GEOMETRY AND MATRIX SPECTRAL PROBLEMS

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By
Murat Altunbulak
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Alexander A. Klyachko (Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Sinan Sertöz

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Alexander S. Shumovsky

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Mehmet Baray
Director of Institute of Engineering and Sciences

ABSTRACT

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Murat Altunbulak
M.S. in Mathematics
Supervisor: Prof. Dr. Alexander A. Klyachko
September, 2002

The aim of this thesis is to give a survey and applications of some recent work of Klyachko, Knutson and Tao that characterizes eigenvalues of sum of Hermitian matrices and decomposition of tensor products of representations of $GL_n(\mathbb{C})$. We also explain related applications to Grassmannian variety and Toric bundles.

Keywords: Hermitian spectral problems, Tensor product, Schubert Calculus, Toric bundles.

ÖZET

GEOMETRİ VE MATRİS SPEKTRAL PROBLEMLERİ

Murat Altunbulak
Matematik Bölümü Yüksek Lisans
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Bu tezin amacı, Klyachko, Knutson ve Tao'nun $GL_n(\mathbb{C})$ temsillerinin tensör çarpımlarının ayrışmasını ve Hermitian matris toplamlarının öz değerlerini karakterize eden bazı güncel çalışmalarının tetkikini ve uygulamalarını vermektir. Ayrıca Grassmannian çokkatlıları ve Torik tutamlarının ilgili uygulamalarını anlatıyoruz.

Anahtar kelimeler: Hermitian spektral problemleri, Tensör çarpımı, Schubert Kalkulus, Toric tutamları.

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Chapter 1

Introduction

1.1 Spectral Problem: Classical Results

The principal characters of this thesis are n by n Hermitian matrices A and B, their sum $C = A+B$, and the eigenvalues of A, B and C enumerated as $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$ and $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$, respectively. Sometimes we would like to emphasize the dependence of the eigenvalues of the matrix. We then use the notation $\lambda_i(A)$ for the i^{th} eigenvalues of A when the eigenvalues are arranged in a weakly decreasing order. This n-tuple of eigenvalues of A as a whole is denoted by $\lambda(A)$.

Our main problem comes from linear algebra and goes back to the 19^{th} century.

Hermitian spectral problem: What can be said about the eigenvalues of a sum of two Hermitian (or real symmetric) matrices, in terms of the eigenvalues of the summands?

If A is real symmetric matrix, then its eigenvalues describe the quadratic form $q_A(x) = x^t Ax$ in an appropriate orthogonal coordinate system. For example if the eigenvalues are positive, the inverses of the square roots of the eigenvalues are half the lengths of the principal axes of the ellipsoid $q_A(x) = 1$.

There is no equation between α, β and γ, except one:

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i, \tag{1.1}$$

trace of $C = A + B$ is the sum of the traces of summands A and B. In fact this is the only equation between α, β and γ. But there are lots of necessary inequalities. For example, Weyl inequalities from 1912

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j \quad \text{for each time } i + j \leq n + 1. \tag{1.2}$$

and K.Fan inequalities from 1949

$$\sum_{i=1}^{p} \gamma_i \leq \sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{p} \beta_i \quad \text{for any} \quad p < n \tag{1.3}$$

are some necessary conditions for the existence of Hermitian matrices with these eigenvalues.

Other inequalities were found, all having the from

$$\sum_{k \in K} \lambda_k(A + B) \leq \sum_{i \in I} \lambda_i(A) + \sum_{j \in J} \lambda_j(B), \tag{IJK}$$

for some subsets I, J, K of $\{1, 2, \ldots, n\}$ of the same cardinality $p < n$.

1.2 New Development

After 50's the inequalities (IJK) were suggested to be a solution for spectral problem. But up to 1998 there was no complete solution of Hermitian spectral problem. During this time new techniques in different subjects, like Schubert calculus, vector bundles and representation theory, have been developed, and relations between spectral problem and these subjects arose. In 1998, A.Klyachko gave a complete solution of spectral problem by use of Schubert calculus, vector bundles and representation theory.

We begin with a problem in representation theory, which is closely related with spectral problem, and give the relations between spectral problem and representation theory, Schubert calculus and vector bundles.

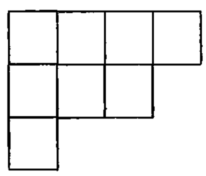
Tensor product problem: Which irreducible representations V_γ can be components of the tensor product $V_\alpha \otimes V_\beta$ of irreducible representations V_α, V_β of $\mathrm{Gl}_n(\mathbb{C})$ with highest weights (=Young diagrams)

$$\alpha: \ \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \ ; \quad \beta: \ \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n.$$

In contrast to spectral problem the coefficients of tensor product decomposition

$$V_\alpha \otimes V_\beta = \sum c^\gamma_{\alpha\beta} V_\gamma$$

can be evaluated algorithmically by famous Littlewood-Richardson rule (described in Section 2.3). It turns out that the spectral problem and tensor product problem are equivalent and have the same answer. To give it, let us associate p-element subsets I of $\{1, 2, \ldots, n\}$ with Young diagram σ_I in a rectangle of format $p \times q$, $p+q$, cut out by polygonal line Γ_I which connects S-W and N-E corners of the rectangle, with i^{th} unit edge running to the North for $i \in I$ and to the East otherwise. For example the diagram



corresponds to the subset $I = \{2, 5, 7\}$ of $\{1, 2, \ldots, 7\}$ in a rectangle of format 3×4. We can formally multiply the diagrams by L-R rule

$$\sigma_I \sigma_J = \sum c^K_{IJ} \sigma_K, \tag{1.4}$$

where c_{IJ}^K are Littlewood-Richardson coefficients.

Geometrically, (1.4) gives the decomposition $s_I \cdot s_J = \sum c_{IJ}^K s_K$ of two Schubert cycles s_I, s_J in cohomology ring of the Grassmannian $G(p,q)$ (see Section 2.4). It follows that s_K is a component of $s_I \cdot s_J$ iff. $c_{IJ}^K \neq 0$. As we will see in the following section if $c_{IJ}^K \neq 0$, then the inequality (IJK) holds. So we have, if s_K is a component of $s_I \cdot s_J$ then (IJK) holds.

The nonvanishing product of Schubert cycles $s_I \cdot s_J \cdot s_K \neq 0$ implies the stability of vector bundle $\mathcal{E}$ corresponding to triplet of filtrations E, F, G which are given by three Schubert cells S_I, S_J, S_K. In this case, if $\mathcal{E}$ is stable, then the inequalities (IJK) hold (see Section 2.5).

