.

Proof: f is f.c. . $\Rightarrow$ f is f.D*.c. $\Rightarrow$f is f.a.D*.c.

Definition 4.1.37: A function f: $X \rightarrow Y$ is said to be fuzzy weakly D*-continuous (f.w.D*.c.) if for each fuzzy open F_{σ}-set β with $f(x_{\alpha}) \in \beta$, there exists a fuzzy open set μ with $x_{\alpha} \in \mu$ such that $f(\mu) \leq \overline{\beta}$.

Theorem 4.1.38: If f: $X \rightarrow Y$ is f.D*c. Then f is f.w.D*.c.

Proof: For a fuzzy set β, since $\beta \leq \overline{\beta}$, the proof is clear.

Corollary 4.1.39: If f: $X \rightarrow Y$ is f.c. Then f is f.w.D*.c.

Proof: f is f.c. . $\Rightarrow$ f is f.D*.c. $\Rightarrow$f is f.w.D*.c.

Corollary 4.1.40: If f: $X \rightarrow Y$ is f.a.D*.c. Then f is f.w.D*.c.

Proof: For a fuzzy set β, since $\beta \leq \overset{o}{\overline{\beta}}$, the proof is clear.

Theorem 4.1.41: If f: $X \rightarrow Y$ is f.w.c. Then f is f.w.D*.c.

Proof: Since a fuzzy open F_{σ}-set is a fuzzy open set, the proof is clear.

4.2. Fuzz y D-Regular Space And Fuzzy D- Hausdorff Space

In this section, fuzzy D- Hausdorff space and fuzzy D- regular space are defined and some comparisons are given with other speration axioms in fuzzy topological space.

Definition 4.2.1: Two fuzzy sets β_1 and β_2 in a f.t.s. (X,τ) are said to be Q- seperated iff there exist fuzzy closed sets μ_i (i=1,2) such that $\mu_i \geq \beta_i$ (i=1,2) and $\beta_2 \wedge \mu_1 = \beta_1 \wedge \mu_2 = 0$. It is obvious that β_1 and β_2 are Q- seperated iff $\overline{\beta}_2 \wedge \beta_1 = \overline{\beta}_1 \wedge \beta_2 = 0$.[5].

Definition 4.2.2: A fuzzy set β in a f.t.s. (X,τ) is called fuzzy disconnected iff there exist two non-zero fuzzy sets A and B in the subspace D_o (i.e; supp $\beta = D_o$) such that A and B Q-seperated and $\beta = A \vee B$. A fuzzy set is called fuzzy connected iff it is not disconnected. [5].

Theorem4.2.3: Two fuzzy sets α and β are Q- seperated iff $A_o \wedge B_o = \varnothing$,

$\overline{\alpha}_{A_o \vee B_o} = \overline{\alpha}_{A_o}$, $\overline{\beta}_{A_o \vee B_o} = \overline{\beta}_{B_o}$

Proof: [5].

Proposition 4.2.4: Two fuzzy sets α and β are Q- seperated iff $\overline{\alpha}_{A_o \vee B_o}$ and $\overline{\beta}_{A_o \vee B_o}$ are Q- separated.

Proof: [5].

Theorem 4.2.5: A f.D.c. image of a fuzzy connected space is fuzzy connected.

Proof: Let f: X→Y be a f.D.c. surjection from a fuzzy connected space X onto a fuzzy topological space Y. Suppose Y is not fuzzy connected. Then from the definition 4.2.2., there are non-zero fuzzy sets α and β such that α and β are Q- seperated and $1 = \alpha \vee \beta$.

By the proposition 4.2.4., $\overline{\alpha}$ and $\overline{\beta}$ are Q-separated. Thus $1 = \overline{\alpha} \vee \overline{\beta}$ and $\overline{\alpha} \wedge \overline{\beta} = 0$. Hence both $\overline{\alpha}$ and $\overline{\beta}$ are fuzzy clopen sets in Y. This means they are fuzzy closed G_δ - sets in Y. Since f is f.D.c., by the theorem 4.1.7., both $f^{-1}(\overline{\alpha})$ and $f^{-1}(\overline{\beta})$ are fuzzy closed sets in X. Then, $1 = f^{-1}(\overline{\alpha}) \vee f^{-1}(\overline{\beta})$ and $f^{-1}(\overline{\alpha}) \wedge f^{-1}(\overline{\beta}) = 0$. Thus $f^{-1}(\overline{\alpha})$ and $f^{-1}(\overline{\beta})$ are Q - seperated and (X,τ) is a fuzzy disconnected space. This is a contradiction to hypothesis.

