CRITICAL PROBABILITIES OF PERCOLATION ON GRAPHS AND RANDOM TREES

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Merve Kaya

August, 2014

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Azer Kerimov (Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Bülent Ünal

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Mehmet Özgür Oktel

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

CRITICAL PROBABILITIES OF PERCOLATION ON GRAPHS AND RANDOM TREES

Merve Kaya
M.S. in Mathematics
Supervisor: Assoc. Prof. Dr. Azer Kerimov
August, 2014

We consider the model of independent percolation on various graphs and random trees. We investigate the critical probabilities of bond and site percolation on these graphs.

Keywords: percolation, critical probability, tree, bond, site.

ÖZET

GRAFLAR VE RASLANTISAL AĞAÇLAR ÜZERİNDEKİ SIZMA KRİTİK DEĞERLERİ

Merve Kaya
Matematik, Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Azer Kerimov
Ağustos, 2014

Çeşitli graflar ve rastlantısal dallanma ağaçı üzerindeki sızma modellerini inceliyoruz. Bu graflar üzerindeki kenar ve köşe modellerinin kritik değerlerini araştıracağız.

Anahtar sözcükler: sızma, kritik olasılıklar, ağaç, kenar, köşe.

Acknowledgement

I would like to express my sincere gratitude to my supervisor Assoc. Prof. Azer Kerimov for his excellent guidance, valuable suggestions, encouragement and patience.

I would also like to thank Assoc. Prof. Bülent Ünal and Assoc. Prof. Mehmet Özgür Oktel for a careful reading of this thesis.

I would like to thank to my family for their great support, endless love and trust. This thesis would never be possible without their countenance.

I am so grateful to Alperen who is always with me whenever I needed.

The thesis is supported financially by TÜBİTAK through the graduate fellowship program, namely "TÜBİTAK-BİDEB 2210-Yurt İçi Yüksek Lisans Burs Programı". I really thank to TÜBİTAK for their support.

Finally, I would like to thank my friends Elif, Berrin and Çisem from the department for their support and encouragement. My thanks also goes to Abdullah who helped me about Latex with all his patience.

Contents

Chapter 1

Introduction

The percolation theory was introduced in the late 1950 by Broadbent and Hammersley [1] to open up to mathematical analysis in the study of random physical process such as the transport of a fluid through a disordered porous medium. It was proved to be a remarkably rich theory, with applications beyond natural phenomena to topics such as the theory of networks. Physicists also have a great interest in it, due to the similar nature of its critical behavior and that of some physical systems, and due to the fact that percolation has so many simple features in comparison with many of those systems.

Percolation is the phenomenon of flow of a fluid through a porous medium. The medium consists of microscopic pores and channels through which the fluid might pass. In a simple case, each channel will be open or closed to the passage of the fluid, depending on characteristics of the medium. The distribution of open and closed channels could be described probabilistically. In the simplest situation, each channel, independently of the others, is open with probability p, the single parameter of the model, and closed with the probability $1-p$. We will model the medium microscopically by the d- dimensional hypercubic lattice, $\mathbb{Z}^d$, whose vertices and (nearest neighbor) edges represent the pores and channels, respectively. This constitutes what we call the *independent bond percolation model (in $\mathbb{Z}^d$).*

This process is called a *bond* percolation model because the random stoppages in the lattice are associated with the edges. Another kind of percolation process is the *site* percolation model, in which the vertices rather than the edges are declared to be open or closed randomly.

In this theory, the fundamental question is the occurrence or not of percolation, that is, the existence of an infinite path, through open bonds or sites. In the next chapters, we will define the model in detail and show its first non-trivial results, establishing the existence of a critical value for the parameter p, $p_c \in (0, 1)$ such that the model does not exhibit percolation almost surely for values of p below p_c, and does exhibit percolation almost surely for values of p above p_c.

With narrow expression, percolation theory is the concept of component structure of random subgraphs of graphs. In general, the underlying graph is a lattice or lattice-like graphs, that may or may not be oriented, and to attain our random subgraph we choose vertices or edges independently. In this paper, we consider the model of independent percolation on various graphs and random trees.

Chapter 2

Fundamental Concepts

2.1 The Model

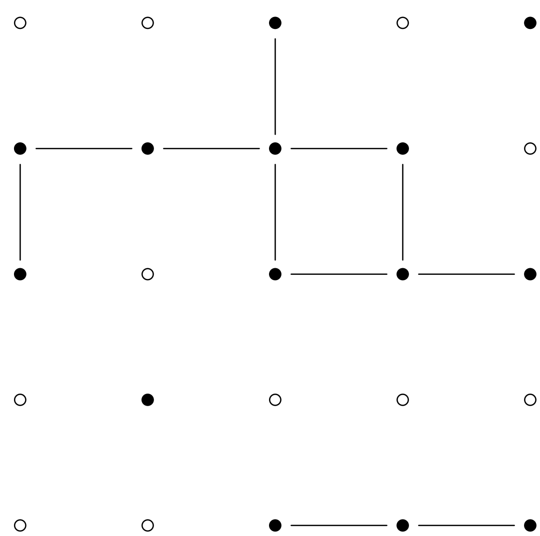
In this part, we give basic definitions and notations of percolation on $\mathbb{Z}^d$. Basically, we shall use the definitions and notations of the graph theory in a standard manner, as the Bollobás [2], for example. In particular, if $\Lambda = (V, E)$ is a graph, then $V(\Lambda)$ and $E(\Lambda)$ denote the sets of vertices and edges of Λ, respectively. An edge $\{x, y\}$ is said to *join* the vertices x and y, and denoted by xy. If $xy \in E(\Lambda)$, then x and y are *adjacent* vertices of Λ. We say $\Lambda' = (V', E')$ is a *subgraph* of $\Lambda = (V, E)$ if $V' \subset V$ and $E' \subset E$. We will write $x \in \Lambda$ instead of $x \in V(\Lambda)$. Moreover, the *order* of Λ is the number of vertices in Λ; it is denoted by $|\Lambda|$. The set of vertices adjacent to a vertex $x \in \Lambda$, the *neighborhood* of x, is denoted by $\Gamma(x)$. The *degree* of x is $d(x) = |\Gamma(x)|$. In addition, the vertex x is called a *leaf* of the graph Λ when $d(x) = 1$. A *path* is a graph P of the form

$$V(P) = \{x_0, x_1, ..., x_l\}\,, \quad E(P) = \{x_0x_1, x_1x_2..., x_{l-1}x_l\} \tag{2.1}$$

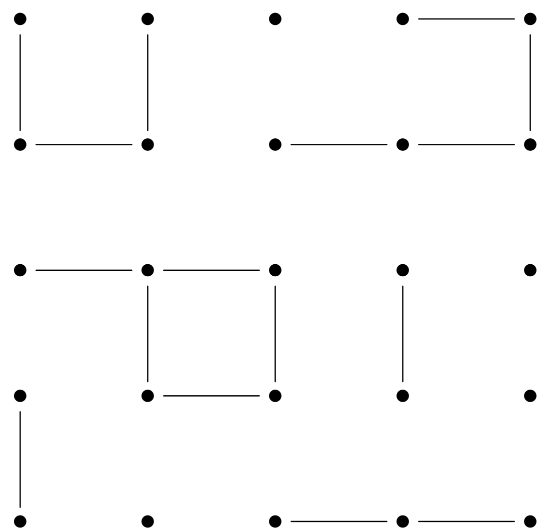
The vertices x_0 and x_l are the *endvertices* of P and l is the *length* of P. We say that P is a path from x_0 to x_l. Lastly, we state the *oriented graph*, which is a directed graph obtained by orienting the edges of a graph, that is by giving the

ab an orientation $\overrightarrow{ab}$ or $\overleftarrow{ab}$. Thus an oriented graph is a directed graph in which at most one of $\overrightarrow{ab}$ or $\overleftarrow{ab}$ occurs.

Although we cling to these definitions and notations, the standard terminology of percolation theory have some differences from that of graph theory. In percolation theory, vertices and edges are called *sites* and *bonds* , and components are called *clusters*. If our random graph is obtained by choosing vertices, we consider *site percolation*; if we choose edges, *bond percolation*. In both cases, the selected sites or bonds are called *open*, and not selected ones are called *closed*; the *state* of a site or bond is open if it is selected, and closed otherwise. The *open subgraph* , in the site percolation, is the graph induced by open sites; in bond percolation, is formed by the edges and all vertices.



