ALGEBRAIC FUNCTIONS IN TERMS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

(GENELLEŞTİRİLMİŞ HİPERGEOMETRİK FONKSİYONLAR CİNSİNDEN CEBİRSEL FONKSİYONLAR)

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List of Symbols

$\Gamma(z)$	: Gamma function (Chapter 1, Section 1)
$B(x,y)$	: Beta function (Chapter 1, Section 2)
${}_2F_1$	: Gauss hypergeometric function (Chapter 2, Section 1)
$(x)_n$	: Pochhammer symbol (Chapter 2, Section 1)
$\pi_1(S,z_0)$	: Fundamental group of topological space S with base point z_0 (Chapter 2, Section 3)
${}_nF_{n-1}$	: Pochhammer hypergeometric function (Chapter 3, Section 1)
$GL(n,\mathbb{C})$	: General linear group of $n \times n$ invertible matrices over the field of complex numbers (Chapter 3, Section 1)
$Y_{psx}^{(i)}(X)$	: Puiseux expansion of the i^{th} branch (Chapter 4, Section 3)
$f(X,Y)$	:Polynomial that give rise to algebraic functions (Chapter 4, Section 1)
$\Delta(f)$	: Carrier of f (Chapter 4, Section 1)
$\tilde{Y}(X)$	: First Mellin inversion solution (Chapter 4, Section 3)
$\tilde{\tilde{Y}}(X)$	: Second Mellin inversion solution (Chapter 4, Section 3)

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Abstract

The aim of this thesis is to describe the solutions of a given general trinomial algebraic equation satisfying some special conditions by means of generalized hypergeometric functions.
Including the introduction, this dissertation consists of five chapters. In the first chapter, the gamma function and the beta function are introduced. Some basic properties of the gamma function and related theorems are reviewed.

In the second chapter, Gauss hypergeometric function(as a model case) is studied in detail. Besides, its differential equation, two different integral representations and monodromy study of Gauss hypergeometric differential equation are given.

In the third chapter, generalized hypergeometric(or Pochhammer) function, which is used as our tool to obtain our results in the fourth chapter and its differential equation are introduced. Moreover, monodromy study of generalized hypergeometric function is briefly stated.

In the fourth chapter, the Newton-Puiseux algorithm which is used to find solutions of an algebraic equation in fractional power series around the origin is explained. However, even though this algorithm enables us to obtain all solutions of an algebraic equation, when it comes to deal with algebraic equations of higher degree, application of the Newton-Puiseux algorithm becomes a hard task due to higher order terms. In the second section of chapter four, the Mellin transformation of a solution of a general trinomial algebraic equation is described. Further, the relation between such a transformation and gamma function is studied. By the aid of this relation solutions of a general trinomial algebraic equation that arise from the Mellin inversion formula are reviewed. In the last section of chapter four, we applied this argument and the Newton-Puiseux algorithm to a given general triniomial algebraic equation satisfying some special conditions and compared the results. Moreover, we obtained hypergeometric type differential equation which is satisfied by the solutions of a given trinomial algebraic equation mentioned above. These applications are supplemented with well-studied examples. In the final chapter, the conclusion of this dissertation is given.

Keywords: Gamma function, beta function, Gauss hypergeometric function, monodromy, generalized hypergeometric functions, algebraic functions, algebraic equations, Newton-Puiseux algorithm.

Özet

Bu tezin amacı, bazı özel koşulları sağlayan genel üç terimli cebirsel denklemlerin çözümlerini hipergeometrik fonskiyonlar aracılığıyla tavsir etmektir.
Bu tez, giriş kısmı ile birlikte beş üniteden oluşmaktadır. İlk ünitede, gamma ve beta fonksiyonları tanıtılmıştır. Gamma ve beta fonskiyonalrının bazı özellikleri ve bu fonksiyonlar ile ilişkili teoremler ele alınmıştır.

İkinci ünitede, Gauss hipergeometrik fonksiyonu(model olarak) ayrıntılı bir şekilde çalışılmıştır. Ayrıca, diferansyiel denklemi, iki farklı integral temsili ve monodromi çalışması verilmiştir.

Üçüncü ünitede, dördüncü ünitedeki sonuçlarımızı elde etmemize aracı olan genelleştirilmiş hipergeometrik fonksiyon(Pochhammer) fonksiyon ve diferansiyel denklemi tanıtılmıştır. Buna ek olarak, genelleştirilmiş hipergeometrik fonksiyonun monodromi çalışması kısaca belirtilmiştir.

Dördüncü ünitede, bir cebirsel denklemin orijin civarındaki kesirli kuvvet serisini bulmak için kullanılan Newton-Puiseux algoritması açıklanmıştır. Fakat bu algoritma, bir cebirsel denklemin tüm çözümlerini elde etmemize olanak tanısa da, yüksek dereceli cebirsel denklemlerle ilgilenildiğinde Newton-Pusieux algoritmasının uygulanması yüksek dereceli terimler yüzünden zor bir hal almaktadır. Dördüncü ünitenin ikinci kısmında, genel üç terimli bir cebirsel denklemin çözümlerinin Mellin dönüşümü tasvir edilmiştir. Dahası, böyle bir dönüşümle gamma fonksiyonu arasındaki ilişki çalışılmıştır. Bu ilişki yardımı ile ters Mellin formülünden ortaya çıkan genel üç terimli cebirsel denklemin çözümleri incelenmiştir. Dördüncü ünitenin son kısmında, bu argümanı ve Newton-Puiseux algoritmasını verilen genel üç terimli özel koşulları sağlayan cebirsel denkleme uyguladık ve elde ettiğimiz sonuçları karşılaştırdık. Buna ek olarak, yukarıda bahsi geçen cebirsel denklemin çözümleri tarafından sağlanan hipergeometrik tip diferansiyel denklemi elde ettik. Bu uygulamalar iyi çalışılmış örneklerle desteklenmiştir. Son ünitede bu tezin sonuç kısmı verilmiştir.

Anahtar Sözcükler: Gamma fonksiyonu, beta fonksiyonu, Gauss hipergeometrik fonskiyonu, monodromi, genelleştirilmiş hipergeometrik fonksiyonlar, cebirsel fonksiyonlar, cebirsel denklemler, Newton-Puiseux algoritması.

Chapter 1

Introduction

The gamma function was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non-integer values. Euler at that time stayed together with D.Bernoulli (1700-1782) in St.Petersburg gave a similar representation of this function. But then, Euler did much more. He gave further representations by integrals and formulated theorems on the properties of this function. It was also studied by other eminent mathematicians like Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901). The notation Γ was introduced by Adrien Marie Legendre (1752-1833). The gamma function appears in various area in mathematics such as asymptotic series, definite integration, hypergeometric series, Riemann-zeta function. As our topic suggests we will be interested mostly in the case of hypergeometric series and thus hypergeometric functions.

In Chapter 2, we study the Gauss hypergeometric function and its differential equation. In fact, this hypergeometric function was first introduced by Euler as a power series expansion of the form

$$1+\frac{\alpha\beta}{\gamma.1}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)1.2}z^2+\cdots$$

with $\alpha,\beta,\gamma\in\mathbb{Q}$. During his studies Euler obtained many classical functions by using the rational parameters α,β,γ. For instance taking $\beta=\gamma=1$, gives generalized binomial series $(1-z)^{-\alpha}$. Also, taking $\alpha=1,\beta=\gamma$, gives the classical geometric series. Euler found the hypergeometric differential equation as well, which is a second order linear differential equation. Years later, Gauss studied solutions of this hypergeometric differential equation in the entire complex plane. And he discovered multivaluedness of these solutions(Monodromy problem). This problem was studied extensively by famous mathematician Bernhand Riemann(1826-1866). For Gauss's contributions, today Euler's hypergeometric series is referred to as Gauss hypergeometric function.

Towards the end of nineteenth century, mathematicians had come up with the idea of generalization of the Gauss hypergeometric function. In Chapter 3, we consider generalized hypergeometric function or often called Pochhammer(1841-1920) hypergeometric function with $2n$ parameters in one variable.

In Chapter 4, firstly we review the Newton-Puiseux algorithm to find the solutions of an algebraic equation in fractional power series around the origin. Apart form this, in considering the Mellin inversion formula for a solution of a general trinomial algebraic equation satisfying some special conditions led us to hypergeometric series. On the other hand, we used the Newton-Puiseux algorithm for the same general trinomial algebraic function and compared the results.

1.1 The Gamma Function

In this chapter, we state definitions and basic properties of two special functions namely, the gamma function and the beta function.

Definition 1.1.1. *The gamma function is defined by the integral:*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{1.1.1}$$

where $z \in \mathbb{C}$ *and* $\Re(z) > 0$.

The above integral makes sense as far as it is convergent. First, let us discuss the convergence of the integral

$$\int_0^\infty t^{z-1} e^{-t} dt$$

We split up the integral as

$$\int_0^\infty t^{z-1} e^{-t} dt = \int_0^1 t^{z-1} e^{-t} dt + \int_1^\infty t^{z-1} e^{-t} dt$$

Notice that, $0 < t^{z-1} e^{-t} \leq t^{z-1}$ for all positive values of t. Thus,

$$\begin{aligned}
\int_0^1 t^{z-1} e^{-t} dt &\leq \lim_{\varepsilon \to 0} \int_\varepsilon^1 t^{z-1} e^{-t} dt \\
&\leq \lim_{\varepsilon \to 0} \int_\varepsilon^1 t^{z-1} dt \\
&= \lim_{\varepsilon \to 0} \left(\frac{t^z}{z} \right) \Bigg|_{t=\varepsilon}^{t=1} = \frac{1}{z} - \frac{\varepsilon^z}{z}
\end{aligned}$$

We see that the limit exist for $\Re(z) > 0$.
For the second integral

$$\int_1^\infty t^{z-1} e^{-t} dt$$

We observe that

$$t^{z-1} e^{-t} = \frac{t^{z-1}}{e^t} = \frac{t^{z-1}}{\sum\limits_{k=0}^{\infty} \frac{t^k}{k!}} \leq \frac{t^{z-1}}{\frac{t^n}{n!}} = \frac{n!}{t^{n-(z-1)}}$$

Hence

$$\begin{aligned}\int_1^\infty t^{z-1}e^{-t}dt &= \lim_{\delta\to\infty}\int_1^\delta t^{z-1}e^{-t}dt\\ &\leq \lim_{\delta\to\infty}\int_1^\delta \frac{n!}{t^{n-(z-1)}}dt\\ &= \lim_{\delta\to\infty}\left(\frac{n!t^{z-n}}{z-n}\right)\Bigg|_{t=1}^{t=\delta}\\ &= \lim_{\delta\to\infty}\frac{n!}{z-n}(\frac{1}{\delta^{n-z}}-1)\end{aligned}$$

Since we can choose arbitrary large δ and n this limit exists. This implies that the integral $\int_1^\infty t^{z-1}e^{-t}dt$, is convergent.
Consequently, the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

is convergent for $\Re(z) > 0$.

Theorem 1.1.1.

$$\Gamma(z+1) = z\Gamma(z) \quad (\Re(z) > 0.)$$

Proof. From the definition of gamma function (1.1.1) we have

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z dt$$

and using integration by parts we obtain

$$\int_0^\infty e^{-t}t^z dt = \left(\frac{-t^z}{e^t}\right)\Bigg|_{t=0}^{t=\infty} + \int_0^\infty e^{-t}zt^{z-1}dt = 0 + z\int_0^\infty e^{-t}t^{z-1}dt = z\Gamma(z)$$

Thus,

$$\Gamma(z+1) = z\Gamma(z).$$

□

Corollary 1.

$$\Gamma(1) = 1$$

Proof. Putting $z = 1$, in the definition of the gamma function we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}dt = \left(-\frac{1}{e^t}\right)\Bigg|_{t=0}^{t=\infty} = 1$$

$\square$

Theorem 1.1.2. *If z is a non-negative integer, then we have*

$$\Gamma(z+1) = z!$$

Proof. We have

$$\begin{aligned}
\Gamma(z+1) &= z\Gamma(z) \\
&= z(z-1)\Gamma(z-1) \\
&= z(z-1)(z-2)\Gamma(z-2) \\
&= z(z-1)(z-2)\ldots 3.2.1.\Gamma(1) \\
&= z!\Gamma(1) = z!
\end{aligned}$$

$\square$

1.1.1 Poles And Residues of The Gamma Function

In theorem (1.1.1) we proved that

$$\Gamma(z+1) = z\Gamma(z)$$

Thus, we can write

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1)$$

The right-hand side of the latter equation is defined for those values of z such that for $z+1 > 0$, by definition of the gamma function.(that is, $z > -1$) We also note that $\Gamma(0) = \infty$.
Hence

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1) \to \infty \quad as \quad z \to 0$$

The right-hand side is defined for $z+1 > 0$ and the left-hand side is defined for $z > 0$. We can use this process to extend the definition of the gamma function as follows: We have defined the gamma function $\Gamma(z)$ for $z > -1$ so that the right-hand side of the equation is defined for $z+1 > -1$ that is, $z > -2$ hence, we may use this to define left-hand side for $z > -2$ and this process can be continued to define gamma function for all negative values of z.

Theorem 1.1.3. $\Gamma(z)$ *has simple poles at* $z=-n$*, where* $n=0,1,2,\ldots$

Proof. We have already noted that $\Gamma(0)=\infty$. By (1.1.1) we have

$$\Gamma(-1)=\frac{1}{-1}\Gamma(0)=\infty$$

$$\Gamma(-2)=\frac{1}{-2}\Gamma(-1)=\infty$$

and etc. $\square$

Now, let us compute the residues of the gamma function
We first note that

$$\operatorname{Res}(\Gamma(z),z=0)=\lim_{z\to 0}z\Gamma(z)=\lim_{z\to 0}\Gamma(z+1)=\Gamma(1)=1$$

Consider,

$$\operatorname{Res}(\Gamma(z),z=-n)=\lim_{z\to -n}(z+n)\Gamma(z)$$

Since

$$\begin{aligned}\Gamma(z+n+1)&=(z+n)\Gamma(z+n)\\&=(z+n)(z+n-1)\Gamma(z+n-1)\\&=(z+n)(z+n-1)(z+n-2)\ldots(z+1)z\Gamma(z).\end{aligned}$$

We write

$$\begin{aligned}\operatorname{Res}(\Gamma(z),z=-n)&=\lim_{z\to -n}(z+n)\frac{\Gamma(z+n+1)}{(z+n)(z+n-1)(z+n-2)\ldots(z+1)z}\\&=\frac{\Gamma(1)}{(-1)(-2)\ldots(-n+1)(-n)}\\&=\frac{(-1)^n}{n!}\end{aligned}$$

Therefore, we conclude that the gamma function $\Gamma(z)$ has simple poles at $z=-n$, where $n=0,1,2,\ldots$ with residue

$$\operatorname{Res}(\Gamma(z),z=-n)=\frac{(-1)^n}{n!}.$$

In future chapters of this thesis we will use the following theorem.

Theorem 1.1.4. *(Euler's reflection formula) For all non-integer $z \in \mathbb{C}$*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Proof. Omitted here. The reader can consult [1, p.26] □

There is another important formula for the gamma function which is called Stirling's formula

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} \tag{1.1.2}$$

which is valid as $|z| \to \infty$, where the notation $f(z) \sim g(z)$ means that

$$\lim_{|z|\to\infty} \frac{f(z)}{g(z)} = 1.$$

The above formula (1.1.2) implies

$$\log\Gamma(z) = z\log z - \frac{1}{2}\log z - z + O(1)$$

For a fixed $s > 0$ and for $t > 0$, we let $z = s + it$, then

$$\log\Gamma(s+it) = (s+it)\log(s+it) - \frac{1}{2}\log(s+it) - (s+it) + O(1) \tag{1.1.3}$$

Since

$$\log(s+it) \sim \log(it) = \log|t| + i\frac{\pi}{2}$$

Rewriting the equation (1.1.3) and taking the real part yields

$$\begin{aligned} \Re(\log\Gamma(s+it)) &= s\log|t| - \frac{\pi}{2}t - \frac{1}{2}\log|t| - s + O(1) \\ &= -\frac{\pi}{2}t - s + (s - \frac{1}{2})\log|t| + O(1) \end{aligned}$$

which implies

$$|\Gamma(s+it)| \sim \sqrt{2\pi} e^{-\frac{\pi}{2}t - s}|t|^{s-\frac{1}{2}}$$

as $t \to \infty$.

Similarly, as $t \to -\infty$ we obtain

$$|\Gamma(s+it)| \sim \sqrt{2\pi} e^{\frac{\pi}{2}t-s}|t|^{s-\frac{1}{2}}$$

Thus, the Stirling's formula for $\Gamma(z)$ can also be given as

$$|\Gamma(s+it)| \sim \sqrt{2\pi} e^{-\frac{\pi}{2}|t|-s}|t|^{s-\frac{1}{2}} \tag{1.1.4}$$

as $|t| \to \infty$.
The formula which is given in (1.1.4) will be needed to show the convergence of the integral in (4.2.6).

1.1.2 The Beta Function

Definition 1.1.2. *The beta function is given by the integral :*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{1.1.5}$$

for $\Re x > 0$ *and* $\Re y > 0$.

The next theorem establishes a connection between the beta function and the gamma function.

Theorem 1.1.5.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Proof. Now that $B(x,y)$ is a function of two variables, we are able to apply the transformation

$$\iint f(x,y)dxdy = \iint f(r\cos(\theta), r\sin(\theta))|J|drd\theta$$

where

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

is a Jacobian matrix.

Consider

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^\infty u^{x-1}e^{-u}du \int_0^\infty v^{y-1}e^{-v}dv \\ &= \int_0^\infty \int_0^\infty v^{y-1}u^{x-1}e^{-(u+v)}dudv \end{aligned}$$

Set $u = r\cos^2(\theta), v = r\sin^2(\theta)$, then

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= \int_0^\infty v^{y-1}u^{x-1}e^{-(u+v)}dudv \\
&= \int_{r=0}^\infty \int_{\theta=0}^{\frac{\pi}{2}} [r\sin^2(\theta)]^{y-1}[r\cos^2(\theta)]^{x-1}e^{-r}2r\sin(\theta)\cos(\theta)drd\theta \\
&= 2\int_{r=0}^\infty \int_{\theta=0}^{\frac{\pi}{2}} e^{-r}r^{x+y-1}\sin^{2y-1}(\theta)\cos^{2x-1}(\theta)drd\theta \\
&= 2\int_{r=0}^\infty e^{-r}r^{x+y-1}dr \int_{\theta=0}^{\frac{\pi}{2}} \sin^{2y-1}(\theta)\cos^{2x-1}(\theta)d\theta \\
&= 2\Gamma(x+y)\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2y-1}(\theta)\cos^{2x-1}(\theta)d\theta
\end{aligned}$$

Now putting $t = \cos^2(\theta)$ and $dt = -2\cos(\theta)\sin(\theta)d\theta$, we get

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= 2\Gamma(x+y)\int_{\theta=0}^{\frac{\pi}{2}} \cos^{2x-1}(\theta)\sin^{2y-1}(\theta)d\theta \\
&= 2\Gamma(x+y)\int_{t=1}^{0} \frac{\cos^{2x}(\theta)\sin^{2y}(\theta)}{\cos(\theta)\sin(\theta)}\frac{dt}{(-2\cos(\theta)\sin(\theta))} \\
&= \Gamma(x+y)\int_{t=0}^{1} \cos^{2x-2}(\theta)\sin^{2y-2}(\theta)dt \\
&= \Gamma(x+y)\int_{t=0}^{1} t^{x-1}(1-t)^{y-1}dt \\
&= \Gamma(x+y)B(x,y)
\end{aligned}$$

Therefore,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$\square$

Chapter 2

Gauss Differential Equation and The Hypergeometric Function

In this chapter, we shall introduce a famous second-order linear differential equation (Gauss's differential equation) which has regular singularities at the points $z = 0, 1, \infty$ and we shall find it's solutions around those singular points by using Frobenius method(infinite series solution) which will lead us to the hypergeometric function.

2.1 Gauss Hypergeometric Function

Definition 2.1.1. *Let* $\alpha, \beta, \gamma \in \mathbb{C}$ *with* $\gamma \notin \mathbb{Z}^- \cup \{0\}$. *We define the Gauss hypergeometric function by the series*

$$ {}_2F_1(\alpha, \beta, \gamma; z) = F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n (1)_n} z^n $$

where the Pochhammer symbol $(x)_n$ *is defined as :*

$$ (x)_n = \begin{cases} 1 & \text{if} \quad n = 0, \\ x(x+1)\dots(x+n-1) & \text{if} \quad n \geq 1. \end{cases} $$

One can easily show by using *D'Alembert Criteria*(the ratio test) that the hypergeometric series converges(uniformly) for $|z| < 1$. Moreover, if $\Re(\gamma - \alpha - \beta) > 0$, then the series converges absolutely on $|z| \leq 1$.

$F(\alpha, \beta, \gamma; z)$ is a holomorphic function in the unit disk $|z| < 1$.

If α or β is a non-positive integer, then $F(\alpha, \beta, \gamma; z)$ becomes a polynomial in z.

It is obvious from the definition of $F(\alpha,\beta,\gamma;z)$ that the *Gauss* hypergeometric function is symmetric in α and β that is,

$$F(\alpha,\beta,\gamma;z) = F(\beta,\alpha,\gamma;z)$$

We remark that many elementary functions can be represented by the hypergeometric function. Let us look at some examples.

Example 2.1.1. *For* $\alpha = 1$ *and* $\beta = \gamma$*,*

$$F(1,\beta,\beta;z) = 1+z+z^2+\cdots = \frac{1}{1-z}$$

Example 2.1.2. *For* $\beta = \gamma = 1$*,*

$$F(\alpha,1,1;z) = (1-z)^{-\alpha}$$

Example 2.1.3. *For* $\alpha = \frac{1}{2}, \beta = 1, \gamma = \frac{3}{2}$ *and replacing z by* z^2 *we obtain*

$$2zF(\frac{1}{2},1,\frac{3}{2};z^2) = log\frac{1+z}{1-z}$$

Example 2.1.4. *For* $\alpha = \gamma = 1$*,*

$$\lim_{\beta\to\infty} F(1,\beta,1;\frac{z}{\beta}) = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z$$

Sometimes it is useful to represent Pochhammer symbol in the definition of the hypergeometric function in terms of gamma function. The following lemma enables us such a representation.

Lemma 2.1.1. *For* $n \geq 0$*,*

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

Proof. If $n = 0$, then clearly $(x)_0 = \frac{\Gamma(x)}{\Gamma(x)} = 1$. If $n \geq 1$, then we write

$$\begin{aligned}\frac{\Gamma(x+n)}{\Gamma(x)} &= \frac{(x+n-1)\Gamma(x+n-1)}{\Gamma(x)} \\ &= \frac{(x+n-1)(x+n-2)\dots(x+1)x\Gamma(x)}{\Gamma(x)} \\ &= x(x+1)\dots(x+n-2)(x+n-1) \\ &= (x)_n\end{aligned}$$

□

At this point, we will discuss the *Gauss* hypergeometric function in detail. It turns out that the hypergeometric function $F(\alpha, \beta, \gamma; z)$ is a solution of a specific type of second order linear differential equation. Even though this differential equation has singularities at some points, we will see that it admits holomorphic solutions around those singular points. But before moving on, we need some definitions and propositions.
Consider the second order linear differential equation given by

$$y'' + p(z)y' + q(z)y = 0$$

where $p(z)$ and $q(z)$ are rational functions. Near every point z_0 which is not pole of $p(z)$ or $q(z)$ there exists two linearly independent Taylor solutions in $(z - z_0)$. If z_0 is a pole of $p(z)$ or $q(z)$ we call z_0 a singularity.

Definition 2.1.2. *The singular point of the differential equation*

$$y'' + p(z)y' + q(z)y = 0$$

is called regular(or regular singular point) if and only if for every solution $y(z)$ of the differential equation there exists a rational number $t \in \mathbb{Q}$ and $c > 0$ such that

$$|y(z)| \leq c|z - z_0|^t$$

or for all $y(z)$ there exists a real number $s \in \mathbb{R}$ such that

$$y(z) = (z - z_0)^s u(z)$$

where $u(z)$ is a holomorphic function around z_0.

Definition 2.1.3. *The differential equation*

$$y'' + p(z)y' + q(z)y = 0$$

is called Fuchsian differential equation, whenever

$$\lim_{z\to z_0}(z - z_0)p(z) \quad \textit{and} \quad \lim_{z\to z_0}(z - z_0)^2 q(z)$$

exists, and this holds for $p(z), q(z)$, where z_0 is a regular singular point.

In other words, the differential equation

$$y'' + p(z)y' + q(z)y = 0$$

is called Fuchsian if every singular point of this differential equation is regular.
After these definitions we address the following proposition.

Proposition 2.1.1 ([3],p 29.)**.** *A point $a \in \mathbb{C}$ is regular or a regular singularity of the differential equation $y'' + p(z)y' + q(z)y = 0$ if and only if $\lim_{z\to a}(z - a)p(z)$ and $\lim_{z\to a}(z - a)^2 q(z)$ exists. The point ∞ is regular or a regular singularity if and only if $\lim_{z\to\infty} zp(z)$ and $\lim_{z\to\infty} z^2 q(z)$ exists.*

Now, let us consider the so-called Gauss differential equation

$$z(1 - z)y'' + [\gamma - (\alpha + \beta + 1)z]y' - \alpha\beta y = 0 \tag{2.1.1}$$

where α, β, γ are constants.
Writing the above differential equation in the standard form we obtain

$$y'' + p(z)y' + q(z)y = 0$$

where

$$p(z) = \frac{\gamma - (\alpha + \beta + 1)z}{z(1 - z)} \quad \textit{and} \quad q(z) = -\frac{\alpha\beta}{z(1 - z)}$$

Thus, by proposition (2.1.1) we see that this differential equation has three regular singular points at $z = 0, 1, \infty$.
Later we will find the solutions of the differential equation around the regular singular points $z = 0, 1, \infty$.

Now, our task is to show that the Gauss hypergeometric function $F(\alpha,\beta,\gamma;z)$ is indeed a solution of the differential equation

$$z(1-z)y'' + [\gamma - (\alpha+\beta+1)z]y' - \alpha\beta y = 0$$

If we take the differential equation of the form

$$z(1-z)y'' + [\gamma - (\alpha+\beta+1)z]y' - \alpha\beta y = 0$$

showing the hypergeometric function as a solution is a bit more challenging. But fortunately, there is another representation of the Gauss differential equation and we give the following proposition.

