CONTROLLING CHAOS IN GENESIO SYSTEM

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FIRAT ÖZCAN

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Prof. Dr. Nihat ÇELEBI

Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Professor Dr. Zafer ERCAN

Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

Assistant Prof. Ömür Umut

Supervisor

Examining Committee Members

Professor Dr. Bülent Karasözen

Assistant Prof. Dr. Ömür Umut

Assistant Prof. Dr. Ismail U. Tiryaki

To My Family

ABSTRACT

CONTROLLING CHAOS IN GENESIO SYSTEM

Özcan, Fırat

Master, Department of Mathematics

Supervisor: Assistant Prof. Dr. Ömür UMUT

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Many chaotic systems, including Genesio system, have been developed and thoroughly analyzed over the past years. A chaotic system is a nonlinear deterministic system behaving complex and unpredictable. Recent efforts have investigated the controlling of chaos and synchronization.

Chaos control problems consist of attempts to stabilize a chaotic system to either a periodic orbit or an equilibrium. The synchronization problem means making two systems oscillate in a synchronize manner. The dynamical behaviors of two copies of a chaotic system may be identical after a transient when the second system is driven by the first one. Several different approaches have already been successfully applied to this problem.

In this thesis, chaos synchronization of two identical Genesio systems is studied. Three methods of chaos synchronization are applied in this study; Pecora and Carroll, active control and linear generalized synchronization. The sufficient conditions for achieving synchronization of two identical Genesio systems are drived by using Lyapunov stability theory. Numerical simulations are presented

to demonstrate the effectiveness of the proposed chaos synchronization schemes.

Keywords: *Genesio system, chaos control, synchranization, method of Pecora and Carroll, active control, linear generalized synchranization.*

ÖZET

GENESIO SİSTEMDE KAOS KONTROLÜ

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Geçmiş yıllarda Genesio sistemi de dahil olmak üzere birçok kaotik sistem geliştirilmiş ve tamamen analiz edilmiştir. Bir kaotik sistem, kompleks ve tahmin edilemeyen davranış gösteren lineer olmayan deterministik bir sistemdir. Son çalışmalar, kaos ve senkronizasyonunun kontrolünü araştırmıştır.

Kaos kontrol problemleri, kaotik bir sistemi periyodik bir yörüngeye yada dengeye sabitleme çabalarından oluşur. Senkronizasyon problemi, iki sistemi eşzamanlı bir şekilde sallandırma anlamına gelir. Bir kaotik sistemin iki kopyasının dinamik hareketi, ikinci sistem birincisiyle harekete geçirildiğinde bir süreden sonra aynı olabilir. Bu probleme birçok farklı yaklaşımlar başarıyla uygulanmıştır.

Bu tezde, iki eş Genesio sistemin kaos senkronizasyonu üzerinde çalışılmıştır. Bu çalışmada üç kaos senkronizasyon metodu uygulandı; Pecora ve Carroll, aktif kontrol ve lineer genelleştirilmiş senkronizasyon. İki eş Genesio sisteminin senkronizasyonunu elde etmek için yeterli şartlar Lyapunov sabitlik teorisinden faydalanarak elde edilmiştir. Sayısal veriler, önerilen kaos senkronizasyon yöntemlerinin etkinliğini göstermek amacıyla sunulmuştur.

Anahtar Kelimeler: *Genesio sistem, kaos kontrol, senkronizasyon, Pecora ve Carroll metodu, aktif kontrol ve lineer genelleştirilmiş senkronizasyon.*

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CHAPTER 1

INTRODUCTION

The term *control of chaos* is used mostly to denote the area of studies lying at the interfaces between the control theory and the theory of dynamic systems studying the methods of control of deterministic systems with nonregular, chaotic behavior. In the ancient mythology and philosophy, the world "$\chi\alpha\omega\sigma$" (chaos) meant the disordered state of unformed matter supposed to have existed before the ordered universe. The combination "control of chaos" assumes a paradoxical sense arousing additional interest in the subject.

The problems of control of chaos attract attention of the researchers and engineers since Hubler published the first paper on chaos control in 1989, [1]. Several thousand publications have appeared over the recent decade.

It seems that T. Li and J.A. Yorke were the first authors who in 1975 introduced in their paper *Period Three Implies Chaos*, [2] the term "chaos" or, more precisely, "deterministic chaos" which is used widely since that. Various mathematical definitions of chaos are known, but all of them express close characteristics of the dynamic systems that are concerned with "supersensitivity" to the initial conditions: even arbitrarily close trajectories diverge with time at a finite distance, that is, long-term forecasts of trajectories are impossible. At that, each trajectory remained bounded, which contradicts the intuitive understanding of instability based on the experience gained with the linear systems.

Nevertheless, the nonlinear deterministic dynamic systems manifesting such characteristics do exist and are not exceptional, "pathological" cases. It also turned out that the methods describing chaotic behavior occur in many areas of science and technology and sometimes are more suitable for describing nonregular oscillations and indeterminacy than the stochastic, probabilistic methods. It suffices to note that a wide class of the chaotic systems is represented by the well-known pseudorandom-number generators that appeared much in advance of the term ”chaos”.

The chaotic systems represent a class of indeterminacy models differing from the stochastic models.Where as with a knowledge of the current system state the deterministic model can predict the future trajectory for an arbitrarily long period and the stochastic model cannot make a precise forecast, generally speaking, even for an arbitrarily short time, the forecast error of the chaotic model grows exponentially and, consequently, a forecast can be made only for a limited time defined by the admissible forecast error. The processes in the chaotic models have the form of nonregular oscillations where both frequency and amplitude vary or ”float”.

Before the XX. century, the linear differential equations were the main mathematical models of oscillations in the mechanical, electrical, and other systems. Yet at the turn of this century it became clear that the linear oscillation models fail to describe adequately the new physical and engineering phenomena and processes. The fundamentals of a new mathematical apparatus, the theory of nonlinear oscillations, were laid by A. Poincaré, B. Van der Pol, A.A. Andronov, N.M. Krylov and N.N. Bogolyubov. The most important notion of this theory is that of the stable limit cycle.

Even the simplest nonlinear models enable one to describe complex oscillations such as the relaxation, that is, close to rectangular, oscillations, take into account variations in the form of oscillations depending on the initial conditions (systems with several limit cycles), and so on. The linear oscillation models and the nonlinear models with limit cycles satisfied the needs of engineers for several decades. It was believed that they describe all possible types of oscillations of the deterministic systems. This conviction was supported by the mathematical findings. For example, the well-known Poincaré-Bendixson theory asserts that the equilibrium state and the limit cycle are the only possible kinds of limited stable motions in continuous systems of the second order.

However, in the middle of the last century the mathematicians M. Cartwright, J. Littlewood, and S. Smale established that this is not the case already for the systems of the third order: very complex motions such as limited nonperiodic oscillations become possible in the system. In 1963, the physicist E. Lorenz revolutionized the situation by his paper, [3] demonstrating that the qualitative nature of atmospheric turbulence which obeys the Navier-Stokes complex partial differential equations is representable by a simple nonlinear model of the third order equation:

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{1.1}$$

For some values of the parameters (for example, $\sigma = 10$, $r = 97$ and $b = 8/3$), the solutions of system (1.1) look like nonregular oscillations as shown in Figure (1.1).