This is an example of the representation graph of site percolation. The filled circles are the open sites; the open subgraph is the subgraph of $\mathbb{Z}^2$ induced by these.



This graph is a kind of the representation of bond percolation. The open subgraph is the spanning subgraph containing all the open bonds.

Throughout the discussion, we shall focus on percolation on an unoriented graph Λ. We suppose that Λ is connected, infinite, and locally finite (i.e., every vertex has finite degree). For the general case, Λ is a *multi-graph*; hence multiple edges are allowed between the same pair of vertices, but not loops. Most of the interesting examples will be simple graphs.

In general, we will choose sites or bonds to be open with the same probability p, independently of each other. This produces a probability measure on the set of subgraphs of Λ. In the bond percolation, we will denote this measure by $\mathbb{P}^b_{\Lambda,p}$ and in the site percolation $\mathbb{P}^s_{\Lambda,p}$. In a similar way, Λ^b_p is the open subgraph in bond percolation, and Λ^s_p in site percolation.

In a formal way, given a graph Λ with E its edge-set, a (bond) *configuration* is a function $\omega : E \to \{0,1\}$, $e \mapsto \omega_e$; we write $\Omega = \{0,1\}^E$ for the set of all (bond) configurations. In the configuration, a bond e is open if and only if $\omega_e = 1$, hence configurations correspond to open subgraphs. Let Σ be the $\sigma - field$ on Ω generated by the cylindrical sets

$$C(F,\sigma) = \{\omega \in \Omega : \omega_f = \sigma_f \ for \ f \in F\} \tag{2.2}$$

where F is a finite subset of E and $\sigma \in \{0,1\}^F$. Let $\mathbf{p} = (p_e)_{e\in E}$ with $0 \leq p_e \leq 1$ for every bond e. We denote by $\mathbb{P}^s_{\Lambda,\mathbf{p}}$ the probability measure on (Ω, Σ) induced by

$$\mathbb{P}^b_{\Lambda,\mathbf{p}}\left(C(F,\sigma)\right) = \prod_{\substack{f\in F\\ \sigma_f=1}} p_f \prod_{\substack{f\in F\\ \sigma_f=0}} (1-p_f) \tag{2.3}$$

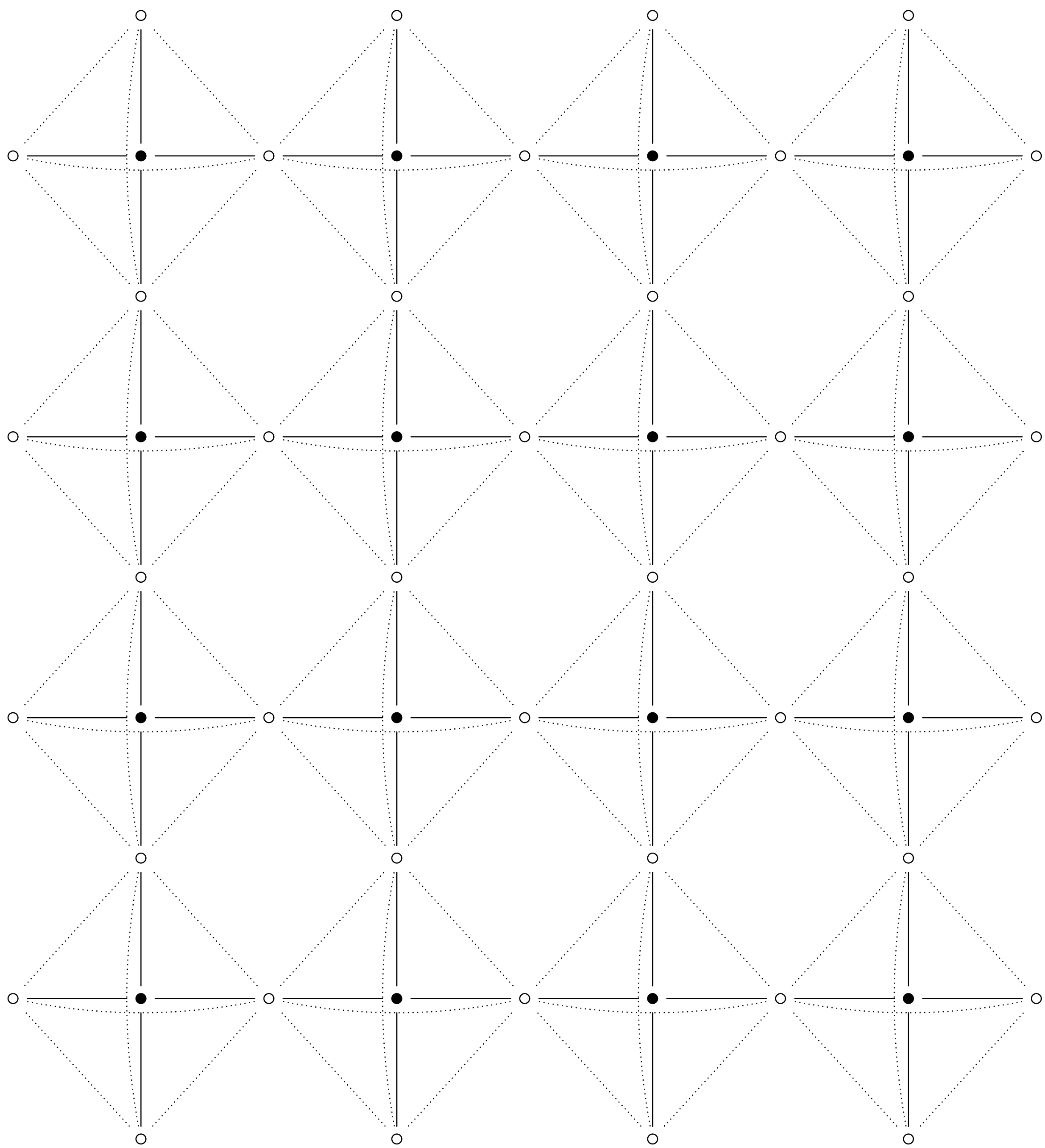
If $p_e = p$ for every edge e, as before, then we write $\mathbb{P}^b_{\Lambda,p}$ instead of $\mathbb{P}^b_{\Lambda,\mathbf{p}}$. In the measure, the states of the bonds are independent, with the probability that e is open equal to p_e; therefore, for two disjoint sets F_0 and F_1 of bonds,

$$\mathbb{P}^b_{\Lambda,\mathbf{p}}(\text{the bonds in } F_1 \text{ are open and those in } F_0 \text{ are closed}) = \prod_{f\in F_1} p_f \prod_{f\in F_0} (1-p_f)$$

This measure $\mathbb{P}^b_{\Lambda,\mathbf{p}}$ is called an *independent bond percolation measure* on Λ. In particular, the case $p_e = p$ for every bond e is exactly the measure $\mathbb{P}^b_{\Lambda,p}$ informally defined before. The formal definitions for *independent site percolation* are also similar.

Next, we importantly remark the close relation between site percolation and bond percolation. In order to see this, we will first give the definition of the line graph.

Definition 2.1. The graph $L(\Lambda)$ is called *line graph* of Λ if its vertices are the edges of Λ; two vertices of $L(\Lambda)$ are adjacent if the corresponding edges of Λ share a vertex.



The solid circles and lines represents a part of the square lattice $\mathbb{Z}^2$ and the hollow circles and dotted lines represents its line graph $L(\mathbb{Z}^2)$. Note that $L(\mathbb{Z}^2)$ is isomorphic to the non-planar graph obtained from $\mathbb{Z}^2$ by adding both diagonals to every other face.