1.3 Solution: Main Theorem

In his paper A.Klyachko [Kl-1] showed that the irreducible representation $V_{N\gamma}$ is a component of tensor product $V_{N\alpha} \otimes V_{N\beta}$ for some N iff. α, β and γ are spectra of Hermitian operators A, B and $C = A + B$. It is not clear whether we can always take $N = 1$. In 1999, A.Knutson and T.Tao [KT] showed that we can always take $N = 1$. After all this results we have our main theorem.

Theorem 1.3.1 *The following conditions are equivalent*

i) There exists Hermitian operators A, B, $C = A + B$ with spectra $\lambda(A)$, $\lambda(B)$, $\lambda(C)$.

ii) Inequality (IJK) holds each time Littlewood-Richardson coefficient $c_{IJ}^K \neq 0$. Here $I, J, K \subset \{1, 2, \ldots, n\}$ are subsets of the same cardinality $p < n$.

iii) For integer spectra $\alpha = \lambda(A)$, $\beta = \lambda(B)$, $\gamma = \lambda(C)$ the above conditions are equivalent to

$$V_\gamma \subset V_\alpha \otimes V_\beta. \tag{1.5}$$

$\square$

1.4 Applications

We give some applications of the above theorem.

1.4.1 Weyl Inequalities

Take $I = \{i\}$ and $J = \{j\}$. Then the corresponding diagrams are

$$\sigma_I = \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{i-1} \quad \text{and} \quad \sigma_J = \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{j-1}$$

Multiplying these diagrams we get $\sigma_I \otimes \sigma_J = \sigma_K$, where

$$\sigma_K = \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{i+j-2}$$

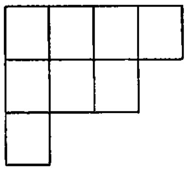
Since $c_{IJ}^K = 1 \neq 0$, by Theorem 1.3.1 we get the Weyl inequality (1.2).

1.4.2 Interlacing Theorem

Theorem 1.4.1 (Cauchy Interlacing Theorem) *Let A be a Hermitian matrix with spectrum $\lambda(A):\ a_1 \geq a_2 \geq \ldots \geq a_n$ and $B \geq 0$ be a nonnegative matrix of rank one with spectrum $\lambda(B):\ b \geq 0 \geq \ldots \geq 0$. Then the spectra of A and $A+B$ satisfy the following interlacing inequalities*

$$\lambda_i(A) \leq \lambda_i(A+B) \leq \lambda_{i-1}(A). \tag{1.6}$$

Proof: Consider the spectra as Young diagrams

$\lambda(A):$  ; $\lambda(B):$ 

Applying Littlewood-Richardson rule we find out that $\lambda(A) \otimes \lambda(B)$ is a sum of diagrams $\gamma : c_1 \geq c_2 \geq \ldots \geq c_n$ satisfying the following interlacing inequalities

$$c_1 \geq a_1 \geq c_2 \geq a_2 \geq \ldots \geq c_n \geq a_n.$$

By Theorem 1.3.1 this implies (1.6).

Remark: Cauchy interlacing theorem for spectra known in mechanics as Rayleigh-Courant-Fisher principle: Let mechanical system S' is obtained from another one S, by imposing a linear constraint, e.g. by fixing a point of a drum. Then the spectrum of S separates spectrum of S'.

1.4.3 A Commutator Relation

As another example where Littlewood-Richardson rule works directly let's consider Heisenberg commutation relation

$$[A, A^*] = AA^* - A^*A = I.$$

It has no finite dimensional representations, since $Tr[A, A^*] = 0 \neq TrI$.

In so called Habbard model of statistical physics it was proposed its modification

$$[A, A^*]^2 = I, \text{ i.e., } [A, A^*] = J = \operatorname{diag}(\underbrace{1, 1, \ldots, 1}_{k}, \underbrace{-1, -1, \ldots, -1}_{k});$$
$$J^2 = 1, \ \ TrJ = 0\ , \ \ n = 2k, \text{ since } Tr[A, A^*] = TrJ = 0.$$

In physical terminology, we have the following

$$\begin{aligned} A &= \text{creation operator}, \\ A^* &= \text{annihilation operator}, \\ A^*A &= \text{operator of number of particles}. \end{aligned}$$

Since A^*A is the operator of number of particles, its spectrum supposed to be integer and $A^*A \geq 0$.

Let

$$H_1 = A^*A \; ; \; H_2 = AA^*. \tag{1.7}$$

Then

i. H_1 and H_2 are both positive $(H_1, H_2 \geq 0)$ and have the same spectra (since $Tr(A^*A)^m = Tr(AA^*)^m \quad \forall m$).

ii. Any pair of such operators can be represented in form (1.7) (write $H_2 = H^2 \geq 0$ then $H_1 = U^*H^2U$ for some unitary operator U, and we get (1.7) with $A = HU$).

So we arrive at the matrix spectral problem

$$H_2 - H_1 = J;\ \lambda(H_1) = \lambda(H_2) = \lambda;\ \lambda(J) = (\overbrace{1, 1, \ldots, 1}^{k}, \overbrace{-1, -1, \ldots, -1}^{k}). \tag{1.8}$$

To deal with nonnegative spectra rewrite (1.8) in the form

$$H_1 + (J + I) = H_2 + I. \tag{1.9}$$

Viewing the spectra as Young diagrams and applying Littlewood-Richardson rule we find out that $\lambda(H_2 + I)$ is a component of $\lambda(H_1) \otimes \lambda(J + I)$ if and only if multiplicity of $\lambda_i(H_2 + I) \leq k = \frac{n}{2}$.

Hence, by Theorem 1.3.1 (1.9) is solvable if and only if the spectrum $\lambda = \lambda(H_1)$ has multiplicity $\leq k = \frac{n}{2}$ (i.e. number of states with given number of particles $\leq k = \frac{n}{2}$).

Chapter 2

Spectral Problems, Representations of $\mathrm{Gl}_n(\mathbb{C})$, Schubert Calculus

In this chapter we'll deal with three apparently disjoint problems:

i) The spectrum of a sum of Hermitian operators

ii) Components of tensor product of irreducible representations of the group $GL_n(\mathbb{C})$

iii) Components of product of Schubert cycles in the cohomology ring of Grassmannian

And finally, we will give the relation between toric bundles and the above problems.

We begin with the most familiar subject.