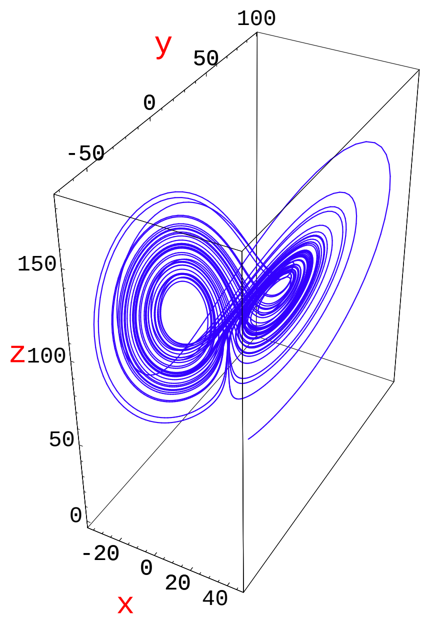

Figure 1.1: Lorenz attractor.

The trajectories in the state (phase) space can approach the limit set (attractor) featuring very sophisticated form. Attention of the physicists and mathematicians and later engineers, was attracted to these models by the work of D. Ruelle and F. Takens, [4] (1971) who called these attractors "strange" and also by the work of Li and Yorke, [2] who introduced the term "chaos" to designate the nonregular phenomena in the deterministic systems. We note that the main result of [2] is a special case of the theorem of a Kievan mathematician A.N. Sharkovrkii that was published in 1964, and the fundamentals of the mathematical apparatus for studying the chaotic phenomena were laid in the 1960's-1970's by the national scientific schools of A.N. Kolmogorov, D.V. Anosov, V.I. Arnol'd, Ya.G. Sinai, V.K. Mel'nikov, Yu.I. Neimark, L.P. Shil'nikov and their collaborators. From this time on, the chaotic behavior was discovered in numerous systems in mechanics, laser and radio physics, chemistry, biology and medicine, electronic circuits, and so on, [5]. The newly developed methods of analytical and numerical study of systems demonstrated that chaos is by no means an exceptional kind of behavior of the nonlinear system. Roughly speaking, chaotic motions arise whenever the system trajectories are globally bounded and locally unstable. In the chaotic system, an arbitrarily small initial divergence of the trajectories does not remain small, but grows exponentially.

The purpose of this thesis is to study on the control of chaos in a chaotic system. The problems of control of chaos are introduced in Chapter 2. In Chapter 3, the Genesio system, [6] is treated as an example and theoretical stability analysis is given. It is also shown that the locking and synchronization of the Genesio system can be achieved by using Pecora Carroll method, [14], active control, [15, 16] and linear generalized synchronization, [17]. The results are verified by numerical simulations.

CHAPTER 2

PROBLEMS OF CONTROL OF CHAOS

The last ten years have seen remarkable developments in the research of chaotic dynamics, particulary with respect to the interaction of chaotic dynamics with other fields of research and with applications. There is now a developed science of chaos that has as an essential underpinning the strong interaction of theory and experiment. This is a departure from earlier times in which theoretical work existed largely in the absence of substantial experimental realization. Along with this new orientation has come increased appreciation and concern for the implication of chaotic dynamics in practical applications.

The presence of chaos in physical systems has been extensively demonstrated and is very common. The main property of chaotic dynamics is its critical sensitivity to initial conditions, which is responsible for initially neighboring trajectories separating from each other exponentially in the course of time. For many years, this feature made chaos undesirable, in so far as the sensitivity to initial conditions of chaotic system reduces their predictability over long time scales. On the other hand, the capability of chaotic dynamics to amplify small perturbations improve their utility for reaching specific desired states with very high flexibility and low energy cost. Indeed, large alternations to achieve a desired behavior are in general experimentally unpractical, nor can a typical system suffer them without substantially changing its main dynamical properties. In contrast, the process of controlling chaos is directed to improving a desired behavior by making only small time-dependent perturbation in an accessible system parameter or dynamical variable. The key observation is that a chaotic attractor typically has embedded within it an infinite, an in fact dense, set of unstable periodic orbits. This means that the system is able to realize an infinite number of possible different kinds of periodic motion apart from chaotic motion. The theoretical and experimental work done on this topic shows that the various methods of control are capable of yielding improved performance in a wide variety of situations.

Controlled chaos is also useful for communications. The naturally occuring chaotic orbits on the attractor can be utilized to carry information by making the symbolic dynamics of the experimental system follow a prescribed symbol sequence: thus one can encode and transmit a prescribed message using chaotic waveform. This process may offer practical advantages over the usual periodic carries, such as the possibility of real time reconstructions of signal dropouts in the communication, or the possibility of warranting higher security levels for confidential communication, or even the possibility of multiplexing.

Chaos control, in a broader sense can be divided into two categories: one is to suppress the chaotic dynamical behavior when it is unwanted, and the other is to generate or enhance chaos when it is benefical, where the latter is known as chaotification or anticontrol of chaos, [7].

The mathematical formulation of the well-known problems of controlling the chaotic processes is preceded by presenting the basic models of the chaotic systems that are used in what follows. The most popular mathematical models encountered in the literature on control of chaos are represented by the systems of ordinary differential equations (state equations)

$$\dot{x}(t) = F(x, u) \tag{2.1}$$

where $x = x(t)$ is the n-dimensional vector of the state variables, $u = u(t)$ is the m-dimensional vector of inputs (controls), $\dot{x} = dx/dt$, and the vector function $F(x, u)$ is usually assumed to be continuous. In the presence of external perturbations, the nonstationary model

$$\dot{x} = F(x, u, t) \tag{2.2}$$

is used. In many cases, a simpler, *affine control* model

$$\dot{x} = f(x) + g(x)u \tag{2.3}$$

may be used.

The measured system output is denoted by $y(t)$. It can be defined as a function of the current system state;

$$y(t) = h(x(t)). \tag{2.4}$$

If the outputs variables are not given explicitly, then we can assume that the entire state vector is observable, that is $y(t) \cong x(t)$.

Now, we proceed immediately to formulating the problems of control of chaotic processes.

1) Suppressing the Choatic Dynamical Behaviour

The problems of stabilization of the unstable periodic solution (orbit) arise in suppression of noise and vibrations of various constructions, elimination of harmonics in the communication systems, electronic devices and so on. These problems are distinguished for the fact that the controlled plant is strongly oscillatory, that is, the eigenvalues of the matrix of the linearized system are close to the imaginary axis. The harmful vibrations can be either regular (quasiperiodic) or chaotic. The problems of suppressing the chaotic oscillations by reducing them to the regular oscillations or suppressing them completely can be formalized as follows:

Let us consider a free (uncontrollable), $u(t) \equiv 0$ motion $x_*(t)$ of system (1.1) to with the initial condition $x_*(0) = x_*0$. Let this motion T-periodic that is, $x_*(t+T) = x_*(t)$ be satisfied for all $t \geq 0$. We need to stabilize it, that is, reduce the solutions $x(t)$ add system (1.1) to $x_*(t)$:

$$\lim_{t\to\infty} [x(t) - x_*(t)] = 0 \tag{2.5}$$

or drive the system output $y(t)$ to the given function $y_*(t)$:

$$\lim_{t\to\infty}[y(t)-y_*(t)]=0 \tag{2.6}$$

for any solution $x(t)$ of system (1.1) under the initial state $x(0)=x_0\in\Omega$ where Ω is the given set of initial conditions.

