GENERALIZED BASKAKOV-KANTOROVICH OPERATORS

by

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

GENERALIZED BASKAKOV-KANTOROVICH OPERATORS

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This thesis consists of third chapter.

The first chapter deals with some basic definitions and a short history of approximation theorems.

In the second chapter, strong approximation properties of a modification of Baskakov-Kantorovich operators via Taylor polynomials are given in polynomial weighted spaces.

The third chapter contains our original results. In this chapter, we introduce a generalization of Baskakov-Kantorovich operators. We first give some direct results with the help of the second order modulus of continuity and the elements of Lipschitz class. Next, we study their weighted approximation properties.

Keywords: Linear positive operator, Kantorovich operator, strong approximation, weighted approximation, modulus of continuity, Lipschitz class.

ÖZET

GENELLEŞTİRİLMİŞ BASKAKOV-KANTOROVICH OPERATÖRLERİ

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Bu tez, üç bölümden oluşmaktadır.

Birinci bölüm, bazı temel tanımlar ve yaklaşım teoremlerini içermektedir.

İkinci bölümde, Taylor polinomu yardımıyla genelleştirilmiş Baskakov- Kantorovich operatörlerinin kuvvetli yaklaşım özellikleri polinom ağırlıklı uzaylarda verilmektedir.

Üçüncü bölüm, bu tezin orijinal sonuçlarını içermektedir. Bu bölümde, Baskakov-Kantorovich operatörlerinin bir genelleştirilmesi tanımlanarak bu operatörler için ilk olarak ikinci süreklilik modulü ve Lipschitz sınıfının elemanları yardımıyla bazı direkt sonuçlar verilmektedir. Daha sonra bu operatörlerin ağırlıklı yaklaşım özellikleri incelenmektedir.

Anahtar Kelimeler: Lineer pozitif operatör, Kantorovich operatör, kuvvetli yaklaşım, ağırlıklı yaklaşım, süreklilik modülü, Lipschitz sınıfı.

To my family, my husband and my supervisor...

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CHAPTER 1

INTRODUCTION AND BASIC FACTS

1.1 Preliminary definitions and auxiliary results

Positive linear operators play an important role in Approximation Theory. As mentioned in [13] when we study Approximation Theory by positive linear operators we meet two main problems:

1- Does a sequence of positive linear operators $L_n(f;x)$ converges uniformly to the continuous function $f(x)$?

2- How quickly does the difference

$$L_n(f;x) - f(x)$$

tend to zero if uniform convergence holds?

In order to answer the above two questions we give some preliminary definitions and fundamental theorems.

Definition 1.1. A nonempty set V is said to be a linear space (or a vector space) over a field $\mathbb{F}$, if it satisfies the following conditions:

(i) $\forall x,\ y \in V,\ we\ have\ x + y \in V$

(ii) $\forall x,\ y \in V,\ we\ have\ x + y = y + x$

(iii) $\forall x,\ y,\ z \in V,\ we\ have\ x + (y + z) = (x + y) + z$

(iv) $\forall x \in V,\ \exists \theta \in V\ such\ that\ x + \theta = \theta + x = x$

(v) $\forall x \in V,\ \exists \tilde{x} \in V\ such\ that\ x + \tilde{x} = \theta$

(vi) $\forall x \in V\ and\ \forall a \in \mathbb{F},\ we\ have\ ax \in V$

(vii) $\forall x \in V\ and\ \forall a,\ b \in \mathbb{F},\ we\ have\ (ab)x = a(bx)$

(viii) $\forall x \in V,\ we\ have\ 1x = x$

(ix) $\forall x \in V,\ \forall a \in \mathbb{F},\ we\ have\ a(x+y) = ax + ay$

(x) $\forall x \in V\ and\ \forall a,\ b \in \mathbb{F},\ we\ have\ (a+b)x = ax + bx.$

Definition 1.2. Let V be a linear space over a field $\mathbb{F}$, a real valued function $||.|| : V \to \mathbb{R}^+$, called a norm, satisfying the following conditions:

(i) $\forall x \in V,\ we\ have\ ||x|| \geq 0$

(ii) $\forall x \in V,\ we\ have\ ||x|| = 0 \Leftrightarrow x = 0$

(iii)$\forall x \in V\ and\ \forall a \in \mathbb{F},\ we\ have\ ||ax|| = |a|\ ||x||$

(iv) $\forall x,\ y \in V,\ we\ have\ ||x+y|| \leq ||x|| + ||y||.$

A linear space on which a norm is defined is then called a "linear normed space".

Definition 1.3. Let X and Y be two linear normed function spaces. An operator $L : X \to Y$ is a rule which assigns to function of X a function of Y. We denote the operators by $L(f;x)$ *or* $L(f(t);x)$.

Definition 1.4. A linear operator L between two linear normed spaces X and Y is a mapping $L : X \to Y$ such that $L(\alpha f + \beta g; x) = \alpha L(f;x) + \beta L(g;x)$ for every $f,\ g \in X$ and for every scalar $\alpha,\ \beta$.

Definition 1.5. An operator L is positive if $f \geq 0$ implies $L(f;x) \geq 0$. If in addition, L is linear we have $L(f;x) \leq L(g;x)$ if $f \leq g$; consequently $|L(f;x)| \leq L(|f|;x)$.

The fundamental development in Approximation Theory was started by K. Weierstrass [31] in 1885 with the following theorem:

Theorem 1.1. *(Weierstrass): Let $f \in C[a,b]$, the space of all continuous functions on $[a,b]$. Given $\varepsilon > 0$, there exists an algebraic polynomial P for which*

$$|f(x) - P(x)| < \varepsilon$$

for all $x \in [a,b]$.

Since, its proof was long and complicated many mathematicians tried to investigate a simpler proof. In 1912, Russian mathematician S. N. Bernstein [3] presented the polynomials

$$B_n(f,x) = \sum_{k=0}^{n} x^k (1-x)^{n-k} f\left(\frac{n}{k}\right), \quad x \in [0,1],$$

which are known as Bernstein polynomials, for $f \in C[0,1]$ and introduced a short proof of Weierstrass's theorem with the help of these linear positive operators.

The history on the contributions of Weierstrass to the development of Approximation Theory may be found in [17].

For every $f \in C[a,b]$, let

$$||f|| = \sup_{x\in[a,b]} |f(x)|.$$

The function $||.||$ is called the supremum norm on $C[a,b]$. If the sequence $f_n \in C[a,b]$ and a function g satisfy $\lim_{n\to\infty} ||f_n - g|| = 0$, then we say that the sequence f_n converges uniformly to the function g.

The following theorem indicates a necessary and sufficient condition for the uniform convergence of a sequence of positive linear operators.

Theorem 1.2. *(Bohman-Korovkin): Let L_n be a sequence of positive linear operators mapping $C[a,b]$ into itself. If*

$$\lim_{n\to\infty} ||L_n(t^\nu, x) - x^\nu|| = 0, \quad \nu = 0,\ 1,\ 2,$$

then for every $f \in C[a,b]$ we have

$$\lim_{n\to\infty} ||L_n(f;x) - f(x)|| = 0.$$

Korovkin type theorem in polynomial weighted spaces of continuous and unbounded functions defined on positive semi-axis proved by Gadjihiev in [8],[9].

To state this theorem we recall some definitions and notations.

Definition 1.6. Let $\rho(x) \geq 1$ be a continuous function on $[0, \infty]$ such that

$$\lim_{x \to \infty} \rho(x) = \infty.$$

$B_\rho[0, \infty)$ be space of all functions having the property

$$|f(x)| \leq M_f \rho(x)$$

where $x \in [0, \infty)$ and M_f is a positive constant depending only f. The space $B_\rho[0, \infty)$ is equipped with the norm

$$||f||_\rho = \sup_{x \geq 0} \frac{|f(x)|}{\rho(x)}.$$

The function $\rho(x)$ is called the weight function and the space $B_\rho[0, \infty)$ is called as weighted space.

When, the weight function is of the form $1 + x^\alpha,\ \alpha \in N$ the weighted space is polynomial weighted space.

Let $C_\rho[0, \infty)$ be space of all continuous functions belonging to $B_\rho[0, \infty)$. By $C_\rho^0[0, \infty)$ we denote the subspace of all functions $f \in C_\rho[0, \infty)$ for which

$$\lim_{x \to \infty} \frac{|f(x)|}{\rho(x)} < \infty.$$

Theorem 1.3. *[8, 9]: Let $\{A_n\}$ be a sequence of positive linear operators acting from $C_\rho[0, \infty)$ to $B_\rho[0, \infty)$ and satisfying the conditions*

$$\lim_{n \to \infty} ||A_n(t^\nu; x) - x^\nu||_\rho = 0, \quad \nu = 0, 1, 2.$$

Then for any function $f \in C^0_\rho[0,\infty)$,

$$\lim_{n\to\infty} ||A_n(f;x) - f(x)||_\rho = 0.$$

Note that a sequence of linear positive operators A_n acts from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ if and only if

$$||A_n(\rho;x)||_\rho \leq M\rho,$$

where $M\rho$ is positive constant. This fact is a simple result of the necessary and sufficient condition that

$$A_n(\rho;x) \leq M\rho(x)$$

given in [8, 9].

One of the main instruments to compute the order of approximation, i.e. the speed of convergence of positive linear operators, is modulus of smoothness.

If I is a real interval, we denote by $C_B(I)$ $(\tilde{C}_B(I))$ the space of all bounded continuous (uniformly continuous) real functions on I.

If $f \in C_B(I)$, for every $x \in I,\ h \in R,\ h \neq 0$ and $k \in N$, the k-th difference $\Delta^k_h f(x)$ of f with step h at the point x is given by

$$\Delta^k_h f(x) := \sum_{m=0}^{k} (-1)^{m+k} \binom{k}{m} f(x+mk)$$

provided that $x + kh \in I$.

