BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

GLOBAL BEHAVIORS OF SOME NONLINEAR DIFFERENCE EQUATIONS AND SOME SYSTEMS OF RATIONAL DIFFERENCE EQUATIONS

DEPARTMENT OF MATHEMATICS

DOCTOR OF PHILOSOPHY THESIS

MEHMET GÜMÜŞ

June 2017

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Mehmet GÜMÜŞ

ADVISOR: Assoc. Prof. Yüksel SOYKAN

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by the academic rules and ethical principles all works that are not result of this study are cited properly."

Mehmet GÜMÜŞ

ABSTRACT

Ph. D. Thesis

GLOBAL BEHAVIORS OF SOME NONLINEAR DIFFERENCE EQUATIONS AND SOME SYSTEMS OF RATIONAL DIFFERENCE EQUATIONS

Mehmet GÜMÜŞ

Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assoc. Prof. Yüksel SOYKAN
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In this thesis, we give a systematic study of the local and global asymptotic stability of equilibrium points, the periodic nature, the boundedness, the oscillation, the semi-cycle analysis of positive solutions of some specific rational non-linear difference equations and systems of difference equations.

The organization of this thesis is as follows:

Chapter 1 is a brief overview of what this thesis is about and also is an introduction of difference equation theory.

Chapter 2 contains some basic definitions and results which are used throughout the thesis. In this regard, this is a self-contained monograph.

In Chapter 3 some results will be given about the periodic character of positive solutions of

ABSTRACT (continued)

the difference equation $x_{n+1} = \alpha + x^p_{n-k} / x_n^q$, for $n = 0,1,...$ where $k > 1$ is an odd integer, $\alpha, p, q \in (0, \infty)$ and the initial conditions $x_{-k},..., x_0$ are arbitrary positive numbers.

In chapter 4 it will be studied the dynamical behaviour of positive solutions for a system of rational difference equations of the following form $u_{n+1} = \alpha u_{n-1} / (\beta + \gamma v^p_{n-2})$, $v_{n+1} = \alpha_1 v_{n-1} / (\beta_1 + \gamma_1 u^p_{n-2})$, for $n = 0,1,...$ where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p$ and the initial values $u_{-2}, u_{-1}, u_0, v_{-2}, v_{-1}, v_0$ are positive real numbers.

In Chapter 5 it will be investigated the local asymptotic stability of equilibria, the periodic nature of solutions, the existence of unbounded solutions and the global behaviour of solutions of the rational difference equation $x_{n+1} = \alpha x_{n-(k+1)} / (\beta + \gamma x^p_{n-k} x^q_{n-(k+2)})$, for $n = 0,1,...$ where the parameters $\alpha, \beta, \gamma, p, q$ are non-negative numbers and the initial values $x_{-(k+2)}, x_{-(k+1)},..., x_{-1}, x_0$ are positive numbers.

Keywords: Difference equation, recursive sequence, equilibrium point, global asymptotic stability, oscillation.

Science Code: 403.06.01

ÖZET

Doktora Tezi

BAZI RASYONEL FARK DENKLEM SİSTEMLERİNİN VE LİNEER OLMAYAN FARK DENKLEMLERİNİN GLOBAL DAVRANIŞLARI

Mehmet GÜMÜŞ

Bülent Ecevit Üniversitesi
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Bu tezde, bazı özel fark denklem sistemlerinin ve rasyonel lineer olmayan fark denklemlerinin çözümlerinin yarı döngü analizi, salınımlılığı, periyodiklik yapısı, sınırlılığı ve denge noktalarının lokal ve global asimptotik kararlılığının sistematik bir çalışması verilmiştir.

Bu tezin organizasyonu aşağıdaki gibidir:

Birinci bölüm bu tezin ne ile alakalı olduğu hakkında kısa bir tanıtım ve aynı zamanda fark denklem teorisinin bir girişidir.

İkinci bölüm tez boyunca kullanılan bazı temel tanımları ve sonuçları içerir. Bu bağlamda, bu bölüm kendi kendine yeten bir bölümdür.

Üçüncü bölümde, keyfi pozitif başlangıç koşulları $x_{-k},...,x_0$, $\alpha, p, q \in (0,\infty)$ ve $k>1$ tek

ÖZET (devam ediyor)

pozitif tamsayı için $x_{n+1}=\alpha+x^{p}_{n-k}/x_n^{q}$, $n=0,1,...$ fark denkleminin periyodiklik karakteri hakkında bazı sonuçlar verilmiştir.

Dördüncü bölümde, keyfi pozitif başlangıç koşulları $u_{-2},u_{-1},u_0,v_{-2},v_{-1},v_0$ ve pozitif α,β,γ,p,q parametreleriyle $u_{n+1}=\alpha u_{n-1}/(\beta+\gamma v^{p}_{n-2})$, $v_{n+1}=\alpha_1 v_{n-1}/(\beta_1+\gamma_1 u^{p}_{n-2})$, $n=0,1,...$ fark denklem sisteminin pozitif çözümlerinin dinamik davranışları çalışılmıştır.

Beşinci bölümde, pozitif başlangıç koşulları $x_{-(k+2)},x_{-(k+1)},...,x_{-1},x_0$ ve α,β,γ,p,q negatif olmayan parametreleri için $x_{n+1}=\alpha x_{n-(k+1)}/(\beta+\gamma x^{p}_{n-k}x^{q}_{n-(k+2)})$, $n=0,1,...$ rasyonel fark denkleminin çözümlerinin global davranışı, sınırsız çözümlerinin varlığı, çözümlerinin periyodiklik yapısı ve denge noktalarının lokal asimptotik kararlılığı araştırılmıştır.

Anahtar Kelimeler: Fark denklem, tekrarlı dizi, denge noktası, global asimptotik kararlı, salınımlılık.

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CHAPTER 1

INTRODUCTION

The investigation of the qualitative analysis of difference equations (or called recursive sequences) and their systems is a fruitful research field and gainly attracts lots of mathematicians. Since lots of real-life phenomenon are modeled via difference equations, the studies related to this field gain high important. Difference equations or discrete dynamical systems theory is a various field which influences nearly every branch of both pure and applied mathematics. Each difference equation $r_{n+1} = h(r_n)$ settles a discrete dynamical system or vice versa. Thus, the study of difference equations is equivalance to the study of discrete dynamical systems. Applications of difference equations have emerged in lots of real-life situations such as ecology, genetics in biology, geometry, physics, population dynamics, psychology, sociology, engineering, economics, statistical problems, combinatorial analysis, electrical networks, neural networks, probability theory, number theory, resource management and so on (see [6, 9, 17]), for instance the difference equation $z_{n+1} = (a + bz_n)/(c + z_n)$ which is known as Riccati Difference Equation, has many applications in mathematical biology and optics (see [1]). That is to say, as special non-linear difference equations have become a research hot spot in mathematical modelling.

Studying of difference equations is not only valuable in its own right, but also it can provide insights into their continuous counterparts, that is, the differential equations. However, difference equations should not only considered as the discrete analogs of differential equations. This is the fact that difference equations appeared much earlier than differential equations and were instrumental in paving the way for the development of the latter. In recent times, difference equations have started receiving the attention they deserve. This is majorly due to the furtherance of computers where differential equations are solved by using their approximate difference equation formulations. Especially, the last quarter of the 20th century has witnessed great development in the theory of discrete dynamical systems and difference equations because of its application in nearly all branches of mathematical and life sciences.

We can state that thanks to some computer programs, for example Mathematica, Matlab etc., researcher can be readily informed about a difference equation and he can readily find out that this equation own fascinating features with a large deal of structure and regularity. We also state that analytically all computer observations and predictions must also be proven.

The difference equations investigation is rather compeller and awarding. There are two ways to study non-linear difference equations, first one is giving the exact expression of all solutions. Second one is studying the qualitative analysis such as asymptotical stability using the linearized method, semicycle analysis, oscillatory behavior, the periodicity, global behavior and so on.

At the same time, more and more attention is paid to systems of rational difference equations composed by two or more rational difference equations. The single equation seems simple, but the coupled ways of systems are diverse and thus such systems have no fixed ways to follow to investigate their behavior.

Rational sequences (or rational recursive sequences) are also called rational (or recursive) difference equations. These equations mostly seem very simple and their properties of solutions can also be observed by computers' simulations, however, it is exceedingly difficult to prove analytically the properties observed and conjectured by computers' simulations. Therefore, the qualitative analysis of recursive difference equations has been the target of recent study.

The aim in this thesis is to research a systematic study of the global behavior of solutions of some specific rational non-linear difference equations and systems of difference equations.

Chapter 1 is a brief presentation of what this thesis is about and also is an introduction and a preliminary of difference equation theory.

Chapter 2 contains some basic definitions and some important results which are used throughout the thesis. In this regard, this is a self-contained section.

In Chapter 3 some results will be given about the periodic character of positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^q}, \text{ for } n = 0, 1, \ldots$$

where

$k > 1$

is an odd integer,

$\alpha, p, q \in (0, \infty)$

and the initial conditions

$x_{-k}, \ldots, x_0$

are arbitrary non-negative numbers.

In Chapter 4 it will be studied the dynamical behavior of positive solutions for a system of rational difference equations of the following form

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma y_{n-2}^{p}},\ y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 + \gamma_1 x_{n-2}^{p}} \text{ for } n = 0, 1, ...$$

where the parameters

$\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p \in (0, \infty)$

and the initial values

$x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$

are non-negative real numbers.

In Chapter 5 it will be investigated the local asymptotic stability of the equilibrium points, the periodic nature of solutions, the existence of unbounded solutions and the global behavior of solutions of the non-linear difference equation

$$x_{n+1} = \frac{\alpha x_{n-(k+1)}}{\beta + \gamma x_{n-k}^{p} x_{n-(k+2)}^{q}}, \text{ for } n = 0, 1, ...$$

where the parameters

$\alpha, \beta, \gamma, p, q \in (0, \infty)$

and the initial values

$x_{-(k+2)}, x_{-(k+1)}, ..., x_{-1}, x_0 \in \mathbb{R}^+.$

The following two subsections contain some recent important studies about difference equations which have taken turn to our studies in this thesis.

1.1 LITERATURE SUMMARY FOR NON-LINEAR DIFFERENCE EQUATIONS

In this subsection, we will give some important studies about the qualitative behavior of nonlinear difference equations in recent literature.

In [21], Amleh et al. studied the global asymptotic stability, the periodicity and the boundedness character of all positive solutions of the recursive sequence

$$x_{n+1} = b + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \dots \tag{1.1}$$

where b is positive, and the initial values x_{-1}, x_0 are positive numbers.

In [22], Stevic investigated the boundedness character, the oscillation, the global asymptotic stability and the periodic nature of all positive solutions of the rational difference equation

$$x_{n+1} = a + \frac{x_{n-1}^r}{x_n^r}, \quad n = 0, 1, \dots \tag{1.2}$$

where $a, r \in (0, \infty)$, and the initial values x_{-1}, x_0 are non-negative numbers (see also [23] for more results on this equation).

In [24], Devault et al. studied dynamical behavior, that is, the boundedness nature, the global asymptotic stability and the periodicity of all solutions of the non-linear difference equation

$$x_{n+1} = A + \frac{x_{n-l}}{x_n}, \quad n = 0, 1, \dots \tag{1.3}$$

where A is positive,

$$l \in \{2, 3, ...\}$$

and the initial conditions

$$x_{-l}, x_{-l+1}, ..., x_{-1}, x_0$$

are non-negative numbers.

In [30], Papaschinopoulos et al. investigated the boundedness character, the global asymptotic stability and the periodicity of all positive solutions of rational recursive sequence

$$x_{n+1} = \beta + \frac{x_{n-1}^k}{x_n^l}, \quad n = 0, 1, \dots \tag{1.4}$$

where β, k, l are positive, and the initial conditions x_{-1}, x_0 are positive numbers.

In [38], El-Owaidy et al. investigated the global behavior of the following rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}, \quad n = 0, 1, \ldots \tag{1.5}$$

with non-negative parameters and non-negative initial values.

By generalizing the results due to El-Owaidy et al. [38], in [34], Chen et al. studied the dynamical behavior of the following rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma x_{n-l}^p}, \quad n = 0, 1, \ldots \tag{1.6}$$

where

$k, l \in \mathbb{N}$,

the parameters are positive real numbers and the initial values

$x_{-\max\{k,l\}}, \ldots, x_{-1}, x_0 \in (0, \infty)$.