The problem lies in determining the control function either as the open-loop control action

$$u(t)=U(t,x_0) \tag{2.7}$$

or the *state feedback*

$$u(t)=U(x(t)) \tag{2.8}$$

or the *output feedback*

$$u(t)=U(y(t)) \tag{2.9}$$

satisfying the control objective (2.5) or (2.6).

This formulation of the problem of stabilization of periodic motion is undistinguishable from the conventional control-theoretic *tracking problem*. Nevertheless, there exists a fundamental distinction lying in that to control the chaotic processes, one needs to reach the objective with a sufficiently, —in theoretic terms, arbitrarily—small level of the control action, [8]. Solvability of this problem is not evident because of instability of the chaotic trajectories. Solvability of this problem is not evident because of instability of the chaotic trajectories x_*(t).

Stabilization of an unstable equilibrium is a special case. Let the right side of (2.1) satisfy $F(x_{*0})=0$. Then, $u(t)\equiv 0$ system (2.1) has the equilibrium state x_0 that should be stabilized in the above sense by choosing an appropriate control. This problem is characterized by an additional requirement on "smallness" of control.

The second important class of the control objectives corresponds to the problem of synchronization. Synchronization is one basic feature in nonlinear science and it finds important applications in vibration technology, communications, biology and biotechnology, and so on.

Historically, the analysis of synchronization phenomena has been a subject of active investigation since the early days of physics. The famous Dutch researcher Christian Huygens was the first to observe and describe this phenomenon is mutually coupled periodic oscillators; namely for two pendulum clocks hanging from a common support, a wooden beam in a half-timber house. Since the early 1990's there has been strongly increased interest in the synchronization of chaotic systems. In the general case, by the synchronization is meant the coordinated variation of the states of two or more systems or, possibly, coordinated variation of some of their characteristics such as oscillation frequencies. If this requirement must be satisfied only asymptotically, then synchronization is said to be *asymptotic*. If without control (for $u = 0$) synchronization cannot arise in a system, then one may pose the problem of determining a control law under which the closed-loop system becomes synchronized. Therefore, synchronization can be used as the aim of control. Full or partial coincidence of the state vectors such as the equality

$$x_1 = x_2 \tag{2.10}$$

can be a formal expression of the synchronous motion of two subsystems with the state vectors $x_1 \in \mathbb{R}^*$ and $x_2 \in \mathbb{R}^*$. In the united state space of the interacting subsystems, equality (2.10) specifies a subspace (diagonal). The requirement of asymptotic synchronization of the states x_1 and x_2 of the two systems can be represented as

$$\lim_{t \to \infty} (x_1(t) - x_2(t)) = 0 \tag{2.11}$$

which implies convergence of $x(t)$ to the diagonal set $\{x : x_1 = x_2\}$ relative to the integral state vector $x = \{x_1, x_2\}$ of the entire system.

The fact that the desired behavior is not uniquely fixed and its characteristics are defined only partially is common to the problems of control of excitation and synchronization of oscillations. In the problem of excitation of oscillations, for example, requirements can be presented only to the oscillation amplitude, their frequency and form being permitted to vary within certain limits. In the synchronization coordinations of the oscillations of all subsystems, whereas the characteristics of each subsystem can vary within wide limits.

In fact, one can distinguish among different types of synchronization for chaotic systems, from complete or identical synchronization, to phase synchronization, lag synchronization, generalized synchronization, and anticipating synchronization. Complete synchronization implies a perfect linking of the chaotic trajectories, so as they remain in step with each other in the course of the time. Generalized synchronization implies the hooking of the output of one system to a given function of the output of the other system.

A weaker form of synchrony is represented by the phase synchronization regime, wherein a perfect locking of the phases from the oscillating subsystems is realized already for small coupling, while the two amplitudes remain almost uncorrelated. To get this kind of synchronization, only low energy is needed; it is abundant in science, nature, engineering, or social life.

A third type of synchronization is lag synchronization, occurring for a stronger coupling of the oscillating subsystems until they become identical in phases and amplitudes, but shifted in the time of a lag time. In the same spirit, anticipating synchronization exploits unidirectional delayed coupling schemes to produce a collective behavior wherein the output of the response system anticipates the output of the drive one.

2) Chaotification

Chaotification is a very attractive theoretical subject which however is quite challenging technically. This is because it involves creating some very complicated but well-organized dynamical behaviors, which usually also involves bifurcations or fractals. For a given system, which may be linear or nonlinear and originally may even be stable, the question is how to generate chaos out of it, by using a simple and easily implementable controller such as a state or output feedback controller, or using an accessible parameter tuner. During the last few years, a great deal of effort has been made toward this goal, not only via computer simulations, but also by development of rigorous mathematical theories. In the endeavor of chaotification, purposefully creating discrete chaos has gained great success, [10, 11]. At the same time, chaotification in continuous-time systems is also developing rapidly. For example, a simple linear partial state-feedback controller is able to drive the Lorenz system, [3] from its nonchaotic state to chaotic state. This engineering design has led to the discovery of the chaotic Chen system, [11], which is a dual of the Lorenz system, and led to the discovery of the transition system between the Lorenz system and the Chen system, [13].

CHAPTER 3

CHAOS SYNCHRONIZATION OF GENESIO SYSTEMS

3.1 Genesio System

The Genesio system proposed by Genesio and Tesi [6] is one of paradigms of chaos since it captures many features of chaotic systems. It includes a simple square part and three simple ordinary differential equations that depend on three positive parameters. The dynamic equations of the system is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -ax - by - cz + x^2 \end{aligned} \tag{3.1}$$

where x, y and z are state variables and a, b, and c are positive constants. When $a = 6, b = 2.92$ and $c = 1.2$, system (3.1) is chaotic as shown in Figure 3.1.

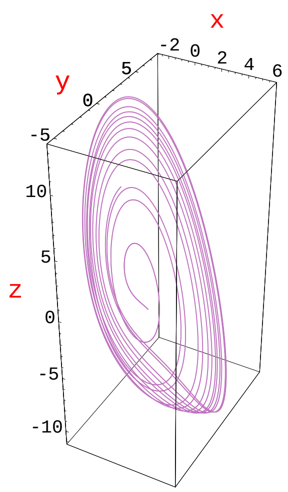

Figure 3.1: Genesio attractor

In system (3.1) equating $\dot{x}$, $\dot{y}$ and $\dot{z}$ to zero we obtain (0,0,0) and (a, 0,0). These are the equilibrium points of that system.