The (k+1)-th difference can be expressed in terms of the k-th difference by means of the following identity:

$$\Delta^{k+1}_h f(x) = \Delta^k_h f(x+h) - \Delta^k_h f(x).$$

Definition 1.7. If $f : I \to R$ is a bounded real function and if $\delta > 0$, the modulus of continuity $\omega(f,\delta)$ of f is defined by

$$\omega(f,\delta) := \sup_{|h|\leq\delta,\ x,x+h\in I} |\Delta_h^1 f(x)| = \sup_{|h|\leq\delta,\ x,x+h\in I} |f(x+h)-f(x)|$$
$$= \sup_{x,y\in I,\ |x-y|\leq\delta} |f(x)-f(y)|.$$

The modulus of continuity of f is also called the first modulus of smoothness of f.

Definition 1.8. If $f: I \to R$ is a bounded real function and if $\delta > 0$ and $k \in N$, we can define the k-th modulus of smoothness $\omega_k(f,\delta)$ of f by

$$\omega_k(f,\delta) := \sup_{|h|\leq\delta,\ x,x+kh\in I} |\Delta_h^k f(x)|.$$

Some elementary properties of modulus of smoothness are below:

1) If $0 < \delta_1 \leq \delta_2$, then $\omega_k(f,\delta_1) \leq \omega_k(f,\delta_2)$.

2) $\lim\limits_{\delta\to 0^+} \omega_k(f,\delta) = 0$ provided $f \in \tilde{C}_B(I)$.

3) For every $\delta > 0$, $\omega_{k+1}(f,\delta) \leq 2\omega_k(f,\delta)$.

4) If f is differentiable and $f' \in C_B(I)$, then $\omega_{k+1}(f,\delta) \leq \delta\omega_k(f',\delta)$ for every $\delta > 0$.

5) For every $\delta > 0$ and $n \in N$, $\omega_k(f,n\delta) \leq n^k\omega_k(f,\delta)$.

6) For every $\delta > 0$ and $\lambda > 0$, $\omega_k(f,\lambda\delta) \leq (1+[\lambda])^k\omega_k(f,\delta)$, where $[\lambda]$ denotes the integer part of λ.

Definition 1.9. Let $\alpha \in N,\ \delta > 0$. For a function $f: [0,\infty) \to R$ the classical weighted modulus of continuity is defined by

$$\omega_\alpha(f,\delta) = \sup_{0<h\leq\delta,\ x\in[0,\infty)} \frac{|f(x+h)-f(x)|}{1+x^\alpha}.$$

This weighted modulus of continuity does not satisfy the some properties of the usual modulus of continuity such as $\omega(f,m\delta) \leq m\omega(f,\delta)$ which implies the inequality weighted $\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta)$ for $\lambda > 0$.

In [15] Lopez-Moreno defined the weighted modulus of continuity

$$\Omega_\alpha(f;\delta) := \sup_{0<h\leq\delta,\, x\in[0,\infty)} \frac{|f(x+h)-f(x)|}{1+(h+x)^\alpha}$$

which has the following similar properties of modulus of continuity

1) $\lim\limits_{\delta\to 0^+} \Omega_\alpha(f;\delta) = 0$ for $f \in C^0_\rho[0,\infty)$, where $\rho(x) = 1 + x^\alpha,\ \alpha \in N$.

2) $\Omega_\alpha(f;m\delta) \leq m\Omega_\alpha(f;\delta)$, for $m \in N$.

3) $\Omega_\alpha(f;\delta) \leq \omega_\alpha(f,\delta)$.

In the last chapter, we compute the rate of convergence of a sequence of positive linear operators by means of the weighted modulus of continuity $\Omega_2(f,\delta)$.

In 1930, L.V. Kantorovich [12] defined the linear positive operators

$$K_n(f,x) = (n+1)\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{m-k} \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(s)ds, \quad x \in [0,1]$$

for any $f \in L_1([0,1])$, which are referred as the Kantorovich operators. It is obvious that these operators derived from the Bernstein polynomials. After that generalization of positive linear operators in the sense of Kantorovich were studied widely by a number of authors. In this thesis, we deal with Kantorovich type Baskakov operators.

CHAPTER 2

A GENERALIZATION OF BASKAKOV-KANTOROVICH OPERATORS VIA TAYLOR POLYNOMIALS

2.1 Introduction

Strong approximation of functions related trigonometric Fourier series was studied in many papers (see[14]). For the examination of the strong approximation involving positive linear operators you can see [19]-[23].

In this chapter, we prove a strong approximation theorem for positive linear operators, defined by Rempulska and Skorupka [18], in polynomial weighted space of differentiable functions. We first recall some definitions and notations appeared in [18] (see also [2]). Let $p \in N_0$. We consider the polynomial weighted space $C_\rho[0,\infty) := C_\rho(R_0) := C_p$ which is associated with the weighted function

$$\rho(x) = w_0(x) = 1, \quad \rho(x) = \frac{1}{w_p(x)} = 1 + x^p, \quad p \geq 1. \tag{2.1}$$

As we mentioned in Chapter 1, the norm on C_p is given by

$$||f||_\rho := ||f||_p = \sup\{w_p(x)|f(x)| : x \in R_0\}. \tag{2.2}$$

Let $r \in N_0$ be a fixed number. C^r denotes the set of all r-times differentiable functions $f \in C_r$ such that $f^{(m)} \in C_{r-m}$ for all $0 \leq m \leq r$.

Let Ω denote the set of all infinite matrices $A = [a_{nk}(.)]$, $n \in N$, $k \in N_0$ of functions $a_{nk} \in C_0$ and having properties:

(i) $a_{nk}(x) \geq 0 \quad x \in R_0,\ n \in N,\ k \in N_0,$

(ii) $\sum_{k=0}^{\infty} a_{nk}(x) = 1 \quad x \in R_0,\ n \in N,$

(iii) the series $\sum_{k=0}^{\infty} k^r a_{nk}(x)$ is uniformly convergent on R_0, $r \in N$ and its sum is a function belonging to C_r,

(iv) for every $r \in N$ there exists positive constant $M_1(r, A)$ independent of $x \in R_0$ and $n \in N$ such that

$$||T_{n,2r}(.; A)||_{2r} \le M_1(r; A)n^{-r}, \tag{2.3}$$

where

$$T_{n,r}(x; A) := \sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{k}{n} - x\right)^r. \tag{2.4}$$

Now, let $A \in \Omega$ and also let $r \in N_0$. We now consider the operators defined by

$$L_{n,r}(f; A; x) := \sum_{k=0}^{\infty} a_{nk}(x) n \int_{I_{nk}} F_r(t, x)dt, \quad x \in R_0,\ n \in N, \tag{2.5}$$

and the strong difference with the power $q > 0$,

$$H^q_{n;r}(f; A; x) := \left(\sum_{k=0}^{\infty} a_{nk}(x) \left| n \int_{I_{nk}} F_r(t, x)dt - f(x)\right|^q\right)^{\frac{1}{q}}; \quad x \in R_0,\ n \in N, \tag{2.6}$$

where $I_{nk} := \left[\frac{k}{n}, \frac{k+1}{n}\right]$ and

$$F_r(t, x) := \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x - t)^j, \tag{2.7}$$

is the Taylor polynomial of $f \in C^r$.

Specially for $r = 0$ and $f \in C^0$, we have

$$L_{n;0}(f; A; x) := \sum_{k=0}^{\infty} a_{nk}(x) n \int_{I_{nk}} f(t)dt,\ x \in R_0,\ n \in N, \tag{2.8}$$

and

$$H_{n;0}^{q}(f;A;x) := \left(\sum_{k=0}^{\infty} a_{nk}(x) \left| n \int_{I_{nk}} f(t)dt - f(x) \right|^{q} \right)^{\frac{1}{q}}. \tag{2.9}$$

Observe that if we choose $a_{nk}(x) = \binom{n+k-1}{k} x^k (1-x)^{-n-k}$ and $r = 0$, the operators defined by (2.5) turn out to be the Baskakov-Kantorovich operators

$$B_n(f,x) = n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1-x)^{-n-k} \int_{I_{nk}} f(t)dt.$$

On the other hand, taking into consideration the property (ii), from (2.5) it follows that

$$L_{n;r}(1;A;x) = 1. \tag{2.10}$$

By (2.5), (2.6) and the properties (i) and (ii), one gets

$$|L_{n;r}(f;A;x) - f(x)| \leq H_{n;r}^{1}(f;A;x) \tag{2.11}$$

and

$$H_{n;r}^{p}(f;A;x) \leq H_{n;r}^{q}(f;A;x) \quad 0 < p < q < \infty. \tag{2.12}$$

for $f \in C^r$, $x \in R_0$ and $n \in N$.

2.2 Auxiliary results

In this part, we prove some auxiliary results which is used in the proof of our main theorem.

Let $p,\ q \in N_0,\ r \in N$ and $p < q$. By the definition of the weight function $w_p(x)$, it is clear that

$$\frac{w_q(x)}{w_p(x)} \leq 2, \quad (w_p(x))^r \leq w_{pr}(x), \quad (w_p(x))^{-r} \leq 2^r (w_{pr}(x))^{-1}, \quad x \in R_0. \tag{2.13}$$

Also, if $p \in N$, then

$$\frac{1}{w_p(t)} \leq 2^p \left(\frac{1}{w_p(x)} + |t-x|^p \right) \qquad t,\ x \in R_0. \tag{2.14}$$

Lemma 2.1. *Let $A \in \Omega$ and $s \in N$. Then there exists a positive constant $M_2(s, A)$ such that*

$$w_s(x) L_{n;0}(|t-x|^s; A; x) \leq M_2(s; A) n^{\frac{-s}{2}}, \quad x \in R_0,\ n \in N. \tag{2.15}$$