In [62], Ahmed studied the dynamical behavior of the following rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \prod_{i=l}^{k} x_{n-2i}^{p_i}}, \quad n = 0, 1, \ldots \tag{1.7}$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

In [27], Erdogan et al. investigated the dynamical behavior of positive solutions of the following higher order difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \sum_{k=1}^{t} x_{n-2k} \prod_{k=1}^{t} x_{n-2k}}, \quad n = 0, 1, \ldots \tag{1.8}$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

In [66], Karatas investigated the global behavior of the equilibria of the following difference equation

$$x_{n+1} = \frac{A x_{n-m}}{B + C \prod_{i=0}^{2k+1} x_{n-i}}, \quad n = 0, 1, \ldots \tag{1.9}$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

1.2 LITERATURE SUMMARY FOR DISCRETE DYNAMICAL SYSTEMS

In this subsection, we will give some studies about dynamical behavior of systems of rational non-linear difference equations in recent literature.
In [46], Papaschinopoulos and Schinas generalized the results concerning equation (1.1) to the system of two nonlinear difference equations

$$x_{n+1} = \alpha + \frac{x_{n-1}}{y_n},\ y_{n+1} = \alpha + \frac{y_{n-1}}{x_n}, \quad n = 0, 1, \ldots \tag{1.10}$$

where α is a non-negative constant and the initial values

$x_{-i}, y_{-i} \in (0, \infty),\ i = 0, 1.$

In [45], Papaschinopoulos and Schinas studied the system of two non-linear difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-p}},\ y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0, 1, \ldots \tag{1.11}$$

where

$A \in (0, \infty)$

and p, q are positive integers.
Taking reciprocals of the fractions (1.11) and letting $p = q$, the following system

$$x_{n+1} = A + \frac{x_{n-p}}{y_n},\ y_{n+1} = A + \frac{y_{n-p}}{x_n}, \quad n = 0, 1, \ldots \tag{1.12}$$

is obtained which was investigated in [55] by Zhang et al.
In [57], Zhang et al. studied the behavior of the symmetrical system of rational difference equations

$$x_{n+1} = A + \frac{y_{n-p}}{y_n},\ y_{n+1} = A + \frac{x_{n-p}}{x_n}, \quad n = 0, 1, \ldots \tag{1.13}$$

where

$A \in (0, \infty)$

and the initial conditions

$x_{-i}, y_{-i} \in (0, \infty),\ i = 0, 1 \ldots, p.$

In [47], Papaschinopoulos and Papadopoulos studied some properties of the following system

$$x_{n+1} = \alpha + \frac{x_n}{y_{n-m}},\ y_{n+1} = \beta + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, \ldots \tag{1.14}$$

where

$A \in (0, \infty)$

and the initial values

$x_{-i}, y_{-i} \in (0, \infty),\ i = 0, 1 \ldots, m;$

while in [59], Camouzis and Papaschinopoulos studied the system (1.14) for the case

$\alpha = \beta = 1,$

in [56], Zhang et al. investigated the system (1.14) taking reciprocals of the fractions and

$\alpha = \beta.$

In [50], Bao studied local stability, oscillatory character of positive solutions to the system of the two non-linear difference equations

$$x_{n+1} = p + \left(\frac{x_{n-1}}{y_n}\right)^q,\ y_{n+1} = p + \left(\frac{y_{n-1}}{x_n}\right)^q, \quad n = 0, 1, \ldots \tag{1.15}$$

where

$p \in (0, \infty),\ q \in [1, \infty),$

the initial conditions

$x_{-i}, y_{-i} \in (0, \infty),\ i = 0, 1.$

In [58], Zhang et al. generalized the results concerning equation (1.14) to the system of rational difference equations

$$x_{n+1} = a + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}},\ y_{n+1} = b + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}}, \quad n = 0, 1, \ldots \tag{1.16}$$

where

$a, b \in (0, \infty)$

and the initial conditions

$x_{-i}, y_{-i} \in (0, \infty),\ i = 0, 1, \ldots k.$

In [53], Yalcinkaya and Cinar investigated the global asymptotic behavior of solutions of the following system of difference equations

$$z_{n+1} = \frac{t_n + z_{n-1}}{t_n z_{n-1} + \alpha},\ t_{n+1} = \frac{z_n + t_{n-1}}{z_n t_{n-1} + \alpha}, \quad n = 0, 1, \ldots \tag{1.17}$$

where

$\alpha \in (0, \infty)$

and the initial values

$z_k, t_k \in (0, \infty)$ for $k = -1, 0.$

In [60], Sroysang studied the solutions, the stability character and the asymptotic behavior of the system of difference equations

$$x_{n+1} = \frac{x_{n-m+1}}{A + y_n y_{n-1} \cdots y_{n-m+1}},\ y_{n+1} = \frac{y_{n-m+1}}{B + x_n x_{n-1} \cdots x_{n-m+1}}, \quad n = 0, 1, \ldots \tag{1.18}$$

where

$A, B \in (0, \infty)$

and the initial conditions are arbitrary positive numbers.

In [41], Khan and Qureshi studied the qualitative behavior of the following two systems of difference equations

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma y_{n-k+1}^2},\ y_{n+1} = \frac{\alpha_1 y_{n-k}}{\beta_1 + \gamma_1 x_{n-k+1}^2}, \quad n = 0, 1, \ldots \tag{1.19}$$

and

$$x_{n+1} = \frac{a y_{n-k}}{b + c x_{n-k+1}^2},\ y_{n+1} = \frac{a_1 x_{n-k}}{b_1 + c_1 y_{n-k+1}^2}, \quad n = 0, 1, \ldots \tag{1.20}$$

where the parameters

$\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, a, b, c, a_1, b_1, c_1$

and the initial conditions are positive real numbers.

In [43], Kurbanlı et al. studied the behavior of positive solutions of the following system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, \; y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}, \quad n = 0, 1, \ldots \tag{1.21}$$

where the initial conditions are arbitrary non-negative real numbers.

In [54], Zhang et al. studied the dynamical behavior of positive solutions for a system for third-order rational difference equations

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \; y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots. \tag{1.22}$$

In [36], Din et al. studied the dynamics of a system of fourth-order rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \; y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}, \quad n = 0, 1, \ldots. \tag{1.23}$$

In [49], Touafek and Elsayed investigated the behavior of solutions of systems of difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}}, \; y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}}, \quad n = 0, 1, \ldots \tag{1.24}$$

with a non-zero real number's initial conditions.

In [48], Papaschinopoulos and Schinas studied the boundedness, the persistence of positive solutions, the existence of a unique positive equilibrium and the global asymptotic stability of the positive equilibrium of the following system of difference equations

$$x_{n+1} = \sum_{i=0}^{k} \frac{A_i}{y_{n-i}^p}, \; y_{n+1} = \sum_{i=0}^{k} \frac{B_i}{x_{n-i}^p}, \quad n = 0, 1, \ldots \tag{1.25}$$

where

$A_i, B_i, \; i \in \{0, 1, ..., k\}$,

$x_i, y_i, i = -k, -k+1, ..., 0$

are positive numbers and

$p_i, q_i, i = 0, 1, ..., k$

are positive constants.

CHAPTER 2

THE GENERAL DEFINITIONS AND THEOREMS ABOUT DIFFERENCE EQUATIONS

In this chapter, we state some definitions and theorems used in this thesis. For details, see [2-17].

Definition 2.1 *(Difference Equation) ([2]) A difference equation of order* $(k+1)$ *is an equation of the following form*

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}),\ n = 0, 1, \dots \tag{2.1}$$

where f *is a continuously differentiable function that maps some set* I^{k+1} *into* I. *The set* I *is usually an interval of real numbers, or a union of intervals, or a discrete set such as the set of integers*

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}.$$

For any conditions

$$x_{-k}, \dots, x_{-1}, x_0 \in I,$$

the $(k+1)-$*order difference equation (2.1) has a unique positive solution* $\{x_n\}_{n=-k}^{\infty}$.

Definition 2.2 *(Discrete Dynamical System) ([2]) An* m*-dimensional discrete dynamical system is a system of the following form*

$$\left.\begin{aligned} x_{n+1} &= f_1(x_n, x_{n-1}, \dots, x_{n-m}, y_n, y_{n-1}, \dots, y_{n-m}), \\ y_{n+1} &= f_2(x_n, x_{n-1}, \dots, x_{n-m}, y_n, y_{n-1}, \dots, y_{n-m}), \end{aligned}\right\} \tag{2.2}$$

where

$$f_1 : I_1^{m+1} \times I_2^{m+1} \to I_1$$

and

$$f_2 : I_1^{m+1} \times I_2^{m+1} \to I_2$$

are continuously differentiable functions and I_1, I_2 are some intervals of real numbers. Also, a solution $\{(x_n, y_n)\}_{n=-m}^{\infty}$ of system (2.2) is uniquely determined by initial values

$(x_{-i}, y_{-i}) \in I_1 \times I_2$ *for* $i = 0, 1, \ldots, m$.

2.1 STABILITY AND GENERAL BACKGROUND

The following two definitions deal with the fixed point of function or the equilibria of equation.

Definition 2.3 *([4]) An equilibrium point of Eq.(2.1) is a point $\overline{x}$ that satisfies*

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

The point $\overline{x}$ is also said to a fixed point of the function f.

Definition 2.4 *([3]) An equilibrium point of system (2.2) is a point $(\overline{x}, \overline{y})$ that satisfies*

$$\begin{aligned} \overline{x} &= f_1(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}) \\ \overline{y} &= f_2(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}). \end{aligned}$$

Together with the system (2.2), if we consider the associated vector map

$$F = (f_1, x_n, x_{n-1}, \ldots, x_{n-m}, f_2, y_n, y_{n-1}, \ldots, y_{n-m}),$$

then, the point $(\overline{x}, \overline{y})$ is also called a fixed point of the vector map F.

Definition 2.5 *([3]) Let $\overline{x}$ be a positive equilibrium of (2.1).*
(i) $\overline{x}$ is stable if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every positive solution $\{x_n\}_{n=-k}^{\infty}$ of (2.1) with $\sum_{i=-k}^{0} |x_i - \overline{x}| < \delta$, $|x_n - \overline{x}| < \varepsilon$ holds for $n \in \mathbb{N}$. If the equilibrium point $\overline{x}$ is not stable, then it is called to be unstable.
(ii) $\overline{x}$ is locally asymptotically stable if $\overline{x}$ is stable and there is $\gamma > 0$ such that

$$\lim x_n = \overline{x}$$

holds for every positive solution $\{x_n\}_{n=-k}^{\infty}$ of (2.1) with

$$\sum_{i=-k}^{0} |x_i - \overline{x}| < \gamma.$$

(iii) $\overline{x}$ *is a global attractor if*

$$\lim x_n = \overline{x}$$

holds for every positive solution $\{x_n\}_{n=-k}^{\infty}$ *of (2.1).*

(iv) $\overline{x}$ *is globally asymptotically stable if* $\overline{x}$ *is both stable and global attractor.*

Definition 2.6 *([5]) Let* $(\overline{x}, \overline{y})$ *be an equilibrium point of the system (2.2).*

(i) An equilibrium point $(\overline{x}, \overline{y})$ *is called to be stable if for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for every initial values* $(x_{-i}, y_{-i}) \in I_1 \times I_2$*, for* $i = 0, 1, \ldots, m$ *with*

$$\begin{aligned} \sum_{i=-m}^{0} |x_i - \overline{x}| &< \delta, \\ \sum_{i=-m}^{0} |y_i - \overline{y}| &< \delta, \end{aligned}$$

implies

$$\begin{aligned} |x_n - \overline{x}| &< \varepsilon, \\ |y_n - \overline{y}| &< \varepsilon, \end{aligned}$$

for all $n \geq 1$.

(ii) If an equilibrium point $(\overline{x}, \overline{y})$ *is not stable, then it is called to be unstable.*

(iii) If an equilibrium point $(\overline{x}, \overline{y})$ *is stable and there exists* $\gamma > 0$ *such that*

$$\begin{aligned} \sum_{i=-m}^{0} |x_i - \overline{x}| &< \gamma, \\ \sum_{i=-m}^{0} |y_i - \overline{y}| &< \gamma \end{aligned}$$

and

$$(x_n, y_n) \to (\overline{x}, \overline{y}) \text{ as } n \to \infty,$$

then, it is said to be asymptotically stable.

(iv) If the following condition

$$\lim_{n\to\infty} (x_n, y_n) = (\overline{x}, \overline{y}),$$

holds, then, the equilibrium point $(\overline{x}, \overline{y})$ *is said to be a global attractor.*

(v) If an equilibrium point $(\overline{x}, \overline{y})$ *is both global attractor and stable, then, it is said to be globally asymptotically stable.*

Definition 2.7 *(Linearization Method) ([2])*
The linearized equation of (2.1) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + \ldots + p_k y_{n-k}, \ n \in \mathbb{N} \tag{2.3}$$

where

$$\begin{aligned} p_0 &= \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \ldots, \overline{x}), \\ p_1 &= \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \ldots, \overline{x}), \\ &\vdots \\ p_k &= \frac{\partial f}{\partial x_{n-k}}(\overline{x}, \overline{x}, \ldots, \overline{x}). \end{aligned}$$

The characteristic equation of (2.3) is

$$\lambda^{k+1} - p_0 \lambda^k - p_1 \lambda^{k-1} - \ldots - p_k = 0. \tag{2.4}$$

Definition 2.8 *(Linearization Method for a discrete dynamical system) ([3])*
If $(\overline{x}, \overline{y})$ be an equilibrium point of a map $F = (f_1, x_n, \ldots, x_{n-m}, f_2, y_n, \ldots, y_{n-m})$ where f_1 and f_2 are continuously differentiable functions at $(\overline{x}, \overline{y})$. The linearized system of system (2.2) about the equilibrium point $(\overline{x}, \overline{y})$ is

$$X_{n+1} = F(X_n) = BX_n$$

where

$$X_n = \begin{pmatrix} x_n \\ \vdots \\ x_{n-m} \\ y_n \\ \vdots \\ y_{n-m} \end{pmatrix}$$

and B is a Jacobian matrix of the system (2.2) about the equilibrium point $(\overline{x}, \overline{y})$.

The following result, known as the Linearized Stability Theorem, is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of equation (2.1).

Theorem 2.1 *(The Linearized Stability Theorem) ([4])*
Assume that the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then, the following statements are true:
(i) If all roots of (2.4) have absolute value less than one, then the equilibrium point $\overline{x}$ of (2.1) is locally asymptotically stable. In this case the locally asymptotically stable equilibrium point $\overline{x}$ is also called a sink.
(ii) If at least one of the roots of (2.4) has absolute value greater than one, then the equilibrium point $\overline{x}$ of (2.1) is unstable. Also, the equilibrium point $\overline{x}$ of (2.1) is called a saddle point if (2.4) has roots both inside and outside the unit disk.
(iii) The equilibrium point $\overline{x}$ of (2.1) is called hyperbolic if no root of (2.4) has absolute value equal to one. If there exists a root of (2.4) with absolute value equal to one, then the equilibrium $\overline{x}$ is called non-hyperbolic.
(iv) An equilibrium point $\overline{x}$ of (2.1) is called a repeller if all roots of (2.4) have absolute value greater than one.