Let $\dot{X} = F(X)$ be the given autonomous system. Let $X^* = (x_1^*, x_2^*, x_3^*, ..., x_n^*)^T$ be an equilibrium point of this system and $\delta X = (\xi_1, \xi_2, \xi_3, ..., \xi_n)^T$ be a small perturbation. Then we consider the following perturbation system

$$X = X^* + \delta X, \qquad \| \delta X \| \ll 1$$

where $\| \delta X \| = \sqrt{\xi_1^2 + \xi_2^2 + ... + \xi_n^2}$ The linearized equation $\delta \dot{X} = \dfrac{\partial F_i}{\partial x}$

$$\delta \dot{X} = \frac{\partial F_i}{\partial x}|_{X=X^*}$$

The coefficient matrix of the linearized system of (3.1) is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a + 2x^* & -b & -c \end{pmatrix}$$

and the charasteristic polynomial is

$$|A - \lambda I| = 0 \rightarrow \lambda^3 + c\lambda^2 + b\lambda + (a - 2x^*) = 0$$

Letting a_0= 1, a_1= c, a_2= b, a_3= $a - 2x^*$ and applying Routh Hurwitz criteria we obtain that when, $a_0 = 1 > 0$, $a_1 = c > 0$, $bc - a + 2x^* > 0$ the corresponding equilibrium point is stable. Namely, for the stability of the point (0,0,0) the last condition implies that $bc - a > 0$ and similarly for the point (a, 0,0) from the last condition we have $bc + a > 0$.

3.2 Synchronization using Pecora and Carroll method

Here, we will apply the most popular method to construct synchronizing systems which was introduced by Pecora and Carroll (PC),[14].

Let us consider an autonomous chaotic dynamical system

$$\dot{u} = f(u)$$

where $u = (u_1, u_2, ..., u_n)^T$ is an n-dimensional state vector with f defining a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. The method of Pecora and Carroll decomposes the dynamical system $\dot{u} = f(u)$ into two subsystems

$$\dot{v} = g(v, w) \quad \text{and} \quad \dot{w} = h(v, w)$$

where $v = (v_1, v_2, ..., v_m)^T$ and $w = (w_{m+1}, w_{m+2}, ..., w_n)^T$ such that any second subsystem (response)

$$\dot{w'} = h(v, w')$$

with the same driving v but with different variable w' synchronizes ($w \to w' \quad as \quad t \to \infty$) with the original w-subsystem. The w-system and the w'-system are called the drive and the response systems, respectively. Let us build a PC drive-response configuration with a drive system given by the Genesio system (with three state variables denoted by subscript 1) and with a response system given by the subspace containing the (y, z) variables. We will use the chaotic signal x_1 to drive the response subsystem whose variables are denoted by subscript 2. The drive system is described by

$$\begin{aligned} \dot{x_1} &= y_1 \\ \dot{y_1} &= z_1 \\ \dot{z_1} &= -ax_1 - by_1 - cz_1 + x_1^2 \end{aligned} \tag{3.2}$$

The response system is given by

$$\begin{aligned} \dot{y}_2 &= z_2 \\ \dot{z}_2 &= -ax_1 - by_2 - cz_2 + x_1^2 \end{aligned} \tag{3.3}$$

Define the synchronization error by

$$e_y = y_2 - y_1, \quad \text{and} \quad e_z = z_2 - z_1$$

Then subtracting system (3.2) from system (3.3), we obtain the error system

$$\begin{aligned} \dot{e}_y &= e_z \\ \dot{e}_z &= -be_y - ce_z \end{aligned} \tag{3.4}$$

The linear system of synchronization error (3.4) has two complex eigenvalues with negative real part

$$\lambda_1 = -\frac{c}{2} + \frac{1}{2}\sqrt{c^2 - 4b}, \quad \text{and} \quad \lambda_2 = -\frac{c}{2} - \frac{1}{2}\sqrt{c^2 - 4b}, \quad (\lambda_2 = \bar{\lambda}_1 \quad \text{since} \quad c^2 < 4b)$$

This implies that the zero solution of system (3.4) satisfies

$$\| e_y \| \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad \| e_z \| \to 0 \quad \text{as} \quad t \to \infty$$

Hence the zero solution of the error system (3.4) is asymptotically stable and then the response system (3.3) with x-drive configuration achieves synchronization.

Remark.1. Only one choice induces the appearance of chaos synchronization namely (y, z) driven by x. For the other possible choices (x, z) driven by y or (x, y) driven by z, the Pecora and Carroll scheme did not succeed in achieving synchronization.

3.3 Synchronization using active control

In this section, the synchronization of two identical Genesio systems is achieved using active control, [15, 16]. We now, assume that we have two Genesio systems and that the drive system with the subscript 1 is to control the response system with subscript 2. The drive and response systems are defined as follows:

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{y}_1 &= z_1 \\ \dot{z}_1 &= -ax_1 - by_1 - cz_1 + x_1^2 \end{aligned} \tag{3.5}$$

$$\begin{aligned} \dot{x}_2 &= y_2 + u_1(t) \\ \dot{y}_2 &= z_2 + u_2(t) \\ \dot{z}_2 &= -ax_2 - by_2 - cz_2 + x_2^2 + u_3(t) \end{aligned} \tag{3.6}$$

We have introduced three control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ in the last system. Our goal is to determine these functions $u_1(t)$, $u_2(t)$ and $u_3(t)$. In order to estimate the control functions, we subtract (3.5) from (3.6). We define the error system as the differences between the Genesio systems (3.5) and (3.6)

$$e_x = x_2 - x_1,\ e_y = y_2 - y_1,\ e_z = z_2 - z_1$$

Using this notation we obtain

$$\begin{aligned} \dot{e}_x &= e_y + u_1(t) \\ \dot{e}_y &= e_z + u_2(t) \\ \dot{e}_z &= -(a - 2x_1)e_x - be_y - ce_z + e_x^2 + u_3(t) \end{aligned} \tag{3.7}$$

System (3.7) is called the *error system* and e_x, e_y, and e_z are called the error states.

The synchronization problem for Genesio system is to achieve the asymptotic stability of the zero solution of the error system (3.7).

We define the active control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ as follows:

$$\begin{aligned} u_1 &= V_1(t) \\ u_2 &= V_2(t) \\ u_3 &= (a - 2x_1)e_x - e_x^2 + V_3(t) \end{aligned} \tag{3.8}$$

This implies

$$\begin{aligned} \dot{e}_x &= e_y + V_1(t) \\ \dot{e}_y &= e_z + V_2(t) \\ \dot{e}_z &= -be_y - ce_z + V_3(t) \end{aligned} \tag{3.9}$$

The error system (3.9) to be controlled is a linear system with the control inputs $V_1(t)$, $V_2(t)$ and $V_3(t)$ as functions of e_x, e_y, and e_z. As long as these feedbacks stabilize the system, e_x, e_y, and e_z converge to zero as time t tends to infinity. This implies that two Genesio systems are synchronized with feedback control. There are many possible choices for the control $V_1(t)$, $V_2(t)$ and $V_3(t)$. We choose

$$\begin{pmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{pmatrix} = B \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \tag{3.10}$$

where B is a 3×3 constant matrix. In order to make the closed loop system will be stable, the proper choice of the elements of the matrix B is such that the feedback system must have all of the eigenvalues with negative real parts.