Definition 4.2.6: A function f: X→is said to be fuzzy connected if $f(\alpha)$ is fuzzy connected for every fuzzy connected set α in X.

Corollary 4.2.7: Every f.D.c. function is a fuzzy connected function.

Proof: This follows from the theorem 4.1.11. and the theorem 4.2.5.

Definition 4.2.8: A f.t.s. (X, τ) is called fuzzy T_1 iff for each $x \in X$ and each $\lambda \in [0, 1]$ there exists a $\beta \in \tau$ such that $\beta(x) = 1 - \lambda$ and $\beta(y) = 1$ for $y \neq x$. [5].

Proposition 4.2.9: A. f.t.s. (X, τ) is fuzzy T_1 iff every fuzzy point is a fuzzy closed set in X. [10].

Definition 4.2.10: A f.t.s. (X, τ) is called fuzzy T_2 (fuzzy Hausdoff) iff for any two fuzzy points e and d satisfying supp$e \neq$ supp$d$ there exist Q- neighborhoods β and α of e and d, respectively, such that $\beta \wedge \alpha = 0$. [5].

Theorem 4.2.11: Let f: X→Y be a one-to-one and f.D.c. function such that each singleton in Y is a fuzzy G_δ - set : if Y is fuzzy T_1, then so X.

Proof : Since f is f.D.c. and injective for a fuzzy point x_α in X, $f(x_\alpha)$ is a fuzzy point in Y and since Y is fuzzy T_1 and $\{f(x_\alpha)\}$ is a fuzzy G_δ - set in Y, then $\{x_\alpha\}$ is fuzzy closed in X. So X is a fuzzy T_1 space.

Theorem 4.2.12: Let f: X→ Y be a f.D.c. and fuzzy closed function from a fuzzy normal space X onto a fuzzy topological space Y such that each singleton in Y is a fuzzy G_δ- set. If either of the spaces X and Y is fuzzy T_1, then Y is fuzzy Hausdorff.

Proof: Case I: The space Y is fuzzy T_1. Let e and d two fuzzy points in Y satisfying suppe ≠ suppd. Then {e} and {d} are fuzzy closed G_δ- set in Y, and so by the theorem 4.1.7. $f^{-1}(e)$ and $f^{-1}(d)$ are fuzzy closed sets in X. By the fuzzy normality of X, there are disjoint fuzzy open sets $\mu_1$and μ_2 such that $f^{-1}(e)q\,\mu_1$ and $f^{-1}(d)q\,\mu_2$ and $\mu_1 \wedge \mu_2=0$. Since f is fuzzy closed, the sets β_1 = 1- f(1- μ_1) and β_2 = 1- f(1- μ_2) are fuzzy open in Y. Also $eq\beta_1$ and $dq\beta_2$ and $\beta_1 \wedge \beta_2=0$, since $f(\mu_1) \le \beta_1$ and $f(\mu_2) \le \beta_2$. Thus Y is fuzzy Hausdorff.

Case II: The space X is fuzzy T_1. Let x_α be a fuzzy point in X . Since the singleton $\{x_\alpha\}$ is fuzzy closed, $\{f(x_\alpha)\}$ is a fuzzy closed set in Y. So Y is fuzzy T_1 and the proof is complete in view of case I.

Definition 4.2.13: We call a space fuzzy D- Hausdorff if each pair of distinct fuzzy points is quasi-coincident with disjoint fuzzy open F_σ - sets.

Definition 4.2.14: A subspace A of a f.t.s X is said to be f.D.c. retract of X if there is a f.D.c. surjection r :X→A such that $r_{/A}$ is the identity map of A.

Theorem 4.2.15: Let f : X→ Y be a f .D.c. injection into a fuzzy D- Hausdorff space Y. Then X is fuzzy Hausdorff.

