ÇUKUROVA UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES

PhD THESIS

Haytham Darweesh Mustafa ABUSARRIS

GENERALIZED TRANSFORMATION SEMIGROUPS AND RESTRICTED RANGE

DEPARTMENT OF MATHEMATICS

ADANA-2022

DEDICATION

TO MY DEAR FAMILY

ABSTRACT

PhD THESIS

GENERALIZED TRANSFORMATION SEMIGROUPS and RESTRICTED RANGE

Haytham Darweesh Mustafa ABUSARRIS

ÇUKUROVA UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

Supervisor : Prof. Dr. Gonca AYIK
Year: 2022, Pages: 89
Jury : Prof. Dr. Gonca AYIK
: Prof. Dr. Hayrullah AYIK
: Prof. Dr. Melis MINISKER
: Assoc. Prof. Dr. Belgin ÖZER
: Assoc. Prof. Dr. Leyla BUGAY

The generalized transformation semigroup $\mathcal{G}T(X,Y;\theta)$ and the semigroup of transformation with restricted range $T(X,Y)$, have been much studied over the last 60 years. Many of algebraic questions were answered over there, as we will see, such as regularity, Green's relations, isomorphism and embedding theorems, generating systems, ranks and some others. One of the important questions which is still unanswered until now is: Determining a minimal generating set and the rank of $\mathcal{G}T(X,Y;\theta)$, and this was the motivation for us to do more investigations.

In this thesis, we give our contributions in this direction. First, we show that $\mathcal{G}T(X,Y;\theta)$ can be embedded in $T(Z,Y) = \{\alpha \in T(Z) : Z\alpha \subseteq Y\}$ where $Z = X \cup Y$. We also show, if $\theta: Y \to X$ is one to one, then $\mathcal{G}T(X,Y;\theta)$ and $T(X, \mathrm{im}(\theta))$ are isomorphic semigroups. After that, we present our main results as follows: If $|X| = m$, $|Y| = n$, and $|\mathrm{im}(\theta)| = r$, where $m, n, r \in \mathbb{N}$, then we show that:

$$\mathrm{rank}(\mathcal{G}T(X,Y;\theta)) = \begin{cases} \displaystyle\sum_{k=r+1}^{\min(m,n)} \frac{n!\,S(m,k)}{(n-k)!} & \text{if } \theta \text{ is neither onto nor } 1-1 \\ \dbinom{n}{m} & \text{if } \theta \text{ is onto but not } 1-1 \\ S(m,r) & \text{if } \theta \text{ is } 1-1 \text{ and } 1 \le r \le m-1 \end{cases}.$$

Key Words: Generalized transformation semigroups, the semigroup of transformation with restricted range, generating set, rank, isomorphism theorems, embedding theorems.

ÖZ

DOKTORA TEZİ

GENELLEŞTİRİLMİŞ DÖNÜŞÜM YARIGRUPLARI ve KISITLANMIŞ İMAJ

Haytham Darweesh Mustafa ABUSARRIS

ÇUKUROVA ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI

Danışman : Prof. Dr. Gonca AYIK
Yıl: 2022, Sayfa: 89
Jüri : Prof. Dr. Gonca AYIK
: Prof. Dr. Hayrullah AYIK
: Prof. Dr. Melis MINISKER
: Doç. Dr. Belgin ÖZER
: Doç. Dr. Leyla BUGAY

X ve Y iki boştan farklı (ayrık) kümeler ve $\mathcal{G}\mathrm{T}(\mathrm{X},\mathrm{Y})$ ile X den Y ye tüm (tam) dönüşümlerin kümesini gösterelim. Herhangi bir sabitlenmiş $\theta: Y \to X$ dönüşümü için, $\mathcal{G}\mathrm{T}(\mathrm{X},\mathrm{Y})$ üzerindeki $*$ sandviç işlemini, her $\alpha,\beta \in \mathcal{G}\mathrm{T}(\mathrm{X},\mathrm{Y})$ için $\alpha * \beta = \alpha \circ \theta \circ \beta$ olarak tanımlayalım. O zaman $\mathcal{G}\mathrm{T}(\mathrm{X},\mathrm{Y})$ kümesi $*$ sandviç işlemi ile birlikte bir yarıgrup olup bu yarıgruba genelleştirilmiş dönüşüm yarıgrubu denir ve $\mathcal{G}T(X,Y;\theta)$ ile gösterilir. $T(Z)$ ile $Z = X \cup Y$ kümesi üzerindeki tüm dönüşümlerin yarıgrubunu gösterelim.

Bu tezde, ilk olarak $\mathcal{G}T(X,Y;\theta)$ kümesinin, $T(Z)$ nin bir alt yarıgrubu olan kısıtlanmış imaj ile dönüşümler yarıgrubu $T(Z,Y) = \{\alpha \in T(Z) : Z\alpha \subseteq Y\}$ nin içine gömülebildiğini gösterdik. Üstelik, eğer $\theta: Y \to X$ birebir ise o zaman $\mathcal{G}T(X,Y;\theta)$ ve $T(X, \mathrm{im}(\theta))$ in izomorfik yarıgruplar olduğunu gösterdik. Bunlara ek olarak aşağıdaki ana sonucumuzu ifade ettik: Eğer $m,n,r \in \mathbb{N}$ ve $S(m,n)$ ikinci tip Stirling sayısı olmak üzere $|X| = m$, $|Y| = n$, ve $|\mathrm{im}(\theta)| = r$ ise

$$\mathrm{rank}\big(\mathcal{G}T(X,Y;\theta)\big) = \begin{cases} \displaystyle\sum_{k=r+1}^{\min(m,n)} \frac{n!\,S(m,k)}{(n-k)!} & \text{eğer } \theta \text{ ne örten nede } 1-1 \\ \dbinom{n}{m} & \text{eğer } \theta \text{ örten fakat } 1-1 \text{ değil} \\ S(m,r) & \text{eğer } \theta\ 1-1 \text{ ve } 1 \le r \le m-1 \end{cases}$$

olduğunu gösterdik.

Anahtar kelemeleri: Genelleştirilmiş dönüşüm yarıgrupları, imajı kısıtlanmış dönüşüm yarıgrubu, doğuray kümesi, rank, izomorfizm teoremleri, gömme teoremleri.

EXTENDED SUMMARY

Before studying transformation semigroups, one must know some history behind it. According to Clifford and Preston (1967), the theory of transformation semigroups really began in 1928 when A. K. Suschkewitsch used transformations of a set to describe the structure of a general finite simple semigroups. The same author, in 1937, published a short book in which T_X played an important role in illustrating various definitions. For example, Suschkewitsch is thought to be the first one who characterized Green's $\mathcal{L}$ and $\mathcal{R}$ relations for T_X, and he proved that every semigroup can be embedded in T_X for some X; and this result means transformation semigroups play the same role for semigroup theory as permutations groups do for group theory.

From 1940s to the early 1960s, the Russian mathematicians produced some major results on transformation semigroups which concerned generators, morphisms, ideals and congruences. The important problem of finding the rank of a semigroup has long been studied in the literature and there are some studies that examined some special kinds of ranks such as idempotent-rank, nilpotent rank, or (m,r) −rank, see Ayık, Ayık, Ünlü, Howie (2008), Ayık, Toker (2018), Garba (1994) and Levi (2006). Just to mention, in 1966, Howie published his famous paper, Howie (1966), and proved that the semigroup $Sing_X$ of all singular transformations of a finite set X (i.e. all non-invertible functions from X to X) is generated by its idempotents. Then, in subsequent works, and with other authors, Howie calculated the rank and idempotent rank of $Sing_X$, see Gomes and Howie (1987) and Howie (1978). Moreover, he calculated the rank and idempotent rank of the ideals of $Sing_X$, see Gomes and Howie (1992). These works have been enormously influential, and have led to the development of several vibrant areas of research covering semigroups of (partial) transformations, matrices, partitions, endomorphisms of various algebraic structures.

It is well-known that $\mathrm{rank}(P_n) = 4$ and $\mathrm{rank}(T_n) = \mathrm{rank}(I_n) = 3$ when $n \geq 3$. After that, the authors tended to generalize these results by defining new transformation semigroups that are, in some sense, generalized from the old ones. For $1 \leq r \leq n$, let $K(n,r) = \{\alpha \in T_n: |\mathrm{im}(\alpha)| \leq r\}$. In Howie and McFadden (1990), the authors showed that for $2 \leq r \leq n-1$, $\mathrm{rank}(K(n,r)) = \mathrm{idrank}(K(n,r)) = S(n,r)$, the Stirling number of the second kind. In the same year, Garba (1990) considered the semigroup $P(n,r) = \{\alpha \in P_n : |\mathrm{im}(\alpha)| \leq r\}$ and showed that for $2 \leq r \leq n-1$, $\mathrm{rank}(P(n,r)) = \mathrm{idrank}(P(n,r)) = S(n+1,r+1)$. For unexplained terms in semigroup theory, see Ganyushkin and Mazorchuk (2009), Howie (1995).

Throughout this thesis we consider two other special kinds of transformation semigroups, which are extensively studied in the last decades. The first one is the transformation semigroups with restricted range and the second is the generalized transformation semigroups. Finally, we attempt to determine the generating sets and rank of the generalized transformation semigroup depending on isomorphism theorems between it and the transformation semigroups with restricted range.

ACKNOWLEDGEMENTS

I would like to sincerely thank my supervisor Prof. Dr. Gonca AYIK for her guidance, understanding, patience, positive encouragement and warm spirit to finish this thesis. It has been a great pleasure and an honor to have her as my supervisor.

I also should not forget all teachers and professors who taught me at all academic levels. In particular, Prof. Dr. Hayrullah AYIK who encouraged me to read, study and research about the awesome topic in abstract algebra, namely semigroup theory.

I acknowledge to thank financial support from the Presidency for Turks Abroad and Related Communities (Yurtdışı Türkler ve Akraba Topluluklar Başkanlığı (YTB)).

CONTENTS **PAGE**

LIST OF TABLES **PAGE**

LIST OF FIGURES **PAGE**

1. INTRODUCTION

An algebraic structure consists of a set and an associative binary operation is called a semigroup. This term was used at the first time in 1904 by the French in the paper: J.-A. de Séguier in Élements de la Théorie des Groupes Abstraits (Elements of the Theory of Abstract Groups). After four years, the term is used in English in the paper: Harold Hinton's Theory of Groups of Finite Order. In 1928, the first non-trivial results about semigroup theory were obtained by Anton Suschkewitsch. When he determined the structure of finite simple semigroups and proved that the minimal ideal or Green's relation $\mathcal{J}$ −class of a finite semigroup is simple.

Indeed, any abstract algebraic study is rooted in concrete examples, and these examples considered as one of the main motivations for the existence of this study. For the case of semigroup theory, trivially, transformation semigroups are such examples. For any set X, one can distinguish three main transformation semigroups: The partial transformation semigroup PT_X, the full transformation semigroup T_X and the symmetric inverse transformation semigroup I_X. If $X = X_n = \{1, \dots, n\}$, we denote the semigroups PT_{X_n} , T_{X_n} , and I_{X_n} by P_n , T_n , and I_n, respectively.

Recall from Cayley's theorem for finite groups that every finite group is isomorphic to a subgroup of a symmetric group S_n. Hence, the finite symmetric group S_n and its subgroups have an important role in finite group theory and also in finite semigroup theory. Similarly, it is well known that every finite semigroup is isomorphic to a subsemigroup of a finite transformation's semigroup T_n, and (Wagner–Preston theorem) that every finite inverse semigroup is isomorphic to a subsemigroup of a finite symmetric inverse semigroup I_n. Moreover, there is an isomorphism between the semigroup P_n and the subsemigroup $P_n{}^*$ of the transformation's semigroup consisting of all self-maps on $X_n \cup \{0\}$ for which

$0\alpha = 0$. Hence, the importance of T_n and P_n to finite semigroup theory and the importance of I_n to finite inverse semigroup theory may be likened to the importance of symmetric group S_n to finite group theory.

In this thesis we concentrate our work at two special kinds of subsemigroups of T_X, namely the generalized transformation semigroup $\mathcal{G}T(X,Y;\theta)$ and the semigroup of transformation with restricted range $T(X,Y)$.

In chapter three we introduced basic definitions and concepts which will be needed in the following chapters. We defined semigroup, monoid, group, subsemigroup, ideal, morphisms, transformations, representations, Green's relations, generating sets, ranks, regularity and some others.

In chapter four we studied the semigroups of transformation with restricted range $PT(X,Y)$, $T(X,Y)$, $I(X,Y)$, and some others. We explained some of their algebraic properties and presented some newly results that have published within the last twenty years.

In chapter five we considered the generalized transformation semigroup $\mathcal{G}T(X,Y;\theta)$ and did some investigations about some algebraic properties on it such as; Symons's and Green's relations, regularity of $PT(X,Y;\theta)$ and $\mathcal{G}T(X,Y;\theta)$ and some embedding and isomorphism theorems.

In chapter six we gave our contributions by trying to answer the questions about the rank and generating sets of $\mathcal{G}T(X,Y;\theta)$, and about some isomorphism theorems between $\mathcal{G}T(X,Y;\theta)$ and $T(X,Y)$.

This thesis is wished to become a ground for more researches and studies about generalized transformation semigroups and semigroups of transformation with restricted range.

2. PREVIOUS STUDIES

Before studying transformation semigroups, one must know some history behind it. According to Clifford and Preston (1967), the theory of transformation semigroups really began in 1928 when A. K. Suschkewitsch used transformations of a set to describe the structure of a general finite simple semigroups. The same author, in 1937, published a short book in which T_X played an important role in illustrating various definitions. For example, Suschkewitsch is thought to be the first one who characterized Green's $\mathcal{L}$ and $\mathcal{R}$ relations for T_X, and he proved that every semigroup can be embedded in T_X for some X; and this result means transformation semigroups play the same role for semigroup theory as permutations groups do for group theory.

From 1940s to the early 1960s, the Russian mathematicians produced some major results on transformation semigroups which concerned generators, morphisms, ideals and congruences. The important problem of finding the rank of a semigroup has long been studied in the literature and there are some studies that examined some special kinds of ranks such as idempotent-rank, nilpotent rank, or (m, r) −rank, see Ayık, Ayık, Ünlü, Howie (2008), Ayık, Toker (2018), Garba (1994) and Levi (2006). Just to mention, in 1966, Howie published his famous paper, Howie (1966), and proved that the semigroup $Sing_X$ of all singular transformations of a finite set X (i.e. all non-invertible functions from X to X) is generated by its idempotents. Then, in subsequent works, and with other authors, Howie calculated the rank and idempotent rank of $Sing_X$, see Gomes and Howie (1987) and Howie (1978). Moreover, he calculated the rank and idempotent rank of the ideals of $Sing_X$, see Gomes and Howie (1992). These works have been enormously influential, and have led to the development of several vibrant areas of research covering semigroups of (partial) transformations, matrices, partitions, endomorphisms of various algebraic structures.

It is well-known that $\mathrm{rank}(P_n) = 4$ and $\mathrm{rank}(T_n) = \mathrm{rank}(I_n) = 3$ when $n \geq 3$. After that, the authors tended to generalize these results by defining new transformation semigroups that are, in some sense, generalized from the old ones. For $1 \leq r \leq n$, let $K(n,r) = \{\alpha \in T_n : |\mathrm{im}(\alpha)| \leq r\}$. In Howie and McFadden (1990), the authors showed that for $2 \leq r \leq n-1$, $\mathrm{rank}(K(n,r)) = \mathrm{idrank}(K(n,r)) = S(n,r)$, the Stirling number of the second kind. In the same year, Garba (1990) considered the semigroup $P(n,r) = \{\alpha \in P_n : |\mathrm{im}(\alpha)| \leq r\}$ and showed that for $2 \leq r \leq n-1$, $\mathrm{rank}(P(n,r)) = \mathrm{idrank}(P(n,r)) = S(n+1, r+1)$. For unexplained terms in semigroup theory, see Ganyushkin and Mazorchuk (2009), Howie (1995).

Throughout this thesis we consider two other special kinds of transformation semigroups, which are extensively studied in the last decades. The first one is the transformation semigroups with restricted range and the second is the generalized transformation semigroups. Finally, we attempt to determine the generating sets and rank of the generalized transformation semigroup depending on isomorphism theorems between it and the transformation semigroups with restricted range.

3. ELEMENTARY CONCEPTS AND THEOREMS

Our aim in this chapter is to study some basic definitions and important theorems that we shall need in the following chapters, such as: semigroup, monoid, group, subsemigroup, ideal, idempotent, transformation, generating set, rank, regularity, Green's relations and others.

3.1. Semigroups, Monoids and Groups

Definition 3.1.1. A *binary operation* $*$ on a set S is a mapping of the Cartesian product $S \times S$ into S. That is;

$$*: S \times S \to S .$$

The image in S of the element (x, y) of $S \times S$ is denoted by $x * y$.

A *groupoid* is a system consisting of a non-empty set S together with a binary operation $*$ and denoted by $(S, *)$. Any groupoid $(S, *)$ is called a *semigroup*, if the operation $*$ is associative. That is; if $\forall\, x, y, z \in S$

$$x * (y * z) = (x * y) * z \ .$$

For ease of notation, we will write just S instead of $(S, *)$ for expressing a semigroup and therefore the associativity will be presented as:

$$(xy)z = x\,(yz)\,.$$

Further, S is a *commutative semigroup*, if $*$ satisfies an additional condition, namely the commutativity property. That is; $\forall\, x, y \in S, xy = yx$.

Example 3.1.2. Let $\mathbb{N}$ be the set of all natural numbers and define $*$ on $\mathbb{N}$ by $x * y = \max\{x, y\}$. Then $(\mathbb{N}, *)$ is a commutative semigroup. The associativity property will be as follows:

$$x * (y * z) = \max\{x, \max\{y, z\}\} = \max\{x, y, z\} = \max\{\max\{x, y\}, z\}$$
$$= (x * y) * z$$

Definition 3.1.3. Let S be a semigroup and $e \in S$. If for all $x \in S$, $ex = x$ then e is called a *left identity* of S. Dually, e is a *right identity* of S if for all $x \in S$, $xe = x$. If e is both a left and a right identity of S, then we call e an *identity* of S.

A *monoid* is a triple $(S, *, e)$. That is a monoid is a semigroup S together with its identity element e.

Lemma 3.1.4. The identity of a monoid S is unique.
Proof. Let $e, f \in S$ be two identity elements of S, then

$$ef = f \quad (\text{since } e \text{ is a left identity})$$
$$ef = e \quad (\text{since } f \text{ is a right identity})$$

Thus, $e = f$. Therefore, any semigroup S has at most one identity. ■

The symbol "1" usually denote the identity of a monoid S. If a semigroup S has no identity element, then we can, by the following definition, add an extra element 1 to S in order to form a monoid.

Definition 3.1.5. Let S be a semigroup and define the set S^1 by

$$S^1 = \begin{cases} S & \text{if S is a monoid} \\ S \cup \{1\} & \text{if S is not a monoid} \end{cases}.$$

The above set with the following binary operation: for all $x, y \in S^1$

$$xy = \begin{cases} xy & if \quad x \neq 1, y \neq 1 \\ x & if \quad x \neq 1, y = 1 \\ y & if \quad x = 1, y \neq 1 \\ 1 & if \quad x = 1, y = 1 \end{cases},$$

is called a *monoid obtained by adjoining an identity* to S.

Next, we define one of the most important concepts in algebra, namely a group. Note that this definition is not the commonest definition but they are equivalent, however this one is frequently used in semigroup theory.

Definition 3.1.6. A semigroup S is a *group* if it satisfies the following condition:

$$\forall\, a \in S, \qquad aS = S \text{ and } Sa = S$$

Equivalently, a semigroup S is a *group* if the following conditions hold

i) $\exists e \in S: \forall\, x \in S, ex = x$ and

ii) $\forall\, x \in S, \exists\, x^{-1} \in S: x^{-1}x = e$

where e is the (group) identity and x^{-1} is the inverse of x.

3.2. Subsemigroups and Ideals

Definition 3.2.1. Let S be a semigroup and $z \in S$. If $\forall x \in S, zx = z$ then z is a *left zero* of S. Dually, z is a *right zero* of S if $\forall x \in S$, $xz = z$. If $zx = xz = z$, $\forall x \in S$, then z is a *two-sided zero* (or simply a *zero*) and denoted by 0.

Lemma 3.2.2. A semigroup S has at most one zero.

Proof. Let $z_1, z_2 \in S$ be two zeros of S, then $z_1 = z_2$ since

$$\begin{aligned} z_1 &= z_1 z_2 && \text{(Since } z_1 \text{ is a left zero)} \\ &= z_2 && \text{(Since } z_2 \text{ is a right zero)}\,.\ \blacksquare \end{aligned}$$

Definition 3.2.3. Let S be a semigroup. An element $e \in S$ is called an *idempotent* if $e^2 = ee = e$.We denote the set of all idempotents of S by $E(S)$. If every element of a semigroup S is an idempotent, then S is called a *band*. Commutative bands are called *semilattices*.

Example 3.2.4. Consider the set of all 2×2 matrices with integer entries

$$\mathbb{Z}^{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}.$$

Under the ordinary multiplication of matrices, $\mathbb{Z}^{2\times 2}$ becomes a semigroup. Furthermore, it forms a monoid with the identity matrix I, and the zero element is the zero matrix 0, which are, respectively:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, $\mathbb{Z}^{2\times 2}$ has many idempotents, one of them for example is:

$$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}.$$

Definition 3.2.5. Let S be a semigroup and $\emptyset \neq T \subseteq S$. We say that T is a *subsemigroup* of S, and denote it by $T \leq S$, if T is closed under the binary operation of S. That is to say: $\forall\, x, y \in T, \quad xy \in T$.

Equivalently:

$$T \leq S \quad \Leftrightarrow \quad T^2 \subseteq S.$$

S, $\emptyset$ and $\{1\}$ are called trivial subsemigroups of S. Any subsemigroup $\emptyset \neq T \subsetneq S$ is called *a proper subsemigroup* of S. A *submonoid* T of S is a subsemigroup that is itself a monoid under the same operation of S. A *subgroup* T of S is a subsemigroup that is itself a group under the same operation of S.