Let the matrix B be chosen in the form

$$B = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & b & c-1 \end{pmatrix}$$

then for this particular choice, the closed loop system (3.9) has the eigenvalues

$$\lambda_1 = -1, \quad \text{and} \quad \lambda_{2,3} = \frac{-(2-c) \pm \sqrt{c^2 - 4b}}{2}, \quad (c < 2 \quad \text{and} \quad c^2 < 4b)$$

With the choices $c < 2$ and $c^2 < 4b$, the error states e_x, e_y, and e_z converge to zero as time t tends to infinity and hence the synchronization of two Genesio systems is achieved.

3.4 Linear generalized synchronization of Genesio system

A dynamical system can be decomposed into two parts

$$\dot{x} = Ax + \Psi(x) \tag{3.11}$$

where A is an $n \times n$ constant matrix and $\Psi : \Re^n \to \Re^n$, [17]. We assume that the drive system transmit the signal $\Psi(x)$ to the response system and consider the following unidirectional synchronization scheme:

$$\begin{aligned} \dot{x} &= Ax + \Psi(x) \longleftarrow \quad \text{drive system} \\ \dot{y} &= Ay + \Lambda\Psi(x) \longleftarrow \quad \text{response system} \end{aligned} \tag{3.12}$$

Theorem 3.1. *(Yang and Chua),[17]: If the matrix Λ commutes with A, then the two dynamical systems in (3.12) are in a state of generalized synchronization via the following generalized synchronization transformation*

$$y(\infty) = H(x) = \Lambda x \tag{3.13}$$

if and only if A is negative definite.

Proof. Let $z = y - \Lambda x$, then the stability of the generalized synchronization between the two dynamical systems in (3.12) via the generalized synchronization transformation $y = H(x) = \Lambda x$ is equivalent to that of the origin of the following system:

$$\begin{aligned} \dot{z} &= [Ay + \Lambda\Psi x] - [\Lambda A x + \Lambda\Psi(x)] \\ &= Ay - \Lambda A x \\ &= A(y - \Lambda A x) \\ &= Az \end{aligned} \tag{3.14}$$

Therefore $z = 0$ is asymptotically stable if and only if A is negative definite. $\square$

The Genesio system can be decomposed into two parts as

$$\dot{x} = Ax + \Psi(x) \tag{3.15}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \qquad \Psi(x) = \begin{pmatrix} 0 \\ 0 \\ x_1^2 \end{pmatrix} \tag{3.16}$$

The matrix A is negative definite if and only if all eigenvalues of A have negative real parts. By Routh-Hurwitz criteria this provided when $bc > a$.

Now, if the driven system is

$$\dot{y} = Ay + \Lambda\Psi(x) \tag{3.17}$$

where the matrix Λ commutes with A, then the drive Genesio system and the response Genesio system are in a state of generalized synchronization.

Simulation 3.1. *In this simulation we take* $\Lambda = constant$*, say 2, then*

$$\Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and the response system becomes

$$\begin{aligned} \dot{x}_2 &= y_2 \\ \dot{y}_2 &= z_2 \\ \dot{z}_2 &= -ax_2 - by_2 - cz_2 + 2x_1^2 \end{aligned} \tag{3.18}$$

The state variables of the drive system (3.15) and the response system (3.18) are related as follows

$$\begin{aligned} x_2 &= 2x_1 \\ y_2 &= 2y_1 \\ z_2 &= 2z_1 \end{aligned} \tag{3.19}$$

Simulation 3.2. *In this simulation we take*

$$\Lambda = A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{pmatrix}$$

then the response system becomes

$$\begin{aligned} \dot{x}_2 &= y_2 \\ \dot{y}_2 &= z_2 + x_1^2 \\ \dot{z}_2 &= -ax_2 - by_2 - cz_2 - cx_1^2 \end{aligned} \tag{3.20}$$

The relation between the state variables of the drive and response systems ((3.15) and (3.16)) is

$$\begin{aligned} x_2 &= y_1 \\ y_2 &= z_1 \\ z_2 &= -ax_1 - by_1 - cz_1 \end{aligned} \tag{3.21}$$

3.5 Numerical Results

In this section numerical simulations are given to verify the effectiveness of Pecora and Carroll method, active control method and linear generalized synchronization to achieve synchronization of two identical Genesio systems. All simulations are carried out using MATHEMATICA. The drive system and the response system are taken as Genesio systems with same parameters but different initial conditions. We select the parameters of Genesio systems as $a = 6$, $b = 2.92$ and $c = 1.2$ to ensure the existence of the chaotic behavior (Figure 3.1).

Figure 3.2 displays the time response of the drive state y_1 and the response state y_2, while Figure 3.3 displays z_1 and z_2 under Pecora and Carroll method where the initial values of the dive system (3.2) and the response system (3.3) are $x_1(0) = 3$, $y_1(0) = -4$, $z_1(0) = 2$, $y_2(0) = 5$ and $z_2(0) = -5$, respectively. The corresponding initial errors are $e_y(0) = 9$ and $e_z(0) = -7$. The chaotic signal $x_1(t)$ is used to drive the response subsystem (3.3). Figures 3.4-3.5 show the synchronization errors e_y and e_z of systems (3.2) and (3.3), respectively, as functions of time, tend to zero asymptotically. From Figures 3.2-3.5 we observe that the synchronization of the drive and response of two identical Genesio system occured after $t = 10$.

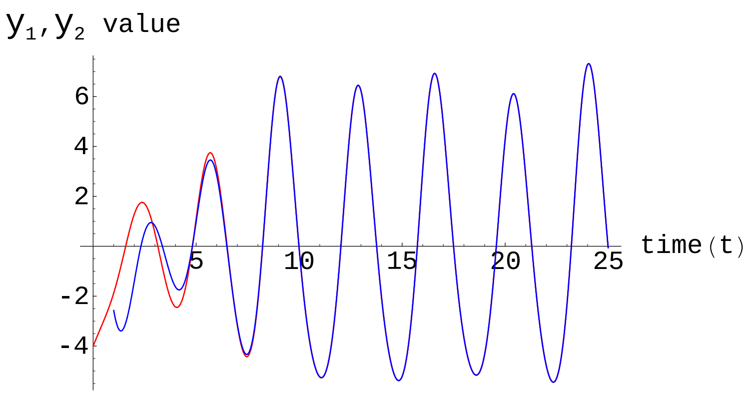

Figure 3.2: Time response of the drive state y_1 and the response state y_2.

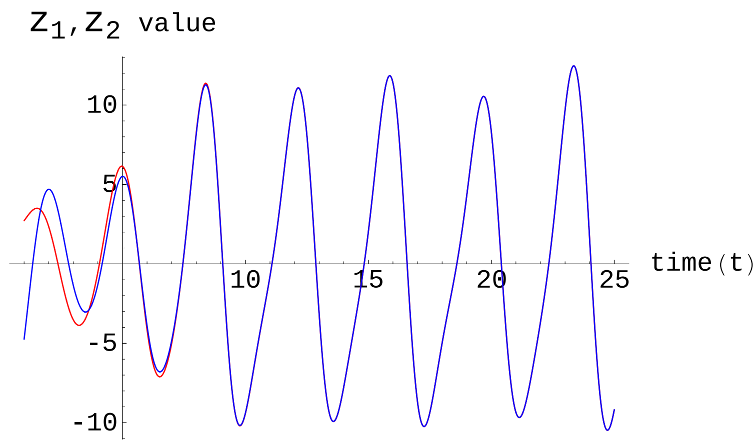

Figure 3.3: Time response of the drive state z_1 and the response state z_2.