Proof. From (2.8), we have

$$\begin{aligned}
L_{n;0}(|t-x|^s; A; x) &= \sum_{k=0}^{\infty} a_{nk}(x) n \int_{I_{nk}} |t-x|^s dt \\
&\leq \sum_{k=0}^{\infty} a_{nk}(x) \left\{ \left| \frac{k+1}{n} - x \right|^s + \left| \frac{k}{n} - x \right|^s \right\} n \int_{I_{nk}} dt \\
&= \sum_{k=0}^{\infty} a_{nk}(x) \left\{ \left| \left(\frac{k}{n} - x \right) + \frac{1}{n} \right|^s + \left| \frac{k}{n} - x \right|^s \right\} \\
&\leq \sum_{k=0}^{\infty} a_{nk}(x) \left\{ \left(\left| \frac{k}{n} - x \right| + \frac{1}{n} \right)^s + \left| \frac{k}{n} - x \right|^s \right\} \\
&\leq \sum_{k=0}^{\infty} a_{nk}(x) \left\{ 2^{(s-1)} \left(\left| \frac{k}{n} - x \right|^s + \left(\frac{1}{n} \right)^s \right) + \left| \frac{k}{n} - x \right|^s \right\} \\
&\leq \sum_{k=0}^{\infty} a_{nk}(x) \left\{ (2^s + 1) \left| \frac{k}{n} - x \right|^s + \left(\frac{2}{n} \right)^s \right\} \\
&= (2^s + 1) \sum_{k=0}^{\infty} a_{nk}(x) \left| \frac{k}{n} - x \right|^s + \left(\frac{2}{n} \right)^s \sum_{k=0}^{\infty} a_{nk}(x).
\end{aligned}$$

By the item (ii) and Hölder's inequality, one gets

$$\begin{aligned}
L_{n;0}(|t-x|^s; A; x) &\leq (2^s + 1) \sum_{k=0}^{\infty} a_{nk}(x) \left| \frac{k}{n} - x \right|^s + \left(\frac{2}{n} \right)^s \\
&= (2^s + 1) \left(\sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{k}{n} - x \right)^{2s} \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} a_{nk}(x) \right)^{\frac{1}{2}} \\
&\quad + \left(\frac{2}{n} \right)^s .
\end{aligned}$$

Then, from (2.4), it follows that

$$L_{n;0}(|t-x|^s;A;x) \leq (2^s+1)\left(T_{n;2s}(x;A)\right)^{\frac{1}{2}} + 2^s n^{-s}$$

and so

$$\begin{aligned} w_s(x)L_{n;0}(|t-x|^s;A;x) &\leq (2^s+1)w_s(x)\left(T_{n;2s}(x;A)\right)^{\frac{1}{2}} + 2^s n^{-s} w_s(x) \\ &= (2^s+1)\sqrt{[w_s(x)]^2\left(T_{n;2s}(x;A)\right)} + 2^s n^{-s} w_s(x). \end{aligned}$$

Since $w_s(x) \leq 1$ and $[w_s(x)]^2 \leq w_{2s}(x)$ we can write

$$w_s(x)L_{n;0}(|t-x|^s;A;x) \leq (2^s+1)\sqrt{w_{2s}(x)\left(T_{n;2s}(x;A)\right)} + 2^s n^{-s}.$$

Now, by means of the inequality (2.3), we obtain

$$\begin{aligned} w_s(x)L_{n;0}(|t-x|^s;A;x) &\leq (2^s+1)\sqrt{M_1(s,A)n^{-s}} + 2^s n^{-s} \\ &\leq (2^s+1)\sqrt{M_1(s;A)}n^{\frac{-s}{2}} + 2^s n^{\frac{-s}{2}} \\ &= n^{\frac{-s}{2}}\left[(2^s+1)\sqrt{M_1(s;A)} + 2^s\right] \\ &= M_2(s;A)n^{\frac{-s}{2}}, \end{aligned}$$

where $M_2(s;A) = (2^s+1)\sqrt{M_1(s;A)} + 2^s$. Thus the proof is completed. □

Lemma 2.2. *Let $A \in \Omega$ and $p,\ s \in N_0$. Then there exists a positive constant $M_3(p,s,A)$ such that*

$$w_{p+s}(x)L_{n;0}\left(\frac{|t-x|^s}{w_p(t)};A;x\right) \leq M_3(s;A)n^{\frac{-s}{2}}, \quad x \in R_0,\ n \in N. \tag{2.16}$$

Proof. For $p=0$ from (2.15) the proof is clear. Now, let $p \in N$ and $s \in N_0$. If we use the inequality (2.14), then we have

$$
\begin{aligned}
L_{n;0}\left(\frac{|t-x|^s}{w_p(t)};A;x\right) &\le L_{n;0}\left(|t-x|^s 2^p\left(\frac{1}{w_p(x)}+|t-x|^p\right);A;x\right)\\
&\le L_{n;0}\left(\frac{2^p}{w_p(x)}|t-x|^s+2^p|t-x|^{s+p};A;x\right)\\
&=\frac{2^p}{w_p(x)}L_{n;0}\left(|t-x|^s;A;x\right)+2^pL_{n;0}(|t-x|^{s+p};A;x).
\end{aligned}
$$

From Lemma 2.1, it follows that

$$
\begin{aligned}
w_{p+s}(x)L_{n;0}\left(\frac{|t-x|^s}{w_p(t)};A;x\right) &\le 2^p\frac{w_{p+s}(x)}{w_p(x)}L_{n;0}(|t-x|^s;A;x)\\
&\quad+2^pw_{p+s}(x)L_{n;0}(|t-x|^{s+p};A;x)\\
&=2^p\frac{w_{p+s}(x)}{w_p(x)w_s(x)}w_s(x)L_{n;0}(|t-x|^s;A;x)\\
&\quad+2^pw_{p+s}(x)L_{n;0}(|t-x|^{s+p};A;x)\\
&=2^p\frac{w_{p+s}(x)}{w_p(x)w_s(x)}M_2(s;A)n^{\frac{-s}{2}}+2^pM_2(s;A)n^{\frac{-(p+s)}{2}}.
\end{aligned}
$$

By (2.13), one gets

$$
\begin{aligned}
w_{p+s}(x)L_{n;0}\left(\frac{|t-x|^s}{w_p(t)};A;x\right) &\le 2^{p+1}M_2(s;A)n^{\frac{-s}{2}}+2^pM_2(s;A)n^{\frac{-s}{2}}\\
&=n^{\frac{-s}{2}}[2^{p+1}M_2(s;A)+2^pM_2(s;A)]\\
&=M_3(s;A)n^{\frac{-s}{2}},
\end{aligned}
$$

where $M_3(s;A)=2^{p+1}M_2(s;A)+2^pM_2(s;A)$. This completes the proof. $\square$

Lemma 2.3. *Let $A\in\Omega$, p, $s\in N_0$ and $q\in N$. Then there exists a positive constant $M_4\equiv M_4(p,q,s,A)$ such that*

$$
w_{(p+s)q}(x)L_{n;0}\left(\left(\frac{|t-x|^s}{w_p(t)}\right)^q;A;x\right)\le M_4(p,q,s,A)n^{\frac{-sq}{2}},\quad x\in R_0,\ n\in N. \tag{2.17}
$$

Proof. By using (2.14), we get

$$\begin{aligned}
&L_{n;0}\left(\left(\frac{|t-x|^s}{w_p(t)}\right)^q;A;x\right)\\
&\quad\le L_{n;0}\left(\left\{|t-x|^s 2^p\left(\frac{1}{w_p(x)}+|t-x|^p\right)\right\}^q;A;x\right)\\
&\quad\le L_{n;0}\left(\left\{2^p\frac{|t-x|^s}{w_p(x)}+2^p|t-x|^{p+s}\right\}^q;A;x\right)\\
&\quad\le L_{n;0}\left(2^{q-1}\left\{\left(2^p\frac{|t-x|^s}{w_p(x)}\right)^q+(2^p|t-x|^{p+s})^q\right\};A;x\right)\\
&\quad\le L_{n;0}\left(\left\{\frac{2^{q(1+p)}}{(w_p(x))^q}|t-x|^{sq}+2^{q(1+p)}|t-x|^{(p+s)q}\right\};A;x\right)\\
&\quad=\frac{2^{q(1+p)}}{(w_p(x))^q}L_{n;0}(|t-x|^{sq};A;x)+2^{q(1+p)}L_{n;0}(|t-x|^{(p+s)q};A;x).
\end{aligned}$$

Multiplying both sides of the above inequality by $w_{(p+s)q}(x)$, we have

$$\begin{aligned}
w_{(p+s)q}(x)L_{n;0}&\left(\left(\frac{|t-x|^s}{w_p(t)}\right)^q;A;x\right)\\
&\le w_{(p+s)q}(x)\frac{2^{q(1+p)}}{(w_p(x))^q}L_{n;0}(|t-x|^{sq};A;x)\\
&\quad+2^{q(1+p)}w_{(p+s)q}(x)L_{n;0}(|t-x|^{(p+s)q};A;x)\\
&=\frac{w_{(p+s)q}(x)}{w_{sq}(x)}\frac{2^{q(1+p)}}{(w_p(x))^q}w_{sq}(x)L_{n;0}(|t-x|^{sq};A;x)\\
&\quad+2^{q(1+p)}w_{(p+s)q}(x)L_{n;0}(|t-x|^{(p+s)q};A;x)
\end{aligned}$$

Now, by Lemma 2.1 and (2.13), we find that

$$\begin{aligned}
w_{(p+s)q}(x)L_{n;0}&\left(\left(\frac{|t-x|^s}{w_p(t)}\right)^q;A;x\right)\\
&\le 2^{1+q(1+p)}\frac{2^q}{w_{pq}(x)}M_2 n^{\frac{-sq}{2}}+2^{q(1+p)}M_2 n^{\frac{-(s+p)q}{2}}\\
&\le 2^{2q+qp+1}M_2 n^{\frac{-sq}{2}}+2^{q(1+p)}M_2 n^{\frac{-sq}{2}}\\
&=(2^{2q+qp+1}+2^{q(1+p)})M_2 n^{\frac{-sq}{2}}\\
&=M_4 n^{\frac{-sq}{2}},
\end{aligned}$$

where $M_4=(2^{2q+qp+1}+2^{q(1+p)})M_2$. $\square$

Lemma 2.4. *Let $k \in N_0$ and $n \in N$ be fixed numbers. Then for every function h continuous on $I_{nk} := \left[\frac{k}{n}, \frac{k+1}{n}\right]$ and $q > 1$ we have*

$$\left| n \int_{I_{nk}} h(t)dt \right|^q \leq n \int_{I_{nk}} |h(t)|^q dt \tag{2.18}$$

Proof. By using Hölder's inequality, we easily obtain

$$\begin{aligned}
\left| n \int_{I_{nk}} h(t)dt \right|^q &= \left| \int_{I_{nk}} nh(t)dt \right|^q \\
&\leq \left| \left(\int_{I_{nk}} n(h(t))^q dt \right)^{\frac{1}{q}} \left(\int_{I_{nk}} n^{\frac{(q-1)p}{q}} dt \right)^{\frac{1}{p}} \right|^q \\
&= \left| \int_{I_{nk}} n(h(t))^q dt \right| \\
&\leq n \int_{I_{nk}} |h(t)|^q dt,
\end{aligned}$$

which completes the proof. □

Lemma 2.5. *Let $A \in \Omega$ and $p \in N_0$. Then there exists a positive constant $M_5(p, A)$ such that*

$$\left|\left| L_{n;0}\left(\frac{1}{w_p(t)}; A; . \right) \right|\right|_p \leq M_5(p, A), \quad n \in N \tag{2.19}$$

and

$$||L_{n;0}(f; A; .)||_p \leq M_5(p, A)||f||_p, \quad n \in N, \tag{2.20}$$

for all $f \in C_p$.