Theorem 2.2 *([5]) For the discrete dynamical systems, the following cases occur;*
(i) For the system

$$X_{n+1} = F(X_n),\ n = 0, 1, ...,$$

of difference equations such that $\overline{X}$ is a fixed point of F. If all eigenvalues of the Jocobian matrix B about $\overline{X}$ lie inside the open unit disk

$$|\lambda| < 1,$$

then, $\overline{X}$ is locally asymtotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.
(ii) For the system

$$X_{n+1} = F(X_n),\ n = 0, 1, ...,$$

of difference equations such that $\overline{X}$ is a fixed point of F. If no eigenvalues of the Jocobian matrix B about $\overline{X}$ has absolute value equal to one, then $\overline{X}$ is called hyperbolic. If there exists an eigenvalue of the Jocobian matrix B about $\overline{X}$ with absolute value equal to one, then $\overline{X}$ is called non-hyperbolic.

We will give in the following theorem which is mentioned Linearized Stability Theorem for a second order difference equations.

Theorem 2.3 *(Linearized Stability Theorem for a second order equation) ([4])*
Consider the difference equation

$x_{n+1} = g(x_n, x_{n-1}),\ n = 0, 1, \ldots.$

The linearized equation of this equation about the equilibrium point $\overline{x}$ *is*

$$y_{n+1} = py_n + qy_{n-1} \tag{2.5}$$

where

$$\begin{aligned} p &= \frac{\partial g}{\partial x_n}(\overline{x}, \overline{x}), \\ q &= \frac{\partial g}{\partial x_{n-1}}(\overline{x}, \overline{x}). \end{aligned}$$

The characteristic equation of the linear equation is

$$\lambda^2 - p\lambda - q = 0. \tag{2.6}$$

Then, the following statements hold;
(a) If both roots of Eq.(2.6) have absolute values less than one, then the equilibrium $\overline{x}$ *of Eq.(2.5) is locally asymptotically stable.*
(b) If at least one of the roots of Eq.(2.6) has an absolute value greater than one, then $\overline{x}$ *is unstable.*
(c) Both roots of Eq.(2.6) have absolute values less than one if and only if

$$|p| < 1 - q < 2.$$

In this case, $\overline{x}$ *is locally asymptotically stable.*
(d) Both roots of Eq.(2.6) have absolute values greater than one if and only if

$$1 < |q| \ \text{and}\ |p| < |1 - q|.$$

In this case, $\overline{x}$ *is a repeller.*
(e) One root of Eq.(1.4) has absolute value less than one while the other root has absolute value greater than one if and only if

$$0 < p^2 + 4q \ \text{and}\ |p| > |1 - q|.$$

In this case, $\overline{x}$ *is unstable and is called a saddle point.*

The following theorems state necessary and sufficient conditions for all the roots of a real polynomial of degree two, three, or four, respectively, to have modulus less than one.

Theorem 2.4 *([2]) Assume that* α_1 *and* α_0 *are real numbers. Then, a necessary and sufficient condition for all roots of the equation*

$$\lambda^2 + \alpha_1\lambda + \alpha_0 = 0$$

to lie inside the unit disk is

$$|\alpha_1| < 1 + \alpha_0 < 2.$$

Theorem 2.5 *([2]) Assume that* α_2*,* α_1*, and* α_0 *are real numbers. Then, a necessary and sufficient condition for all roots of the equation*

$$\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0$$

to lie inside the unit disk is

$$\begin{aligned} |\alpha_2 + \alpha_0| &< 1 + \alpha_1, \\ |\alpha_2 - 3\alpha_0| &< 3 - \alpha_1, \end{aligned}$$

and

$$\alpha_0^2 + \alpha_1 - \alpha_0\alpha_2 < 1.$$

Theorem 2.6 *([2]) Assume that* α_3*,* α_2*,* α_1*, and* α_0 *are real numbers. Then, a necessary and sufficient condition for all roots of the equation*

$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0$$

to lie inside the unit disk is

$$\begin{aligned} |\alpha_1 + \alpha_3| &< 1 + \alpha_0 + \alpha_2, \\ |\alpha_1 - \alpha_3| &< 2(1 - \alpha_0), \\ \alpha_2 - 3\alpha_0 &< 3, \end{aligned}$$

and

$$\alpha_0 + \alpha_2 + \alpha_0^2 + \alpha_1^2 + \alpha_0^2\alpha_2 + \alpha_0\alpha_3^2 < 1 + 2\alpha_0\alpha_2 + \alpha_1\alpha_3 + \alpha_0\alpha_1\alpha_3 + \alpha_0^3.$$

Theorem 2.7 *(Clark theorem) ([20]) Assume that* $p_1, p_2, \ldots, p_k \in \mathbb{R}$ *and* $k \in \{1, 2, \ldots\}$. *Then,*

$$\sum_{i=1}^{k} |p_i| < 1$$

is sufficient condition for the asymptotic stability of the difference equation (2.3).

Theorem 2.8 *(Clark theorem) ([20]) Assume that* $a, b \in \mathbb{R}$ *and* $k \in \{0, 1, \ldots\}$. *Then,*

$$|a| + |b| < 1$$

is a sufficient condition for the asymptotic stability of the delay difference equation

$$z_{n+1} - az_n + bz_{n-k} = 0,\ n = 0, 1, \ldots.$$

Suppose in addition that one of the following two cases holds;
(a) k *is odd and* $b < 0$.
(b) k *is even and* $ab < 0$.
Then,

$$|a| + |b| < 1$$

is a also necessary condition.

Theorem 2.9 *([20]) Assume that* $q \in \mathbb{R}$ *and* $k \in \{0, 1, 2, \ldots\}$. *Then, the delay difference equation*

$$x_{n+1} - x_n + qx_{n-k} = 0,\ n = 0, 1, \ldots$$

is asymptotically stable if and only if

$$0 < q < 2\cos\frac{k\pi}{2k+1}.$$

Theorem 2.10 *([20]) Assume that* $b > 0$ *and* k *is even. Then, the delay difference equation*

$$z_{n+1} + bz_n - bz_{n-k} = 0,\ n = 0, 1, \ldots$$

is asymptotically stable if and only if

$$0 < b < \frac{1}{2\cos\left(\frac{\pi}{k+2}\right)}.$$

The so-called Schur-Cohn criterion provides necessary and sufficient conditions for all roots of the equation

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0 \tag{2.7}$$

with real coefficients to lie in the open disk $|\lambda| < 1$.
Before we can explain the Schur-Cohn criterion, we need the so-called Routh-Hurwitz criterion.

Theorem 2.11 *(Routh-Hurwitz criterion) ([3])*
Assume that

$$X_{n+1} = F(X_n),\ n = 0, 1, ...,$$

is a system of difference equations and $\overline{X}$ is a fixed point of F, the characteristic polynomial of this system about the equilibrium point $\overline{X}$ is (2.7) with real coefficients and $a_0 > 0$. Then all roots of the polynomial $P(\lambda)$ lie inside the open unit disk $|\lambda| < 1$ if and only if

$$\Delta_k > 0 \text{ for } k = 1, 2, \ldots, n$$

where Δ_k is the principal minor of order k of the $n \times n$ matrix

$$\Delta_n = \begin{pmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}.$$

Theorem 2.12 *(Schur-Cohn criterion) ([3])*
The equation

$$P(\lambda) = \alpha_0\lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0$$

has all its roots in the open unit disk $|\lambda| < 1$ if and only if the equation

$$P\left(\frac{z+1}{z-1}\right) = 0$$

has all its roots in the left-half plane

$$\mathrm{Re}(z) < 0.$$

2.2 PERIODICITY, BOUNDEDNESS AND INVARIANT INTERVALS

In this section, we state the definitions of bounedness, periodicity, invariant intervals and basin of attraction in the theory of difference equations which are some dynamics of difference equations.

Definition 2.9 *([4]) A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) is bounded and persists if there exist positive constant P and Q such that*

$P \leq x_n \leq Q$

for $n = -k, -k+1, \ldots$.

Definition 2.10 *(Basin of Attraction) ([3]) Consider the equilibrium point $\overline{x}$ of Eq.(2.1). The basin of attraction of the equilibrium $\overline{x}$ is the set M of all initial conditions*

$(x_{-k}, \ldots, x_0)$

through which the solution of Eq.(2.1) converges to the equilibrium.

Definition 2.11 *(Invariant Intervals) ([17]) I is an invariant interval for Eq.(2.1) if*

$x_{-k}, \ldots, x_0 \in I,$

then,

$x_n \in I.$

An interval might eventually an invariant. That is, J is an invariant interval for Eq.(2.1) if

$x_N, x_{N+1} \ldots, x_{N+k} \in J$

for some $N \geq 0$, then,

$x_n \in J$

for every $n > N$.

Definition 2.12 *([4]) Any solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) is said to periodic with period p if there exists an integer $p \geq 1$ such that*

$x_{n+p} = x_n$*, for all $n \geq -k$.*

Any solution is called periodic with prime period p if p is the smallest positive integer for which the condition above holds.

2.3 OSCILLATION AND SEMICYCLE ANALYSIS

In this section we will give some definitions of various notion of oscillation and statements of all the results in oscillation theory which we will need throughout the thesis. Also, we state that a semi-cycle analysis of the solutions of a difference equation is a powerful tool for a detailed understanding of the entire character of solutions and often leads to straightforward proofs of their long term behavior.

Definition 2.13 *([3]) Any solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) is said to non-oscillatory solution if there exists $n_0 \geq -k$ such that either*

$x_n > \overline{x}$ for all $n \geq n_0$

or

$x_n < \overline{x}$ for all $n \geq n_0$.

Also, $\{x_n\}_{n=-k}^{\infty}$ is said to an oscillatory solution if it is not a non-oscillatory solution.

Definition 2.14 *([3]) A positive semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) consists of a "string" of terms*

$\{x_l, x_{l+1}, \ldots, x_m\}$

all greater than or equal to $\overline{x}$, with $l \geq -k$ and $m \leq \infty$, such that

either $l = -k$ or $l > -k$ and $x_{l-1} < \overline{x}$

and

either $m = \infty$ or $m < \infty$ and $x_{m+1} < \overline{x}$.

Definition 2.15 *([3]) A negative semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) consists of a "string" of terms*

$\{x_l, x_{l+1}, \ldots, x_m\}$

all less than $\overline{x}$, with $l \geq -k$ and $m \leq \infty$, such that

either $l = -k$ or $l > -k$ and $x_{l-1} \geq \overline{x}$

and

either $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} \geq \overline{x}$.

Definition 2.16 *([9]) Let* $\overline{x}$ *be an equilibrium point of Eq.(2.1). Then, the solution* $\{x_n\}_{n=-k}^{\infty}$ *of Eq.(2.1) with*

$x_n = \overline{x}$ *for all* $n \geq -k$

is called a trivial solution. A trivial solution has only one semi-cycle, namely, the positive semi-cycle

$\{\overline{x}, \overline{x}, \ldots\}$.

This semi-cycle is called a trivial semi-cycle.

Definition 2.17 *([5]) A function* x_n *(resp.* y_n*) oscillates about* $\overline{x}$ *(resp.* $\overline{y}$*) if for every* $\zeta \in \mathbb{N}$ *there exist* $\mu, \nu \in \mathbb{N}$, $\mu \geq \zeta$, $\nu \geq \zeta$ *such that*

$(x_\mu - \overline{x})(x_\nu - \overline{x}) \leq 0$ *(resp.*$(y_\mu - \overline{y})(y_\nu - \overline{y}) \leq 0$*).*

A solution $\{(x_n, y_n)\}$ *of system (2.2) oscillates about* $(\overline{x}, \overline{y})$ *if* x_n *oscillates about* $\overline{x}$ *or* y_n *oscillates about* $\overline{y}$.

Definition 2.18 *([53]) A "string" of sequential terms*

$\{x_\mu, \ldots, x_\nu\}$ *(resp.* $\{y_\mu, \ldots, y_\nu\}$*),*

$\mu \geq -1, \nu \leq \infty$ *is called to be a positive semi-cycle if* $x_i \geq \overline{x}$ *(resp.* $y_i \geq \overline{y}$*),* $i \in \{\mu, \ldots, \nu\}$, $x_{\mu-1} < \overline{x}$ *(resp.* $y_{\mu-1} < \overline{y}$*), and* $x_{\nu+1} < \overline{x}$ *(resp.* $y_{\nu+1} < \overline{y}$*).*
A "string" of sequential terms

$\{x_\mu, \ldots, x_\nu\}$ *(resp.* $\{y_\mu, \ldots, y_\nu\}$*),*

$\mu \geq -1$, $\nu \leq \infty$ *is called to be a negative semi-cycle if* $x_i < \overline{x}$ *(resp.* $y_i < \overline{y}$*),* $i \in \{\mu, \ldots, \nu\}$, $x_{\mu-1} \geq \overline{x}$ *(resp.* $y_{\mu-1} \geq \overline{y}$*), and* $x_{\nu+1} \geq \overline{x}$ *(resp.* $y_{\nu+1} \geq \overline{y}$*).*
A "string" of sequential terms

$\{(x_\mu, y_\mu), \ldots, (x_\nu, y_\nu)\}$,

$\mu \geq -1$, $\nu \leq \infty$ *is called to be a positive semi-cycle (resp. negative semi-cycle) if both*

$$\{x_\mu, \ldots, x_\nu\}$$

and

$$\{y_\mu, \ldots, y_\nu\}$$

are positive semi-cycles (resp. negative semi-cycles).
A "string" of sequential terms

$$\{(x_\mu, y_\mu), \ldots, (x_\nu, y_\nu)\},$$

$\mu \geq -1$, $\nu \leq \infty$ *is called to be a positive semi-cycle (resp. negative semi-cycle) with respect to* x_n *and negative semi-cycle (resp. positive semi-cycle) with respect to* y_n *if*

$$\{x_\mu, \ldots, x_\nu\}$$

is a positive semi-cycle (resp. negative semi-cycle) and

$$\{y_\mu, \ldots, y_\nu\}$$

is a negative semi-cycle (resp. positive semi-cycle).