Proposition 2.1.2. *Gauss differential equation*

$$z(1-z)y'' + [\gamma - (\alpha+\beta+1)z]y' - \alpha\beta y = 0$$

can be written in the operator form as follows:

$$z(\theta+\alpha)(\theta+\beta)y = \theta(\theta+\gamma-1)y$$

where, $\theta = z\dfrac{d}{dz}$ *is known as Euler's differential operator.*

Proof. First, let us compute the right hand side

$$\theta(\theta+\gamma-1)y = \theta^2 y + \gamma\theta y - \theta y$$

Note that

$$\theta^2 = z\frac{d}{dz}\left(z\frac{d}{dz}\right) = z\left(\frac{d}{dz} + z\frac{d^2}{dz^2}\right)$$

Hence,

$$\theta(\theta+\gamma-1)y = \theta^2 y + \gamma\theta y - \theta y = z^2\frac{d^2y}{dz^2} + \gamma z\frac{dy}{dz}$$

On the other hand, the left hand side gives

$$\begin{aligned} z(\theta+\alpha)(\theta+\beta)y &= z(\theta^2 + \beta\theta + \alpha\theta + \alpha\beta)y \\ &= z\left(z\frac{dy}{dz} + z^2\frac{d^2y}{dz^2} + \beta z\frac{dy}{dz} + \alpha z\frac{dy}{dz} + \alpha\beta y\right) \end{aligned}$$

Thus, subtracting $z(\theta+\alpha)(\theta+\beta)y$ from $\theta(\theta+\gamma-1)y$ gives the required result. $\square$

Proposition 2.1.3. *The Gauss hypergeometric function*

$$ {}_2F_1(\alpha,\beta,\gamma;z) = F(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n} z^n $$

satisfies Gauss differential equation

$$ z(1-z)y'' + [\gamma-(\alpha+\beta+1)z]y' - \alpha\beta y = 0 $$

Proof. We shall use operator form of Gauss differential equation to prove :
$z(\theta+\alpha)(\theta+\beta)F = \theta(\theta+\gamma-1)F$. We have

$$ \begin{aligned} \theta(\theta+\gamma-1)F &= \sum_{n=0}^{\infty} n(n+\gamma-1)\frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n} z^n \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_{n-1}(n-1)!} z^n \\ &= \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_{n-1}(n-1)!} z^n \end{aligned} $$

On the other hand,

$$ \begin{aligned} (\theta+\alpha)(\theta+\beta)F &= \sum_{n=0}^{\infty} (n+\alpha)(n+\beta)\frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n} z^n \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1}(\beta)_{n+1}}{(\gamma)_n n!} z^n \\ &= \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_{n-1}(n-1)!} z^{n-1} \end{aligned} $$

Therefore, $z(\theta+\alpha)(\theta+\beta)F - \theta(\theta+\gamma-1)F = 0$, where $F = {}_2F_1(\alpha,\beta,\gamma;z)$. $\square$

2.1.1 Solutions Around Regular Singular Points

In this section, we shall obtain solutions of the Gauss differential equation around the regular singular points by the aid of *Frobenius* method.

Solutions around the regular singular point $z=0$**:** Suppose that our solutions are of the form

$$ y(z) = \sum_{n=0}^{\infty} a_n z^{n+r} $$

where $r \in \mathbb{C}$.
Note that

$$y' = \sum_{n=0}^{\infty} a_n(n+r)z^{n+r-1} \quad and \quad y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)z^{n+r-2}$$

Plugging these series into the Gauss differential equation (2.1.1), we obtain

$$\begin{aligned} &z(1-z)\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)z^{n+r-2} + [\gamma-(\alpha+\beta+1)z]\sum_{n=0}^{\infty} a_n(n+r)z^{n+r-1} \\ &-\alpha\beta\sum_{n=0}^{\infty} a_n z^{n+r} = 0. \end{aligned} \tag{2.1.2}$$

Expanding out the terms yields

$$\begin{aligned} &\sum_{n=0}^{\infty}\{(n+r)(n+r-1)+\gamma(n+r)\}a_n z^{n+r-1} \\ -&\sum_{n=0}^{\infty}\{(n+r)^2+(\alpha+\beta)(n+r)+\alpha\beta\}a_n z^{n+r} = 0. \end{aligned} \tag{2.1.3}$$

And hence

$$\sum_{n=0}^{\infty}\{(n+r)(n+r-1+\gamma)\}a_n z^{n+r-1} - \sum_{n=0}^{\infty}(n+r+\alpha)(n+r+\beta)a_n z^{n+r} = 0.$$

Whence

$$r(r-1+\gamma)a_0 z^{r-1} + \sum_{n=1}^{\infty}\{(n+r)(n+r-1+\gamma)\}a_n z^{n+r-1} \tag{2.1.4}$$

$$-\sum_{n=1}^{\infty}(n-1+r+\alpha)(n-1+r+\beta)a_{n-1}z^{n+r-1} = 0. \tag{2.1.5}$$

Solving the indicial equation $r(r-1+\gamma) = 0$ ($a_0 \neq 0$, $z \neq 0$), we obtain $r = 0$ and $r = 1-\gamma$. Now, rearranging the series in (2.1.4) and (2.1.5) we obtain

$$\sum_{n=1}^{\infty}\{a_n(n+r)(n-1+r+\gamma) - a_{n-1}(n-1+r+\alpha)(n-1+r+\beta)\}z^{n+r-1} = 0.$$

Since the right hand side is zero and $z \neq 0$, we must have

$$a_n(n+r)(n-1+r+\gamma) - a_{n-1}(n-1+r+\alpha)(n-1+r+\beta) = 0.$$

which gives us the following recurrence relation

$$a_n = \frac{(n-1+r+\alpha)(n-1+r+\beta)}{(n+r)(n-1+r+\gamma)} a_{n-1}$$

for $n \geq 1$. Now, let us put $r = 0$ in the above recurrence relation and compute the first few terms. We have

$$a_n = \frac{(n-1+\alpha)(n-1+\beta)}{n(n-1+\gamma)} a_{n-1}$$

for $n = 1$,

$$a_1 = \frac{\alpha\beta}{\gamma} a_0$$

for $n = 2$,

$$a_2 = \frac{(1+\alpha)(1+\beta)}{2(1+\gamma)} a_1 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)1.2} a_0$$

for $n = 3$,

$$a_3 = \frac{(2+\alpha)(2+\beta)}{3(2+\gamma)} a_2 = \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)1.2.3} a_0$$

Thus, if we continue to compute the terms for those values of n and letting $a_0 = 1$, we obtain

$$a_n = \frac{(n-1+\alpha)(n-1+\beta)}{n(n-1+\gamma)} a_{n-1} = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}. \tag{2.1.6}$$

Likewise, putting $r = 1-\gamma$ in the recurrence relation gives,

$$a_n = \frac{(n-\gamma+\alpha)(n-\gamma+\beta)}{n(n+1-\gamma)} a_{n-1}$$

for $n \geq 1$. Similarly, computing the first few terms from the recurrence relation and letting $a_0 = 1$, one can easily obtain that

$$a_n = \frac{(n-\gamma+\alpha)(n-\gamma+\beta)}{n(n+1-\gamma)} a_{n-1} = \frac{(1-\gamma+\alpha)_n(1-\gamma+\beta)_n}{(2-\gamma)_n n!} \tag{2.1.7}$$

Therefore, we have obtained two solutions(for $r = 0$ and $r = 1-\gamma$) namely,

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

$$y_2(z) = \sum_{n=0}^{\infty} a_n z^{n+(1-\gamma)} = z^{1-\gamma} \sum_{n=0}^{\infty} a_n z^n.$$

Since our solutions were of the form $y(z) = \sum_{n=0}^{\infty} a_n z^{n+r}$ hence, using (2.1.6) and (2.1.7) we can write more explicitly,

$$y_1(z) = F(\alpha, \beta, \gamma; z)$$

$$y_2(z) = z^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta, 2-\gamma; z)$$

Consequently, we have obtained two linearly independent holomorphic solutions around the regular singular point $z = 0$. A general solution near $z = 0$ can be given by

$$y(z) = c_1 F(\alpha, \beta, \gamma; z) + c_2 z^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta, 2-\gamma; z) \tag{2.1.8}$$

where c_1, c_2 are arbitrary constants.

Solutions around the regular singular point $z = 1$:
In order to study the solutions around $z = 1$, we use the change of variables $t = 1 - z$. Now, studying around $z = 1$ corresponds to study around $t = 0$. But, we know how to deal with the case $t = 0$ from the previous case.
We rewrite Gauss differential equation (2.1.1) with respect to t, we also note that in the original differential equation y' denotes $\frac{dy}{dz}$.
We perform chain rule as

$$\frac{dy}{dz} = \frac{dt}{dz}\frac{dy}{dt}$$

and

$$\frac{d^2y}{dz^2} = \frac{d^2t}{dz^2}\frac{dy}{dt} + \left(\frac{dt}{dz}\right)^2 \frac{d^2y}{dt^2}$$

In our case we have

$$\frac{dy}{dz} = -\frac{dy}{dt} \quad \textit{and} \quad \frac{d^2y}{dz^2} = \frac{d^2y}{dt^2}$$

Now, rewriting the differential equation (2.1.1) with respect to t yields

$$t(1-t)\frac{d^2y}{dt^2} + [\alpha+\beta+1-\gamma-(\alpha+\beta+1)t]\frac{dy}{dt} - \alpha\beta y = 0 \tag{2.1.9}$$

Let, $\tilde{\gamma} = \alpha+\beta+1-\gamma$ from the previous case($z = 0$) we obtained two linearly independent holomorphic solutions $F(\alpha,\beta,\gamma;z)$ and $F(1-\gamma+\alpha,1-\gamma+\beta,2-\gamma;z)$. Hence using equation (2.1.8) and replacing γ by $\tilde{\gamma}$ we get the following solutions around $z=1$.

$$y_1(z) = F(\alpha,\beta,\alpha+\beta+1-\gamma;1-z)$$

and

$$y_2(z) = (1-z)^{\gamma-\alpha-\beta}F(\gamma-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;1-z)$$

Therefore, we can write a general solution around the regular singular point $z=1$ as follows

$$y(z) = c_1F(\alpha,\beta,\alpha+\beta+1-\gamma;1-z)+c_2(1-z)^{\gamma-\alpha-\beta}F(\gamma-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;1-z) \tag{2.1.10}$$

where c_1,c_2 are constants.

Solutions around the regular singular point $z=\infty$:

In this case, we let $t=\frac{1}{z}$, then solutions around $z=\infty$ correspond to $t=0$. Note that

$$\frac{dy}{dz} = \frac{dt}{dz}\frac{dy}{dt} = -\frac{1}{z^2}\frac{dy}{dt} = -t^2\frac{dy}{dt}$$

and

$$\frac{d^2y}{dz^2} = \frac{d^2t}{dz^2}\frac{dy}{dt}+\left(\frac{dt}{dz}\right)^2\frac{d^2y}{dt^2} = 2t^3\frac{dy}{dt}+t^4\frac{d^2y}{dt^2}$$

plugging the above equations into the original differential equation gives

$$t^2(t-1)\frac{d^2y}{dt^2}+[t^2(2-\gamma)+(\alpha+\beta-1)t]\frac{dy}{dt}-\alpha\beta y = 0 \tag{2.1.11}$$

Now using a similar argument let us assume that our solutions are of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$$

where $r\in\mathbb{C}$. Then using equation (2.1.11)

$$(t^3-t^2)\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)t^{n+r-2}+[t^2(2-\gamma)+(\alpha+\beta-1)t]\sum_{n=0}^{\infty} a_n(n+r)t^{n+r-1} - \alpha\beta\sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

Rearranging the series yields

$$\sum_{n=1}^{\infty}[a_{n-1}\{(n+r-1)(n+r-\gamma)\}+a_n(n+r)(\alpha+\beta-n-r)-\alpha\beta]t^{n+r}=0 \qquad (2.1.12)$$

with indicial equation $r^2-(\alpha+\beta)r+\alpha\beta=0$, which implies $r=\alpha,\beta$. It follows from (2.1.12) that we have the following recurrence relation

$$a_n=\frac{(n+r-1)(n+r-\gamma)}{(n+r-\alpha)(n+r-\beta)}a_{n-1}$$

for $n\geq 1$.
Plugging $r=\alpha$, and computing the first few terms of the recurrence relation for those values of n we get

$$a_n=\frac{(\alpha)_n(1+\alpha-\gamma)_n}{(1+\alpha-\beta)_n n!}$$

where $a_0=1$.
Likewise, for $r=\beta$ we get

$$a_n=\frac{(\beta)_n(1+\beta-\gamma)_n}{(1+\beta-\alpha)_n n!}$$

where $a_0=1$. Therefore, the first solution can be given by

$$\begin{aligned} y_1(t)=\sum_{n=0}^{\infty}a_nt^{n+\alpha} &= \sum_{n=0}^{\infty}\frac{(\alpha)_n(1+\alpha-\gamma)_n}{(1+\alpha-\beta)_n n!}t^{n+\alpha} \\ &= t^{\alpha}F(\alpha,1+\alpha-\gamma,1+\alpha-\beta;t) \\ &= z^{-\alpha}F(\alpha,1+\alpha-\gamma,1+\alpha-\beta;\frac{1}{z}) \end{aligned}$$

and the second solution can be given by

$$\begin{aligned} y_2(t)=\sum_{n=0}^{\infty}a_nt^{n+\beta} &= \sum_{n=0}^{\infty}\frac{(\beta)_n(1+\beta-\gamma)_n}{(1+\beta-\alpha)_n n!}t^{n+\beta} \\ &= t^{\beta}F(\beta,1+\beta-\gamma,1+\beta-\alpha;t) \\ &= z^{-\beta}F(\beta,1+\beta-\gamma,1+\beta-\alpha;\frac{1}{z}) \end{aligned}$$

Consequently, a general solution around the singular point $z=\infty$ can be given by

$$y(z)=c_1z^{-\alpha}F(\alpha,1+\alpha-\gamma,1+\alpha-\beta;\frac{1}{z})+c_2z^{-\beta}F(\beta,1+\beta-\gamma,1+\beta-\alpha;\frac{1}{z}) \quad (2.1.13)$$

2.2 Local Monodromy

Consider the Gauss' differential equation

$$z(1-z)y'' + [\gamma - (\alpha+\beta+1)z]y' - \alpha\beta y = 0 \tag{2.2.1}$$

with the regular singularities $z = 0, 1, \infty$.
For any point z_0 not equal to 0 or 1 in the complex plane. Cauchy's fundamental theorem asserts that there exists two linearly independent holomorphic solutions around z_0. Or equivalently, the set of solutions around z_0 forms a 2-dimensional vector space over $\mathbb{C}$.
We consider analytic continuations of these solutions along a path in $\mathbb{C} - \{0,1\}$. Let ℓ be a loop in $\mathbb{C} - \{0,1\}$ starting and ending at z_0, and let $u_1(z)$ and $u_2(z)$ be two linearly independent solutions at z_0. Furthermore, let $\ell_* u_1(z)$ and $\ell_* u_2(z)$ be the solutions obtained by the analytic continuation of u_1 and u_2 along the loop ℓ. Then, we have

$$(\ell)_* \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = M(\ell) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

for some $M(\ell) \in GL(2, \mathbb{C})$.
If ℓ_1 and ℓ_2 are loops starting and ending at z_0, then we define their product $\ell_1\ell_2$ to be the curve first ℓ_1 and then joining ℓ_2. We have

$$\begin{aligned}
(\ell_1\ell_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= (\ell_2)_*(\ell_1)_* \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= (\ell_2)_* M(\ell_1) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= M(\ell_1)(\ell_2)_* \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= M(\ell_1)M(\ell_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\end{aligned}$$

We note that if ℓ_1 can be deformed continuously to ℓ_2 that is, ℓ_1 and ℓ_2 are *homotopic* in $X = \mathbb{C} - \{0,1\}$, then we have $M(\ell_1) = M(\ell_2)$.
Let us denote by $\pi_1(X, z_0)$ the *fundamental group*[1] of X with base point z_0, then the

[1]This group is defined as the set of homotopy equivalence classes of loops starting and ending at z_0.

image of $\pi_1(X,z_0)$ under the homomorphism

$$\begin{aligned} \phi : \pi_1(X,z_0) &\longrightarrow GL(2,\mathbb{C}) \\ [\ell] &\longmapsto M(\ell) \end{aligned}$$

is called the **monodromy representation** of the differential equation (2.2.1). We remark that this map depends on both u_1 and u_2 as well as on z_0. The monodromy group of the *Gauss* differential equation is now defined as the image of the monodromy representation in $GL(2,\mathbb{C})$.
We remark that as we also want to study the solution around $z=\infty$, it is reasonable to consider the extended complex plane $\mathbb{C}\cup\{\infty\}$ which is homeomorphic to the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$.
Consider three loops $[\ell_0],[\ell_1],[\ell_\infty] \in \pi_1(\mathbb{P}^1_{\mathbb{C}}-\{0,1,\infty\},z_0)$
satisfying the condition $\ell_0\ell_1\ell_\infty = 1_{id}$. where "$1_{id}$" denotes the identity loop.

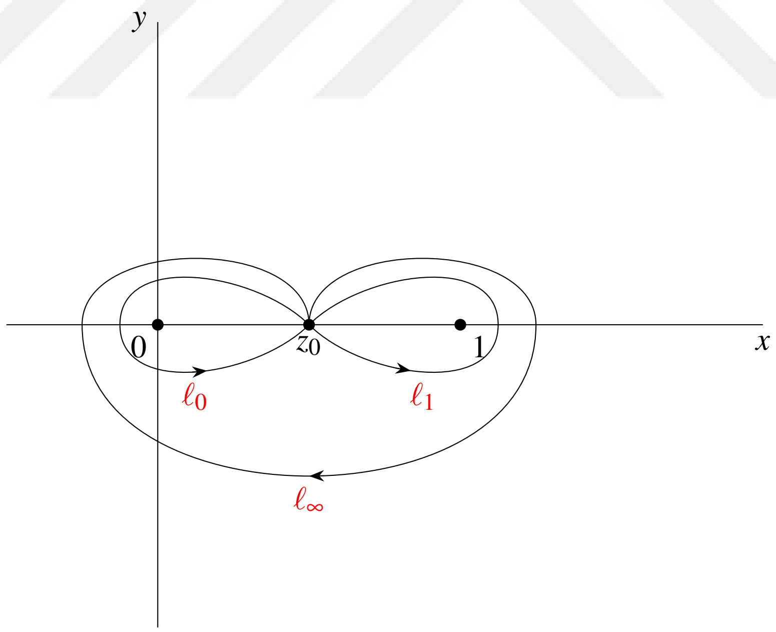


Figure 2.1: Loops around the singular points

Let us denote the corresponding monodromy matrices by M_0, M_1, M_∞ satisfying $M_0M_1M_\infty = Id_{2\times 2}$. We notice that the fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}}-\{0,1,\infty\},z_0)$ is in fact generated by two loops as we have $[\ell_0][\ell_1][\ell_\infty] = [1_{id}]$ and this implies that $[\ell_\infty] = [\ell_0\ell_1]^{-1} = [\ell_1]^{-1}[\ell_0]^{-1}$.
By the same reason, if we denote the monodromy group by H, then H is indeed

generated by the matrices M_0, M_1 as we have $M_\infty = (M_0 M_1)^{-1}$ and we write $H =< M_0, M_1 >$.
We now return to the Gauss hypergeometric differential equation

$$z(\theta+\alpha)(\theta+\beta)-\theta(\theta+\gamma-1)=0,$$

where $\theta = z\frac{d}{dz}$. We have found that there exists two linearly independent holomorphic solutions around each regular singular point of this differential equation. We shall be interested in branching behaviour of those solutions.
Around the point $z = 0$, we have shown that

$$y_1(z) = F(\alpha,\beta,\gamma;z)$$

and

$$y_2(z) = z^{1-\gamma}F(1-\gamma+\alpha,1-\gamma+\beta,2-\gamma;z)$$

are two solutions. Notice that the solution $y_2(z)$ has branching behaviour due to the term $z^{1-\gamma}$.
Let ℓ_0 be the simple loop around $z = 0$. Then as z traverses once along the loop ℓ_0 in the counter clockwise direction(z turns once around 0 in the positive direction) that is, $z \mapsto e^{2\pi i}z$, then the solutions $y_1(z)$ and $y_2(z)$ around $z = 0$, become

$$y_1(z) = F(\alpha,\beta,\gamma;z)$$

and

$$y_2(z) = e^{-2\pi i\gamma}z^{1-\gamma}F(1-\gamma+\alpha,1-\gamma+\beta,2-\gamma;z)$$

Thus we obtain the monodromy representation of the solutions $y_1(z)$ and $y_2(z)$ as

$$(\ell_0)_*\begin{pmatrix} y_1(z)\\ y_2(z)\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & e^{-2\pi i\gamma}\end{pmatrix}\begin{pmatrix} y_1(z)\\ y_2(z)\end{pmatrix}$$

Likewise, if ℓ_1 is a loop around the singular point $z = 1$, then after a counter clockwise turn around $z = 1$, that is $z \mapsto 1+e^{2\pi i}(z-1)$, the solutions $\widetilde{y_1}(z)$ and $\widetilde{y_2}(z)$ around $z = 1$, become

$$\widetilde{y_1}(z) = F(\alpha,\beta,\alpha+\beta+1-\gamma;1-z)$$

and

$$\widetilde{y_2}(z) = e^{2\pi i(\gamma-\alpha-\beta)}(1-z)^{\gamma-\alpha-\beta}F(\gamma-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;1-z)$$

Thus, we obtain the monodromy representation

$$(\ell_1)_* \begin{pmatrix} \widetilde{y_1}(z) \\ \widetilde{y_2}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(\gamma-\alpha-\beta)} \end{pmatrix} \begin{pmatrix} \widetilde{y_1}(z) \\ \widetilde{y_2}(z) \end{pmatrix}$$

where

$$\widetilde{y_1}(z) = F(\alpha,\beta,\alpha+\beta+1-\gamma;1-z)$$

and

$$\widetilde{y_2}(z) = (1-z)^{\gamma-\alpha-\beta}F(\gamma-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;1-z).$$

Lastly, we look at the loop ℓ_∞ around the singular point $z=\infty$.

The solutions around $z=\infty$, are

$$\widetilde{\widetilde{y_1}}(z) = z^{-\alpha}F(\alpha,1+\alpha-\gamma,1+\alpha-\beta;\frac{1}{z})$$

and

$$\widetilde{\widetilde{y_2}}(z) = z^{-\beta}F(\beta,1+\beta-\gamma,1+\beta-\alpha;\frac{1}{z}).$$

If we put $\frac{1}{z}=t$ then, as t traverses around the point $t=0$ in the positive direction, z traverses around the point $z=\infty$, in the clockwise sense. Hence the solutions become

$$\widetilde{\widetilde{y_1}}(z) = e^{-2\pi i\alpha}\widetilde{\widetilde{y_1}}(z)$$

and

$$\widetilde{\widetilde{y_2}}(z) = e^{-2\pi i\beta}\widetilde{\widetilde{y_2}}(z)$$

Therefore, we obtain the monodromy representation

$$(\ell_\infty)_* \begin{pmatrix} \widetilde{\widetilde{y_1}}(z) \\ \widetilde{\widetilde{y_2}}(z) \end{pmatrix} = \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix} \begin{pmatrix} \widetilde{\widetilde{y_1}}(z) \\ \widetilde{\widetilde{y_2}}(z) \end{pmatrix}.$$

The matrices $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i\gamma} \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(\gamma-\alpha-\beta)} \end{pmatrix}$ and $A_\infty = \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix}$ are called the local monodromy matrices for the *Gauss* hypergeometric differential equation with bases $B_0=\{y_1(z),y_2(z)\}$, $B_1=\{\widetilde{y_1}(z),\widetilde{y_2}(z)\}$, $B_\infty=\{\widetilde{\widetilde{y_1}}(z),\widetilde{\widetilde{y_2}}(z)\}$ respectively. However, for a suitable change of basis matrix we can choose a common basis say, $B_0=\{y_1(z),y_2(z)\}$, then we write

$$\begin{pmatrix}\widetilde{y_1}(z)\\ \widetilde{y_2}(z)\end{pmatrix}=C_1\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}$$

where C_1 is the connection(change of basis) matrix. This implies that

$$(\ell_1)_*C_1\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}=\begin{pmatrix}1&0\\ 0&e^{2\pi i(\gamma-\alpha-\beta)}\end{pmatrix}C_1\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}$$

and

$$(\ell_1)_*\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}=C_1^{-1}\begin{pmatrix}1&0\\ 0&e^{2\pi i(\gamma-\alpha-\beta)}\end{pmatrix}C_1\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}.$$

In a similar fashion,

$$\begin{pmatrix}\widetilde{\widetilde{y_1}}(z)\\ \widetilde{\widetilde{y_2}}(z)\end{pmatrix}=C_\infty\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}$$

where C_∞ is the connection(change of basis) matrix. It follows that

$$(\ell_\infty)_*C_\infty\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}=\begin{pmatrix}e^{-2\pi i\alpha}&0\\ 0&e^{-2\pi i\beta}\end{pmatrix}C_\infty\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}.$$

Therefore,

$$(\ell_\infty)_*\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}=C_\infty^{-1}\begin{pmatrix}e^{-2\pi i\alpha}&0\\ 0&e^{-2\pi i\beta}\end{pmatrix}C_\infty\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}.$$

After such a change of basis we obtain the matrices M_0,M_1,M_∞ so that $M_0M_1M_\infty=Id_{2\times 2}$, where

$$M_0=\begin{pmatrix}1&0\\ 0&e^{-2\pi i\gamma}\end{pmatrix}$$

and

$$M_1=C_1^{-1}\begin{pmatrix}1&0\\ 0&e^{2\pi i(\gamma-\alpha-\beta)}\end{pmatrix}C_1$$

and

$$M_\infty=C_\infty^{-1}\begin{pmatrix}e^{-2\pi i\alpha}&0\\ 0&e^{-2\pi i\beta}\end{pmatrix}C_\infty$$

Remark 2.2.1. *Finding the connection matrices in general requires a lot of work and* *will not be given here. See Section 2.5.*

2.3 Euler Integral Representation

There are two integral representations of the Gauss hypergeometric function. Namely, Euler and Barnes integral representations. In this section we introduce an integral representation due to Euler.