2.1 Hermitian Operators and Spectral Problems

Recall that a Hermitian matrix is a matrix which coincides with its complex conjugate matrix, i.e., $A = A^* = \overline{A^t}$. It is a basic fact of linear algebra that all of the eigenvalues of any Hermitian matrices are real. Let

$$\lambda(A): \ \lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A) \tag{2.1}$$

be the eigenvalues (spectrum) of A.

With this notation the Hermitian matrix spectral problem that we mentioned in Introduction becomes:

> What are the relations between spectra $\lambda(A)$, $\lambda(B)$ and $\lambda(C)$, where $C = A + B$?

First of all we have a simple relation between $\lambda(A)$, $\lambda(B)$ and $\lambda(C) = \lambda(A+B)$: the trace of C is the sum of the traces of A and B. The trace of A, denoted by trA, is the sum of the diagonal entries of A and also the sum of eigenvalues of A. So $trC = trA + trB$ and hence

$$\sum_{i=1}^{n} \lambda_i(A+B) = \sum_{i=1}^{n} \lambda_i(A) + \sum_{i=1}^{n} \lambda_i(B). \tag{2.2}$$

And there are classical inequalities [Kl-1], due to

(1) *Herman Weyl*

$$\lambda_{i+j-1}(A+B) \leq \lambda_i(A) + \lambda_j(B), \ \text{for} \ \ i+j \leq n+1$$
$$\lambda_{i+j-1}(A+B) \geq \lambda_i(A) + \lambda_j(B), \ \text{for} \ \ i+j \geq n+1$$

(2) *Ky Fan*

$$\sum_{i \leq p} \lambda_i(A+B) \leq \sum_{i \leq p} \lambda_i(A) + \sum_{i \leq p} \lambda_i(B)$$

(3) *Lidskii and Wielandt*

$$\sum_{i\in I} \lambda_i(A+B) \leq \sum_{i\in I} \lambda_i(A) + \sum_{i\leq p} \lambda_i(B)$$

where I is any subset of $\{1, 2, \ldots, n\}$ of cardinality p,

and more.

The inequalities (1) - (3) give a complete list of restrictions for $n = \dim V \leq 3$. For example, when $n = 2$ the statement (1) contains three inequalities

$$\lambda_1(A+B) \leq \lambda_1(A) + \lambda_1(B), \ \lambda_2(A+B) \leq \lambda_1(A) + \lambda_2(B), \\ \lambda_2(A+B) \leq \lambda_2(A) + \lambda_1(B). \tag{2.3}$$

It turns out that, together with the trace identity (2.2), these three inequalities are sufficient to characterize the possible eigenvalues of A, B and $A+B$, i.e., if three pairs of real numbers $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$, $\{\gamma_1, \gamma_2\}$ each ordered decreasingly ($\alpha_1 \leq \alpha_2$, etc.) satisfy relations (2.2) and (2.3), then these pairs are the eigenvalues (spectrum) of A, B and $A+B$.

But in higher dimensions there are lots of others. In 1998 A.Klyachko [Kl-1] showed that all of them are of the form

$$\sum_{k\in K} \lambda_k(A+B) \leq \sum_{i\in I} \lambda_i(A) + \sum_{j\in J} \lambda_j(B), \tag{IJK}$$

for some triple of subsets $I, J, K \subset \{1, 2, \ldots, n\}$ of the same cardinality.

A.Horn defined this triple of subsets I, J, K inductively [Ho]. We will give Horn's description for these triples in Chapter 4. But now let us give how Klyachko described these triples and give the connection between Schubert calculus and the above problem.

Fix a decomposition $n = p + q$. There is a bijection between subsets $I \subset \{1, 2, \ldots, n\}$ of cardinality p and Young diagrams $\sigma = \sigma_I$ (see Section

2.3) in a rectangular box of dimension p(North) by q(East) which can be given as follows.

[Kl-1] Let $\Gamma = \Gamma_I$ be a polygonal line with unit edge that runs from the South-West corner of the box to the East-North corner with the i^{th} edge running to the North for $i \in I$ and to the East otherwise. The line $\Gamma = \Gamma_I$ cuts out from the box a Young diagram $\sigma = \sigma_I \subset p \times q$ situated in its North-West angle. The Young diagram σ_I in the usual way [GH] corresponds to a Schubert cycle s_I in a cohomology ring of the Grassmannian(see Section 2.4)

$$\mathrm{Gr}(p, q) = \{V \subset E | \dim V = p,\ \mathrm{codim} V = q\}.$$

The following theorem gives the connection between spectral inequalities and the Schubert calculus problem: Find the components of the product of Schubert cycles in the cohomology ring of the Grassmannian? (For proof see [Kl-1]).

Theorem 2.1.1 (Klyachko) *Consider a triple of subsets $I, J, K \subset \{1, 2, \ldots, n\}$ such that the Schubert cycle s_K is a component of $s_I \cdot s_J$. Then*

i) The inequality (IJK) holds.

ii) In union with the trace identity, this inequalities form a complete and independent set of restrictions on spectra of A, B and $A + B$. □

2.2 Representation Theory

In this section we will give the relation between matrix spectral problem and tensor product problem in representation theory. First, we need some preliminaries.

2.2.1 Preliminaries

A *representation* of the group G in a finite dimensional complex vector space V is a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ of G to the group of automorphisms of V. For simplicity, we call V itself a representation of G and write $g \cdot v$ for $\rho(g)(v)$. We denote the representation V of G by $G : V$.

A *subrepresentation* of a representation V is called irreducible if there is no proper nonzero invariant subspace W of V, i.e., $\forall w \in W$ and $g \in G$, $g \cdot w \in W$. A representation V is called *irreducible* if there is no proper nonzero invariant subspace W of V.

A character of the representation $G : V$ is a complex valued function χ on G defined by $\chi(g) = Tr(g|_V)$, the trace of g on V. The importance of this function comes primarily from the fact that it characterizes the representation $G : V$.

If V and W are two representations, the *direct sum* $V \oplus W$ and the *tensor product* $V \otimes W$ are also representations, the latter via

$$g(v \otimes w) = gv \otimes gw.$$

2.2.2 Highest Weights

Now, we will introduce a significant subject in representation theory, namely highest weights.