Proof. It is clear that for $s = 0$ Lemma 2.2 gives the inequality (2.19). Now, by (2.8) and (2.1) one gets

$$\begin{aligned}
|L_{n;0}(f; A; x)| &= \left| \sum_{k=0}^{\infty} a_{nk}(x) n \int_{I_{nk}} f(t)dt \right| \\
&\leq \sum_{k=0}^{\infty} a_{nk}(x) n \int_{I_{nk}} |f(t)|dt
\end{aligned}$$

$$
\begin{aligned}
&= ||f||_p \sum_{k=0}^{\infty} a_{nk}(x) n \int_{I_{nk}} \frac{1}{w_p(t)} dt \\
&= ||f||_p L_{n;0}\left(\frac{1}{w_p(t)}; A; x\right).
\end{aligned}
$$

Then, multiplication of both sides the final inequality by $w_p(x)$ leads to

$$
w_p(x)|L_{n;0}(f;A;x)| \leq ||f||_p w_p(x) L_{n;0}\left(\frac{1}{w_p(t)}; A; x\right),
$$

and thus

$$
\begin{aligned}
||L_{n;0}(f;A;x)||_p &\leq ||f||_p \left|\left| L_{n;0}\left(\frac{1}{w_p(t)}; A; x\right)\right|\right| \\
&\leq ||f||_p M_5(p, A).
\end{aligned}
$$

So the proof is completed. □

Lemma 2.6. *Let $A \in \Omega$ and $r \in N$. Then there exists a positive constant $M_6(r, A)$ such that for every $f \in C^r$ and $n \in N$ we have*

$$
\|L_{n;r}(f;A;.)\|_r \leq M_6(r, A) \sum_{j=0}^{r} ||f^{(j)}||_{r-j}. \tag{2.21}
$$

Proof. By (2.1), (2.2), (2.5) and (2.7), we have

$$
\begin{aligned}
|L_{n;r}(f;A;x)| &= \left| \sum_{k=0}^{\infty} a_{nk}(x) n \int_{I_{nk}} F_r(t,x) dt \right| \\
&= \left| \sum_{k=0}^{\infty} a_{nk}(x) n \int_{I_{nk}} \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!} (x-t)^j dt \right| \\
&\leq \sum_{k=0}^{\infty} a_{nk}(x) \sum_{j=0}^{r} \frac{n}{j!} \int_{I_{nk}} \mid f^{(j)}(t) \mid |x-t|^j dt \\
&\leq \sum_{k=0}^{\infty} a_{nk}(x) \sum_{j=0}^{r} \frac{n}{j!} \int_{I_{nk}} ||f^{(j)}(t)||_{r-j} \frac{|x-t|^j}{w_{r-j}(t)} dt
\end{aligned}
$$

$$\leq \sum_{j=0}^{r} \frac{1}{j!} ||f^{(j)}(t)||_{r-j} \sum_{k=0}^{\infty} a_{nk}(x) n \int_{I_{nk}} \frac{|x-t|^j}{w_{r-j}(t)} dt$$
$$= \sum_{j=0}^{r} \frac{1}{j!} ||f^{(j)}(t)||_{r-j} L_{n;0} \left(\frac{|x-t|^j}{w_{r-j}(t)}; A; x \right),$$

and so

$$w_r(x)|L_{n;r}(f;A;x)| \leq \sum_{j=0}^{r} \frac{1}{j!} ||f^{(j)}(t)||_{r-j} w_r(x) L_{n;0} \left(\frac{|x-t|^j}{w_{r-j}(t)}; A; x \right)$$

Using the inequality (2.16), one gets

$$||L_{n;r}(f;A;x)||_r \leq M_6(r,A) \sum_{j=0}^{r} ||f^{(j)}(t)||_{r-j}.$$

□

Lemma 2.7. *Let $A \in \Omega$, $r \in N$ and $q > 0$. Then there exists a positive constant $M_7(q,r,A)$ such that*

$$||H^q_{n,r}(f;A;.)||_r \leq M_7 \sum_{j=0}^{r} ||f^{(j)}||_{r-j}, \quad f \in C^r, for \quad n \in N. \tag{2.22}$$

Proof. If $r = 0$, then by means of (2.1), (2.2), (2.9) and items (i) and (ii) we have,

$$H^q_{n;0}(f;A;x) = \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left| n \int_{I_{nk}} f(t)dt - f(x) \right|^q \right\}^{\frac{1}{q}}$$
$$\leq \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(\left| n \int_{I_{nk}} f(t)dt \right| + |f(x)| \right)^q \right\}^{\frac{1}{q}}$$
$$\leq \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(n \int_{I_{nk}} \frac{w_0(x)|f(t)|}{w_0(x)} dt + \frac{w_0(x)|f(x)|}{w_0(x)} \right)^q \right\}^{\frac{1}{q}}$$
$$= \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(||f||_0 n \int_{I_{nk}} dt + ||f||_0 \right)^q \right\}^{\frac{1}{q}}$$
$$= 2||f||_0.$$

If $r \in N$, then by (2.1), (2.2) and (2.7), we get

$$
\begin{aligned}
\left| n\int_{I_{nk}} F_r(t;x)dt - f(x)\right| &\le \left| n\int_{I_{nk}} \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j \right| + |f(x)| \\
&\le n\sum_{j=0}^{r} \frac{1}{j!}||f^{(j)}||_{r-j}\int_{I_{nk}} \frac{|x-t|^j}{w_{r-j}(t)}dt + \frac{||f||_r}{w_r(x)}.
\end{aligned} \tag{2.23}
$$

Now, using (2.6), (2.23), Minkowski's inequality and properties (i), (ii) of A, we obtain

$$
\begin{aligned}
H^q_{n;r}(f;A;x) &\le \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(n\sum_{j=0}^{r} \frac{1}{j!}||f^{(j)}||_{r-j}\int_{I_{nk}} \frac{|x-t|^j}{w_{r-j}(t)}dt + \frac{||f||_r}{w_r(x)} \right)^q \right\}^{\frac{1}{q}} \\
&\le \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(n\sum_{j=0}^{r} \frac{1}{j!}||f^{(j)}||_{r-j}\int_{I_{nk}} \frac{|x-t|^j}{w_{r-j}(t)}dt \right)^q \right\}^{\frac{1}{q}} \\
&\quad + \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{||f||_r}{w_r(x)} \right)^q \right\}^{\frac{1}{q}} \\
&\le \sum_{j=0}^{r} ||f^{(j)}||_{r-j} \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(n\int_{I_{nk}} \frac{|x-t|^j}{w_{r-j}(t)}dt \right)^q \right\}^{\frac{1}{q}} + \frac{||f||_r}{w_r(x)}
\end{aligned}
$$

which gives

$$
\begin{aligned}
&w_r(x)H^q_{n;r}(f;A;x) \\
&\qquad \le \sum_{j=0}^{r} ||f^{(j)}||_{r-j} w_r(x) \left(\sum_{k=0}^{\infty} a_{nk}(x) \left(n\int_{I_{nk}} \frac{|x-t|^j}{w_{r-j}(t)}dt \right)^q \right)^{\frac{1}{q}} + ||f||_r \\
&\qquad \le \sum_{j=0}^{r} ||f^{(j)}||_{r-j} w_r(x) \left(\sum_{k=0}^{\infty} a_{nk}(x) n\int_{I_{nk}} \frac{|x-t|^{jq}}{w_{(r-j)q}(t)}dt \right)^{\frac{1}{q}} + ||f||_r.
\end{aligned}
$$

Then equations (2.8) and (2.13) leads to

$$
\begin{aligned}
w_r(x)H^q_{n;r}(f;A;x) &\le \sum_{j=0}^{r} ||f^{(j)}||_{r-j} w_r(x) \left\{ L_{n;0}\left(\frac{|x-t|^{jq}}{w_{(r-j)q}(t)};A;x \right) \right\}^{\frac{1}{q}} + ||f||_r \\
&\le 4\sum_{j=0}^{r} ||f^{(j)}||_{r-j} \left\{ w_{rq}(x) L_{n;0}\left(\frac{|x-t|^{jq}}{w_{(r-j)q}(t)};A;x \right) \right\}^{\frac{1}{q}} + ||f||_r.
\end{aligned}
$$

Using Lemma 2.3 we reach to desired result.

Let $[q]$ denote the integral part of q if $0 < q \notin N$, then $[q]+1$ belongs to N and $q < [q]+1$. This reality and the inequality (2.12) shows that

$$||H^q_{n;r}(f;A;.)||_r \leq ||H^{[q]+1}_{n;r}(f;A;.)||_r, \quad n \in N$$

and by (2.22) with power $[q]+1$ we find (2.22) for $0 < q \notin N$. Thus the proof is completed. □

2.3 Main theorem

Theorem 2.1. *Let $A \in \Omega$, $r \in N_0$ and $q > 0$. Then there exists a positive constant $M_8 = M_8(q,r,A)$ such that for every $f \in C^r$ and $n \in N$ we have*

$$||H^q_{n;r}(f;A;.)||_{r+1} \leq M_8 n^{\frac{-r}{2}} \omega(f^{(r)}; n^{\frac{-1}{2}}) \tag{2.24}$$

where ω usual modulus of continuity of f^r.