Theorem 2.3.1 ([9],Chapter 2)**.** *For* $|z| < 1$, $\gamma > \beta > 0$, $\alpha > 0$ *we have*

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt \qquad (2.3.1)$$

Proof. Consider

$$\begin{aligned}
F(\alpha,\beta,\gamma;z) &= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+k)} \frac{z^k}{k!} \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)\Gamma(\gamma-\beta)}{\Gamma(\alpha)\Gamma(\gamma+k)} \frac{z^k}{k!} \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_{k=0}^{\infty} B(\beta+k,\gamma-\beta) \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \frac{z^k}{k!}
\end{aligned}$$

□

Note that

$$\begin{aligned}
(1-zt)^{-\alpha} &= \sum_{k=0}^{\infty} \frac{-\alpha(-\alpha-1)\ldots(-\alpha-(k-1))(-zt)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{2k}\alpha(\alpha+1)\ldots(\alpha+k-1)(zt)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}(zt)^k \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \frac{(zt)^k}{k!}.
\end{aligned}$$

This series is uniformly convergent for every $t \in [0,1]$, with $0 \leq |zt| \leq |z| < 1$. Then

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_{k=0}^{\infty} B(\beta+k,\gamma-\beta)\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\frac{z^k}{k!} \tag{2.3.2}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_{k=0}^{\infty} \int_0^1 t^{\beta+k-1}(1-t)^{\gamma-\beta-1}dt\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\frac{z^k}{k!} \tag{2.3.3}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\frac{(zt)^k}{k!}dt \tag{2.3.4}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt \tag{2.3.5}$$

Here (2.3.4) is obtained by exchanging the order of summation and integration by using uniform convergence of the series

$$\sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\frac{z^k}{k!}$$

and (2.3.5) by the generalized binomial theorem.
Euler integral representation of $F(\alpha,\beta,\gamma;z)$ allows choices of z such that $|z| > 1$ and the condition $\gamma > \beta > 0$ is necessary for convergence of the beta integral.
However, we can drop the restriction $\gamma > \beta > 0$, if we consider the so-called *Pochhammer* contour C as an integration path

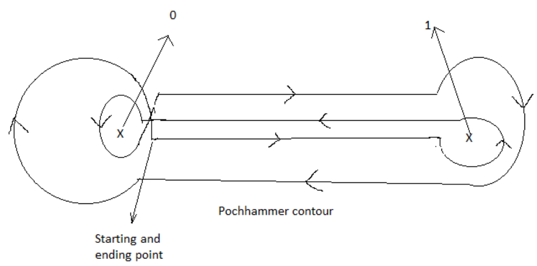


Figure 2.2: Pochhammer Contour

For simplicity, let us denote $t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}$ by $f(z,t)=f$, then we have

$$\begin{aligned}\int_C f dt &= \int_0^1 f dt + e^{2\pi i(\gamma-\beta)}\int_1^0 f dt + e^{2\pi i\beta}e^{2\pi i(\gamma-\beta)}\int_0^1 f dt \\ &+ e^{2\pi i\beta}e^{2\pi i(\gamma-\beta)}e^{-2\pi i(\gamma-\beta)}\int_1^0 f dt \\ &= (1-e^{2\pi i(\gamma-\beta)})(1-e^{2\pi i\beta})\int_0^1 f dt \\ &= (1-e^{2\pi i(\gamma-\beta)})(1-e^{2\pi i\beta})\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt\end{aligned}$$

Hence, we obtain

$$\int_0^1 f(z,t)dt = \frac{1}{(1-e^{2\pi i(\gamma-\beta)})(1-e^{2\pi i\beta})}\int_C t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$$

Therefore, we obtain

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 f(z,t)dt$$

We remark that this representation is valid for any $\gamma,\beta \in \mathbb{C}\setminus\mathbb{Z}$

2.4 Barnes Integral Representation

As we mentioned in Section 2.3 there is another integral representation of $F(\alpha,\beta,\gamma;z)$ which is called **Barnes** integral representation. First of all, we give the construction of such a representation.
Let $f(x)$ be a function defined by a power series

$$f(x) = \sum_{m=0}^{\infty} a_m x^m$$

Assume that there is a function $g(t)$ satisfying the followings:

(i) $g(t)$ is meromorphic in $|t| < \infty$

(ii) $g(t)$ is holomorphic at $t = 0, 1, 2, \ldots$

(iii) $g(m) = a_m$

Let us define the function

$$h(t) := g(t)\frac{\pi}{\sin(\pi t)}(-x)^t$$

Then $h(t)$ has simple poles at $t \in \mathbb{Z}_{\geq 0}$ and the condition $g(m) = a_m$ implies that

$$\mathrm{Res}(h(t), t = m) = a_m x^m$$

for $m \in \mathbb{Z}_{\geq 0}$
If C is a loop which encircles the poles $t = 0, 1, 2, \ldots, N$ in the counter clockwise sense, then Cauchy's residue theorem asserts that

$$\frac{1}{2\pi i}\int_C h(t)dt = \sum_{m=0}^{N} a_m x^m$$

If C' denotes the inverse loop of C, then we have

$$-\frac{1}{2\pi i}\int_{C'} h(t)dt = \sum_{m=0}^{N} a_m x^m$$

For convenience, we replace z by x in $F(\alpha,\beta,\gamma;z)$.
Now, we go back to the hypergeometric function given by

$$F(\alpha,\beta,\gamma;x) = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m(1)_m} x^m$$

where $\gamma \neq 0, -1, -2, \dots$
We rewrite the series in terms of Γ-function as

$$F(\alpha,\beta,\gamma;x) = \sum_{m=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)\Gamma(1+m)} x^m$$

where α, β, γ are non-negative integers.
Hence, if we take $g(s)$ as the function

$$g(t) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)\Gamma(1+t)}$$

then $g(t)$ is meromorphic in $|t| < \infty$ thus, the condition (i) is satisfied. Poles of $g(t)$ are at

$$t = -\alpha, -\alpha - 1, \dots$$

and

$$t = -\beta, -\beta - 1, \dots$$

and if $\alpha, \beta \neq 0, -1, -2, \dots$ then none of these points is zero or a positive integer. Thus $g(t)$ is holomorphic. Hence the condition (ii) is satisfied. Also,

$$g(m) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m(1)_m} = a_m$$

for $m = 0, 1, 2, \dots$ which implies condition (iii).
Now consider

$$h(t) = g(t)\frac{\pi}{\sin(\pi t)}(-x)^t = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)\Gamma(1+t)} \frac{\pi}{\sin(\pi t)}(-x)^t$$

has simple poles at the points

$$t = -\alpha, -\alpha - 1, \dots, -\beta, -\beta - 1, \dots, 0, 1, 2, \dots$$

We now employ the Euler's reflection formula

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)}$$

From this we have

$$\Gamma(-t)\Gamma(1+t) = \frac{\pi}{\sin(-\pi t)} = -\frac{\pi}{\sin(\pi t)}$$

thus

$$\Gamma(-t) = -\frac{\pi}{\Gamma(1+t)\sin(\pi t)}$$

Hence, we obtain

$$-h(t) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-x)^t$$

We have the following theorem

Theorem 2.4.1 ([9],Chapter 2, Section 3.5)**.** *Suppose that* $\alpha, \beta, \gamma \neq 0, -1, -2, \ldots$. *Then the equality*

$$\begin{aligned} F(\alpha,\beta,\gamma,x) &= -\frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_C \frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)\Gamma(1+t)}\frac{\pi}{\sin(\pi t)}(-x)^t dt \\ &= \frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_C \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-x)^t dt \end{aligned}$$

holds for $|x| < 1$ and $|\arg(-x)| < \pi$. The integration path C runs from $-i\infty$ to $+i\infty$ in such a way that, if $|t|$ is sufficiently large, then $t \in C$ lies on the imaginary axis, the poles of $\Gamma(\alpha+t)\,\Gamma(\beta+t)$ lie to the left of C and those of $\Gamma(-t)$ lie to the right of C. See the figure below.

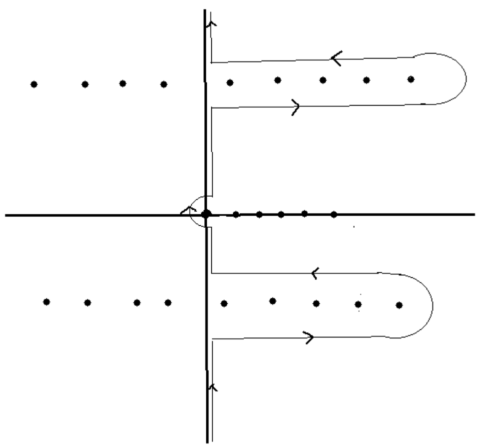


Figure 2.3: The contour C

Proof. ([9], [Chapter 2, Section 3.5]) In order to prove the theorem we need to estimate the integrand by the aid of Stirling's formula for the gamma function.(1.1.2)

$$\log\Gamma(z) = z\log z - \frac{1}{2}\log z - z + O(1).$$

For convenience let us replace z by t. From the above equality we have

$$\log\Gamma(a+t) = (a+t)\log(t(1+\frac{a}{t})) - (a+t) - \frac{1}{2}\log(t(1+\frac{a}{t})) \tag{2.4.1}$$

Since

$$\log(1-y) = -\sum_{k=1}^{\infty}\frac{y^k}{k} = -(y+\frac{y^2}{2}+\frac{y^3}{3}+\cdots)$$

putting $y = -\frac{a}{t}$, we see that

$$\log\Gamma(a+t) = t\log t - t + (a-\frac{1}{2})\log t + O(1)$$

From this we can write

$$\log\Gamma(\alpha+t) = t\log t - t + (\alpha-\frac{1}{2})\log t + O(1)$$

$$\log\Gamma(\beta+t) = t\log t - t + (\beta-\frac{1}{2})\log t + O(1)$$

$$\log\Gamma(\gamma+t) = t\log t - t + (\gamma-\frac{1}{2})\log t + O(1)$$

$$\log\Gamma(-t) = -t\log(-t) + t + -\frac{1}{2}\log(-t) + O(1)$$

Let us put $t = is$, where s is a positive sufficiently large real number. Then we get

$$\begin{aligned}\log\Gamma(\alpha+t) &= -\frac{\pi}{2}s + (\Re\alpha-\frac{1}{2})\log s - \frac{\pi}{2}Im\alpha + PI + O(1)\\ &= -\frac{\pi}{2}s + (\Re\alpha-\frac{1}{2})\log s + PI + O(1).\end{aligned}$$

Since

$$\log(is) = \log s + i\frac{\pi}{2}$$

and where PI denotes a purely imaginary quantity.

Likewise

$$\log\Gamma(\beta+t) = -\frac{\pi}{2}s + (\Re\beta - \frac{1}{2})\log s + PI + O(1)$$

$$\log\Gamma(\gamma+t) = -\frac{\pi}{2}s + (\Re\gamma - \frac{1}{2})\log s + PI + O(1)$$

and

$$\begin{aligned}\log\Gamma(-t) &= -is(\log s - i\frac{\pi}{2}) + is - \frac{1}{2}(\log s - i\frac{\pi}{2}) + O(1)\\ &= -\frac{\pi}{2}s - \frac{1}{2}\log s + PI + O(1)\end{aligned}$$

also,

$$\begin{aligned}\log(-x)^t &= t\log(-x) = is(\log|x| + i\arg(-x))\\ &= -s\arg(-x) + PI.\end{aligned}$$

Combining the above relations we derive

$$\begin{aligned}\log\frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-x)^t = (\Re\alpha - \frac{1}{2} + \Re\beta - \frac{1}{2} - \Re\gamma + \frac{1}{2} - \frac{1}{2})\log s\\ - s\arg(-x) - \pi s + PI + O(1)\end{aligned}$$

Thus we obtain

$$\left|\frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-x)^t\right| = O(s^{\Re(\alpha+\beta-\gamma-1)}e^{-s(\pi+\arg(-x))})$$

Therefore, we see that the quantity

$$s^{\Re(\alpha+\beta-\gamma-1)}e^{-s(\pi+\arg(-x))}$$

decays exponentially as $s \to \infty$. In the same way, one can obtain the same result when $t = -is$,

$$\left|\frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-x)^t\right| = O(s^{\Re(\alpha+\beta-\gamma-1)}e^{-s(\pi-\arg(-x))})$$

Again we see that the quantity

$$s^{\Re(\alpha+\beta-\gamma-1)}e^{-s(\pi-\arg(-x))}$$

decays exponentially as $s \to \infty$.
Now to investigate what happens as $\Re t \to \infty$, we consider the closed path $C = C_N^{(1)} + C_N^{(2)}$, where N is a positive integer which is sufficiently large and $C_N^{(1)}$ is the arc of C in the strip between $Imt = N + \frac{1}{2}$ and $Imt = -(N + \frac{1}{2})$ and $C_N^{(2)}$ is a semi-circle defined by $t = (N + \frac{1}{2})e^{i\theta}$, $|\theta| \leq \frac{\pi}{2}$.
By virtue of Euler's reflection formula we can rewrite the quantity

$$\frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-x)^t$$

as

$$\frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)\Gamma(1+t)}\frac{\pi}{\sin(\pi t)}(-x)^t = h(t)$$

It is enough to show that

$$\int_{C_N^{(2)}} h(t)dt$$

decays exponentially as $N \to \infty$.
Let $t = N + \frac{1}{2} + is$. Applying the Stirling's formula to $h(t)$ eventually yields to the following expression

$$\log\frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)\Gamma(1+t)}(-x)^t = (\alpha+\beta-\gamma-1)\log(s) - \arg(-x)s) + PI + O(1).$$

From this we get

$$\left|\frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)\Gamma(1+t)}(-x)^t\right| = O(s^{(\alpha+\beta-\gamma-1)}e^{-\arg(-x)s}).$$

Also, since

$$\left|\frac{\pi}{sin(\pi t)}\right| = \left|\frac{2\pi i}{e^{i\pi(N+\frac{1}{2}+is)} - e^{-i\pi(N+\frac{1}{2}+is)}}\right| \leq 2\pi e^{-\pi|s|},$$

we derive

$$\left|\frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)\Gamma(1+t)}(-x)^t\frac{\pi}{sin(\pi t)}\right| = O(s^{(\alpha+\beta-\gamma-1)}e^{-\pi|s|-\arg(-x)s}).$$

But the quantity

$$s^{(\alpha+\beta-\gamma-1)}e^{-\pi|s|-\arg(-x)s}$$

decays exponentially as $N \to \infty$. This shows that $h(t)$ decays exponentially on all of $C_N^{(2)}$. Therefore completes the proof of the theorem (2.4.1). $\square$

2.5 Global Monodromy via Barnes Integral

In Section 2.1.1, we have found two linearly independent local solutions at each singular point given by the hypergeometric function $F(\alpha,\beta,\gamma;x)$. But somehow these solutions are related.
In particular, we can find this relation between the local solutions at $x=0$ and $x=\infty$ by the aid of **Barnes** representation.
Recall that Barnes representation of $F(\alpha,\beta,\gamma;x)$ is given by

$$F(\alpha,\beta,\gamma;x) = \frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_C \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-x)^t dt$$

Let us put

$$\frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-x)^t = \psi(x,t)$$

then by Cauchy's residue theorem we have

$$F(\alpha,\beta,\gamma;x) = \frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_C \psi(x,t)dt$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\left(\sum_{n=0}^{N}\mathrm{Res}(\psi(x,t),t=-\alpha-n)+\sum_{n=0}^{N}\mathrm{Res}(\psi(x,t),t=-\beta-n)+I_N(x)\right)$$

Let us calculate the first sum

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{N}\mathrm{Res}(\psi(x,t),t=-\alpha-n)$$

We have

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{N}\mathrm{Res}(\psi(x,t),t=-\alpha-n) = \sum_{n=0}^{N}\frac{(-1)^n}{n!}\frac{\Gamma(\beta-\alpha-n)\Gamma(\alpha+n)(-x)^{-\alpha-n}}{\Gamma(\gamma-\alpha-n)} \tag{2.5.1}$$

Note that

$$\Gamma(\beta-\alpha) = (\beta-\alpha)(\beta-\alpha-1)\dots(\beta-\alpha-n)\Gamma(\beta-\alpha-n)$$

From this we get

$$\Gamma(\beta-\alpha-n) = \frac{\Gamma(\beta-\alpha)}{(-1)^n(\alpha-\beta+1)_n}$$

and similarly we get

$$\Gamma(\gamma-\alpha-n)=\frac{\Gamma(\gamma-\alpha)}{(-1)^n(\alpha-\gamma+1)_n}$$

Thus, the equation (2.5.1) becomes

$$e^{-\pi i\alpha}\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}x^{-\alpha}\sum_{n=0}^{N}\frac{(\alpha-\gamma+1)_n(\alpha)_n}{(\alpha-\beta+1)_n n!}x^{-n}$$

If we apply the same argument to the second sum

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{N}\mathrm{Res}(\psi(x,t),t=-\beta-n)$$

then we eventually obtain

$$e^{-\pi i\beta}\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}x^{-\beta}\sum_{n=0}^{N}\frac{(\beta-\gamma+1)_n(\beta)_n}{(\beta-\alpha+1)_n n!}x^{-n}$$

In assuming none of $\alpha,\beta,\gamma,\gamma-\alpha,\gamma-\beta$ and $\beta-\alpha$ is an integer. Then we let

$$e^{-\pi i\alpha}\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}=c(\alpha,\beta,\gamma)$$

and

$$e^{-\pi i\beta}\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}=c(\beta,\alpha,\gamma)$$

then we have

$$f_0(x;0)=c(\alpha,\beta,\gamma)f_{\infty,N}(x;\alpha)+c(\beta,\alpha,\gamma)f_{\infty,N}(x;\beta)+I_N(x)$$

where

$$I_N(x)=\frac{1}{2\pi i}\int_{C_N}\psi(x,t)dt$$

$$f_{\infty,N}(x;\alpha)=x^{-\alpha}\sum_{n=0}^{N}\frac{(\alpha-\gamma+1)_n(\alpha)_n}{(\alpha-\beta+1)_n n!}x^{-n}$$

and

$$f_{\infty,N}(x;\beta)=x^{-\beta}\sum_{n=0}^{N}\frac{(\beta-\gamma+1)_n(\beta)_n}{(\beta-\alpha+1)_n n!}x^{-n}$$

where $f_a(x;\lambda)$:=local solution at $x=a$ with the local exponent λ.

Furthermore, notice that as $N \to \infty$, $f_{\infty,N}(x;\nu) \to f_\infty(x;\nu)$ for $\nu = \alpha, \beta$ and $I_N(x) \to 0$. Therefore, we obtain

$$f_0(x;0) = c(\alpha,\beta,\gamma) f_\infty(x;\alpha) + c(\beta,\alpha,\gamma) f_\infty(x;\beta) \tag{2.5.2}$$

If we replace (α,β,γ) by $(\alpha',\beta',\gamma') = (\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma)$ in (2.5.2) and multiply by $x^{1-\gamma}$, we get

$$f_0(x;1-\gamma) = c(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma) f_\infty(x;\alpha) + c(\beta-\gamma+1,\alpha-\gamma+1,2-\gamma) f_\infty(x;\beta).$$

Remark 2.5.1. *The other solution around $x = 0$ was found in Section[2.1.1] and we replaced the parameters (α,β,γ) with respect to the parameters in* (2.1.8).

Therefore, we obtain

$$f_0(x;0) = c(\alpha,\beta,\gamma) f_\infty(x;\alpha) + c(\beta,\alpha,\gamma) f_\infty(x;\beta)$$

and

$$f_0(x;1-\gamma) = c(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma) f_\infty(x;\alpha) + c(\beta-\gamma+1,\alpha-\gamma+1,2-\gamma) f_\infty(x;\beta).$$

and thus,

$$\begin{pmatrix} f_0(x;0) \\ f_0(x;1-\gamma) \end{pmatrix} = C_\infty \begin{pmatrix} f_\infty(x;\alpha) \\ f_\infty(x;\beta) \end{pmatrix}$$

where

$$C_\infty = \begin{pmatrix} c(\alpha,\beta,\gamma) & c(\beta,\alpha,\gamma) \\ c(\alpha',\beta',\gamma') & c(\beta',\alpha',\gamma') \end{pmatrix}$$

is the connection matrix.
In this case, the monodromy group with respect to the fundamental system of solutions f is generated by

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i\gamma} \end{pmatrix}$$

and

$$M_\infty = C_\infty^{-1} \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix} C_\infty.$$

Chapter 3

Pochhammer Hypergeometric Function

In Chapter 2, we defined the *Gauss* hypergeometric function by the series

$$F(\alpha,\beta,\gamma;z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}$$

Here we study the higher order case of this definition.

3.1 Generalized Hypergeometric Function

Definition 3.1.1. *The higher order hypergeometric function also known as Pochhammer hypergeometric function* ${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;z)$ *is defined by the series*

$${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_n)_k}{(\beta_1)_k\ldots(\beta_{n-1})_k}\frac{z^k}{k!}$$

with $(\alpha_1,\ldots,\alpha_n)$ *being numerator and* $(\beta_1,\ldots,\beta_n)$ *being denominator parameters and sometimes denoted by* $F(\overrightarrow{\alpha},\overrightarrow{\beta};z)$*, where*

$$\overrightarrow{\alpha}=(\alpha_1,\ldots,\alpha_n),\ \overrightarrow{\beta}=(\beta_1,\ldots,\beta_n)\in\mathbb{C}^n$$

with $\beta_n=1$*. Also, the Pochhammer symbol*

$$(\alpha)_k=(\alpha)(\alpha+1)\cdots(\alpha+k-1)=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

as before.

The Pochhammer hypergeometric series converges in the unit disk $|z| < 1$ and for $n = 2$, it reduces to the Gauss hypergeometric series and thus, $F(\alpha, \beta, \gamma; z)$.
The hypergeometric differential equation of order n

$$z^n(z-1)\frac{d^n y}{dz^n} + \sum_{k=0}^{n-1}(a_k - b_k z)z^k \frac{d^k y}{dz^k} = 0$$

has regular singular points at $z = 0, 1, \infty$. It can be written in the operator form

$$z(\theta+\alpha_1)\dots(\theta+\alpha_n) - \theta(\theta+\beta_1-1)\dots(\theta+\beta_{n-1}-1) = 0$$

where $\theta = z\frac{d}{dz}$.

Lemma 3.1.1. *The Pochhammer hypergeometric function $F(\overrightarrow{\alpha}, \overrightarrow{\beta}; z)$ is a solution of the hypergeometric differential equation*

$$z(\theta+\alpha_1)\dots(\theta+\alpha_n) - \theta(\theta+\beta_1-1)\dots(\theta+\beta_{n-1}-1) = 0$$

where $\theta = z\frac{d}{dz}$.

Proof. We need to show that

$$[z(\theta+\alpha_1)\dots(\theta+\alpha_n) - \theta(\theta+\beta_1-1)\dots(\theta+\beta_{n-1}-1)]F(\overrightarrow{\alpha}, \overrightarrow{\beta}; z) = 0.$$

Consider

$$\begin{aligned}
&\theta(\theta+\beta_1-1)\dots(\theta+\beta_{n-1}-1)F(\overrightarrow{\alpha}, \overrightarrow{\beta}; z) \\
&\qquad = \sum_{k=0}^{\infty} k(k+\beta_1-1)\dots(k+\beta_{n-1}-1)\frac{(\alpha_1)_k\dots(\alpha_n)_k}{(\beta_1)_k\dots(\beta_{n-1})_k}\frac{z^k}{k!} \\
&\qquad = \sum_{k=1}^{\infty} \frac{(\alpha_1)_k\dots(\alpha_n)_k}{(\beta_1)_{k-1}\dots(\beta_{n-1})_{k-1}}\frac{z^k}{(k-1)!} \\
&\qquad = \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k+1}\dots(\alpha_n)_{k+1}}{(\beta_1)_{k-1}\dots(\beta_{n-1})_{k-1}}\frac{z^{k+1}}{k!} \\
&\qquad = z(k+\alpha_1)\dots(k+\alpha_n)\sum_{k=0}^{\infty} \frac{(\alpha_1)_k\dots(\alpha_n)_k}{(\beta_1)_k\dots(\beta_{n-1})_k}\frac{z^k}{k!} \\
&\qquad = z(\theta+\alpha_1)\dots(\theta+\alpha_n)F(\overrightarrow{\alpha}, \overrightarrow{\beta}; z).
\end{aligned}$$

Hence,

$$\theta(\theta+\beta_1-1)\dots(\theta+\beta_{n-1}-1)F(\overrightarrow{\alpha},\overrightarrow{\beta};z)=z(\theta+\alpha_1)\dots(\theta+\alpha_n)F(\overrightarrow{\alpha},\overrightarrow{\beta};z)$$

which we wanted to show. □

If we denote the hypergeometric differential equation

$$z(\theta+\alpha_1)\dots(\theta+\alpha_n)-\theta(\theta+\beta_1-1)\dots(\theta+\beta_{n-1}-1)=0$$

by $D(\overrightarrow{\alpha},\overrightarrow{\beta})$, then by using Frobenius method we obtain the following local exponents coming from indicial equation

$z=0$	$z=1$	$z=\infty$
$1-\beta_j$	$0,1,2,\dots,n-2,\gamma=-1+\sum_{j=1}^{n}(\beta_j-\alpha_j)$	α_j

for $j=1,2,\dots,n$

If the numbers $\beta_1,\dots,\beta_n$ are not integers, then n-linearly independent solutions of $D(\overrightarrow{\alpha},\overrightarrow{\beta})$ are given by

$$z^{1-\beta_i}{}_nF_{n-1}(1+\alpha_1-\beta_i,\dots,1+\alpha_n-\beta_i,1+\beta_1-\beta_i,\widehat{\dots},1+\beta_n-\beta_i;z) \qquad (3.1.1)$$

around the point $z=0$, for $i=1,2,\dots,n$ where $\widehat{\ }$ denotes omission of the term $1+\beta_i-\beta_i$.

$$z^{-\alpha_i}{}_nF_{n-1}(\alpha_i,1+\alpha_i-\beta_1,\dots,1+\alpha_i-\beta_{n-1},1+\alpha_i-\alpha_2,\widehat{\dots},1+\alpha_i-\alpha_n;\frac{1}{z}) \qquad (3.1.2)$$

around the point $z=\infty$, for $i=1,2,\dots,n$ where $\widehat{\ }$ denotes omission of the term $1+\alpha_i-\alpha_i$.