Let $G = \mathrm{Gl}_n(\mathbb{C})$ be the group of invertible n by n complex matrices and $G : V$ be a representation of G and

$$T = \{\mathrm{diag}(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}\}$$

be the diagonal subgroup (subgroup containing the diagonal matrices) of G (maximal torus). Clearly, $T \simeq \mathbb{C}^* \times \mathbb{C}^* \times \ldots \mathbb{C}^*$. Since T is abelian, $T : V$ splits into 1-dimensional components

$$V = \bigoplus_i V_i \ ; \ \dim V = 1$$

and since $\dim V_i = 1$, T acts on V as multiplication by scalars, i.e., $\forall t \in T$, $t : v \mapsto \chi(t)v$, $\forall v \in V$, where $\chi(t) \in \mathbb{C}^*$ and

$$\chi(t_1 t_2) = \chi(t_1).\chi(t_2), \qquad \forall t_1, t_2 \in T.$$

The homomorphism, just defined, $\chi : T \to \mathbb{C}^*$ is the character of T.

Since any homomorphism $\chi : \mathbb{C}^* \to \mathbb{C}^*$ is of the form $z \mapsto z^a$, we have $\chi(t) = z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n}$; $a_i \in \mathbb{Z}$, $t = \mathrm{diag}(z_1, z_2, \ldots, z_n) \in T$.

Define $\alpha = (a_1, a_2, \ldots, a_n)$ and $\chi_\alpha : T \to \mathbb{C}^*$ the corresponding character. α is called a *weight* of $G : V$ if the following subspace $V_\alpha = \{x \in V | \ t \cdot x = \chi_\alpha(t) \cdot x\}$ of V is nonempty. The subspace V_α is called weight space.

Now consider the normalizer of T

$$N_G(T) := \{x \in V \mid g^{-1}tg \in T, \ \forall \ t \in T\}.$$

It is the maximal normal subgroup containing T. From the definition of $N_G(T)$ one can see that g is obtained from diagonal matrix by permutation of rows, where $g \in N_G(T)$. So for $g \in N_G(T)$ $gV_\alpha = V_{\alpha'}$, where α' is obtained from α by permutation of columns. From the fact that the Weyl group $W_G = N_G(T)/T$ is isomorphic to the symmetric group S_n of n letters, we have the following proposition:

Proposition 2.2.1 *The set of weights of $G : V$ is invariant under S_n, i.e., if $\alpha = (a_1, a_2, \ldots, a_n)$ is a weight then $\alpha' = (a_{i_1}, a_{i_2}, \ldots, a_{i_n})$ is also a weight.* $\square$

The *highest weight* of $G : V$ is maximal weight in lexicographical order (i.e., $\alpha > \beta$ if the first nonvanishing $a_i - b_i$ is positive).

Corollary 2.2.1 *If* $\alpha = (a_1, a_2, \ldots, a_n)$ *is highest weight of* $G : V$, *then* $a_1 \geq a_2 \geq \ldots \geq a_n$.

Proof: Follows from definition of highest weight. $\square$

The significant role of highest weight in representation theory can be seen in the following theorem,

Theorem 2.2.1 *Let* $G : V$ *be irreducible representation of* $G = \mathrm{Gl}_n(\mathbb{C})$ *with highest weight* α. *Then*

i. multiplicity of $\alpha := \dim V = 1$,

ii. $V \simeq_G W \Leftrightarrow$ *highest weights coincides.* $\square$

2.2.3 Tensor Product Problem

In previous subsection we saw that an irreducible, finite dimensional, holomorphic representation of $\mathrm{Gl}_n(\mathbb{C})$ is characterized by its highest weight $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n)$, where $\alpha_i \in \mathbb{Z}$.

For example, the representation $\bigwedge^k \mathbb{C}^n$ corresponding to the sequence $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ consisting of k 1's and $n-k$ 0's, and the representation $\mathrm{Sym}^k\mathbb{C}^n$ has highest weight $(k, 0, \ldots, 0)$.

As before we denote the irreducible representation with highest weight α by V_α. It is a basic fact of representation theory that $\mathrm{Gl}_n(\mathbb{C})$ is reductive. This means that any finite dimensional, holomorphic representation decomposes into a direct sum of irreducible representations, and the number of times that a given irreducible representation V_γ appears in the sum is independent of the choice of the decomposition. In particular, for any α and β, the tensor product $V_\alpha \otimes V_\beta$ decomposes into a direct sum of representations V_γ. Let us define $c^\gamma_{\alpha\beta}$ to be the number of copies of V_γ in an irreducible decomposition of $V_\alpha \otimes V_\beta$. In this situation the problem of interest is the following:

When does V_γ occur in $V_\alpha \otimes V_\beta$; i.e., when is $c^\gamma_{\alpha\beta} > 0$?

It follows immediately from the definition of highest weights that a necessary condition for this is that $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$. Other conditions are more difficult to find.

A simple case of this problem is when $\beta = (1, 1, \ldots, 1)$, so V_β is the one dimensional determinant representation [Fu]. In this case $V_\alpha \otimes V_\beta = V_{(\alpha_1+1, \alpha_2+1, \ldots, \alpha_n+1)}$. In particular, the problem is unchanged if each of the representations is tensored by this determinant representation several times. Therefore we may assume that each of α, β and γ consists of nonnegative integers, i.e., is partition.

In contrast to the spectral problem the coefficients of the tensor product decomposition

$$V_\alpha \otimes V_\beta = \sum c^\gamma_{\alpha\beta} V_\gamma$$

can be evaluated algorithmically by the Littlewood-Richardson rule, which is described in the following section.

It turns out that these two problems are essentially equivalent and have the same answer. To give it, we use the bijection between subsets $I \subset \{1, 2, \ldots, n\}$ of cardinality p and Young diagrams σ_I in a rectangular box of dimension p by q (see Section 2.1). One can formally multiply the diagrams by Littlewood-Richardson rule

$$\sigma_I \cdot \sigma_J = \sum_K c^K_{IJ} \sigma_K$$

where $c^K_{IJ} := c^{\sigma_K}_{\sigma_I \sigma_J}$ are the Littlewood-Richardson coefficients.