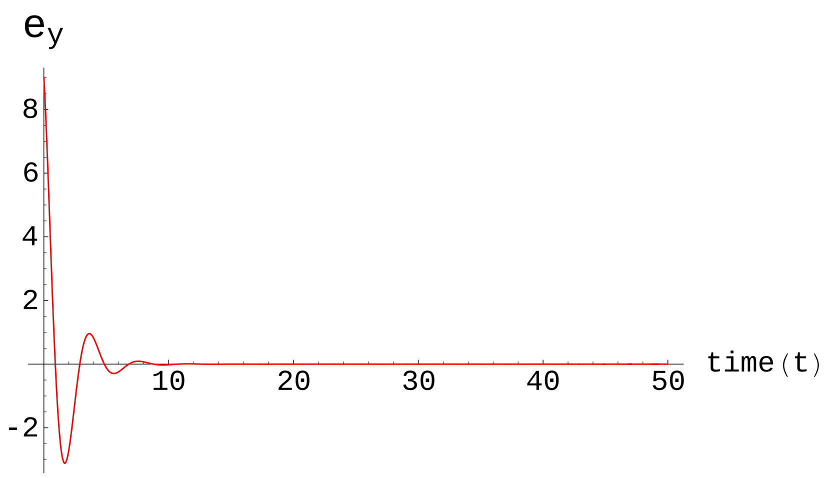

Figure 3.4: The synchronization error e_y as a function of time t.

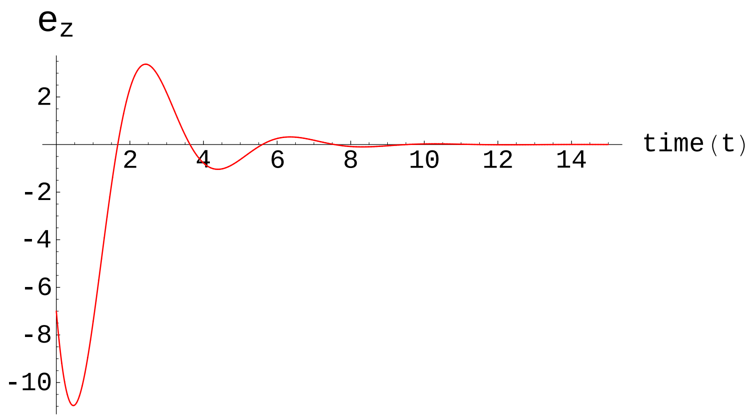

Figure 3.5: The synchronization error e_z as a function of time t.

The numerical simulations of the drive and response systems ((3.5) and (3.6)), respectively, are presented to show the effect of chaos synchronization by using active control method. The control functions are chosen as in (3.8) and (3.10) and the initial states of the drive and response systems are taken as $x_1(0) = 3$, $y_1(0) = -4$, $z_1(0) = 2$, $x_2(0) = 3.4$, $y_2(0) = -5$, $z_2(0) = 2.3$, respectively. Then the initial values for the error states are $e_x(0) = 0.4$, $e_y(0) = -1$ and $e_z(0) = 0.3$.

Figure 3.6 displays the time response of the drive state x_1 and the response state x_2, Figure 3.7 displays the time response of the drive state y_1 and the response y_2 and Figure 3.8 displays the time response of the drive state z_1 and the response state z_2. Figures 3.9-3.11 show the synchronization errors e_x, e_y, and e_z, respectively as functions of time, tend to zero asymptotically. We observe in numerical simulations that the synchronization of drive and response of two identical Genesio systems occured after $t = 10$.

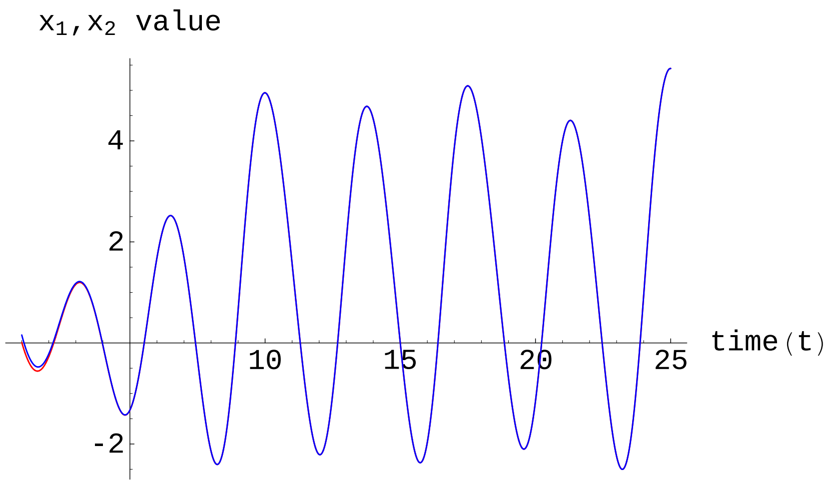

Figure 3.6: Time response of the drive state x_1 and the response state x_2.

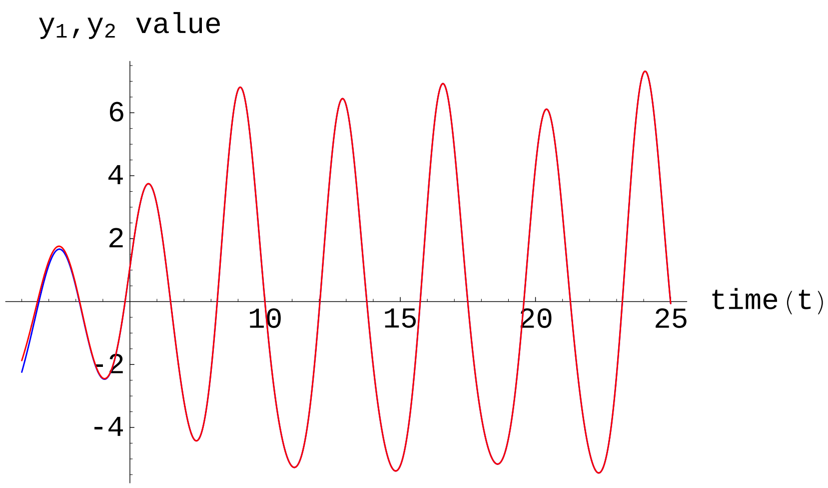

Figure 3.7: Time response of the drive state y_1 and the response state y_2.