Lemma 3.1.2 ([5], p 327.). *For $\delta\in\mathbb{C}$ we have*

$$(\theta+\delta-1)D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)=D(\alpha_1,\dots,\alpha_n,\delta;\beta_1,\dots,\beta_n,\delta)$$

and

$$D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)(\theta+\delta)=D(\alpha_1,\dots,\alpha_n,\delta;\beta_1,\dots,\beta_n,\delta+1)$$

Proof. We have

$$
\begin{aligned}
(\theta+\delta-1)D(\overrightarrow{\alpha},\overrightarrow{\beta}) &= (\theta+\delta-1)[z(\theta+\alpha_1)\dots(\theta+\alpha_n)-(\theta+\beta_1-1)\dots(\theta+\beta_n-1)]\\
&= z(\theta+\delta)(\theta+\alpha_1)\dots(\theta+\alpha_n)-(\theta+\delta-1)(\theta+\beta_1-1)\dots(\theta+\beta_n-1)\\
&= z(\theta+\alpha_1)\dots(\theta+\alpha_n)(\theta+\delta)-(\theta+\beta_1-1)\dots(\theta+\beta_n-1)(\theta+\delta-1)\\
&= D(\alpha_1,\dots,\alpha_n,\delta;\beta_1,\dots,\beta_n,\delta).
\end{aligned}
$$

By the same argument we deduce that

$$
D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)(\theta+\delta) = D(\alpha_1,\dots,\alpha_n,\delta;\beta_1,\dots,\beta_n,\delta+1)
$$

where $D(\overrightarrow{\alpha},\overrightarrow{\beta}) = D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)$. □

Corollary 2 ([5], p 328.)**.** *We have*

$$
D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)(\theta+\alpha_j-1) = (\theta+\alpha_j-1)D(\alpha_1,\dots,\alpha_j-1,\dots,\alpha_n;\beta_1,\dots,\beta_n)
$$

and

$$
D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)(\theta+\beta_j) = (\theta+\beta_j-1)D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_j+1,\dots,\beta_n)
$$

for $j=1,2,\dots,n$

Proof. First let us prove the first equality. We have

$$
\begin{aligned}
D(\overrightarrow{\alpha},\overrightarrow{\beta})(\theta+\alpha_j-1) &= [z(\theta+\alpha_1)\dots(\theta+\alpha_j-1)\dots(\theta+\alpha_n)(\theta+\alpha_j-1)\\
&\quad -(\theta+\beta_1-1)\dots(\theta+\beta_n-1)](\theta+\alpha_j-1)\\
&= (\theta+\alpha_j-1)[z(\theta+\alpha_1)\dots,(\theta+\alpha_j-1),\dots(\theta+\alpha_n)-(\theta+\beta_1-1)\dots(\theta+\beta_n-1)]\\
&= (\theta+\alpha_j-1)D(\alpha_1,\dots,\alpha_j-1,\dots,\alpha_n;\beta_1,\dots,\beta_n).
\end{aligned}
$$

By a similar argument we can show the second equality. We have

$$
\begin{aligned}
&D(\overrightarrow{\alpha},\overrightarrow{\beta})(\theta+\beta_j) = z(\theta+\alpha_1)\dots(\theta+\alpha_n)(\theta+\beta_j)\\
&-(\theta+\beta_1-1)\dots(\theta+\beta_j-1)\dots(\theta+\beta_n-1)(\theta+\beta_j)\\
&=(\theta+\beta_j-1)z(\theta+\alpha_1)\dots(\theta+\alpha_n)-(\theta+\beta_1-1)\dots(\theta+\beta_j-1)\dots(\theta+\beta_n-1)(\theta+\beta_j)\\
&=(\theta+\beta_j-1)[z(\theta+\alpha_1)\dots(\theta+\alpha_n)-(\theta+\beta_1-1)\dots(\theta+\beta_j)\dots(\theta+\beta_n-1-1)]\\
&=(\theta+\beta_j-1)D(\alpha_1,\dots,\alpha_n;\beta_1,\beta_j+1,\dots,\beta_n).
\end{aligned}
$$

$\square$

3.2 Monodromy Representation

Choose a base point $z_0 \in \mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$. We consider the fundamental group $G = \pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}, z_0)$. Then G is generated by the loops $\ell_0, \ell_1, \ell_\infty$ with $\ell_0\ell_1\ell_\infty = 1$ as in *Gauss* case.
Let us denote by $V(\overrightarrow{\alpha}, \overrightarrow{\beta})$ the local solution space of the hypergeometric equation $D(\overrightarrow{\alpha}, \overrightarrow{\beta})$ around z_0. Then the monodromy representation of the hypergeometric equation $D(\overrightarrow{\alpha}, \overrightarrow{\beta})$ is given by the following map

$$\begin{aligned} M(\overrightarrow{\alpha}, \overrightarrow{\beta}) : G &\longrightarrow GL(V(\overrightarrow{\alpha}, \overrightarrow{\beta})) \\ [\ell_i] &\longmapsto M(\overrightarrow{\alpha}, \overrightarrow{\beta})(\ell_i) \end{aligned}$$

for $i = 0, 1, \infty$.

Corollary 3. *If $\alpha_j - \beta_k \notin \mathbb{Z}$ for all $j,k = 1,2,\ldots,n$ then the representations $M(\alpha_1 + k_1, \ldots, \alpha_n + k_n; \beta_1 + l_1, \ldots, \beta_n + l_n)$ and $M(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ are equivalent for any $k_1, \ldots, k_n, l_1, \ldots, l_n \in \mathbb{Z}$.*

Proposition 3.2.1 ([5], p 329.). *If $\alpha_j - \beta_k \in \mathbb{Z}$ for some $j,k = 1,2,\ldots,n$ then the monodromy representation is reducible.*

The following proof is taken from ([5], p 329.)

Proof. Without loss of generality, let $\alpha_n - \beta_n = m \in \mathbb{Z}$. If $m = -1$, then

$$\begin{aligned} D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n) &= z(\theta + \alpha_1)\ldots(\theta + \alpha_n) - (\theta + \beta_1 - 1)\ldots(\theta + \beta_n - 1) \\ &= z(\theta + \alpha_1)\ldots(\theta + \alpha_n) - (\theta + \beta_1 - 1)\ldots(\theta + \alpha_n) \\ &= D(\alpha_1, \ldots, \alpha_{n-1}; \beta_1, \ldots, \beta_{n-1})(\theta + \alpha_n) \end{aligned}$$

Hence, $z^{-\alpha_n} \in V(\alpha_1, \ldots, \alpha_n)$ generates a one dimensional invariant subspace.
If $m \geq 0$, then consider the sequence
$V(\alpha_1, \ldots, \alpha_{n-1}, \beta_n - 1; \beta_1, \ldots, \beta_n) \xrightarrow{\theta+\beta_n-1} V(\alpha_1, \ldots, \alpha_{n-1}, \beta_n; \beta_1, \ldots, \beta_n) \xrightarrow{\theta+\beta_n}$
$\ldots \xrightarrow{\theta+\beta_n+m-2} V(\alpha_1, \ldots, \alpha_{n-1}, \beta_n + m - 1; \beta_1, \ldots, \beta_n) \xrightarrow{\theta+\beta_n+m-1} V(\alpha_1, \alpha_n; \beta_1, \beta_n)$.
Obviously $\theta + \beta_n - 1$ has a non-trivial kernel. Choose $j \in \{-1, 0, \ldots, m-1\}$ maximal such that $(\theta + \beta_n + j)$ has a non-trivial kernel. Then this generates a one dimensional invariant subspace.
If $m \leq -2$ then consider the sequence

$V(\alpha_1,\dots,\alpha_{n-1},\alpha_n=\beta_n+m;\beta_1,\dots,\beta_n)\xrightarrow{\theta+\beta_n+m}$
$V(\alpha_1,\dots,\alpha_{n-1},\beta_n+m+1;\beta_1,\dots,\beta_n)\xrightarrow{\theta+\beta_n+m+1}\dots\xrightarrow{\theta+\beta_n-2}$
$V(\alpha_1,\dots,\alpha_{n-1},\beta_{n-1};\beta_1,\dots,\beta_n)\xrightarrow{\theta+\beta_n+m-1}V(\alpha_1,\dots,\alpha_{n-1},\beta_n;\beta_1,\dots,\beta_n).$
Since $\theta+\beta_n-1$ has a non-trivial kernel, choose $j\in\{m,m+1,\dots,-1\}$ minimal such that $(\theta+\beta_n+j)$ has a non-trivial kernel. Then this generates one dimensional invariant subspace. □

Definition 3.2.1 ([5], p 330.)**.** *Let V be a finite dimensional complex vector space. A linear map $g\in GL(V)$ is called a quasi-reflection if $g-Id$ has rank one. The determinant of a quasi-reflection is called the special eigenvalue of g.*

Proposition 3.2.2 ([5], p 330.)**.** *If $\alpha_j-\beta_k\notin\mathbb{Z}$ for all $k,j=1,2,\dots,n$, then the monodromy matrix $M(\ell_1)$ around $z=1$ is a quasi-reflection with special eigenvalue $c=e^{2\pi i\gamma}$.*

Proof. Let $z-1=x$ and recall the local exponents $0,1,2,\dots,n-2,\gamma=-1+\sum_{j=1}^{n}(\beta_j-\alpha_j)$ at $z=1$. We can write the holomorphic solutions $F_0,\dots,F_{n-1}$ as

$$F_0(x)=x^0h_0(x),F_1(x)=xh_1(x)\quad,\dots,\quad F_{n-2}(x)=x^{n-2}h_{n-2}(x),\quad F_{n-1}(x)=x^{\gamma}h_{n-1}(x)$$

where $h_j(x)$ are holomorphic functions with $h_j(0)\neq 0$ for all $j=0,1,\dots,n-1$.
If we let $x\mapsto e^{2\pi i}x$, then we obtain

$$\begin{aligned}F_0(e^{2\pi i}x)&=x^0h_0(x)=F_0(x)\\F_1(e^{2\pi i}x)&=xh_1(x)=F_1(x)\\&\vdots\\F_{n-2}(e^{2\pi i}x)&=xe^{2\pi i(n-2)}h_{n-2}(x)=xh_{n-2}F_{n-2}(x)\\F_{n-1}(e^{2\pi i}x)&=xe^{2\pi i\gamma}h_{n-1}(x)=e^{2\pi i\gamma}F_{n-1}(x)\end{aligned}$$

Therefore, we obtain

$$(\ell_0)_*\begin{pmatrix}F_0\\F_1\\\vdots\\F_{n-2}\\F_{n-1}\end{pmatrix}=\begin{pmatrix}1&0&0&\dots&0\\0&1&0&\dots&0\\\vdots&0&\ddots&\dots&0\\0&0&\dots&\ddots&0\\0&0&\dots&\dots&e^{2\pi i\gamma}\end{pmatrix}\begin{pmatrix}F_0\\F_1\\\vdots\\F_{n-2}\\F_{n-1}\end{pmatrix}$$

The rank of the matrix $M(\overrightarrow{\alpha},\overrightarrow{\beta})(\ell_1)-Id$ is at most one. If $M(\overrightarrow{\alpha},\overrightarrow{\beta})(\ell_1)=Id$, then this contradicts the assumption of $\alpha_j-\beta_k\notin\mathbb{Z}$, where

$$M(\overrightarrow{\alpha},\overrightarrow{\beta})(\ell_1)=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \dots & 0 \\ 0 & 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & \dots & e^{2\pi i\gamma} \end{pmatrix}$$

□

3.3 The Hypergeometric Group

Definition 3.3.1. *[5] Suppose $a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{C}^*$ with $a_j \neq b_k$ for all j and $k = 1,2,\dots,n$. A hypergemoetric group with numerator parameters $a_1,\dots,a_n$ and denominator parameters $b_1,\dots,b_n$ is a subgroup of $GL(n,\mathbb{C})$ generated by elements*

$$h_0,h_1,h_\infty \in GL(n,\mathbb{C})$$

such that $h_\infty h_1 h_0 = Id$.
Furthermore,

$$det(t-h_\infty) = \prod_{j=1}^{n}(t-a_j) \quad ,det(t-h_0^{-1}) = \prod_{j=1}^{n}(t-b_j)$$

and h_1 is a quasi-reflection in the sense of Definition (3.2.1)

Proposition 3.3.1 ([5], p 330.)**.** *Suppose $a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{C}^*$ with $a_j \neq b_k$ for all j and $k = 1,2,\dots,n$. Let $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n \in \mathbb{C}$ be such that $a_j = e^{2\pi i\alpha_j}$ and $b_j = e^{2\pi i\beta_j}$ for $j = 1,2,\dots,n$. Then the monodromy group of the hypergeometric equation*

$$D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)F = 0$$

is a hypergeometric group with parameters $a_1,\dots,a_n,b_1,\dots,b_n$.

Proof. Let us denote by
$H(a;b) = H(a_1,\dots,a_n;b_1,\dots,b_n) = M(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)(G)$ the monodromy group of $D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)F$. Notice that by Corollary (3) this gorup depends only on the numbers $a_1,\dots,a_n,b_1,\dots,b_n$ and not on the choice of their logarithms $\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n$. Also we write

$$h_0 = M(\ell_0), \quad h_1 = M(\ell_1), \quad h_\infty = M(\ell_\infty)$$

for the corresponding monodromy matrices around $z = 0,1,\infty$ where $M = M(\vec{\alpha},\vec{\beta})$. Then by using Corollary (3) and Proposition(3.2.1) it follows that $H(a;b)$ is a hypergeometric group with numerator parameters $a_1,\dots,a_n$ and denominator parameters $b_1,\dots,b_n$. □

Proposition 3.3.2 ([5], p 330.)**.** *Any hypergeometric group H generated by h_0,h_1,h_∞ as in Definiton* (3.2.1) *is an irreducible subgroup of $GL(n,\mathbb{C})$.*

Proof. Let $V_1 \subset \mathbb{C}^n$ be a proper H-invariant linear subspace and $V_2 := {}^{\mathbb{C}^n}/_{V_1}$. Then we get induced groups $H_1 \subset GL(V_1)$ and $H_2 \subset GL(V_2)$. Since h_1 is a quasi-reflection, either h_1 restricted to V_1 or h_1 restricted to V_2 is the identity. But since V_1 and V_2 are proper we get a contradiction with the assumption $a_j \neq b_k$ for all j , $k = 1,\ldots,n$. $\square$

The following theorem was obtained by Levelt in his PhD thesis.

Theorem 3.3.1 ([5], p 331.)**.** *(Levelt) Suppose that $a_1,\ldots,a_n,b_1,\ldots,b_k \in \mathbb{C}^*$ with $a_j \neq b_k$ for all $j,k = 1,\ldots,n$. Let $A_1,\ldots,A_n,B_1,\ldots,B_n \in \mathbb{C}$ be defined by*

$$\prod_{j=1}^{n}(t-a_j) = t^n + A_1t^{n-1} + \cdots + A_n, \quad \prod_{j=1}^{n}(t-b_j) = t^n + B_1t^{n-1} + \cdots + B_n \tag{3.3.1}$$

and let $A,B \in GL(n,\mathbb{C})$ be given by

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & -A_n \\ 1 & 0 & \ldots & 0 & -A_{n-1} \\ 0 & 1 & \ldots & 0 & -A_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -A_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \ldots & 0 & -B_n \\ 1 & 0 & \ldots & 0 & -B_{n-1} \\ 0 & 1 & \ldots & 0 & -B_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -B_1 \end{pmatrix} \tag{3.3.2}$$

Then the matrices $h_\infty = A, h_0 = B^{-1}, h_1 = A^{-1}B$ generate a hypergeometric group with parameters $a_1,\ldots,a_n,b_1,\ldots,b_n$. Moreover, any hypergeometric group with the same parameters is conjugated inside $GL(n,\mathbb{C})$.

Proof. ([5], p 332.) $\square$

Chapter 4

Algebraic Functions in terms of Pochhammer Hypergeometric Functions

In this chapter, we will be interested in algebraic functions. In the first section of this chapter we introduce the so-called Newton-Puiseux algorithm. Further, in the last two sections of this chapter we make us of the hypergeometric function theory which we studied in Chapter 2 and Chapter 3 so as to acquire our results.

4.1 Algebraic Functions, Newton Polygon Method and Puiseux Series

Definition 4.1.1. *A function $y(x)$ is said to be an algebraic function if it satisfies $f(x,y(x)) = 0$, where $f(x,y) \in \mathbb{C}[x,y]$ is a polynomial in x and y.*

We want to describe the zero set of f that is, the set of solutions of the equation $f(x,y) = 0$ in a suitable neighbourhood of the origin.
Newton begins by viewing $f(x,y) = 0$ as an implicit equation for y as a function of x and computes the solution y by an approximation process which yields an expansion of the solution y in powers of x. To obtain such solutions we shall make use of Newton's method but before introducing this method, let us recall the implicit function theorem.

Theorem 4.1.1. *(Special Implicit Function Theorem) Suppose that $F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ has continuous partial derivatives. Denote the points in $\mathbb{R}^{n+1}$ by (x,y) where $x \in \mathbb{R}^n$ and*

$y \in \mathbb{R}$, assume that (x_0, y_0) satisfy

$$F(x_0, y_0) = 0 \quad and \quad \frac{\partial F}{\partial y}(x_0, y_o) \neq 0.$$

Then there is an open ball $U \subset \mathbb{R}^n$ containing x_0 and an open interval $V \subset \mathbb{R}$ containing y_0 such that there is a unique function $y = g(x)$ defined over $x \in U$ and $y \in V$ that satisfies $F(x, bg(x)) = 0$. If $x \in U$ and $y \in V$ satisfy $F(x, y) = 0$, then $y = g(x)$.

Example 4.1.1. *Suppose that we are given the following equations*

$$x^2 - y^2 - u^3 + v^2 + 4 = 0$$

$$2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$

Can we solve u and v in terms of x and y in a neighbourhood of the solution $(x, y, u, v) = (2, -1, 2, 1)$?

We write

$$F(x, y, u, v) = (x^2 - y^2 - u^3 + v^2 + 4, 2xy + y^2 - 2u^2 + 3v^4 + 8)$$

so that $F(2, -1, 2, 1) = (0, 0)$.
We consider the Jacobian matrix

$$J = \begin{pmatrix} -3u^2 & 2v \\ -4u & 12v^2 \end{pmatrix} |_{(2,1)} = \begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix}$$

Since $det(J) \neq 0$, we can solve $f(u, v)$ in a neighbourhood of $(x_0, y_0) = (2, -1)$. More precisely, there are neighbourhoods U of $(2, -1)$ and W of $(2, 1)$ so that for each $(x, y) \in U$ there is a unique $(u, v) \in W$ such that $F(x, y, u, v) = 0$.
Thus, there is a unique function $f : U \longrightarrow W$ such that $F(x, y, f_1(x, y), f_2(x, y)) = 0$.
In our case the implicit function theorem asserts that if $f(x, y) = C$ where $f(x, y)$ is a continuous function and C is a constant and $\frac{\partial f}{\partial y} \neq 0$ at some point p, then y may be expressed as a function of x in some domain about p i.e there exits a unique function over that domain such that $y = g(x)$.

Remark 4.1.1. *If $f(x, y) = 0$ and y is a function of x, then we have*

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

which implies

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

Definition 4.1.2. *(Homogeneous polynomials) A non-zero polynomial $f \in A[x_1, \ldots, x_k]$ is homogeneous of degree n. If*

$$f(x_1, \ldots, x_k) = t^n f(x_1, \ldots, x_k).$$

Example 4.1.2. *For example, the polynomial $f(x,y) = x^2 + y^2$ is a homogeneous polynomial of degree 2. Since*

$$f(tx,ty) = t^2 f(x,y).$$

We now want to describe Newton polygon method for an algebraic function $f(x,y) \in \mathbb{C}[x,y]$, we begin with the following simple example:

Example 4.1.3 ([6],p 376.)**.** *$f(x,y) = x^p - y^q = 0$ has the solution $y = x^{\frac{p}{q}}$. In fact, $'q'$ independent solutions to the algebraic function $f(x,y) = x^p - y^q = 0$, are given by*

$$y_k(x) = e^{2\pi i \frac{p}{q} k} x^{\frac{p}{q}} \quad for \quad k = 0, \ldots, q-1.$$

where we assume p and q are coprime.

More generally, we can find a solution of the form

$$y = tx^{\mu},$$

where $t \in \mathbb{C}$ and $\mu = \frac{p}{q} \in \mathbb{Q}^+$, whenever $f(x,y)$ is a polynomial of the form

$$f(x,y) = \sum_{\alpha + \mu\beta = \nu} a_{\alpha\beta} x^{\alpha} y^{\beta},$$

one can find a solution $y = tx^{\mu}$ by substituting in $f(x,y)$

$$f(x,tx^{\mu}) = \sum_{\alpha+\mu\beta=\nu} a_{\alpha\beta} x^{\alpha} t^{\beta} x^{\mu\beta} = \sum_{\alpha+\mu\beta=\nu} a_{\alpha\beta} x^{\alpha+\mu\beta} t^{\beta} = x^{\nu} \sum_{\alpha+\mu\beta=\nu} a_{\alpha\beta} t^{\beta} = x^{\nu} g(t)$$

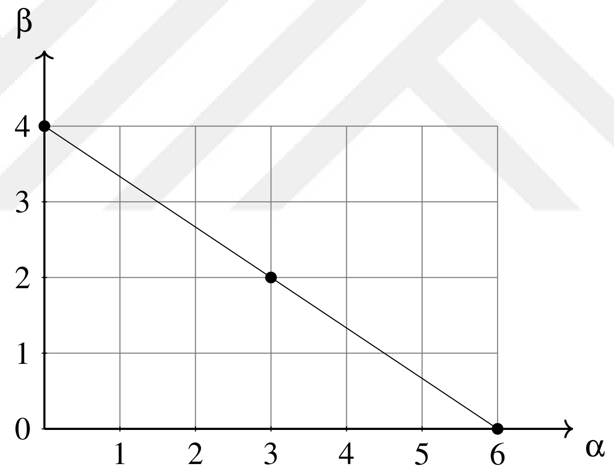
If t_0 is a zero of the polynomial $g(t)$, then $y = t_0 x^{\mu}$ is a solution of $f(x,y)$. We can choose $t_0 \neq 0$ if $g(t) \neq ct^m$, that is when $f(x,y)$ contains two distinct monomials.

The condition that $f(x,y)$ contains only monomials $x^\alpha y^\beta$ for which $\alpha+\mu\beta = v$ can be described -this is an essential part of Newton's idea- in the following geometric way: Each monomial $x^\alpha y^\beta$ corresponds to the pair (α,β) of natural numbers, hence to a point of $\mathbb{N}^2$ of points (α,β) with integer coordinates in the plane $\mathbb{R}^2$.
When $f(x,y) = \sum a_{\alpha\beta} x^\alpha y^\beta$ is an arbitrary power series, we consider the set of lattice points (α,β) whose monomials appear in the power series i.e those for which the coefficients $a_{\alpha\beta} \neq 0$. We call this set of lattice points the carrier of f and denote it by $\Delta(f)$,

$$\Delta(f) = \{(\alpha,\beta) \in \mathbb{N}^2 | \quad a_{\alpha\beta} \neq 0\}.$$

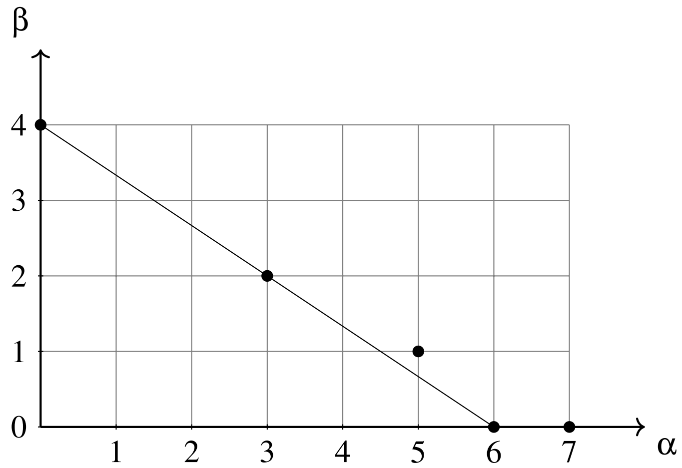
Let us look at some examples.

Example 4.1.4 ([6],p 377.). $\quad f(x,y) = y^4 - 2x^3y^2 + x^6$



In this case, $\quad \Delta(f) = \{(0,4),(3,2),(6,0)\}$.