In the light of this definition, we deduce that site percolation is more general, in the sense that bond percolation on a graph Λ is equivalent to site percolation on line graph of Λ, $L(\Lambda)$. Due to this generalization, the critical probability for the site percolation is greater than or equal to the critical probability for the bond percolation. In this thesis, we are mostly given the examples that have the equal critical probability for both cases. However, there are also some examples that have greater critical probability for site percolation. For example, for the square lattice $\mathbf{Z}^2$ in two dimensions, the critical probability is equal to 1/2 for bond percolation, a fact which was an open question for more than 20 years and was finally resolved by Harry Kesten in the early 1980s,[3]. On the other hand, the exact value of the critical probability for site percolation is still an open question. Several authors have given rigorous upper and lower bonds. From the general arguments of Hammersley [4], it followed that the above-mentioned critical probability is larger than 1/3. The main result of Harris [5] (combined with a comparison result of Hammersley [6]) yields that it is at least 1/2. About 20 years later, rigorously proved that 1/2 is also strict lower bound [7]. After this, improvements were made more frequently: 0.503478 (T*óth* [8]), 0.522105 (Zuev [9]), and finally, 0.5416 (Menshikov and Pelikh [10]).

Even though we shall make some remarks about general infinite graphs, the main applications are always to 'lattice-like' graphs. These graphs have a finite number of 'types' of vertices and edges. On occasion, we may choose vertices or edges of different types with different probabilities.

For a fixed underlying graph Λ, there is a natural coupling of the measures $\mathbb{P}^b_{\Lambda,\mathbf{p}}$, $0 \leq p \leq 1$ as follows: take X_e independent random variable for each bond e of Λ, with X_e uniformly distributed on $[0, 1]$. We may consider Λ^b_p as the spanning subgraph of Λ containig all bonds e with $X_e \leq p$. When $p_1 < p_2$, $\Lambda^b_{p_1}$ is a subgraph of $\Lambda^b_{p_2}$. For site percolation, similar coupling is also possible.

A path in the open subgraph is called an *open path.* For sites x and y, the event that there is an open path from x to y will be shown by '$x \to y$' and the probability of this event will be denoted by $\mathbb{P}(x \to y)$ in the measure under consideration. We also write '$x \to \infty$' for the event that there is an infinite open path starting at x.

An *open cluster* is a component of the open subgraph. The graphs under considerations are locally finite, therefore an open cluster is infinite if and only if, for every site x in the cluster, the event $x \to \infty$ holds. Given a site x, C_x is written for the open cluster containing x, if there is one; otherwise, we take C_x to be empty. Hence, $C_x = \{y \in \Lambda : x \to y\}$ is the set of sites y for which there is an open $x \to y$ path. In bond percolation, C_x always contains x, and in site percolation, $C_x = \emptyset$ if and only if x is closed.

2.2 The Critical Probability

One of the main interests in percolation theory is the percolation probability which is the probability that a given vertex belongs to an infinite open cluster. Let $\theta_x(p)$ be the probability that C_x is infinite, thus $\theta_x(p) = \mathbb{P}_p(x \to \infty)$. Clearly, $\theta_x(p)$ depends on the underlying graph Λ, and whether we take bond or site percolation. In a formal way, for bond percolation, we write

$$\theta_x(p) = \theta_x(\Lambda_p^b) = \theta_x^b(\Lambda, p) = \mathbb{P}^b_{\Lambda,p}(|C_x| = \infty) \tag{2.4}$$

where $|C_x| = |V(C_x)|$ is the number of sites in C_x. Whichever form of the notationis clearest in any given context, we will use it. Two sites x and y of a graph Λ are said to be *equivalent* if there is an automorphism of Λ mapping x to y. If all sites are equivalent, then we write $\theta(p)$ instead of $\theta_x(p)$ for any site x. The quantity $\theta(p)$, or $\theta_x(p)$, is called as the *percolation probability.*

If x and y are sites at distance d, then $\theta_x(p) \geq p^d\theta_y(p)$. Hence, either $\theta_x(p) = 0$ for every x, or $\theta_x(p) > 0$ for every x. By the coupling described above, $\theta_x(p)$ is clearly an increasing function of p. Thus, there is a critical value p_H, $0 \leq p_H \leq 1$, such that

$$\theta_x(p) \begin{cases} = 0, & \text{for every } x \quad \text{if} \quad p < p_H \\ > 0, & \text{for every } x \quad \text{if} \quad p > p_H \end{cases}$$

p_H is called the *critical probability* and is defined by

$$p_H = sup\,\{p : \theta_x(p) = 0\} \tag{2.6}$$

The notation p_H is in honour of Hammersley. If the model under consideration is not clear from the context, then we write $p_H^s(\Lambda)$ for site percolation on Λ and $p_H^b(\Lambda)$ for bond percolation.

2.3 The First Result

The component structure of the open subgraph undergoes a dramatic change as p increases past p_H as follows: if $p < p_H$ then the probability that there is an infinite open cluster is 0, while for $p > p_H$ this probability is 1. In order to show this, we will use one of the most basic result in probability theory, which is known as Kolmogorov's 0-1 law.

Theorem 2.2. *([11]) Let $X = (X_1, X_2, ...)$ be a sequence of independent random variables, and let A be an event in the σ- field generated by X. Suppose that, for every n, the event A is independent of $X_1, ..., X_n$. Then $\mathbb{P}(A)$ is 0 or 1.*

An event A with the property described above is known as a *tail event*: This theorem states that any tail event in a product probability space has probability 0 or 1.

Indeed, this theorem was not Kolmogorov's original formulation of this result. What he showed was that, if $X = (X_1, X_2, ...)$ is a sequence of real-valued random variables, $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a Baire function, and

$$\mathbb{P}(f(X) = 0 | X_1, ..., X_n) = \mathbb{P}(f(X) = 0) \tag{2.7}$$

then $\mathbb{P}(f(X) = 0)$ is 0 or 1. As Kolmogorov noted, these assumptions are satisfied if the X_i are independent and the value of the function $f(X)$ remains unchanged when only a finite number of variables are changed.

By using this theorem, we will present that the probability that there is an infinite open cluster is either 0 or 1.

Theorem 2.3. *([?]) The probability $\psi(p)$ that there exists an infinite open cluster satisfies*

$$\psi(p) = \begin{cases} = 0, & \textit{if} \quad \theta_x(p) = 0 \\ = 1, & \textit{if} \quad \theta_x(p) > 0 \end{cases}$$

Proof. Let E be the event that there is an infinite open cluster. Observe that E does not depend upon the states of any finite set of bonds or sites. Thus, Kolmogorov's 0-1 law implies that $\mathbb{P}_p(E)$ is either 0 or 1. If $p < p_H$, so that $\theta_x(p) = 0$ for every x, then

$$\psi(p) = \mathbb{P}_p(E) \leq \sum_x \theta_x(p) = 0 \tag{2.9}$$

and if $p > p_H$, then

$$\psi(p) = \mathbb{P}_p(E) \geq \theta_x(p) > 0 \tag{2.10}$$

for some site x (and so for all sites), implying that $\psi(p) = 1$. □

We says that *percolation occurs* in a certain model if $\theta_x(p) > 0$, so $\psi(p) = 1$.

Chapter 3

Percolation On Rooted Trees

In this chapter, we study the percolation on rooted trees and investigate the critical probabilities at which the percolation occurs.

The percolation theory deals with infinite graphs, and many of the basic events studied (such as the occurrence of percolation) require the states of infinitely many bonds. Nonetheless, it is enough to study events in *finite* probability spaces, since, for instance,

$$\theta_x(p) = \lim_{n\to\infty} \mathbb{P}_p(|C_x| \geq n) \tag{3.1}$$

When $p < p_H$, the open cluster C_x is finite with probability 1, however its expected size need not be finite. This leads us to another critical probability, p_T, named in honour of Temperley. Repeatedly, we write $p_T^s(\Lambda)$ for site percolation and $p_T^b(\Lambda)$ for bond percolation on Λ. For a site x,

$$\chi_x(p) = \mathbb{E}_p(|C_x|) \tag{3.2}$$

where $\mathbb{E}_p$ is the expectation associated to $\mathbb{P}_p$. When all sites are equivalent, we simply write $\chi(p)$. Clearly, $\chi_x(p)$ is increasing with p, and, as before, $\chi_x(p)$ is finite for some site x if and only if it is finite for all sites. Therefore there is a critical probability

$$p_T = \sup\{p : \chi_x(p) < \infty\} = \inf\{p : \chi_x(p) = \infty\} \tag{3.3}$$

which does not depend on x. By definition, $p_T \leq p_H$.