Definition 3.2.6. Let S be a semigroup and $\emptyset \neq I \subseteq S$.

i) If $\forall s \in S$ and $\forall a \in I$, $sa \in I$ then I is called a *left ideal* of S and we write $SI \subseteq I$.

ii) If $\forall s \in S$ and $\forall a \in I$, $as \in I$ then I is called a *right ideal* of S and we write $IS \subseteq I$.

iii) If I is both left and right ideal of S then I is called an *ideal* of S and denoted by $I \trianglelefteq S$.

Every ideal, whether a left, a right or a two-sided, is a subsemigroup. $S, \emptyset$ are called trivial ideals of the semigroup S. If I is not a trivial ideal of S ($I \neq \emptyset$, $I \neq S$) then I is called *a proper ideal.*

Definition 3.2.7. Let S and T be semigroups and $\alpha: (S, *) \to (T, \circ)$ is a mapping, where $*$ and $\circ$ are any binary operations. If $\forall\, x, y \in S$

$$(x * y)\alpha = (x\alpha) \circ (y\alpha)$$

then α is said to be *homomorphism*.

If S and T are monoids, then α is a *monoid homomorphism* if

i) $(x * y)\alpha = (x\alpha) \circ (\alpha y)$, $\forall\, x, y \in S$

ii) $(1_S)\alpha = 1_T$.

Definition 3.2.8. Let S and T be two semigroups and $\alpha: (S, *) \to (T, \circ)$ be a homomorphism. Then

i) α is a *monomorphism* or (*embedding*), if it is injective,

ii) α is an *epimorphism*, if it is surjective,

iii) α is an *isomorphism*, if it is both monomorphism and epimorphism,

iv) α is an *endomorphism*, if $S = T$,

v) α is an *automorphism*, if it is both an isomorphism and an endomorphism.

A semigroup S is *embeddable* in a semigroup T, if there exists an embedding $\alpha: S \to T$, and we write $S \hookrightarrow T$. A semigroup S is *isomorphic* to T if there exists an isomorphism $\alpha: S \to T$ and we write $S \cong T$.

Lemma 3.2.9. If $\alpha: S \to T$ is an isomorphism then so is the inverse map $\alpha^{-1}: T \to S$.

Proof. Since α is a bijection (injective and surjective) then α^{-1} exists. Moreover, $\alpha^{-1}\alpha = I$ (the identity map). Thus;

$$\big((x)\alpha^{-1}(y)\alpha^{-1}\big)\alpha = (x\alpha^{-1})\alpha\,(y\alpha^{-1})\alpha = xy \ .$$

Therefore,

$$(x)\alpha^{-1}(y)\alpha^{-1} = (xy)\alpha^{-1} \ . \qquad \blacksquare$$

Lemma 3.2.10. If $\alpha: S \to T, \beta: T \to R$ are homomorphisms, then so is $\alpha\beta: S \to R$.

Proof. $\forall\, x, y \in S$

$$(xy)\alpha\beta = \big((xy)\alpha\big)\beta = \big((x\alpha)(y\alpha)\big)\beta = \big((x\alpha)\big)\beta\big((y\alpha)\big)\beta = (x)\alpha\beta(y)\alpha\beta. \qquad \blacksquare$$

3.3. Relations, Transformations and Representations

Definition 3.3.1. Let X and Y be two non-empty sets. The *Cartesian product* of X and Y is defined by

$$X \times Y = \{(x, y): x \in X, y \in Y\}\,.$$

Any subset of $X \times Y$ is called a *relation* from X to Y. Thus, a relation ρ on X is any subset of $X \times X$.

Definition 3.3.2. The set of all relations on a set X is denoted by B_X and defined as

$$B_X = \{\rho : \rho \subseteq X \times X\} .$$

This is a semigroup with the composition of relations operation, defined as follows

$$\rho \circ \delta = \{(x, y) \in X \times X : \exists z \in X , (x, z) \in \rho \text{ and } (z, y) \in \delta\}$$

forms a semigroup, and with the identity relation $\Delta_X = 1_X = \{ (x, x) : x \in X \times X\}$ forms a monoid. The empty set $\emptyset$, the identity relation Δ_X and the universal relation $X \times X$ are all trivial examples of relations on a set X.

Definition 3.3.3. Let $\rho \in \beta_X$, then the sets
i) $\rho^{-1} = \{(y, x) : (x, y) \in \rho\}$,
ii) $\mathrm{dom}(\rho) = X\rho = \{x \in X : \exists y \in Y, (x, y) \in \rho \}$,
iii) $\mathrm{im}(\rho) = \{y \in X : \exists x \in Y, (x, y) \in \rho \}$,
iv) $x\rho = \{y \in X : (x, y) \in \rho \}$,
v) $y\rho^{-1} = \{x \in X : (x, y) \in \rho\}$
are called the *inverse relation* of ρ, the *domain* of ρ, the *images* of ρ, the *image of* x *under* ρ and *the pre-image* of $y \in Y$, respectively.

Definition 3.3.4. Let X be a non-empty set and $\rho \in \beta_X$. Then
i) ρ is called a *reflexive* relation if $\Delta_X \subseteq \rho$,
ii) ρ is called a *symmetric* relation if $\rho^{-1} \subseteq \rho$,
iii) ρ is called an *anti-symmetric* relation if $\rho^{-1} \cap \rho \subseteq \Delta_X$ and
iv) ρ is called a *transitive* relation if $\rho \circ \rho \subseteq \rho$.

If ρ is a reflexive, symmetric and transitive relation then ρ is called an *equivalence relation*. However, if ρ is a reflexive, anti-symmetric and transitive relation then ρ is called a *partial order*.

Definition 3.3.5. Let ρ be an equivalence relation on a set X. For any $x \in X$, an *equivalence class* of x is defined by

$$x\rho = \{y \in X : (x, y) \in \rho\}.$$

The set of all equivalence classes of x is written as

$$X/\rho = \{x\rho : x \in X\}.$$

The set X/ρ is called the *quotient set* of X by ρ.

Definition 3.3.6. Let X be a non-empty set and consider the collection of some subsets of X, say

$$\mathbb{A} = \{A_i : i \in I\}.$$

The collection $\mathbb{A}$ is said to be a *partition* of X, if

i) $A_i \neq \emptyset, \forall i \in I$,

ii) $\forall\, i, j \in I$; either $A_i \cap A_j = \emptyset$ or $A_i = A_j$ and

iii) $\bigcup_{i \in I} A_i = X$.

It is important to know that the quotient set X/ρ forms a partition of X.

Definition 3.3.7. Let $\Phi: X \rightarrow Y$ be a map between any two non-empty sets, then the relation

$$\Phi \circ \Phi^{-1} = \{(x, y) \in X \times Y : x\Phi = y\Phi\}$$

is an equivalence relation. This relation is called the *kernel* of Φ and denoted by $\ker(\Phi)$.

If $\{\rho_i : i \in I\}$ is a collection of equivalence relations on a set X, then it is clear that $\bigcap_{i\in I} \rho_i$ is also an equivalence relation.

Definition 3.3.8. Let S be a semigroup and ρ be any relation on S. If $\forall a \in S$ and $\forall\ (s,t),(s',t') \in \rho$

i) $(as, at) \in \rho$ then ρ is called a *left compatible*,

ii) $(sa, ta) \in \rho$ then ρ is called a *right compatible* and

iii) $(ss', tt') \in \rho$ then ρ is called a *compatible*.

A left compatible equivalence relation is called a *left congruence*, a right compatible equivalence relation is called a *right congruence* and a compatible equivalence relation is called a *congruence*.

Lemma 3.3.9. A relation ρ on a semigroup S is a congruence if and only if it is both a left and a right congruence.

Proof. Howie (1995), Proposition 1.5.1. ■

Lemma 3.3.10. Let $\Phi: X \to Y$ be a semigroup homomorphism, then $\ker(\Phi) = \Phi \circ \Phi^{-1}$ is a congruence on S.

Proof. Howie (1995), Theorem 1.5.2. ■

Definition 3.3.11. For any $x \in X$ consider the equivalence class $x\rho = \{y \in X : (x,y) \in \rho\}$. If $|\rho| \leq 1$, $\forall\ x \in X$ then ρ is called a *partial map*. P_X denote the set of all partial maps. When $x\rho = \emptyset$, we say that $x\rho$ is *undefined*. When $x\rho = \{y\}$ (a singleton), we say that $x\rho$ is *defined* and we write $x\rho = y$.

Definition 3.3.12. If Φ, ψ are partial maps of X and $\Phi \subseteq \psi$ then

i) Φ is called a *restriction* of ψ and we write $\Phi = \psi|_A$ (where $A \subseteq X$),

ii) $\mathrm{dom}(\Phi) \subseteq \mathrm{dom}(\psi)$ and $\mathrm{im}(\Phi) \subseteq \mathrm{im}(\psi)$,

iii) ψ is called an *extension* of Φ.

Lemma 3.3.13. P_X is a subsemigroup of B_X under the composition operation.

Proof. Howie (1995), Proposition 1.4.2. ■

Theorem 3.3.14. If $\Phi, \psi \in P_X$ then

i) $\mathrm{dom}(\Phi \circ \psi) = [\mathrm{im}(\Phi) \cap \mathrm{dom}(\psi)]\Phi^{-1}$,

ii) $\mathrm{im}\,(\Phi \circ \psi) = [\mathrm{im}(\Phi) \cap \mathrm{dom}(\psi)]\psi$.

Proof. Howie (1995), Proposition 1.4.3. ■

Definition 3.3.15. Let X, Y be any non-empty sets. For any $\rho \in B_X$ and $x \in X$, consider the equivalence class $x\rho = \{y \in X: (x, y) \in \rho\}$, if $|x\rho| = 1$, for each $x \in X$, then ρ is called a *(full) map* (*full transformation*) or a map (*transformation*) from X to Y.

It is easy to show that the set of all maps (transformations) from X into itself (denoted by T_X) is a subsemigroup also of B_X. Moreover, the set of all bijections on X (denoted by S_X) is also a subsemigroup of B_X. One can clearly see that

$$S_X \leq T_X \leq P_X \leq B_X \quad .$$

Example 3.3.16. Let $X = \{1, 2\}$. Then

i) There are two elements in S_X, which are:

$$I_d = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

ii) There are four elements in T_X, which are I_d, σ_2 and

$$\sigma_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \sigma_4 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

iii) There are nine elements in P_X, which are I_d, σ_2, σ_3, σ_4, and

$$\sigma_5 = \begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix}, \sigma_6 = \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix}, \sigma_7 = \begin{pmatrix} 1 & 2 \\ - & 1 \end{pmatrix}, \sigma_8 = \begin{pmatrix} 1 & 2 \\ - & 2 \end{pmatrix}, \sigma_9 = \begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix}$$

iv) There are 16 possible elements of B_X, which are all the subsets of $X \times X$.

The following theorem presents the cardinality of each one of them.

Corollary 3.3.17. Let $X = \{1, 2, ..., n\}$. Then
i) $|S_X| = n!$,
ii) $|T_X| = n^n$,
iii) $|P_X| = (n+1)^n$,
iv) $|B_X| = 2^{n^2}$.
Proof. It is easy to prove because of the definitions of S_X, T_X, P_X, B_X. ■

Definition 3.3.18. Let S be a semigroup and T_X denotes the full transformations semigroup. A map $\psi: S \to T_X$ is called a *representation* of S if it is homomorphism. If ψ is an embedding, then it is said to be a *faithful representation* of S.

Next theorem shows that every semigroup S is isomorphic to a subsemigroup of a full transformation semigroup, and this theorem is very close to the well-known theorem, which is **Cayley's theorem** for groups.

Theorem 3.3.19. Every semigroup S has a faithful representation.
Proof. Take $X = S^1$ and consider the full transformation semigroup T_X. For each $a \in S$, define the map $\sigma_a: X \to X$ by $x\sigma_a = xa$, $\forall\, x \in X$. Thus $\sigma_a \in T_X$.

Now, define the map $\psi: S \rightarrow T_X$ by $a\psi = \sigma_a$ for any $a \in S$. Then, for any $a, b \in S$ and $x \in X$ we have

$$\begin{aligned} a = b &\Rightarrow xa = xb && \forall x \in S \\ &\Rightarrow x\sigma_a = x\sigma_b && \forall x \in S \\ &\Rightarrow \sigma_a = \sigma_b \\ &\Rightarrow a\psi = b\psi\,. \end{aligned}$$

Hence, ψ is a well-defined map.

Now, we show that ψ is an injective map. Indeed

$$\begin{aligned} a\psi = b\psi &\Rightarrow \sigma_a = \sigma_b \\ &\Rightarrow xa = xb, && \forall x \in S \\ &\Rightarrow a = b, && x = 1 \in X\,. \end{aligned}$$

Finally, $\forall\, a, b \in S$ and $x \in X$

$$x(\sigma_a\sigma_b) = (x\sigma_a)\sigma_b = (xa)\sigma_b = xa\mathrm{b} = x(\sigma_{ab})\,.$$

Thus, $(ab)\psi = (a)\psi\,(b\psi)$, and hence ψ is a homomorphism. Therefore, ψ is a faithful representation of S. ■

Lemma 3.3.20. Let $\alpha \in P_X$. Then α is an idempotent element if and only if $\forall x \in \mathrm{im}\alpha$, $x\alpha = x$.

Proof. ($\Longrightarrow$) Suppose that α is an idempotent and assume that there exists an element $x \in \mathrm{im}\alpha$ such that $x\alpha \neq x$. So there exists an element $y \in X$ such that $y\alpha = x$, and $x \neq y$. It follows that

$$y\alpha^2 = (y\alpha)\alpha = x\alpha \neq x = y\alpha\,,$$

and this contradicts with the fact that α is an idempotent. Thus, $\forall x \in \mathrm{im}\alpha$, $x\alpha = x$.

($\Longleftarrow$) Let $\alpha \in P_X$ and $\forall x \in \mathrm{im}\alpha$, $x\alpha = x$.

Case (1): If $x \in \mathrm{im}\alpha \subseteq \mathrm{dom}\alpha$ then

$$x\alpha^2 = (x\alpha)\alpha = x\alpha \,.$$

Case (2): If $\forall y \in \text{dom}\alpha \backslash \text{im}\alpha$, there exist $x \in \text{im}\alpha$ such that $y\alpha = x$ then

$$y\alpha^2 = (y\alpha)\alpha = x\alpha = x = y\alpha \,.$$

Thus, α is an idempotent as required. ■

3.4. Generating Sets and Ranks

Definition 3.4.1. Let S be a semigroup and let A be a nonempty subset of S. Then the collection $\mathbb{A} = \{T: A \subseteq T \text{ and } T \leq S\}$ is not empty (since at least $S \in \mathbb{A}$) and hence it has a minimum. This element is called *a generating set* A and denoted by $< A >$. That is to say, $< A >$ is the smallest subsemigroup of S that containing A.

In other words, let S be a semigroup and A be a non-empty subset of S. Then

i) $< A > = \cap \, \{T: A \subseteq T \leq S\}$,

ii) $< A >= \{a_1a_2 \cdots a_n : n \in \mathbb{Z}^+ \text{ and } a_1, a_2, \ldots, a_n \in A\}$.

Remark 3.4.2.

i) If S is a monoid or a group then $< \emptyset >= \{1\}$.

ii) If S is a semigroup and A, B are two non-empty subsets of S with $A \subseteq B$. Then $< A > \subseteq < B >$.

iii) If S is a monoid and $A \subseteq S$, then the submonoid generated by A is defined as

$$< A > = \{a_1a_2 \cdots a_n : n \in \mathbb{Z}^+ \text{ and } a_1, a_2, \ldots, a_n \in A\} \cup \{1\} \,.$$

iv) If S is a group and $A \subseteq S$, then the subgroup generated by A is defined as

$$< A >= \{a_1a_2 \cdots a_n : n \in \mathbb{Z}^+ \text{ and } a_1, a_2, \ldots, a_n \in A \cup A^{-1}\} \,.$$

Equivalently

$$< A > = \{a_1{}^{\epsilon} a_2{}^{\epsilon} \cdots\ a_n{}^{\epsilon} : n \in \mathbb{Z}^+ \text{ and } a_i \in A \text{ and } \epsilon_i = \pm 1, \qquad 1 \leq i \leq n\}.$$

Definition 3.4.3. Let S be a semigroup and A be a non-empty subset of S. If $S =< A >$ then

i) A is called *a generating set* of S,

ii) S is called the *semigroup generated* by A.

If S is a semigroup and A is a finite subset of S such that $S = < A >$. Then S is called *a finitely generated semigroup*. In particular, if S is a finite semigroup then it is a finitely generated semigroup since $S = < S >$. Moreover, if there is an element $a \in S$ such that $S =< \{a\} >$ then S is called *a monogenic semigroup* and denoted by $S = < a >$. S is called *periodic* semigroup if all its elements are of finite order. In particular, if S is a finite semigroup then it is obviously periodic.

Definition 3.4.4. let S be a finitely generated semigroup, then the number

$$\text{rank}(S) = \min\{|A| : < A > = S \text{ and } A \subseteq S\}$$

is called the *rank of* S.

If $\text{rank}(S) = |A|$ where $< A > = S$ then A is called *the minimal generating set* of S.

Definition 3.4.5. Let S be a semigroup. If $S = < A >$ where $\emptyset \neq A \subseteq E(S)$, and $E(S)$ is the set of all idempotents of S, then A is called *the idempotent generating set*, and S is called *the idempotent generated semigroup*. Moreover, if A is finite subset then S is called *a finitely idempotent generated semigroup*.

Definition 3.4.6. If S is a finitely idempotent generated semigroup then the idempotent rank, denoted by $\mathrm{idrank}(S)$, is defined as

$$\mathrm{idrank}(S) = \min\{|A|: < A > = S \text{ and } A \subseteq \mathrm{E}(S)\}.$$

Definition 3.4.7. Let S be semigroup and G be a fixed subset of S. If A is any non-empty subset of S and $< A \cup G > = S$ then A is called a *relative generating set of* S *modulo* G. However, if $\emptyset \neq A \subseteq E(S)$ and $< A \cup G > = S$ then A is called a *relative idempotent generating set of* S *modulo* G.

Definition 3.4.8. Let S be a finitely generated semigroup and G is a specific subset of S. Then the *relative rank of* S *modulo* G, denoted by $\mathrm{rank}(S{:}\,G)$, is defined as:

$$\mathrm{rank}(S{:}\,\mathrm{G}) = \min\{|A|: < A \cup \mathrm{G} > = S \text{ and } A \subseteq S\}.$$

Whereas, *the idempotent relative rank of* S *modulo* G, denoted by $\mathrm{idrank}(S{:}\,G)$, is defined as

$$\mathrm{idrank}(S{:}\,G) = \min\{|A|: < A \cup G > = S \text{ and } A \subseteq \mathrm{E}(S)\}.$$

3.5. Green's Relations

In order to understand the structure of any semigroup and how its elements are related we need to study the well-known Green's relations on that semigroup. The definition of these relations depends on the principal ideals that generate. Let S be a semigroup and $a \in S$, then the *principal left ideal*, the *principal right ideal* and *the principal ideal* generated by a are defined, respectively, as

i) $S^1a = Sa \cup \{a\} = \{xa: x \in S^1\}$,

ii) $aS^1 = aS \cup \{a\} = \{ay: y \in S^1\}$ and

iii) $S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\} = \{xay: x, y \in S^1\}$.

Actually, there are five Green's relations on a semigroup: $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$. We begin by defining the first three relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$, and then consequently we define $\mathcal{H}$ and $\mathcal{D}$.

Definition 3.5.1. Let S be a semigroup. We define the following relations

i) $\mathcal{L} = \{(a,b) \in S \times S: S^1a = S^1b\}$,

ii) $\mathcal{R} = \{(a,b) \in S \times S: aS^1 = bS^1\}$ and

iii) $\mathcal{J} = \{(a,b) \in S \times S: S^1aS^1 = S^1bS^1\}$

which are called a *left Green's relation* ($\mathcal{L}$ −Green), a *right Green's relation* ($\mathcal{R}$ −Green) and *Green's relation* ($\mathcal{J}$ −Green), respectively.

Lemma 3.5.2. The Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ are equivalence relations on S. Moreover, $\mathcal{L}$ is a right congruence whereas $\mathcal{R}$ is a left congruence on S.

Proof. Howie (1995), Proposition 2.1.2. ■

There is an alternative description of Green's relation given by the following lemma.

Lemma 3.5.3. Let S be a semigroup and $a, b \in S$. Then

i) $a\mathcal{L}b \Leftrightarrow a = sb$ and $b = ta$ where $s, t \in S^1$,

ii) $a\mathcal{R}b \Leftrightarrow a = bs$ and $b = at$ where $s, t \in S^1$ and

iii) $a\mathcal{J}b \Leftrightarrow a = sbt$ and $b = uav$ where $s, t, u, v \in S^1$.

Proof. Howie (1995), Proposition 2.1.1. ■

Lemma 3.5.4. The relations $\mathcal{L}$ and $\mathcal{R}$ commute, that is $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

Proof. Howie (1995), Proposition 2.1.3. ■

Definition 3.5.5. Let S be a semigroup then $\mathcal{D}$ −Green and $\mathcal{H}$ −Green relations on S are defined as

i) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, the largest equivalence relation of S contained in both $\mathcal{L}$ and $\mathcal{R}$.

ii) $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$, the smallest equivalence relation of S containing both $\mathcal{L}$ and $\mathcal{R}$.

Another characterization for the Green's relation $\mathcal{D}$ −Green, given by the following lemma.

Lemma 3.5.6. $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$.