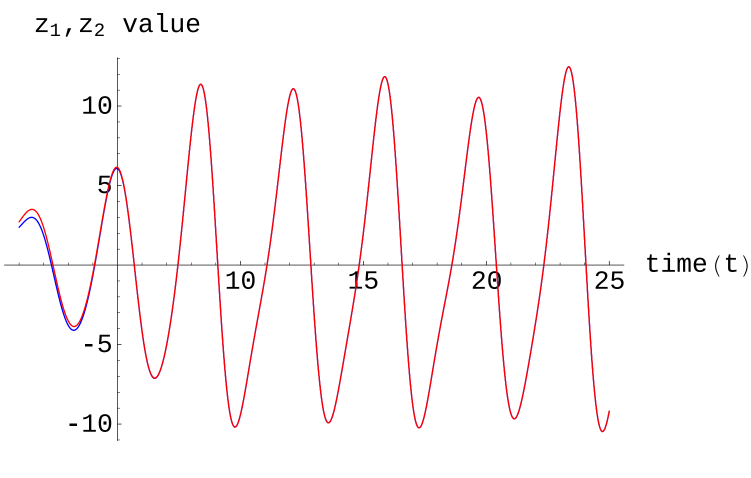

Figure 3.8: Time response of the drive state z_1 and the response state z_2.

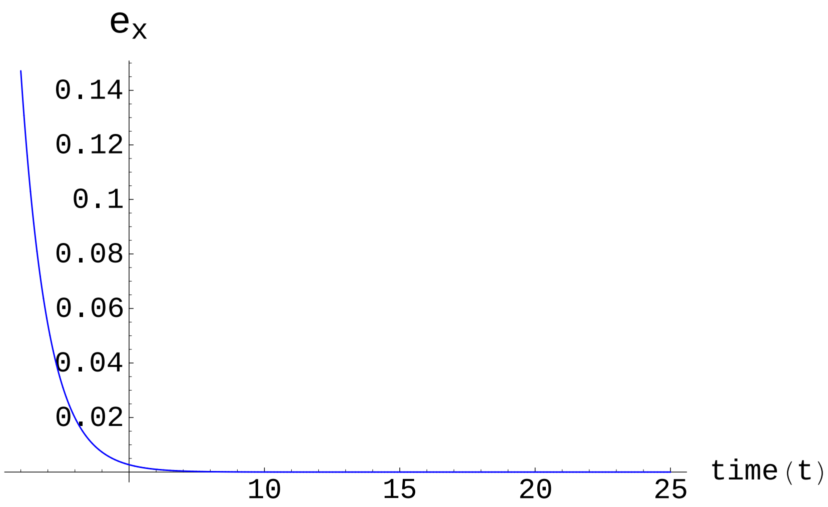

Figure 3.9: The synchronization error e_x as a function of time t.

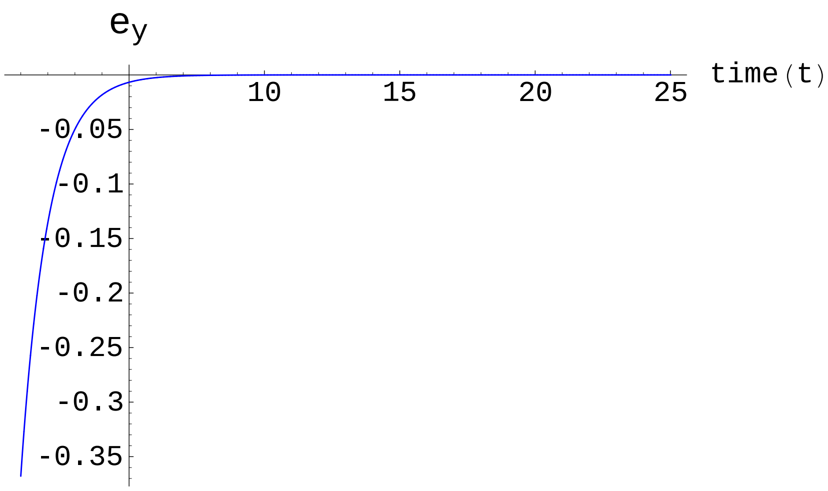

Figure 3.10: The synchronization error e_y as a function of time t.

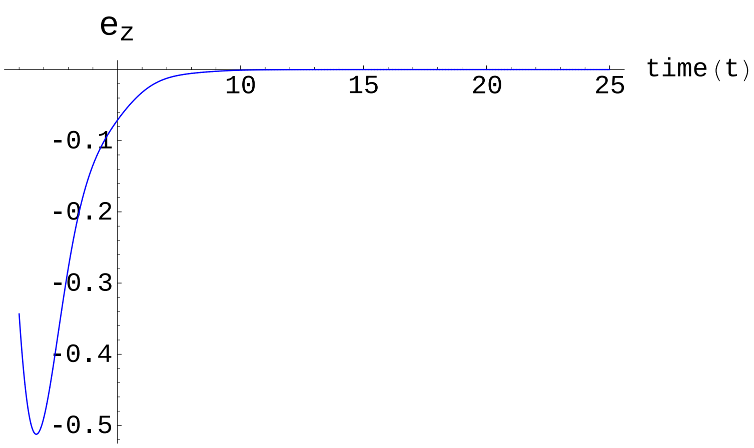

Figure 3.11: The synchronization error e_z as a function of time t.

We present two simulations for the linear generalized synchronization. In the first simulation, we take $\Lambda = diag(2,2,2)$. The initial values of the drive and the response systems ((3.15) and (3.18)) are taken as $x_1(0) = 3$, $y_1(0) = -4$, $z_1(0) = 2$, $x_2(0) = 6$, $y_2(0) = -8$ and $z_2(0) = 4$. Figures 3.12-3.14 show the projections of the synchronized attractors in the coupled Genesio systems onto xy, xz and yz-planes, respectively.

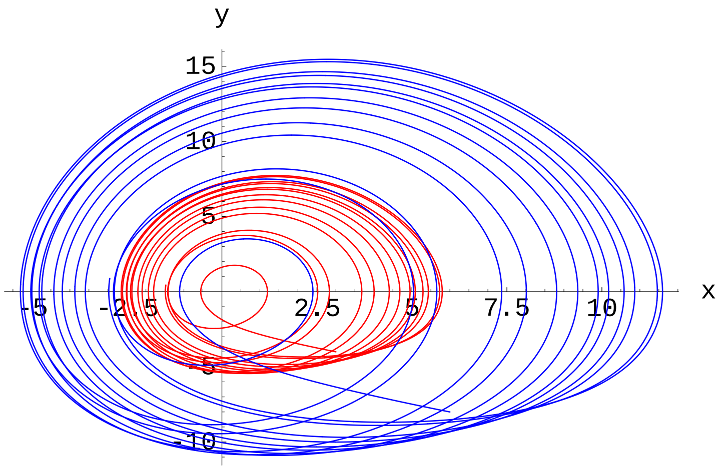

Figure 3.12: Projections of the synchronized attractors onto xy-plane.

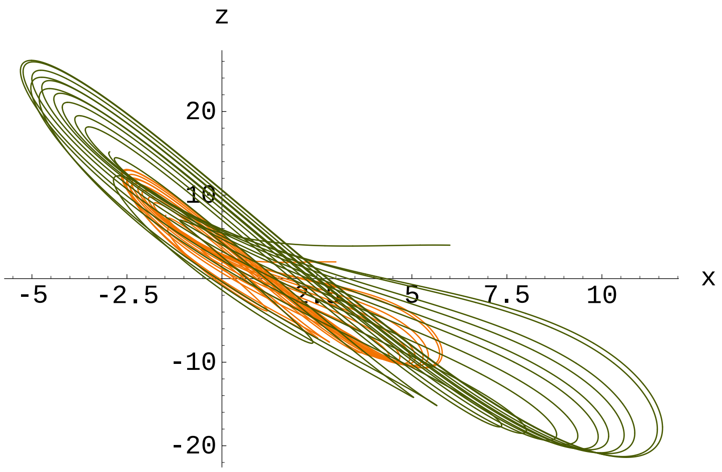

Figure 3.13: Projections of the synchronized attractors onto xz-plane.