3.1 Percolation On k-Branching Trees

Calculating the critical probabilities p_H and p_T is not very easy excepting a few cases. The prime example is the d-regular infinite tree.

Definition 3.1. A *d-regular infinite tree* is a connected graph with the property that there exists a unique path connecting any two of its vertices and the degree of each vertex is d.

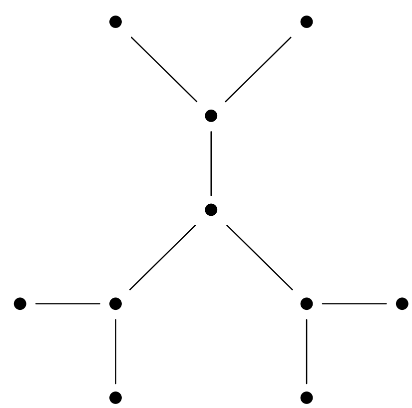
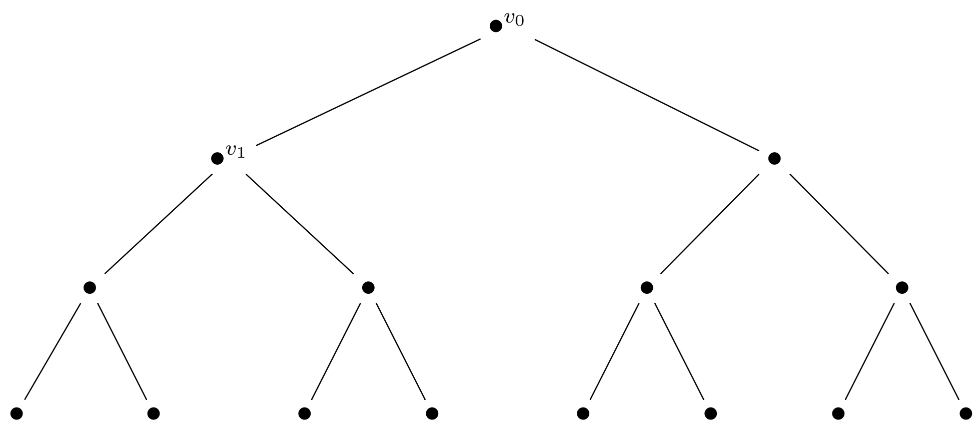


Figure 3.1: The 3-regular tree, for which $p_T^b = p_H^b = p_T^s = p_H^s = 1/2$. Deleting an edge, this tree falls into two components, each of which is a 2-branching tree.

For the purposes of calculation, it is more convenient to consider k-branching tree.

Definition 3.2. A *k-branching tree* (or Bethe lattice) T_k is an infinite rooted tree in which each vertex has k children, so all sites have degree $k+1$ except for one whose degree is k. This particular vertex is the *root* of the tree, denoted by v_0.

Firstly, we investigate the critical probabilities for the k-branching tree. Let $T_{k,n}$ be the section of this tree up to height (or, the usual mathematical convention of planting trees with the root at the top, depth) n.



....

This graphs represents the tree $T_{2,3}$, with the root v_0, and the following theorem states that the critical probabilities for percolation on this tree is $1/k$.

Theorem 3.3. $p_H^b(T_k) = p_T^b(T_k) = p_H^b(T_k) = p_T^b(T_k) = 1/k$

Proof. Let $\pi_n = \pi_{k,n}(p)$ be the probability that $T_{k,n}$ contains an open path of length n from the root to a leaf. Taking the bonds to be open independently with probability p, such a path exists if and only if, for some child v_1 of v_0, the bond v_0v_1 is open and there is an open path of length $n-1$ from v_1 to a leaf. Thus, we get the equation

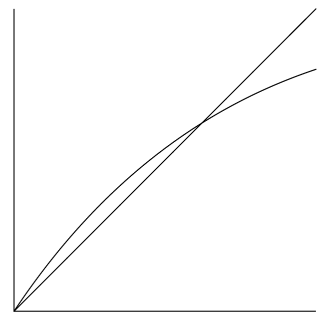
$$1 - \pi_n = (1 - p.\pi_{n-1})^k \tag{3.4}$$

This equation gives a function such that

$$\pi_n = 1 - (1 - p.\pi_{n-1})^k = f_{k,p}(\pi_{n-1}) \tag{3.5}$$

On the interval $[0, 1]$, the function $f_{k,p}(\pi_{n-1})$ is increasing and concave, with $f_{k,p}(0) = 0$ and $f_{k,p}(1) < 1$. This gives that $f_{k,p}(x_0) = x_0$ for some $0 < x_0 < 1$

if and only if $f'_{k,p}(0) = kp > 1$; moreover, the fixed point x_0 is unique when it exists.



Hence, if $p > 1/k$, then, appealing to (3.4), we get that $\pi_{n-1} \geq x_0$ implies $\pi_n \geq x_0$. Since $\pi_0 = 1$, it follows that $\pi_n \geq x_0$ for every n, thus, $\theta^b_{v_0}(T_k; p) \geq x_0 > 0$. So we should have $p^b_H(T_k) \leq 1/k$.

In addition, if $p \leq 1/k$, then π_n converges to 0, the unique fixed point of $f_{k,p}(x_0)$, and so $\theta^b_{v_0}(T_k; p) = 0$. This means that the critical probability $p^b_H(T_k)$ is equal to $1/k$.

Now we will calculate the other critical probability, p_T. Notice that the probability that a site y at graph distance l from the root v_0 belongs to C_{v_0} is exactly p^l. Thus, we get

$$\chi^b_{v_0}(T_k; p) = \mathbb{E}_p(|C_{v_0}|) = \sum_{y \in T_k} \mathbb{P}(y \in C_{v_0}) = \sum_{l=0}^{\infty} k^l p^l \tag{3.6}$$

This sum is finite for $p < 1/k$ and infinite for $p \geq 1/k$. This means that the critical probability $p^b_T(T_k)$ is also equal to $1/k$.

After conditioning on the root x being open, the open clusters containing x in site and bond percolation have exactly the same distribution, for any infinite tree. In fact, each child of a site in the open cluster lies in the open cluster with probability p. Hence, for the k-branching tree T_k, we get the result

$$p_H^s(T_k) = p_H^b(T_k) = p_T^s(T_k) = p_T^b(T_k) = 1/k \tag{3.7}$$

□

Corollary 3.4. *It will be deduced that the four critical probabilities associated to the $(k+1)$-regular tree are also equal to $1/k$.*

The argument leads us to a comparison between percolation on T_k and a certain branching process. We shall consider a slightly less trivial example of such a comparison shortly. If Λ is any graph with maximum degree Δ, then a 'one-way' comparison with branching process gives that all critical probabilities associated to Λ are at least $1/(\Delta - 1)$. In order to see this, notice that for every $y \in C_x$ there is at least one open path in Λ from x to y. So $\chi_x(p) = \mathbb{E}_p(|C_x|)$ is at most the expected number of open (finite) paths in Λ starting at x. Since there are at most $\Delta(\Delta-1)^{l-1}$ paths in Λ of length l starting at x, we have

$$\chi_x^b(p) \leq 1 + \sum_{l \geq 1} \Delta(\Delta-1)^{l-1} p^l \tag{3.8}$$

and

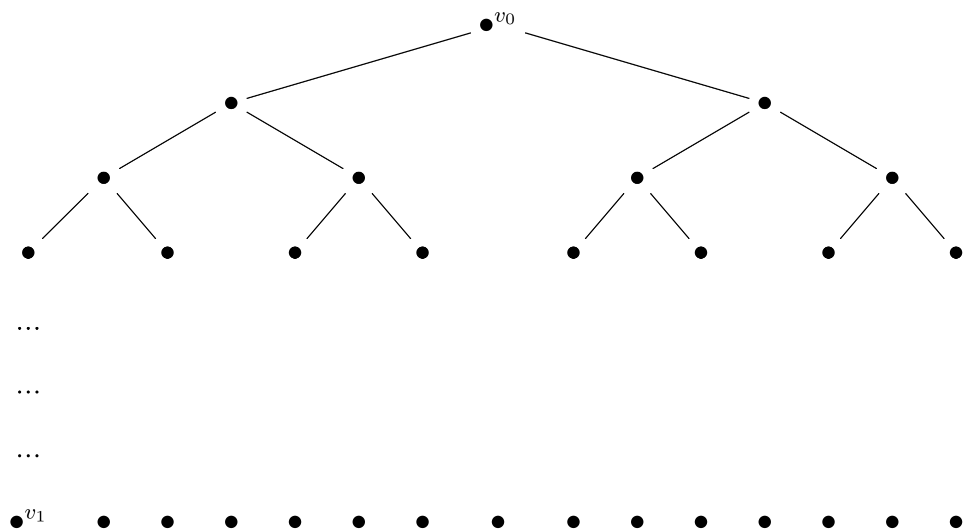
$$\chi_x^s(p) \leq p + \sum_{l \geq 1} \Delta(\Delta-1)^{l-1} p^{l+1} \tag{3.9}$$

for bond and site percolation respectively. For any $p < 1/(\Delta-1)$, both sums converge; thus $p_T^b(\Lambda), p_T^s(\Lambda) \geq 1/(\Delta-1)$.