Proof. Since $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations then $(\mathcal{L} \cap \mathcal{R})^{-1} = \mathcal{L} \cap \mathcal{R}$, $1_S \subseteq \mathcal{L} \cap \mathcal{R}$ and so we have

$$(\mathcal{L} \cap \mathcal{R}) \cup (\mathcal{L} \cap \mathcal{R})^{-1} \cup 1_S \subseteq (\mathcal{L} \cap \mathcal{R}) .$$

Therefore, $\mathcal{L} \cap \mathcal{R}$ also an equivalence relation. Moreover, since $\mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$, $\mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R}$ and $\mathcal{L}, \mathcal{R}$ are commute, then

$$\begin{aligned}(\mathcal{L} \cup \mathcal{R}) \circ (\mathcal{L} \cup \mathcal{R}) &= (\mathcal{L} \circ \mathcal{L}) \cup (\mathcal{L} \circ \mathcal{R}) \cup (\mathcal{R} \circ \mathcal{L}) \cup (\mathcal{R} \circ \mathcal{R}) \\ &= \mathcal{L} \cup (\mathcal{L} \circ \mathcal{R}) \cup (\mathcal{L} \circ \mathcal{R}) \cup \mathcal{R} \\ &= (\mathcal{L} \circ \mathcal{R}) \cup (\mathcal{L} \circ \mathcal{R}) \cup (\mathcal{L} \circ \mathcal{R}) \\ &= (\mathcal{L} \circ \mathcal{R}) .\end{aligned}$$

Thus,

$$(\mathcal{L} \cup \mathcal{R}) \circ (\mathcal{L} \circ \mathcal{R}) = [\mathcal{L} \circ (\mathcal{L} \circ \mathcal{R})] \cup [\mathcal{R} \circ (\mathcal{L} \circ \mathcal{R})] = (\mathcal{L} \circ \mathcal{R}).$$

Therefore,

$$\begin{aligned}(\mathcal{L} \circ \mathcal{R}) \circ (\mathcal{L} \cup \mathcal{R}) &= \mathcal{L} \cup [\mathcal{R} \cup (\mathcal{L} \circ \mathcal{R})] \\ &= (\mathcal{L} \circ \mathcal{R}) .\end{aligned}$$

Inductively, we can show that for $n \geq 2$, $(\mathcal{L} \cup \mathcal{R})^n = \mathcal{L} \circ \mathcal{R}$ and therefore $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. ■

For any semigroup S, the Green's relation $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ −Green on S are related as

$$1_S \subseteq \mathcal{H} \subseteq \mathcal{L}\,, \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J} \subseteq S \times S\,.$$

The following diagram summarizing this relationship, which is called Hasse's diagram.

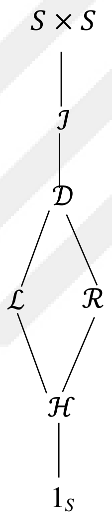


Figure 3.1: Hasse's diagram

Lemma 3.5.7. If S is a group then $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = S \times S$.

Proof. For $a, b \in S$, $a = a\,(b^{-1}b) = (ab^{-1})\,b$ and $b = b(a^{-1}a) = (ba^{-1})a$. If we take $x = ab^{-1}$ and $y = ba^{-1}$ then $a = xb$ and $b = yb$ where $x, y \in S$ .Thus, $\forall\, a, b \in S$, $a\,\mathcal{L}\,b$ and so $\mathcal{L} = S \times S$. Dually, one can show that $\mathcal{R} = S \times S$. Now, by knowing that $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ then obviously $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = S \times S$. Moreover, if $(a, b) \in S \times S$ then $SaS = S$ and $SbS = S$. Hence $(a, b) \in \mathcal{J}$ and so $\mathcal{J} = S \times S$. ■

Lemma 3.5.8. If S is a commutative semigroup then $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$.

Proof. For $a, b \in S$ we have

$$\begin{aligned}(a,b) \in \mathcal{L} &\Leftrightarrow S^1 a = S^1 b \\ &\Leftrightarrow aS^1 = bS^1 \\ &\Leftrightarrow (a,b) \in \mathcal{R} \\ &\Leftrightarrow \mathcal{L} = \mathcal{R}.\end{aligned}$$

Now,

$$\begin{aligned}(a,b) \in \mathcal{J} &\Leftrightarrow S^1 a\, S^1 = S^1 b\, S^1 \\ &\Leftrightarrow S^1 S^1 a = S^1 S^1 b \\ &\Leftrightarrow S^1 a = S^1 b \\ &\Leftrightarrow (a,b) \in \mathcal{L} = \mathcal{R} \quad .\end{aligned}$$

Thus, $\mathcal{L} = \mathcal{R} = \mathcal{J}$. Now, since $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{L} \circ \mathcal{L} = \mathcal{L} = \mathcal{R}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \mathcal{L} = \mathcal{L} = \mathcal{R}$. Therefore, $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$. ■

Lemma 3.5.9. If S is a periodic semigroup then $\mathcal{D} = \mathcal{L}$.

Proof. Howie (1995), Proposition 2.1.4. ■

Let S be a semigroup and $a \in S$. Then the $\mathcal{H}-$, $\mathcal{L}-$, $\mathcal{R}-$, $\mathcal{D}-$ and $\mathcal{J}-$Green's classes of a are denoted, respectively, by $\mathcal{H}_a$, $\mathcal{L}_a$, $\mathcal{R}_a$, $\mathcal{D}_a$ and $\mathcal{J}_a$. Depending on the containment between Green's relations described in Hasse's diagram, one can see that

$$\begin{aligned}&\mathcal{H}_a \subseteq \mathcal{L}_a \subseteq \mathcal{D}_a \\ &\mathcal{H}_a \subseteq \mathcal{R}_a \subseteq \mathcal{D}_a \\ &\mathcal{H}_a = \mathcal{L}_a \cap \mathcal{R}_a \quad .\end{aligned}$$

There are natural partial orders on the collection of $\mathcal{L}$ −classes $S/\mathcal{L}$, the collection of $\mathcal{R}$ −classes $S/\mathcal{R}$, and the collection of $\mathcal{J}$ −classes $S/\mathcal{J}$ given by: For $a, b \in S$

$$S^1a \subseteq S^1b \implies \mathcal{L}_a \leq \mathcal{L}_b$$
$$aS^1 \subseteq bS^1 \implies \mathcal{R}_a \leq \mathcal{R}_b$$
$$S^1aS^1 \subseteq S^1bS^1 \implies \mathcal{J}_a \leq \mathcal{J}_b \quad .$$

Lemma 3.5.10. For any semigroup S and $a, b \in S$

i) $a\mathcal{D}b \iff \mathcal{R}_a \cap \mathcal{L}_b \neq \emptyset \iff \mathcal{L}_a \cap \mathcal{R}_b \neq \emptyset$,

ii) $\mathcal{D}_a = \bigcup_{b\in \mathcal{D}_a} \mathcal{L}_b = \bigcup_{b\in \mathcal{D}_a} \mathcal{R}_b$.

Proof. i) $a\mathcal{D}b \iff \exists c \in S$: $a\mathcal{L}c$ and $c\mathcal{R}b \iff \exists e \in S$: $a\mathcal{R}e$ and $e\mathcal{L}b$,

ii) This claim is obvious, since $\mathcal{L} \subseteq \mathcal{D}$ and $\mathcal{R} \subseteq \mathcal{D}$. ∎

Thus, the structure of $\mathcal{D}$ −classes can visualize as an egg-box diagram. Which is described as

		$\mathcal{L}_y$			
$\mathcal{R}_z$		$\mathcal{L}_y \cap \mathcal{R}_z$			

Figure 3.2. The egg-box visualization of a $\mathcal{D}$ −class

In this diagram, the rows represent $\mathcal{R}$ −classes, the columns represent $\mathcal{L}$ −classes and the cells represent $\mathcal{H}$ −classes.

Theorem 3.5.11. (*Green's theorem*) Let S be a semigroup. For any $s \in S$ define the map

$$\rho_s : S^1 \to S^1 \text{ by: } x\rho_s = xs\,, \qquad \forall x \in S^1\,.$$

If $a\mathcal{R}b$ then there exist $s, t \in S$ such that $as = b$ and $bt = a$, and thus

i) $\rho_s : \mathcal{L}_a \longrightarrow \mathcal{L}_b$ is a bijection, and $\rho_t : \mathcal{L}_b \longrightarrow \mathcal{L}_a$ is a bijection also,

ii) $\rho_t = {\rho_s}^{-1}$ and $\rho_s = {\rho_t}^{-1}$ (each one is the inverse of the other) and

iii) ρ_s and ρ_t are 1-1 correspondence from an $\mathcal{H}$ −class onto another $\mathcal{H}$ −class.

Proof. Howie (1995), Lemma 2.2.1. ■

The dual version of the pervious theorem is given by the next lemma.

Lemma 3.5.12. Let S be a semigroup. For any $s \in S$ define the map

$$\lambda_u : S^1 \to S^1 \text{ by: } x\lambda_u = us\,, \qquad \forall x \in S^1\,.$$

If $a\mathcal{L}b$ then there exist $u, v \in S$ such that $ua = b$ and $vb = a$, and thus

i) $\lambda_u : \mathcal{R}_a \longrightarrow \mathcal{R}_b$ is a bijection, and $\lambda_v : \mathcal{R}_b \longrightarrow \mathcal{R}_a$ is a bijection also,

ii) $\lambda_v = {\lambda_u}^{-1}$ and $\lambda_u = {\lambda_v}^{-1}$(each one is the inverse of the other) and

iii) λ_u and λ_v are 1-1 correspondence from an $\mathcal{H}$ −class onto another $\mathcal{H}$ −class.

Proof. Howie (1995), Lemma 2.2.2. ■

Lemma 3.5.13. the $\mathcal{H}$ −classes ($\mathcal{L}$ −classes and $\mathcal{R}$ −classes) inside a $\mathcal{D}$ −class contain the same number of elements. That is; if $a\mathcal{D}b$ then $|\mathcal{H}_a| = |\mathcal{H}_b|$ ($|\mathcal{L}_a| = |\mathcal{L}_b|$ and $|\mathcal{R}_a| = |\mathcal{R}_b|$)

Proof. Howie (1995), Lemma 2.2.3. ■

Theorem 3.5.14. let S be a semigroup. If $\mathcal{H}$ is an $\mathcal{H}$ −class of S, then either $\mathcal{H}$ is a group or $\mathcal{H}^2 \cap \mathcal{H} = \emptyset$.

Proof. Howie (1995), Lemma 2.2.5. ■

Corollary 3.5.15. Let S be a semigroup and $\mathcal{H}$ is an $\mathcal{H}$ −class of S. Then

1) If $\mathcal{H}$ contains an idempotent then $\mathcal{H}$ is a subgroup of S .
2) $\mathcal{H}$ −class has at most one idempotent element.

3.6. Regular Semigroups

Definition 3.6.1. An element a of a semigroup S is called a *regular element* if $a = axa$ for some $x \in S$. A semigroup S is called a *regular semigroup* if every element of S is regular.

Idempotent elements are regular, since $e = eee$ and so all idempotents of a semigroup are regular elements. Therefore, $E(S)$, the set of all idempotents of S, is a regular subsemigroup of S. Moreover, if S is a group then it is a regular semigroup, for if $a \in S$ is any element then $a = aa^{-1}a$, where a^{-1} is the group inverse of a. The converse is not true in general. However, we have the following characterization.

Lemma 3.6.2. A regular semigroup S is a group if and only if it has a unique idempotent element.

Proof. (⇐) Assume that e is the unique idempotent element of S. Since S is a regular semigroup, for any $a \in S$ there exists $x \in S$ such that $axa = a$, this means $(axa)\, x = ax$ and this implies that $(ax)^2 = ax$. Thus, ax is an idempotent of S, by the uniqueness property of e. Then we conclude that $ax = e$, and so $a = axa = ea$. Hence, e is a left identity. Analogously, one can show that e is a right identity. So e is the identity of S. Since $ax = e$ and $xa = e$, then S is a group by the definition of groups.

(⇒) Suppose that S be a group and that $e, f \in S$ are both idempotents in S. Then $ef = eeff$. If we cancel f from the right we have $e = eef = ef$ and cancel e from the left we have $f = eff = ef$. Hence, $e = f$ and the uniqueness holds. ■

Theorem 3.6.3. The semigroups T_n and P_n are regular semigroups.

Proof. Ganyushkin and Mazorchuk (2009), Theorem 2.6.3. ■

The next lemma provides a relation between the regular elements and the idempotent elements.

Lemma 3.6.4. Let $a \in S$ be a regular element, that is $a = axa$ for some $x \in S$. Then $ax \in E(S)$.

Proof. It is easy to see that $(ax)^2 = axax = ax$. Thus ax is an idempotent element of S. ■

Definition 3.6.5. An element x of a semigroup S is called an *inverse element* of $a \in S$ provided that $a = axa$ and $x = xax$.

An inverse element of a, if exists, need not to be unique. However, if the uniqueness property holds, then we encounter a new semigroup called the *inverse semigroup*. As an example, any group S is an inverse semigroup.

Lemma 3.6.6. Every regular element of S has an inverse element.

Proof. Suppose that $a \in S$ is a regular element, so there is an element $x \in S$ such that $a = axa$.

Now, consider $b = xax$, then

$$aba = a(xax)a = (axa)xa = axa = a$$

and

$$bab = (xax)a(xax) = x(xax)xax = x(axa)x = xax = b\,.$$

Thus, b is an inverse of a. ■

Theorem 3.6.7. A semigroup S is an inverse semigroup if and only if it is regular and idempotents are commute.

Proof. Suppose that S be an inverse semigroup, thus the inverses are unique. Let e, f are idempotents and take x to be the inverse of ef. That is

$$efxef = ef \text{ and } xefx = x$$

Claim: fxe is also an inverse of ef. Indeed

$$ef\,(fxe)\,ef = e(ff)\,x\,(ee)f = (ef)\,x\,(ef) = ef$$

and

$$fxe\,(ef)fxe = fx\,(ee)(ff)xe = f\,(xefx)e = fxe\,.$$

Moreover, fxe is an idempotent since $(fxe)^2 = f(xefx)e = fxe$. Thus, fxe is its own unique inverse and so $fxe = ef$. Similarly, one can show that $fxe = fe$. Hence $fe = ef$.

Conversely, let x, y be inverses of a. Then we have

$$x = xax = x(aya)x = (xa)(ya)x = y(axa)x = yax = y(aya)x$$
$$= y(ay)(ax) = yaxay = y(axa)y = yay = y\,.$$

Thus, the uniqueness property holds. ■

Now, we study one of the important semigroups which is the symmetric inverse semigroup. One of the main types of evidence for the importance of this semigroup is their universality property. Which says that “Every inverse semigroup S can be embedded in a symmetric inverse semigroup”.

Definition 3.6.8. A partial map $\alpha: X \longrightarrow X$ is called *injective*, if

$$x\alpha \neq y\alpha \; for\; all\; x \neq y \;\; with\; x, y \in dom\alpha\, .$$

The set of all injective partial map under the composition operation forms a semigroup. This semigroup is called the *symmetric inverse semigroup*, and denoted by I_X.

Next theorem is analogous to Theorem 3.3.19. which is called the *Wagner-Preston representation*, characterizes the universality property of I_X.

Theorem 3.6.9. Every inverse semigroup S can be embedded in a symmetric inverse semigroup.

Proof. Howie (1995), Theorem 5.1.7. ■

Theorem 3.6.10. The symmetric inverse semigroup I_X is an inverse semigroup, while P_X and T_X are not.

Proof. Ganyushkin and Mazorchuk (2009), Theorems 2.6.5 and 2.6.7. ■

Theorem 3.6.11. Let α, β ∈ I_X Then

i) $\alpha\mathcal{L}\beta \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)$,

ii) $\alpha\mathcal{R}\beta \Leftrightarrow \mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$,

iii) $\alpha\mathcal{D}\beta \Leftrightarrow |\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$,

iv) $\alpha\mathcal{H}\beta \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)$ and $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$.

Proof. Ganyushkin and Mazorchuk (2009), Theorem 4.5.1. ■

Theorem 3.6.12. If $X = \{1,2,\ldots,n\}$, then the symmetric inverse semigroup I_X is written as I_n and its cardinality is given by

$$|I_n| = \sum_{k=0}^{n} \mathrm{k}! \binom{n}{k}$$

where $k = \mathrm{im}(\alpha)$ for any $\alpha \in I_n$.

Proof. Ganyushkin and Mazorchuk (2009), Theorem 2.5.1. ■

3.7. Some Finite Transformations Semigroups

In section 3.3. we defined the semigroups B_X, P_X, T_X and S_X, where X is a non-empty set. Here, in the current section, it is the time to shed some light on the case when X is a finite set. That is; $X = X_n = \{1,2,\ldots,n\}$ and we write P_n, T_n, I_n and S_n instead of writing P_X, T_X, I_X and S_X. We will do more investigations on their algebraic properties first, and then we study some finite subsemigroups of the main semigroups as well.

Lemma 3.7.1. Let $\alpha, \beta \in P_n$, then

i) $\mathrm{dom}(\alpha\beta) \subseteq \mathrm{dom}(\alpha)$,

ii) $\mathrm{im}(\alpha\beta) \subseteq \mathrm{im}(\beta)$.

Proof. i) Let $x \in \mathrm{dom}(\alpha\beta)$. Then there exists $y \in X_n$ such that $x\alpha\beta = y = (x\alpha)\beta$ and hence $x\alpha \in \mathrm{dom}(\beta) \subseteq X_n$. Thus $x \in \mathrm{dom}(\alpha)$. Therefore, $\mathrm{dom}(\alpha\beta) \subseteq \mathrm{dom}(\alpha)$.

ii) Let $y \in \mathrm{im}(\alpha\beta)$. Then there exists $x \in X_n$ such that $x\alpha\beta = y = (x\alpha)\beta$ and so $y \in \mathrm{im}(\beta)$. Thus, $\mathrm{im}(\alpha\beta) \subseteq \mathrm{im}(\beta)$. ■

Definition 3.7.2. For any $\alpha \in P_n$, the kernel of α is defined by

$$\ker(\alpha) = \{(x,y) \in X_n \times X_n : (x,y \in \mathrm{dom}(\alpha) \text{ and } x\alpha = y\alpha\ ) \\ \text{or } (x,y \in X_n \backslash \mathrm{dom}(\alpha))\}\,.$$

In particular, if $\alpha \in T_n$. Then

$$\ker(\alpha) = \{(x, y) \in X_n \times X_n : x\alpha = y\alpha\}.$$

For all $\alpha \in P_n$, $\ker(\alpha)$ is an equivalence relation on $\mathrm{dom}(\alpha)$ and the set of all equivalence classes of this equivalence relation is $\{x\alpha^{-1} : x \in \mathrm{im}(\alpha)\}$. Moreover, this set is a partition of $\mathrm{dom}(\alpha)$. Therefore, the collection the $\{x\alpha^{-1} : x \in \mathrm{im}(\alpha)\} \cup \{X_n \backslash \mathrm{dom}(\alpha)\}$ is a partition of X_n.

For any $\alpha \in P_n$, let $\mathrm{im}(\alpha) = \{a_1, a_2, ..., a_r\}$ where $1 \leq r \leq n$, and $A_1 = a_1\alpha^{-1}$, ... , $A_r = a_r\alpha^{-1}$ and $A_{r+1} = X_n \backslash \bigcup_{i=1}^{r} A_i$, then we write

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r & A_{r+1} \\ a_1 & a_2 & \cdots & a_r & - \end{pmatrix}.$$

In this case $\ker(\alpha) = \bigcup_{i=1}^{r+1} A_i \times A_i$.

If $\alpha \in T_n$ then $\mathrm{dom}(\alpha) = X_n$ and we write

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$$

and in this case $\ker(\alpha) = \bigcup_{i=1}^{r} A_i \times A_i$.

It is clear that for any $\alpha \in P_n$, α is an idempotent element if and only if $A_\mathrm{i}\, \alpha \in A_\mathrm{i}$, $\forall\, 1 \leq i \leq r$.

Example 3.7.3. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & - & 2 & - \end{pmatrix} \in P_5$. Then $A_1 = 2\alpha^{-1} = \{1, 4\}$, $A_2 = 3\alpha^{-1} = 2$ and $\mathrm{im}(\alpha) = \{2, 3\}$. Thus, we write

$$\alpha = \begin{pmatrix} \{1,4\} & \{2\} & \{3,5\} \\ 2 & 3 & - \end{pmatrix}$$

and here we notice that α is not an idempotent element since, for example

$$3 = \{2\}\alpha \notin \{2\} .$$

Lemma 3.7.4. For any $\alpha, \beta \in P_n$, $\ker(\alpha) \subseteq \ker(\alpha\beta)$.

Proof. Let $(x, y) \in \ker(\alpha)$, then $(x,\ y \in \mathrm{dom}(\alpha)$ and $x\alpha = y\alpha)$ or $(x,y \notin \mathrm{dom}(\alpha)$.

Case (1): If $x, y \in \mathrm{dom}(\alpha)$ and $x\alpha = y\alpha$ then

$$(x\alpha)\beta = (y\alpha)\beta \implies x(\alpha\beta) = y(\alpha\beta)$$

If $x\alpha, y\alpha \in \mathrm{dom}(\beta)$ then $x, y \in \mathrm{dom}(\alpha\beta)$ and so $(x, y) \in \ker(\alpha\beta)$. On the other hand, if $x\alpha, y\alpha \notin \mathrm{dom}(\beta)$ then $x, y \in X_n\backslash\mathrm{dom}(\alpha\beta)$ and hence $(x, y) \in \ker(\alpha\beta)$.

Case (2): If $x, y \in X_n\backslash\mathrm{dom}(\alpha)$, then $x, y \in X_n\backslash\mathrm{dom}(\alpha\beta)$ and so $(x, y) \in \ker(\alpha\beta)$. In both cases $\ker(\alpha) \subseteq \ker(\alpha\beta)$. ■

Definition 3.7.3. A partial transformation $\alpha \in P_n$ is called an *order-preserving map* if it satisfies the following property;

$$\forall\, x,y \in \mathrm{dom}(\alpha),\ \ x \le y \implies x\alpha \le y\alpha .$$

An order-preserving map is called an *order-isomorphism* if it is bijection (injective and surjective). The set of all order-preserving partial transformations is a subsemigroup of P_n and denoted by OP_n. In particular, the set of all order-preserving (full) transformations is a subsemigroup of T_n and denoted by

$$O_n = PO_n \cap T_n .$$

Moreover, the set of all injective order-preserving partial transformation is a subsemigroup of I_n and denoted by

$$IO_n = PO_n \cap I_n \,.$$

For example, if $X = \{1, 2\}$,

i) $PO_2 = \left\{ \begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix} \begin{pmatrix} 1 & 2 \\ - & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix} \begin{pmatrix} 1 & 2 \\ - & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\}$,

ii) $O_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\}$,

iii) $IO_2 = \left\{ \begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix} \begin{pmatrix} 1 & 2 \\ - & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix} \begin{pmatrix} 1 & 2 \\ - & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \right\}$.