Theorem 2.13 *([3]) Consider the linear homogeneous difference equation*

$$x_{n+k} + \sum_{i=1}^{k} q_i x_{n+k-i} = 0,\ n = 0, 1, \ldots \tag{2.8}$$

where k *is a nonnegative integer and*

$$q_i \in \mathbb{R},\ \textit{for}\ i = 1, \ldots, k.$$

Then, the following statements are equivalent:

(i) Every solution of Eq.(2.8) oscillates.

(ii) The characteristic equation of Eq.(2.8)

$$\lambda^k + \sum_{i=1}^{k} q_i \lambda^{k-i} = 0$$

has no positive roots.

Corollary 2.1 *([3]) Assume that $p \in \mathbb{R}$ and k is a nonnegative integer. Then, every solution of the delay difference equation*

$$y_{n+1} - y_n + py_{n-k} = 0, 1, \ldots$$

oscillates if and only if

$$p \geq 1 \text{ if } k = 0$$

and

$$p > \frac{k^k}{(k+1)^{k+1}} \text{ if } k \geq 1.$$

2.4 GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINTS

In this section we present some important theorems about the global behavior which will be useful in the sequel.
Especially, the following theorem is a very useful result which is mentioned Negative Feedback Condition about the global behavior of solutions to prove the global attractivity of the equilibrium points.

Theorem 2.14 *([3]) Consider the difference equation (2.1) where*

$$f \in C[(0, \infty)^{k+1}, (0, \infty)]$$

is increasing in each of its arguments and where the initial conditions

$$x_{-k}, \ldots, x_0$$

are positive real numbers. Assume that Eq.(2.1) has a unique positive equilibrium $\overline{x}$ and suppose that the function h defined by

$$h(x) = g(x, \ldots, x), \ x \in (0, \infty)$$

satisfies the negative feedback condition:

$$(h(x) - x)(x - \overline{x}) < 0, \ \text{for } x \neq \overline{x}.$$

Then, $\overline{x}$ is a global attractor of all positive solutions of Eq.(2.1).

The following comparison result is a very useful tool in establishing bounds for solutions of nonlinear difference equations in terms of the solutions of equations with known behavior, for example, linear or riccati.

Theorem 2.15 *(Comparison Theorem) ([2])*
Let m be a non-negative integer, let $\alpha_0, \alpha_1, ..., \alpha_m$ be non-negative real numbers and let β be a real number. Suppose that $\{x_n\}_{n=-m}^{\infty}$, $\{y_n\}_{n=-m}^{\infty}$ and $\{z_n\}_{n=-m}^{\infty}$ are sequences of real numbers such that

$$x_n \leq y_n \leq z_n, \text{ for all } -m \leq n \leq 0$$

and such that

$$\begin{aligned} x_{n+1} &\leq \alpha_0 x_n + \alpha_1 x_{n-1} + ... + \alpha_m x_{n-m} + \beta \\ y_{n+1} &= \alpha_0 y_n + \alpha_1 y_{n-1} + ... + \alpha_m y_{n-m} + \beta \\ z_{n+1} &\geq \alpha_0 z_n + \alpha_1 z_{n-1} + ... + \alpha_m z_{n-m} + \beta \end{aligned}$$

for all $n = 0, 1, ...,$ then

$$x_n \leq y_n \leq z_n, \text{ for all } -m \leq n.$$

Lemma 2.16 *[2] Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Suppose that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are sequences of real numbers such that*

$$u_0 \leq v_0 \leq w_0$$

and

$$\left.\begin{aligned} u_{n+1} \leq \alpha u_n + \beta \\ v_{n+1} \leq \alpha v_n + \beta \\ w_{n+1} \leq \alpha w_n + \beta \end{aligned}\right\}, \ n = 0, 1, \ldots .$$

Then

$$u_n \leq v_n \leq w_n, \ n = 0, 1, \ldots .$$

Theorem 2.17 *([3]) Assume that all the roots of the polynomial*

$$P(t) = t^N - s_1 t^{N-1} - ... - s_N$$

where $s_1, s_2, ..., s_N \geq 0$, *have absolute value less than* 1. *If* $\{x_n\}$ *is a nonnegative solution of the inequality*

$$x_{n+N} \leq s_1 x_{n+N-1} + ... + s_N x_n + y_n$$

where $y_n \geq 0$, *for* $n = 0, 1, ...,$ *then the following statements are true:*

i) If $\sum_{n=0}^{\infty} y_n$ *converges, then* $\sum_{n=0}^{\infty} x_n$ *converges.*

ii) If y_n *is bounded, then* x_n *is bounded.*

iii) If $\lim_{n\to\infty} y_n = 0$, *then* $\lim_{n\to\infty} x_n = 0$.

CHAPTER 3

THE PERIODIC CHARACTER OF ALL SOLUTIONS OF A RATIONAL DIFFERENCE EQUATION WITH DELAY TERMS

Firstly, we state that the results of this chapter are cited from [29] which has been published by us. See also [21-30] for more results.

3.1 INTRODUCTION AND PRELIMINARIES

In this chapter, we will give some important results about the periodic character of positive solutions of the non-linear delay difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^q}, \quad n = 0, 1, \dots, \tag{3.1}$$

where

$k > 1$

is an odd integer,

$\alpha, p, q \in (0, \infty)$

and the initial conditions

$x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$

are arbitrary positive real numbers. Eq.(3.1) was studied by many authors for different cases of k, p, q. There exist many papers related with Eq.(3.1) and on its extensions (see [21, 22, 24, 28, 30]).

In [30], the authors studied the special case of $k = 1$ of Eq.(3.1), that is, the recursive sequence

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, \dots, \tag{3.2}$$

where α, p, q are positive, and the initial values x_{-1}, x_0 are positive real numbers.
In [28], the authors have obtained the periodicity results of positive solutions of Eq.(3.1). They have investigated the existence of a prime periodic solution of Eq.(3.1). But they have not investigated the positive solutions of Eq.(3.1) which converge to a prime two periodic solution.
Our aim in this chapter is to obtain some results about the periodic character of all positive solutions of Eq.(3.1). The most important finding here is that for the odd number k, all positive solutions of Eq.(3.1) converge to a prime two periodic solution.
The linearized equation for Eq.(3.1) about the positive equilibrium $\overline{x}$ is

$$y_{n+1} + q\overline{x}^{p-q-1}y_n - p\overline{x}^{p-q-1}y_{n-k} = 0, \quad n = 0, 1, \ldots. \tag{3.3}$$

We have the following three lemmas which are essentially given from [28, 30].

Lemma 3.1 *Assume that*

$$p \in (0, 1). \tag{3.4}$$

Then, every positive solution of Eq.(3.1) is bounded.

Lemma 3.2 *Let k be odd. Then, Eq.(3.1) has prime two periodic solutions if and only if*

$$0 < p < 1 < q \tag{3.5}$$

and there exists a sufficient small positive number ε_1, such that

$$\left.\begin{array}{c} \frac{1}{(\alpha+\varepsilon_1)^{q-p}} > \varepsilon_1 \\ (\alpha+\varepsilon_1)^{p/q}\varepsilon_1^{-1/q} < \alpha + \varepsilon_1^{-p/q}(\alpha+\varepsilon_1)^{p^2-q^2/q} \end{array}\right\}. \tag{3.6}$$

Lemma 3.3 *If either*

$$0 < q < p < 1 \tag{3.7}$$

or

$$0 < p < q \tag{3.8}$$

hold, then, Eq.(3.1) has a unique equilibrium point $\overline{x}$.

3.2 MONOTONIC CHARACTER OF SOLUTIONS

In the following theorem, we will obtain the monotonic character of solutions of Eq.(3.1). This result will be a very powerful instrument in our main result.

Theorem 3.4 *Let k be odd in Eq.(3.1). For every positive solution $\{x_n\}$ of Eq.(3.1) which satisfies the following initial conditions, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are eventually monotone.*

Proof. From Eq.(3.1), we obtain

$$x_{2n+1} - x_{2n-1} = \frac{x_{2n-2}^q(x_{2n-k}^p - x_{2n-k-2}^p) + x_{2n-k-2}^p(x_{2n-2}^q - x_{2n}^q)}{(x_{2n}x_{2n-2})^q} \tag{3.9}$$

and

$$x_{2n+2} - x_{2n} = \frac{x_{2n-1}^q(x_{2n-k+1}^p - x_{2n-k-1}^p) + x_{2n-k-1}^p(x_{2n-1}^q - x_{2n+1}^q)}{(x_{2n+1}x_{2n-1})^q}. \tag{3.10}$$

If

$$x_{-k+2l} \geq x_{-k+2l-2} \quad \forall l = 1, 2, ..., \frac{k-1}{2}$$

and

$$x_{-k+2s+1} \leq x_{-k+2s-1} \quad \forall s = 1, 2, ..., \frac{k+1}{2},$$

from which (3.9) and (3.10) we obtain

$$x_3 \geq x_1$$

and consequently

$$x_4 \leq x_2.$$

By induction we obtain

$$x_0 \geq x_2 \geq ... \geq x_{2n} \geq ...$$

and

$$... \geq x_{2n+1} \geq x_{2n-1} \geq ... \geq x_1.$$

Similarly if

$$x_{-k+2l} \leq x_{-k+2l-2} \quad \forall l = 1, 2, ..., \frac{k-1}{2}$$

and

$$x_{-k+2s+1} \geq x_{-k+2s-1} \quad \forall s = 1, 2, ..., \frac{k+1}{2},$$

from which (3.9) and (3.10) using induction we obtain

$$x_0 \leq x_2 \leq ... \leq x_{2n} \leq ...$$

and

$$... \leq x_{2n+1} \leq x_{2n-1} \leq ... \leq x_1.$$

If

$$x_{-k+2l} \geq x_{-k+2l-2} \quad \forall l = 1, 2, ..., \frac{k-1}{2}$$

and

$$x_{-k+2s+1} \geq x_{-k+2s-1} \quad \forall s = 1, 2, ..., \frac{k+1}{2}$$

and

$$x_1 \geq x_3,$$

we can obtain from (3.9) and (3.10)

$$x_0 \leq x_2 \leq ... \leq x_{2n} \leq ...$$

and

$$... \leq x_{2n+1} \leq x_{2n-1} \leq ... \leq x_1.$$

Hence, we may assume that

$$x_{-k+2l} \geq x_{-k+2l-2} \quad \forall l = 1, 2, ..., \frac{k-1}{2},$$

$$x_{-k+2s+1} \geq x_{-k+2s-1} \quad \forall s = 1, 2, ..., \frac{k+1}{2},$$

$$x_3 \geq x_1$$

and

$x_2 \geq x_4,$

then,

$x_2 \geq ... \geq x_{2n} \geq ...$

and

$... \geq x_{2n+1} \geq x_{2n-1} \geq ... \geq x_1.$

So, we may assume that

$$x_{-k+2l} \geq x_{-k+2l-2} \quad \forall l = 1, 2, ..., \frac{k-1}{2},$$

$$x_{-k+2s+1} \geq x_{-k+2s-1} \quad \forall s = 1, 2, ..., \frac{k+1}{2},$$

$x_3 \geq x_1$

and

$x_2 \leq x_4.$

By induction we obtain the result in this case. The case

$$x_{-k+2l} \leq x_{-k+2l-2} \quad \forall l = 1, 2, ..., \frac{k-1}{2}$$

and

$$x_{-k+2s+1} \leq x_{-k+2s-1} \quad \forall s = 1, 2, ..., \frac{k+1}{2}$$

can be treated similarly. This completes the proof. ■

3.3 AN OSCILLATION RESULT

The following result is essentially proved in [30] for $k = 1$ of Eq.(3.1). We will obtain the same result for odd integer k which is greater than 1.