Theorem 2.2.2 ([Kl-2]) *The following conditions are equivalent*

i) There exists Hermitian operators A, B, $C = A + B$ with spectra $\lambda(A)$, $\lambda(B)$, $\lambda(C)$.

ii) Inequality (IJK) holds each time Littlewood-Richardson coefficient $c_{IJ}^K \neq 0$. Here $I, J, K \subset \{1, 2, \ldots, n\}$ are subsets of the same cardinality $p < n$.

iii) For integer spectra $\alpha = \lambda(A)$, $\beta = \lambda(B)$, $\gamma = \lambda(C)$ the above conditions are equivalent to

$$V_\gamma \subset V_\alpha \otimes V_\beta. \tag{2.4}$$

□

Remarks from [Kl-2]:
(1) The last claim iii) implies a recurrence procedure to generate all α, β, γ with $c_{\alpha\beta}^\gamma \neq 0$.

$$c_{\alpha\beta}^\gamma \neq 0 \Longleftrightarrow V_\gamma \subset V_\alpha \otimes V_\beta \Longleftrightarrow \gamma_K \leq \alpha_I + \beta_J \text{ each time } c_{IJ}^K \neq 0,$$

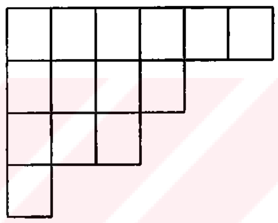
where $\gamma_K = \sum_{k \in K} \gamma_k$, $\alpha_I = \sum_{i \in I} \alpha_i$, $\beta_J = \sum_{j \in J} \beta_k$.
Here $c_{\alpha\beta}^\gamma$ are Littlewood-Richardson coefficients for group $\mathrm{Gl}_n(\mathbb{C})$ while c_{IJ}^K are Littlewood-Richardson coefficients for group $\mathrm{GL}_p(\mathbb{C})$ of smaller rank $p < n$. An explicit form of this recurrence has been conjectured by A.Horn [Ho] in the framework of Hermitian spectral problem.
(2) Inequalities (IJK) for $c_{IJ}^K \neq 0$ define a cone in the space of triplets of spectra, and the facets of this cone correspond to $c_{IJ}^K = 1$. P.Belkale [Be] was first to note that all inequalities (IJK) follows from those with $c_{IJ}^K = 1$, and in recent preprint A.Knutson, T.Tao, and Ch.Woodward [KTW] proved their independence. In [Kl-1] condition (2.4) appears in weaker form

$$V_{N\gamma} \subset V_{N\alpha} \otimes V_{N\beta} \text{ for some } N > 0, \tag{2.5}$$

and its equivalence to (2.4), known as saturation conjecture, was later proved by A.Knutson and T.Tao [KT], and in more general quiver context by H.Derksen and J.Weyman [DW].

2.3 Young Diagrams and Littlewood-Richardson Rule

The *Young diagram* α is an array of boxes, lined up at the left, with α_i boxes in the i^{th} row, with the rows arranged from top to bottom. For example



is the Young diagram of $(6, 4, 3, 1)$.

In the 1934 Littlewood and Richardson gave a remarkable combinatorial formula for the number $c^{\gamma}_{\alpha\beta}$ (defined in Section 2.2).

Their rule may be described as follows:

Let $\alpha = (\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n)$, $\beta = (\beta_1 \geq \beta_2 \geq ... \geq \beta_n)$ and $\gamma = (\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_n)$. Fill i^{th} row of the diagram β by numbers i.

1	1	1	1
2	2	2	
3	3		
4			

Then the Littlewood-Richardson coefficients $c^{\gamma}_{\alpha\beta}$ is the number of ways to produce γ by adding cells from β to α in such a way that:

i) The entries in any row are weakly increasing from left to right.

ii) The entries in each column are strictly increasing from top to bottom.

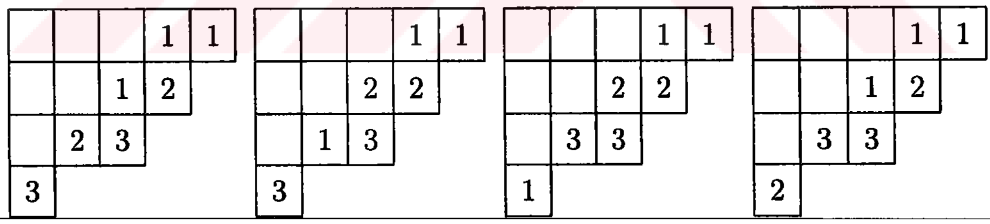
iii) The integer i occurs exactly β_i times.

iv) Reading all symbols from right to left and top to bottom we get a lattice permutation. (i.e. For any p with $1 \leq p < \sum \beta_i$ and any i with $1 \leq i < n$, the number of times i occurs in the first p boxes of the ordering is at least as large as the number of times that $i+1$ occurs in these first p boxes).

According to this rule we can formally multiply the Young diagrams α and β as

$$\alpha \otimes \beta = \sum c^{\gamma}_{\alpha\beta} \gamma.$$

Example: For $\alpha = (3, 2, 1)$, $\beta = (3, 2, 2)$ and $\gamma = (5, 4, 3, 1)$, the following are some of the ways to produce γ by adding cells from β satisfying the first three conditions:



The first three examples satisfy the forth condition. The forth one does not, since the first six boxes in ordering have more 3's than 2's. One can easily see that the first three are the only possibilities satisfying all four conditions, so $c^{\gamma}_{\alpha\beta} = 3$.

Two special cases of this rule were known to Pieri in the context of Schubert calculus [Fu]. Let $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n)$. If $\beta = (p)$ then the

possible γ for which $c^{\gamma}_{\alpha\beta} \neq 0$ are those of the form $(\gamma_1, \gamma_2, ..., \gamma_{n+1})$ with $\gamma_1 \geq \alpha_1 \geq \gamma_2 \geq \alpha_2 \geq ... \geq \gamma_n \geq \alpha_n \geq \gamma_{n+1} \geq 0$ with $\sum \gamma_i = \sum \alpha_i + p$.
In these cases $c^{\gamma}_{\alpha\beta} = 1$ (For representations of $GL_n(\mathbb{C})$ only those with $\gamma_{n+1} = 0$ are allowed). In terms of Young diagrams, β consists of a row p boxes, and the diagram of γ is obtained from that of α by adding p boxes, with no two in any column.

The other Pieri rule is for $\beta = (1, 1, ..., 1)$ consisting of p 1's. The possible γ with $c^{\gamma}_{\alpha\beta} \neq 0$ also have $c^{\gamma}_{\alpha\beta} = 1$, and these have the form $\gamma = (\gamma_1, \gamma_2, ..., \gamma_t)$ with $\alpha_i + 1 \geq \gamma_i \geq \alpha_i$ for all i and $\sum \gamma_i = \sum \alpha_i + p$.
Here β is a column of p boxes, and the diagram of γ is obtained from that of α by adding p boxes, with no two in any row.

2.4 Schubert Calculus

R.C. Thompson seems to have been the first one to realize that there are deep connections, we have mentioned in Section 2.1, between the spectral inequalities and a topic in algebraic geometry called Schubert calculus. Now we will give an exposition (not in details) of these ideas.