Proof: Let x_α and y_λ be two fuzzy points satisfying supp $x_\alpha \neq$ supp y_λ. Then f(x) ≠ f(y). Since Y is fuzzy D- Hausdorff, there are disjoint fuzzy open F_σ - sets β_1 and β_2 with $\beta_1 q\, f(x_\alpha)$ and $\beta_2 q\, f(y_\lambda)$, respectively. By the theorem 4.1.7., $f^{-1}(\beta_1)$ and $f^{-1}(\beta_2)$ are disjoint fuzzy open sets with $x_\alpha q\, f^{-1}(\beta_1)$ and $y_\lambda\, q\, f^{-1}(\beta_2)$, respectively. Thus X is fuzzy Hausdorff.

Let (X, τ) be a f.t.s. and let B denote the collection of all fuzzy open F_σ - sets in (X, τ). Since the intersection of two fuzzy open F_σ - sets is a fuzzy open F_σ - sets , the collection B is a base for a fuzzy topology τ^* on X. Clearly $\tau^* \subset \tau$. Moreover, if each singleton in X is a fuzzy G_δ- set, then (X, τ^*) is fuzzy T_1 whenever (X, τ) is.

Definition 4.2.16: A f.t.s. (X,τ) is called fuzzy D- regular if for each fuzzy point x_α in X and each fuzzy open set μ with $x_\alpha q\mu$ there is a fuzzy open F_σ - set μ^* such that $x_\alpha q\mu^* \leq \mu$.

Corollary 4.2.17: A f.t.s (X, τ) is fuzzy D- regular if and only if $\tau = \tau^*$.

Proof : (⇒): Let (X, τ) be a fuzzy D- regular space and let $\beta \in \tau$. If $\beta \notin \tau^*$, there is a fuzzy point x_α in X such that for every fuzzy open F_σ - set μ^* in X with $x_\alpha q\mu^*$, $\mu^* \not\leq \beta$. But since X is fuzzy D- regular, this is contradiction. Thus $\beta \in \tau^*$ and $\tau = \tau^*$.

(⇐) : Let $\tau=\tau^*$ and let x_α be a fuzzy point in X. Suppose μ be a fuzzy open set in X with $x_\alpha q\mu$. Since $\tau = \tau^*$, $\mu \in \tau^*$ and $x_\alpha q\mu \leq \mu$. Thus X is a fuzzy D- regular space.

Theorem 4.2.18: Let (X, τ) be a f.t.s. Then the following statements are equivalent.

(a) (Y, τ') is fuzzy D-regular.

(b) Every f.D.c. function from a f.t.s. (X, τ) into a f.t.s. Y is f.c.

(c) The identitiy mapping I from (Y, τ'^*) onto (Y, τ') is f.c.

Proof: (a) ⇒ (b): Let $f(x_\alpha)\ q\ \beta$ and β be a fuzzy open set in Y. Since Y is fuzzy D- regular, there is a fuzzy open F_σ -set β^* such that $x_\alpha q\ \beta^* \leq \beta$. By the theorem 4.1.7. $f^{-1}(\beta^*)$ is fuzzy open and $x_\alpha q f^{-1}(\beta^*)$ and $f(f^{-1}(\beta^*)) \leq \beta$. Thus f is f.c.

(b) ⇒ (c): Let $f = I: (Y,\tau'^*) \rightarrow (Y, \tau')$ be the identity mapping. Let $f(x_\alpha)q\beta$ and β be a fuzzy open F_σ - set in (Y, τ'). Then $x_\alpha q f^{-1}(\beta)$ and $f^{-1}(\beta) \in \tau'^*$. So $f(f^{-1}(\beta)) \leq \beta$. Thus f is f.D.c. From (b), f is f.c.

(c) ⇒ (a): Let x_α be a fuzzy point and β be a fuzzy open set in (Y, τ') with $x_\alpha\ q\beta$. Since $f=I: (Y, \tau'^*) \rightarrow (Y, \tau')$ is f.c., there is a fuzzy open F_σ - set μ in (Y, τ'^*) such that $x_\alpha q\mu$ and $f(\mu) \leq \beta$. Thus $x_\alpha q\ \mu \leq \beta$ and (Y, τ') is fuzzy D- regular.