Note that

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \in P_2 \text{ but } \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \notin PO_2 \,.$$

In order to find the cardinalities, ranks and count the number of idempotent elements of some finite semigroups, first we need to recall some important expressions in combinatorics theory, namely the *Stirling number of the second kind* and *Fibonacci numbers* F_n .

Definition 3.7.4. The *Stirling number of the second kind,* which is denoted by $S(n, k)$, defined as the number of ways of partitioning a set of n elements into k non-empty subsets.

For example, the set $\{1, 2, 3\}$ can be partition into

i) 3 subsets in 1 way: $\{\{1\}, \{2\}, \{3\}\}$,

ii) 2 subsets in 3 ways: $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$ and

iii) 1 subset in 1 way: {{1, 2, 3}}.

Thus, we write $S(3, 1) = 1$, $S(3, 2) = 3$ and $S(3,3) = 1$.

Lemma 3.7.5. For any set of n elements and $1 \leq k \leq n$, then

i) $S(n, 1) = S(n, n) = 1$

ii) $S(n, 2) = 2^{n-1} - 1$

iii) $S(n, 3) = \frac{1}{6}(3^n - 3.2^n + 3)$

iv) $S(n, k) = S(n-1, k-1) + k.S(n-1, k)$.

The next table views the triangle of Stirling numbers of the second kind:

Table 3.3. The triangle of Stirling numbers of the second kind

	k =1	k = 2	k = 3	k = 4	k = 5	k = 6
S(1, k)	1					
S(2, k)	1	1				
S(3, k)	1	3	1			
S(4, k)	1	7	6	1		
S(5, k)	1	15	25	10	1	
S(6, k)	1	31	90	65	15	1

Definition 3.7.6. The *Fibonacci numbers* F_n is defined as follows

i) $F_1 = F_2 = 1$,

ii) $\forall\, n \geq 3,\ F_n = F_{n-1} + F_{n-2}$.

In general, the cardinalities of the last three defined semigroups are given by the following lemma.

Lemma 3.7.7. For a finite set $X = \{1, 2, \dots, n\}$, we have

1) $|PO_n| = \sum_{m=0}^{n} \binom{n}{m}\binom{n+m-1}{m}$, where $m = |\mathrm{dom}(\alpha)|$,

2) $|O_n| = \binom{2n-1}{n}$,

3) $|IO_n| = 2|O_n| = \binom{2n}{n}$.

Proof. Ganyushkin and Mazorchuk (2009), Propositions 14.2.3, 14.2.4 and 14.2.9. ■

Lemma 3.7.8. The number of idempotents of PO_n, O_n and IO_n is given by

1) $|E(PO_n)| = 1 + \sum_{m=1}^{n} \binom{n}{m} F_{2m}$, where $m = |\mathrm{dom}(\alpha)|$,
2) $|E(O_n)| = F_{2n}$,
3) $|E(IO_n)| = 2^n$.

Proof. Ganyushkin and Mazorchuk (2009), Propositions 14.3.1, 14.3.6 and 14.3.7. ■

Next lemma presents the ranks and idempotent ranks of each semigroup of PO_n, O_n and IO_n in terms of $|X|$.

Lemma 3.7.9. For $n \geq 3$, we have

1) $\mathrm{rank}(O_n) = n$ and $\mathrm{idrank}(O_n) = 2n - 2$,
2) $\mathrm{rank}(PO_n) = 2n - 1$ and $\mathrm{idrank}(PO_n) = 3n - 2$.

Proof. Gomes and Howie (1992), Theorems 2.7., 2.8., 3.2. and 3.13. ■

Now, we look at Green's description for some finite transformation semigroups.

Theorem 3.7.10. Let $\alpha, \beta \in P_n$, then

i) $\alpha\mathcal{L}\beta \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)$,
ii) $\alpha\mathcal{R}\beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$ and $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$,
iii) $\alpha\mathcal{D}\beta \Leftrightarrow |\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$,
iv) $\alpha\mathcal{H}\beta \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)$, $\ker(\alpha) = \ker(\beta)$ and $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$.

Proof. Ganyushkin and Mazorchuk (2009), Theorem 4.5.1. ■

Theorem 3.7.11. Let $\alpha, \beta \in T_n$, then

i) $\alpha\mathcal{L}\beta \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)$,
ii) $\alpha\mathcal{R}\beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$,

iii) $\alpha\mathcal{D}\beta \Leftrightarrow |\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$,

iv) $\alpha\mathcal{H}\beta \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)$ and $\ker(\alpha) = \ker(\beta)$.

Proof. Ganyushkin and Mazorchuk (2009), Theorem 4.5.1. ■

Indeed, Green's relations on the remaining semigroups PO_n and IO_n are just restrictions of the Green's relations on P_n and I_n, respectively.

Corollary 3.7.12. The semigroup P_n has $n+1$ $\mathcal{D}$ −Green equivalence classes, namely $\mathcal{D}_r = \{\alpha \in P_n: |\mathrm{im}(\alpha)| = r\}$, where $0 \leq r \leq n$. ■

Corollary 3.7.13. The semigroup T_n has n $\mathcal{D}$ −Green equivalence classes, namely $\mathcal{D}_r = \{\alpha \in T_n: |\mathrm{im}(\alpha)| = r\}$, where $1 \leq r \leq n$. ■

In the previous section, we mentioned that the semigroups P_n, T_n and I_n are regular semigroups. Now, with regard to the semigroup O_n, in 1992, Higgins also proved the regularity in his book (Techniques of semigroup theory, page 203, see Higgins (1992)). Furthermore, recently, in Kemprasit and Changphas (2000), the authors there showed that PO_n and IO_n are regular.

Now, we talk about some important subsemigroups of the main semigroups. These subsemigroups have been studied very much in the last decades.

Definition 3.7.14. Let $X = \{1, 2, \ldots, n\}$ and, for $1 \leq r \leq n$, define some subsemigroups of T_n, P_n and I_n, respectively, as follows

i) $K(n,r) = \{\alpha \in T_n: |\mathrm{im}(\alpha)| \leq r\} \leq T_n$,

ii) $P(n,r) = \{\alpha \in P_n : |\mathrm{im}(\alpha)| \leq r\} \leq P_n$,

iii) $I(n,r) = \{\alpha \in I_n : |\mathrm{im}(\alpha)| \leq r\} \leq I_n$.

In particular, $K(n,n) = T_n$, $K(n,n-1) = ST_n$, $P(n,n) = P_n$ and $I(n,n) = I_n$. Thus, one can consider the last semigroups $K(n,r)$, $P(n,r)$ and $I(n,r)$ as a generalization of T_n , P_n and I_n, respectively.

Lemma 3.7.15. For $2 \leq r \leq n-1$, rank$(K(n,r))$ = idrank$(K(n,r)) = S(n,r)$, the Stirling number of the second kind.

Proof. Howie and Mcfadden (1990), Theorem 5. ■

Lemma 3.7.16. For $2 \leq r \leq n-1$, rank$(P(n,r))$ = idrank$(P(n,r)) = S(n+1, r+1)$, the Stirling number of the second kind.

Proof. Garba (1990), Theorem 4.2. ■

The description of Green's relations for $K(n,r), P(n,r)$ and $I(n,r)$ has also determined lately.

There are some other transformations subsemigroups of the main semigroups T_n, P_n and I_n. Here, we just mention some of them and give the definitions, then we suggest some references to study their algebraic properties:

1. $ST_n = Sing_n$: the singular transformation semigroup $T_n \backslash S_n$.
2. SPO_n: the order-preserving partial transformation semigroup $PO_n \backslash O_n$.
3. OP_n: the orientation-preserving transformation semigroup.
4. OR_n: the orientation-preserving or orientation-reversing transformation semigroup.
5. $L(n,r) = \{ \alpha \in O_n : |\mathrm{im}(\alpha)| \leq r\}$.
6. $M(n,r) = \{ \alpha \in PO_n : |\mathrm{im}(\alpha)| \leq r\}$.
7. $N(n,r) = \{ \alpha \in SPO_n : |\mathrm{im}(\alpha)| \leq r\}$.
8. DP_n: the semigroup of all partial isometries of X_n.
9. ODP_n: the semigroup of order-preserving partial isometries of X_n.
10. $DP_{n,r} = \{\alpha \in DP_n : |\mathrm{im}(\alpha)| \leq r\}$.
11. $ODP_{n,r} = \{ \alpha \in ODP_n : |\mathrm{im}(\alpha)| \leq r\}$.

In 1966, Howie proved that the semigroup ST_n is generated by its idempotents, see Howie (1966). Moreover, Howie with other author calculated its rank and idempotent rank, see Howie and McFadden (1990). One of the recent works, related to this semigroup, is Ayık, Ayık, Ünlü and Howie (2008), where the authors reprove some known results and give several new examples of generating systems for ST_n. In the paper Gomes and Howie (1992), showed that the semigroup SPO_n has no idempotent rank (it is not idempotent generated semigroup), but they calculated the rank to be $2n-2$. After two years, in Garba (1994), he considered the semigroups $L(n,r)$, $M(n,r)$ and $N(n,r)$ and showed that both the rank and idrank of each one of them are equal and he calculated them.

The notation of an orientation-preserving transformation was introduced in McAlister (1998) and Catarino and Higgins (1999). Several properties of OP_n and OR_n have been studied in their articles. Recently, Al-Karousi, Kehinde and Umar (2014) proved that the semigroups DP_n and ODP_n are inverse subsemigroups of I_n, they also calculated their cardinalities over there. Then, the same authors, described the Green's relations on DP_n and ODP_n. Furthermore, they calculated their ranks with respect to the floor function $\lfloor x \rfloor$. Lately, in Bugay, Yağcı and Ayık (2018), they introduced the subsemigroups $DP_{n,\,r}$ and $ODP_{n,\,r}$ of DP_n, and they determined the cardinalities and the ranks for each one of them.

In the following chapters, we will concentrate our study at two special kinds of transformation semigroups, namely *the semigroups of transformations with restricted range* and *generalized transformation semigroup.*

4. TRANSFORMATION SEMIGROUPS WITH RESTRICTED RANGE

In 1975, Symons was the first author who considered the notion of transformation semigroups with restricted range, denoted by $T(\mathrm{X}, \mathrm{Y})$. He described all the automorphisms of $T(\mathrm{X}, \mathrm{Y})$ and he determined when $T(\mathrm{X}_1, \mathrm{Y}_1) \cong T(\mathrm{X}_2, \mathrm{Y}_2)$, see Symons (1975b). After that, series of articles about this type of semigroups started to appear. Some of those articles dealt with the normal setting and some other with the linear counter part of those semigroups.

The full transformation semigroup T_{X}, the partial transformation semigroup P_{X}, the inverse semigroup I_{X}, the order-related semigroups and others have been studied in a new way under the light of "restricted range". Many algebraic properties, which we had studied in the previous chapter, were investigated on those semigroups. For example, regularity, a description of Green's relations, ideals, idempotent elements, isomorphic theorems, generating sets, ranks, idempotent ranks, relative ranks and some others.

For a non-empty set X and a fixed nonempty subset Y of X, let $PT(X)$, $T(X)$ and $I(X)$ denote the semigroup of all partial transformations, the semigroup of all (full) transformations and the semigroup of all injective partial transformations on X, respectively. Then, define the following subsemigroups of transformations with restricted range:

i) $PT(X,Y) = \{\alpha \in PT(X): X\alpha \subseteq Y\} \leq PT(X)$,
ii) $T(X,Y) = \{\alpha \in T(X): X\alpha \subseteq Y\} \leq T(X)$ and
iii) $I(X,Y) = \{\alpha \in I(X): X\alpha \subseteq Y\} \leq I(X)$.

It is clear from the definition that $PT(X) = PT(X,X)$, $T(X) = T(X,X)$ and $I(X) = I(X,X)$. Therefore, in some sense, we may also regard these subsemigroups as a generalization of the main semigroups $PT(X)$, $T(X)$ and $I(X)$, respectively.

In this chapter, we look at the headlines of each one of the main transformation semigroups with restricted range $PT(X,Y), T(X,Y), I(X,Y)$, and some others then we explain some of their algebraic properties. Further, in each property we will do a simple comparing study between the old semigroups and the restricted range ones. This chapter involving three sections named, depending the algebraic property, by

i) Regularity and Green's relations,

ii) Generating systems and ranks,

iii) Isomorphism and embedding theorems.

4.1. Regularity and Green's Relations

Let S denote any semigroup of the following semigroups: $PT(X)$, $T(X)$ and $I(X)$. It is well known that S is a regular semigroup, but $PT(X,Y), T(X,Y)$ and $I(X,Y)$ need not to be regular. Just to mention, we derive a counter example to show one of the cases.

Example 4.1.1. Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{3, 4, 5\}$ and

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} \in I(X,Y).$$

Assume that α is a regular element in $I(X,Y)$, thus there is an element $\beta \in I(X,Y)$ such that $\alpha = \alpha\beta\alpha$. Hence, $1\alpha = (1\alpha)\beta\alpha$ and since $\alpha \in I(X,Y)$, that is α is an injective map, then we conclude that $1 = 3\beta$ but $1 \notin Y$ and this contradicts with the assumption.

Recently, some authors like J. Sanwong, V. Fernandes and others considered the regularity of the transformation semigroups with restricted range in many papers (see, for examples, Symons (1975a), Sanwong and Sommanee (2008), Sanwong (2011), Sommanee and Sanwong (2013) and (2015), Fernandes, Honyam, Quinterio and Singha (2014) and (2016), Fernandes and Sanwong (2014), Sun and Sun (2016), Sangkhanan and Sanwong (2020)). They characterized the regular elements and determined the largest regular subsemigroup, which is one of the important events because it leads then to the description of Green's relations and other applications. We will arrange these events based on the historical development. First, in Nenthein, Youngkhong and Kemprasit (2005), the authors characterized the regular elements of $T(X,Y)$ by the following lemma.

Lemma 4.1.2. The semigroup $T(X,Y)$ is regular if and only if $X = Y$ or $|Y| = 1$.
Proof. If $X = Y$ then $T(X,Y) = T(X,X) = T(X)$ which is regular. Moreover, if $|Y| = 1$ then $|T(X,Y)| = 1$ and so it is obviously regular.

Conversely, assume that $T(X,Y)$ is a regular semigroup and suppose that $Y \subsetneq X$ and $|Y| \geq 2$. Then there exist at least two elements in Y, say $\{y_0, y_1\}$, where $y_0 \neq y_1$. Then define a transformation α as follows

$$\alpha = \begin{pmatrix} Y & X\backslash Y \\ y_0 & y_1 \end{pmatrix}$$

It is clear that $\alpha \in T(X,Y)$, and since $T(X,Y)$ is regular, then there exists a transformation $\beta \in T(X,Y)$ such that $\alpha\beta\alpha = \alpha$. But since

$$(y_1)\alpha\beta\alpha = (y_0)\beta\alpha = (y_0\beta)\alpha = y_0 = (y_0)\alpha,$$

We have $y_0 = y_1$, which contradicts with the assumption. ■

In addition, the authors computed the cardinality of the set of all regular elements, denoted by $RegT(X,Y)$, when X is a finite set.

Lemma 4.1.3. If $|X| = n$ and $|Y| = m$, then

$$|RegT(X,Y)| = \sum_{r=1}^{m} \binom{r}{m} r!\, S(m,r) r^{n-m}$$

Proof. Nenthein, Youngkhong and Kemprasit (2005), Theorem 2.6. ■

After that, Sanwong and Sommanee (2008) obtained the largest regular subsemigroup, which is defined by

$$F(X,Y) = \{\alpha \in T(X,Y) : X\alpha = Y\alpha\}.$$

This set contains all the regular elements of $T(X,Y)$. That is to say

$$F(X,Y) = RegT(X,Y).$$

In fact, $F(X,Y)$ is not just a subsemigroup, it is a right ideal also but not a left ideal of $T(X,Y)$.

Example 4.1.4. Let $X = \mathbb{Z}^+$ and $Y = 2\mathbb{Z}^+$. Then define the following maps

$$\alpha = \begin{pmatrix} k \\ 2k \end{pmatrix} \quad \text{and } \beta = \begin{pmatrix} 2k & \mathrm{X\backslash Y} \\ 2k & 2 \end{pmatrix}.$$

It is clear that $\alpha \in T(X,Y)\backslash F(X,Y)$ and $\beta \in F(X,Y)$, but $\alpha\beta = \alpha \notin F(X,Y)$. Hence this shows that $F(X,Y)$ is not a left ideal.

After the definition of $F(X,Y)$, they were able to describe Green's relations of $T(X,Y)$ as we will see in the following lemma.

Lemma 4.1.5. For $\alpha, \beta \in T(X,Y)$. Then

i) $\alpha\mathcal{L}\beta \Leftrightarrow (\alpha, \beta \in F(X,Y)$ and $X\alpha = X\beta)$ or $(\alpha, \beta \in T(X,Y) \backslash F(X,Y)$ and $\alpha = \beta)$,

ii) $\alpha\mathcal{R}\beta \Leftrightarrow (\alpha, \beta \in F(X,Y)$ and $\ker(\alpha) = \ker(\beta)$) or $(\alpha, \beta \in T(X,Y) \backslash F(X,Y)$ and $\ker(\alpha) = \ker(\beta))$,

iii) $\alpha\mathcal{D}\beta \Leftrightarrow (\alpha, \beta \in F(X,Y)$ and $|X\alpha| = |X\beta|)$ or $(\alpha, \beta \in T(X,Y) \backslash F(X,Y)$ and $\ker(\alpha) = \ker(\beta))$ and

iv) $\alpha\mathcal{J}\beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$ or $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$.

Proof. Sanwong and Sommanee (2008), Theorem 3.2., Theorem 3.3., Theorem 3.7. and Theorem 3.9. ■

In 2011, the first author limited her research on the subsemigroup $F(X,Y)$ and proved that every regular semigroup S can be embedded in $F(S^1, S)$. That means, $F(X,Y)$ can be consider as a model for all regular semigroups, see Sanwong (2011). Furthermore, she described the ideals of $F(X,Y)$ as follows.

Lemma 4.1.6. For $2 \leq r \leq |Y|$, the sets

$$Q_r = \{\alpha \in F(X,Y) : |X\alpha| < r\}$$

considered as the proper ideals of $F(X,Y)$.

Proof. Sanwong (2011), Theorem 3.4. ■

Moreover, in the same paper, the author characterized the $\mathcal{H}$-classes and $\mathcal{J}$-classes of $F(X,Y)$ and by that she was able to describe the maximal regular subsemigroup of $F(X,Y)$. Later, in Fernandes and Sanwong (2014), they concentrated their studies on the semigroups $PT(X,Y)$ and $I(X,Y)$, and presented analogous results to $T(X,Y)$. For example, they determined the largest regular subsemigroup of $PT(X,Y)$ to be

$$PF = \{\alpha \in PT(X,Y) \colon X\alpha = Y\alpha\}.$$

On the other hand, they showed that the symmetric inverse semigroup $I(Y)$ is the largest regular subsemigroup of $I(X,Y)$, and the largest inverse subsemigroup of $I(X,Y)$ as well. Consequently, the description of Green's relations on $PT(X,Y)$ and $I(X,Y)$ can be given by the following two lemmas.

Lemma 4.1.7. For $\alpha, \beta \in PT(X,Y)$,

i) $\alpha\mathcal{L}\beta \Leftrightarrow$ ($\alpha, \beta \in PF$ and $X\alpha = X\beta$) or ($\alpha, \beta \in PT(X,Y)\backslash PF$ and $\alpha = \beta$),

ii) $\alpha\mathcal{R}\beta \Leftrightarrow \mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ and $\ker(\alpha) = \ker(\beta)$,

iii) $\alpha\mathcal{D}\beta \Leftrightarrow$ ($\alpha, \beta \in PF$ and $|X\alpha| = |X\beta|$) or ($\alpha, \beta \in PT(X,Y)\backslash PF$ $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ and $\ker(\alpha) = \ker(\beta)$) and

iv) $\alpha\mathcal{J}\beta \Leftrightarrow$ ($\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ and $\ker(\alpha) = \ker(\beta)$) or ($|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$).

Proof. Fernandes and Sanwong (2014), Theorems 1.4., 1.5., 1.7. and 1.9. ■

Lemma 4.1.8. For $\alpha, \beta \in I(X,Y)$,

i) $\alpha\mathcal{L}\beta \Leftrightarrow$ ($\alpha, \beta \in I(Y)$ and $X\alpha = X\beta$) or ($\alpha, \beta \in I(X,Y)\backslash I(Y)$ and $\alpha = \beta$),

ii) $\alpha\mathcal{R}\beta \Leftrightarrow \mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$,

iii) $\alpha\mathcal{D}\beta \Leftrightarrow$ ($\alpha, \beta \in I(Y)$ and $|\mathrm{dom}(\alpha)| = |\mathrm{dom}(\beta)|$) or ($\alpha, \beta \in I(X,Y)\backslash I(Y)$ and $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$) and

iv) $\alpha\mathcal{J}\beta \Leftrightarrow \mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ or $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$.

Proof. Fernandes and Sanwong (2014), Theorems 3.3., 3.4., 3.6. and 3.7. ■

Definition 4.1.9. For a chain X and $\emptyset \neq Y \subseteq X$, consider the order-preserving transformation semigroup $O(X)$ and define the subsemigroup

$$O(X,Y) = \{\alpha \in O(X): X\alpha \subseteq Y\}$$

to be the *order-preserving transformation subsemigroup with restricted range*. If X is a finite chain, say $|X| = n$, then we denote $O(X,Y)$ by $O_n(Y)$.

Although, the semigroup $O(X)$ is a regular semigroup its subsemigroup $O(X,Y)$ need not to be. This promoted the authors in Mora and Kemprasit (2010) to describe the regular elements of $O(X,Y)$ and to characterize the regular semigroups of that type. In Fernandes, Honyam, Quinterio and Singha (2014), the authors gave a description for the largest regular subsemigroup of $O(X,Y)$ as follows.

$$FO(X,Y) = \{\alpha \in O(X,Y): X\alpha = Y\alpha\}$$

which is also a right ideal of $O(X,Y)$.

As we saw in the case of $T(X,Y)$, after this description of the largest regular subsemigroup of $O(X,Y)$, traditionally, the next step was the characterization of Green's relations on $O(X,Y)$ and on $O(X)$ in particular. Nevertheless, we need some definitions first.