Proof. a)We assume that $r=0$ and $q \in N$. For $f \in C_0$ from (2.9) we have

$$H^q_{n;0}(f;A;x) \leq \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(n \int_{I_{nk}} |f(t)-f(x)| dt \right)^q \right\}^{\frac{1}{q}}.$$

On the other hand, we can write

$$|f(t)-f(x)| \leq \omega(f,|t-x|) \leq (\sqrt{n}|t-x|+1)\omega\left(f;\frac{1}{\sqrt{n}}\right)$$

for $x,\ t \in R_0$ and $n \in N$. Thus, we obtain that

$$\begin{aligned} H^q_{n;0}(f;A;x) &\leq \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(n \int_{I_{nk}} (\sqrt{n}|t-x|+1)\omega\left(f;\frac{1}{\sqrt{n}}\right) dt \right)^q \right\}^{\frac{1}{q}} \\ &\leq \omega\left(f;\frac{1}{\sqrt{n}}\right) \left\{ \sum_{k=0}^{\infty} a_{nk}(x) \left(n \int_{I_{nk}} (\sqrt{n}|t-x|+1) dt \right)^q \right\}^{\frac{1}{q}} \end{aligned}$$

$$= \omega\left(f; \frac{1}{\sqrt{n}}\right)\left\{\sum_{k=0}^{\infty} a_{nk}(x)\left(n^{\frac{3}{2}}\int_{I_{nk}} |t-x|dt + 1\right)^q\right\}^{\frac{1}{q}}.$$

Use of Minkowski's inequality, equations (2.8) and (2.10), and Lemma 2.4, yields

$$H_{n;0}^q(f;A;x) \leq \omega\left(f; \frac{1}{\sqrt{n}}\right)\left\{\left(\sqrt{n}\sum_{k=0}^{\infty} a_{nk}(x) n\int_{I_{nk}} |t-x|^q dt\right)^{\frac{1}{q}} + 1\right\}$$
$$\leq \omega\left(f; \frac{1}{\sqrt{n}}\right)\left\{\sqrt{n}(L_{n;0}(|t-x|^q; A; x))^{\frac{1}{q}} + 1\right\}$$

and so

$$w_1(x)H_{n;0}^q(f;A;x) \leq \omega\left(f; \frac{1}{\sqrt{n}}\right)(\sqrt{n}w_1(x)(L_{n;0}(|t-x|^q; A; x))^{\frac{1}{q}} + w_1(x)).$$

Then, by (2.1) and Lemma 2.1, we obtain

$$w_1(x)H_{n;0}^q(f;A;x) \leq \omega\left(f; \frac{1}{\sqrt{n}}\right)(\sqrt{n}(w_q(x)L_{n;0}(|t-x|^q; A; x))^{\frac{1}{q}} + 1)$$
$$= \omega\left(f; \frac{1}{\sqrt{n}}\right)M_9(q, A).$$

This gives (2.24) for $r = 0$ and $q \in N$.

b)Let $r \in N$ and $q \in N$. We consider modified Taylor formula for $f \in C^r$ at a point $t \in R_0$ (see [7] and [11]):

$$f(x) = \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^r}{(r-1)!}Z_r(x,t), \quad x \in R_0, \tag{2.25}$$

where

$$Z_r(x,t) := \int_0^1 (1-u)^{r-1}(f^{(r)}(t+u(x-t)) - f^{(r)}(t))du.$$

Equations (2.6), (2.7) and (2.25) imply that

$$H_{n;r}^{q}(f;A;x)=\left\{\sum_{k=0}^{\infty}a_{nk}(x)\left|n\int_{I_{nk}}\frac{(x-t)^{r}}{(r-1)!}Z_{r}(x,t)dt\right|^{q}\right\}^{\frac{1}{q}}. \tag{2.26}$$

Since $f^{(r)}\in C_0$, using the modulus of continuity we get

$$\begin{aligned}|Z_r(x,t)| &\le \int_0^1 (1-u)^{r-1}|f^{(r)}\left(t+u(x-t)\right)-f^{(r)}(t)|du\\ &=\int_0^1 (1-u)^{r-1}\omega(f^{(r)};u|x-t|)du\\ &\le \omega\left(f^{(r)};|x-t|\right)\int_0^1(1-u)^{r-1}du\\ &\le \frac{1}{r}\left(\sqrt{n}|t-x|+1\right)\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right).\end{aligned} \tag{2.27}$$

By means of the equations (2.26), (2.27) and Minkowski's inequality and Lemma 2.4, we may write

$$\begin{aligned}&H_{n;r}^{q}(f;A;x)\\ &\quad\le \frac{1}{r!}\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right)\left\{\sum_{k=0}^{\infty}a_{nk}(x)\left(n\int_{I_{nk}}|t-x|^{r}(1+\sqrt{n}|t-x|)dt\right)^{q}\right\}^{\frac{1}{q}}\\ &\quad=\frac{1}{r!}\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right)\left\{\sum_{k=0}^{\infty}a_{nk}(x)\left(n\int_{I_{nk}}(|t-x|^{r}+\sqrt{n}|t-x|^{r+1})dt\right)^{q}\right\}^{\frac{1}{q}}\\ &\quad\le \frac{1}{r!}\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right)\left\{\left(\sum_{k=0}^{\infty}a_{nk}(x)\left(n\int_{I_{nk}}|t-x|^{r}dt\right)^{q}\right)^{\frac{1}{q}}\right.\\ &\qquad\left.+\left(\sum_{k=0}^{\infty}a_{nk}(x)\left(n^{\frac{3}{2}}\int_{I_{nk}}(|t-x|^{r+1}dt)\right)^{q}\right)^{\frac{1}{q}}\right\}\\ &\quad=\frac{1}{r!}\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right)\left((L_{n;0}(|t-x|^{rq};A;x))^{\frac{1}{q}}+\sqrt{n}(L_{n;0}(|t-x|^{(r+1)q};A;x))^{\frac{1}{q}}\right).\end{aligned}$$

Now, in terms of (2.1), (2.13) and Lemma 2.1, we have

$$\begin{aligned}w_{r+1}(x)\left(L_{n;0}(|t-x|^{rq};A;x)\right)^{\frac{1}{q}} &\le \left(w_{rq}(x)L_{n;0}(|t-x|^{rq};A;x)\right)^{\frac{1}{q}}\\ &\le M_{10}(q,r,A)n^{\frac{-r}{2}}\end{aligned}$$

and

$$w_{r+1}(x)\left(L_{n;0}(|t-x|^{(r+1)q};A;x)\right)^{\frac{1}{q}} \leq M_{11}(q,r,A)n^{\frac{-(r+1)}{2}}.$$

Using the above inequality, we easily find (2.24) for $n \in N$ and $q \in N$.

c)Let $r \in N_0$ and $0 < q \notin N$. Also let $q < [q]+1$. By (2.12), (2.2) and (2.24) we can write

$$||H^{q}_{n;r}(f;A;.)||_{r+1} \leq ||H^{[q]+1}_{n;r}(f;A;.)||_{r+1} \leq M_8 n^{\frac{-r}{2}} \omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right)$$

for every $f \in C^r$ and $n \in N$. Therefore, the proof is completed. □

Corollary 2.1. *We assume that A, r and q satisfy assumption of Theorem 2.1. If $f \in C^r$ and $f^{(r)}$ is uniformly continuous on R_0, then*

$$\lim_{n\to\infty} n^{\frac{r}{2}}||H^{q}_{n;r}(f;A;.)||_{r+1} = 0$$

which implies that

$$\lim_{n\to\infty} n^{\frac{r}{2}} H^{q}_{n;r}(f;A;x) = 0 \quad \text{at every} \quad x \in R_0.$$

Corollary 2.2. *Let $A \in \Omega$ and $r \in N_0$. Then there exists a positive constant $M_{12}(r,A)$ such that for every $f \in C^r$ and $n \in N$ we have*

$$\| L_{n;r}(f;A;.) - f(.) \|_{r+1} \leq \| H^{1}_{n;r}(f;A;.) \|_{r+1} \leq M_{12}(r,A)n^{\frac{-r}{2}}\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right).$$

Moreover if $f^{(r)}$ is uniformly continuous on R_0, then

$$\lim_{n\to\infty} n^{\frac{r}{2}}\|L_{n;r}(f;A;x) - f(x)\|_{r+1} = 0.$$

From this follows

$$\lim_{n\to\infty} n^{\frac{r}{2}}(L_{n;r}(f;A;x)-f(x))=0$$

at every $x \in R_0$.

Remark 2.1. *Theorem 2.1. and corollaries are also valid for the Baskakov-Kantorovich operators*

$$\tilde{V}_{n;r}(f;x) := \sum_{k=0}^{\infty} \binom{n-1+k}{k} x^k (1-x)^{n-k} n \int_{I_{nk}} F_r(t,x)dt$$

of functions $f \in C^r$, $r \in N_0$ *(see[24]).*

CHAPTER 3

A MODIFICATION OF GENERALIZED BASKAKOV -KANTOROVICH OPERATORS

3.1 Introduction

In the identity

$$(1-t)^{-x}e^{at} = \sum_{k=0}^{\infty} P_k(x,a)\frac{t^k}{k!},$$

where $a \geq 0$ is any constant and

$$P_k(x,a) = \sum_{i=0}^{k} \binom{k}{i}(x)_i a^{k-i}$$

with $(x)_0 = 1$, $(x)_i = x(x+1)\cdots(x+i-1)$ for $i \geq 1$ by setting $x = n$ and $t = \frac{x}{1+x}$, Miheşan [16] constructed the generalized Baskakov operators

$$B_n^a(f;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0, n \in \mathbb{N}$$

for every $f \in C[0,\infty)$. He showed that these operators converge uniformly on $[0,b]$ for functions having exponential growth on positive x-axis and obtained the order of approximation with the help of the usual modulus of continuity. After that these operators and their integral modifications were studied in [5], [6] and [25]-[30]. We now deal with only the works which are necessary for this thesis. In [29], by proposing integral type modification of the operators B_n^a in the sense of Kantorovich as follows:

$$V_n^a(f;x) = ne^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \quad x \geq 0, n \in \mathbb{N}$$

Wafi and Khatoon proved a Voronovskaya type theorem in polynomial weight spaces for these operators. In 2010, Erençin and Başcanbaz-Tunca [5] presented the following generalization of the operators $B_n^a(f;x)$

$$L_n(f;x) = e^{-\frac{a_n x}{1+x}} \sum_{k=0}^{\infty} f\left(\frac{k}{b_n}\right) \frac{P_k(n,a_n)}{k!} \frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0, n \in \mathbb{N}, \tag{3.1}$$

where (a_n) are (b_n) are two sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{n}{b_n} = 1, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0,$$

and investigated their weighted approximation properties and also introduced an application to functional differential equations which gives a recurrence relation for the monomials of that operators.