The set of all 1-dimensional subspaces of $\mathbb{C}^{n+1}$ is known as the *complex projective space* $\mathbb{P}^n$ of dimension n (for more details see [GH]). Any nonzero vector of $\mathbb{C}^{n+1}$ determines a point in $\mathbb{P}^n$; two points $(z_0, z_1, \ldots, z_n)$, $(z'_0, z'_1, \ldots, z'_n)$ determine the same point of $\mathbb{P}^n$ if and only if there is a nonzero $w \in \mathbb{C}$ such that $z'_i = w \cdot z_i$ for each $i = 0, 1, \ldots, n$. The point ℓ of $\mathbb{P}^n$ determined by $(z_0, z_1, \ldots, z_n)$ is denoted by $[z_0 : z_1 : \cdots : z_n]$ and these are called the *homogeneous coordinates of ℓ*. Note that the homogeneous coordinates of $\ell \in \mathbb{P}^n$ is not uniquely determined; they are determined up to multiplication by nonzero constants $w \in \mathbb{C}$.

The generalization of the notion of projective space is the *Grassmannian*

$G(p,q)$, which is defined to be the set of p-dimensional and q-codimensional subspaces of $\mathbb{C}^n$, where $p+q=n$. Given a p-dimensional subspace V of $\mathbb{C}^n$, we may represent V by a set of p row vectors in $\mathbb{C}^n$ spanning V, i.e., by a $p \times n$ matrix

$$A = \begin{pmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & \ddots & \vdots \\ e_{p1} & \cdots & e_{pn} \end{pmatrix}$$

of rank p. Clearly, any such matrix represents an element of $G(p,q)$ and any two such matrix A, B represent the same element of $G(p,q)$ if and only if $A = gB$ for some $g \in GL_p\mathbb{C}$. From the fact of linear algebra that any $m \times n$ matrix is row equivalent to a row reduced echelon matrix

$$\begin{pmatrix} 0 & 1 & * & 0 & * & 0 & * & \cdots & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * & \cdots & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots & 0 & * \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & * \end{pmatrix}, \tag{2.6}$$

where $*$ stands for arbitrary elements of $\mathbb{C}$, we have the following correspondence

$$p - \text{subspaces } V \subset \mathbb{C}^n \longleftrightarrow p \times n \text{ row reduced echelon matrices.}$$

Associates (2.6) with p-element subsets $\{j_1, j_2, \ldots, j_p\} \subset \{1, 2, \ldots, n\}$, where j_i's are indices of columns containing leading entries of rows $1, 2, \ldots p$, and after this we can also associate (2.6) with Young diagrams σ_{J^*} corresponding to subset $J^* = \{n+1-i \mid i \in I\}$.

Define X_J = the set of row reduced echelon matrices (2.6) with given p-element subset J of $\{1, 2, \ldots, n\}$ (or given Young diagram σ_{J^*}). Clearly $X_J \cong \mathbb{C}^N$, where $N = (j_2 - j_1 - 1) + 2(j_3 - j_2 - 1) + 3(j_4 - j_3 - 1) + \ldots + p(n - j_p) =$ area of the Young diagram σ_{J^*} corresponding to subset J^*. This set X_J

corresponds a subset in $G(p,q)$ which we call Schubert cell. Let us define the Schubert cell precisely.

A complete flag $\mathcal{F}$ is a chain $0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^n$ of subspaces of $\mathbb{C}^n$, with $\dim(F_i) = i$ for all i. Let $J = \{j_1, \ldots, j_p\}$ be any subset of cardinality p in $\{1, 2, , \ldots, n\}$. The Schubert cell is defined as

$$S_J = S(J, \mathcal{F}) = \{V \in G(p,q) \mid \dim(V \cap F_{j_i}) = i \ \text{ for } \ 1 \leq i \leq n\}.$$

The closure of S_J

$$\overline{S}_J = \overline{S(J, \mathcal{F})} = \{V \in G(p,q) \mid \dim(V \cap F_{j_i}) \leq i \ \text{ for } \ 1 \leq i \leq n\}$$

is called Schubert cycle.

It is a basic fact that the classes $s_J = [\overline{S}_J]$ of Schubert cycles $\overline{S}_J$, we call them also Schubert cycles, form a $\mathbb{Z}$-basis for the cohomology ring of the Grassmannian. It follows that for any p-subset I and J there is a unique expression

$$s_I \cdot s_J = \sum d_{IJ}^K s_K,$$

for integers d_{IJ}^K. It is a consequence of the fact that $\mathrm{GL_p}(\mathbb{C})$ acts transitively on $G(p,q)$ that all these coefficients are nonnegative. By Theorem 2.2.1 instead of Hermitian spectral problem, we can deal with the following problem.

When does s_K appear in the product $s_I \cdot s_J$; i.e. when is the coefficient d_{IJ}^K positive?

In 1947 L.Lesieur [Le] proved that the Littlewood-Richardson coefficients c_{IJ}^K are the same as the coefficients d_{IJ}^K which describe the multiplication in the cohomology ring of the Grassmannian. In fact, $d_{IJ}^K = \sharp\{S(I^*, \mathcal{F}) \cap S(J^*, \mathcal{G}) \cap S(K, \mathcal{H})\}$, where $S(I^*, \mathcal{F})$, $S(J^*, \mathcal{G})$, $S(K, \mathcal{H})$ are Schubert cells corresponding to any three complete flags $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$. So we have

Theorem 2.4.1 *For the triple (I, J, K) of subsets I, J, K of $\{1, 2, \ldots, n\}$, the inequality (IJK) holds if and only if for any three complete flags $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ the intersection of the Schubert cells $S(I^*, \mathcal{F})$, $S(J^*, \mathcal{G})$, $S(K, \mathcal{H})$ is nonempty.* $\square$

2.5 Vector Bundles

A $\mathbb{C}^\infty$ *complex vector bundle* on a differentiable manifold M consists of a family $\{E_x\}_{x \in M}$ of complex vector spaces parametrized by M together with a $\mathbb{C}^\infty$ manifold structure on $E = \cup_{x \in M} E_x$ such that

(1) The projection map $\pi : E \to M$ taking E_x to x is $\mathbb{C}^\infty$, and

(2) For every $x_0 \in M$, there exists an open set U in M containing x_0 and a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$ taking the vector space E_x isomorphically onto $\{x\} \times \mathbb{C}^k$ for each $x \in U$;

φ_U is called a trivialization of E over U and E_x is called the fiber corresponding to the point x.