Definition 4.1.10. Let $\theta: A \longrightarrow B$ be an order-preserving map from a subchain A of X into a subchain B of Y. Then θ is called *completable* in $O(X,Y)$, if there exists $\lambda \in O(X,Y)$ such that $a\lambda = a\theta$, for all $a \in A$.

Definition 4.1.11. Let $\theta: A \longrightarrow B$ be an order-isomorphism map, where A and B are subchains of Y. Then θ is called *bicompletable* in $O(X,Y)$ if θ and its inverse $\theta^{-1}: B \longrightarrow A$ are completable in $O(X,Y)$.

Now, we give the description of Green's relations on $O(X,Y)$ by the following lemma.

Lemma 4.1.12. For $\alpha, \beta \in O(X,Y)$,
1. $\alpha\mathcal{L}\beta \Leftrightarrow (\alpha,\beta \in FO(X,Y)$ and $X\alpha = X\beta)$ or $(\alpha,\beta \in O(X,Y)\backslash FO(X,Y)$ and $\alpha = \beta)$,
2. $\alpha\mathcal{R}\beta \Leftrightarrow (\alpha,\beta \in FO(X,Y)$ and $\ker(\alpha) = \ker(\beta))$ or $(\alpha,\beta \in O(X,Y)\backslash FO(X,Y)$ and either $\alpha = \beta$ or $\ker(\alpha) = \ker(\beta))$ and the canonical order-isomorphism $\theta: \mathrm{im}(\alpha) \longrightarrow \mathrm{im}(\beta)$ is bicompletable in $O(X,Y)$,
3. $\alpha\mathcal{D}\beta$ if and only if one of the following statements holds:
i) $\alpha\mathcal{L}\beta$,
ii) $\alpha,\beta \in FO(X,Y)$ and there exists order-isomorphism such that bicompletable in $O(X,Y)$ $\theta: \mathrm{im}(\alpha) \longrightarrow \mathrm{im}(\beta)$,
iii) $\alpha,\beta \in O(X,Y)\backslash FO(X,Y)$ and $\alpha\mathcal{R}\beta$.
4. $\alpha\mathcal{J}\beta$ if and only if one of the following statements holds:
i) $\alpha\mathcal{L}\beta$ in $O(X,Y)$,
ii) $\alpha\mathcal{R}\beta$ in $O(X,Y)$,
iii) there exist injective order-preserving functions $\theta: \mathrm{im}(\alpha) \longrightarrow Y\beta$ and $\tau: \mathrm{im}(\beta) \longrightarrow Y\alpha$ admitting completable inverses in $O(X,Y)$.

Proof. Fernandes, Honyam, Quinteiro and Singha (2014), Propositions 2.5., 2.8., 2.13. and 2.15. ■

Of course, there are more other transformations subsemigroups of the main semigroups $OPT(X)$, $T(X)$ and $I(X)$ with restricted range, for example, the orientation-preserving transformation semigroup OP_n, the orientation-preserving or orientation-reversing transformation semigroup OR_n. There, also the description of Greens relations and regularity are investigated in analogous ways. For more details about these topics, we refer the reader to consult the paper Fernandes, Honyam, Quinterio and Singha (2016).

4.2. Generating Systems and Ranks

In the previous chapter, we explained, in general, what is the meaning of generating systems and ranks of a semigroup. Here, we specify our talking about generating sets and ranks of the semigroups $PT(X,Y), T(X,Y), I(X,Y)$ and $O(X,Y)$ when X is a finite set. Moreover, we study the rank of the regular part of $T(X,Y)$, namely $F(X,Y)$, and its ideals. Finally, we consider the relative ranks of $T(X,Y)$ modulo specific subsets.

First, we need to remember some facts about generating systems and ranks of the main transformation semigroups PT_n, T_n and I_n. For this, let S be any one of the semigroups PT_n, T_n or I_n, then remember that:

1) All the generating sets of S must contain a generating set of the symmetric group S_n.

2) If G is a minimal generating set of S then $G \cap S_n$ is a minimal generating set of S_n.

3) Any generating set of S must have at least one element of height $n-1$, where the height of an element α of S is defined as $h(\alpha) = |im\alpha|$.

4) For $n \geq 3$, $\text{rank}(PT_n) = 4$ and $\text{rank}(T_n) = \text{rank}(I_n) = 3$.

5) For $n \geq 2$, $\text{rank}(PO_n) = 2n-1$ and $\text{idrank}(PO_n) = 3n-2$.

6) For $n \geq 2$, $\text{rank}(O_n) = n$ and $\text{idrank}(O_n) = 2n-2$.

Let $X = \{1, 2, ..., \text{n}\}$ and $Y = \{1, 2, ..., \text{r}\}$ where $1 \leq r \leq n$ and $n \in \mathbb{N}$. Then we denote the (full) transformation semigroup, the (partial) transformation semigroup and the inverse semigroup with restricted range by $PT_{n,r}$, $T_{n,r}$ and $I_{n,r}$, respectively.

In Fernandes, Honyam, Quinterio and Singha (2014), the authors proved the following lemmas.

Lemma 4.2.1. For $n \geq 3$ and $2 \leq r \leq n-1$, the semigroups $PT_{n,r}$, $T_{n,r}$ and $I_{n,r}$ are generated by their own elements of height r.

Proof. Let $\alpha \in PT_{n,r}$, with $|\mathrm{im}(\alpha)| = s$ for some $2 \leq s \leq r-1$, and let $b \in Y\backslash\mathrm{im}(\alpha)$. Then we have three cases:

Case (1): If $\alpha \in PT_{n,r}\backslash T_{n,r}$, then for $a \in X\backslash\mathrm{dom}(\alpha)$, define $\beta, \gamma \in PT_{n,r}$ with $\mathrm{dom}(\beta) = \mathrm{dom}(\alpha) \cup \{a\}$ and $\mathrm{dom}(\beta) = X\backslash b$, by

$$x\beta = \begin{cases} x\alpha & if \quad x \neq a \\ b & if \quad x = a \end{cases} \qquad \text{and} \quad x\gamma = \begin{cases} x & if \quad x \in \mathrm{im}(\alpha) \\ b & if \quad \text{elsewhere} \end{cases}$$

It is clear that $|\mathrm{im}(\beta)| = |\mathrm{im}(\gamma)| = s+1$ and $\beta\gamma = \alpha$, as required.

Case (2): If $\alpha \in T_{n,r}$ then we choose an element $a \in X = \mathrm{dom}(\alpha)$, where a lies in a nontrivial kernel class of α, and define $\beta, \gamma \in T_{n,r}$ by:

$$x\beta = \begin{cases} x\alpha & if \quad x \neq a \\ b & if \quad x = a \end{cases} \qquad \text{and} \quad x\gamma = \begin{cases} x & if \quad x \in \mathrm{im}(\alpha) \\ a\alpha & if \quad x = b \\ b & if \quad \text{elsewhere} \end{cases} .$$

Thus, we have $|\mathrm{im}(\beta)| = |\mathrm{im}(\gamma)| = s+1$ and $\beta\gamma = \alpha$, as required.

Case (3): If $\alpha \in I_{n,r}$, then for $a \in X\backslash\mathrm{dom}(\alpha)$, define $\beta, \gamma \in I_{n,r}$, by:

$$x\beta = \begin{cases} x\alpha & if \quad x \in \mathrm{dom}(\alpha) \\ b & if \quad x = a \end{cases} \qquad \text{and} \quad x\gamma = \begin{cases} x & if \quad x \in \mathrm{im}(\alpha) \\ b & if \quad x = n \end{cases} .$$

Thus, we have $|\mathrm{im}(\beta)| = |\mathrm{im}(\gamma)| = s+1$ and $\beta\gamma = \alpha$, as required. ■

Lemma 4.2.2. For $n \geq 2$ and $2 \leq r \leq n-1$, we have

i) $\mathrm{rank}(PT_{n,r}) = S(n+1, r+1)$,

ii) $\mathrm{rank}(T_{n,r}) = S(n,r)$,

iii) $\mathrm{rank}(\,I_{n,r}) = \begin{cases} \binom{n}{r} & \text{if} \quad r \in \{1,2\} \\ \binom{n}{r}+1 & \text{if} \quad 3 \leq r \leq n-1 \end{cases}$.

Proof. Fernandes and Sanwong (2014), Theorems 2.3., 2.4. and 4.2. ■

Now, let $F_{n,r}$ denote the largest regular subsemigroup of $T_{n,r}$. Then in the paper Sommanee and Sanwong (2013), the authors proved that $F_{n,r}$ is not idempotent generated semigroup. However, they determined that its rank is equal to

$$\mathrm{rank}(\,F_{n,r}) \;= \begin{cases} 3, & n = r \\ r^{n-r}+1\,, & 2 \leq r \leq n-1 \end{cases} .$$

In addition, the ideals of $F_{n,r}$, for $1 \leq k \leq r$, is defined as

$$Q(F;k) = \{\alpha \in F_{n,r} : |X\alpha| \leq k\}.$$

The rank of $Q(F;k)$ and its idempotent rank are equal and given by the following lemma.

Lemma 4.2.3. For $1 \leq k \leq r-1$

$$\mathrm{rank}(Q(F;k)) = \mathrm{idrank}(Q(F;k)) = S(r,k)k^{n-r}.$$

Proof. Sommanee and Sanwong (2013), Theorem 4.4. ■

On the other hand, in order to compute the rank of $O_n(Y)$, we need to define the notion of a "*captive element*", which is due to Fernandes and his team in Fernandes, Honyam, Quinterio and Singha (2014).

Definition 4.2.4. Let $X = \{1, 2, ..., n\}$ and Y be any subset of X. Then an element $a \in Y$ is said to be captive if either $a \in \{1, 2\}$ or $1 < a < n$ and $a - 1, a + 1 \in Y$. The set of all captive elements of Y is denoted by $Y^{\#}$.

Example 4.2.5. Let $X = \{1, 2, ..., 7\}$, $Y = \{1, 3, 4, 5\}$ and $Z = \{2, 4, 6\}$. Then
i) $Y^{\#} = \{1, 4\}$,
ii) $Z^{\#} = \{2\}$.

Now, the rank of $O_n(Y)$ is given by the following lemma.

Lemma 4.2.6. For $1 \leq r \leq n$,

$$\operatorname{rank}(O_n(Y)) = \binom{n-1}{r-1} + |Y^{\#}|.$$

Proof. Fernandes, Honyam, Quinteiro and Singha (2014), Theorem 4.3. ■

Relative ranks of the transformation semigroups with restricted range have also studied recently, for example in Tinpun and Koppitz (2016a) and (2016b) they computed the relative ranks for both cases, when X is a finite or infinite set. For the finite case, the authors determined the relative rank of the following semigroups
i) $T(X, Y)$ modulo $S(X, Y)$,
ii) $T(X, Y)$ modulo $O(X, Y)$,

iii) $O(X,Y)$ modulo $EO(X,Y)$,

where $EO(X,Y)$ denotes the set of all the idempotent elements of $O(X,Y)$ and $S(X,Y)$ is defined by

$$S(X,Y) = \{\alpha \in T(X,Y) : \alpha|_Y \text{ is a bijection on } Y\}.$$

For the infinite case, they determined the relative rank for these semigroups

i) $T(X,Y)$ modulo $S(X,Y)$ and

ii) $T(X,Y)$ modulo $E(X,Y)$.

The cardinalities of each one of the semigroups $PT_{n,r}$, $T_{n,r}$ and $I_{n,r}$ are given by the following lemma.

Lemma 4.2.7. Let $n \in \mathbb{N}$ where $n \geq 2$ and $1 \leq r \leq n$. Then

i) $|PT_{n,r}| = (r+1)^n$,

ii) $|T_{n,r}| = (r)^n$,

iii) $|I_{n,r}| = \sum_{k=0}^{r} \binom{k}{n} \binom{k}{r} k!$.

Proof. Fernandes and Sanwong (2014). ■

On the other hand, relating to the cardinality of an order-preserving semigroup with restricted range $O_n(Y)$ it differs according to the value of $|Y|$

i) if $|Y| = 1$ then $|O_n(Y)| = 1$,

ii) if $|Y| = n$ then $|O_n(Y)| = |O_n| = \binom{n-1}{2n-1}$ and

iii) if $|Y| = r$ where $1 < r < n$ then $|O_n(Y)| = \binom{n-1}{n+r-1}$.

In the first chapter, we mentioned that an element $\alpha \in P_X$ is an idempotent element if and only if $\forall x \in \mathrm{im}\alpha,\ x\alpha = x$. Here one may ask the same question for the "restricted range" semigroups. In Fernandes and Sanwong (2014) they have answered this question for both semigroups $PT(X,Y)$ and $I(X,Y)$ as we see in the following lemmas.

Lemma 4.2.8. If $\alpha \in PT(X,Y)\backslash\emptyset$, then the followings are equivalent:

1) $\alpha = \binom{y}{y}$, for some $y \in Y$

2) α is an idempotent and $\forall\ \beta \in PT(X,Y)$ either $(\alpha\beta)^2 = \emptyset$ or $(\alpha\beta)^2 = \alpha$.

Proof. Fernandes and Sanwong (2014), Lemma 1.10. ■

As $T(X,Y) \leq T(X)$ and the set of idempotent elements of $T(X)$ is defined as

$$\{\alpha \in T(X)\colon x\alpha = x\ ,\forall x\ \in\ X\alpha\}\,,$$

the set of idempotent elements of $T(X,Y)$ is described in a similar way.

Lemma 4.2.9. The idempotent elements of $I(X,Y)$ are the elements of the set $\{id_A\colon A \subseteq Y\}$ where id_A denote the identity map on A.

Proof. Fernandes and Sanwong (2014), Lemma 3.8. ■

4.3. Isomorphism and Embedding Theorems

Recall that an automorphism of a semigroup S is an isomorphism $\varphi\colon S \to S$. Suppose that S be a monoid and for all $a \in S\backslash\{0\}$ consider the automorphism $\varphi_a\colon S \to S$ defined by: $x \longmapsto a^{-1}xa\ ,\ \forall a \in S$. Then this automorphism is called an *inner automorphism* of S and the set $Inn(S)$ denote all inner automorphisms of S.

A. Malcev, showed that any automorphism of $T(X)$ is inner, see Malcev (1952). After that Symons, in his famous paper "Some results concerning a transformation semigroup" generalized this result by considering the subsemigroup $T(X,Y)$, see Symons (1975). The description of the set of all automorphism of $T(X,Y)$ was given as follows

$$Aut\big(T(X,Y)\big) = Aut_Y\big(T(X,Y)\big) \cdot Inn_Y\big(T(X,Y)\big).$$

At the same paper, he tried to answer the question of "When are two semigroups of the form $T(X,Y)$ isomorphic?" Thus, in order to find an answer, he proved the following related theorems.

Theorem 4.3.1. If $T(X_1,Y_1)$ isomorphic to $T(X_2,Y_2)$ then $|Y_1| = |Y_2|$.
Proof. Symons (1975b), Theorem 5.1. ∎

Theorem 4.3.2. If $|Y_1| = |Y_2|$ then $T(X_1,Y_1)$ and $T(X_2,Y_2)$ are isomorphic.
Proof. Symons (1975b), Theorem 5.1. ∎

Theorem 4.3.3. If $|Y_1| = 2 = |Y_2|$ then $T(X_1,Y_1)$ and $T(X_2,Y_2)$ are isomorphic if and only if $|X_1 \backslash Y_1| = |X_2 \backslash Y_2|$.
Proof. Symons (1975b), Theorem 5.1. ∎

Recently, in Sanwong (2011), the author interested in the regular part of $T(X,Y)$, namely $F(X,Y)$, and proved some embedding and isomorphism theorems related to that part, for instance she proved the following theorems.

Theorem 4.3.4. $T(Y)$ is isomorphic to $F(X,Y)$ if and only if $|Y| = 1$ or $X = Y$.
Proof. Sanwong (2013), Lemma 2.2. ∎

Theorem 4.3.5. For some set Z, $T(Z)$ is isomorphic to $F(X,Y)$ if and only if $|Y| = |Z| = 1$ or $X = Y$ and $|Y| = |Z|$.

Proof. Assume that $T(Z) \cong F(X,Y)$, for some set Z. Then $i_d \in F(X,Y)$ (where i_d denote the identity map) and we have two cases:

Case (1): If $|Y| = 1$ then $|F(X,Y)| = 1$ also and by the isomorphism assumption we have shown that $|Y| = |Z| = 1$.

Case (2): If $|Y| > 1$ then we claim that $X = Y$ .To show that we suppose the opposite, that is $X \supsetneq Y$ and this means that $i_d \notin F(X,Y)$ which contradicts with the above assumption. This contradiction led us to conclude that $X = Y$, and since $T(Y) = F(X,Y) \cong T(Z)$ then $|Y| = |Z|$. ■

Theorem 4.3.6. $T(Y)$ can be embedded in $F(X,Y)$.

Proof. Fix an element $a \in Y$ and for all $\alpha \in T(Y)$ define $\tilde{\alpha} \in T(X)$ as

$$x\tilde{\alpha} = \begin{cases} x\alpha & , \quad x \in Y \\ a\alpha & , \quad x \notin Y \end{cases}.$$

Now, consider the mapping $\sigma_a : T(Y) \to F(X,Y)$ defined as $\alpha\sigma_a = \tilde{\alpha}$ for all $\alpha \in T(Y)$.

If $X = Y$ or $|Y| = 1$ then by the last theorem $T(Y)$ is isomorphic to $F(X,Y)$, and thus the embedding follows. Assume that $X \supsetneq Y$ and $|Y| \geq 2$, we need to show that σ_a is a one to one homomorphism. First, for all $x \in X$, if $x \in Y$ then

$$x(\widetilde{\alpha\beta}) = x(\alpha\beta) = (x\alpha)\beta = (x\tilde{\alpha})\beta = (x\tilde{\alpha})\tilde{\beta}.$$

If $x \notin Y$, since $x\tilde{\alpha} \in Y$, then

$$x(\widetilde{\alpha\beta}) = a(\alpha\beta) = (a\alpha)\beta = (x\tilde{\alpha})\beta = (x\tilde{\alpha})\tilde{\beta}.$$

Thus, $\widetilde{\alpha\beta} = \tilde{\alpha} \circ \tilde{\beta}$ and this proves that σ_a is a homomorphism.

For all $x \in X$, suppose that $\alpha\sigma_a = \beta\sigma_a$ then, it does not matter $x \in Y$ or $x \notin Y$, we have

$$x\tilde{\alpha} = x\tilde{\beta} \Longrightarrow x\alpha = x\beta\,.$$

Thus, $\tilde{\alpha} = \tilde{\beta}$ leads to conclude that $\alpha = \beta$ and this proves that σ_a is a one to one map. ∎

After that, she was able to generalize the previous theorem by proving that:

Theorem 4.3.7. For some set $X \supseteq S^1$, every semigroup S can be embedded in $F(X, S^1)$.
Proof. Sanwong (2013), Corollary 2.2. ∎

Theorem 4.3.8. Every regular semigroup S can be embedded in $F(S^1, S)$.
Proof. Sanwong (2013), Theorem 2.3. ∎

Regarding with the isomorphism theorems for both semigroups $PT(X,Y)$ and $I(X,Y)$ proved them in a way similar to that in $T(X,Y)$, see Fernandes and Sanwong (2014).

Theorem 4.3.9. For a finite set X, let Y_1, Y_2 be two non-empty subsets of X. Then
i) $PT(X, Y_1)$ isomorphic to $PT(X, Y_2)$ if and only if $|Y_1| = |Y_2|$.
ii) $I(X, Y_1)$ isomorphic to $I(X, Y_2)$ if and only if $|Y_1| = |Y_2|$.
Proof. Fernandes and Sanwong (2014), Theorem 1.11. and Theorem 3.10. ∎

Moreover, in subsequent works, and with other authors, Fernandes, Honyam, Quinterio and Singha (2014), proved some isomorphism theorems for the order-preserving semigroups with restricted range $O(X,Y)$. One of those theorems was proving when $O(\mathrm{X}_1,\mathrm{Y}_1) \cong O(\mathrm{X}_2,\mathrm{Y}_2)$ where $\mathrm{X}_1,\ \mathrm{X}_2$ are finite chains. Likewise, in Sommanee and Sanwong (2015), they proved some embedding and isomorphism theorems for the regular part of $O(X,Y)$, namely $OF(X,Y)$. For example, they proved that $O(Y)$ can be embedded in $FO(X,Y)$ and in order to be isomorphic they provided necessary and sufficient conditions. In addition, they have proved when $FO(X_1,Y_1) \cong FO(X_2,Y_2)$ where $\mathrm{X}_1,\mathrm{X}_2$ are finite chains.

5. GENERALIZED TRANSFORMATION SEMIGROUPS

K.D. Magill, in several papers started to consider the generalized transformation semigroup, denoted by $\mathcal{GT}(X,Y;\theta)$, which we will define soon. Magill in his paper, Magill (1967), under the hypothesis that θ is surjective, was able to describe all the automorphisms of $\mathcal{GT}(X,Y;\theta)$ and he determined when $\mathcal{GT}(X_1,Y_1;\theta_1) \cong \mathcal{GT}(X_2,Y_2;\theta_2)$. After that, Symons came up and generalized the same results without need to the assumption of surjectivity of θ, see Symons (1975a). He also suggested new relations on $\mathcal{GT}(X,Y;\theta)$ which are partially analogues to Green's relations and he gave some structure results for $\mathcal{GT}(X,Y;\theta)$.

The generalized transformation semigroup $\mathcal{GT}(X,Y;\theta)$ is a natural generalization of the usual transformation semigroup $T(X)$ and it is defined as follows. Let X and Y be non-empty sets, and let $\mathcal{GT}(X,Y)$ denote the set of all full transformations from X to Y. For any fixed $\theta \in \mathcal{GT}(Y,X)$ and any $\alpha,\beta \in \mathcal{GT}(X,Y)$ define a sandwich operation $*$ by

$$\alpha * \beta = \alpha \circ \theta \circ \beta$$

with this sandwich operation $*$, $\mathcal{GT}(X,Y)$ is a semigroup that is called a *generalized transformation semigroup* and denoted by $\mathcal{GT}(X,Y;\theta)$. In particular, if $X = Y$ and $\theta = i_d$, the identity transformation, then $\mathcal{GT}(X,Y;\theta)$ become $T(X)$, the full transformation semigroup.