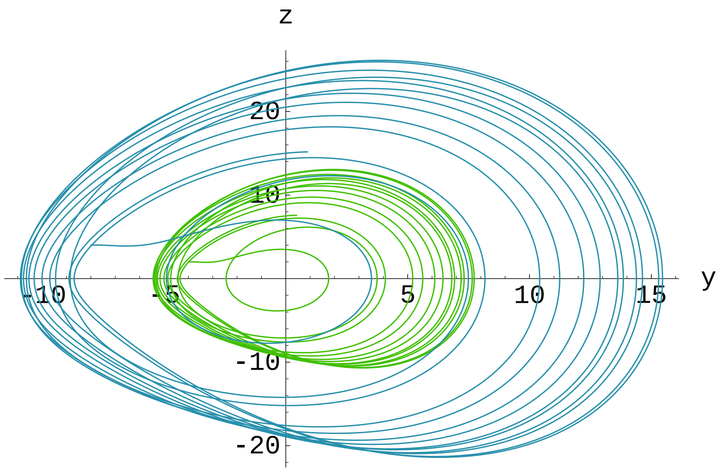

Figure 3.14: Projections of the synchronized attractors onto yz-plane.

In the second simulation, we take $\Lambda = A$. The initial values of the drive and response systems ((3.15) and (3.20))are taken as $x_1(0) = 3$, $y_1(0) = -4$, $z_1(0) = 2$, $x_2(0) = -4$, $y_2(0) = 2$, $z_2(0) = -8.72$. Figures 3.15-3.17 display the projections of the synchronized attractors in the coupled Genesio systems onto xy, xz and yz-planes, respectively.

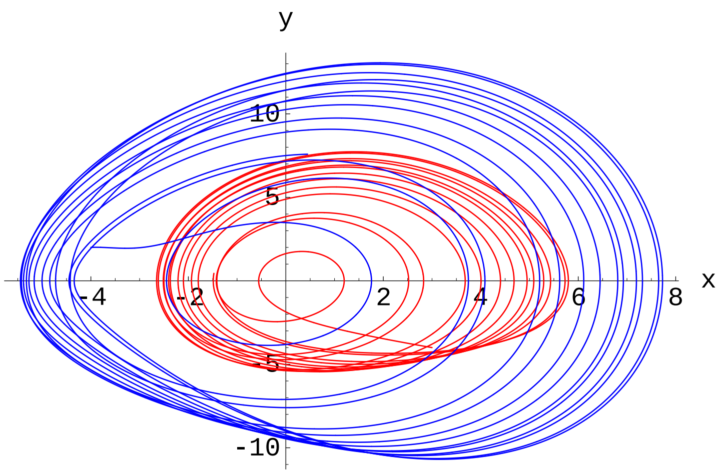

Figure 3.15: Projections of the synchronized attractors onto xy-plane.

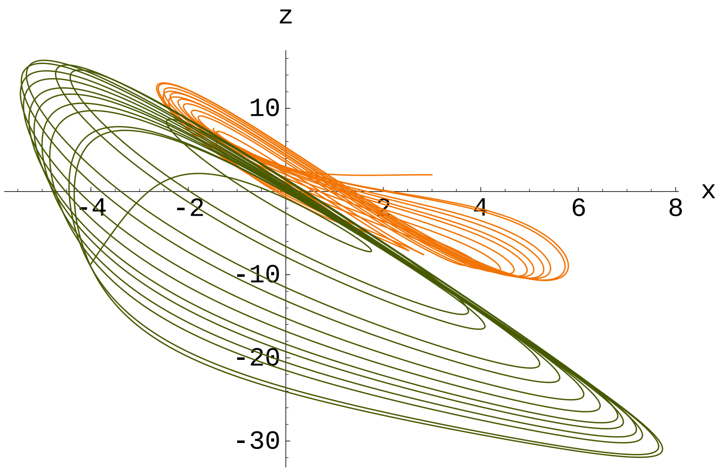

Figure 3.16: Projections of the synchronized attractors onto xz-plane.

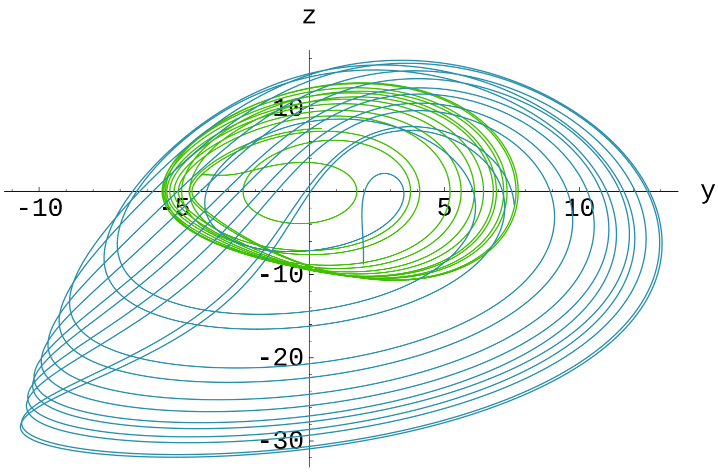

Figure 3.17: Projections of the synchronized attractors onto yz-plane.

3.6 Conclusion

In this thesis, we show that synchronization of two identical Genesio systems can be achieved by using Pecora and Carroll method, active control and linear generalized synchronization.

We construct a response system to carry out each method.

We obtain a perfect linking of the chaotic trajectories in Pecora and Carroll method and active control. In both methods, the distance between the corresponding states of drive and response systems converges to zero as time goes to zero as seen in Figures 3.2-3.2 and Figures 3.6-3.8. So, the components of the synchronization error converge to zero asymptotically as shown in Figures 3.4-3.5 and Figures 3.6-3.8.

In linear generalized synchronization, we consider two unidirectionally coupled Genesio systems and assume that the states of the drive and response systems are related by the linear transformation $y = \Lambda x$, where Λ is a 3x3 constant matrix. Using the strict conditions, that the linear part matrix A of the drive system commutes with linear transformation matrix Λ and A is negative definite for $a = 6.0$, $b = 2.92$ and $c = 1.2$ we obtain hooking of the outputs of response system with outputs of the drive system.

Here, two simulations are given. In the first simulation, we choose $\Lambda = diag(2, 2, 2)$. As easily seen from Figures 3.12-3.14, we obtain coupling of the trajectories of drive and response systems. This follows from the similarity of response system (3.18) to drive system (3.15) and (3.16). In the second simulation, the choice of $\Lambda = A$ results coupling of the trajectories of drive and response systems, but with a small distortion of trajectories of response system as shown in Figures 3.15-3.16. This is due to the existence of the quadratic nonlinear term in the second equation of the response system (3.20).

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