Since we have $p_H \geq p_T$ by definition, the corresponding inequalities for p_H are also valid. Therefore we may state the following corollary:

Corollary 3.5. *Among all graphs with maximum degree Δ, the Δ-regular tree has the lowest critical probabilities.*

Now, we will introduce another kind of an infinite branching tree. Let T be a finite rooted tree with height (depth) h, with l leaves. Let $T^1 = T$, and T^n be the rooted tree of height hn formed from T^{n-1} by identifying each leaf with the root of a copy of T. Let T^∞ be the 'limit' of the trees T^n, defined in the obviously.



This graph represents the tree $T = T^1$, with the root v_0 and v_1 is a leaf of T^1. The next theorem states the critical probabilities for this graph.

Theorem 3.6. $p_H^s(T^\infty) = p_H^b(T^\infty) = p_T^s(T^\infty) = p_T^b(T^\infty) = l^{-1/h}$

Proof. Note that the probability that there is an open path from the root to a leaf of T is p^h, taking the bonds to be open independently with probability p. Assume that v_0 is the root of T^1, so also T^n, and v_1 is a leaf of T^1. Let π_n be the probability that T^n contains an open path of length hn from the root to a leaf of T^n. Such a path exists if and only if there is an open path of length $h(n-1)$ from v_1 to a leaf of T^n and also an open path from v_0 to v_1. Under these considerations, we get the relation

$$1 - \pi_n = (1 - p^h \pi_{n-1})^l \tag{3.10}$$

Then we will consider the function,

$$\pi_n = 1 - (1 - p^h \pi_{n-1})^l = f_{l,h,p}(\pi_{n-1}) \tag{3.11}$$

This function is again increasing and concave on the interval $[0, 1]$. Furthermore, $f_{l,h,p}(0) = 0$ and $f_{l,h,p}(1) < 1$, so $f_{l,h,p}(x_0) = x_0$ for some $0 < x_0 < 1$ if and only if $f'_{l,h,p}(0) = p^h l > 1$. Thus, by the similar arguments above, when $p > l^{-1/h}$, we get that $\theta^b_{v_0}(T^\infty) > 0$; implying that $p^b_H(T^\infty) \leq l^{-1/h}$. Additionally, when $p < l^{-1/h}$, π_n converges to 0; so $\theta^b_{v_0}(T^\infty) = 0$. Therefore, the critical probability $p^b_H(T^\infty)$ is equal to $l^{-1/h}$.

Turning to p_T, we will use very effective property of expectation.

Theorem 3.7. *Let $X_1, ..., X_n$ be any finite collection of discrete random variables and let $X = \sum_{i=1}^n X_i$. Then we have*

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i] \tag{3.12}$$

It can be deduced that the linearity of expectation also holds for countably infinite summations in certain cases. For example, it holds that

$$\mathbb{E}[\sum_{i=1}^\infty X_i] = \sum_{i=1}^\infty \mathbb{E}[X_i] \tag{3.13}$$

if $\sum_{i=1}^\infty \mathbb{E}[|X_i|] < \infty$.

Notice that the number of leaves of T joined to the root by open paths has a certain distribution X with expectation $p^h l$. Let X_n be the number of sites of T^∞ at distance hn from the root by open paths. Then the sequence $(X_0, X_1, X_2, ...)$ is a branching process: we have $X_0 = 1$ and each X_n is the sum of X_{n-1} independent copies of the distribution X. As X is bounded, we have

$$\chi^b_{v_0}(T^n; p) = \mathbb{E}_p(|C_{v_0}|) = \mathbb{E}[\sum_{i=1}^\infty X_i] = \sum_{i=1}^\infty \mathbb{E}[X_i] = \sum_{i=1}^\infty (p^h l)^i \tag{3.14}$$

which is finite for $p < l^{-1/h}$ and infinite for $p \geq l^{-1/h}$. Hence, the critical probability $p_T^b(T^\infty)$ is equal to $l^{-1/h}$.

As we state above, after conditioning on the root x being open, the open clusters containing x in site and bond percolation have exactly same distribution. Thus, we get

$$p_H^s(T^\infty) = p_H^b(T^\infty) = p_T^s(T^\infty) = p_T^b(T^\infty) = l^{-1/h} \tag{3.15}$$

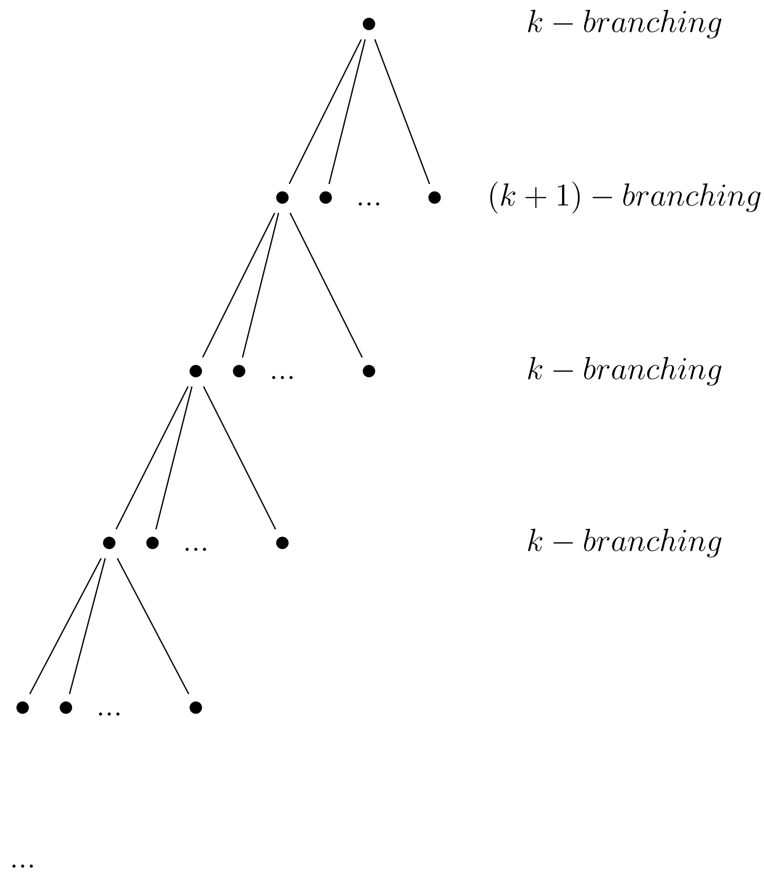
$\square$

There is also a fascinating graph which will leads us to a surprising theorem. Suppose that $k \geq 1$ and $1/(k+1) < \pi < 1/k$. We will define $0 < \alpha < 1$ by $(k+1)^\alpha k^{1-\alpha} = 1/\pi$. Let $\mathbf{a} = (a_i)_{i=1}^\infty$ be the 0-1 sequence with density α constructed in the following way: whenever 2^{j-1} divides i but 2^j does not, set $a_i = 1$ if and only if the jth bit in the binary expansion of α is 1. Let $T_\mathbf{a}$ be the rooted tree in which each site at distance i from the root has $k + a_{i+1}$ children. The following theorem asserts the critical probabilities:

Theorem 3.8. $p_H^s(T_{\boldsymbol{a}}) = p_H^b(T_{\boldsymbol{a}}) = p_T^s(T_{\boldsymbol{a}}) = p_T^b(T_{\boldsymbol{a}}) = \frac{1}{(k+1)^\alpha k^{1-\alpha}}$

Firstly, we will give some numerical examples in order to see this construction more clearly. For example, let $\alpha = 0,01$, we need to examine $\mathbf{a} = (a_i)_{i=1}^\infty$. By the construction above, in order to see a_1, we will look at 1st bit in the binary expansion of α. Since it is 0, we say $a_1 = 0$. Similarly, for a_2, we look at 2nd bit in the binary expansion of α. 2nd bit is 1, which gives $a_2 = 1$. And for a_3, we again consider 1st bit, then $a_3 = 0$. For a_4, we will look at 3rd bit and it gives $a_4 = 0$. By proceeding in this way, we get the general form

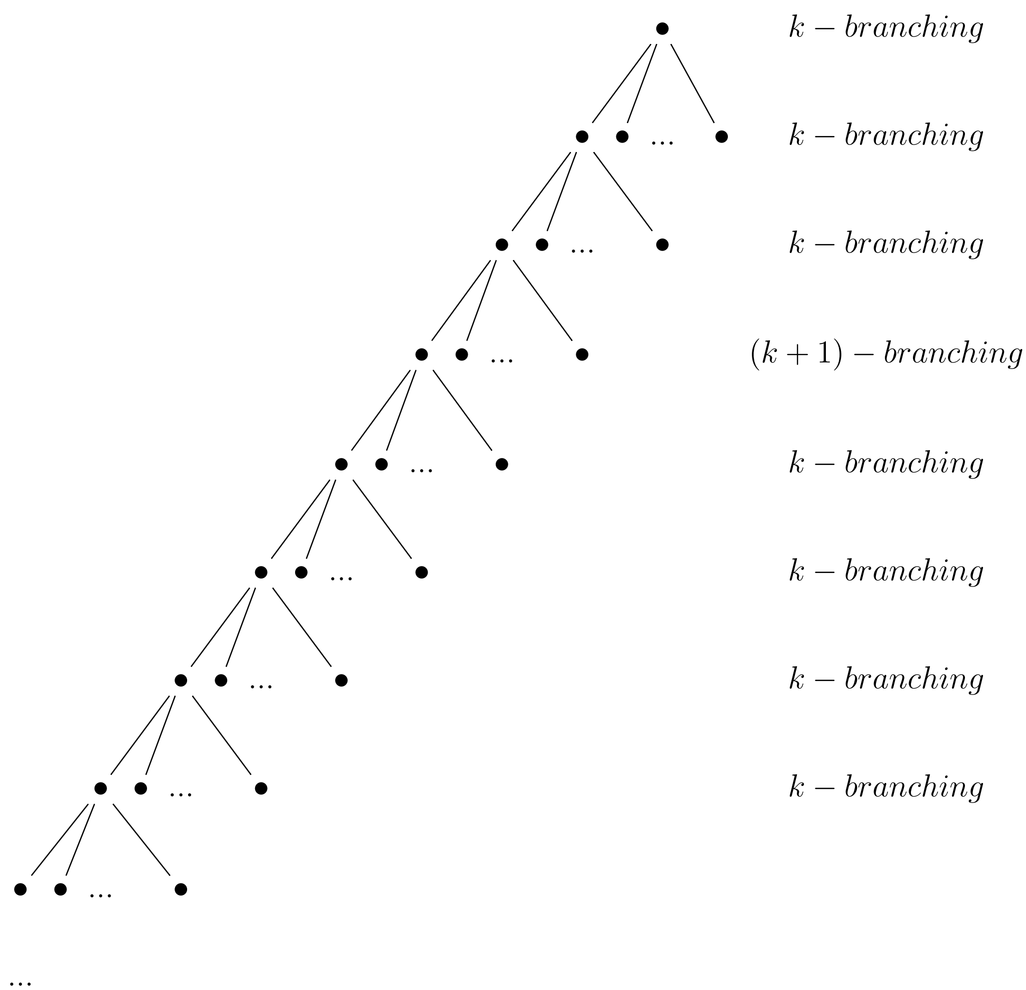
$$\begin{aligned}
&j = 1 \Rightarrow a_i = 0 \; for \; i \; such \; that \; i \; is \; odd \\
&j = 2 \Rightarrow a_i = 1 \; for \; i \; such \; that \; 2|i \; but \; 4 \nmid i \\
&j = 3 \Rightarrow a_i = 0 \; for \; i \; such \; that \; 4|i \; but \; 8 \nmid i \\
&\quad \ldots
\end{aligned} \tag{3.16}$$



Then we get the sequence $\mathbf{a} = (01000100....)$, which gives the tree $T_{\mathbf{a}}$ such that the root has k children and at distance 1 from the root each sites has $k+1$ children. So at distance 2 from the root, the tree has $k(k+1)$ children. And the construction continues like in this way. After the fourth stage, the process repeats itself. Then we come into the case in the previous one, where T becomes a finite rooted tree with height (depth) $h = 4$ and with $l = k^3(k+1)$. In the previous example, we find out that the critical probabilities $p_c(T) = l^{-1/h}$. So the critical probabilities for this tree $T_{\mathbf{a}}$, $p_c(T_{\mathbf{a}})$, is equal to $\frac{1}{(k+1)^{1/4}k^{3/4}}$, which is actually $\frac{1}{(k+1)^{\alpha}k^{1-\alpha}}$.

Alternatively, we can also consider another example by taking $\alpha = 0,001$. This time, we have $a_1 = 0$ since 1st bit is 0, $a_2 = 0$ since 2nd bit is 0, $a_3 = 0$ since again 1st bit 0, $a_4 = 1$ since 3rd bit is 1, ... By proceeding in this way, we get

$$
\begin{aligned}
&a_i = 0 \;\; for\ i\ such\ that\ i\ is\ odd\ since\ 1st\ bit\ is\ 0 \\
&a_i = 0 \;\; for\ i\ such\ that\ 2|i\ but\ 4 \nmid i \;\; since\ 2nd\ bit\ is\ 0 \\
&a_i = 1 \;\; for\ i\ such\ that\ 4|i\ but\ 8 \nmid i \;\; since\ 3rd\ bit\ is\ 1 \\
&a_i = 0 \;\; for\ i\ such\ that\ 8|i\ but\ 16 \nmid i \;\; since\ 4th\ bit\ is\ 0 \\
&\quad ...
\end{aligned}
\tag{3.17}
$$



Then we get the sequence $\mathbf{a} = (0001000000010000....)$, which gives the tree $T_{\mathbf{a}}$ such that the root has k children and at distance 1 from the root each sites has also k children. So at distance 2 from the root, the tree has k^2 children. And the process conclude in this way. After the eighth stage, the process repeats itself.

Then we come into the case in the previous one, where T becomes a finite rooted tree with height (depth) 4 and with $l = (k+1)k^7$. As we find out in the previous case, the critical probabilities $p_c(T_{\mathbf{a}})$ for this tree is equal to $\frac{1}{(k+1)^{1/8}k^{7/8}}$, which is equal to $\frac{1}{(k+1)^{\alpha}k^{1-\alpha}}$.

For the general case, as we see in the above examples, the number of $(k+1)$-branching depends on the 1's in α. Since the tree $T_{\mathbf{a}}$ is constructed with $\mathbf{a}$ that has density α, there are α times $(k+1)$-branching steps and $1-\alpha$ times k-branching steps. Then it follows, from the previous results, that the critical probabilities becomes

$$p_H^s(T_{\mathbf{a}}) = p_H^b(T_{\mathbf{a}}) = p_T^s(T_{\mathbf{a}}) = p_T^b(T_{\mathbf{a}}) = \frac{1}{(k+1)^{\alpha}k^{1-\alpha}} \tag{3.18}$$

Since k can be chosen any number, we reach the below remarkable theorem:

Theorem 3.9. *Any $0 < \pi < 1$ is the critical probability for some graph, indeed, for some tree.*

So, for any π, we may draw a branching tree such that percolation occurs on this tree; ie; it contains an open infinite cluster from the root to a leaf of the tree.