Example 4.1.5. $\quad f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$



In this case, $\Delta(f) = \{(0,4),(3,2),(5,1),(6,0),(7,0)\}$

The condition that there be rational numbers μ and v so that $\alpha + \mu\beta = v$ for all $(\alpha, \beta) \in \Delta(f)$ means that all points of $\Delta(f)$ lie on a line. This line then has slope $-\frac{1}{\mu}$ and it meets the α-axis at $\alpha = v$.
In example (4.1.4), the condition is satisfied with $\mu = \frac{3}{2}$ and $v = 6$. We can therefore find a solution of the form $y = tx^{\frac{3}{2}}$, $t \in \mathbb{C}$. Substituting this in f gives,

$$\begin{aligned} f(x, tx^{\frac{3}{2}}) &= t^4x^6 - 2x^3t^2x^3 + x^6 \\ &= x^6(t^4 - 2t^2 + 1). \end{aligned}$$

Denote $t^4 - 2t^2 + 1$ by $g(t)$. Then the zeroes of $g(t)$ are given by $t = \pm 1$. Thus, we obtain two solutions of our equation $y = tx^{\frac{3}{2}}$, that is $y = x^{\frac{3}{2}}$ and $y = -x^{\frac{3}{2}}$.
However, in example (4.1.5), the points of $\Delta(f)$ do not all lie on a line and therefore we cannot find a solution as simply as in example (4.1.4). Let us view the function f in example (4.1.5) as a sum

$$f = \tilde{f} + h, \quad \textit{where} \quad \tilde{f}(x,y) = y^4 - 2x^3y^2 + x^6 \quad \textit{and} \quad h(x,y) = -4x^5y - x^7$$

Here the order of monomial $x^{\alpha}y^{\beta}$ is not $\alpha + \beta$ as usual but $\alpha + \mu\beta$ with respect to this definition of order, all monomials of $\tilde{f}$ have order 6, x^5y has order $\frac{13}{2}$ and x^7 has order 7.$(\mu = \frac{3}{2})$
Now, when we simply substitute the solution $y = x^{\frac{3}{2}}$ in $\tilde{f}(x,y) = 0$ in $f(x,y)$, then f does not vanish but at least the terms of lower order do. This means that we can regard the solution $y = x^{\frac{3}{2}}$ of $\tilde{f}(x,y) = 0$ as an approximate solution to $f(x,y) = 0$. We express the solution as $y = x^{\frac{3}{2}} + \quad \textit{terms of higher order} \quad = y = x^{\frac{3}{2}}(1 + y_1)$.
In order to avoid calculating with fractional powers of x we substitute $x^{\frac{1}{2}} = x_1$.
Now, we substitute in $f(x,y)$ for

$$y = x_1^3(1 + y_1) \quad x = x_1^2$$

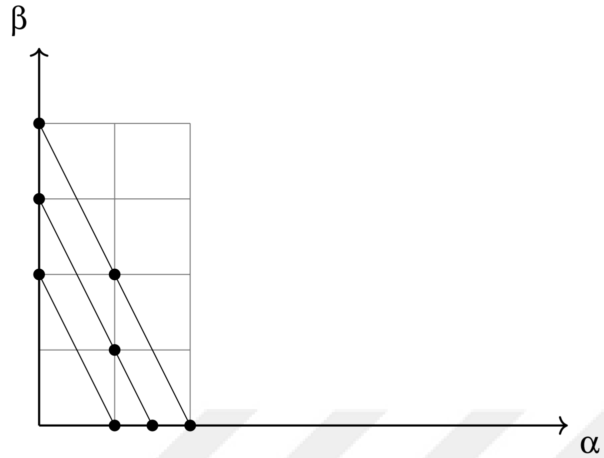
Then we obtain

$$f(x_1^2, x_1^3(1 + y_1)) = x_1^{12} f_1(x_1, y_1)$$

where

$$f_1(x_1, y_1) = y_1^4 + 4y_1^3 + 4y_1^2 - 4x_1y_1 - 4x_1 - x_1^2.$$

Again we consider the carrier $\Delta(f_1)$:



We see that $\Delta(f_1)$ does not lie on one line, however it lies on a system of parallel lines of slope -2. Thus , we seek a solution of $f_1(x_1,y_1)=0$ of the form

$$y_1 = t_1 x_1^{\frac{1}{2}}$$

Substitution of in the terms of lower order yields

$$4y_1^2 - 4x_1 = x_1(4t_1^2 - 4) = 0,$$

this gives $t_1 = \pm 1$. If we choose $t_1 = 1$ and substitute $y_1 = x_1^{\frac{1}{2}}$ in $f_1(x_1,y_1)$, then we get

$$f_1(x_1, x_1^{\frac{1}{2}}) = x_1^2 + 4x_1^{\frac{3}{2}} + 4x_1 - 4x_1^{\frac{3}{2}} - 4x_1 - x_1^2 = 0.$$

Hence, $y_1 = x_1^{\frac{1}{2}}$ is a solution of $f_1(x_1,y_1)=0$. Because of this,

$$y = x^{\frac{3}{2}}(1+y_1) = x^{\frac{3}{2}}(1+x_1^{\frac{1}{2}}) = x^{\frac{3}{2}}(1+x^{\frac{1}{4}})$$

is a solution of $f(x,y)=0$. Therefore, we found a solution $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$ of the original equation $f(x,y)=0$.
After these examples, we now describe the general Newton process:
Let $f(x,y) = \sum a_{\alpha\beta} x^{\alpha} y^{\beta}$ be a convergent power series and without loss of generality, assume that f is y-general (say of order $m>0, i.e a_{0m} \neq 0$ and $a_{0i}=0$ for $i<m$). Define the carrier of f as

$$\Delta(f) = \{(\alpha,\beta) \in \mathbb{N}^2 | \quad a_{\alpha\beta} \neq 0\}.$$

We now want to find the line which we begin the process. It is the steepest possible line

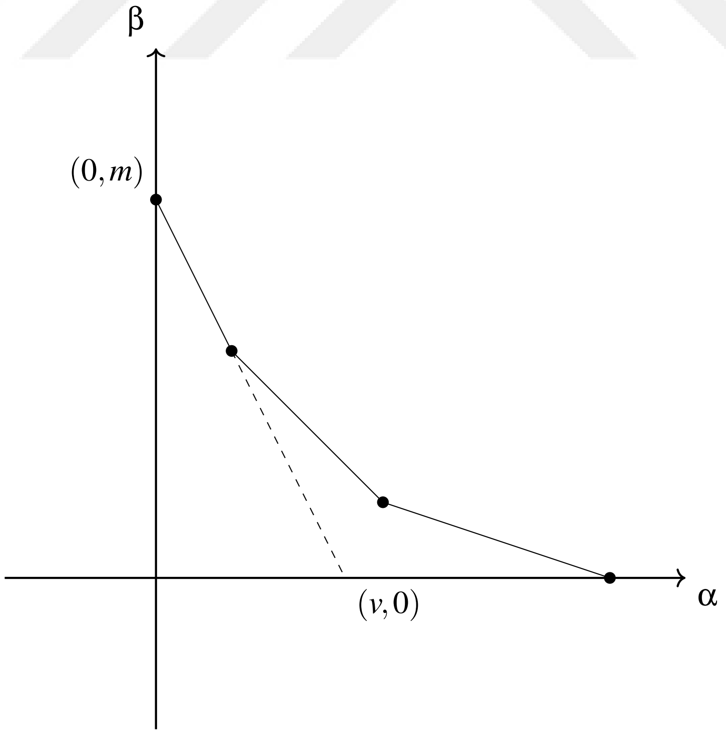
through the lowest point of the carrier on the β-axis.
For each point p of the carrier of f we consider the positive quadrant $p+(\mathbb{R}^+)^2$ moved up to p. From the union of all these displaced with quadrants we construct the convex hull

$$conv(\cup_{p\in\Delta(f)}(p+(\mathbb{R}^+)^2))$$

The boundary consists of a compact polygonal path (where all the segments have negative slope) and two half lines. This compact polygonal path is called the Newton polygon of f. The steepest segment of the Newton polygon is the starting line that we want. We let $-\frac{1}{\mu_0}$ be the slope of the steepest segment of the Newton polygon.
Now, consider $f(x,y)$ given by

$$f(x,y) = \sum_{\alpha+\mu_0\beta=v} a_{\alpha\beta}x^{\alpha}y^{\beta} + \sum_{\alpha+\mu_0\beta>v} a_{\alpha\beta}x^{\alpha}y^{\beta} = \tilde{f}(x,y)+h(x,y)$$

Such an $f(x,y)$ has the following Newton polygon:



Here v is the intercept on the α-axis of the line through $(0,m)$ with slope $-\frac{1}{\mu_0}$ (hence $v=m\mu_0$), and by construction there are at least two number pairs $(\alpha,\beta)\in\Delta(f)$ with $\alpha+\mu_0\beta$.
We apply the perturbation method to $\tilde{f}+h$.

First we find Puiseux monomial $y = y_0(x)$ such that $\tilde{f}(x, y_0) = 0$.
Second we look for $y_1(x) = y(1+\tilde{y}_1)$, we substitute $y_0 = t_0 x^{\mu_0}$ into $\tilde{f}$ and get

$$\tilde{f}(x, t_0 x^{\mu_0}) = (\sum_{\alpha+\mu_0\beta=\nu} a_{\alpha\beta} t_0^{\beta}) x^{\nu},$$

so t_0 must be a root of an algebraic equation say $g(t) = 0$, where $g(t) = \sum_{\alpha+\mu_0\beta=\nu} a_{\alpha\beta} t^{\beta}$.
As $\mu_0 = \frac{\nu}{m}$ is rational it admits an expression

$$\mu_0 = \frac{p_0}{q_0}, \quad gcd(p_0, q_0) = 1$$

Thus, q_0 is uniquely determined.
We substitute $x_1 := x^{\frac{1}{q_0}}$ so that the first approximate solution is $y_0 = t_0 x_1^{p_0}$.
We make a fresh start with

$$y = x_1^{p_0}(t_0 + y_1)$$

and plug this throughout the equation $f(x, y) = 0$.
We plug

$$f(x_1^{q_0}, x_1^{p_0}(t_0 + y_1)) = \tilde{f}(x_1^{q_0}, x_1^{p_0}(t_0 + y_1)) + h(x_1^{q_0}, x_1^{p_0}(t_0 + y_1))F_1$$

As $\tilde{f}(x_1^{q_0}, t_0 x_1^{p_0}) = 0$ and $F_1 = x_1^{\nu q_0} f_1(x_1, y_1)$,
we shall look for y_1 satisfying $f_1(x_1, y_1) = 0$, where degree of f_1 in y_1 is $m_1 \leq m$.
We draw the new Newton polygon of f_1 and repeat the same procedure to find $\alpha + \mu_1\beta = \nu_1$.
Now find y_1 by solving $f_1(x_1, y_1)$,

$$y_1 = x_1^{\frac{p_1}{q_1}}(t_1 + y_2)$$

Now put $x_2 = x_1^{\frac{1}{q_1}}$ so that $y_1 = x_2^{p_1}(t_1 + y_2)$. We plug it into $f_1(x_1, y_1)$, then

$$\tilde{f}(x_2^{q_1}, x_2^{p_1}(t_1 + y_2)) = x_2^{\nu_1 q_1} f_2(x_2, y_2).$$

In this way we get a sequence of approximate solutions

$$y_0 = t_0 x^{\mu_0}$$

$$y = x^{\mu_0}(t_0 + y_1) = x^{\frac{p_0}{q_0}}(t_0 + y_1) = x_1^{p_0}(t_0 + y_1)$$
$$y_1 = x_1^{\mu_1}(t_1 + y_2) = x_1^{\frac{p_1}{q_1}}(t_1 + y_2) = x_2^{p_1}(t_1 + y_2)$$
$$y_2 = x_1^{\mu_2}(t_2 + y_3) = x_1^{\frac{p_2}{q_2}}((t_2 + y_3)) = x_3^{p_2}(t_2 + y_3)$$

Note that
$$x_3 = x_2^{\frac{1}{q_2}} = x_1^{\frac{1}{q_1 q_2}} = x^{\frac{1}{q_0 q_1 q_2}}$$

Therefore, we obtain

$$\begin{aligned} y &= x^{\mu_0}(t_0 + x_1^{\mu_1}(t_1 + x_2^{\mu_2}(t_2 + \cdots))) \\ &= t_0 x^{\mu_0} + t_1 x^{\mu_0} x_1^{\mu_1} + t_2 x^{\mu_0} x_1^{\mu_1} x_2^{\mu_2} + \cdots \end{aligned}$$

The last expression for y is also known as the Puiseux expansion of $f(x,y) = 0$. A formal definition is as follows :

Definition 4.1.3 ([6],p 385.)**.** *The fractional power series* $y(x) = \sum_{i=0}^{\infty} a_i x^{\frac{i}{n}}$ *is the Puiseux expansion for the curve with equation* $f(x,y) = 0$*, where* $a_i \in \mathbb{C}$ *and* $n = q_0 q_1 \ldots q_n$*.*

We end this section with the following remark.

Remark 4.1.2. *Puiseux expansion of an algebraic function* $f(x,y)$ *will be very important and will play a significant role in the last section.*

4.2 Trinomial Algebraic Function

In this section, we shall be interested in a specific type of algebraic functions(trinomial equations).
Let us consider a general trinomial equation given by

$$\varphi(x,y) = y^n + xy^p - 1 = 0, \quad (n > p) \tag{4.2.1}$$

Remark 4.2.1. *The singular points of the equation* (4.2.1) *are given by*

$$x^n = (-1)^p \frac{n^n}{p^p q^q}, \quad (q = n - p)$$

Proof. We have

$$\varphi(x,y) = y^n + xy^p - 1 = 0 \tag{4.2.2}$$

and by the implicit function theorem we have

$$\frac{d\varphi}{dy} = ny^{n-1} + xpy^{p-1} = (ny^n + xpy^p)y^{-1} = 0, \tag{4.2.3}$$

then using equations (4.2.2) and (4.2.3) we obtain

$$(p-n)xy^p + n = 0 \Leftrightarrow x = \frac{n}{q}\frac{1}{y^p}$$

On the other hand, by the equation (4.2.3) we see that

$$ny^n + xpy^p = 0 \Leftrightarrow xpy^p = -ny^n \Leftrightarrow x = -\frac{n}{p}y^{n-p}$$

Combining these one can have

$$\frac{n}{q}\frac{1}{y^p} = -\frac{n}{p}\frac{y^n}{y^p} \Leftrightarrow \frac{-p}{q} = y^n,$$

then it follows that

$$x^n = \frac{n^n}{q^n}\left(\frac{1}{y^p}\right)^n = \frac{n^n}{q^n}\frac{q^p}{(-p)^p}$$
$$x^n = \frac{(-1)^p n^n q^{-q}}{p^p}$$
$$x^n = (-1)^p \frac{n^n}{p^p q^q}$$

□

Example 4.2.1. *($n = 2, p = 1$) The singular points of the following algebraic equation*

$$\varphi(x,y) = y^2 + xy - 1 = 0$$

are given by

$$x^2 = -4$$

or

$$x = \pm 2i.$$

In the next sections we shall be interested in solutions of algebraic functions (4.2.1) rather than their singular points. We note that if $y(x)$ is a solution of (4.2.1) then

$$z = \varepsilon_k y((\varepsilon_k)^p x)$$

is also a solution to $\varphi(x,y) = y^n + xy^p - 1 = 0$, where $\varepsilon_k = e^{\frac{2\pi i k}{n}}$ for $k = 0, 1, \ldots n-1$. The following theorem will carry great importance for our further discussions.

Theorem 4.2.1 ([7], p 103.)**.** *(The Mellin Inversion Theorem) Let $z = \sigma + it \in \mathbb{C}$. Let $f(z)$ be analytic in the strip $\alpha < \sigma < \beta$ and let $\int_{-\infty}^{\infty} |f(\sigma + it)|dt$, converge in this strip. Further, assume that the function $f(z)$ tends uniformly to 0 as $|t| \to \infty$ in every strip $\alpha + \delta \leq \sigma \leq \beta - \delta$, for a positive number δ. If for a real positive x and fixed σ we define*

$$g(x) := \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} x^{-z} f(z)dz$$

then

$$f(z) = \int_0^{\infty} x^{z-1} g(x)dx$$

Proof. Our aim is to show that $f(z) = \int_0^{\infty} x^{z-1} g(x)dx$ in the strip $\alpha < \sigma < \beta$.

Since $f(z)$ converges uniformly to 0 in the strip $\alpha+\delta \le \sigma \le \beta-\delta$, as $|t| \to \infty$, we can displace the path of integration of $g(x)$ with parallel to itself as long as it stays within the strip $\alpha < \sigma < \beta$. Hence, $g(x)$ does not depend on σ.
Choose σ_1 and σ_2 so that $\alpha < \sigma_1 < \sigma < \sigma_2 < \beta$, then consider

$$\int_0^\infty x^{z-1} g(x)dx = \int_0^1 x^{z-1} dx \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1+i\infty} x^{-z_1} f(z_1)dz_1 + \int_1^\infty x^{z-1} dx \frac{1}{2\pi i} \int_{\sigma_2 - i\infty}^{\sigma_2+i\infty} x^{-z_2} f(z_2)dz_2$$

As $f(z)$ is uniformly convergent in the strip $\alpha < \sigma_1 < \sigma < \sigma_2 < \beta$, the order of integration can be interchanged. Hence

$$\begin{aligned}
\int_0^\infty x^{z-1} g(x)dx &= \int_0^1 x^{z-1-z_1} dx \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1+i\infty} f(z_1)dz_1 + \int_1^\infty x^{z-1-z_2} dx \frac{1}{2\pi i} \int_{\sigma_2 - i\infty}^{\sigma_2+i\infty} f(z_2)dz_2 \\
&= \left(\frac{x^{z-z_1}}{z-z_1}\right)\Bigg|_{x=0}^{x=1} \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1+i\infty} f(z_1)dz_1 + \left(\frac{x^{z-z_2}}{z-z_2}\right)\Bigg|_{x=1}^{x=\infty} \frac{1}{2\pi i} \int_{\sigma_2 - i\infty}^{\sigma_2+i\infty} f(z_2)dz_2 \\
&= \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1+i\infty} \frac{f(z_1)dz_1}{z-z_1} - \frac{1}{2\pi i} \int_{\sigma_2 - i\infty}^{\sigma_2+i\infty} \frac{f(z_2)dz_2}{z-z_2} \\
&= \frac{1}{2\pi i} \int_{\sigma_2 - i\infty}^{\sigma_2+i\infty} \frac{f(z_2)dz_2}{z_2-z} + \frac{1}{2\pi i} \int_{\sigma_1 + i\infty}^{\sigma_1-i\infty} \frac{f(z_1)dz_1}{z_1-z}
\end{aligned}$$

But Cauchy's integral theorem asserts that the last expression is equal to $f(z)$, and thus completes the proof of the theorem. □

Moreover, $f(z)$ is called the Mellin transformation of the function $g(x)$ and $g(x)$ is called the inverse Mellin transformation of the function $f(z)$.

Now in order to find solutions of the algebraic equation $\varphi(x,y) = y^n + xy^p - 1 = 0$ we utilize the Mellin Transformation. If $y(x)$ is a solution of (4.2.1), then the Mellin Transform of $y(x)$ is given by

$$\int_0^\infty y^\mu x^{z-1} dx$$

where μ is a positive real number and z is a complex variable.
The reason why we use such a transformation is because the Mellin Transformation of such a solution can be written in terms of the Γ-function. The following lemma shows the relation.

Lemma 4.2.1 ([2], p 38.). *For a solution $y(x)$ of* (4.2.1),*the Mellin Transformation of*

$y(x)$ reads

$$\int_0^\infty y^\mu x^{z-1}dx = \frac{\mu}{n}\frac{\Gamma(z)\Gamma(\frac{\mu-pz}{n})}{\Gamma(\frac{\mu+qz}{n}+1)}$$

Proof. If we substitute $x = \frac{1-y^n}{y^p}$, then we get $dx = -(py^{p-1}+(n-p)y^{n-p-1})dy$, and the boundaries become $x=\infty, y=0$ and $x=0, y=1$.
Consider

$$\begin{aligned}
&\int_0^\infty y^\mu x^{z-1}dx = \int_1^0 y^\mu (1-y^n)^{z-1} y^{-pz+p}[-(py^{p-1}+(n-p)y^{n-p-1})]dy \\
&= -p\int_1^0 y^{\mu-pz+p-p-1}(1-y^n)^{z-1}dy - (n-p)\int_1^0 y^{\mu-pz+p+n-p-1}(1-y^n)^{z-1}dy \\
&= p\int_0^1 y^{\mu-pz-1}(1-y^n)^{z-1}dy + (n-p)\int_0^1 y^{\mu-pz+n-1}(1-y^n)^{z-1}dy. \qquad (4.2.4)
\end{aligned}$$

Let $y^n = t$, $y = t^{\frac{1}{n}}$, $dy = \frac{1}{n}t^{\frac{1-n}{n}}dt$ and recall the Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

for $x, y > 0$. Then the equation (4.2.4) becomes

$$\begin{aligned}
&\frac{p}{n}\int_0^1 t^{\frac{\mu-pz}{n}-1}(1-t)^{z-1}dt + \frac{(n-p)}{n}\int_0^1 t^{\frac{\mu-pz}{n}}(1-t)^{z-1}dt \\
&= \frac{p}{n}B(\frac{\mu-pz}{n}, z) + \frac{(n-p)}{n}B(\frac{\mu-pz}{n}+1, z)
\end{aligned}$$

We know that from the theorem (1.1.5)

$$\frac{p}{n}B(\frac{\mu-pz}{n}, z) = \frac{p}{n}\frac{\Gamma(\frac{\mu-pz}{n})\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+z)}$$

Likewise,

$$\frac{(n-p)}{n}B(\frac{\mu-pz}{n}+1, z) = \frac{(n-p)}{n}\frac{\Gamma(\frac{\mu-pz}{n}+1)\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+1+z)}$$

Thus, we have

$$\frac{p}{n}\frac{\Gamma(\frac{\mu-pz}{n})\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+z)} - \frac{p}{n}\frac{\Gamma(\frac{\mu-pz}{n}+1)\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+1+z)} + \frac{\Gamma(\frac{\mu-pz}{n}+1)\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+1+z)}. \qquad (4.2.5)$$

Note that

$$\Gamma(\frac{\mu-pz}{n}+1) = (\frac{\mu-pz}{n})\Gamma(\frac{\mu-pz}{n}),$$

and

$$\Gamma(\frac{\mu-pz}{n}+1+z)=(\frac{\mu-pz}{n}+z)\Gamma(\frac{\mu-pz}{n}+z).$$

Then the equation (4.2.5) yields

$$\begin{aligned}\left[\frac{pz}{\mu-pz+nz}\right]\frac{\Gamma(\frac{\mu-pz}{n})\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+z)}+\left[\frac{\frac{\mu-pz}{n}}{\frac{\mu-pz}{n}+z}\right]\frac{\Gamma(\frac{\mu-pz}{n})\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+z)}\\ =\frac{\Gamma(\frac{\mu-pz}{n})\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+z)}\left[\frac{pz}{\mu-pz+nz}+\frac{\frac{\mu-pz}{n}}{\frac{\mu-pz}{n}+z}\right]\end{aligned}$$

This gives us

$$\begin{aligned}\frac{\Gamma(\frac{\mu-pz}{n})\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+z)}\frac{\mu}{\mu-pz+nz}&=\frac{\mu}{\frac{n}{n}(\mu-pz+nz)}\frac{\Gamma(\frac{\mu-pz}{n})\Gamma(z)}{\Gamma(\frac{\mu-pz}{n}+z)}\\ &=\frac{\mu}{n}\frac{\Gamma(\frac{\mu-pz}{n})\Gamma(z)}{(\frac{\mu-pz+nz}{n})\Gamma(\frac{\mu-pz}{n}+z)}\\ &=\frac{\mu}{n}\frac{\Gamma(\frac{\mu-pz}{n})\Gamma(z)}{\Gamma(\frac{\mu-pz+nz}{n}+1)}\end{aligned}$$

Consequently, we obtain

$$\int_0^\infty y^\mu x^{z-1}dx=\frac{\mu}{n}\frac{\Gamma(z)\Gamma(\frac{\mu-pz}{n})}{\Gamma(\frac{\mu+(n-p)z}{n}+1)}.$$

□

Example 4.2.2. *For $n=2, p=1$,*

$$\int_0^\infty\left(\frac{-x+\sqrt{x^2+4}}{2}\right)^\mu x^{z-1}dx=\frac{\mu}{2}\frac{\Gamma(z)\Gamma(\frac{\mu-z}{2})}{\Gamma(\frac{\mu+z}{2}+1)}.$$

Example 4.2.3. *For $n=3, p=1$,*

$$\int_0^\infty\left(\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{4}+\left(\frac{x}{3}\right)^3}}+\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{4}+\left(\frac{x}{3}\right)^3}}\right)^\mu x^{z-1}dx=\frac{\mu}{3}\frac{\Gamma(z)\Gamma(\frac{\mu-z}{3})}{\Gamma(\frac{\mu+2z}{3}+1)}.$$

Remark 4.2.2. *From now on, we choose μ to be always* 1.

Taking into account the theorem (4.2.1), we introduce the following proposition

Proposition 4.2.1 ([2], p 39.). *A special solution of the algebraic equation* $\varphi(x,y) = y^n + xy^p - 1 = 0$*, is given by the Mellin Inversion formula*

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{n}\frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)} x^{-z}dz, \quad 0 < c < \frac{1}{p}. \tag{4.2.6}$$

Proof. In order to show the convergence of the integral (4.2.6) we need to estimate the integrand

$$\frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)} x^{-z} \tag{4.2.7}$$

by the aid of Stirling's formula (1.1.4).

$$|\Gamma(s+it)| \sim \sqrt{2\pi} e^{-\frac{\pi}{2}|t|-s}|t|^{s-\frac{1}{2}} \tag{4.2.8}$$

$$\left|\Gamma(\frac{1-p(s+it)}{n})\right| \sim \sqrt{2\pi} e^{-\frac{\pi}{2}|\frac{pt}{n}|-(\frac{1-ps}{n})}\left|\frac{pt}{n}\right|^{\frac{1-ps}{n}-\frac{1}{2}} \tag{4.2.9}$$

$$\left|\Gamma(\frac{1+q(s+it)}{n}+1)\right| \sim \sqrt{2\pi} e^{-\frac{\pi}{2}|\frac{qt}{n}|-(1+\frac{qs}{n})}\left|\frac{qt}{n}\right|^{\frac{qs}{n}+\frac{1}{2}} \tag{4.2.10}$$

and

$$x = |x|e^{-i\theta} \tag{4.2.11}$$

Then neglecting s, it follows that

$$\begin{aligned}
\frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)} x^{-z} &\sim \sqrt{2\pi}\frac{e^{-\frac{\pi}{2}|t|}e^{-\frac{\pi}{2}|\frac{pt}{n}|}}{e^{-\frac{\pi}{2}|\frac{qt}{n}|}} e^{-\theta t} \\
&= \sqrt{2\pi} e^{-\frac{\pi}{2}|t|(1+\frac{p}{n}-\frac{q}{n})} e^{-\theta t} \\
&= \sqrt{2\pi} e^{-|t|\frac{\pi p}{n}} e^{-\theta t}
\end{aligned}$$

(i) If $t < 0$, then $e^{t\frac{\pi p}{n}-\theta t} = e^{t(\frac{\pi p}{n}-\theta)}$ implies that $\frac{\pi p}{n} - \theta \geq 0$, as $t \to -\infty$.

(ii) If $t > 0$, then $e^{-t\frac{\pi p}{n}-\theta t} = e^{t(-\frac{\pi p}{n}-\theta)}$ implies that $\frac{-\pi p}{n} - \theta \leq 0$, as $t \to \infty$.