Very recently, Altomare, Montano and Leonessa [1] presented the modification of Szasz-Mirakyan-Kantorovich operators defined by

$$C_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left[\frac{n}{b_n - a_n} \int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)dt\right], \quad x \geq 0, n \in \mathbb{N},$$

where (a_n) and (b_n) are sequences of real numbers such that $0 \leq a_n < b_n \leq 1$. They studied some approximation properties of these operators on continuous function spaces, weighted continuous function spaces and Lebesgue spaces and also obtained some estimates for the rate of convergence.

Inspired by that work, we consider the generalization of the operators V_n^a as follows:

$$K_n(f;x) = \sum_{k=0}^{\infty} S_{n,a_n}(k,x) \frac{b_n}{d_n - c_n} \int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}} f(t)dt, \quad x \geq 0, n \in \mathbb{N}, \tag{3.2}$$

where

$$S_{n,a_n}(k,x) = e^{-\frac{a_n x}{1+x}} \frac{P_k(n,a_n)}{k!} \frac{x^k}{(1+x)^{n+k}}$$

and (a_n), (b_n), (c_n) and (d_n) are sequences having the properties:

(i) $a_n \geq 0, \quad b_n \geq 1, \quad 0 \leq c_n < d_n \leq 1$

(ii) $\lim_{n\to\infty} \frac{n}{b_n} = 1, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$

In this chapter, we first give some direct results. Next, we prove a weighted Korovkin type theorem and compute the order of approximation with the help of the weighted modulus of continuity for these operators.

3.2 Auxiliary results

By [5], we have

$$L_n(1;x) = 1 \tag{3.3}$$

$$L_n(t;x) = \frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} \tag{3.4}$$

$$L_n(t^2;x) = \frac{n(n+1)}{b_n^2}x^2 + \frac{2a_n n}{b_n^2}\frac{x^2}{1+x} + \frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2} + \frac{n}{b_n^2}x + \frac{a_n}{b_n^2}\frac{x}{1+x}, \tag{3.5}$$

where $L_n(f;x)$ is defined by (3.1).

In the sequel, we shall need to following lemmas.

Lemma 3.1. *The following equalities hold:*

$$\begin{aligned} L_n(t^3;x) =& \frac{n(n+1)(n+2)}{b_n^3}x^3 + \frac{3a_n n(n+1)}{b_n^3}\frac{x^3}{1+x} + \frac{3a_n^2 n}{b_n^3}\frac{x^3}{(1+x)^2} + \frac{a_n^3}{b_n^3}\frac{x^3}{(1+x)^3} \\ &+ \frac{3n(n+1)}{b_n^3}x^2 + \frac{6a_n n}{b_n^3}\frac{x^2}{1+x} + \frac{3a_n^2}{b_n^3}\frac{x^2}{(1+x)^2} + \frac{n}{b_n^3}x + \frac{a_n}{b_n^3}\frac{x}{1+x}, \end{aligned}$$

and

$$\begin{aligned} L_n(t^4;x) =& \frac{n(n+1)(n+2)(n+3)}{b_n^n}x^4 + \frac{4a_n n(n+1)(n+2)}{b_n^4}\frac{x^4}{1+x} \\ &+ \frac{6a_n^2 n(n+1)}{b_n^4}\frac{x^4}{(1+x)^2} + \frac{4a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3} + \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} \\ &+ \frac{6n(n+1)(n+2)}{b_n^4}x^3 + \frac{18a_n n(n+1)}{b_n^4}\frac{x^3}{1+x} + \frac{18a_n^2 n}{b_n^4}\frac{x^3}{(1+x)^2} \end{aligned}$$

$$+\frac{6a_n^3}{b_n^4}\frac{x^3}{(1+x)^3}+\frac{7n(n+1)}{b_n^4}x^2+\frac{14a_n n}{b_n^4}\frac{x^2}{1+x}+\frac{7a_n^2}{b_n^4}\frac{x^2}{(1+x)^2}$$
$$+\frac{n}{b_n^4}x+\frac{a_n}{b_n^4}\frac{x}{1+x}.$$

It can be proved in a similar way that of the proof of Lemma 2.1 in [16] or by using the recurrence relation given in [5].

Lemma 3.2. *For the operators $K_n(f;x)$ defined by (3.2), we have*

$$\begin{aligned}
K_n(1;x) =\ & 1,\\
K_n(t;x) =\ & \frac{n}{b_n}x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}\\
K_n(t^2;x) =\ & \frac{n(n+1)}{b_n^2}x^2+\frac{2a_n n}{b_n^2}\frac{x^2}{1+x}+\frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2}+\frac{nm_1(n)}{b_n^2}x\\
& +\frac{a_n m_1(n)}{b_n^2}\frac{x}{1+x}+\frac{m_2(n)}{3b_n^2},\\
K_n(t^3;x) =\ & \frac{n(n+1)(n+2)}{b_n^3}x^3+\frac{3a_n n(n+1)}{b_n^3}\frac{x^3}{1+x}+\frac{3a_n^2 n}{b_n^3}\frac{x^3}{(1+x)^2}\\
& +\frac{a_n^3}{b_n^3}\frac{x^3}{(1+x)^3}+\frac{3n(n+1)m_3(n)}{2b_n^3}x^2+\frac{3a_n n m_3(n)}{b_n^3}\frac{x^2}{1+x}\\
& +\frac{3a_n^2 m_3(n)}{2b_n^3}\frac{x^2}{(1+x)^2}+\frac{nm_4(n)}{2b_n^3}x+\frac{a_n m_4(n)}{2b_n^3}\frac{x}{1+x}+\frac{m_5(n)}{4b_n^3},
\end{aligned}$$

and

$$\begin{aligned}
K_n(t^4;x) =& \frac{n(n+1)(n+2)(n+3)}{b_n^4}x^4+\frac{4a_n n(n+1)(n+2)}{b_n^4}\frac{x^4}{1+x}\\
& +\frac{6a_n^2 n(n+1)}{b_n^4}\frac{x^4}{(1+x)^2}+\frac{4a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3}+\frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4}\\
& +\frac{2n(n+1)(n+2)m_6(n)}{b_n^4}x^3+\frac{6a_n n(n+1)m_6(n)}{b_n^4}\frac{x^3}{1+x}\\
& +\frac{6a_n^2 n m_6(n)}{b_n^4}\frac{x^3}{(1+x)^2}+\frac{2a_n^3 m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3}+\frac{n(n+1)m_7(n)}{b_n^4}x^2\\
& +\frac{2a_n n m_7(n)}{b_n^4}\frac{x^2}{1+x}+\frac{a_n^2 m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2}\\
& +\frac{nm_8(n)}{b_n^4}x+\frac{a_n m_8(n)}{b_n^4}\frac{x}{1+x}+\frac{m_9(n)}{5b_n^4},
\end{aligned}$$

where $m_0(n)=c_n+d_n$, $m_1(n)=c_n+d_n+1$, $m_2(n)=c_n^2+c_nd_n+d_n^2$, $m_3(n)=c_n+d_n+2$, $m_4(n)=2(c_n^2+c_nd_n+d_n^2)+3(c_n+d_n)+2$, $m_5(n)=$

$c_n^3+c_n^2d_n+c_nd_n^2+d_n^3$, $\quad m_6(n)=c_n+d_n+3$, $\quad m_7(n)=2(c_n^2+c_nd_n+d_n^2)+6(c_n+d_n)+7$, $\quad m_8(n)=c_n^3+c_n^2d_n+c_nd_n^2+d_n^3+2(c_n^2+c_nd_n+d_n^2)+2(c_n+d_n)+1$ *and* $m_9(n)=c_n^4+c_n^3d_n+c_n^2d_n^2+c_nd_n^3+d_n^4$.

By using the definition of K_n, the equalities (3.3)-(3.5) and Lemma 3.1, it can be proved easily. So, we omit them.

Now in terms of the linearity of the operators K_n and Lemma 3.2 we can state the following lemma.

Lemma 3.3. *For the operators $K_n(f;x)$ defined by (3.2), we have*

$$\begin{aligned} K_n((t-x)^2;x) = {} & \left(\frac{n(n+1)}{b_n^2}-\frac{2n}{b_n}+1\right)x^2+\frac{2a_n}{b_n}\left(\frac{n}{b_n}-1\right)\frac{x^2}{1+x}+\frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2} \\ & +\left(\frac{nm_1(n)}{b_n^2}-\frac{m_0(n)}{b_n}\right)x+\frac{a_nm_1(n)}{b_n^2}\frac{x}{1+x}+\frac{m_2(n)}{3b_n^2} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & K_n((t-x)^4;x) \\ & = \left(\frac{n(n+1)(n+2)(n+3)}{b_n^4}-\frac{4n(n+1)(n+2)}{b_n^3}+\frac{6n(n+1)}{b_n^2}-\frac{4n}{b_n}+1\right)x^4 \\ & +\frac{4a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3}-\frac{3n(n+1)}{b_n^2}+\frac{3n}{b_n}-1\right)\frac{x^4}{1+x}+\frac{6a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2}\right. \\ & \left.-\frac{2n}{b_n}+1\right)\frac{x^4}{(1+x)^2}+\frac{4a_n^3}{b_n^3}\left(\frac{n}{b_n}-1\right)\frac{x^4}{(1+x)^3}+\frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} \\ & +2\left(\frac{n(n+1)(n+2)m_6(n)}{b_n^4}-\frac{3n(n+1)m_3(n)}{b_n^3}+\frac{3nm_1(n)}{b_n^2}-\frac{m_0(n)}{b_n}\right)x^3 \\ & +\frac{6a_n}{b_n}\left(\frac{n(n+1)m_6(n)}{b_n^3}-\frac{2nm_3(n)}{b_n^2}+\frac{m_1(n)}{b_n}\right)\frac{x^3}{1+x}+\frac{6a_n^2}{b_n^2}\left(\frac{nm_6(n)}{b_n^2}\right. \\ & \left.-\frac{m_3(n)}{b_n}\right)\frac{x^3}{(1+x)^2}+\frac{2a_n^3m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3}+\left(\frac{n(n+1)m_7(n)}{b_n^4}-\frac{2nm_4(n)}{b_n^3}\right. \\ & \left.+\frac{2m_2(n)}{b_n^2}\right)x^2+\frac{2a_n}{b_n}\left(\frac{nm_7(n)}{b_n^3}-\frac{m_4(n)}{b_n^2}\right)\frac{x^2}{1+x}+\frac{a_n^2m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2} \\ & +\left(\frac{nm_8(n)}{b_n^4}-\frac{m_5(n)}{b_n^3}\right)x+\frac{a_nm_8(n)}{b_n^4}\frac{x}{1+x}+\frac{m_9(n)}{5b_n^4}, \end{aligned} \tag{3.7}$$

where $m_0(n)$,$m_1(n)$, $m_2(n)$, $m_3(n)$, $m_4(n)$,$m_5(n)$, $m_6(n)$, $m_7(n)$, $m_8(n)$ and $m_9(n)$ are given as in Lemma 3.2.