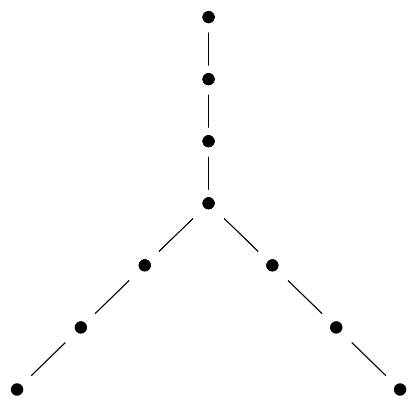
3.2 The Changing In Critical Probabilities By Modification Of Trees

We will make some changes to a graph and investigate the effects of these changes on the critical probabilities. First of all, let Λ be any graph and $\Lambda^{(l)}$ be obtained from Λ by subdividing each edge $l-1$ times, then the critical probability for $\Lambda^{(l)}$ becomes

$$p_c^b(\Lambda^{(l)}) = p_c^b(\Lambda)^{1/l} \tag{3.19}$$

where p_c^b is p_H^b or p_T^b.

Example 3.10. Let Λ is a 2-branching tree, which has the critical probability $1/2$ by the previous result. Assume $\Lambda^{(3)}$ is obtained from Λ by subdividing each edge 2 times. Then we calculate the critical probability for $\Lambda^{(3)}$.



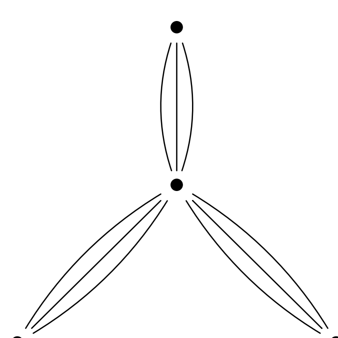
$$p_c^b(\Lambda^{(l)}) = (1/2)^{1/3} \tag{3.20}$$

Moreover, if $\Lambda^{[k]}$ is obtained from Λ by replacing each edge by k parallel edges, then we get

$$1 - p_c^b(\Lambda^{[k]}) = (1 - p_c^b(\Lambda))^{1/k} \tag{3.21}$$

where again p_c^b is p_H^b or p_T^b.

Example 3.11. Let Λ is again a 2-branching tree. Assume $\Lambda^{[3]}$ is obtained from Λ by replacing each edge by 3 parallel edges. Then the critical probability for $\Lambda^{(3)}$



$$
\begin{aligned}
1 - p_c^b(\Lambda^{[3]}) &= (1-1/2)^{1/3} = (1/2)^{1/3} && (3.22)\\
p_c^b(\Lambda^{[3]}) &= 1-(1/2)^{1/3} && (3.23)
\end{aligned}
$$

Clearly, there is no change in site percolation, so $p_c^s(\Lambda^{[k]}) = p_c^s(\Lambda)$.

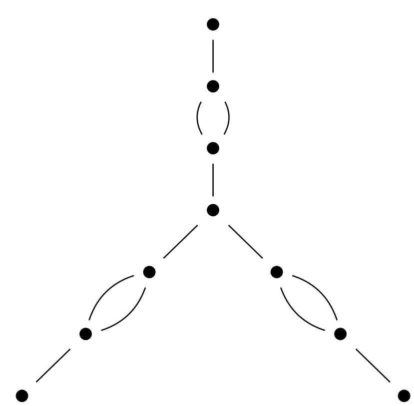
Combining these operations, we may get a new graph by replacing each bond of a graph by k independent paths of length l. For bond percolation, the critical probabilities p_{old} and p_{new} satisfy

$$
1-(1-p_{new}^l)^k = p_{old} \tag{3.24}
$$

In this wise, by a trivial operation on the graph, a critical probability in the interval $(0,1)$ can be moved very close to any point of $(0,1)$.

This technique guides us to calculate the critical probabilities for a family of graphs Λ' when we know the critical probability for a graph Λ.

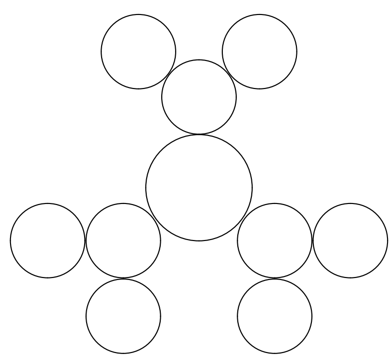
Example 3.12.



Let the first graph has critical probability p and the second one has critical probability r. Then the relation between these critical probabilities is that $r^3(2-r)=p$. It is simply deduce by considering the relation that $r^2(1-(1-r)^2)=p$.

3.3 Percolation On Tree-like Graphs

In general, applying the same procedure, we may calculate the critical probability for any 'tree-like' graph. For $l \geq k \geq 3$, let $C_{k,l}$ be the *cactus*, which is the graph formed by replacing each vertex of k-branching tree T_k by a complete graph on l vertices, and joining each pair of complete graphs corresponding to an edge of T_k by identifying a vertex of one with a vertex of the other, using no vertex in more than one identification. Resulting from these identifications, the vertices are called as *attachment vertices.* Although $C_{k,l}$ contains many circles, it still has the global structure of a tree, and percolation on $C_{k,l}$ may again be compared with a branching process.



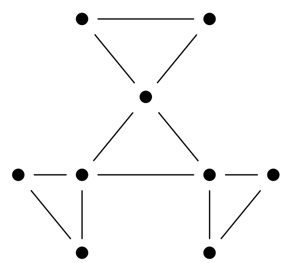
We will state the critical probabilities for this graph in the following theorem:

Theorem 3.13. *For the tree $C_{k,l}$, we have the relation*

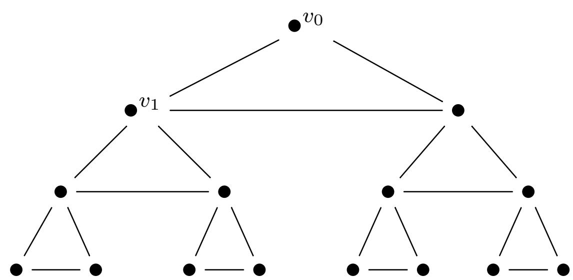
$$p_c^{l-l/k} + p_c^l/k - p_c^l = 1/k \tag{3.25}$$

where p_c is $p_H^b(C_{k,l})$ or $p_T^b(C_{k,l})$.

Proof. We begin with taking $l = 3$ and $k = 2$ in order to see the simplest situation. In this case, we have the following picture:



Moreover, it is similar to a 2-branching tree in a such way that



....

Notice that the probability that there is an open path from v_0 to v_1 is $1 - [(1 - p)(1 - p^2)]$, taking the bonds to be open independently with probability p. Let π_n be the probability that $C_{2,3}$ contains an open path of length n from the root to a leaf. This kind of a path exists if and only if there is an open path of length $(n-1)$ from v_0 to a leaf and also an open path from v_0 to v_1. So, we get

$$1 - \pi_n = \left\{1 - [1 - (1-p)(1-p^2)]\pi_{n-1}\right\}^2 \tag{3.26}$$

Then we have the function,

$$\pi_n = 1 - \left\{1 - [1 - (1-p)(1-p^2)]\pi_{n-1}\right\}^2 = f_p(\pi_{n-1}) \tag{3.27}$$

This function is again increasing and concave on the interval $[0, 1]$. Furthermore, $f_p(0) = 0$ and $f_p(1) < 1$, so $f_p(x_0) = x_0$ for some $0 < x_0 < 1$ if and only if $f_p'(0) = 2[1-(1-p)(1-p^2)] > 1$. Thus, by the similar arguments above, we have the relation $(p_H^b)^2 + (p_H^b) - (p_H^b)^3 = 1/2$, as we want.