Lemma 3.5 *Consider Eq.(3.1) where (3.5) and (3.6) hold and k is odd. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(3.1) such that either*

$$\left.\begin{array}{c}\alpha < x_{-k}, x_{-k+2}, ..., x_{-1} < \alpha + \varepsilon_1, \\ (\alpha+\varepsilon_1)^{p/q}\varepsilon_1^{-1/q} < x_{-k+1}, x_{-k+3}, ..., x_0\end{array}\right\} \tag{3.11}$$

or

$$\left.\begin{array}{c}\alpha < x_{-k+1}, x_{-k+3}, ..., x_0 < \alpha + \varepsilon_1, \\ (\alpha+\varepsilon_1)^{p/q}\varepsilon_1^{-1/q} < x_{-k}, x_{-k+2}, ..., x_{-1}\end{array}\right\} \tag{3.12}$$

Then, if (3.11) holds, we have

$$\left.\begin{array}{c}\alpha < x_{2n-1} < \alpha + \varepsilon_1, \\ x_{2n} > (\alpha+\varepsilon_1)^{p/q}\varepsilon_1^{-1/q},\end{array}\right\},\ n = 0, 1, ..., \tag{3.13}$$

and if (3.12) holds, we have

$$\left.\begin{array}{c}\alpha < x_{2n} < \alpha + \varepsilon_1, \\ x_{2n-1} > (\alpha+\varepsilon_1)^{p/q}\varepsilon_1^{-1/q},\end{array}\right\},\ n = 0, 1, ...,. \tag{3.14}$$

Proof. Firstly, assume that (3.11) holds. Then from (3.6) we have

$$\begin{aligned}
\alpha &< x_1 = \alpha + \frac{x_{-k}^p}{x_0^q} < \alpha + \varepsilon_1 \frac{(\alpha+\varepsilon_1)^p}{(\alpha+\varepsilon_1)^p} = \alpha + \varepsilon_1 \\
x_2 &= \alpha + \frac{x_{-k+1}^p}{x_1^q} > \alpha + (\alpha+\varepsilon_1)^{(p^2-q^2)/q}\varepsilon_1^{-p/q} > (\alpha+\varepsilon_1)^{p/q}\varepsilon_1^{-1/q} \\
\alpha &< x_k = \alpha + \frac{x_{-1}^p}{x_{k-1}^q} < \alpha + \varepsilon_1 \frac{(\alpha+\varepsilon_1)^p}{(\alpha+\varepsilon_1)^p} = \alpha + \varepsilon_1 \\
x_{k+1} &= \alpha + \frac{x_0^p}{x_k^q} > \alpha + (\alpha+\varepsilon_1)^{(p^2-q^2)/q}\varepsilon_1^{-p/q} > (\alpha+\varepsilon_1)^{p/q}\varepsilon_1^{-1/q}.
\end{aligned}$$

Working inductively we can easily prove relations (3.13). Similarly if (3.12) holds, we can prove that (3.14) is satisfied. ■

3.4 BOUNDEDNESS OF SOLUTIONS

Theorem 3.6 *If $\alpha \neq 1$, then every positive solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(3.1) satisfies the following inequalities;*

$$\left.\begin{array}{c}\alpha < x_{2n+k} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_{2n-1}^p \\ \alpha < x_{2n+k+1} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_{2n}^p,\end{array}\right\},\ \textit{for}\ n = 1, 2, \ldots \tag{3.15}$$

and here the cases

$$\left.\begin{array}{c}\alpha < x_{2n} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_{-k+1}^p \\ \alpha < x_{2n-1} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_{-k}^p,\end{array}\right\},\ \textit{for}\ n = 1, 2, \ldots \tag{3.16}$$

$$\beta = \frac{1}{\alpha^q} \tag{3.17}$$

hold.

Proof. We have

$$\alpha < x_{n+1} < \alpha + \beta x_{n-k}^p \ \forall n = 1, 2, ...$$

By induction, we obtain (3.15) and (3.16). If here $\alpha > 1$, we also see that

$$\begin{aligned} \alpha &< x_{2n} < \frac{\alpha}{1-\beta} + x_{-k+1}^p \ n = 1, 2, ... \\ \alpha &< x_{2n-1} < \frac{\alpha}{1-\beta} + x_{-k}^p \ n = 1, 2, ... \end{aligned}$$

Now, we are ready for the main result of this chapter. ∎

3.5 MAIN RESULT

In the following theorem, we will find solutions for k is odd of Eq.(3.1) which converge two a prime to periodic solution.

Theorem 3.7 *Consider Eq.(3.1) where (3.5) and (3.6) hold and k is odd. Suppose that*

$$\alpha + \varepsilon_1 < 1, \tag{3.18}$$

Then, every positive solution $\{x_n\}_{n=-k}^{\infty}$ with initial values

$$x_{-k}, x_{-k+1}, ..., x_{-1}, x_0$$

which satisfy conditions of Theorem 3.4 and either (3.11) or (3.12), converges to a prime two periodic solution.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution with initial values

$$x_{-k}, x_{-k+1}, ..., x_{-1}, x_0$$

which satisfy conditions of Theorem 3.4 and either (3.11) or (3.12). Using Lemma 3.1 and Theorem 3.4, we have that there exist finete limits

$$\left.\begin{array}{c} \lim_{n\to\infty} x_{2n+1} = L, \\ \lim_{n\to\infty} x_{2n} = S \end{array}\right\}. \tag{3.19}$$

Besides, from Lemma 3.5, we have that either L or S belongs to the interval

$(\alpha, \alpha + \varepsilon_1)$.

Furthermore from Lemma 3.3, we have that Eq.(3.1) has a unique equilibrium $\overline{x}$ such that

$1 < \overline{x} < \infty$.

Therefore, from (3.18) we have that

$L \neq S$.

So, $\{x_n\}_{n=-k}^{\infty}$ converges to a prime two periodic solution. This completes the proof. ■

CHAPTER 4

GLOBAL ANALYSIS OF A TWO DIMENSIONAL RATIONAL SYSTEM OF DIFFERENCE EQUATIONS

We state that the results of this chapter are cited from [31, 32] which has been published by us. See also for [33-60] more results.

4.1 INTRODUCTION

In [38], El-Owaidy et al. investigated the global behavior of the following difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}, \; n = 0, 1, ... \tag{4.1}$$

with non-negative parameters and non-negative initial values.

Inspire of this work, in this chapter we study the equilibrium points, the local and global asymptotic stability of these points, the existence unbounded solutions and the existence of two prime periodic solutions of the following system

$$\left.\begin{aligned} u_{n+1} &= \frac{\alpha u_{n-1}}{\beta+\gamma v_{n-2}^{p}}, \\ v_{n+1} &= \frac{\alpha_1 v_{n-1}}{\beta_1+\gamma_1 u_{n-2}^{p}}, \end{aligned}\right\}, \; n = 0, 1, ... \tag{4.2}$$

where the parameters

$$\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p$$

and the initial values

$$u_{-2}, u_{-1}, u_0, v_{-2}, v_{-1}, v_0$$

are positive real numbers. Our results extend and complement some results in the literature.

Note that the system (4.2) can be reduced to the following system of difference equations

$$\left.\begin{array}{l} x_{n+1} = \frac{rx_{n-1}}{1+y_{n-2}^{p}}, \\ y_{n+1} = \frac{sy_{n-1}}{1+x_{n-2}^{p}}, \end{array}\right\}, \; n = 0, 1, ... \tag{4.3}$$

by the change of variables

$$u_n = \left(\frac{\beta_1}{\gamma_1}\right)^{1/p} x_n$$

and

$$v_n = \left(\frac{\beta}{\gamma}\right)^{1/p} y_n$$

with

$$r = \frac{\alpha}{\beta}$$

and

$$s = \frac{\alpha_1}{\beta_1}.$$

So, in order to study the system (4.2), we investigate the system (4.3). Both studies are equivalent.

By using the induction and the equations in (4.3) we see that if x_{-i}, y_{-i} are positive real numbers for $i \in \{0, 1, 2\}$, then

$$\min\{x_n, y_n\} > 0, \; n \geq -2, \tag{4.4}$$

which means that positive initial values generate positive solutions of system (4.3).

There are two possible cases that need to be considered. If the initial conditions in system (4.3) provide the equalities $x_{-i} = y_{-i}$ for $i = 0, 1, 2$, then the system (4.3) reduces into Eq.(4.1) which has been studied by Owaidy in [38]. Therefore, here we deal with the case $x_{-i} \neq y_{-i}$ for $i = 0, 1, 2$ and we study system (4.3) basing on this condition thorought this chapter.

4.2 THE EQUILIBRIUM POINTS

Theorem 4.1 *We have the following cases for the equilibrium points of (4.3)*;

i $(\overline{x}_0, \overline{y}_0) = (0, 0)$ is always the equilibrium point of system (4.3).

ii If $r > 1$ and $s > 1$, then system (4.3) has the equilibrium point

$(\overline{x}_1, \overline{y}_1) = ((s-1)^{1/p}, (r-1)^{1/p})$.

iii If $r > 1$ and $s = 1$, then system (4.3) has the equilibrium point

$(\overline{x}_2, \overline{y}_2) = (0, (r-1)^{1/p})$.

iv if $r = 1$ and $s > 1$, then system (4.3) has the equilibrium point

$(\overline{x}_3, \overline{y}_3) = ((s-1)^{1/p}, 0)$.

v if $r \in (0, 1)$, $s = 1$ and $\frac{1}{p}$ is an even positive integer, then system (4.3) has the equilibrium point

$(\overline{x}_4, \overline{y}_4) = (0, (r-1)^{1/p})$.

vi If $r = 1$, $s \in (0, 1)$ and $\frac{1}{p}$ is an even positive integer, then system (4.3) has the equilibrium point

$(\overline{x}_5, \overline{y}_5) = ((s-1)^{1/p}, 0)$.

vii If $r, s \in (0, 1)$ and $\frac{1}{p}$ is an even positive integer, then it also has the positive equilibrium point

$(\overline{x}_6, \overline{y}_6) = ((s-1)^{1/p}, (r-1)^{1/p})$.

Proof. A point $(\overline{x}, \overline{y})$ is an equilibrium point of system (4.3) if and only if it is a root for the following functions

$$\begin{aligned} f(x, y) &= x(1 + y^p - r), \\ g(x, y) &= y(1 + x^p - s). \end{aligned}$$

Hence, it is clear that $(0, 0)$ is a root both f and g. This completes proof of part (i). Now we consider the case

$x \neq 0, y \neq 0$.

For this case we have from the definition of f and g,

$$\begin{aligned}(1+y^p-r) &= 0,\\ (1+x^p-s) &= 0.\end{aligned}$$

So, one obtain

$$x=(s-1)^{1/p}$$

and

$$y=(r-1)^{1/p}.$$

According to the values

$$r, s, p,$$

x and y are identified. From there, the proof of parts $(ii), (iii), (iv), (v), (vi), (vii)$ is completed. ■

4.3 STABILITY CHARACTER OF EQUILIBRIUM POINTS

Before we give the following theorems about the local asymptotic stability of the equilibrium points above, we build the corresponding linearized form of the system (4.3) and consider the following transformation

$$(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \to (f, f_1, f_2, g, g_1, g_2)$$

where

$$\begin{aligned} f &= \frac{rx_{n-1}}{1+y_{n-2}^p},\\ f_1 &= x_n,\\ f_2 &= x_{n-1},\\ g &= \frac{sy_{n-1}}{1+x_{n-2}^p},\\ g_1 &= y_n,\\ g_2 &= y_{n-1}.\end{aligned}$$

The Jacobian matrix about the fixed point $(\overline{x}, \overline{y})$ under the above transformation is as follows:

$$B(\overline{x}, \overline{y}) = \begin{pmatrix} 0 & \frac{r}{1+\overline{y}^p} & 0 & 0 & 0 & -\frac{rp\overline{x}\overline{y}^{p-1}}{(1+\overline{y}^p)^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{sp\overline{y}\overline{x}^{p-1}}{(1+\overline{x}^p)^2} & 0 & \frac{s}{1+\overline{x}^p} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where

$r, s, p \in (0, \infty)$.

Theorem 4.2 *(i) If $r < 1$ and $s < 1$, then the zero equilibrium point*

$(\overline{x}_0, \overline{y}_0)$

is locally asymptotically stable.
(ii) If $r > 1$ or $s > 1$, then the zero equilibrium point

$(\overline{x}_0, \overline{y}_0)$

is locally unstable.

Proof. (i) The linearized system of system (4.3) about the equilibrium point

$(\overline{x}_0, \overline{y}_0) = (0, 0)$

is given by

$X_{n+1} = B(\overline{x}_0, \overline{y}_0)X_n,$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and

$$B(\overline{x}_0, \overline{y}_0) = \begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_0, \overline{y}_0)$ is as follows:

$$P(\lambda) = \lambda^6 - (r+s)\lambda^4 + rs\lambda^2 = 0.$$

The roots of $P(\lambda)$ are

$$\begin{aligned} \lambda_{1,2} &= \pm\sqrt{s}, \\ \lambda_{3,4} &= 0, \\ \lambda_{5,6} &= \pm\sqrt{r}. \end{aligned}$$

Since all eigenvalues of the Jacobian matrix B about

$$(\overline{x}_0, \overline{y}_0) = (0,0)$$

lie inside the open unit disk

$$|\lambda| < 1,$$

the zero equilibrium point is locally asymptotically stable.
(ii) It is easy to see that if $r > 1$ or $s > 1$, then there exists at least one root λ of $P(\lambda)$ such that

$$|\lambda| > 1.$$

Hence by Linearized Stability Theorem, if $r > 1$ or $s > 1$, then

$$(\overline{x}_0, \overline{y}_0) = (0,0)$$

is unstable. Thus, the proof is complete. ■

Theorem 4.3 *(i) If $r > 1$ and $s > 1$, then the positive equilibrium point $(\overline{x}_1, \overline{y}_1)$ is locally unstable.*

(ii) If $r < 1$, $s < 1$ and $\frac{1}{p}$ is an even positive integer, then the positive equilibrium point $(\overline{x}_6, \overline{y}_6)$ is locally unstable.

Proof. (i) The linearized system of (4.3) about the equilibrium point

$$(\overline{x}_1, \overline{y}_1) = ((s-1)^{1/p}, (r-1)^{1/p})$$

is given by

$$X_{n+1} = B(\overline{x}_1, \overline{y}_1)X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and

$$B(\overline{x}_1, \overline{y}_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -\frac{p(s-1)^{\frac{1}{p}}(r-1)^{\frac{p-1}{p}}}{r} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{p(r-1)^{\frac{1}{p}}(s-1)^{\frac{p-1}{p}}}{s} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_1, \overline{y}_1)$ is as follows:

$$P(\lambda) = \lambda^6 - 2\lambda^4 + \lambda^2 - \frac{p^2}{rs}(r-1)(s-1) = 0.$$

It is clear that $P(\lambda)$ has a root in the interval $(1, \infty)$ since

$$P(1) = -\frac{p^2}{rs}(r-1)(s-1) < 0$$

and

$$\lim_{\lambda\to\infty} P(\lambda) = \infty.$$

This completes the proof.