Note that all the following definitions and results are from [Kl-2] and [Kl-1].

Consider the projective plane

$$\mathbb{P}^2 = \{(x^\alpha : x^\beta : x^\gamma) \mid x \in \mathbb{C}\}$$

on which diagonal torus

$$T = \{(t_\alpha : t_\beta : t_\gamma) \mid t \in \mathbb{C}^*\}$$

acts by the formula $t \cdot x = (t_\alpha x^\alpha : t_\beta x^\beta : t_\gamma x^\gamma)$.

Orbits of this action are vertices , sides and complement of the coordinate triangle. In particular there is unique dense orbit, consisting of points with nonzero coordinates.

The objects of our interest are T-equivariant (or toric for short) vector bundles $\mathcal{E}$ over $\mathbb{P}^2$. This means that $\mathcal{E}$ is endowed with an action $T : \mathcal{E}$ which is linear on fibers and makes the following diagram commutative

$$\begin{tikzcd} \mathcal{E} \arrow[r, "t"] \arrow[d, "\pi"'] & \mathcal{E} \arrow[d, "\pi"] \\ \mathbb{P}^2 \arrow[r, "t"] & \mathbb{P}^2 \end{tikzcd}, \qquad t \in T.$$

Let us fix a generic point $p_0 \in \mathbb{P}^2$ not in a coordinate line, and denote by

$$E := \mathcal{E}(p_0)$$

the corresponding generic fiber. There is no action of torus T on the fiber E. Instead, the equivariant structure produces some distinguished subspaces in E by the following construction. Let us choose a generic point $p_\alpha \in X^\alpha$ in coordinate line $X^\alpha : x^\alpha = 0$. Since T-orbit of p_0 is dense in $\mathbb{P}^2$, we can vary $t \in T$ so that tp_0 tends to p_α. Then for any vector $e \in E = \mathcal{E}(p_0)$, we have $t \cdot e \in \mathcal{E}(tp_0)$ and can try the limit

$$\lim_{tp_0 \to p_\alpha} (te),$$

which either exists or not. Let us denote by $E^\alpha(0)$ the set of vectors $e \in E$ for which the limit exists:

$$E^\alpha(0) := \{e \in E \mid \lim_{tp_0 \to p_\alpha} (te) \quad \text{exists}\}.$$

Evidently $E^\alpha(0)$ is a vector subspace of E, independent of p_0 and p_α. An easy modification of the previous construction allows to define for integer $m \in \mathbb{Z}$, the subspace

$$E^\alpha(m) := \{e \in E \mid \lim_{tp_0 \to p_\alpha} \left(\frac{t_\alpha}{t_\beta}\right)^{-m} (te) \quad \text{exists}\}.$$

Roughly speaking $E^\alpha(m)$ consists of vectors $e \in E$ for which te vanishes up to order m as tp_0 tends to coordinate line X^α. The subspaces $E^\alpha(m)$ form a non-increasing $\mathbb{Z}$-filtration:

$$E^\alpha : \cdots \supset E^\alpha(m-1) \supset E^\alpha(m) \supset E^\alpha(m+1) \supset \cdots$$

which is exhaustive

$$\begin{aligned} E^\alpha(m) &= 0\,, \quad \text{for } m \gg 0, \\ E^\alpha(m) &= E\,, \quad \text{for } m \ll 0. \end{aligned}$$

Applying this construction to other coordinate lines, we get a triple of filtrations $E^\alpha,\ E^\beta,\ E^\gamma$ in generic fiber $E = \mathcal{E}(p_0)$, associated with toric bundle $\mathcal{E}$.

Theorem 2.5.1 [Kl-3] *The correspondence*

$$\mathcal{E} \mapsto (E^\alpha,\ E^\beta,\ E^\gamma)$$

establishes an equivalence between category of toric bundles on $\mathbb{P}^2$ and category of triply filtered vector spaces. $\square$

Denote the toric bundle corresponding to triplet of filtrations $E^\alpha,\ E^\beta,\ E^\gamma$ by $\mathcal{E}(E^\alpha,\ E^\beta,\ E^\gamma)$.

Recall that, the toric bundle $\mathcal{E} = \mathcal{E}(E^\alpha,\ E^\beta,\ E^\gamma)$ is stable iff for every proper subspace $F \subset E$ the following inequality holds

$$\frac{1}{\dim F} \sum_{\nu=\alpha,\beta,\gamma} i \dim F^{[\nu]}(i) < \frac{1}{\dim E} \sum_{\nu=\alpha,\beta,\gamma} i \dim E^{[\nu]}(i), \qquad (2.7)$$

where $F^\nu(i) = F \cap E^\nu(i)$ is induced filtration with composition factors $F^{[\nu]}(i) = F^\nu(i)/F^\nu(i+1)$, and semi-stable if weak inequalities hold (with sign $\leq$ instead of $<$).

If filtrations E^α, E^β, E^γ are in general position then they are given by three Schubert cells S_α, S_β, S_γ as follows: Fixing $\dim F = p$ we get $F \in G(p,q)$. So subspaces with given $\dim(F \cap E^{[\nu]}(i))$, $i \in \mathbb{Z}$, form a Schubert cell S_ν. Hence stability inequality (2.7) holds if and only if

$$S_\alpha \cap S_\beta \cap S_\gamma \neq \emptyset. \tag{2.8}$$

For filtrations in general position (2.8) is equivalent to nonvanishing of the product of Schubert cycles $s_\alpha \cdot s_\beta \cdot s_\gamma \neq 0$ in cohomology ring of the Grassmannian. From previous section the inequality (2.8) holds iff $c^\gamma_{\alpha\beta} \neq 0$. So if filtrations E^α, E^β, E^γ are in general position then stability inequality (2.7) amount to inequalities (IJK).

Chapter 3

Classical Inequalities and Applications

3.1 Classical Inequalities

In this section we will deduce some classical inequalities, and for this we used [Kl-1].

3.1.1 Weyl Inequalities

Let us take $p = 1$. Then $G(p, q) = \mathbb{P}^{n-1}$. In this case the Schubert cycle s_i corresponding to one element subset $I = \{i\}$ is just H^{i-1}, where H is the class of a hyperplane. So we have the equation

$$s_i \cdot s_j = s_{i+j-1} \ , \qquad \text{for } \ i + j \leq n + 1$$

which implies, by Theorem 2.1.1, the Weyl inequality $\lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B)$.