In this chapter, we consider the semigroup $\mathcal{GT}(X,Y;\theta)$ and do some investigations about some algebraic properties on it such as:

i) Symons's relations and Green's relations,

ii) Regularity of $P(X,Y;\theta)$ and $\mathcal{GT}(X,Y;\theta)$,

iii) Embedding and isomorphism theorems,

5.1. Symons's Relations and Green's Relations

In chapter three we considered Green's relations $\mathcal{L},\ \mathcal{R},\ \mathcal{D},\ \mathcal{H}\ and\ \mathcal{J}$ for any semigroup in general and we studied some related theorems. It is the purpose of this section to do a comparing study between Symons's relations on $\mathcal{G}T(X,Y;\theta)$ and Green's.

Let's start by defining Symons's relations, which are denoted by small letters (ℓ, $\mathfrak{r}$, h and d), then we present some theorems concerning with the structure of the semigroup $\mathcal{G}T(X,Y;\theta)$.

Definition 5.1.1. For $\alpha,\beta\in\mathcal{G}T(X,Y;\theta)$, define

i) $\alpha\ \ell\ \beta$ to mean $\theta\alpha=\theta\beta$,

ii) $\alpha\ \mathfrak{r}\ \beta$ to mean $\alpha\theta=\beta\theta$,

iii) $\alpha\ d\ \beta$ to mean $\theta\alpha\theta=\theta\beta\theta$ and

iv) $h=\ell\cap\ \mathfrak{r}$.

Lemma 5.1.2. The relations $\ell,\mathfrak{r}$, and d are congruences on the semigroup $\mathcal{G}T(X,Y;\theta)$.

Proof. It is easy to show that $\ell,\mathfrak{r}$, and d are equivalence relations. Thus, the remaining part is showing that they are compatible. Let $\alpha,\beta\in\mathcal{G}T(X,Y;\theta)$ and suppose that $\alpha\ d\ \beta$ then

$$\theta\alpha\theta=\theta\beta\theta\ .$$

Therefore, for each $\lambda\in\mathcal{G}T(X,Y;\theta)$ we have

$$\lambda\theta\alpha\theta=\lambda\theta\beta\theta\ ,$$

which implies that:

$$\theta(\lambda * \alpha)\theta = \theta\lambda\theta\alpha\theta = \theta\lambda\theta\beta\theta = \theta(\lambda * \beta)\theta .$$

This means $(\lambda * \alpha)d(\lambda * \beta)$. Likewise, $(\alpha * \lambda)d(\beta * \lambda)$ and this means d is compatible and hence a congruence on $\mathcal{G}T(X, Y; \theta)$. In a similar way, one can show the cases of ℓ and $\mathfrak{r}$. ■

Lemma 5.1.3. The relations $\mathfrak{r}$ and ℓ commute and $d = \ell \circ \mathfrak{r} = \mathfrak{r} \circ \ell = l \vee \mathfrak{r}$.

Proof. We begin by showing that $d = \ell \circ \mathfrak{r}$. Let $(\alpha, \beta) \in d$ this means $\alpha\ d\ \beta$ and thus $\theta\alpha\theta = \theta\beta\theta$. Then define the mapping $\lambda : X \to X$ by

$$x\lambda = \begin{cases} x\beta , & x \in Y\theta \\ x\alpha , & \text{elsewhere} \end{cases} .$$

Clearly, one can show that $\theta\lambda = \theta\beta$ and hence $\lambda\ell\beta$.

On the other hand, we claim that $\lambda\theta = \alpha\theta$. To see that, assume that $x \notin Y\theta$ then $x\lambda = x\alpha$ and so $x\lambda\theta = x\alpha\theta$. Otherwise, $x = y\theta$ for some $y \in Y$. Then

$$x\lambda\theta = x\beta\theta = y\theta\beta\theta = y\theta\alpha\theta = x\alpha\theta ,$$

therefore, $\alpha\mathfrak{r}\lambda$ and so $\alpha(\ell \circ \mathfrak{r})\beta$.

Conversely, we aim to show that $\alpha\ d\ \beta$. Let $\alpha, \beta \in \mathcal{G}T(X, Y; \theta)$ such that $\alpha(\ell \circ \mathfrak{r})\beta$. Then $\exists\delta \in \mathcal{G}T(X, Y; \theta)$ such that $\theta\alpha = \theta\delta$ and $\beta\theta = \delta\theta$. Now, since

$$(\theta\delta)\theta = \theta(\delta\theta) \text{ and so } (\theta\alpha)\theta = \theta(\beta\theta) ,$$

we have that $d = \ell \circ \mathfrak{r}$. Now, since

$$\ell \circ \mathfrak{r} = d = d^{-1} = (\ell \circ \mathfrak{r})^{-1} = \mathfrak{r}^{-1} \circ \ell^{-1} = \mathfrak{r} \circ \ell ,$$

we have ℓ and $\mathfrak{r}$ commute and hence $l \vee \mathfrak{r}$ by Corollary 1.5.12 Howie's book (1965). ■

Now, we state two theorems without proof since they are very similar to the case of Green's relations for arbitrary semigroup, see (Clifford and Preston (1967), pages 47-49).

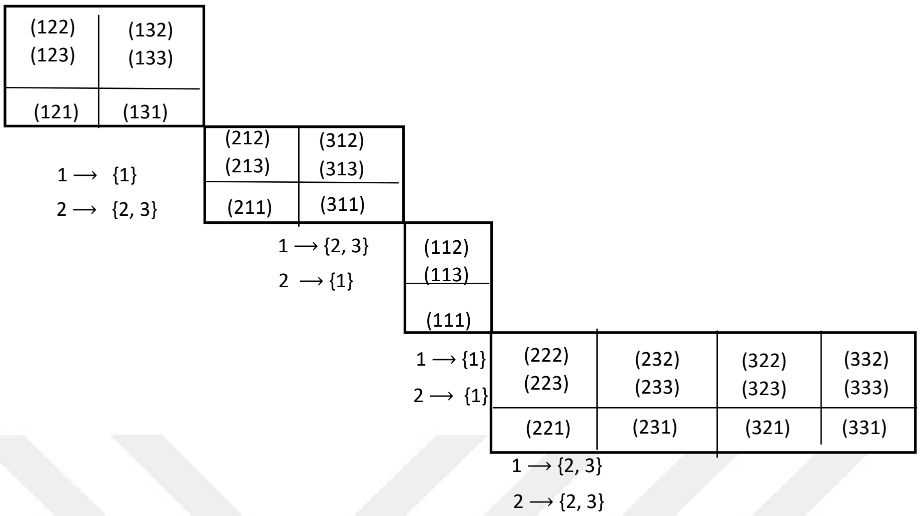
Theorem 5.1.4. If $\mathfrak{r}$ is an $\mathfrak{r}$-class and ℓ is an ℓ-class then $\mathfrak{r}$ meets ℓ in $\mathcal{GT}(X, Y; \theta)$ if and only if they are in the same d-class.

Proof. Symons (1975a), Theorem 1.3. ■

Theorem 5.1.5. The set product of any ℓ-class and any $\mathfrak{r}$-class of $\mathcal{GT}(X, Y; \theta)$ is always contained in a single d-class.

Proof. Symons (1975a), Theorem 1.4. ■

The last theorems show that the relations $\mathfrak{r}$, ℓ and d separate the semigroup $\mathcal{GT}(X, Y; \theta)$ into rows, columns and cells, in a way similar to Green's relations ($\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$) do on an arbitrary semigroup. That is to say, $\mathcal{GT}(X, Y; \theta)$ have its own egg-box picture. Look at figure below, which explains an example of a class structure of $\mathcal{GT}(X, Y; \theta)$.



	$\ell - classes$	$r - classes$
	2	2
	2	2
	1	2
	4	2
Total	$9=3^2$	$8=2^3$

Figure 5.1. A class structure of $\mathcal{G}T(X, Y; \theta)$

Symons, also proved a theorem which is partially similar to the well-known lemma (*Green's lemma*), see Symons (1975a).

Theorem 5.1.6. The ℓ-classes of $\mathcal{G}T(X, Y; \theta)$ have the same size, and any two h-classes contained in a single $\mathfrak{r}$-class have the same size.

Proof. Symons (1975a), Theorem 1.5. ■

One of the differences between Green's relations and Symons's relations is that in the first case all $\mathcal{H}$-classes in one single $\mathcal{D}$-class have the same size, but this is not always true for Symons's case. Another difference is that Symons's relations are special for the semigroup $\mathcal{G}T(X,Y;\theta)$ while Green's relations are for any arbitrary semigroup.

After 40 years of the last theorems the problem of "Describing Green's relations on $\mathcal{G}T(X,Y;\theta)$ " was solved in Mendes-gonçalves and Sullivan (2013). Actually, they concentrated at the semigroup $P(X,Y;\theta)$, a *generalized partial transformation semigroup*, and as a result the description for $\mathcal{G}T(X,Y;\theta)$ followed.

Definition 5.1.7. Let $\theta \in P(Y,X)$, where $\mathrm{dom}(\theta) = Y$, and for all $\alpha, \beta \in P(X,Y)$ define a sandwich operation $*$ as follows

$$\alpha * \beta = \alpha \circ \theta \circ \beta$$

$(P(X,Y), *)$ is called a *generalized partial transformation semigroup* and denoted by $P(X,Y;\theta)$.

Theorem 5.1.8. For $\alpha, \beta \in P(X,Y;\theta)$, the description of Green's relations on this semigroup is given by

i) $\alpha\mathcal{L}\beta \Leftrightarrow \alpha = \beta$ or $\mathrm{im}(\alpha) = \mathrm{im}(\theta\alpha) = \mathrm{im}(\theta\beta) = \mathrm{im}(\beta)$,

ii) $\alpha\mathcal{R}\beta \Leftrightarrow \alpha = \beta$ or $(\mathrm{dom}(\alpha) = \mathrm{dom}(\theta\alpha) = \mathrm{dom}(\theta\beta) = \mathrm{dom}(\beta)$ and

$$\alpha \circ \alpha^{-1} = \alpha\theta \circ (\alpha\theta)^{-1} = \beta\theta \circ (\beta\theta)^{-1} = \beta \circ \beta^{-1}) \ ,$$

iii) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

Proof. Mendes-Gonçalves and Sullivan (2013), Theorem 2.6. ■

For the case of $\mathcal{D}$ and $\mathcal{J}$, the authors characterized them in a different way, as we will see next theorem.

Theorem 5.1.9. If the semigroup $P(X,Y;\theta)$ denoted by P. Then

i) $\mathcal{J} = \mathcal{L} \cup \mathcal{R} \cup \mathbb{J}$, where

$$\alpha \mathbb{J} \beta \Leftrightarrow |\mathrm{im}(\alpha)| = |\mathrm{im}(\theta\alpha\theta)| = |\mathrm{im}(\theta\beta\theta)| = |\mathrm{im}(\beta)|$$

ii) $\mathcal{D} = \mathcal{L} \cup \mathcal{R} \cup \mathbb{D}$ where $\alpha \mathcal{D} \beta \Leftrightarrow \alpha = \binom{A_i}{y_i}$ and $= \binom{B_i}{z_i}$,

1) $\forall i, A_i\theta^{-1} \neq \emptyset$ and $B_i\theta^{-1} \neq \emptyset$,

2) $\mathrm{im}(\alpha) \cup \mathrm{im}(\beta) \subseteq \mathrm{dom}(\alpha)$ and the restrictions of θ to $\mathrm{im}(\alpha)$ and to $\mathrm{im}(\beta)$ are injective.

Proof. Mendes-Gonçalves and Sullivan (2013), Theorems 2.7. and 2.8. ■

Furthermore, Sullivan with his team gave a special description of Green's relations for the regular elements of $P(X, Y; \theta)$. We will mention that in the next section.

Remark 3.1.10. The same comments can be made for the semigroups $\mathcal{G}T(X, Y; \theta)$ and $I(X, Y; \theta)$, (the subsemigroups of $P(X, Y; \theta)$ consisting of all full and injective mappings, respectively), with some more notations.

5.2. Regularity of $\boldsymbol{P(X, Y; \theta)}$ and $\boldsymbol{\mathcal{G}T(X, Y; \theta)}$

K.D. Magill and S. Subbiah, characterized the regularity of $P(X, Y; \theta)$, see Magill and Subbiah (1975). Another simpler description was given recently by Mendes-gonçalves and Sullivan (2013), which is given as follows.

Theorem 5.2.1. The semigroup $P(X, Y; \theta)$ is regular if and only if one of the following statements holds

i) X or Y is an empty set,

ii) $|X| = 1$ and $\mathrm{dom}(\theta) = Y \neq \emptyset$,

iii) $\mathrm{dom}(\theta) = Y$ and θ is a bijective map.

Proof. Mendes-Gonçalves and Sullivan (2013), Theorem 2.1. ■

An analogues result, due to Chinram (2008a), for the semigroup $I(X,Y;\theta)$. In order to do the same for the semigroup $\mathcal{G}T(X,Y;\theta)$ he defined the following semigroup

$$T_k(X,Y;\theta) = \{\alpha \in \mathcal{G}T(X,Y;\theta) \colon |\mathrm{im}(\alpha)| \leq k, 1 \leq k \leq |Y|\}$$

which is, in some sense, a generalization of $\mathcal{G}T(X,Y;\theta)$.

In addition, the author also described the regular elements of $P(X,Y;\theta)$, $I(X,Y;\theta)$ and $\mathcal{G}T(X,Y;\theta)$ in Chinram (2008a) and (2008b). Alternatively, Sullivan came up and gave the following description for the regular elements $P(X,Y;\theta)$ and $T_k(X,Y;\theta)$.

Theorem 5.2.2. Let $\alpha \in P(X,Y;\theta)$. Then α is a regular element if and only if

i) $\mathrm{dom}(\alpha) = \mathrm{dom}(\alpha\theta)$ and $\mathrm{im}(\alpha) = \mathrm{im}(\alpha\theta)$,

ii) $\alpha \circ \alpha^{-1} = \alpha\theta \circ (\alpha\theta)^{-1}$.

Proof. Mendes-Gonçalves and Sullivan (2013), Theorem 2.2. ■

We can made similar comments for the semigroup $T_k(X,Y;\theta)$. However, if $\alpha \in I(X,Y;\theta)$ then α is a regular element of $I(X,Y;\theta)$ if and only if the first condition of last theorem holds. Furthermore, Sullivan and his team gave a special description of Green's relations for the regular elements of $P(X,Y;\theta)$, $\mathcal{G}T(X,Y;\theta)$ and $I(X,Y;\theta)$ as follows.

Theorem 5.2.3. For $\alpha, \beta \in P(X,Y;\theta)$, $\mathcal{G}T(X,Y;\theta)$ or $I(X,Y;\theta)$, suppose α is a regular element then

i) $\alpha\mathcal{L}\beta \Leftrightarrow \beta$ is regular and $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$,

ii) $\alpha\mathcal{R}\beta \Leftrightarrow \beta$ is regular and $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ and $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$,

iii) $\alpha\mathcal{D}\beta \Leftrightarrow \beta$ is regular and $|\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$.

Proof. Mendes-Gonçalves and Sullivan (2013), Corollary 2.1. ■

Concerning with $\mathcal{J}$, it is well known that $\mathcal{J} = \mathcal{D}$ for the regular semigroup P(X), but this is not always the case here. Indeed, $\mathcal{J} = \mathcal{D}$ just for the regular elements of $P(X, Y; \theta)$ with finite heights.

Definition 5.2.4. Let S be a semigroup. Then S is called

i) *Unit-regular* if identity map $i_d \in S$ and $\forall a \in S$, $\exists u \in S$ such that $a = aua$.

ii) *Completely regular* if $\forall a \in S$, $\exists x \in S$ such that $a = axa$ and $ax = xa$.

Next theorems, the authors also gave necessary and sufficient conditions to prove that $P(X, Y; \theta)$ are both unit-regular and completely regular semigroup.

Theorem 5.2.5. $P(X, Y; \theta)$ is unit-regular if and only if one of the following statements holds

i) X or Y is an empty set,

ii) X is finite, $\mathrm{dom}(\theta) = Y$ and θ is a bijective map.

Proof. Mendes-Gonçalves and Sullivan (2013), Theorem 2.3. ■

Theorem 5.2.6. $P(X, Y; \theta)$ is completely regular if and only if $|\mathrm{X}| = 1$ and $\mathrm{dom}(\theta) = Y$.

Proof. Mendes-Gonçalves and Sullivan (2013), Theorem 2.4. ■

Theorem 5.2.7. $\mathcal{GT}(X, Y; \theta)$ is completely regular if and only if one of the following statements holds

i) $|X| = 1$ or $|Y| = 1$,

ii) $|X| \leq 2$ and θ is a bijective map.

Proof. Mendes-Gonçalves and Sullivan (2013), Theorem 2.5. ■

5.3. Embedding and Isomorphism Theorems

This section is devoted to consider some embedding and isomorphism theorems for $P(X,Y;\theta)$ and $\mathcal{G}T(X,Y;\theta)$ by presenting some efforts of some authors like Magill, Symons and Sullivan in showing these theorems. Next chapter, we will present our contributions in the same direction but with other semigroups.

We start with Sullivan and his work, in Sullivan (1975), they defined two embedding maps. The first map embeds $P(X,Y;\theta)$ in $(T_{X\cup a},\theta_1)$ when $|X|\geq|Y|$, and the second embeds $P(X,Y;\theta)$ in $(T_{Y\cup a},\theta_2)$ otherwise, where $a\notin X\cup Y$ and $\theta_1\in T_{X\cup a}$, $\theta_2\in T_{Y\cup a}$ and $a\theta_i=a$, for all i =1, 2.

Definition 5.3.1. For $|X|\geq|Y|$ let λ_1 be an injective map of Y into X and let $a\ \notin X\cup Y$. Then for any $\alpha\in P(X,Y;\theta)$ define the maps α_1 and θ_1 in $(T_{X\cup a})$ by

$$x\alpha_1=\begin{cases} x\alpha\lambda_1 , & x\in \mathrm{dom}(\theta) \\ a , & \text{elsewhere} \end{cases},$$

$$x\theta_1=\begin{cases} x{\lambda_1}^{-1}\theta , & x\in (\mathrm{dom}(\theta))\,\lambda_1 \\ a , & \text{elsewhere} \end{cases}.$$

Definition 5.3.2. For $|X|<|Y|$ let λ_2 be an injective map of X into $Y$and let $a\ \notin X\cup Y$. Then for any $\alpha\in P(X,Y;\theta)$ define the maps α_2 and θ_2 in $T_{Y\cup a}$ by

$$y\alpha_2=\begin{cases} y\,{\lambda_2}^{-1}\alpha , & y\in (\mathrm{dom}(\theta))\,\lambda_2 \\ a , & \text{elsewhere} \end{cases},$$

$$y\theta_2=\begin{cases} y\theta\lambda_2 , & y\in \mathrm{dom}(\theta) \\ a , & \text{elsewhere} \end{cases}.$$

Using these definitions, we have the following isomorphism theorem for $P(X,Y;\theta)$.

Theorem 5.3.3. For any $\alpha \in P(X,Y;\theta)$

i) If $|X| \geq |Y|$ then $P(X,Y;\theta) \cong P(X,Y;\theta_1)$ and the isomorphism maps α onto α_1.

ii) If $|X| < |Y|$ then $P(X,Y;\theta) \cong P(X,Y;\theta_2)$ and the isomorphism maps α onto α_2.

Proof. Sullivan (1975), Theorem 2.1. ■

After that, Symons proved the following isomorphism theorems, see Symons (1975a).

Theorem 5.3.4. If $\mathcal{G}T(X,Y;\theta_1) \cong \mathcal{G}T(X,Y;\theta_2)$ then the isomorphism maps each $\ell-$ (r $-$,$h-$ and $d-$) class of $\mathcal{G}T(X,Y;\theta_1)$ onto a corresponding $\ell-$ (r $-$,$h-$ and $d-$) class of $\mathcal{G}T(X,Y;\theta_2)$.

Proof. Symons (1975a), Theorem 2.1. ■

Theorem 5.3.5. Let $\varphi: \mathcal{G}T(X,Y;\theta_1) \to \mathcal{G}T(X,Y;\theta_2)$ be an isomorphism and define the following maps

$$\rho : \mathcal{G}T((X,Y)\theta_1,\circ) \to \mathcal{G}T((X,Y)\theta_2,\circ) ,$$

and

$$\lambda : \mathcal{G}T(\theta_1(X,Y),\circ) \to \mathcal{G}T(\theta_2(X,Y),\circ) ,$$

by $(\alpha\theta_1)\rho = \alpha\varphi\theta_2$ and $(\theta_1\alpha)\lambda = \theta_2\alpha\varphi$, respectively. Then ρ and λ are isomorphisms.

Proof. Symons (1975a), Theorem 2.2. ■

Theorem 5.3.6. $\mathcal{G}T(X,Y;\theta_1) \cong \mathcal{G}T(X,Y;\theta_2)$ if and only if $\theta_1 h = g\theta_2$ where $g \in \mathcal{G}(Y)$ and $h \in \mathcal{G}(X)$.

Proof. Symons (1975a), Theorem 2.5. ■

However, Magill has proved the last results under the condition that θ is surjective, see Magill (1967). For more information about generalized transformation semigroups see,for example, Ittharat and Sullivan (2005), Sullivan (2012), Kemprasit and Jaidee (2005), Dolinka, Durev, East, Honyam, Sangkhanan, Sanwong and Sommanee (2018), East (2020).

6. ON THE RANK OF GENERALIZED TRANSFORMATION SEMIGROUPS

The generalized transformation semigroup $\mathcal{GT}(X,Y;\theta)$ and *the semigroup of transformation with restricted range* $T(X,\ Y)$, which we introduced in last two chapters, have been extensively studied over the last years. Many of algebraic questions were answered over there such as regularity, Green's relations, isomorphism and embedding theorems, generating systems, ranks and some others. One of the important questions which is related to both algebra and combinatorics is: Determining the rank of $\mathcal{GT}(X,Y;\theta)$ by considering different situations for θ. This was the motivation for us to do more investigations.