(ii) The linearized system of system (4.3) about the equilibrium point

$$(\overline{x}_6, \overline{y}_6) = ((s-1)^{1/p}, (r-1)^{1/p})$$

is given by

$$X_{n+1} = B(\overline{x}_6, \overline{y}_6) X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and

$$B(\overline{x}_6, \overline{y}_6) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -\frac{p(s-1)^{\frac{1}{p}}(r-1)^{\frac{p-1}{p}}}{r} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{p(r-1)^{\frac{1}{p}}(s-1)^{\frac{p-1}{p}}}{s} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_6, \overline{y}_6)$ is as follows:

$$P(\lambda) = \lambda^6 - 2\lambda^4 + \lambda^2 - \frac{p^2}{rs}(r-1)(s-1) = 0.$$

It is clear that $P(\lambda)$ has a root in the interval $(1, \infty)$ since

$$P(1) = -\frac{p^2}{rs}(r-1)(s-1) < 0$$

and

$$\lim_{\lambda\to\infty} P(\lambda) = \infty.$$

This completes the proof.

■

Theorem 4.4 *(i) If $r > 1$ and $s = 1$, then the equilibrium point $(\overline{x}_2, \overline{y}_2)$ is non-hyperbolic point.*
(ii) If $r = 1$ and $s > 1$, then the equilibrium point $(\overline{x}_3, \overline{y}_3)$ is non-hyperbolic point.
(iii) If $r < 1$, $s = 1$ and $\frac{1}{p}$ is an even positive integer, then the equilibrium point $(\overline{x}_4, \overline{y}_4)$ is non-hyperbolic point.
(iv) If $r = 1$, $s < 1$ and $\frac{1}{p}$ is an even positive integer, then the equilibrium point $(\overline{x}_5, \overline{y}_5)$ is non-hyperbolic point.

Proof. (i) The linearized system of system (4.3) about the equilibrium point

$(\overline{x}_2, \overline{y}_2) = (0, (r-1)^{1/p})$

is given by

$X_{n+1} = B(\overline{x}_2, \overline{y}_2)X_n,$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and

$$B(\overline{x}_2, \overline{y}_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_2, \overline{y}_2)$ is as follows:

$$P(\lambda) = \lambda^6 - (s+1)\lambda^4 + s\lambda^2 = 0.$$

The roots of $P(\lambda)$ are

$$\begin{aligned}
\lambda_{1,2} &= \pm\sqrt{s}, \\
\lambda_{3,4} &= 0, \\
\lambda_{5,6} &= \pm 1.
\end{aligned}$$

Thus, the equilibrium point

$$(\overline{x}_2, \overline{y}_2) = (0, (r-1)^{1/p})$$

is non-hyperbolic point.

(ii) The linearized system of system (4.3) about the equilibrium point

$$(\overline{x}_3, \overline{y}_3) = (0, (r-1)^{1/p})$$

is given by

$$X_{n+1} = B(\overline{x}_3, \overline{y}_3)X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and

$$B(\overline{x}_2, \overline{y}_2) = \begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_3, \overline{y}_3)$ is as follows:

$$P(\lambda) = \lambda^6 - (r+1)\lambda^4 + r\lambda^2 = 0.$$

The roots of $P(\lambda)$ are

$$\begin{aligned} \lambda_{1,2} &= \pm\sqrt{r}, \\ \lambda_{3,4} &= 0, \\ \lambda_{5,6} &= \pm 1. \end{aligned}$$

Thus, the equilibrium point

$$(\overline{x}_3, \overline{y}_3) = (0, (r-1)^{1/p})$$

is non-hyperbolic point.

(iii) The linearized system of system (4.3) about the equilibrium point

$$(\overline{x}_4, \overline{y}_4) = (0, (r-1)^{1/p})$$

is given by

$$X_{n+1} = B(\overline{x}_4, \overline{y}_4) X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and

$$B(\overline{x}_4, \overline{y}_4) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_4, \overline{y}_4)$ is as follows:

$$P(\lambda) = \lambda^6 - (s+1)\lambda^4 + s\lambda^2 = 0.$$

The roots of $P(\lambda)$ are

$$\begin{aligned}
\lambda_{1,2} &= \pm\sqrt{s}, \\
\lambda_{3,4} &= 0, \\
\lambda_{5,6} &= \pm 1.
\end{aligned}$$

Thus, the equilibrium point

$$(\overline{x}_4, \overline{y}_4) = (0, (r-1)^{1/p})$$

is non-hyperbolic point.

(iv) The linearized system of system (4.3) about the equilibrium point $(\overline{x}_5, \overline{y}_5) = (0, (r-1)^{1/p})$ is given by

$$X_{n+1} = B(\overline{x}_5, \overline{y}_5)X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and

$$B(\overline{x}_5, \overline{y}_5) = \begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_5, \overline{y}_5)$ is as follows:

$$P(\lambda) = \lambda^6 - (r+1)\lambda^4 + r\lambda^2 = 0.$$

The roots of $P(\lambda)$ are

$$\begin{aligned}
\lambda_{1,2} &= \pm\sqrt{r}, \\
\lambda_{3,4} &= 0, \\
\lambda_{5,6} &= \pm 1.
\end{aligned}$$

Thus, the equilibrium point

$$(\overline{x}_5, \overline{y}_5) = (0, (r-1)^{1/p})$$

is non-hyperbolic point.

■

4.4 GLOBAL BEHAVIOR OF SYSTEM (4.3)

In this section, we will study the global behavior of zero equilibrium point.

Theorem 4.5 *If $r < 1$ and $s < 1$, then the zero equilibrium point $(\overline{x}_0, \overline{y}_0)$ is globally asymptotically stable.*

Proof. We know by Theorem 4.2 that the equilibrium point $(\overline{x}_0, \overline{y}_0)$ of the system (4.3) is locally asymptotically stable. Hence, it suffices to show for any $\{(x_n, y_n)\}$ solution of system (4.3) that

$$\lim_{n\to\infty} (x_n, y_n) = (\overline{x}_0, \overline{y}_0).$$

Since

$$0 \leq x_{n+1} = \frac{r x_{n-1}}{1 + y_{n-2}^p} < r x_{n-1}$$

and

$$0 \leq y_{n+1} = \frac{s y_{n-1}}{1 + x_{n-2}^p} < s y_{n-1},$$

we have by induction

$$x_{2n-1} < r^n x_{-1} \text{ and } x_{2n} < r^n x_0$$

and

$$y_{2n-1} < s^n y_{-1} \text{ and } y_{2n} < s^n y_0.$$

Thus, for $r<1$ and $s<1$, we obtain

$$\lim_{n\to\infty}(x_n,y_n)=(0,0).$$

This completes the proof. ■

4.5 INVARIANT INTERVALS

Theorem 4.6 *Consider system (4.3) and suppose that*

$r>1$ *and* $s>1$.

Then, we obtain the following invariant intervals, for $i\in\{0,1,2\}$;

i) $(x_{-i},y_{-i})\in(0,(s-1)^{1/p})\times((r-1)^{1/p},\infty)\Rightarrow(x_n,y_n)\in(0,(s-1)^{1/p})\times((r-1)^{1/p},\infty)$ *for* $n\geq 1$.

ii) $(x_{-i},y_{-i})\in((s-1)^{1/p},\infty)\times(0,(r-1)^{1/p})\Rightarrow(x_n,y_n)\in((s-1)^{1/p},\infty)\times(0,(r-1)^{1/p})$ *for* $n\geq 1$.

Proof. i) Let

$(x_{-i},y_{-i})\in(0,(s-1)^{1/p})\times((r-1)^{1/p},\infty)$ for $i\in\{0,1,2\}$.

From system (4.3), we have

$$x_1=\frac{rx_{-1}}{1+y_{-2}^p}<\frac{r\overline{x}_1}{1+\overline{y}_1^p}=\overline{x}_1=(s-1)^{1/p}$$

and

$$y_1=\frac{sy_{-1}}{1+x_{-2}^p}>\frac{s\overline{y}_1}{1+\overline{x}_1^p}=\overline{y}_1=(r-1)^{1/p}.$$

We prove by induction that

$$(x_n,y_n)\in(0,(s-1)^{1/p})\times((r-1)^{1/p},\infty),\text{ for all } n>1. \tag{4.5}$$

Suppose that (4.5) is true for

$n=k>1$.

Then, from system (4.3), we have

$$x_{k+1}=\frac{rx_{k-1}}{1+y_{k-2}^p}<\frac{r\overline{x}_1}{1+\overline{y}_1^p}=\overline{x}_1=(s-1)^{1/p}$$

and

$$y_{k+1} = \frac{sy_{k-1}}{1 + x_{k-2}^p} > \frac{s\overline{y}_1}{1 + \overline{x}_1^p} = \overline{y}_1 = (r-1)^{1/p}.$$

Therefore, (4.5) is true for all $n \geq -2$. This completes the proof of (i).

(ii) Let

$$(x_{-i}, y_{-i}) \in ((s-1)^{1/p}, \infty) \times (0, (r-1)^{1/p}) \text{ for } i \in \{0, 1, 2\}.$$

From system (4.3), we have

$$x_1 = \frac{rx_{-1}}{1 + y_{-2}^p} > \frac{r\overline{x}_1}{1 + \overline{y}_1^p} = \overline{x}_1 = (s-1)^{1/p}$$

and

$$y_1 = \frac{sy_{-1}}{1 + x_{-2}^p} < \frac{s\overline{y}_1}{1 + \overline{x}_1^p} = \overline{y}_1 = (r-1)^{1/p}.$$

We prove by induction that

$$(x_n, y_n) \in ((s-1)^{1/p}, \infty) \times (0, (r-1)^{1/p}), \text{ for all } n > 1. \tag{4.6}$$

Suppose that (4.6) is true for

$$n = k > 1.$$

Then from system (4.3), we have

$$x_{k+1} = \frac{rx_{k-1}}{1 + y_{k-2}^p} > \frac{r\overline{x}_1}{1 + \overline{y}_1^p} = \overline{x}_1 = (s-1)^{1/p}$$

and

$$y_{k+1} = \frac{sy_{k-1}}{1 + x_{k-2}^p} < \frac{s\overline{y}_1}{1 + \overline{x}_1^p} = \overline{y}_1 = (r-1)^{1/p}.$$

Therefore, (4.6) is true for all $n \geq -2$. This completes the proof of (ii). ■

Corollary 4.1 *Consider system (4.3) and suppose that*

$$r < 1, \; s < 1 \text{ and } \frac{1}{p} \in 2\mathbb{Z}^+$$

Then, we obtain the following invariant intervals, for $i \in \{0, 1, 2\}$;

i) $(x_{-i}, y_{-i}) \in (0, (s-1)^{1/p}) \times ((r-1)^{1/p}, \infty) \Rightarrow (x_n, y_n) \in (0, (s-1)^{1/p}) \times ((r-1)^{1/p}, \infty)$ *for* $n \geq 1$.

ii) $(x_{-i}, y_{-i}) \in ((s-1)^{1/p}, \infty) \times (0, (r-1)^{1/p}) \Rightarrow (x_n, y_n) \in ((s-1)^{1/p}, \infty) \times (0, (r-1)^{1/p})$ *for* $n \geq 1$.

4.6 UNBOUNDED SOLUTIONS

Theorem 4.7 *Assume that $r, s \in (1, \infty)$, then, there exists unbounded solutions of system (4.3).*

Proof. From Theorem 4.6, we can assume without loss of generality that the solution $\{x_n, y_n\}$ of system (4.3) is such that

$x_n < \overline{x}_1 = (s-1)^{1/p}$, for $n \geq -2$

and

$y_n > \overline{y}_1 = (r-1)^{1/p}$, for $n \geq -2$.

Then,

$$x_{n+1} = \frac{rx_{n-1}}{1+y_{n-2}^p} < \frac{rx_{n-1}}{1+(r-1)} = x_{n-1}$$

and

$$y_{n+1} = \frac{sy_{n-1}}{1+x_{n-2}^p} > \frac{sy_{n-1}}{1+(s-1)} = y_{n-1},$$

from which it follows that

$$\lim_{n\to\infty} x_n = 0$$

and

$$\lim_{n\to\infty} y_n = \infty.$$

This completes the proof. ■

Corollary 4.2 *Assume that $r, s \in (0, 1)$ and $\frac{1}{p} \in 2\mathbb{Z}^+$, then, there exists unbounded solutions of system (4.3).*

4.7 PERIODIC CHARACTER OF SYSTEM (4.3)

Theorem 4.8 *If $r = s = 1$, then system (4.3) possesses the prime period two solution which is one of the following forms;*

$\ldots, (0, b), (0, d), (0, b), (0, d), \ldots$

$\ldots, (0, b), (c, 0), (0, b), (c, 0), \ldots$

$\ldots, (a, 0), (c, 0), (a, 0), (c, 0), \ldots$

$\ldots, (a, 0), (0, d), (a, 0), (0, d), \ldots$

with $a, b, c, d > 0$.

Proof. Let

$\ldots, (a, b), (c, d), (a, b), (c, d), \ldots$

be a period two solution of system (4.3). Then, we have

$$a = \frac{ra}{1 + d^p}, \; b = \frac{rb}{1 + c^p} \tag{4.7}$$

and

$$c = \frac{sc}{1 + b^p}, \; d = \frac{sd}{1 + a^p} \tag{4.8}$$

such that

$a \neq b$

and

$c \neq d$.