Lemma 3.4. *For the operators $K_n(f;x)$ defined by (3.2), we have*

$$K_n((t-x)^4;x) \leq 12m_7(n)A(n)(x^4+x^3+x^2+x+1),$$

where $m_7(n)$ given as in Lemma 3.2 and $A(n) = \max\{A_1(n), A_2(n)\}$ with

$$\begin{aligned} A_1(n) =& \left|\frac{n(n+1)(n+2)(n+3)}{b_n^4} - \frac{4n(n+1)(n+2)}{b_n^3} + \frac{6n(n+1)}{b_n^2} - \frac{4n}{b_n} + 1\right| \\ &+ \frac{a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3} + \frac{n}{b_n}\right) + \frac{a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2} + 1\right) + \frac{a_n^3}{b_n^3}\left(\frac{n}{b_n} + \frac{a_n}{b_n}\right), \end{aligned}$$

$$A_2(n) = \frac{n(n+1)(n+2)}{b_n^4} + \frac{n}{b_n^2} + \frac{a_n}{b_n^2}\left(\frac{n(n+1)}{b_n^2} + 1\right) + \frac{a_n^2}{b_n^3}\left(\frac{n}{b_n} + \frac{a_n}{b_n}\right).$$

Proof. From (3.7) we may write

$$\begin{aligned} &K_n((t-x)^4;x) \\ \leq& \left(\frac{n(n+1)(n+2)(n+3)}{b_n^4} - \frac{4n(n+1)(n+2)}{b_n^3} + \frac{6n(n+1)}{b_n^2} - \frac{4n}{b_n} + 1\right)x^4 \\ &+ \frac{4a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3} + \frac{3n}{b_n}\right)\frac{x^4}{1+x} + \frac{6a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2} + 1\right)\frac{x^4}{(1+x)^2} \\ &+ \frac{4a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3} + \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} + 2\left(\frac{n(n+1)(n+2)m_6(n)}{b_n^4} + \frac{3nm_1(n)}{b_n^2}\right)x^3 \\ &+ \frac{6a_n}{b_n}\left(\frac{n(n+1)m_6(n)}{b_n^3} + \frac{m_1(n)}{b_n}\right)\frac{x^3}{1+x} + \frac{a_n^2 nm_6(n)}{b_n^4}\frac{x^3}{(1+x)^2} \\ &+ \frac{2a_n^3 m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3} + \left(\frac{n(n+1)m_7(n)}{b_n^4} + \frac{2m_2(n)}{b_n^2}\right)x^2 + \frac{2a_n nm_7(n)}{b_n^4}\frac{x^2}{1+x} \\ &+ \frac{a_n^2 m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2} + \frac{nm_8(n)}{b_n^4}x + \frac{a_n m_8(n)}{b_n^4}\frac{x}{1+x} + \frac{m_9(n)}{5b_n^4} \\ \leq& 12\Bigg\{\left|\frac{n(n+1)(n+2)(n+3)}{b_n^4} - \frac{4n(n+1)(n+2)}{b_n^3} + \frac{6n(n+1)}{b_n^2} - \frac{4n}{b_n} + 1\right|x^4 \\ &+ \frac{a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3} + \frac{n}{b_n}\right)\frac{x^4}{1+x} + \frac{a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2} + 1\right)\frac{x^4}{(1+x)^2} \\ &+ \frac{a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3} + \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} + \left(\frac{n(n+1)(n+2)m_6(n)}{b_n^4} + \frac{nm_1(n)}{b_n^2}\right)x^3 \end{aligned}$$

$$
\begin{aligned}
&+\frac{a_n}{b_n}\left(\frac{n(n+1)m_6(n)}{b_n^3}+\frac{m_1(n)}{b_n}\right)\frac{x^3}{1+x}+\frac{a_n^2 n m_6(n)}{b_n^4}\frac{x^3}{(1+x)^2}\\
&+\frac{a_n^3 m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3}+\left(\frac{n(n+1)m_7(n)}{b_n^4}+\frac{m_2(n)}{b_n^2}\right)x^2+\frac{a_n n m_7(n)}{b_n^4}\frac{x^2}{1+x}\\
&+\frac{a_n^2 m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2}+\frac{n m_8(n)}{b_n^4}x+\frac{a_n m_8(n)}{b_n^4}\frac{x}{1+x}+\frac{m_9(n)}{b_n^4}\Bigg\}.
\end{aligned}
$$

Since $\dfrac{x^s}{(1+x)^l}\leq x^s$ for all $x\geq 0,\ l\leq s\ (l,s=1,2,3,4)$ and $m_1(n),m_2(n),m_6(n),$ $m_8(n),m_9(n)<m_7(n)$ for all $n\in\mathbb{N}$ one gets

$$
K_n((t-x)^4;x)\leq 12\Big\{A_1(n)x^4+m_7(n)\left[A_2(n)x^3+A_3(n)x^2+A_4(n)x+A_5(n)\right]\Big\},
$$

where

$$
A_3(n)=\frac{n(n+1)}{b_n^4}+\frac{1}{b_n^2}+\frac{a_n}{b_n^3}\left(\frac{n}{b_n}+\frac{a_n}{b_n}\right),A_4(n)=\frac{1}{b_n^3}\left(\frac{n}{b_n}+\frac{a_n}{b_n}\right),A_5(n)=\frac{1}{b_n^4}.
$$

Finally, since $A_5(n)\leq A_4(n)<A_3(n)<A_2(n)$ for all $n\in\mathbb{N}$ we can write

$$
\begin{aligned}
K_n((t-x)^4;x)&\leq 12\Big\{A_1(n)x^4+m_7(n)A_2(n)\left(x^3+x^2+x+1\right)\Big\}\\
&\leq 12m_7(n)\Big\{A_1(n)x^4+A_2(n)\left(x^3+x^2+x+1\right)\Big\}
\end{aligned}
$$

which gives the desired result. □

3.3 Direct results

Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions f on the interval $[0,\infty)$, endowed with the norm

$$
\|f\|=\sup_{0\leq x<\infty}|f(x)|.
$$

For any $\delta>0$, Peetre's K-functional is defined by

$$
K_2(f;\delta)=\inf_{g\in C_B^2[0,\infty)}\left\{\|f-g\|+\delta\,\|g''\|\right\},
$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By DeVore and Lorentz ([4], p.177, Theorem 2.4) there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{3.8}$$

where the second order modulus of smoothness $\omega_2(f;\delta)$ of $f \in C_B[0,\infty)$ and usual modulus of continuity ω of $f \in C_B[0,\infty)$ is defined as in Chapter 1. Now consider the following operator

$$\widehat{K}_n(f;x) = K_n(f;x) - f\left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) + f(x),$$

where $m_0(n)$ given as Lemma 3.2.

Lemma 3.5. *Let $g \in C_B^2[0,\infty)$. Then we have*

$$\left|\widehat{K}_n(g;x) - g(x)\right| \leq \delta_n(x)\|g''\|,$$

where

$$\delta_n(x) = K_n((t-x)^2;x) + \left[\left(\frac{n}{b_n} - 1\right)x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right]^2.$$

Proof. By the definition of the operators $\widehat{K}_n$ and Lemma 3.2 we get

$$\begin{aligned}
\widehat{K}_n(t-x;x) &= K_n(t-x,x) - \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - x\right) \\
&= K_n(t,x) - xK_n(1,x) - \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - x\right) \\
&= 0.
\end{aligned}$$

Let $g \in C_B^2[0,\infty)$ and $x \in [0,\infty)$. By Taylor's formula of g

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du, \quad t \in [0,\infty)$$

one may write

$$
\begin{aligned}
\widehat{K}_n(g;x) - g(x) =& g'(x)\widehat{K}_n(t-x,x) + \widehat{K}_n\left(\int_x^t (t-u)g''(u)du;x\right)\\
=& \widehat{K}_n\left(\int_x^t (t-u)g''(u)du;x\right)\\
=& K_n\left(\int_x^t (t-u)g''(u)du;x\right)\\
& - \int_x^{\frac{n}{b_n}x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}} \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - u\right) g''(u)du.
\end{aligned}
$$

Now using the following inequalities

$$
\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2 \|g''\|
$$

and

$$
\begin{aligned}
&\left|\int_x^{\frac{n}{b_n}x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}} \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - u\right) g''(u)du\right|\\
&\leq \left[\left(\frac{n}{b_n}-1\right)x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right]^2 \|g''\|
\end{aligned}
$$

we can write

$$
\begin{aligned}
&\left|\widehat{K}_n(g;x) - g(x)\right|\\
&\leq \left\{K_n((t-x)^2;x) + \left[\left(\frac{n}{b_n}-1\right)x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right]^2\right\} \|g''\|\\
&= \delta_n(x)\|g''\|.
\end{aligned}
$$

□

Theorem 3.1. *Let $f \in C_B[0,\infty)$. Then for all $x \in [0,\infty)$ there exists a constant $A > 0$ such that*

$$
|K_n(f;x) - f(x)| \leq A\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{n}{b_n}-1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right),
$$

where $\delta_n(x)$ defined as in Lemma 3.5.