Therefore, the integral (4.2.6) converges in the angular domain

$$-\frac{\pi p}{n} \leq \theta \leq \frac{\pi p}{n}$$

as $|t| \to \infty$. □

Example 4.2.4. *Let us try to find a solution of the algebraic equation* $\varphi(x,y) = y^3 + xy - 1 = 0$, *by using Mellin Inversion formula*

We have

$$y(x) = \frac{1}{3}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} x^{-z} dz.$$

Let $\vartheta_x = x\frac{d}{dx}$ be Euler's differential operator and observe that

$$(-\vartheta_x + 2)(-\vartheta_x + 1)(-\vartheta_x)y(x) = \lambda \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z+3)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} x^{-z} dz \tag{4.2.12}$$

where $\lambda = \frac{1}{6\pi i}$. Now, put $z + 3 = z'$ and $z' - 3 = z$. Then we have

$$(-\vartheta_x + 2)(-\vartheta_x + 1)(-\vartheta_x)y(x) = \lambda \int_{(c+3)-i\infty}^{(c+3)+i\infty} \frac{\Gamma(z')\Gamma(\frac{1-z'}{3}+1)}{\Gamma(\frac{4+2z'}{3}-2)} x^{-(z'-3)} dz' \tag{4.2.13}$$

$$= \lambda x^3 \int_{(c+3)-i\infty}^{(c+3)+i\infty} \frac{\Gamma(z')(\frac{1+2z'}{3})(\frac{-2+2z'}{3})(\frac{1-z'}{3})\Gamma(\frac{1-z'}{3})}{\Gamma(\frac{4+z'}{3})} x^{-z'} dz' \tag{4.2.14}$$

$$= x^3(\frac{1-2\vartheta_x}{3})(\frac{-2-2\vartheta_x}{3})(\frac{1+\vartheta_x}{3})\lambda \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z')\Gamma(\frac{1-z'}{3})}{\Gamma(\frac{4+2z'}{3})} x^{-z'} dz'. \tag{4.2.15}$$

We note that in the equation (4.2.13) since $z' = 1$ is not a pole any more of the factor $\Gamma(\frac{1-z'}{3}+1)$ we can deform the integral given in (4.2.13) into the integral (4.2.15) to obtain the same result.

Then we obtain

$$\left[x^3(\frac{1-2\vartheta_x}{3})(\frac{-2-2\vartheta_x}{3})(\frac{1+\vartheta_x}{3}) + \vartheta_x(\vartheta_x-2)(\vartheta_x-1)\right] y(x) = 0 \tag{4.2.16}$$

Substitute $x^3 = t$, then $\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} = 3x^2\frac{d}{dt} \Rightarrow \vartheta_x = 3\vartheta_t$. It follows that

$$\left[t(\frac{1}{3} - 2\vartheta_t)(-\frac{2}{3} - 2\vartheta_t)(\frac{1}{3} + \vartheta_t) + 3(\vartheta_t)(3\vartheta_t - 2)(3\vartheta_t - 1)\right] y(t^{\frac{1}{3}}) = 0$$

or

$$\left[2^2t(\vartheta_t-\frac{1}{6})(\vartheta_t+\frac{1}{3})(\vartheta_t+\frac{1}{3})+3^3(\vartheta_t)(\vartheta_t-\frac{2}{3})(\vartheta_t-\frac{1}{3})\right]y(t^{\frac{1}{3}})=0. \tag{4.2.17}$$

Now, put $-\frac{2^2}{3^3}t=s$, then $\vartheta_t=\vartheta_s$, then the equation (4.2.17) becomes

$$\left[-s(\vartheta_s-\frac{1}{6})(\vartheta_s+\frac{1}{3})^2+(\vartheta_s)(\vartheta_s-\frac{2}{3})(\vartheta_s-\frac{1}{3})\right]y((-\frac{3^3}{2^2}s)^{\frac{1}{3}})=0 \tag{4.2.18}$$

The latter equation looks familiar to us. In fact, the equation (4.2.18) is a Pochhammer hypergeometric differential equation. Furthermore, $y((\frac{3^3}{2^2}s)^{\frac{1}{3}})$ satisfies this differential equation. From the previous chapter we know that this differential equation (4.2.18) has a solution. Namely, a hypergeometric function of the form

$$F(-\frac{1}{6},\frac{1}{3},\frac{1}{3},-\frac{2}{3},-\frac{1}{3};s)=\sum_{k=0}^{\infty}\frac{(-\frac{1}{6})_k(\frac{1}{3})_k^2}{(-\frac{2}{3})_k(-\frac{1}{3})_k k!}s^k$$

Hence we deduce that

$$y((-\frac{3^3}{2^2}s)^{\frac{1}{3}})=F(-\frac{1}{6},\frac{1}{3},\frac{1}{3},-\frac{2}{3},-\frac{1}{3};s)$$

Therefore if we substitute backwards, then we obtain our solution to the differential equation (4.2.16) and thus to $\varphi(x,y)=y^3+xy-1=0$, as

$$y(x)=\sum_{k=0}^{\infty}\frac{(-\frac{1}{6})_k(\frac{1}{3})_k^2}{(-\frac{2}{3})_k(-\frac{1}{3})_k k!}\left(-\frac{2^2}{3^3}x^3\right)^k$$

Example 4.2.5. *Let us find a solution to the algebraic equation* $\varphi(x,y)=y^5+xy-1=0$ *By Mellin Inversion formula we have*

$$y(x)=\frac{1}{5}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z)\Gamma(\frac{1-z}{5})}{\Gamma(\frac{6+4z}{5})}x^{-z}dz.$$

We observe that

$$\begin{aligned}(-\vartheta_x+4)(-\vartheta_x+3)&(-\vartheta_x+2)(-\vartheta_x+1)(-\vartheta_x)y(x)\\ &=\lambda\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z+5)\Gamma(\frac{1-z}{5})}{\Gamma(\frac{6+4z}{5})}x^{-z}dz\\ &=\lambda\int_{(c+5)-i\infty}^{(c+5)+i\infty}\frac{\Gamma(z')\Gamma(\frac{1-z'}{5}+1)}{\Gamma(\frac{6+4z'}{5}-4)}x^{-(z'-5)}dz'\end{aligned}$$

where $\lambda=\frac{1}{10\pi i}$. *Thus, we have*

$$x^5\lambda\int_{(c+5)-i\infty}^{(c+5)+i\infty}\frac{\Gamma(z')(\frac{1-z'}{5})(\frac{1+4z'}{5})(\frac{-4+4z'}{5})(\frac{-9+4z'}{5})(\frac{-14+4z'}{5})\Gamma(\frac{1-z'}{5})}{\Gamma(\frac{6+4z'}{5})}x^{-z'}dz'$$

or equivalently

$$x^5(\frac{1+\vartheta_x}{5})(\frac{1-4\vartheta_x}{5})(\frac{-4-4\vartheta_x}{5})(\frac{-9-4\vartheta_x}{5})(\frac{-14-4\vartheta_x}{5})\lambda\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z')\Gamma(\frac{1-z'}{5})}{\Gamma(\frac{6+4z'}{5})}x^{-z'}dz'$$

This leads us to the following Pochhammer hypergeometric differential equation

$$\begin{aligned}&[x^5(\frac{1+\vartheta_x}{5})(\frac{1-4\vartheta_x}{5})(\frac{-4-4\vartheta_x}{5})(\frac{-9-4\vartheta_x}{5})(\frac{-14-4\vartheta_x}{5})\\ &\qquad+(\vartheta_x)(\vartheta_x-4)(\vartheta_x-3)(\vartheta_x-2)(\vartheta_x-1)]y(x)=0\end{aligned}\tag{4.2.19}$$

Setting $x^5=t$ *and* $\vartheta_x=5\vartheta_t$, *gives*

$$\begin{aligned}&[t(\frac{1}{5}+\vartheta_t)(\frac{1}{5}-4\vartheta_t)(-\frac{4}{5}-4\vartheta_t)(-\frac{9}{5}-4\vartheta_t)(-\frac{14}{5}-4\vartheta_t)\\ &\qquad+(5\vartheta_t)(5\vartheta_t-4)(5\vartheta_t-3)(5\vartheta_t-2)(5\vartheta_t-1)]y(t^{\frac{1}{5}})=0\end{aligned}$$

Now, let $-\frac{4^4}{5^5}t=s\Rightarrow\vartheta_t=\vartheta_s$, *rewriting the above differential equation we obtain*

$$\begin{aligned}&[-s(\vartheta_s+\frac{1}{5})(\vartheta_s-\frac{1}{20})(\vartheta_s+\frac{1}{5})(\vartheta_s+\frac{9}{20})(\vartheta_s+\frac{14}{20})\\ &\qquad+(\vartheta_s)(\vartheta_s-\frac{4}{5})(\vartheta_s-\frac{3}{5})(\vartheta_s-\frac{2}{5})(\vartheta_s-\frac{1}{5})]y((-\frac{5^5}{4^4}s)^{\frac{1}{5}})=0\end{aligned}$$

Since $y(s)$ is a solution of the above differential equation, by the same reason in the previous example we conclude that

$$y((-\frac{5^5}{4^4}s)^{\frac{1}{5}})=F(\frac{1}{5},-\frac{1}{20},\frac{1}{5},\frac{9}{20},\frac{14}{20},-\frac{4}{5},-\frac{3}{5},-\frac{2}{5},-\frac{1}{5};s)=\sum_{k=0}^{\infty}\frac{(\frac{1}{5})_k^2(-\frac{1}{20})_k(\frac{9}{20})_k(\frac{14}{20})_k}{(-\frac{4}{5})_k(-\frac{3}{5})_k(-\frac{2}{5})_k(-\frac{1}{5})_k k!}s^k$$

Therefore we obtain

$$y(x)=\sum_{k=0}^{\infty}\frac{(\frac{1}{5})_k^2(-\frac{1}{20})_k(\frac{9}{20})_k(\frac{14}{20})_k}{(-\frac{4}{5})_k(-\frac{3}{5})_k(-\frac{2}{5})_k(-\frac{1}{5})_k k!}\left(\frac{4^4}{5^5}x^5\right)^k$$

as a solution to the differential equation (4.2.19) *and a solution to the algebraic function* $\varphi(x,y)=y^5+xy-1=0$.

Proposition 4.2.2. *In general, for a given algebraic equation of the form* $\varphi(x,y)=y^n+xy^p-1=0$*, the general hypergeometric differential equation satisfied by* $y(x)$ *can be given by*

$$\left[\prod_{l=0}^{n-1}(\vartheta_x-l)-\frac{(-p)^p q^q}{n^n}x^n\prod_{j=0}^{p-1}(\vartheta_x+\frac{1+nj}{p})\prod_{k=0}^{q-1}(\vartheta_x+\frac{-1+nk}{q})\right]y(x)=0$$

and after the change of variables $s=\frac{(-p)^p q^q}{n^n}x^n$*, and* $\vartheta_x=n\vartheta_s$ *one gets*

$$\left[\prod_{l=0}^{n-1}(\vartheta_s-\frac{l}{n})-s\prod_{j=0}^{p-1}(\vartheta_s+\frac{1}{np}+\frac{j}{p})\prod_{k=0}^{q-1}(\vartheta_s-\frac{1}{nq}+\frac{k}{q})\right]Y(s)=0$$

Thus,

$$Y(s)={}_nF_{n-1}\left(\overrightarrow{\alpha},\overrightarrow{\beta};s\right) \tag{4.2.20}$$

where

$$\overrightarrow{\alpha}=\left(\frac{1}{np},\frac{1}{np}+\frac{1}{p},\ldots,\frac{1}{np}+\frac{p-1}{p},-\frac{1}{nq},-\frac{1}{nq}+\frac{1}{q},\ldots,-\frac{1}{nq}+\frac{q-1}{q}\right),$$

$$\overrightarrow{\beta}=\left(-\frac{1}{n},\ldots,-\frac{n-1}{n}\right),$$

$s=\frac{(-p)^p q^q}{n^n}x^n$ *and* $Y(s)=y(s^{\frac{1}{n}})$.

Proof. In considering only the integral part of the Mellin inversion solution

$$y(x)=\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)}x^{-z}dz$$

We have

$$(-\vartheta_x+n-1)\dots(-\vartheta_x)y(x)=\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z+n)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)}x^{-z}dz \tag{4.2.21}$$

Let $z+n=z'$, then the equation (4.2.21) becomes

$$(-\vartheta_x+n-1)\dots(-\vartheta_x)y(x)=x^n\int_{(c+n)-i\infty}^{(c+n)+i\infty}\frac{\Gamma(z')\Gamma(\frac{1-pz'}{n}+p)}{\Gamma(\frac{1+qz}{n}-q)}x^{-z'}dz' \tag{4.2.22}$$

Now that $z'=\frac{1}{p}$ is not a pole of the factor $\Gamma(\frac{1-pz'}{n}+p)$ we can deform the integral in (4.2.22) and by using the theorem (1.1.5) we get

$$\begin{aligned}
&(-\vartheta_x+n-1)\dots(-\vartheta_x)y(x)\\
&=x^n\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z')\prod_{j=0}^{p-1}(\frac{1-pz'}{n}+j)\prod_{k=0}^{q-1}(\frac{1+qz'}{n}-k)\Gamma(\frac{1-pz'}{n})}{\Gamma(\frac{1+qz'}{n})}x^{-z'}dz' &(4.2.23)\\
&=x^n\prod_{j=0}^{p-1}(\frac{1+p\vartheta_x}{n}+j)\prod_{k=0}^{q-1}(\frac{1-q\vartheta_x}{n}-k)\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z')\Gamma(\frac{1-pz'}{n})}{\Gamma(\frac{1+qz'}{n})}x^{-z'}dz' &(4.2.24)
\end{aligned}$$

Thus, it follows that

$$\left[(-\vartheta_x)_n-x^n(\frac{1+p\vartheta_x}{n})_p(\frac{1-q\vartheta_x}{n})_q\right]y(x)=0 \tag{4.2.25}$$

where

$$(-\vartheta_x)_n=(-\vartheta_x)(-\vartheta_x+1)\dots(-\vartheta+n-1)=(-1)^n(\vartheta_x-n+1)_n$$

and

$$(\alpha)_p=(\alpha)(\alpha+1)\dots(\alpha+p-1)$$

Notice that

$$\prod_{j=0}^{p-1}(\frac{1+p\vartheta_x}{n}+j)=\left(\frac{p}{n}\right)^p\prod_{j=0}^{p-1}(\vartheta_x+\frac{1}{p}+\frac{nj}{p})$$

and

$$\prod_{k=0}^{q-1}(\frac{1-q\vartheta_x}{n}-k)=\left(\frac{-q}{n}\right)^q\prod_{k=0}^{q-1}(\vartheta_x+\frac{nk}{q}-\frac{1}{q})$$

Hence the equation (4.2.25) can be rewritten as

$$\left[(\vartheta_x-n+1)_n-\frac{(-p)^p q^q}{n^n}x^n\prod_{j=0}^{p-1}(\vartheta_x+\frac{1+nj}{p})\prod_{k=0}^{q-1}(\vartheta_x+\frac{nk-1}{q})\right]y(x)=0 \qquad (4.2.26)$$

In making use of change of variable $s=\frac{(-p)^p q^q}{n^n}x^n$ hence $\vartheta_x=n\vartheta_s$ we obtain

$$\left[\prod_{l=0}^{n-1}(n\vartheta_s-n+1+l)-s\prod_{j=0}^{p-1}(n\vartheta_s+\frac{1+nj}{p})\prod_{k=0}^{q-1}(n\vartheta_s+\frac{nk-1}{q})\right]y(s^{\frac{1}{n}})=0 \qquad (4.2.27)$$

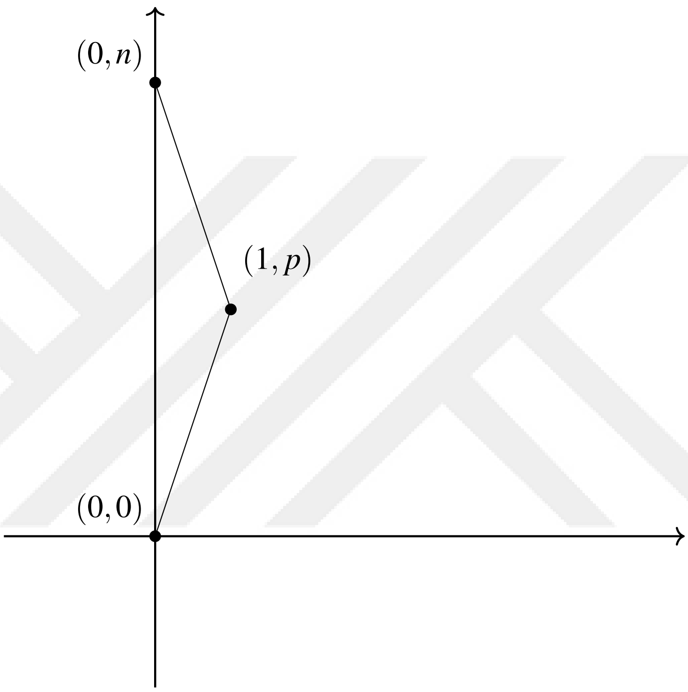
which implies

$$\left[\prod_{l=0}^{n-1}(\vartheta_s-\frac{l}{n})-s\prod_{j=0}^{p-1}(\vartheta_s+\frac{1}{np}+\frac{j}{p})\prod_{k=0}^{q-1}(\vartheta_s-\frac{1}{qn}+\frac{k}{q})\right]y(s^{\frac{1}{n}})=0.$$

□

4.3 Algebraic Functions in terms of Hypergeometric Series

In Section 4.2, we have dealt with the algebraic equation $\varphi(x,y) = y^n + xy^p - 1 = 0$, whose Newton polygon is of the form:



In this case, the Newton polygon of the algebraic function $\varphi(x,y)$ is not appropriate for the Newton's algorithm. However, as we described in [Section 4.1], we can apply Newton polygon method to a general trinomial algebraic equation of the form $f(X,Y) = Y^n - XY^p + X^r = 0$, with $(n-p) > \frac{p}{r-1}$. It is clear that the functions $\varphi(x,y)$ and $f(X,Y)$ look similar but they are not the same. Nevertheless, for a suitable change of variables we can convert the equation $\varphi(x,y) = 0$ into the equation $f(X,Y) = 0$.
So in this section, we shall try to express the solutions of a general trinomial algebraic equation of the form

$$f(X,Y) = Y^n - XY^p + X^r = 0, \quad (n-p) > \frac{p}{r-1}, \quad q = (n-p), \quad gcd(p, r-1) = 1. \tag{4.3.1}$$

in terms of hypergeometric series.
In order to obtain such an expression we will make use of change of variables and Mellin inversion formula which is given in Section 4.2. Furthermore, once we we obtain the solutions in terms of hypergeometric series we will also apply Newton polygon method

to the general trinomial algebraic equation $f(X,Y) = Y^n - XY^p + X^r = 0$, and compare the results(hypergeometric series and Puiseux expansion of $f(X,Y)$) to check that such an expression indeed holds.
Suppose that we are given a general algebraic equation of the form

$$f(X,Y) = Y^n - XY^p + X^r = 0 \quad with \quad (n-p) > \frac{p}{r-1}, \quad gcd(p, r-1) = 1.$$

Employ the roots to the equation (4.2.1) with the new variables

$$y = \lambda Y \quad and \quad x = \mu X, \quad (\lambda, \mu \neq 0)$$

and consider

$$\lambda^n Y^n + \lambda^p \mu XY^p - 1 \quad \Longrightarrow \quad Y^n - XY^p + X^r$$

Hence

$$Y^n + \lambda^{p-n} \mu XY^p - \lambda^{-n} = Y^n - XY^p + X^r$$

Then one can precisely get the following equalities

$$Y(X) = (-1)^{\frac{1}{n}} X^{\frac{r}{n}} y(x) \tag{4.3.2}$$

and

$$x = (-1)^{\frac{p}{n}} X^{\frac{n-qr}{n}} \tag{4.3.3}$$

Let $n - qr = N$. We observe that $N < 0$ for every n, q, r as in the equation (4.3.1), due to the convexity of the Newton polygon of $f(X,Y)$. (See Figure (4.1)). Let us denote by $\tilde{Y}(X)$ one of the solutions to the equation (4.3.1), then in general in order to obtain the general hypergeometric differential equation satisfied by $\tilde{Y}(X)$, we utilize proposition (4.2.2)

$$\left[\prod_{l=0}^{n-1} (\vartheta_x - l) - \frac{(-p)^p q^q}{n^n} x^n \prod_{j=0}^{p-1} (\vartheta_x + \frac{1+nj}{p}) \prod_{k=0}^{q-1} (\vartheta_x + \frac{-1+nk}{q}) \right] y(x) = 0$$

and by using (4.3.3) we may write

$$\vartheta_x = \frac{n}{N} \vartheta_X$$

and hence

$$\Big[\prod_{l=0}^{n-1}((\frac{n}{N})\vartheta_X - l)$$
$$-\frac{p^p q^q}{n^n}X^N\prod_{j=0}^{p-1}((\frac{n}{N})\vartheta_X+\frac{1+nj}{p})\prod_{k=0}^{q-1}((\frac{n}{N})\vartheta_X+\frac{-1+nk}{q})\Big]y((-1)^{\frac{p}{n}}X^{\frac{N}{n}})=0 \quad (4.3.4)$$

Now putting $\frac{p^p q^q}{n^n}X^N=\xi$, gives $\vartheta_X = N\vartheta_\xi$. From this we can rewrite the latter hypergeometric differential equation as

$$\left[\prod_{l=0}^{n-1}(n\vartheta_\xi - l)-\xi\prod_{j=0}^{p-1}(n\vartheta_\xi+\frac{1+nj}{p})\prod_{k=0}^{q-1}(n\vartheta_\xi+\frac{-1+nk}{q})\right]\psi(\xi)=0$$

which implies

$$\left[\prod_{l=0}^{n-1}(\vartheta_\xi-\frac{l}{n})-\xi\prod_{j=0}^{p-1}(\vartheta_\xi+\frac{1}{np}+\frac{j}{p})\prod_{k=0}^{q-1}(\vartheta_\xi+\frac{-1}{nq}+\frac{k}{q})\right]\psi(\xi)=0.$$

Therefore,

$$\psi(\xi)={}_nF_{n-1}\left(\overrightarrow{\alpha},\overrightarrow{\beta};\xi\right)$$

and by using (4.3.2), we obtain our solution to the hypergeometric differential equation as

$$\tilde{Y}(\xi)=(-X^r)^{\frac{1}{n}}\psi(\xi)=(-X^r)^{\frac{1}{n}}{}_nF_{n-1}\left(\overrightarrow{\alpha},\overrightarrow{\beta};\xi\right).$$

It follows that

$$\tilde{Y}(X)=(-X^r)^{\frac{1}{n}}{}_nF_{n-1}\left(\overrightarrow{\alpha},\overrightarrow{\beta};\frac{p^p q^q}{n^n}X^N\right), \quad (4.3.5)$$

where

$$\overrightarrow{\alpha}=\left(\frac{1}{np},\frac{1}{np}+\frac{1}{p},\ldots,\frac{1}{np}+\frac{p-1}{p},-\frac{1}{nq},-\frac{1}{nq}+\frac{1}{q},\ldots,-\frac{1}{nq}+\frac{q-1}{q}\right),$$

$$\overrightarrow{\beta}=\left(-\frac{1}{n},\ldots,-\frac{n-1}{n}\right),$$

and $N=n-qr$.

Remark 4.3.1. *Notice that in equation* (4.3.5)*, we obtained a solution $\tilde{Y}(X)$ around $X=\infty$, as $N<0$. However, since one cannot compare the Puiseux series with a series around $X=\infty$, we need to make a small change in our differential equation* (4.3.4) *in order to obtain a series solution around $X=0$.*

In considering the above remark, we rewrite the hypergeometric differential equation (4.3.4) as follows

$$[\frac{n^n}{p^p q^q} X^{-N} \prod_{l=0}^{n-1} (\frac{n}{N}\vartheta_X - l) \\ - \prod_{j=0}^{p-1} (\frac{n}{N}\vartheta_X + \frac{1+nj}{p}) \prod_{k=0}^{q-1} (\frac{n}{N}\vartheta_X + \frac{-1+nk}{q})] y((-1)^{\frac{p}{n}} X^{\frac{N}{n}}) = 0.$$

Applying the change of variable $\frac{n^n}{p^p q^q} X^{-N} = s$, hence $\vartheta_X = -N\vartheta_s$ yields the following hypergeometric differential equation

$$\left[s \prod_{l=0}^{n-1} (-n\vartheta_s - l) - \prod_{j=0}^{p-1} (-n\vartheta_s + \frac{1+nj}{p}) \prod_{k=0}^{q-1} (-n\vartheta_s + \frac{-1+nk}{q}) \right] \varphi(s) = 0,$$

or equivalently

$$\left[s \prod_{l=0}^{n-1} (\vartheta_s + \frac{l}{n}) - \prod_{j=0}^{p-1} (\vartheta_s - \frac{1}{np} - \frac{j}{p}) \prod_{k=0}^{q-1} (\vartheta_s + \frac{1}{nq} - \frac{k}{q}) \right] \varphi(s) = 0. \tag{4.3.6}$$

But we observe that the above hypergeometric differential equation (4.3.6) is not in the Pochhammer form.