Firstly, we consider the case

$a \neq 0, \;\; b \neq 0$.

Then, we obtain from (4.7) that

$c = d = (r - 1)^{1/p}$

which is a contradiction. Similarly, the case

$c \neq 0, \;\; d \neq 0$

is impossible with (4.8). Thus, one of them must be equal to zero. Now we assume that

$a = 0, b \neq 0, c = 0, d \neq 0$.

In this sense, we obtain from (4.7) and (4.8) that

$r = s = 1$.

Therefore,

$\ldots, (0, b), (0, d), (0, b), (0, d), \ldots$

is a two periodic solution of system (4.3) with $b, d > 0$. The other cases are similar and the proofs of them are omitted. ■

4.8 RATE OF CONVERGENCE

In this section, we will give exact results about the rate of convergence of positive solutions that converge to the equilibrium point of the system (4.3), in the regions of parameters described in Theorem 4.5. In addition to this, we will present the use of *Poincaré's Theorem* and a devolopment of *Perron's Theorem* to conclude the precise asymptotics of positive solutions that converge to the equilibrium.

Consider the following system of difference equations

$$\left.\begin{aligned} x_{n+1} &= f_1(x_n, y_n),\ n = 0, 1, \ldots \\ y_{n+1} &= f_2(x_n, y_n),\ n = 0, 1, \ldots \end{aligned}\right\} \tag{4.9}$$

where f_1, f_2 are continuous functions that maps some set I into I. The set I is an interval of real numbers. System (4.9) is competitive if $f_1(x, y)$ is non-decreasing in x and non-increasing in y and $f_2(x, y)$ is non-increasing in x and non-decreasing in y. System (4.9) is called anti-competitive system, if the functions f_1 and f_2 have monotonic character opposite to the monotonic character in competitive system.

We state that the following theorems give precise information about the asymptotics of linear non-autonomous difference equations. Consider the scalar mth-order linear difference equation

$$y_{n+m} + p_1(n)y_{n+m-1} + p_m(n)y_n = 0 \tag{4.10}$$

where m is a positive integer and $p_i : \mathbb{Z}^+ \to \mathbb{C}$ for $i \in \{1, \ldots, m\}$. Suppose that

$$q_i = \lim_{n\to\infty} p_i(n), \text{ for } i = 1, 2, \ldots, m, \tag{4.11}$$

exist in $\mathbb{C}$. For the following limitting equation of (4.10)

$$y_{n+m} + q_1 y_{n+m-1} + \ldots + q_m y_n = 0, \tag{4.12}$$

the asymptotics of solutions of (4.10) are given the following results. See [19].

Theorem 4.9 *(Poincaré's Theorem) Consider (4.10) based on the condition (4.11). Let λ_i for $i = 1, \ldots, m$ be the roots of the characteristic equation*

$$\lambda^m + q_1\lambda^{m-1} + \ldots + q_m = 0 \tag{4.13}$$

of the limiting equation (4.12) under the condition that $|\lambda_i| \neq |\lambda_j|$ *for* $i \neq j$*. If* x_n *is a positive solution of (4.10), then either* $x_n = 0$ *for all large* n *or there exists an index* $j \in \{1, \ldots, m\}$ *such that*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_j.$$

The releated results were obtained by Perron, and one of Perron's results was improved by Pituk, see [19].

Theorem 4.10 *Assume that (4.11) holds. If* x_n *is a positive solution of (4.10), then either eventually* $x_n = 0$ *or*

$$\lim_{n\to\infty} \sup(|x_{nj}|)^{1/n} = |\lambda_j|,$$

where $\lambda_1, \ldots, \lambda_m$ *are the roots (not necessarily distinct) of the characteristic equation (4.13).*

Consider

$$Y_{n+1} = [A + B(n)]Y_n \tag{4.14}$$

where Y_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix and

$$B : \mathbb{Z}^+ \to C^{m\times m}$$

is a matrix function satisfying

$$\|B(n)\| \to 0, \text{ when } n \to \infty, \tag{4.15}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm $\|.\|$. See [18].

Theorem 4.11 *(Pituk) Suppose that condition (4.15) holds for system (4.14). If* Y_n *is a solution of (4.14), then either*

$$Y_n = 0$$

for all large n *or*

$$\theta = \lim_{n\to\infty} \|Y_n\|^{1/n}$$

exists and θ *is equal to the modulus of one the eigenvalues of the matrix* A*.*

Theorem 4.12 *(Pituk) Suppose that condition (4.15) holds for system (4.14). If Y_n is a solution of (4.14), then either*

$$Y_n = 0$$

for all large n or

$$\theta = \lim_{n\to\infty} \frac{\|Y_{n+1}\|}{\|Y_n\|}$$

exists and θ is equal to the modulus of one the eigenvalues of the matrix A.

Using Theorem 4.11 and 4.12, we obtain the following rate of convergence result.

Theorem 4.13 *Suppose that $r < 1$ and $s < 1$. Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be any positive solution of the system (4.3) such that*

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \overline{x}_1, \\
\lim_{n\to\infty} y_n &= \overline{x}_2
\end{aligned}$$

where $M = (\overline{x}_1, \overline{x}_2)$ and M is globally asymptotically stable. Then, the error vector

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix}_{6\times 1} = \begin{pmatrix} x_n^{(1)} - \overline{x}_1 \\ x_{n-1}^{(1)} - \overline{x}_1 \\ x_{n-2}^{(1)} - \overline{x}_1 \\ x_n^{(2)} - \overline{x}_2 \\ x_{n-1}^{(2)} - \overline{x}_2 \\ x_{n-2}^{(2)} - \overline{x}_2 \end{pmatrix}_{6\times 1}$$

of every positive solution of the system (4.3) satisfies both of the following asymptotic relations:

$$\begin{aligned}
\lim_{n\to\infty} \|E_n\|^{1/n} &= |\lambda_i J_F(M)|\,, \text{ for some } i = 1, 2, \ldots, 6 \\
\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} &= |\lambda_i J_F(M)|\,, \text{ for some } i = 1, 2, \ldots, 6
\end{aligned}$$

where

$$|\lambda_i J_F(M)|$$

is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point M.

Proof. Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be any positive solution of the system (4.3) such that

$$\lim_{n\to\infty} x_n = \overline{x}_1$$

and

$$\lim_{n\to\infty} y_n = \overline{x}_2.$$

To find the error terms, we have

$$\begin{aligned} x_{n+1} - \overline{x}_1 &= \textstyle\sum_{i=0}^{2} A_i(x_{n-i} - \overline{x}_1) + \sum_{i=0}^{2} B_i(y_{n-i} - \overline{x}_2) \\ y_{n+1} - \overline{x}_2 &= \textstyle\sum_{i=0}^{2} C_i(x_{n-i} - \overline{x}_1) + \sum_{i=0}^{2} D_i(y_{n-i} - \overline{x}_2). \end{aligned}$$

Set

$$\begin{aligned} e_n^1 &= x_n - \overline{x}_1, \\ e_n^2 &= y_n - \overline{x}_2; \end{aligned}$$

therefore, it follows that

$$\begin{aligned} e_{n+1}^1 &= \textstyle\sum_{i=0}^{2} A_i e_{n-i}^1 + \sum_{i=0}^{2} B_i e_{n-i}^2 \\ e_{n+1}^2 &= \textstyle\sum_{i=0}^{2} C_i e_{n-i}^1 + \sum_{i=0}^{2} D_i e_{n-i}^2 \end{aligned}$$

where

$$\begin{aligned} A_0 &= 0,\ A_1 = \frac{r}{1+(y_{n-2})^p},\ A_2 = 0, \\ B_0 &= 0,\ B_1 = 0,\ B_2 = -\frac{rpx_{n-1}(y_{n-2})^{p-1}}{(1+(y_{n-2})^p)^2}, \\ C_0 &= 0,\ C_1 = 0,\ C_2 = -\frac{spy_{n-1}(x_{n-2})^{p-1}}{(1+(x_{n-2})^p)^2}, \\ D_0 &= 0,\ D_1 = \frac{s}{1+(x_{n-2})^p},\ D_2 = 0. \end{aligned}$$

Taking the limits, it is clear that

$$\begin{aligned} \lim_{n\to\infty} A_0 &= 0,\ \lim_{n\to\infty} A_1 = \frac{r}{1+\overline{x}_2^p} \text{ and } \lim_{n\to\infty} A_2 = 0 \\ \lim_{n\to\infty} B_0 &= 0,\ \lim_{n\to\infty} B_1 = 0 \text{ and } \lim_{n\to\infty} B_2 = -\frac{rp\overline{x}_1\overline{x}_2^{p-1}}{(1+\overline{x}_2^p)^2}, \\ \lim_{n\to\infty} C_0 &= 0,\ \lim_{n\to\infty} C_1 = 0 \text{ and } \lim_{n\to\infty} C_2 = -\frac{sp\overline{x}_2\overline{x}_1^{p-1}}{(1+\overline{x}_1^p)^2}, \\ \lim_{n\to\infty} D_0 &= 0,\ \lim_{n\to\infty} D_1 = \frac{s}{1+\overline{x}_1^p} \text{ and } \lim_{n\to\infty} D_2 = 0. \end{aligned}$$

That is

$$A_1 = \frac{r}{1+\overline{x}_2^p} + \alpha_n, \quad B_2 = -\frac{rp\overline{x}_1\overline{x}_2^{p-1}}{(1+\overline{x}_2^p)^2} + \beta_n, \quad C_2 = -\frac{sp\overline{x}_2\overline{x}_1^{p-1}}{(1+\overline{x}_1^p)^2} + \gamma_n, \quad D_1 = \frac{s}{1+\overline{x}_1^p} + \delta_n$$

where $\alpha_n \to 0$, $\beta_n \to 0$, $\gamma_n \to 0$, $\delta_n \to 0$ for $n \to \infty$.

Thus, the limitting system of error terms about the equilibrium M can be written as follows:

$E_{n+1} = (C + D(n))E_n,$

where $E_n = (e_n^1, e_{n-1}^1, e_{n-2}^1, e_n^2, e_{n-1}^2, e_{n-2}^2)^T$,

$$C = \begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{6\times 6},$$

$$D_n = \begin{pmatrix} 0 & \alpha_n & 0 & 0 & 0 & \beta_n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_n & 0 & \delta_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 6}$$

and $\|D(n)\| \to 0$, when $n \to \infty$. As desired. ■

Corollary 4.3 *Assume that $r < 1$ and $s < 1$. Then, the error vector of every non-trivial solution of system (4.3) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \|E_n\|^{1/n} = |\lambda_i J_F(M)|, \text{ for some } i = 1, 2, \ldots, 6,$$
$$\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda_i J_F(M)|, \text{ for some } i = 1, 2, \ldots, 6$$

where $|\lambda_i J_F(M)|$ is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point M, i.e. $\{0, \sqrt{r}, \sqrt{s}\}$.

CHAPTER 5

THE DYNAMICS OF POSITIVE SOLUTIONS OF A HIGHER ORDER DIFFERENCE EQUATION WITH ARBITRARY POWERS AND DELAYS

We state that the results of this chapter are cited from [61] which was published by us. See also [62-68] for more results.

5.1 INTRODUCTION

The purpose of this chapter is to study the local asymptotic stability of equilibrium points, the periodic nature of solutions, the existence of unbounded solutions and the global behavior of solutions of the following non-linear difference equation

$$x_{n+1} = \frac{\alpha x_{n-(k+1)}}{\beta + \gamma x_{n-k}^{p} x_{n-(k+2)}^{q}}, \ n \in \mathbb{N} \tag{5.1}$$

where the parameters

$\alpha, \beta, \gamma, p, q$

are non-negative numbers and the initial values

$x_{-(k+2)}, x_{-(k+1)}, ..., x_{-1}, x_0$

are arbirary positive numbers such that the denominator is always positive.
If some parameters of Eq.(5.1) are zero, then five equations emerge, that is, if

$\alpha = 0$

in Eq.(5.1), then, it is trivial, if

$\beta = 0$

in Eq.(5.1), then it can be reduced to a linear difference equation by the change of variables

$x_n = e^{y_n}.$

If

$\gamma = 0$

in Eq.(5.1), then it is linear and finally, the case

$p = 0$

or

$q = 0$

was investigated in [34].

Note that Eq.(5.1) can be reduced to the following non-linear difference equation

$$y_{n+1} = \frac{r y_{n-(k+1)}}{1 + y_{n-k}^{p} y_{n-(k+2)}^{q}}, \; n = 0, 1, ... \tag{5.2}$$

by the change of variables

$$x_n = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{p+q}} y_n$$

with

$$r = \frac{\alpha}{\beta}.$$

So, in order to study Eq.(5.1), we will investigate Eq.(5.2). Both studies are equivalent.

5.2 ASYMPTOTIC BEHAVIOR OF SOLUTIONS

Theorem 5.1 *We have the following cases for the equilibrium points of Eq.(5.2);*

i $\overline{y}_0 = 0$ is always the equilibrium point of Eq.(5.2).

ii If $r > 1$, then, Eq.(5.2) has the positive equilibrium

$$\overline{y}_1 = (r-1)^{\frac{1}{p+q}}.$$

iii If $r < 1$ and $\frac{1}{p+q}$ is an even positive integer, then, Eq.(5.2) has the positive equilibrium

$$\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$$

which is always in the interval $(0, 1)$.