In this chapter, we give our contributions in this direction. First, we show that $\mathcal{GT}(X,Y;\theta)$ can be embedded in $T(Z,Y) = \{\alpha \in T(Z): Z\alpha \subseteq Y\}$, where $Z = X \cup Y$. We also show, if $\theta: Y \to X$ is one to one, then $\mathcal{GT}(X,Y;\theta)$ and $T(X, \mathrm{im}(\theta))$ are isomorphic semigroups. Moreover, if $|X| = m$, $|Y| = n$, and $|\mathrm{im}(\theta)| = r$ where $m, n, r \in \mathbb{N}$ then we show that:

$$\mathrm{rank}(\mathcal{GT}(X,Y;\theta)) = \begin{cases} \displaystyle\sum_{k=r+1}^{\min(m,n)} \frac{n!\,S(m,k)}{(n-k)!} & \text{if } \theta \text{ is neither onto nor } 1-1 \\ \dbinom{n}{m} & \text{if } \theta \text{ is onto but not } 1-1 \\ S(m,r) & \text{if } \theta \text{ is } 1-1 \text{ and } 1 \le r \le m-1 \end{cases}.$$

This chapter involving three sections titled by

i) Isomorphism and Embedding theorems,

ii) Regularity and Green's Relations,

iii) Generating sets and Ranks.

6.1. Isomorphism and Embedding Theorems

For non-empty (disjoint) sets X and Y, let θ be any fixed transformation from Y to X. In this section, we investigate some connections between the semigroup of generalized transformations $\mathcal{G}T(X,Y;\theta)$ and the semigroup of transformations with restricted range $T(Z,Y)$ where $Z = X \cup Y$. With these notations, we have the following lemma.

Lemma 6.1.1. $\mathcal{G}T(X,Y;\theta)$ can be embedded in $T(X \cup Y, Y)$.

Proof. Let $Z = X \cup Y$. For each $\alpha \in \mathcal{G}T(X,Y;\theta)$, consider the transformation $\hat{\alpha} \in T(Z,Y)$ defined by

$$x\hat{\alpha} = \begin{cases} x\alpha, & \text{if} \quad x \in X \\ (x\theta)\alpha, & \text{if} \quad x \in Y \end{cases},$$

for any $\alpha, \beta \in \mathcal{G}T(X,Y;\theta)$, first we show that

$$\widehat{\alpha * \beta} = \hat{\alpha} \circ \hat{\beta}$$

for any $x \in Z$, if $x \in X$, then we have

$$x(\widehat{\alpha * \beta}) = x(\widehat{\alpha \circ \theta \circ \beta}) = \big((x\alpha)\theta\big)\beta = (x\alpha)\hat{\beta} = (x\hat{\alpha})\hat{\beta} = x(\hat{\alpha} \circ \hat{\beta})\,.$$

Suppose that $x \in Y$. Since $x\hat{\alpha} = (x\theta)\alpha \in Y$, it follows that

$$x(\widehat{\alpha * \beta}) = x(\widehat{\alpha \circ \theta \circ \beta}) = (x\theta)(\alpha \circ \theta \circ \beta) = (((x\theta)\alpha)\theta)\beta = ((x\hat{\alpha})\theta)\beta$$
$$= (x\hat{\alpha})\hat{\beta} = x\big(\hat{\alpha} \circ \hat{\beta}\big)\,.$$

Hence, $\widehat{\alpha * \beta} = \hat{\alpha} \circ \hat{\beta}$.

Now consider the map $\psi: \mathcal{G}T(X,Y;\theta) \to T(Z,Y)$ defined by $\alpha\psi = \hat{\alpha}$ for each $\alpha \in \mathcal{G}T(X,Y;\theta)$. For $\alpha,\beta \in \mathcal{G}T(X,Y;\theta)$, if $\alpha\psi = \beta\psi$, then $x\hat{\alpha} = x\hat{\beta}$ for all $x \in Z$, in particular, $x\hat{\alpha} = x\alpha = x\beta = x\hat{\beta}$ for all $x \in X$ and so $\alpha = \beta$. Thus ψ is one to one. Moreover, for any $\alpha,\beta \in \mathcal{G}T(X,Y;\theta)$, we have $(\alpha * \beta)\psi = \widehat{\alpha * \beta} = \hat{\alpha} \circ \hat{\beta} = (\alpha)\psi \circ (\beta)\psi$. Therefore, ψ is a one to one homomorphism, as required. ■

We suppose that $|X| \geq |Y|$ for the rest of this section.

Theorem 6.1.2. If $\theta: Y \to X$ is one to one, then $\mathcal{G}T(X,Y;\theta)$ is isomorphic to $T(X, \mathrm{im}(\theta))$. In particular, if θ is both one to one and onto, then $\mathcal{G}T(X,Y;\theta)$ is isomorphic to $T(X)$.

Proof. Let $\phi : \mathcal{G}T(X,Y;\theta) \to T(X, \mathrm{im}(\theta))$ be defined by $\alpha\phi = \alpha \circ \theta$ for each $\alpha \in \mathcal{G}T(X,Y;\theta)$. Then clearly ϕ is well defined and one to one, since θ is one to one. Moreover, for all $\alpha,\beta \in \mathcal{G}T(X,Y;\theta)$ we have

$$\begin{aligned}(\alpha * \beta)\phi &= (\alpha * \beta) \circ \theta = (\alpha \circ \theta \circ \beta) \circ \theta \\ &= ((\alpha \circ \theta)\beta)\theta = (\alpha \circ \theta) \circ (\beta \circ \theta) \\ &= (\alpha\phi) \circ (\beta\phi)\end{aligned}$$

and so ϕ is a homomorphism.

Now consider the map $\breve{\theta}: Y \to \mathrm{im}(\theta)$ defined by $y\breve{\theta} = y\theta$, for each $y \in Y$. It is clear that $\breve{\theta}$ is one to one and onto. For each $\beta \in T(X, \mathrm{im}(\theta))$, we have $\beta \circ \breve{\theta}^{-1} \in \mathcal{G}T(X,Y)$ and

$$(\beta \circ \breve{\theta}^{-1})\phi = \beta \circ \breve{\theta}^{-1} \circ \theta = \beta \circ 1_{\mathrm{im}(\theta)} = \beta \ ,$$

where $1_{\mathrm{im}(\theta)}$ is the identity map on $\mathrm{im}(\theta)$. Therefore, ϕ is onto, and so an isomorphism, as required. Moreover, if $\theta: Y \to X$ is bijective, then $\mathrm{im}(\theta) = X$, and so $\mathcal{GT}(X, Y; \theta)$ is isomorphic to $T(X, X) = T(X)$. ■

Definition 6.1.3 For a semigroup S, let a be a fixed element of S. With the (sandwich) operation $\star_a$ defined by $x \star_a y = xay$ for all $x, y \in S$, S is a semigroup. This semigroup is called the *variant* of S with respect to a and denoted by S^a.

For every $x \in \mathrm{im}(\theta)$, consider the subset $x\theta^{-1}$ of Y. From the axiom of choice, we fix an element $\mathrm{y}_x \in x\theta^{-1}$ and then, let $\mathrm{Y}_1 = \{ \mathrm{y}_x : x \in \mathrm{im}(\theta)\}$. Since $|X| \geq |Y|$ and $|\mathrm{Y}_1| = |\mathrm{im}(\theta)|$, it follows that $|X \backslash \mathrm{im}(\theta)| \geq |Y \backslash \mathrm{Y}_1|$, and so there exists a one to one mapping $\rho_1 : Y \backslash \mathrm{Y}_1 \to X \backslash \mathrm{im}(\theta)$. Consider the map $\rho: Y \to X$ defined by

$$y\rho = \begin{cases} x & \text{if } \mathrm{y} = \mathrm{y}_x \text{ for } \mathrm{x} \in \mathrm{im}(\theta) \\ \mathrm{y}\rho_1 & \text{otherwise} \end{cases} .$$

for each $y \in Y$. Then it is clear that $\rho: Y \to X$ is a one to one mapping with $\mathrm{im}(\theta) \subseteq \mathrm{im}(\rho)$. For each $\alpha \in \mathcal{GT}(X, Y; \theta)$, consider the transformation $\tilde{\alpha} \in T(X)$ defined by

$$x\tilde{\alpha} = (x\alpha)\rho ,$$

for each $x \in X$. In fact, $\tilde{\alpha} \in T(X, \mathrm{im}(\rho))$. Moreover, for any $\theta \in \mathcal{GT}(Y, X)$, first we fix an element $x_0 \in \mathrm{im}(\theta)$, and then, we consider the transformation $\Theta \in T(X)$ defined by

$$x\Theta = \begin{cases} (x\rho^{-1})\theta & \text{if } \mathrm{x} \in \mathrm{im}(\rho) \\ x_0 & \text{otherwise} \end{cases} ,$$

for each $x \in X$. Now it is clear that $\Theta \in T(X, \mathrm{im}(\rho))$ since $\mathrm{im}(\theta) \subseteq \mathrm{im}(\rho)$. With these notations, we have the following result.

Theorem 6.1.4. If $|X| \geq |Y|$ then $\mathcal{G}T(X, Y; \theta)$ isomorphic to $T(X, \mathrm{im}(\rho))^{\Theta}$, the variant of $T(X, \mathrm{im}(\rho))$ with respect to Θ.

Proof. For any α,$\beta \in \mathcal{G}T(X, Y; \theta)$, first we show that

$$\widetilde{\alpha * \beta} = \tilde{\alpha} \star_{\Theta} \tilde{\beta} \,,$$

since $|X| \geq |Y|$ and $\rho : Y \to X$ is one to one such that $\mathrm{im}(\theta) \subseteq \mathrm{im}(\rho)$, we have that ρ^{-1} is a bijection from $\mathrm{im}(\rho)$ to Y and $\rho \circ \rho^{-1} = 1_Y$. For each $x \in X$, since $(x\alpha)\theta \in X$, $\big((x\alpha)\rho\big)\rho^{-1} = x\alpha$, and $(x\alpha)\rho \in \mathrm{im}(\rho)$, it follows that

$$\begin{aligned} x\big(\widetilde{\alpha * \beta}\big) &= (x(\alpha * \beta))\rho = x(\alpha \circ \theta \circ \beta)\rho = (((x\alpha)\theta)\beta)\rho = ((x\alpha)\theta)\tilde{\beta} \\ &= \Big(\Big(\big((x\alpha)\rho\big)\rho^{-1}\Big)\,\theta\Big)\,\tilde{\beta} = \Big(\big((x\alpha)\rho\big)\Theta\Big)\,\tilde{\beta} = \big((x\tilde{\alpha})\Theta\big)\tilde{\beta} = x\big(\tilde{\alpha} \circ \Theta \circ \tilde{\beta}\big) \\ &= x\big(\tilde{\alpha} \star_{\Theta} \tilde{\beta}\big) \,, \end{aligned}$$

as required.

Now consider the map $\varphi : \mathcal{G}T(X, Y; \theta) \to T(X, \mathrm{im}(\rho))^{\Theta}$ defined by $\alpha\varphi = \tilde{\alpha}$ for each $\alpha \in \mathcal{G}T(X, Y; \theta)$. It is clear that φ is well defined. For any $\alpha, \beta \in \mathcal{G}T(X, Y; \theta)$, since ρ is one to one, it follows that $(x\alpha)\rho = (x\beta)\rho$ implies $x\alpha = x\beta$ for each $x \in X$, and so $\alpha = \beta$. Moreover, we have $(\alpha * \beta)\varphi = \widetilde{\alpha * \beta} = \tilde{\alpha} \star_{\Theta} \tilde{\beta} = (\alpha)\varphi \star_{\Theta} (\beta)\varphi$. Therefore, φ is a one to one homomorphism.

Let $\beta \in T(X, \mathrm{im}(\rho))^{\Theta}$. Define $\breve{\beta}: X \to Y$ defined by $(x\beta^{-1})\breve{\beta} = x\rho^{-1}$ for each $x \in \mathrm{im}(\beta)$. Notice that $\{x\beta^{-1}: x \in \mathrm{im}(\beta)\}$ is a partition of X. Moreover, for each $x \in \mathrm{im}(\beta)$, since $\mathrm{im}(\beta) \subseteq \mathrm{im}(\rho)$ and ρ is one to one, there exists a unique element $y \in Y$ such that $x\rho^{-1} = y$. Now it is clear that $\breve{\beta} \in \mathcal{G}T(X, Y; \theta)$. For each $x \in \mathrm{im}(\beta)$, since

$$(x\beta^{-1})(\breve{\beta}\varphi) = (x\beta^{-1})\breve{\breve{\beta}} = \left((x\beta^{-1})\breve{\beta}\right)\rho = (x\rho^{-1})\rho = x = (x\beta^{-1})\beta \ ,$$

it follows that $\breve{\beta}\varphi = \beta$. Hence, φ is surjective, and so φ is an isomorphism. ■

6.2. Regularity and Green's Relations

In this section, we give a necessary and sufficient condition for $T(Z, \mathrm{im}(\theta))$, which we have mentioned in the previous section, to be regular. Consequently, we have a description of Green's relations for that. First recall that the description of Green's relations on $T(X, Y)$ was depending on the largest regular subsemigroup of it, namely $F(X, Y)$. Hence, to describe Green's relation on $T(Z, \mathrm{im}(\theta))$ and study the regularity, we need to know the relationship between the two subsemigroups $F(Z, Y)$ and $T(Z, \mathrm{im}(\theta))$.

It is easy to show that $T(Z, \mathrm{im}(\theta)) \nsubseteq F(Z, Y)$ in general, because we can easily find an example of an element $\hat{\alpha} \in T(Z, \mathrm{im}(\theta))$ which does not satisfy the condition of $F(Z, Y)$ (that is $Z\hat{\alpha} = Y\hat{\alpha}$). However, in this section, we give a necessary and sufficient condition for $T(Z, \mathrm{im}(\theta))$ that solving the problem.

First, we need some notations. Namely, if $\alpha \in \mathcal{G}T(X, Y; \theta)$ and $\theta : Y \to X$ then $\hat{\alpha} \in T(Z,\mathrm{Y})$ where $Z = X \cup Y$. So, we write:

$$\alpha = \begin{pmatrix} X_1 & X_2 & \cdots & X_r \\ y_1 & y_2 & \cdots & y_r \end{pmatrix}$$

$$\theta = \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_s \\ x_1 & x_2 & \cdots & x_s \end{pmatrix}$$

where $X_i = y_i\alpha^{-1}$, for each $1 \le i \le r$. $Y_j = x_j\theta^{-1}$, for each $1 \le j \le s$ and

$$\hat{\alpha} = \begin{pmatrix} X_1 & X_2 & \cdots & X_r & Y_1 & Y_2 & \cdots & Y_s \\ y_1 & y_2 & \cdots & y_r & x_1\alpha & x_2\alpha & \cdots & x_s\alpha \end{pmatrix}$$

Theorem 6.2.1. $T(Z, \operatorname{im}(\theta))$ is regular if and only if for all $\hat{\alpha} \in T(Z, \operatorname{im}(\theta))$, the following condition are satisfied

i) $s \geq r$,

ii) For every $1 \leq i \leq r$; there exists at least one $1 \leq j \leq s$ such that $X_j\alpha = y_i$ or equivalently $X_j \in X_i$.

Proof. $X\hat{\alpha} = Y\hat{\alpha}$ if and only if the conditions are satisfied. ■

Thus, by providing this condition we get $T(X, \operatorname{im}(\theta)) \subseteq F(X, Y)$, and so $T(X, \operatorname{im}(\theta))$ is a regular semigroup because $F(X, Y)$ is the largest regular subsemigroup of $T(X, Y)$. Therefore, we immediately have the following corollary.

Corollary 6.2.2. For $\widehat{\alpha}, \hat{\beta} \in T(X, \operatorname{im}(\theta))$ we have

i) $\hat{\alpha}\mathcal{L}\hat{\beta}$ if and only if $X\hat{\alpha} = X\hat{\beta}$,

ii) $\hat{\alpha}\mathcal{R}\widehat{\beta}$ if and only if $\ker(\hat{\alpha}) = \ker(\hat{\beta})$,

iii) $\hat{\alpha}\mathcal{D}\hat{\beta}$ if and only if $|X\hat{\alpha}| = |X\hat{\beta}|$,

iv) $\mathcal{J} = \mathcal{D}$.

We end this section by providing an example that applying the last theorem:

Example 6.2.3. Let $X = \{1, 2, 3, 4\}$ and $Y = \{5, 6, 7, 8, 9\}$ and define

$$\alpha = \begin{pmatrix} \{1, 4\} & \{2\} & \{3\} \\ 5 & 6 & 7 \end{pmatrix}$$

$$\theta = \begin{pmatrix} \{5, 9\} & \{6, 8\} & \{7\} \\ 1 & 2 & 3 \end{pmatrix}$$

Then

$$\hat{\alpha} = \begin{pmatrix} 1\,2\,3\,4 & 5\,6\,7\,8\,9 \\ 5\,6\,7\,5 & 5\,6\,7\,6\,5 \end{pmatrix}$$

It is clear that $X\hat{\alpha} = Y\hat{\alpha}$.

6.3. Generating Sets and Ranks

For a non-empty subset T of a semigroup S, let $T^2 = \{ st : s, t \in T \}$. For any non-empty (disjoint) sets X and Y, suppose that $\theta : Y \to X$ has a finite height, that is the cardinality of $\mathrm{im}(\theta)$ is finite. From now on we fix $S = \mathcal{G}T(X, Y; \theta)$ and $|\mathrm{im}(\theta)| = r \in \mathbb{N}$. With the above notations, we have the following lemma.

Lemma 6.3.1. $S^2 = \{ \alpha \in S: |\mathrm{im}(\alpha)| \leq r\}$

Proof. Suppose that $\mathrm{im}(\theta) = \{x_1,..., x_r\}$, and $T = \{ \alpha \in S: |\mathrm{im}(\alpha)| \leq r\}$. Then it is clear that for each $\alpha, \beta \in S$, $|\mathrm{im}(\alpha * \beta)| = |\mathrm{im}(\alpha \circ \theta \circ \beta)| \leq |\mathrm{im}(\theta)| \leq r$ and so $S^2 \subseteq T$.

For any $\alpha \in T$, let $\mathrm{im}(\alpha) = \{y_1,...,y_s\}$ $(1 \leq s \leq r)$, and let $A_i = y_i\alpha^{-1}$ for each $1 \leq i \leq s$. Then we choose a unique element $z_i \in x_i\theta^{-1}$ for each $1 \leq i \leq s$, and we consider the transformations

$$\beta = \begin{pmatrix} A_1 & \cdots & A_{s-1} & A_s \\ z_1 & \cdots & z_{s-1} & z_s \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} x_1 & \cdots & x_{s-1} & B \\ y_1 & \cdots & y_{s-1} & y_s \end{pmatrix},$$

where $B = X \backslash \{x_1, \dots, x_{s-1}\}$. Then it is clear that $\beta, \gamma \in S$. Since $x_s \in B$, it follows that $\alpha = \beta * \gamma \in S^2$, and so $T \subseteq S^2$. Therefore, $S^2 = T$ as required. ■

For each $1 \leq s \leq r + 1$, let

$$D_s = \{ \alpha \in S: |\mathrm{im}(\alpha)| = s\} .$$

Now with the above notation, we have the following result:

Proposition 6.3.2. With the above notations we have:
i) If θ is not one to one, then $D_s \subseteq D^2{}_{s+1}$ for every $1 \leq s \leq r-1$.
ii) If θ is neither one to one nor onto, then $D_r \subseteq D^2{}_{r+1}$.
Proof. Let $\mathrm{im}(\theta) = \{x_1,...,x_r\}$ and $B_i = x_i\theta^{-1} \subseteq Y$ for each $1 \leq i \leq r$. Since θ is not one to one, there exists at least one set B_i such that $|B_i| \geq 2$, and so $|Y| \geq r+1$. Without loss of generality, suppose that $|B_1| \geq 2$. Then we fix two different elements z_0 and z_1 in B_1, and an element z_i in B_i for each $2 \leq i \leq r$.

i) For $1 \leq s \leq r-1$, let $\alpha \in D_s$ with $\mathrm{im}(\alpha) = \{y_1,...,y_s\}$, and let $A_i = y_i\alpha^{-1}$ for each $1 \leq i \leq s$. If $|A_i| = 1$ for all $1 \leq i \leq s$, then $|X| = |\bigcup_{i=1}^{s} A_i| = s$. It follows that $s \leq r-1 < r = |im(\theta)| \leq |X| = s$, a contradiction. Hence $|A_i| \geq 2$ for some $1 \leq i \leq s$. We may suppose that $|A_1| \geq 2$. Then we fix an element a_0 in A_1, and we consider the following transformations

$$\beta_s = \begin{pmatrix} \{a_0\} & A_1\backslash\{a_0\} & A_2 & \cdots & A_s \\ z_0 & z_1 & z_2 & \cdots & z_s \end{pmatrix} \quad \text{and} \quad \gamma_s = \begin{pmatrix} x_1 & \cdots & x_s & C \\ y_1 & \cdots & y_s & y_{s+1} \end{pmatrix}$$

where $y_{s+1} \in Y\backslash\{y_1, \dots, y_s\} \neq \emptyset$ and $C_s = X\backslash\{x_1, \dots, x_s\} \neq \emptyset$ since $s \leq r-1$. Now it is clear that $\beta_s, \gamma_s \in D_{s+1}$, and that $\alpha = \beta_s\theta\gamma_s = \beta_s * \gamma_s \in D^2{}_{s+1}$.

ii) Suppose that θ is neither one to one nor onto. For any $\alpha \in D_r$, let $\mathrm{im}(\alpha) = \{y_1,...,y_r\}$. Since $|\mathrm{im}(\theta)| = r$ is finite and $\mathrm{im}(\theta) \neq X$, there exists $1 \leq i \leq r$ such that $|y_i\alpha^{-1}| \geq 2$. Notice that $|Y| \geq r+1$, and so $Y\backslash\{y_1,...,y_r\} \neq \emptyset$. Since θ is not onto, $C_r = X\backslash\{x_1, \dots, x_r\} \neq \emptyset$. If we replace s by r above, then we see that $\beta_r, \gamma_r \in D_{r+1}$, and that $\beta_r * \gamma_r = \alpha$, as required. ■

Notice that $S\backslash S^2 \neq \emptyset$ if θ is not bijective. Indeed, we have $\gamma_r \in D_{r+1} \subseteq S$, and if $\gamma_r \in S^2$, then by lemma 6.3.1 we get that $r+1 = |\mathrm{im}(\gamma_r)| \leq r$, a contradiction. Hence $\gamma_r \in S\backslash S^2$. For any semigroup S, if A is a generating set of S, then it is clear that A contains $S\backslash S^2$. If $S\backslash S^2 \neq \emptyset$ is a generating set of S, then it is now clear that $S\backslash S^2$ is the unique minimum generating set of S.