Theorem 4.2.19: The product of any family $\{X_\alpha: \alpha \in D\}$ of fuzzy D- regular spaces is fuzzy D-regular.

Proof: To show that $X=\Pi X_\alpha$ is fuzzy D-regular, in view of the theorem 4.1.15., it is sufficient to show that every f.D.c. function f: $Y \to X$ is f.c. Thus, it suffices to show that P_α of is f.c., for each α, where P_α denotes the projection onto the α - co - ordinate space.

Let σ be a fuzzy closed G_δ - set in X_i. Then $p_\alpha^{-1}(\sigma x \prod_{i \neq \alpha} 1\alpha)$ is a fuzzy closed G_δ- set in X. Since $(P_\alpha \text{ of})^{-1}(\sigma) = f^{-1}(P_\alpha^{-1}(\sigma))$ is fuzzy closed in Y, $P_\alpha$of is f.D.c. In view of fuzzy D-regularity of X_α $(\alpha \in D)$, $P_\alpha$of is f.c. and the proof of the theorem is complete.

Let f:$(X, \tau) \rightarrow (Y, \tau')$ be a function. Then we have the followings:

f is f.c.	$\Rightarrow$ f is f.D.c.
f is f.c.	$\Rightarrow$ f is f.D*.c.
f is f.c.	$\Rightarrow$ f is f.a.D.c.
f is f.c.	$\Rightarrow$ f is f.a.D*.c.
f is f.c.	$\Rightarrow$ f is f.w.D.c.
f is f.c.	$\Rightarrow$ f is f.w.D*.c.
f is f.a.c.	$\Rightarrow$ f is f.a.D.c.
f is f.a.c.	$\Rightarrow$ f is f.w.D.c.
f is f.D.c.	$\Rightarrow$ f is f.a.D.c.
f is f.D.c.	$\Rightarrow$ f is f.w.D.c.
f is f.D*.c.	$\Rightarrow$ f is f.a.D*.c.
f is f.D*.c.	$\Rightarrow$ f is f.w.D*.c.
f is f.a.D.c.	$\Rightarrow$ f is f.w.D.c.
f is f.a.D*.c.	$\Rightarrow$ f is f.w.D*.c.
f is f.w**.c.	$\Rightarrow$ f is f.w.D.c.
f is f.w.c.	$\Rightarrow$ f is f.w.D*.c.

If (Y, τ') is fuzzy D-regular space;

f is f.D.c.	$\Rightarrow$ f is f.c.

CHAPTER 5

CONSLISION

In fuzzy topological spaces, weak and strong forms of fuzzy continuity have been studied by some outhers since 1965 (Zadeh). In this thesis, we have defined two different forms of fuzzy weakly continuity (Definition 2.1.1. and definition 2.1.3, Chapter 2) and proved that they are independettı each other. Some theorems and corollaries are obtained using fuzzy almost continuity (In the sense of Singal and Husain). Later, fuzzy almost continuity in the sense of stallings is defined and seen that fuzzy continuity implies fuzzy almost continuity (stallings). Also, weakly and almost forms of fuzzy quasi continuity are defined and compared fuzzy weakly, fuzzy almost and fuzzy continuities. Then, it is seen that if (Y,τ') is fuzzy semi-regular space, fuzzy almost quasi continuity and fuzzy quasi continuity are same and if (Y,τ') is fuzzy almost regular space, fuzzy almost quasi continuity and fuzzy weakly quasi continuity are same and if (Y,τ') is fuzzy regular space, fuzzy quasi continuity and fuzzy weakly quasi continuity are same.

In the third chapter, same properties of fuzzy rarely continuity studied. Fuzzy rarely* continuity and fuzzy strongly θ^* - continuity of a function are defined and some relations between them are investigated.

In the forth chapter, D-continuity, D-Housdorff space and D-regular space concepts which are given by Kohli [9] are defined in fuzzy topology. Later, some characterizations of fuzzy D-continuity is given by the theorem 4.1.7. and fuzzy D*-continuity is defined and some comparizons are given.

My works will be continue about fuzzy separation axioms and several fuzzy continuities according to the developments in general topology.

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