Taking $q = 1$, we get in a similar way the inequality

$$\lambda_{i+j-n}(A + B) \geq \lambda_i(A) + \lambda_j(B) \ , \qquad \text{for } \ i + j \geq n + 1$$

also due to Weyl.

3.1.2 Ky Fan Inequalities

Now let p be arbitrary and

$$I = J = K = \{1, 2, \ldots, p\}.$$

Then $\sigma_I = \sigma_J = \sigma_K = \emptyset$ and

$$s_I = s_J = s_K = (\text{the fundamental class of } G(p, q)).$$

Hence $s_K = s_I \cdot s_J$, and we get the Ky Fan inequality

$$\sum_{i \leq p} \lambda_i(A + B) \leq \sum_{i \leq p} \lambda_i(A) + \sum_{i \leq p} \lambda_i(B).$$

3.1.3 Lidskii-Wielandt Inequalities

We can extend the previous example by taking $I = \{1, 2, \ldots, p\}$ to be the initial interval and $J = K$ to be arbitrary. Then again s_I is the fundamental cycle and therefore

$$s_K = s_I \cdot s_J.$$

This gives us the Lidskii-Wielandt inequality

$$\sum_{i \in I} \lambda_i(A + B) \leq \sum_{i \in I} \lambda_i(A) + \sum_{i \leq p} \lambda_i(B).$$

For fixed dimensions n, one can easily find all the components of the product $s_I \cdot s_J$ by making use of the Littlewood-Richardson rule [Ja], [Mac] and then write down the corresponding inequalities (IJK). Most of them seem to be new.

3.1.4 Complementary Cycles

Let us take a pair of complementary diagrams σ_I and σ_J, so that the central symmetry of the $p \times q$-box maps σ_I onto the complement of σ_J. In this case

$$s_I \cdot s_J = (\text{class of a point}) = s_{\{q+1,q+2,\dots,n\}}.$$

Complementary diagrams σ_I, σ_J correspond to subsets

$$I = \{i_1 < i_2 < \dots < i_p\}$$
$$J = \{j_1 > j_2 > \dots > j_p\}$$

such that

$$i_k + j_k = n + 1.$$

Theorem 2.1.1 gives us in this case the inequality

$$\sum_{k \geq q} \lambda_k(A + B) \leq \sum_{i \in I} \lambda_i(A) + \sum_{i \in I} \lambda_{n+1-i}(B),$$

for any subset $I \subset \{1, 2, \dots, n\}$ of cardinality $p = n - q$.

3.2 Generalizing Interlacing Theorem.

Theorem 3.2.1 *In any interval $[a, b]$, the number of eigenvalues of A in $[a, b]$ and the number of eigenvalues of $A + \delta A$ in $[a, b]$ differ no more than by $rk(\delta A)$ (rank of δA).*

Proof: We can prove by induction on $r = rk(\delta A)$. When $r = 1$ this is an interlacing theorem: between any odd numbered (or even numbered) eigenvalues of A there is at least one eigenvalue of $A + \delta A$. Assume it is true for $r - 1$. We can write δA as a sum of two matrices B and C such that $rkB = r - 1$ and $rkC = 1$. By induction assumption, the number of eigenvalues of A in $[a, b]$ and the number of eigenvalues of $A + B$ in $[a, b]$

differ no more than by $rkB = r - 1$. By applying Theorem 3.2.1 to $A + B$ and C, we get Theorem 3.4.1.

An alternating proof can be given as follows:
Let A and B be two Hermitian n by n matrices that differ by a matrix of rank at most r, then applying the Weyl inequalities (1.2) to the triples $(B, A - B, A)$ and $(A, B - A, B)$ with $j = r + 1$, we get

$$\alpha_{k+r} \leq \beta_k \text{ and } \beta_{k+r} \leq \alpha_k \text{ for } 1 \leq k, \; k + r \leq n. \tag{3.1}$$

where α and β are eigenvalues of A and B, respectively, which implies the theorem. $\square$

Chapter 4

Horn's Conjecture

We have seen that the inequalities (IJK) give the complete and independent set of restrictions on eigenvalues of Hermitian matrices A, B and $C = A+B$, and that the following recurrence procedure generates all α, β, γ with $c^{\gamma}_{\alpha\beta} \neq 0$

$$c^{\gamma}_{\alpha\beta} \neq 0 \overset{L-R}{\Longleftrightarrow} V_{\gamma} \subset V_{\alpha} \otimes V_{\beta} \overset{Thm.1.3.1}{\Longleftrightarrow} \gamma_K \leq \alpha_I + \beta_J \text{ each time } c^K_{IJ} \neq 0, \quad (4.1)$$

where $c^{\gamma}_{\alpha\beta}$ are Littlewood-Richardson coefficients for group $\mathrm{GL}_n(\mathbb{C})$ and c^K_{IJ} are L-R coefficients for group $\mathrm{GL}_p(\mathbb{C})$ of smaller rank $p < n$. An explicit form of this recurrence has been conjectured by A.Horn [Ho] in 1962. Now, we give this conjecture briefly. To avoid the confusion we use different notation for Horn's settings.

Horn defined sets T^n_p of triples (K, L, M) of subsets of $\{1, 2, \dots, n\}$ of the same cardinality p, by the following recurrence procedure. Let us write $K = \{k_1 < k_2 < \dots < k_p\}$ and likewise for L and M. Set

$$U^n_p = \{(K, L, M) \mid \sum_{k\in K} k + \sum_{l\in L} l = \sum_{m\in M} m + \frac{p(p+1)}{2}\}$$

When $r = 1$, set $T^n_1 = U^n_1$ (The inequalities specified by (K, L, M) in T^n_1 are Weyl inequalities (1.2)). In general,

$$T_p^n = \{(K,L,M) \in U_p^n \mid \text{ for all } r < p \text{ and all } (F,G,H) \text{ in } T_r^p, \sum_{f\in F} k_f + \sum_{g\in G} l_g = \sum_{h\in H} m_h + \frac{r(r+1)}{2}\}.$$

Horn's Conjecture. *A triple (α, β, γ) occurs as eigenvalues of Hermitian n by n matrices A, B, and C with $C = A+B$ if and only if $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$ and the inequalities (IJK) hold for every (K, L, M) in T_p^n, for all $p < n$.*

In 1999, A.Knutson and T.Tao [KT] proved Horn's conjecture. They showed that the Horn's setting is equivalent to (4.1). Hence, we can write it as a theorem.

Theorem 4.0.2 *Horn's conjecture is true.* $\square$

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