Proposition 4.3.1. *In general, solutions of the algebraic equation*

$$f(X,Y) = Y^n - XY^p + X^r = 0 \quad with \quad (n-p) > \frac{p}{r-1}, \quad gcd(p, r-1) = 1.$$

are of the form

$$\tilde{Y}_t(X) = \tilde{Y}(\varepsilon^t X^{\frac{1}{p}}) \quad \varepsilon^t = e^{\frac{2\pi i t}{p}} \quad for \quad t = 0, 1, \ldots, p-1.$$

and

$$-\tilde{\tilde{Y}}_s(X) = \tilde{\tilde{Y}}(\omega^s X^{\frac{1}{q}}) \quad \omega^s = e^{\frac{2\pi i s}{q}} \quad for \quad s = 0, 1, \ldots, q-1.$$

where, $\tilde{Y}_t$ and $\tilde{\tilde{Y}}_s$ are the solutions of the hypergeometric differential equation (4.3.6)*, derived from the Mellin inversion formula.*

Proof. The proof of this proposition relies on comparison with the Mellin inversion solutions and the Puiseux expansions of the algebraic equation

$$f(X,Y) = Y^n - XY^p + X^r = 0, \quad gcd(p, r-1) = 1.$$

In considering the Mellin inversion formula for $y(x)$, by proposition (4.2.1)

$$y_0(x) = \frac{1}{n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)} x^{-z} dz$$

one obtains

$$y_0(x) = -\frac{1}{n}\sum_{k=0}^{\infty} \operatorname*{Res}_{z=\frac{1+nk}{p}} \frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)} x^{-z} \tag{4.3.7}$$

$$= \frac{1}{p}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{1+nk}{p})(-1)^k}{\Gamma(\frac{p+q+qnk+pn}{pn})k!} x^{-(\frac{1+nk}{p})} \tag{4.3.8}$$

Combining the equations (4.3.2),(4.3.3) and (4.3.8) yields the following series expansion of a solution to the equation (4.3.1)

$$\tilde{Y_0}(X) = \frac{1}{p}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{1+nk}{p})(X^{\frac{1}{p}})^{r-1+(qr-n)k}}{\Gamma(\frac{p+q+qnk+pn}{pn})k!} = X^{\frac{r-1}{p}} + \frac{1}{p}X^{\frac{r-1+qr-n}{p}} + \cdots$$

Since we have p- branches the remaining $(p-1)$- solutions can be obtained by using the remark (4.3.2) as follows

$$\tilde{Y_1}(X) = \frac{1}{p}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{1+nk}{p})(e^{\frac{2\pi i}{p}}X^{\frac{1}{p}})^{r-1+(qr-n)k}}{\Gamma(\frac{p+q+qnk+pn}{pn})k!} = e^{\frac{2\pi i(r-1)}{p}}X^{\frac{r-1}{p}} + \frac{1}{p}e^{\frac{2\pi i(r-1+qr-n)}{p}}X^{\frac{r-1+qr-n}{p}} + \cdots$$

$$\vdots$$

$$\vdots$$

$$\tilde{Y}_{p-1}(X) = \frac{1}{p}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{1+nk}{p})(e^{\frac{2\pi i(p-1)}{p}}X^{\frac{1}{p}})^{r-1+(qr-n)k}}{\Gamma(\frac{p+q+qnk+pn}{pn})k!}$$

$$= e^{\frac{2\pi i(p-1)(r-1)}{p}}X^{\frac{r-1}{p}} + \frac{1}{p}e^{\frac{2\pi i(p-1)(r-1+qr-n)}{p}}X^{\frac{r-1+qr-n}{p}} + \cdots$$

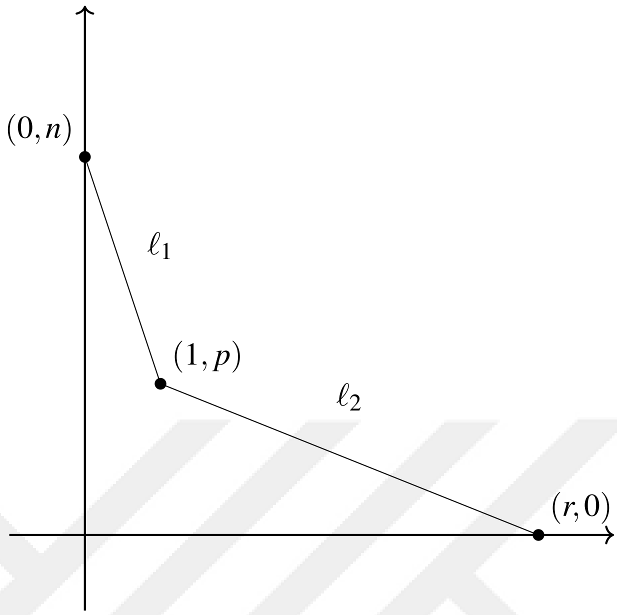


Figure 4.1: $N(f(X,Y))$

At this point, let us perform Newton's algorithm for the algebraic function $f(X,Y) = Y^n - XY^p + X^r = 0$, having the above Newton polygon see figure (4.1). On the line $\ell_2(\mu_0 = \frac{r-1}{p})$, starting with $Y_0 = c_0 X^{\frac{r-1}{p}}$ gives

$$-Xc_0^p X^{r-1} + X^r = 0 \quad \Longrightarrow \quad (1-c_0^p)X^r = 0 \quad \Longrightarrow \quad c_0 = e^{\frac{2\pi it}{p}}$$

for $t = 0, 1, \dots, p-1$.
For the second approximation, let $X^{\frac{r-1}{p}} = X_1$ and hence $X_1^{\frac{p}{r-1}} = X$.
Consider

$$\begin{aligned} f(X_1^{\frac{r-1}{p}}, X_1(e^{\frac{2\pi it}{p}} + Y_1)) &= X_1^n(e^{\frac{2\pi it}{p}} + Y_1)^n - X_1^{\frac{r-1}{p}} X_1^p(e^{\frac{2\pi it}{p}} + Y_1)^p + X_1^{\frac{pr}{r-1}} = 0 \\ &= X_1^{\frac{pr}{r-1}} f_1(X_1, Y_1) = 0 \end{aligned}$$

where $f(X_1,Y_1) = 1 - (e^{\frac{2\pi it}{p}} + Y_1)^p + X^{\frac{rq-n}{r-1}}(e^{\frac{2\pi it}{p}} + Y_1)^n$.
Solving for Y_1 in $f_1(X_1,Y_1)$ gives

$$Y_1 = \frac{1}{p} e^{\frac{2\pi it(p-1)}{p}} X_1^{\frac{rq-n}{r-1}}$$

which implies that $c_1 = \frac{1}{p} e^{\frac{2\pi it(p-1)}{p}}$.
Therefore, one can obtain the following Puiseux expansion of $f(X,Y)$.

$$
\begin{aligned}
&X^{\frac{r-1}{p}}(e^{\frac{2\pi it}{p}} + X^{\frac{rq-n}{p}}(\frac{1}{p}e^{\frac{2\pi it(p-1)}{p}}) + \cdots) \\
&= e^{\frac{2\pi it}{p}} X^{\frac{r-1}{p}} + \frac{1}{p} e^{\frac{2\pi it(p-1)}{p}} X^{\frac{r-1+qr-n}{p}} + \cdots
\end{aligned}
$$

for $t = 0, 1, \ldots, p-1$.
Let us introduce the following fact

Fact 1. *There exists an integer t such that*

$$\bar{t} \equiv (r-1)t \pmod{p}$$

and

$$\bar{t} \equiv (p-1)t \pmod{p}$$

where $gcd(r-1,p) = 1$ and $gcd(p-1,p) = 1$ and $\bar{t} \in \{0, 1, \ldots, p-1\}$.

From this fact we can rewrite the Puiseux expansion as

$$\tilde{Y}^{(t)}_{psx}(X) = (e^{2\pi it}X)^{\frac{r-1}{p}} + \frac{1}{p}(e^{2\pi it}X)^{\frac{r-1+qr-n}{p}} + \cdots$$

Now, it suffices to compare with the first two terms of the Puiseux expansion with the first two terms of the hypergeometric series $\tilde{Y}_t(X)$ for $1 \leq t \leq p-1$.
For $t = 0$,

$$\tilde{Y}_0(X) = X^{\frac{r-1}{p}} + \frac{1}{p} X^{\frac{r-1+qr-n}{p}} + \cdots = \tilde{Y}^{(0)}_{psx}(X)$$

For $t = 1$,

$$
\begin{aligned}
\tilde{Y}_1(X) = e^{\frac{2\pi i(r-1)}{p}} X^{\frac{r-1}{p}} + \frac{1}{p} e^{\frac{2\pi i(r-1+qr-n)}{p}} X^{\frac{r-1+qr-n}{p}} + \cdots \\
= e^{\frac{2\pi i}{p}} X^{\frac{r-1}{p}} + \frac{1}{p} e^{\frac{2\pi i(p-1)}{p}} X^{\frac{r-1+qr-n}{p}} + \cdots = \tilde{Y}^{(1)}_{psx}(X)
\end{aligned}
$$

$$\vdots$$

For $t = p-1$,

$$\tilde{Y}_{p-1}(X) = e^{\frac{2\pi i(p-1)}{p}} X^{\frac{r-1}{p}} + \frac{1}{p} e^{\frac{2\pi i(p-1)(r-1+qr-n)}{p}} X^{\frac{r-1+qr-n}{p}} + \cdots$$
$$= e^{\frac{2\pi i(p-1)}{p}} X^{\frac{r-1}{p}} + \frac{1}{p} e^{\frac{2\pi i(p-1)(r-1+qr-n)}{p}} X^{\frac{r-1+qr-n}{p}} + \cdots = \tilde{Y}_{psx}^{(p-1)}(X)$$

In a similar fashion, after using Euler's reflection formula with a periodic function $\phi(z+n) = \phi(z)$, to be more precise, at this point, recall the Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

and consider

$$\Gamma(z)\Gamma(1-z)(-1)^{-z} = \frac{\pi}{\sin(\pi z)}(-1)^{-z}$$

we claim that

$$\rho(z) := \frac{\pi}{\sin(\pi z)}(-1)^{-z}$$

is a periodic function. Moreover, $\rho(z)$ is a rational function of $e^{2\pi i z}$. To see this fact, we consider

$$sin(\pi z) = \frac{e^{\pi i z} - e^{-\pi i z}}{2i} \quad and \quad (-1)^{-z} = e^{-\pi i z}$$

then one can easily obtain

$$\frac{\pi}{\sin(\pi z)}(-1)^{-z} = 2\pi i \frac{e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = 2\pi i \frac{1}{(e^{2\pi i z} - 1)}.$$

It is easy to see that $\rho(z+1) = \rho(z)$, and therefore $\rho(z)$ is a periodic function of period 1. Also, from the last equality it can be seen that it is a rational function of $e^{2\pi i z}$.
We now introduce the following claim,

Claim 1. *There exists a periodic function $\phi(z)$ such that*

$$\frac{\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)}\phi(z) = \frac{\Gamma(\frac{-1-qz}{n})}{\Gamma(\frac{n-1+pz}{n})}(-1)^{-z} \tag{4.3.9}$$

with $\phi(z+n) = \phi(z)$.

Proof. Setting $\phi(z) = \dfrac{\sin(\pi(\frac{1-pz}{n}))}{\sin(\pi(\frac{1+qz}{n}+1))}(-1)^{-z}$ in the equation (4.3.9) gives the required result.

□

Therefore, one can obtain the second Mellin inversion solution as

$$\tilde{y_0}(x) = \frac{1}{n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z)\Gamma(\frac{-1-qz}{n})}{\Gamma(\frac{n-1+pz}{n})}(-x)^{-z}dz$$

Hence

$$\begin{aligned}\tilde{y_0}(x) &= -\frac{1}{n}\sum_{k=0}^{\infty}\operatorname*{Res}_{z=\frac{nk-1}{q}}\frac{\Gamma(z)\Gamma(\frac{-1-qz}{n})}{\Gamma(\frac{n-1+pz}{n})}(-x)^{-z}\\ &= \frac{1}{q}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{nk-1}{q})(-1)^k}{\Gamma(\frac{q+pk-1}{q})k!}(-x)^{\frac{1-nk}{q}} \qquad (4.3.10)\end{aligned}$$

Now, it follows from (4.3.2), (4.3.3) and (4.3.10) that

$$\tilde{\tilde{Y_0}}(X) = \frac{1}{q}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{nk-1}{q})(-1)^{\frac{2-2pk}{q}}}{\Gamma(\frac{q+pk-1}{q})k!}(X^{\frac{1}{q}})^{1+k(qr-n)} = -e^{\frac{2\pi i}{q}}X^{\frac{1}{q}} + \frac{1}{q}e^{\frac{2\pi i(1-p)}{q}}X^{\frac{1-n+qr}{q}} + \cdots$$

Now that we have q-branches by the virtue of the remark (4.3.2) the other $(q-1)$-solutions can be given by

$$\tilde{\tilde{Y_1}}(X) = \frac{1}{q}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{nk-1}{q})(-1)^{\frac{2-2pk}{q}}(e^{\frac{2\pi i}{q}}X^{\frac{1}{q}})^{1+k(qr-n)}}{\Gamma(\frac{q+pk-1}{q})k!} = -e^{\frac{4\pi i}{q}}X^{\frac{1}{q}} + \frac{1}{q}e^{\frac{4\pi i(1-p)}{q}}X^{\frac{1-n+qr}{q}} + \cdots$$

$$\vdots$$

$$\vdots$$

$$\tilde{\tilde{Y}}_{q-1}(X) = \frac{1}{q}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{nk-1}{q})(-1)^{\frac{2-2pk}{q}}(e^{\frac{2\pi i(q-1)}{q}}X^{\frac{1}{q}})^{1+k(qr-n)}}{\Gamma(\frac{q+pk-1}{q})k!} = -X^{\frac{1}{q}} + \frac{1}{q}X^{\frac{1-n+qr}{q}} + \cdots$$

We now perform Newton's algorithm for $f(X,Y)$ on the line $\ell_1(\mu_0 = \frac{1}{n-p})$, see figure (4.1).
Starting the process with $Y_0 = c_0X^{\frac{1}{n-p}}$, gives

$$c_0^nX^{\frac{n}{n-p}} - c_0^pX^{\frac{n}{n-p}} = 0 \quad\Longrightarrow\quad c_0^p(c_0^q-1)X^{\frac{n}{q}} = 0$$

since $c_0 \neq 0$, we must have

$$c_0^q = 1 \quad \Longrightarrow \quad c_0 = e^{\frac{2\pi i(s+1)}{q}}$$

for $s = 0, 1, \ldots, q-1$.
For the second approximation, let $X_1 = X^{\frac{1}{n-p}} = X^{\frac{1}{q}}$ and consider

$$f(X_1^q, X_1(e^{\frac{2\pi i(s+1)}{q}} + Y_1)) = X_1^n f_1(X_1, Y_1) = 0$$

where $f_1(X_1, Y_1) = (e^{\frac{2\pi i(s+1)}{q}} + Y_1)^n - (e^{\frac{2\pi i(s+1)}{q}} + Y_1)^p + X_1^{rq-n}$.
Solving for Y_1 in $f_1(X_1, Y_1)$ gives

$$(ne^{\frac{2\pi i(s+1)(n-1)}{q}} - pe^{\frac{2\pi i(p-1)}{q}})Y_1 + X_1^{rq-n} = 0 \tag{4.3.11}$$

Here note that

$$\begin{aligned} ne^{\frac{2\pi i(s+1)(n-1)}{q}} &= (p+q)e^{\frac{2\pi i(s+1)(n-1)}{q}} \\ &= pe^{\frac{2\pi i(s+1)(p+q-1)}{q}} + qe^{\frac{2\pi i(s+1)(p+q-1)}{q}} \\ &= pe^{\frac{2\pi i(s+1)(p+q-1)+2\pi iqs}{q}} + qe^{\frac{2\pi i(s+1)(p+q-1)+2\pi iqs}{q}} \\ &= pe^{\frac{2\pi i(s+1)(p-1)}{q}} + qe^{\frac{2\pi i(s+1)(p-1)}{q}} \end{aligned}$$

Thus the equation (4.3.11) becomes

$$qe^{\frac{2\pi i(s+1)(p-1)}{q}} Y_1 + X_1^{rq-n} = 0$$

Hence

$$Y_1 = -\frac{1}{q} e^{-\frac{2\pi i(s+1)(p-1)}{q}} X_1^{rq-n}$$

which implies $c_1 = -\frac{1}{q} e^{-\frac{2\pi i(s+1)(p-1)}{q}}$.
Therefore, the Puiseux expansion is given by

$$\begin{aligned} \tilde{\tilde{Y}}_{psx}^{(s)}(X) &= X^{\frac{1}{n-p}}(e^{\frac{2\pi i(s+1)}{q}} + X^{\frac{r(n-p)-n}{n-p}}(-\frac{1}{q}e^{-\frac{2\pi i(s+1)(p-1)}{q}} + \cdots)) \\ &= e^{\frac{2\pi i(s+1)}{q}} X^{\frac{1}{q}} - \frac{1}{q} e^{\frac{2\pi i(s+1)(1-p)}{q}} X^{r+\frac{1-n}{q}} + \cdots \end{aligned}$$

for $s = 0, 1, \ldots, q-1$.

It again suffices to compare the first two terms of the Puiseux expansion and the first two terms of the hypergeometric series $\tilde{\tilde{Y}}_s(X)$ for $0 \leq s \leq q-1$.
For $s = 0$,

$$-\tilde{\tilde{Y}}_0(X) = e^{\frac{2\pi i}{q}} X^{\frac{1}{q}} + e^{\frac{2\pi i(1-p)}{q}} X^{r+\frac{1-n}{q}} + \cdots = \tilde{\tilde{Y}}^{(0)}_{psx}(X)$$

For $s = 1$,

$$-\tilde{\tilde{Y}}_1(X) = e^{\frac{4\pi i}{q} X^{\frac{1}{q}}} + \frac{1}{q} e^{\frac{4\pi i(1-p)}{q}} X^{\frac{1-n+qr}{q}} + \cdots = \tilde{\tilde{Y}}^{(1)}_{psx}(X)$$

$$\vdots$$

$$\vdots$$

And for $s = q-1$,

$$-\tilde{\tilde{Y}}_{q-1}(X) = X^{\frac{1}{q}} - \frac{1}{q} X^{r+\frac{1-n}{q}} + \cdots = \tilde{\tilde{Y}}^{(q-1)}_{psx}(X)$$

This completes the proof of the proposition.

$\square$

Remark 4.3.2. *We remark that if $\tilde{Y}_0(X)$ is a solution of $f(X,Y) = 0$, then $\tilde{Y}_t(X) = Y_0(\varepsilon^t X^{\frac{1}{p}})$ is also a solution of $f(X,Y) = 0$, and if $\tilde{\tilde{Y}}_0(X)$ is a solution of $f(X,Y) = 0$, then $\tilde{\tilde{Y}}_s(X) = \tilde{\tilde{Y}}_0(\omega^s X^{\frac{1}{q}})$ is also a solution of $f(X,Y) = 0$, where $\varepsilon^t = e^{\frac{2\pi i t}{p}}$ for $t = 0,1,\ldots,p-1$ and $\omega^s = e^{\frac{2\pi i s}{q}}$ for $s = 0,1,\ldots,q-1$.*

Remark 4.3.3. *([8], **Theorem 4.3**, p 1289). Remark on* (4.3.6),

(i) If $gcd(n,p) = d > 1$, then solutions to the hypergeometirc differential equation in (4.3.6) *are all algebraic functions with associated equation $y^n + xy^p - 1 = 0$.*

(ii) If $d = 1$ and $p < n-1$, then there exists 1-dimensional logarithmic solution space of the hypergeometric differential equation in (4.3.6) *over $\mathbb{C}$, spanned by $\sum_{k=0}^{n-1} Y_k \log(Y_k)$.*

(iii) If $d = 1$ and $p = n-1$, then the set $\{Y_0,\ldots,Y_{n-2},X\}$ gives $n-$dimensional solution space over $\mathbb{C}$ to the hypergeometric differential equation in (4.3.6).

If $p < n-1$,

$$\sum_{t=0}^{p-1} \tilde{Y}_t(X) - \sum_{s=0}^{q-1} \tilde{\tilde{Y}}_s(X) = 0$$

These algebraic solutions span a $(n-1)-$*dimensional solution subspace to* (4.3.6) *over* $\mathbb{C}$.
If $p = n-1$, *then*

$$\sum_{t=0}^{p-1} \tilde{Y}_t(X) - \sum_{s=0}^{q-1} \tilde{\tilde{Y}}_s(X) = X.$$

Now we examine examples illustrating the proposition (4.3.1).

Example 4.3.1. *(*$n=3, p=1, r=4$*) Assume that we are given the following algebraic equation*

$$f(X,Y) = Y^3 - XY + X^4 = 0.$$

We want to find solutions of the equation $f(X,Y) = 0$.
Firstly, recall that in [Section 4.2], we have found a solution $y(x)$ of the algebraic equation $\varphi(x,y) = y^3 + xy - 1 = 0$, by using Mellin inversion formula.
We let

$$y_0(x) = \frac{1}{3}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} x^{-z} dz$$

By the virtue of the above claim (4.3.9) and Euler's reflection formula we consider

$$y_1(x) = \frac{1}{3}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} (\phi(z))\, x^{-z} dz$$

in this case with a periodic function $\phi(z+3) = \phi(z)$. Thus, we get

$$\tilde{y_0}(x) = \frac{1}{3}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{-1-2z}{3})}{\Gamma(\frac{2+z}{3})} (-x)^{-z} dz$$

which is also a solution of the differential equation (4.2.16) obtained in [Section 4.2].
Now, when we simply calculate the residue we get,

$$\begin{aligned} y_0(x) &= -\frac{1}{3}\sum_{k=0}^{\infty} \operatorname*{Res}_{z=3k+1} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} x^{-z} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(3k+1)(-1)^k}{\Gamma(2k+2)k!} x^{-3k-1} \end{aligned}$$

Since

$$\operatorname*{Res}_{z=3k+1} \Gamma(\frac{1-z}{3}) dz = \operatorname*{Res}_{s=-k} \Gamma(s)(-3) ds$$

Also

$$\tilde{y_0}(x) = -\frac{1}{3}\sum_{k=0}^{\infty}\operatorname*{Res}_{z=\frac{3k-1}{2}} \frac{\Gamma(z)\Gamma(\frac{-1-2z}{3})}{\Gamma(\frac{2+z}{3})}(-x)^{-z}$$
$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3k-1}{2})(-1)^k}{\Gamma(\frac{k+1}{2})k!}(-x)^{\frac{1-3k}{2}}$$

Since

$$\operatorname*{Res}_{z=\frac{3k-1}{2}} \Gamma(\frac{-1-2z}{3})dz = \operatorname*{Res}_{s=-k}\Gamma(s)\left(-\frac{3}{2}\right)ds$$

We are now in a position to make use of change of variables as we mentioned earlier. So we introduce the new variables

$$y = \lambda Y \quad and \quad x = \mu X, \quad (\lambda, \mu \neq 0) \tag{4.3.12}$$

and we consider

$$y^3 + xy - 1 \quad \Longrightarrow \quad Y^3 - XY + X^4$$

We simply put y and x in (4.3.12) throughout the equation $y^3 + xy - 1 = 0$, and obtain

$$\lambda^3 Y^3 + \lambda\mu XY - 1 \quad \Longrightarrow \quad Y^3 - XY + X^4 \tag{4.3.13}$$

which implies

$$Y^3 + \lambda^{-2}\mu XY - \lambda^{-3} = Y^3 - XY + X^4$$

thus, we get two equalities,

$$-\lambda^{-3} = X^4 \quad and \quad \lambda^{-2}\mu = -1$$

It is straightforward to see that

$$\lambda = (-1)^{-\frac{1}{3}}X^{-\frac{4}{3}} \quad and \quad \mu = -(-1)^{-\frac{2}{3}}X^{-\frac{8}{3}}$$

and hence

$$y(x) = (-1)^{-\frac{1}{3}}X^{-\frac{4}{3}}Y(X), \quad x = -(-1)^{-\frac{2}{3}}X^{-\frac{8}{3}}X$$

or equivalently,

$$Y(X) = (-1)^{\frac{1}{3}}X^{\frac{4}{3}}y(x) \tag{4.3.14}$$

and

$$x = (-1)^{\frac{1}{3}} X^{-\frac{5}{3}} \tag{4.3.15}$$

We put the above variables in $y_0(x)$ and $\tilde{y_0}(x)$, then

$$\begin{aligned}\tilde{Y_0}(X) &= (-1)^{\frac{1}{3}} X^{\frac{4}{3}} \sum_{k=0}^{\infty} \frac{\Gamma(3k+1)(-1)^k}{\Gamma(2k+2)k!}((-1)^{\frac{1}{3}} X^{-\frac{5}{3}})^{-3k-1} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(3k+1)}{\Gamma(2k+2)k!} X^{5k+3}\end{aligned}$$

$$\begin{aligned}\tilde{\tilde{Y_0}}(X) &= (-1)^{\frac{1}{3}} X^{\frac{4}{3}} \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})(-1)^k}{\Gamma(\frac{k+1}{2})k!}(-(-1)^{\frac{1}{3}} X^{-\frac{5}{3}})^{\frac{1-3k}{2}} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})(-1)^{1-k}}{\Gamma(\frac{k+1}{2})k!} X^{\frac{5k+1}{2}}\end{aligned}$$

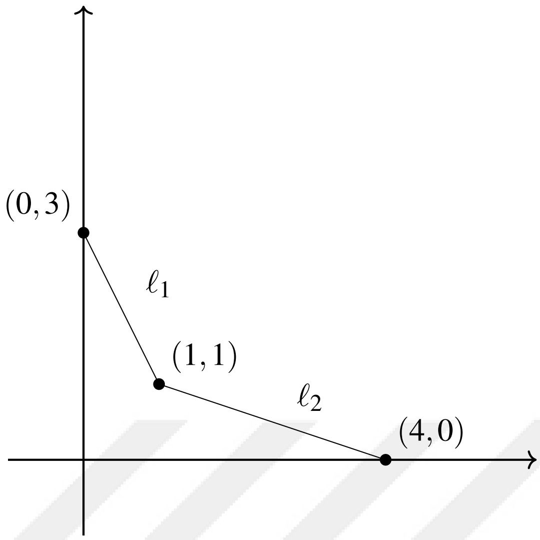
It follows from the remark (4.3.2) that

$$\begin{aligned}\tilde{\tilde{Y_1}}(X) &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})(-1)^{1-k}}{\Gamma(\frac{k+1}{2})k!}(e^{\pi i} X^{\frac{1}{2}})^{5k+1} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})}{\Gamma(\frac{k+1}{2})k!} X^{\frac{5k+1}{2}}\end{aligned}$$

is also a solution.
On the other hand, we apply Newton polygon method to the algebraic equation $f(X,Y) = Y^3 - XY + X^4 = 0$.
In other words, we will obtain the Puiseux expansion of $f(X,Y)$ of the form

$$\begin{aligned}Y(X) &= c_0 X^{\mu_0} + c_1 X^{\mu_0 \mu_1} + c_2 X^{\mu_0 \mu_1 \mu_2} + \cdots \\ &= X^{\mu_0}(c_0 + X^{\mu_1}(c_1 + c_2 X^{\mu_2} + \cdots))\end{aligned}$$

So we start with drawing Newton polygon of $f(X,Y)$:



On the line ℓ_1:
$\mu_0 = \frac{1}{2}$, so we start with putting $Y_0 = c_0 X^{\frac{1}{2}}$ into the equation $\tilde{f}(X,Y) = Y^3 - XY = 0$, and get

$$c_0^3 X^{\frac{3}{2}} - c_0 X^{\frac{3}{2}} = c_0(c_0^2 - 1)X^{\frac{3}{2}} = 0,$$