Proof. A point $\overline{y}$ is an equilibrium point of Eq.(5.2) if and only if it is a root for the function

$$f(y) = y^{p+q+1} + (1-r)y.$$

Hence, it is clear that 0 is a root of f. Hence,

$$\overline{y}_0 = 0$$

is an equilibrium point for Eq.(5.2). This completes proof of part (i). Now we consider the case $y \neq 0$. For this case we have from the definition of f that

$$1 + y^{p+q} = r.$$

So, one obtain

$$y = (r-1)^{\frac{1}{p+q}}.$$

According to the values r, p, q, the value of y is identified. From there, the proofs of parts $(ii), (iii)$ are completed, as desired. ■

In the following theorems, we will investigate the local asymptotic behavior of the equilibria and the global behavior of solutions of Eq.(5.2) with

$$r, p, q > 0$$

and positive initial conditions.

Theorem 5.2 *For the local asymptotic stability of equilibria of Eq.(5.2), we obtain the following results;*

(i) If $r < 1$, then, the zero equilibrium point is locally asymptotically stable.

(ii) If $r > 1$, then, the zero equilibrium point is locally unstable.

(iii) If $r = 1$, then, the zero equilibrium point is non-hyperbolic point.

(iv) If $r > 1$ and $k \in 2\mathbb{Z}^+$, then, the positive equilibrium point

$$\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$$

is locally unstable.

(v) If $r \in (0,1)$ and $\frac{1}{p+q} \in 2\mathbb{Z}^+$, then, the positive equilibrium point

$$\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$$

is locally unstable.

Proof. The linearized equation associated with Eq.(5.2) about zero equilibrium is

$$z_{n+1} - rz_{n-(k+1)} = 0, \ n \in \mathbb{N}.$$

The characteristic polynomial of Eq.(5.2) about zero equilibrium is

$$\lambda^{k+3} - r\lambda = 0.$$

So, the proof of (i), (ii) and (iii) follows immediately from Linearized Stability Theorem. For the proof (iv) suppose that $r > 1$, then the linearized equation associated with Eq.(5.2) about

$$\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$$

is

$$z_{n+1} + p(1-\frac{1}{r})z_{n-k} - z_{n-(k+1)} + q(1-\frac{1}{r})z_{n-(k+2)} = 0, \ n \in \mathbb{N}.$$

Therefore, the characteristic equation of Eq.(5.2) about the equilibrium

$$\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$$

is

$$\lambda^{k+3} + p(1-\frac{1}{r})\lambda^2 - \lambda + q(1-\frac{1}{r}) = 0.$$

If we set the function as follows;

$$h(\lambda) = \lambda^{k+3} + p(1-\frac{1}{r})\lambda^2 - \lambda + q(1-\frac{1}{r}) = 0,$$

then, it is clear that

$$h(-1) = \frac{(p+q)(r-1)}{r} > 0$$

and

$$\lim_{\lambda \to -\infty} h(\lambda) = -\infty,$$

so, $h(\lambda)$ has at least a root in the interval $(-\infty, -1)$. This completes the proof.
For the proof (v) we assume that $r < 1$, then the linearized equation associated with Eq.(5.2) about

$$\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$$

is

$$t_{n+1}+p(1-\frac{1}{r})t_{n-k}-t_{n-(k+1)}+q(1-\frac{1}{r})t_{n-(k+2)}=0,\ n\in\mathbb{N}.$$

Therefore, the characteristic equation of Eq.(5.2) about the equilibrium

$$\overline{y}_2=(r-1)^{\frac{1}{p+q}}$$

is

$$\lambda^{k+3}+p(1-\frac{1}{r})\lambda^2-\lambda+q(1-\frac{1}{r})=0.$$

If we set the function as follows;

$$g(\lambda)=\lambda^{k+3}+p(1-\frac{1}{r})\lambda^2-\lambda+q(1-\frac{1}{r})=0,$$

then, it is clear that

$$g(1)=\frac{(p+q)(r-1)}{r}<0$$

and

$$\lim_{\lambda\to\infty}g(\lambda)=\infty,$$

so, $g(\lambda)$ has at least a root in the interval $(1,\infty)$. This completes the proof. ■

Theorem 5.3 *Assume that $r<1$, then the zero equilibrium point of Eq.(5.2) is globally asymptotically stable.*

Proof. We know by Theorem 5.2(i) that the zero equilibrium point of Eq.(5.2) is locally asymptotically stable, hence, it suffices to show that

$$\lim_{n\to\infty}y_n=0$$

for any positive solution $\{y_n\}_{n=-(k+2)}^{\infty}$ of Eq.(5.2).
From Eq.(5.2), we have the following inequlity for all $n\geq 0$.

$$0\leq y_{n+1}=\frac{ry_{n-(k+1)}}{1+y_{n-k}^{p}y_{n-(k+2)}^{q}}\leq ry_{n-(k+1)}.$$

In this sense, we have the following inequlities for $i \in \{0, 1, 2, ...\}$,

$$\begin{aligned} y_{i(k+2)+1} &\leq r^{i+1} y_{-k-1}, \\ y_{i(k+2)+2} &\leq r^{i+1} y_{-k}, \\ &\vdots \\ y_{i(k+2)+k+2} &\leq r^{i+1} y_0. \end{aligned}$$

Since $r < 1$, we have

$$\lim_{i\to\infty} r^{i+1} = 0,$$

hence, we obtain that

$$\lim_{n\to\infty} y_n = 0.$$

Thus, the proof is complete. ■

5.3 PERIODICITY OF SOLUTIONS

Theorem 5.4 *Assume k is an even positive integer, then, Eq.(5.2) possesses eventual prime period two solutions if and only if $r = 1$.*

Proof. Let

$$..., \Phi, \Psi, \Phi, \Psi, \Phi, \Psi, ...$$

a period two solution of Eq.(5.2). Then, we eventually have

$$\Phi = \frac{r\Phi}{1 + \Psi^{p+q}} \text{ and } \Psi = \frac{r\Psi}{1 + \Phi^{p+q}}. \tag{5.3}$$

such that

$$\Phi \neq \Psi.$$

If both Φ and Ψ are non-zero, then we obtain from (5.3) that

$$\Phi = \Psi = (r-1)^{\frac{1}{p+q}},$$

which is a contradiction. Hence, either Φ or Ψ must be equal to zero. Assume that

$$\Phi = 0$$

which implies that

$(r-1)\Psi = 0,$

so

$r = 1.$

In contrast, if

$r = 1,$

then, choose the initial conditions such as

$y_{-(k+2)} = y_{-k} = \ldots = y_0 = 0$

and

$y_{-(k+1)} = y_{-(k-1)} = y_{-1} = \delta > 0$

or such as

$y_{-(k+2)} = y_{-k} = \ldots = y_0 = \delta > 0$

and

$y_{-(k+1)} = y_{-(k-1)} = y_{-1} = 0.$

We can see by induction that

$\ldots, 0, \delta, 0, \delta, \ldots$

is the prime period two solution of Eq.(5.2). ■

Theorem 5.5 *Assume that k is an odd positive integer, then, Eq.(5.2) has no eventual prime period two solutions.*

Proof. Assume that Eq.(5.2) has the prime period two solution

$..., x, y, x, y, ...$

then, we eventually have

$$x = \frac{ry}{1+x^{p+q}} \text{ and } y = \frac{rx}{1+y^{p+q}} \tag{5.4}$$

such that $x \neq y$. Obviously, $x = 0$ implies $y = 0$ or vice versa. This case is impossible. So we obtain that both x and y are greater than zero. So we have

$$\begin{aligned} x^{p+q}(1+x^{p+q})^{p+q+1} + r^{p+q}(1-r^2+x^{p+q}) &= 0, \\ y^{p+q}(1+y^{p+q})^{p+q+1} + r^{p+q}(1-r^2+y^{p+q}) &= 0, \end{aligned}$$

that is, x and y are two distinct positive roots of

$$f(z) = z^{p+q}(1+z^{p+q})^{p+q+1} + r^{p+q}(1-r^2+z^{p+q}) = 0.$$

Obviously, when

$$r \leq 1,$$

then $f(z)$ has no positive roots.

Now, let

$$r > 1$$

and set

$$1 + z^{p+q} = w.$$

Then, the function,

$$g(w) = w^{p+q+2} - w^{p+q+1} + wr^{p+q} - r^{p+q+2}, w > 1,$$

has at least two distinct positive roots. However,

$$g'(w) = w^{p+q}\left[(p+q+2)w - (p+q+1)\right] + r^{p+q} > 0$$

for any

$$w \in (1, \infty),$$

which indicates that $g(w)$ is strictly increasing in the interval $(1, \infty)$. This implies that the function $g(w)$ does not have two distinct positive roots at all in the interval $(1, \infty)$. Thus, Eq.(5.2) does not have the prime period two solution when $r > 1$. This completes the proof. ■

5.4 OSCILLATION OF SOLUTIONS

For the oscillatory solution of Eq.(5.2) , we have the following results.

Theorem 5.6 *Suppose that*

$r > 1, \;\; k \in 2\mathbb{Z}^+$

and let $\{y_n\}_{n=-(k+2)}^{\infty}$ *be any solution of Eq.(5.2) such that*

$$y_{-(k+2)}, y_{-k}, ..., y_{-2}, y_0 \geq \overline{y}_1 \text{ and } y_{-(k+1)}, y_{-(k-1)}, ..., y_{-3}, y_{-1} < \overline{y}_1 \tag{5.5}$$

or

$$y_{-(k+2)}, y_{-k}, ..., y_{-2}, y_0 < \overline{y}_1 \text{ and } y_{-(k+1)}, y_{-(k-1)}, ..., y_{-3}, y_{-1} \geq \overline{y}_1 \tag{5.6}$$

holds. Then, $\{y_n\}_{n=-(k+2)}^{\infty}$ *oscillates about the positive equilibrium point*

$\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$

with semicycles of length one.

Proof. Suppose that $r > 1$ and the case (5.5) holds for the solution $\{y_n\}_{n=-(k+2)}^{\infty}$. From Eq.(5.2), it is clear that

$$y_1 = \frac{ry_{-(k+1)}}{1 + y_{-k}^p y_{-(k+2)}^q} < \overline{y}_1 = (r-1)^{\frac{1}{p+q}}, \;\; y_2 = \frac{ry_{-k}}{1 + y_{-(k-1)}^p y_{-(k+1)}^q} > \overline{y}_1 = (r-1)^{\frac{1}{p+q}},$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$y_{k+1} = \frac{ry_{-1}}{1 + y_0^p y_{-2}^q} < \overline{y}_1 = (r-1)^{\frac{1}{p+q}}, \;\; y_{k+2} = \frac{ry_0}{1 + y_1^p y_{-1}^q} > \overline{y}_1 = (r-1)^{\frac{1}{p+q}}.$$

So, the proof follows by induction. For the case (5.6) the proof is similar and will be omitted. ■

Corollary 5.1 *Suppose that*

$r < 1, \;\; \dfrac{1}{p+q}, k \in 2\mathbb{Z}^+,$

and let $\{y_n\}_{n=-(k+2)}^{\infty}$ *be any solution of Eq.(5.5) such that*

$$y_{-(k+2)}, y_{-k}, ..., y_{-2}, y_0 \geq \overline{y}_2 \text{ and } y_{-(k+1)}, y_{-(k-1)}, ..., y_{-3}, y_{-1} < \overline{y}_2$$

or

$y_{-(k+2)}, y_{-k}, \ldots, y_{-2}, y_0 < \overline{y}_2$ *and* $y_{-(k+1)}, y_{-(k-1)}, \ldots, y_{-3}, y_{-1} \geq \overline{y}_2$

holds. Then, $\{y_n\}_{n=-(k+2)}^{\infty}$ *oscillates about the positive equilibrium point*

$\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$

with semicycles of length one.

5.5 UNBOUNDED SOLUTIONS

In respect of the unbounded solutions of Eq.(5.2), the following result is reproduced.

Theorem 5.7 *Assume*

$r > 1, \ k \in 2\mathbb{Z}^+,$

then, Eq.(5.2) possesses unbounded solutions. Especially, every solution of Eq.(5.2) which oscillates about the positive equilibrium point

$\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$

with semicycles of length one is unbounded.

Proof. From Theorem 5.6, we can assume without loss of generality that the solution $\{y_n\}_{n=-(k+2)}^{\infty}$ of Eq.(5.2) is such that

$y_{2n+1} > \overline{y}_1 = (r-1)^{\frac{1}{p+q}}$ and $y_{2n+2} < \overline{y}_1 = (r-1)^{\frac{1}{p+q}}$, for all $n \geq 0$.

Then,

$$y_{2n+2} = \frac{ry_{2n-k}}{1 + y_{2n-k+1}^p y_{2n-k-1}^q} < \frac{ry_{2n-k}}{1+(r-1)} = y_{2n-k}$$

and

$$y_{2n+3} = \frac{ry_{2n-k+1}}{1 + y_{2n-k+2}^p y_{2n-k}^q} > \frac{ry_{2n-k+1}}{1+(r-1)} = y_{2n-k+1}.$$

Thus, we obtain that

$$\lim_{n\to\infty} y_{2n+1} = \infty$$

and

$$\lim_{n\to\infty} y_{2n+2} = 0$$

which completes the proof. ■

Corollary 5.2 *Assume*

$$r < 1, \ \frac{1}{p+q}, k \in 2\mathbb{Z}^{+},$$

then, Eq.(5.2) possesses unbounded solutions. Especially, every solution of Eq.(5.2) which oscillates about the positive equilibrium point

$$\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$$

with semicycles of length one is unbounded.

Open Problem Investigate the dynamical behavior of Eq.(5.1) where the parameters

$$\alpha, \beta, \gamma, p, q$$

are non-negative numbers, k is an odd number and the initial values

$$x_{-(k+2)}, x_{-(k+1)}, \ldots, x_{-1}, x_0$$

are non-negative numbers.

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