Proof. By means of the definitions of the operators $\widehat{K}_n$ and K_n we have

$$\begin{aligned}|K_n(f;x)-f(x)| \leq& \left|\widehat{K}_n(f-g;x)\right| + |(f-g)(x)| + \left|\widehat{K}_n(g;x)-g(x)\right| \\ &+ \left|f\left(\frac{n}{b_n}x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}\right)-f(x)\right|\end{aligned}$$

and

$$\left|\widehat{K}_n(f;x)\right| \leq |K_n(f;x)| + 2\|f\| \leq \|f\|K_n(1;x) + 2\|f\| = 3\|f\|.$$

Thus we may conclude that

$$\begin{aligned}|K_n(f;x)-f(x)| \leq& 4\|f-g\| + \left|\widehat{K}_n(g;x)-g(x)\right| \\ &+ \left|f\left(\frac{n}{b_n}x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}\right)-f(x)\right|.\end{aligned}$$

In the light of Lemma 3.5 one gets

$$\begin{aligned}|K_n(f;x)-f(x)| \leq& 4\|f-g\| + \delta_n(x)\|g''\| \\ &+ \omega\left(f;\left|\frac{n}{b_n}-1\right|x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}\right).\end{aligned}$$

Therefore taking the infimum over all $g \in C_B^2[0,\infty)$ on the right-hand side of the last inequality and considering (3.8), we reach to

$$\begin{aligned}|K_n(f;x)-f(x)| &\leq 4K_2(f;\delta_n(x)) + \omega\left(f;\left|\frac{n}{b_n}-1\right|x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}\right) \\ &\leq 4C\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{n}{b_n}-1\right|x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}\right) \\ &= A\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{n}{b_n}-1\right|x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}\right)\end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let $0<\gamma\leq 1$ and $f \in C_B[0,\infty)$. Then if $f \in Lip_M(\gamma)$, that is, the inequality*

$$|f(t)-f(x)| \leq M\,|t-x|^{\gamma}\,, \quad x,t \in [0,\infty)$$

holds, then for each $x \in [0,\infty)$ we have

$$|K_n(f;x) - f(x)| \leq \delta_n^{\frac{\gamma}{2}}(x),$$

where $\delta_n(x) = K_n((t-x)^2;x)$ and $M > 0$ is a constant.

Proof. Let $f \in C_B[0,\infty) \cap Lip_M(\gamma)$. By the linearity and monotonicity of the operators K_n we get

$$\begin{aligned}
|K_n(f;x) - f(x)| &\leq K_n\left(|f(t) - f(x)|\,;x\right) \\
&\leq MK_n\left(|t-x|^{\gamma}\,;x\right) \\
&= M\sum_{k=0}^{\infty} S_{n,a_n}(k,x)\frac{b_n}{d_n - c_n}\int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}} |t-x|^{\gamma}\,dt.
\end{aligned}$$

Now applying the Hölder inequality two times successively with $p = \frac{2}{\gamma}$, $q = \frac{2}{2-\gamma}$, we obtain

$$\begin{aligned}
|K_n(f;x) - f(x)| &\leq M\sum_{k=0}^{\infty} S_{n,a_n}(k,x)\left\{\frac{b_n}{d_n - c_n}\int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}} (t-x)^2 dt\right\}^{\frac{\gamma}{2}} \\
&\leq MK_n\left((t-x)^2;x\right)^{\frac{\gamma}{2}} \\
&= M\delta_n^{\frac{\gamma}{2}}(x).
\end{aligned}$$

This completes the proof. □

3.4 Weighted approximation properties

Now, we introduce convergence properties of the operators K_n via Theorem 1.3 mentioned in Chapter 1.

Theorem 3.3. *Let $\{K_n\}$ be the sequence of linear positive operators defined by (3.2). Then for each $f \in C_\rho^0[0,\infty)$, we have*

$$\lim_{n\to\infty} ||K_n(f;x) - f(x)||_\rho = 0,$$

where $\rho(x) = 1 + x^2$.

Proof. Using Lemma 3.2, we may write

$$\sup_{x\geq 0} \frac{|K_n(\rho; x)|}{1+x^2} \leq 1 + \frac{n(n+1)}{b_n^2} + \frac{2a_n n}{b_n^2} + \frac{a_n^2}{b_n^2} + \frac{nm_1(n)}{b_n^2} + \frac{a_n m_1(n)}{b_n^2} + \frac{m_2(n)}{3b_n^2}$$

Under the conditions (i) and (ii), there exists a positive constant M^* such that

$$\frac{n(n+1)}{b_n^2} + \frac{2a_n n}{b_n^2} + \frac{a_n^2}{b_n^2} + \frac{nm_1(n)}{b_n^2} + \frac{a_n m_1(n)}{b_n^2} + \frac{m_2(n)}{3b_n^2} < M^*$$

for each n. Hence we get

$$||K_n(\rho; x)||_\rho \leq 1 + M^*$$

which shows that $\{K_n\}$ is a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$.

In order to complete the proof, it is enough to prove that the conditions of Theorem 1.3

$$\lim_{n\to\infty} ||K_n(t^\nu; x) - x^\nu||_\rho = 0, \quad \nu = 0, 1, 2$$

are satisfied. It is clear that

$$\lim_{n\to\infty} ||K_n(1; x) - 1||_\rho = 0.$$

By Lemma 3.2, we have

$$\begin{aligned} ||K_n(t; x) - x||_\rho &= \sup_{x\geq 0} \left| \left(\frac{n}{b_n} - 1\right) \frac{x}{1+x^2} + \frac{a_n}{b_n} \frac{x}{(1+x)(1+x^2)} + \frac{m_0(n)}{2b_n} \frac{1}{1+x^2} \right| \\ &\leq \left| \frac{n}{b_n} - 1 \right| + \frac{a_n}{b_n} + \frac{m_0(n)}{b_n}. \end{aligned}$$

Thus taking into consideration the conditions (i) and (ii) we can conclude that

$$\lim_{n\to\infty} ||K_n(t; x) - x||_\rho = 0.$$

Similarly, one gets

$$\begin{aligned}
||K_n(t^2;x)-x^2||_\rho =& \sup_{x\geq 0}\Bigg| \left(\frac{n(n+1)}{b_n^2}-1\right)\frac{x^2}{1+x^2}+\frac{2a_n n}{b_n^2}\frac{x^2}{(1+x)(1+x^2)} \\
&+\frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2(1+x^2)}+\frac{nm_1(n)}{b_n^2}\frac{x}{1+x^2} \\
&+\frac{a_n m_1(n)}{b_n^2}\frac{x}{(1+x)(1+x^2)}+\frac{m_2(n)}{3b_n^2}\frac{1}{(1+x^2)}\Bigg| \\
&\leq \left|\frac{n(n+1)}{b_n^2}-1\right|+\frac{2a_n n}{b_n^2}+\frac{a_n^2}{b_n^2}+\frac{nm_1(n)}{b_n^2}+\frac{a_n m_1(n)}{b_n^2}+\frac{m_2(n)}{b_n^2}
\end{aligned}$$

which leads to

$$\lim_{n\to\infty}||K_n(t^2;x)-x^2||_\rho = 0.$$

Thus the proof is completed. □

Now we compute the order of approximation of the operators K_n in terms of the weighted modulus of continuity $\Omega_2(f,\delta)$ defined in Chapter 1

$$\Omega_2(f,\delta)=\sup_{x\geq 0, 0<h\leq\delta} = \frac{|f(x+h)-f(x)|}{1+(x+h)^2}, \quad f\in C_\rho^0[0,\infty).$$

Theorem 3.4. *Let $\{K_n\}$ be the sequence of linear positive operators defined by (3.2). Then for each $f\in C_\rho^0[0,\infty)$, we have*

$$\sup_{x\geq 0}\frac{|K_n(f;x)-f(x)|}{(1+x^2)^3}\leq C\Omega_2\left(f,[m_7(n)A(n)]^{\frac{1}{4}}\right),$$

where C is positive constant and $m_7(n)$ and $A(n)$ defined as in Lemma 3.2 and Lemma 3.4, respectively.

Proof. For $x\geq 0$ and $t\geq 0$, by the definition of $\Omega_2(f,\delta)$ and the property (3), we may write

$$\begin{aligned}
|f(t)-f(x)| &\leq \left(1+(x+|t-x|)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_n) \\
&\leq 2(1+x^2)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_n).
\end{aligned}$$

By using the monotonicity of K_n and the following inequality (see [10])

$$\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right) \leq 2(1+\delta_n^2)\left(1+\frac{(t-x)^4}{\delta_n^4}\right)$$

one gets

$$\begin{aligned}
|K_n(f;x)-f(x)| &\leq 2(1+x^2)K_n\left(\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right);x\right)\Omega_2(f,\delta_n) \\
&\leq 4(1+\delta_n^2)(1+x^2)K_n\left(1+\frac{(t-x)^4}{\delta_n^4};x\right)\Omega_2(f,\delta_n) \\
&= 4(1+\delta_n^2)(1+x^2)\left[1+\frac{1}{\delta_n^4}K_n((t-x)^4;x)\right]\Omega_2(f,\delta_n) \\
&\leq C_1(1+x^2)\left[1+\frac{1}{\delta_n^4}K_n((t-x)^4;x)\right]\Omega_2(f,\delta_n).
\end{aligned}$$

With the help of the Lemma 3.4 this inequality leads to

$$|K_n(f;x)-f(x)| \leq 12C_1(1+x^2)\left[1+\frac{m_7(n)A(n)}{\delta_n^4}(x^4+x^3+x^2+x+1)\right]\Omega_2(f,\delta_n)$$

which gives the required result. □

We observe that in Theorem 3.3 we have showed that K_n converges to f in the weighted space $C\rho[0,\infty)$. But in Theorem 3.4 we have computed the rate of convergence for these operators in the weighted space $C\rho^3[0,\infty)$.

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