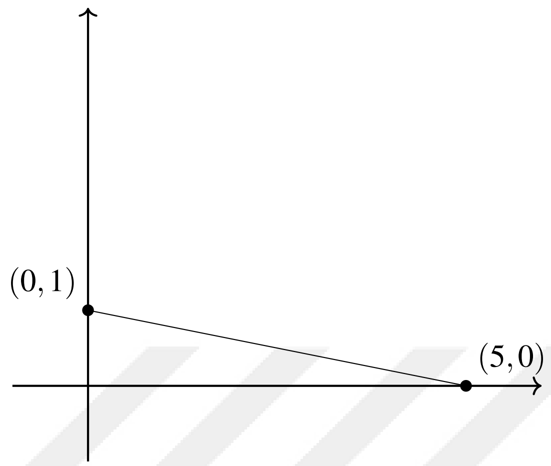
which implies $c_0 = \pm 1$.
Choose the branch $c_0 = 1$, then $Y_0 = X^{\frac{1}{2}}$.
We let $X^{\frac{1}{2}} = X_1$, and compute the second approximation

$$\begin{aligned} f(X_1^2, X_1(1+Y_1)) &= X_1^3(1+Y_1)^3 - X_1^3(1+Y_1) + X_1^8 \\ &= X_1^3 f_1(X_1, Y_1) \end{aligned}$$

where

$$f_1(X_1,Y_1) = Y_1^3 + 3Y_1^2 + 2Y_1 + X_1^5$$

Consider the Newton polygon of $f_1(X_1,Y_1)$:



Thus, on the above line we obtain

$$2Y_1 + X_1^5 = 0 \quad \Rightarrow \quad Y_1 = -\frac{X_1^5}{2}, \quad c_1 = -\frac{1}{2}$$

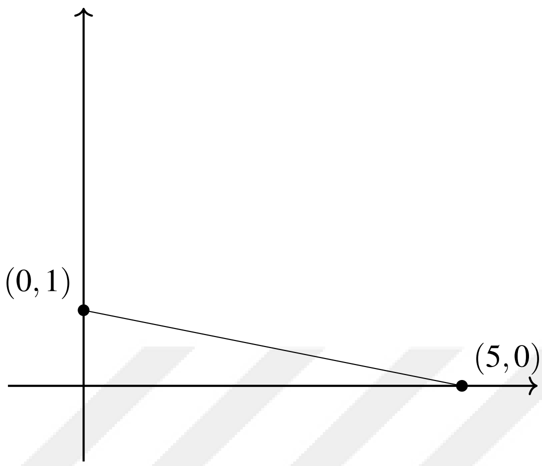
For the third approximation we put $f_1(X_1, X_1^5(-\frac{1}{2} + Y_2))$, then we have

$$\begin{aligned} f_1(X_1, X_1^5\left(-\frac{1}{2} + Y_2\right)) &= X_1^{15}\left(-\frac{1}{2} + Y_2\right)^3 + 3X_1^{10}\left(-\frac{1}{2} + Y_2\right)^2 + 2X_1^5\left(-\frac{1}{2} + Y_2\right) + X_1^5 \\ &= X_1^5 f_2(X_1, Y_2), \end{aligned}$$

where

$$f_2(X_1, Y_2) = X_1^{10}\left(-\frac{1}{2} + Y_2\right)^3 + 3X_1^5\left(-\frac{1}{2} + Y_2\right)^2 + 2Y_2.$$

Again we consider Newton polygon of f_2:



This implies that

$$2Y_2+\frac{3}{4}X_1^5=0 \quad \Rightarrow \quad Y_2=-\frac{3}{8}X_1^5, \quad c_2=-\frac{3}{8}$$

Lastly, we apply the algorithm one step further that is, we put

$$\begin{aligned} f_2(X_1,X_1^5(-\frac{3}{8}+Y_3)) &= X_1^{10}[2Y_3-3X_1^5(-\frac{3}{8}+Y_3)+higher \quad order \quad terms] \\ &= X_1^{10}f_3(X_1,Y_3) \end{aligned}$$

where

$$f_3(X_1,Y_3)=2Y_3+\frac{9}{8}X_1^5-\frac{1}{8}X_1^5+higher \quad order \quad terms$$

Notice that the Newton polygon of f_3 is the same as f_2, hence it follows that

$$2Y_3+X_1^5=0 \quad \Rightarrow \quad Y_3=-\frac{1}{2}X_1^5, \quad c_3=-\frac{1}{2}.$$

Therefore, we get a sequence of approximate solutions as

$$\begin{aligned} Y(X) &= X_1(1+X_1^5(-\frac{1}{2}+X_1^5(-\frac{3}{8}+-\frac{1}{2}X_1^5+\cdots))) \\ &= X^{\frac{1}{2}}(1+X^{\frac{5}{2}}(-\frac{1}{2}+X^{\frac{5}{2}}(-\frac{3}{8}-\frac{1}{2}X^{\frac{5}{2}}+\cdots))) \\ &= X^{\frac{1}{2}}-\frac{1}{2}X^3-\frac{3}{8}X^{\frac{11}{2}}-\frac{1}{2}X^8+\cdots \end{aligned}$$

We can therefore write

$$\tilde{\tilde{Y}}_{psx}^{(1)}(X) = X^{\frac{1}{2}} - \frac{1}{2}X^3 - \frac{3}{8}X^{\frac{11}{2}} - \frac{1}{2}X^8 + \cdots$$

Recall that there is another branch on the line ℓ_1. Namely, $\mu_0 = \frac{1}{2}$ and $c_0 = -1$. Thus, one can apply a similar process for the second branch starting with

$$Y_0 = c_0 X^{\mu_0} = -X^{\frac{1}{2}},$$

and hence can obtain the second solution as

$$\tilde{\tilde{Y}}_{psx}^{(0)}(X) = -X^{\frac{1}{2}} - \frac{1}{2}X^3 + \frac{3}{8}X^{\frac{11}{2}} - \frac{1}{2}X^8 + \cdots$$

Lastly, for the third branch we look at the line ℓ_2 ($\mu_0 = 3$, $c_0 = 1$), so starting the process with

$$Y_0 = c_0 X^{\mu_0} = X^3,$$

eventually yields the third solution

$$\tilde{Y}_{psx}^{(0)}(X) = X^3 + X^8 + 3X^{13} + 12X^{18} + \cdots$$

Notice that

$$-\tilde{\tilde{Y}}_1(X) = -\frac{1}{2}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})}{\Gamma(\frac{k+1}{2})k!} X^{\frac{5k+1}{2}} = X^{\frac{1}{2}} - \frac{1}{2}X^3 - \frac{3}{8}X^{\frac{11}{2}} - \frac{1}{2}X^8 + \cdots = \tilde{\tilde{Y}}_{psx}^{(1)}(X),$$

$$-\tilde{\tilde{Y}}_0(X) = -\frac{1}{2}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})(-1)^{1-k}}{\Gamma(\frac{k+1}{2})k!} X^{\frac{5k+1}{2}} = -X^{\frac{1}{2}} - \frac{1}{2}X^3 + \frac{3}{8}X^{\frac{11}{2}} - \frac{1}{2}X^8 + \cdots = \tilde{\tilde{Y}}_{psx}^{(0)}(X),$$

and

$$\tilde{Y}_0(X) = \sum_{k=0}^{\infty} \frac{\Gamma(3k+1)}{\Gamma(2k+2)k!} X^{5k+3} = X^3 + X^8 + 3X^{13} + \cdots = \tilde{Y}_{psx}^{(0)}(X).$$

As a result,

(i) $-\tilde{\tilde{Y}}_0(X)$ coincides with the Puiseux expansion $\tilde{\tilde{Y}}_{psx}^{(0)}(X)$ on the line ℓ_1.

(ii) $-\tilde{\tilde{Y}}_1(X)$ coincides with the Puiseux expansion $\tilde{\tilde{Y}}_{psx}^{(1)}(X)$ on the line ℓ_1.

(iii) $\tilde{Y}_0(X)$ coincides with the Puiseux expansion $\tilde{Y}_{psx}^{(0)}(X)$ on the line ℓ_2.

Finally, let us derive the differential equation satisfied by $\tilde{Y}_0, \tilde{\tilde{Y}}_0, \tilde{\tilde{Y}}_1$.

In particular

$$y_0(x) = \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} x^{-z} dz,$$

consider

$$\begin{aligned}(-1)^{\frac{1}{3}} X^{\frac{4}{3}} y_0(x) &= (-1)^{\frac{1}{3}} X^{\frac{4}{3}} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} ((-1)^{\frac{1}{3}} X^{-\frac{5}{3}})^{-z} dz \\ &= X^{\frac{4}{3}} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} (-1)^{\frac{1-z}{3}} X^{\frac{5z}{3}} dz = \tilde{Y}_0(X).\end{aligned}$$

Observe that

$$\vartheta_X \tilde{Y}_0(X) = \left(\frac{5z+4}{3}\right) X^{\frac{4}{3}} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} (-1)^{\frac{1-z}{3}} X^{\frac{5z}{3}} dz,$$

thus,

$$\vartheta_X \quad \leftrightarrow \quad \frac{5z+4}{3} \quad \Rightarrow \quad z = \frac{3\vartheta_X - 4}{5}.$$

Therefore, we can write

$$(\frac{3\vartheta_X - 4}{5})(\frac{3\vartheta_X - 4}{5} + 1)(\frac{3\vartheta_X - 4}{5} + 2)\tilde{Y}_0(X) = X^{\frac{4}{3}} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z+3)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} (-1)^{\frac{1-z}{3}} X^{\frac{5z}{3}} dz$$

Now setting $z+3 = z'$ gives,

$$X^{\frac{4}{3}} \int_{(c+3)-i\infty}^{(c+3)+i\infty} \frac{\Gamma(z')\Gamma(\frac{1-z'}{3}+1)}{\Gamma(\frac{4+2z'}{3}-2)} (-1)^{\frac{4-z'}{3}} X^{\frac{5z'}{3}} X^{-5} dz'$$

which can be rewritten as

$$X^{-5} X^{\frac{4}{3}} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z')\Gamma(\frac{1-z'}{3})(\frac{1-z'}{3})(\frac{4+2z'}{3}-1)(\frac{4+2z'}{3}-2)}{\Gamma(\frac{4+2z'}{3})} (-1)^{1+\frac{1-z'}{3}} X^{\frac{5z'}{3}} dz', \qquad (4.3.16)$$

putting $z' = \frac{3\vartheta_X - 4}{5}$ in (4.3.16), we get

$$(\frac{3\vartheta_X - 4}{5})(\frac{3\vartheta_X + 1}{5})(\frac{3\vartheta_X + 6}{5})\tilde{Y}_0(X) = -X^{-5}(\frac{3-\vartheta_X}{5})(\frac{2\vartheta_X - 1}{5})(\frac{2\vartheta_X - 6}{5})\tilde{Y}_0(X).$$

Or equivalently,

$$\left[X^5(\frac{3\vartheta_X-4}{5})(\frac{3\vartheta_X+1}{5})(\frac{3\vartheta_X+6}{5})+(\frac{3-\vartheta_X}{5})(\frac{2\vartheta_X-1}{5})(\frac{2\vartheta_X-6}{5})\right]\tilde{Y}_0(X)=0. \tag{4.3.17}$$

The above hypergeometric differential equation can be rewritten in the form (4.3.6). Note that for any non-zero a and for any b, we have the following relation

$$\begin{aligned}(a\vartheta_X+b)(X^\alpha Y(X)) &= a\alpha X^\alpha Y(X)+a\vartheta_X X^\alpha Y(X)+bX^\alpha Y(X)\\ &= X^\alpha(a(\vartheta_X+\alpha)+b)Y(X).\end{aligned}$$

From the above relation we can rewrite the equation (4.3.17) as follows

$$\left[\frac{3^3}{2^2}X^5\vartheta_X(\vartheta_X+\frac{5}{3})(\vartheta_X+\frac{10}{3})-(\vartheta_X-\frac{5}{3})(\vartheta_X+\frac{5}{6})(\vartheta_X-\frac{5}{3})\right]\tilde{Y}_0(X)=0.$$

Putting $\frac{3^3}{2^2}X^5=s$ and hence $\vartheta_X=5\vartheta_s$ in the above equation yields the following hypergeometric differential equation

$$\left[s\vartheta_s(\vartheta_s+\frac{1}{3})(\vartheta_s+\frac{2}{3})-(\vartheta_s-\frac{1}{3})(\vartheta_s+\frac{1}{6})(\vartheta_s-\frac{1}{3})\right]\varphi_1(s)=0.$$

Therefore, we conclude that

$$\left[\frac{3^3}{2^2}X^5\vartheta_X(\vartheta_X+\frac{5}{3})(\vartheta_X+\frac{10}{3})-(\vartheta_X-\frac{5}{3})(\vartheta_X+\frac{5}{6})(\vartheta_X-\frac{5}{3})\right]\tilde{\tilde{Y}}_j(X)=0$$

for $\quad 0\le j\le 1.$

Example 4.3.2. *($n=5,p=2,r=6$) We consider the algebraic equation*

$$f(X,Y)=Y^5-XY^2+X^6=0.$$

Our aim is to find solutions of the algebraic equation

$$f(X,Y)=Y^5-XY^2+X^6=0$$

by using a similar argument as in previous example. In this case, in considering the algebraic equation

$$\varphi(x,y) = y^5 + xy^2 - 1 = 0$$

with the Mellin inversion formula

$$y_0(x) = \frac{1}{5}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-2z}{5})}{\Gamma(\frac{6+3z}{5})} x^{-z} dz$$

one can obtain

$$\begin{aligned} y_0(x) &= -\frac{1}{5}\sum_{k=0}^{\infty} \operatorname*{Res}_{z=\frac{5k+1}{2}} \frac{\Gamma(z)\Gamma(\frac{1-2z}{5})}{\Gamma(\frac{6+3z}{5})} x^{-z} \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{5k+1}{2})(-1)^k}{\Gamma(\frac{3k+3}{2})k!} x^{\frac{-1-5k}{2}} \end{aligned}$$

In a similar fashion as in the previous example, one can get precisely the following new variables

$$Y(X) = (-1)^{\frac{1}{5}} X^{\frac{6}{5}} y(x) \quad and \quad x = (-1)^{\frac{2}{5}} X^{-\frac{13}{5}},$$

and therefore one obtains

$$\tilde{Y_0}(X) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{5k+1}{2})}{\Gamma(\frac{3k+3}{2})k!} X^{\frac{13k+5}{2}} = X^{\frac{5}{2}} + \frac{1}{2}X^9 + \frac{9}{8}X^{\frac{31}{2}} + \cdots$$

Then by virtue of the remark we obtain (4.3.2)

$$\tilde{Y_1}(X) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{5k+1}{2})(-1)^{13k+5}}{\Gamma(\frac{3k+3}{2})k!} X^{\frac{13k+5}{2}} = -X^{\frac{5}{2}} + \frac{1}{2}X^9 - \frac{9}{8}X^{\frac{31}{2}} + \cdots$$

Furthermore, since

$$\tilde{y_0}(x) = \frac{1}{5}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{-1-3z}{5})}{\Gamma(\frac{4+2z}{5})} (-x)^{-z} dz$$

is also a solution, one can have

$$\tilde{y_0}(x) = -\frac{1}{5}\sum_{k=0}^{\infty} \operatorname*{Res}_{z=\frac{5k-1}{3}} \frac{\Gamma(z)\Gamma(\frac{-1-3z}{5})}{\Gamma(\frac{4+2z}{5})}(-x)^{-z}$$
$$= \frac{1}{3}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{5k-1}{3})(-1)^{\frac{1-2k}{3}}}{\Gamma(\frac{2k+2}{3})k!} x^{\frac{1-5k}{3}}.$$

Again by using change of variables and remark (4.3.2) we get

$$\tilde{\tilde{Y}}_0(X) = \frac{1}{3}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{5k-1}{3})(-1)^{\frac{2-4k}{3}}}{\Gamma(\frac{2k+2}{3})k!} x^{\frac{13k+1}{3}} = -e^{\frac{2\pi i}{3}} X^{\frac{1}{3}} + \frac{1}{3}e^{-\frac{2\pi i}{3}} X^{\frac{14}{3}} + \frac{1}{3}X^9 + \cdots$$

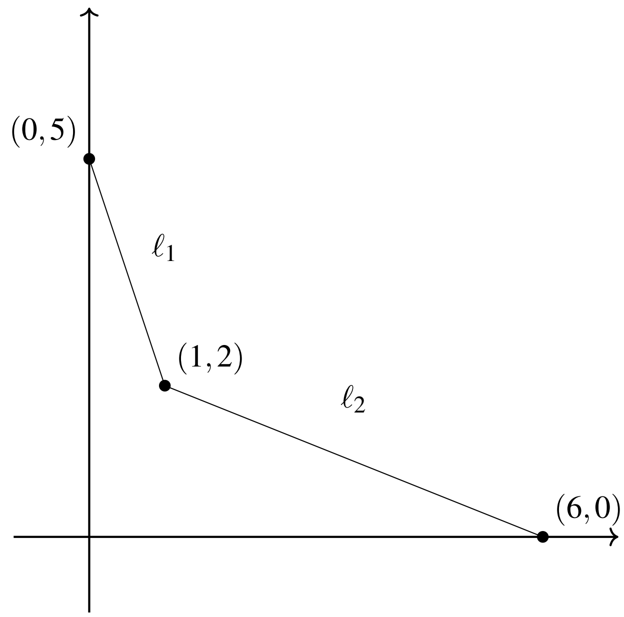
$$\tilde{\tilde{Y}}_1(X) = \frac{1}{3}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{5k-1}{3})e^{\frac{\pi i(22k+4)}{3}}}{\Gamma(\frac{2k+2}{3})k!} x^{\frac{13k+1}{3}} = -e^{\frac{4\pi i}{3}} X^{\frac{1}{3}} + \frac{1}{3}e^{\frac{26\pi i}{3}} X^{\frac{14}{3}} + \frac{1}{3}X^9 + \cdots$$

$$\tilde{\tilde{Y}}_2(X) = \frac{1}{3}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{5k-1}{3})}{\Gamma(\frac{2k+2}{3})k!} x^{\frac{13k+1}{3}} = -X^{\frac{1}{3}} + \frac{1}{3}X^{\frac{14}{3}} + \frac{1}{3}X^9 + \cdots$$

On the other hand, we apply Newton's algorithm to the algebraic equation

$$f(X,Y) = Y^5 - XY^2 + X^6 = 0 \tag{4.3.18}$$

which has the following Newton polygon:



In this case, on the line ℓ_1 ($\mu_0 = \frac{1}{3}$) we have three branches.
More precisely, starting with

$$Y_0 = c_0 X^{\frac{1}{3}}$$

gives

$$\tilde{f}(X, Y_0) = c_0^5 X^{\frac{5}{3}} - X c_0^2 X^{\frac{2}{3}} = c_0^2(c_0^3 - 1) X^{\frac{5}{3}} = 0$$

where $\tilde{f}(X,Y) = Y^5 - XY^2 = 0$. From this we obtain

$$c_0 = 1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}.$$

If we apply consecutive approximation process for each branch, then we eventually obtain the following solutions

$$\tilde{\tilde{Y}}^{(0)}_{psx}(X) = e^{\frac{2\pi i}{3}} X^{\frac{1}{3}} - \frac{1}{3} e^{-\frac{2\pi i}{3}} X^{\frac{14}{3}} - \frac{1}{3} X^9 + \cdots$$

$$\tilde{\tilde{Y}}^{(1)}_{psx}(X) = e^{\frac{4\pi i}{3}} X^{\frac{1}{3}} - \frac{1}{3} e^{-\frac{4\pi i}{3}} X^{\frac{14}{3}} - \frac{1}{3} X^9 + \cdots$$

$$\tilde{\tilde{Y}}^{(2)}_{psx}(X) = X^{\frac{1}{3}} - \frac{1}{3} X^{\frac{14}{3}} - \frac{1}{3} X^9 + \cdots$$

with respect to the branches $c_0 = e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}, 1$.
Likewise, on the line $\ell_2(\mu_0 = \frac{5}{2})$ one can immediately see that there are two branches.
More precisely, starting with

$$Y_0 = c_0 X^{\frac{5}{2}} \tag{4.3.19}$$

gives

$$\tilde{h}(X, c_0 X^{\frac{5}{2}}) = -X c_0^2 X^5 + X^6 = (1 - c_0^2) X^6 = 0$$

From this one can get

$$c_0 = \pm 1.$$

where $\tilde{h}(X,Y) = -XY^2 + X^6 = 0$. Applying the process for the branches $c_0 = 1, -1$ respectively we obtain the following solutions.

$$\tilde{Y}^{(0)}_{psx}(X) = X^{\frac{5}{2}} + \frac{1}{2} X^9 + \frac{9}{8} X^{\frac{31}{2}} + \cdots$$

$$\tilde{Y}^{(1)}_{psx}(X) = -X^{\frac{5}{2}} + \frac{1}{2} X^9 - \frac{9}{8} X^{\frac{31}{2}} + \cdots$$

Consequently,

(i) $\tilde{Y}_0(X)$ coincides with the Puiseux expansion $\tilde{Y}^{(0)}_{psx}(X)$ on the line ℓ_2.

(ii) $\tilde{Y}_1(X)$ coincides with the Puiseux expansion $\tilde{Y}^{(1)}_{psx}(X)$ on the line ℓ_2.

(iii) $-\tilde{\tilde{Y}}_0(X)$ coincides with the Puiseux expansion $\tilde{\tilde{Y}}^{(0)}_{psx}(X)$ on the line ℓ_1.

(iv) $-\tilde{\tilde{Y}}_1(X)$ coincides with the Puiseux expansion $\tilde{\tilde{Y}}^{(1)}_{psx}(X)$ on the line ℓ_1.

(v) $-\tilde{\tilde{Y}}_2(X)$ coincides with the Puiseux expansion $\tilde{\tilde{Y}}^{(2)}_{psx}(X)$ on the line ℓ_1.

Finally, we give the hypergeometric differential equation satisfied by $\tilde{Y}_i(X)$ and $\tilde{\tilde{Y}}_j(X)$, for $0 \le i \le 1$ and $0 \le j \le 2$.
After some computations we obtain

$$
\begin{aligned}
&[X^{13}(\frac{5\vartheta_X - 6}{13})(\frac{5\vartheta_X + 7}{13})(\frac{5\vartheta_X + 20}{13})(\frac{5\vartheta_X + 33}{13})(\frac{5\vartheta_X + 46}{13}) \\
&-(\frac{18 - 2\vartheta_X}{13})(\frac{5 - 2\vartheta_X}{13})(\frac{3\vartheta_X - 1}{13})(\frac{3\vartheta_X - 14}{13})(\frac{3\vartheta_X - 27}{13})]\tilde{Y}_i(X) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&[X^{13}(\frac{5\vartheta_X - 6}{13})(\frac{5\vartheta_X + 7}{13})(\frac{5\vartheta_X + 20}{13})(\frac{5\vartheta_X + 33}{13})(\frac{5\vartheta_X + 46}{13}) \\
&-(\frac{18 - 2\vartheta_X}{13})(\frac{5 - 2\vartheta_X}{13})(\frac{3\vartheta_X - 1}{13})(\frac{3\vartheta_X - 14}{13})(\frac{3\vartheta_X - 27}{13})]\tilde{\tilde{Y}}_j(X) = 0.
\end{aligned}
$$

Also we can convert the above differential equation into the form (4.3.6).
Applying a similar procedure which we did in the previous example to the above differential equations yields the following

$$
\begin{aligned}
&[\frac{5^5}{2^2 3^3}X^{13}(\vartheta_X)(\vartheta_X + \frac{13}{5})(\vartheta_X + \frac{26}{5})(\vartheta_X + \frac{39}{5})(\vartheta_X + \frac{52}{5}) \\
&-(\frac{39}{5} - \vartheta_X)(\frac{13}{5} - \vartheta_X)(\vartheta_X + \frac{13}{15})(\vartheta_X - \frac{52}{15})(\vartheta_X - \frac{39}{5})]\tilde{Y}_i(X) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&[\frac{5^5}{2^2 3^3}X^{13}(\vartheta_X)(\vartheta_X + \frac{13}{5})(\vartheta_X + \frac{26}{5})(\vartheta_X + \frac{39}{5})(\vartheta_X + \frac{52}{5}) \\
&-(\frac{39}{5} - \vartheta_X)(\frac{13}{5} - \vartheta_X)(\vartheta_X + \frac{13}{15})(\vartheta_X - \frac{52}{15})(\vartheta_X - \frac{39}{5})]\tilde{\tilde{Y}}_j(X) = 0.
\end{aligned}
$$

If we put $\frac{5^5}{2^2 3^3} X^{13} = s$, then we eventually obtain the following hypergeometric differential equation

$$[s\vartheta_s(\vartheta_s+\frac{1}{5})(\vartheta_s+\frac{2}{5})(\vartheta_s+\frac{3}{5})(\vartheta_s+\frac{4}{5})$$
$$-(\vartheta_s-\frac{3}{5})(\vartheta_s-\frac{1}{10})(\vartheta_s+\frac{1}{15})(\vartheta_s-\frac{4}{15})(\vartheta_s-\frac{3}{5})]\varphi_i(s)=0$$

and

$$[s\vartheta_s(\vartheta_s+\frac{1}{5})(\vartheta_s+\frac{2}{5})(\vartheta_s+\frac{3}{5})(\vartheta_s+\frac{4}{5})$$
$$-(\vartheta_s-\frac{3}{5})(\vartheta_s-\frac{1}{10})(\vartheta_s+\frac{1}{15})(\vartheta_s-\frac{4}{15})(\vartheta_s-\frac{3}{5})]\psi_j(s)=0$$

for $0 \le i \le 1$ and $0 \le j \le 2$.

Chapter 5

Conclusion

In general when n, p, r are large positive integers, finding the solutions of a given algebraic equation of the form

$$f(X,Y) = Y^n - XY^p + X^r = 0 \quad with \quad (n-p) > \frac{p}{r-1}$$

by means of Newton's algorithm requires much more complicated calculations because of the higher order terms which occur at each approximation step. On the other hand, the hypergeometric series in which we proposed proposition (4.3.1) do not require such tough computations. Furthermore, writing the solutions of the algebraic equation $f(X,Y) = 0$ in terms of hypergeometric series provides us global solutions(i.e infinitely many coefficients and powers of X are known).
In conclusion, one does not have to perform Newton's algorithm for a given algebraic equation with the conditions mentioned above in virtue of proposition (4.3.1). And thus one can obtain the solutions of the algebraic equation $f(X,Y) = 0$ by using hypergeometric series with less effort.

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