Lemma 6.3.3. If θ is neither one to one nor onto, then $S\backslash S^2$ is the minimum generating set of S.

Proof. Let α be any element of S. If $\alpha \in S^2$, then $|\mathrm{im}(\alpha)| \leq r$ by Lemma 6.3.1. So $\alpha \in D_s$ for some $1 \leq s \leq r$, and it follows from Proposition 6.3.2. that $\alpha \in {D^2}_{s+1}$. Thus, we can write $\alpha = \lambda * \lambda'$ where $\lambda, \lambda' \in D_{s+1}$. We continue in this process to obtain $\alpha = \lambda_1 * \lambda_2 * \dots * \lambda_t$ where $\lambda_1, \lambda_2, \dots, \lambda_t \in D_{r+1}$, that is $\lambda_i \notin S^2$ for all $1 \leq i \leq t$. Hence $\langle S\backslash S^2 \rangle = S$. From the above fact, we conclude that $S\backslash S^2$ is the minimum generating set. ■

For $m, n, r \in \mathbb{N}$, we also fix $|X| = m$ and $|Y| = n$ for the rest of the section, and so $m \geq |\mathrm{im}(\theta)| = r$.

Theorem 6.3.4. If θ is neither onto nor one to one, i.e. $r < m$ and $r < n$, then

$$\mathrm{rank}(\mathcal{GT}(X, Y; \theta)) = \sum_{k=r+1}^{\min(m,n)} \frac{n!\, S(m,k)}{(n-k)!}$$

where $S(m,k)$ is *the Stirling number of the second kind.*

Proof. First notice that if $m > r$ since θ is not onto, and $n > r$ since θ is not one to one. Moreover, for every $\alpha \in \mathcal{GT}(X, Y; \theta)$, $|\mathrm{im}(\alpha)| \leq \min(m,n)$ since both $|\mathrm{im}(\alpha)| \leq |X|$ and $|\mathrm{im}(\alpha)| \leq |Y|$. From Lemma 6.3.3. it follows that $S\backslash S^2$ is the minimum generating set of S. From Lemma 6.3.1., we have

$$S\backslash S^2 = \{\, \alpha \in S : |\mathrm{im}(\alpha)| \geq r+1 \} = \bigcup_{k=r+1}^{\min(m,n)} \{\, \alpha \in \mathcal{GT}(X, Y; \theta) : |\mathrm{im}(\alpha)| = k \}\,.$$

For $r+1 \leq k \leq n$, let $P_k(Y)$ be the set of all subsets of Y with k elements. Recall that for each $U \in P_k(Y)$, there exist $k!\,S(m,k)$ surjective functions from X to U where $S(m,k)$ is *the Stirling number of the second kind*. Since $|P_k(Y)| = \binom{n}{k}$, we obtain

$$\operatorname{rank}(\mathcal{G}T(X,Y;\theta)) = |S \backslash S^2| = \left| \bigcup_{k=r+1}^{\min(m,n)} \{ \alpha \in \mathcal{G}T(X,Y;\theta) : |\operatorname{im}(\alpha)| = k\} \right|$$

$$= \sum_{k=r+1}^{\min(m,n)} |\{ \alpha \in \mathcal{G}T(X,Y;\theta) : |\operatorname{im}(\alpha)| = k\}|$$

$$= \sum_{k=r+1}^{\min(m,n)} \binom{n}{k} k!\,S(m,k) = \sum_{k=r+1}^{\min(m,n)} \frac{n!\,S(m,k)}{(n-k)!} \;,$$

as required. ■

Suppose that θ is onto, but not one to one. Then $\operatorname{im}(\theta) = X$ and $m = |X| = |\operatorname{im}(\theta)| = r < r+1$. So, we have $D_m = \{ \alpha \in \mathcal{G}T(X,Y;\theta) : |\operatorname{im}(\theta)| = m\}$. If we take $X = \{1, \ldots, m\}$, then we have $\operatorname{im}(\theta) = \{1, \ldots, m\}$ and

$$D_m = \left\{ \begin{pmatrix} 1 & \cdots & m \\ y_1 & \cdots & y_m \end{pmatrix} : \{y_1, \ldots, y_m\} \text{ is a subset of } Y \text{ with } m \text{ elements} \right\},$$

which is the set of all one to one transformations from X to Y. Notice that $|Y| \geq m+1$. Now, for each $1 \leq i \leq m$, let $B_i = i\theta^{-1}$. Thus, $\{B_1, \ldots, B_m\}$ is the set of all kernel classes of θ, which is called the kernel partition of θ, and denoted by $\operatorname{Ker}(\theta)$. Then, for any subset $\{y_1, \ldots, y_m\}$ of Y with m elements, if $|\{y_1, \ldots, y_m\} \cap B_i| = 1$, then $\{y_1, \ldots, y_m\}$ is called a transversal of $\operatorname{Ker}(\theta) = \{B_1, \ldots, B_m\}$. We note that $\operatorname{im}(\alpha) \subseteq Y = \bigcup_{i=1}^{m} B_i$ for all $\alpha \in \mathcal{G}T(X,Y;\theta)$. Now let

$$D_m{}^{\theta} = \{\, \gamma \in D_m \colon \mathrm{im}(\gamma) \text{ is a transversal of } \mathrm{Ker}(\theta)\} \,.$$

With the above notations, we have the following result:

Proposition 6.3.5. For any $\alpha, \beta \in D_m$, the following statements are equivalent

i) $\alpha * \beta \in D_m$,

ii) $\alpha \in D_m{}^{\theta}$,

iii) $\mathrm{im}(\alpha * \beta) = \mathrm{im}(\beta)$.

Proof. i) $\Longrightarrow$ ii) Suppose that $\alpha * \beta \in D_m$. To show $\alpha \in D_m{}^{\theta}$ assume that there are two different elements $i, j \in X = \{1, \dots, m\}$ such that both $i\alpha$ and $j\alpha$ are in the same kernel class of θ, that is $(i\alpha)\theta = (j\alpha)\theta$. Then, for any $\beta \in D_m$, since

$$i(\alpha * \beta) = \big((i\alpha)\theta\big)\beta = \big((j\alpha)\theta\big)\beta = j(\alpha * \beta) \,,$$

it follows that $\alpha * \beta$ is not one to one, and so $\alpha * \beta \notin D_m$ which is a contradiction. Therefore, $\alpha \in D_m{}^{\theta}$, as required.

ii) $\Longrightarrow$ iii) Suppose that $\alpha \in D_m{}^{\theta}$. Since $\mathrm{im}(\alpha)$ is a transversal of $\mathrm{Ker}(\theta)$, and $\mathrm{im}(\theta) = X$, it follows that $\mathrm{im}(\alpha\theta) = X$, and so $\mathrm{im}(\alpha * \beta) = \mathrm{im}(\beta)$.

iii) $\Longrightarrow$ i) It is clear. ∎

Theorem 6.3.6. If θ is onto, but not one to one, i.e. $r = m < n$, then

$$\mathrm{rank}\big(\mathcal{GT}(X, Y; \theta)\big) = \binom{n}{m}.$$

Proof. Since θ is not one to one, there exists $B_j \in \mathrm{Ker}(\theta)$ such that $|B_j| \geq 2$. We may suppose that $|B_1| \geq 2$ and $y_1,\ y_{m+1} \in B_1$ such that $y_1 \neq y_{m+1}$ and $y_i \in B_i$ for each $2 \leq i \leq m$. Also let $\mathcal{P}_m(Y) = \{Y_i \colon 1 \leq i \leq p\}$ be the set of all subsets of Y

with m elements where $p = \binom{n}{m}$. Moreover, for each $1 \leq i \leq p$, we fix an element $\alpha_i \in D_m$ such that $\mathrm{im}(\alpha_i) = Y_i$. In particular, we fix

$$\alpha_1 = \begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ y_2 & y_3 & \cdots & y_m & y_1 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} 1 & 2 & 3 & \cdots & m \\ y_2 & y_{m+1} & y_3 & \cdots & y_m \end{pmatrix},$$

for $\mathrm{Y}_1 = \{y_1, y_2 \dots, y_m\}$ and $\mathrm{Y}_2 = \{y_{m+1}, y_2 \dots, y_m\}$, respectively.

Now we show that the set $A = \{\alpha_i : 1 \leq i \leq r\}$ is a minimal generating set of $\mathcal{GT}(X, Y; \theta)$. First, observe that

$$\alpha_1 \circ \theta = \begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \end{pmatrix} \quad \text{and} \quad \alpha_2 \circ \theta = \begin{pmatrix} 1 & 2 & 3 & \cdots & m \\ 2 & 1 & 3 & \cdots & m \end{pmatrix}.$$

Thus, $\alpha_1 \circ \theta = (12 \dots m)$ is an m-cycle and $\alpha_2 \circ \theta = (12)$ is a transposition in S_m. It is a well-known fact that $\{(12 \dots m), (12)\}$ is a generating set for S_m, that is $S_m = \langle \alpha_1 \circ \theta, \alpha_2 \circ \theta \rangle = \langle (12 \dots m), (12) \rangle$.

First notice that since $|\mathrm{im}(\beta)| \leq X = m$ for every $\beta \in \mathcal{GT}(X, Y; \theta)$, it is enough to consider D_m. For any $\beta \in D_m$, there exists an element $\alpha \in A$ such that $\mathrm{im}(\beta) = \mathrm{im}(\alpha)$. Since $\alpha: X_m \to \mathrm{im}(\beta) = \{i\beta : 1 \leq i \leq m\}$ is a bijection, the map $\lambda: X_m \to X_m$ defined by $i\lambda = (i\beta)\alpha^{-1}$ for every $i \in X_m$ is a permutation in S_m such that $\beta = \lambda\alpha$. Since $\lambda \in \langle \alpha_1 \circ \theta, \alpha_2 \circ \theta \rangle$, there exists $\mathrm{t} \in \mathbb{N}$ such that

$$\lambda = \left(\alpha_{i_1} \circ \theta\right) \circ \left(\alpha_{i_2} \circ \theta\right) \circ \dots \circ \left(\alpha_{i_t} \circ \theta\right),$$

where $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_t} \in \{\alpha_1, \alpha_2\}$. Moreover, since $\beta = \lambda \circ \alpha$, it follows that

$$\begin{aligned} \beta &= \left(\alpha_{i_1} \circ \theta\right) \circ \left(\alpha_{i_2} \circ \theta\right) \circ \dots \circ \left(\alpha_{i_t} \circ \theta\right) \circ \alpha \\ &= \alpha_{i_1} * \alpha_{i_2} * \dots * \alpha_{i_t} * \alpha \quad , \end{aligned}$$

and so $\beta \in \langle A \rangle$. Therefore, it follows from Propositions 6.3.2. (i) and 6.3.5. that A is a minimal generating set for $\mathcal{G}T(X,Y;\theta)$. Moreover, it follows from Proposition 6.3.2. (iii) that $\mathrm{im}(\beta) = \mathrm{im}(\alpha)$. Thus, for any subset U of Y with m elements, there exists $\alpha \in A \subseteq D_m$ such that $\mathrm{im}(\alpha) = U$, and so A is a minimal generating set for $\mathcal{G}T(X,Y;\theta)$. Therefore, $\mathrm{rank}\big(\mathcal{G}T(X,Y;\theta)\big) = |A| = p = \binom{n}{m}$. ■

Finally, if $\theta : Y \to X$ is one to one, it follows from Theorem 6.1.2. that $\mathcal{G}T(X,Y;\theta)$ and $T(X, \mathrm{im}(\theta))$ are isomorphic. In addition, if θ is also onto, then $\mathcal{G}T(X,Y;\theta)$ and $T(X)$ are isomorphic, and so

$$\mathrm{rank}\big(\mathcal{G}T(X,Y;\theta)\big) = \mathrm{rank}(T(X)) = 3\,,$$

for $m \geq 3$. When θ is one to one but not onto we have the following corollary.

Corollary 6.3.7. If θ is one to one but not onto, i.e. $r = n < m$, then

$$\mathrm{rank}\big(\mathcal{G}T(X,Y;\theta)\big) = S(m,r)\,,$$

where $S(m,r)$ is the Stirling number of the second kind. ■

Finally, we have our main results as follows: if $|X| = m, |Y| = n$, and $|\mathrm{im}(\theta)| = r$ where $m,n,r \in \mathbb{N}$ then;

$$\mathrm{rank}\big(\mathcal{G}T(X,Y;\theta)\big) = \begin{cases} \sum_{k=r+1}^{\min(m,n)} \frac{n!S(m,k)}{(n-k)!} & \text{if } \theta \text{ is neither onto nor } 1-1 \\ \binom{n}{m} & \text{if } \theta \text{ is onto but not } 1-1 \\ S(m,r) & \text{if } \theta \text{ is } 1-1 \text{ and } 1 \leq r \leq m-1 \end{cases}.$$

REFERENCES

Al-Kharousi, F., Kehinde, R., Umar, A., 2014. Combinatorial Results for Certain Semigroups of Partial Isometries of a Finite Chain. Australas. J. Combin, 58(3), 365-375.

Ayık, G., Ayık, H., Ünlü, Y., Howie, J.M., 2008. Rank Properties of The Semigroup of Singular Transformations on a Finite Set, Communications in Algebra, 36: 2581-2587.

Ayık, H., Toker, K., 2018. On the Rank of Transformation Semigroup $T(n,m)$. Turkish Journal of Mathematics, 42, 1970-1977.

Bugay, L., Yağcı, M., Ayık, H., 2018. The Ranks of Certain Semigroups of Partial Isometries, Semigroup Forum, 97(2), 214-222.

Catarino, P., Higgins, P.M., 1999. The Monoid of Orientation-Preserving Mappings on a Chain, Semigroup Forum, 58(2), 190–206.

Chinram, R., 2008a. Regularity and Green's Relations of Generalized One-To-One Partial Transformation Semigroups, Far East J. Math. Sci., 30(3). 513–521.

Chinram, R, 2008b. Regularity and Green's Relations of Generalized Partial Transformation Semigroups, Asian-Eur. J. Math., 1(3). 295–302.

Clifford, A.H., Preston, G.B., 1967. The Algebraic Theory of Semigroups. Vol. 1. Mathematical Surveys, Vol. 7, American Mathematical Society, Providence.

Dolinka, I., Durdev, I., East, J., Honyam, P., Sangkhanan, K., Sanwong, J., Sommanee, W., 2018. Sandwich Semigroups in Locally Small Categories II: Transformations., Algebra Universalis, 79(3), Paper No. 76, 53 pp.

East, J., 2020. Transformation Representations of Sandwich Semigroups. Experimental Mathematics, 29(3), 291-295.

Fernandes, V.H., Honyam, P., Quinterio, T.M., Singha, B., 2014. On Semigroups of Endomorphisms of a Chain with Restricted Range, Semigroup Forum, 89 ,77-104.

Fernandes, V.H., Honyam, P., Quinterio, T.M., Singha, B, 2016. On Semigroups of Orientation-preserving Transformations with Restricted Range, Communications in Algebra, 44:1, 253-264.

Fernandes, V.H., Sanwong, J., 2014. On the Ranks of Semigroups of Transformations on a Finite Set with Restricted Range, Algebra Colloq., 21(3), 497- 510.

Ganyushkin, O., Mazorchuk, V., 2009. Classical Finite Transformation Semigroups, Springer-Verlag, London, 314 p.

Garba, G.U., 1990. Idempotents in Partial Transformation Semigroups. Proc. Royal Soc. Edinburgh, 116A:359--366.

Garba, G.U, 1994. On the Idempotent Ranks of Certain Semigroups of Order-preserving Transformations, Portugal. Math., 51: 185--204.

Gomes, G.M.S., Howie, J.M. 1987. On the Ranks of Certain Finite Semigroups of Transformations, Math. Proc. Cambridge Philos. Soc., 101(3):395-403.

Gomes, G.M.S., Howie, J.M, 1992. On the Ranks of Certain Semigroups of Order-Preserving Transformations, Semigroup Forum, 45: 272--282.

Higgins, P.M., 1992. Techniques of Semigroup Theory, Oxford University Press, New York.

Howie, J.M., 1966. The Subsemigroup Generated by The Idempotents of a Full Transformation Semigroup, J. London Math. Soc., 41, 707-716.

Howie, J.M, 1978. Idempotent Generators in Finite Full Transformation Semigroups, Proc. Roy. Soc. Edinburgh Sect. A, 81(3-4):317-323.

Howie, J.M, 1995. Fundamentals of Semigroup Theory. Oxford University Press, New York, 351 p.

Howie, J.M., McFadden, R.B., 1990. Idempotent Rank in Finite Full Transformation Semigroups, Proc. Royal Soc. Edinburgh, 114A:161--167.

Ittharat, J., Sullivan, R.P., 2005. Factorisable Semigroups of Generalized Transformations, Communications in Algebra, 33, 3179-3193.

Kemprasit, Y., Changphas, T., 2000. Regular Order-Preserving Transformation Semigroups, Bull. Aust. Math. Soc., 62(03), 511-524.

Kemprasit, Y., Jaidee, S., 2005. Regularity and Isomorphism Theorems of Generalized Order–Preserving Transformation Semigroups, Vietnam Journal of Mathematics, 33:3. 253–260.

Laysiriku, E., 2016. Semigroups of Full Transformations with Restriction on the Fixed Set is Bijective, Thai Journal of Mathematics Volume 14, Number 2: 497–50.

Levi, I., 2006. Nilpotent Ranks of Semigroups of Partial Transformations, Semigroup Forum, 72: 459-476.

Mal'cev, A., 1952. Symmetric Groupoids. Mat. Sb. N.S., 31(73), 136–151.

Magill, K.D. Jr., 1967. Semigroup Structures for Families of Functions, I. Some Homomorphism Theorems, J. Austral. Math. Soc. 7 (1), 81-94.

Magill, K.D. Jr., Subbiah, S., 1975. Green’s Relations for Regular Elements of Sandwich Semigroups, I. General Results, Proc. London Math. Soc., 31 (2)194-210.

Mc-Alister, D.B., 1998. Semigroups Generated by a Group and an Idempotent, Comm. Algebra, 26:515–547.

Mendes-Gonçalves, S., Sullivan, R.P., 2010. Semigroups of Transformations Restricted by an Equivalence, Cent. Eur. J. Math., 8(6), 1120-1131.

Mendes-Gonçalves, S., Sullivan, R.P, 2013. Regular Elements and Green’s Relations in Generalised Transformation, Asian-European Journal of Mathematics, Vol. 6, No. 1. 1350006 (11p).

Mora, W., Kemprasit, Y., 2010. Regular Elements of Some Order-Preserving Transformation Semigroups. Int. J. Algebra, 4(13-16), 631-641.

Nenthein, S., Youngkhong, P., Kemprasit, Y., 2005. Regular Elements of Some Transformation Semigroups, Pure Math. Appl. ,16 (3) 307-314.

Pookpienlert, C., Honyam, P., Sanwong, J., 2018. Green's Relations on a Semigroup of Transformations with Restricted Range that Preserves an Equivalence Relation and a Cross-Section, Mathematics, 6, 134.

Sangkhanan, K., 2016. Green's Relations on Semigroup of Regressive Transformations with Restricted Range, International Journal of Pure and Applied Mathematics, Volume 108 No. 2, 467-476.

Sangkhanan, K., Sanwong, J., 2020. Regularity and Greens Relations on Semigroups of Transformations with Restricted Range that Preserve an Equivalence, Semigroup Forum, 87(1), 568-584.

Sanwong, J., 2011. The Regular Part of a Semigroup of Transformations with Restricted Range, Semigroup Forum, 83(1), 134-146.

Sanwong, J., Sommanee, W., 2008. Regularity and Green's Relations on a Semigroup of Transformations with Restricted Range, Int. J. Math. Math. Sci., Art. ID 794013, 11 p.

Sommanee, W., Sanwong, J., 2013. Rank and Idempotent Rank of Finite Full Transformation Semigroups with Restricted Range, Semigroup Forum, 87:230–242.

Sommanee, W., Sanwong, J, 2015. Order-Preserving Transformations with Restricted Range: Isomorphism Theorems and Ranks, Algebra Universalis, 74(3-4), 277-291.

Sullivan, R.P., 1975. Generalised Partial Transformation Semigroups, J. Aust. Math. Soc., 19. 470–473.

Sullivan, R.P, 2012. Embedding Theorems for Semigroups of Generalised Linear Transformations, Southeast Asian Bulletin of Mathematics, 36, 547-552.

Sun, L., Sun, J. 2016. A Natural Partial Order on Certain Semigroups of Transformations with Restricted Range, Semigroup Forum, 92(1), 135-141.

Symons, J.S.V., 1975a. On a Generalization of The Transformation Semigroup, J. Austral. Math. Soc., 19 (series A), 47-61.

Symons, J.S.V, 1975b. Some Results Concerning a Transformation Semigroup, J. Aust. Math. Soc. (Series A), 19(4), 413-425.

Tinpun, K., Koppitz, J., 2016a. Relative Rank of the Finite Full Transformations with Restricted Range, Acta Mathematica Universatatis Comenianae, 85(2), 347-356.

Tinpun, K., Koppitz, J., 2016b. Generating Sets of Infinite Full Transformation Semigroups with Restricted Range, Acta Sci. Math., 82, 55–63.

CURRICULUM VITAE

I completed my primary, secondary and high school education in Nablus. In 2012, I completed the Bachelor's degree in mathematics at AN-Naiah University in Nablus. At the same year, I have accepted to a master's degree in pure mathematics at the same university and at the end of this period I published, with my supervisor, my first paper which is in " functional analysis". After finishing my master's degree, I accepted as an international student for a PhD program with a full scholarship from the Presidency for Turks Abroad and Related Communities (YTB) at Çukurova University in Adana-Turkey. I am married and father of two children.