Turning to p_T, note that the number of leaves of $C_{2,3}$ joined to the root by open paths has a certain distribution X with expectation $2[1-(1-p)(1-p^2)]$. Let X_n be the number of sites of T^∞ at distance n from the root by open paths. Then the sequence $(X_0, X_1, X_2, ...)$ is a branching process with $X_0 = 1$. As X is bounded, we have

$$\chi_{v_0}^b(T_2;p) = \mathbb{E}_p(|C_{v_0}|) = \sum_{i=1}^{\infty}[2(1-(1-p)(1-p^2))]^i \tag{3.28}$$

which gives $(p_T^b)^2 + (p_T^b) - (p_T^b)^3 = 1/2$.

As we state above, after conditioning on the root x being open, the open clusters containing x in site and bond percolation have exactly same distribution. So this relation is valid for all critical values, as we want.

We may generalize the result for any k and l, by considering the same calculations. Firstly, note that the probability that there is an open path from the root v_0 to a v_1 is $1-[(1-p^{l/k})(1-p^{l-l/k})]$ for this time, taking the bonds to be open independently with probability p. Let π_n be the probability that T_3 contains an open path of length n from the root to a leaf. Such a path exists if and only if there is an open path of length $(n-1)$ from v_0 to a leaf and also an open path from v_0 to v_1. So, we get

$$1-\pi_n = \left\{1-[1-(1-p^{l/k})(1-p^{l-l/k})]\pi_{n-1}\right\}^k \tag{3.29}$$

Then we have the function,

$$\pi_n = 1-\left\{1-[1-(1-p^{l/k})(1-p^{l-l/k})]\pi_{n-1}\right\}^k = f_{k,l,p}(\pi_{n-1}) \tag{3.30}$$

This function is again increasing and concave on the interval $[0,1]$. By the similar calculations above, we have the relation

$$[1 - (1 - p_H^{l/k})(1 - p_H^{l-l/k})] = 1/k \quad \Rightarrow \quad p_H^{l-l/k} + p_H^{l/k} + p_H^{l} = 1/k$$

For p_T, notice that the number of leaves of $C_{k,l}$ joined to the root by open paths has a certain distribution X with expectation $k[1 - (1 - p_H^{l/k})(1 - p_H^{l-l/k})]$. Let X_n be the number of sites of $C_{k,l}$ at distance n from the root by open paths. Then the sequence $(X_0, X_1, X_2, ...)$ is a branching process with $X_0 = 1$. As X is bounded, we have

$$\chi_{v_0}^{b}(T_2; p) = \mathbb{E}_p(|C_{v_0}|) = \sum_{i=1}^{\infty}(k[1 - (1 - p_H^{l/k})(1 - p_H^{l-l/k})])^i \tag{3.31}$$

So we get $p_T^{l-l/k} + p_T^{l/k} + p_T^{l} = 1/k$.

As we state above, after conditioning on the root x being open, the open clusters containing x in site and bond percolation have exactly same distribution. Then

$$p_c^{l-l/k} + p_c^{l}/k - p_c^{l} = 1/k \tag{3.32}$$

where p_c is $p_H^b(C_{k,l})$ or $p_T^b(C_{k,l})$.

3.4 Percolation on Random Rooted Trees

We will illustrate another type of trees in which we need to consider additional randomness. Take T be the random rooted tree in which each site has $k+1$ children with probability r and k children with probability $1-r$, with the choices made independently for each site and with probability p.

Theorem 3.14. $p_H^s(T) = p_H^b(T) = p_T^s(T) = p_T^b(T) = \frac{1}{(k+r)}$

Proof. In order to calculate the critical probabilities, we will follow the same procedures in the previous examples in a slightly different way. Firstly, let v_0 is the root of this tree and v_1 is a child of v_0. Also, let π_n be the probability that T contains an open path of length n from the root to a leaf. This kind of path exists if and only if the bond v_0v_1 is open and there is an open path from v_1 to a leaf of length $n-1$. By considering that v_0 has $k+1$ children with probability r and k children with probability $1-r$, we get the following equation

$$1-\pi_n = r(1-p\pi_{n-1})^{k+1} + (1-r)(1-p\pi_{n-1})^k \tag{3.33}$$

so we will have the function

$$\pi_n = 1-(r(1-p\pi_{n-1})^{k+1} + (1-r)(1-p\pi_{n-1})^k) = f_{k,r,p}(\pi_{n-1}) \tag{3.34}$$

Note that this function is again increasing and concave on the interval $[0,1]$. Moreover, $f_{k,r,p}(0) = 0$ and $f_{k,r,p}(1) < 1$, so $f_{k,r,p}(x_0) = x_0$ for some $0 < x_0 < 1$ if and only if $f'_{k,r,p}(0) = p(r(k+1) + (1-r)k) = p(k+r) > 1$. Thus, by the same arguments in previous cases, when $p > \frac{1}{(k+r)}$, we get that $\theta^b_{v_0}(T) > 0$; implying that $p^b_H(T) \leq \frac{1}{(k+r)}$. Additionally, when $p < \frac{1}{(k+r)}$, π_n converges to 0; so $\theta^b_{v_0}(T) = 0$. Therefore, the critical probability $p^b_H(T)$ is equal to $\frac{1}{(k+r)}$.

Next, in order to illustrate the other critical probability p_T, we crucially need to consider conditional expectation. There is a very practical property of conditional expectation which is given in the following theorem.

Theorem 3.15. *([12])* $E(E(X|Y)) = E(X)$

According to this theorem, the expectation (averaged over all Y's) of the conditional expectation of X given Y is the plain old expectation of X.

Since the branching of each steps in this tree depends on the branching of the previous steps and each time we have extra randomness on the branching, we need to use the conditional expectation.

Let X_n be the number of sites of T at distance n from the root joined to the root by open paths. So the sequence $(X_0, X_1, X_2, ...)$ is branching process with $X_0 = 1$. Firstly, we calculate the expectation of X_1;

$$E(X_1) = p[r(k+1) + (1-r)k] = p(k+r) \tag{3.35}$$

For the other expectations, we need to consider conditional expectations. By the theorem above, we get

$$
\begin{aligned}
E(E(X_2|X_1)) &= E(X_2) = p^2[r(k+1)(k+r) + (1-r)k(k+r)] = p^2(k+r)^2 \\
E(E(X_3|X_2)) &= E(X_3) = p^3[r(k+1)(k+r)^2 + (1-r)k(k+r)^2] = p^3(k+r)^3 \\
&\dots \\
E(E(X_n|X_{n-1})) &= E(X_n) = p^n[r(k+1)(k+r)^{n-1} + (1-r)k(k+r)^{n-1}] \\
&= p^n(k+r)^n
\end{aligned}
$$

Then we get

$$\chi^b_{v_0}(T;p) = \mathbb{E}_p(|C_{v_0}|) = \mathbb{E}[\sum_{i=1}^{\infty} X_i] = \sum_{i=1}^{\infty} \mathbb{E}[X_i] = \sum_{i=1}^{\infty} p^i(k+r)^i \tag{3.36}$$

This equation is finite for $p < 1/(k+r)$ and infinite for $p \geq 1/(k+r)$. Hence, the critical probability $p^b_T(T)$ is equal to $1/(k+r)$.

As we state above, after conditioning on the root x being open, the open clusters containing x in site and bond percolation have exactly same distribution. Thus, we get

$$p^s_H(T) = p^b_H(T) = p^s_T(T) = p^b_T(T) = \frac{1}{(k+r)} \tag{3.37}$$

□

We may also generalize this tree into n-branching tree. Let T be a random rooted tree in which each site has k_1 children with probability p_1, k_2 children with probability p_2, ..., and k_n children with probability p_n with the choices made independently for each site.

Theorem 3.16. $p^s_H(T) = p^b_H(T) = p^s_T(T) = p^b_T(T) = \frac{1}{p_1k_1+p_2k_2+...+p_nk_n}$

Proof. By following the same method, we will calculate the critical probabilities. Again we consider the probability that T contains an open path of length n from the root to a leaf as π_n and this type of path exists if and only if there is an open path from a child v_1 of the root to a leaf of the tree and the bond from the root to v_1 is open. That gives the equation

$$1-\pi_n = p_1(1-p\pi_{n-1})^{k_1} + p_2(1-p\pi_{n-1})^{k_2} + ... + p_n(1-p\pi_{n-1})^{k_n} \qquad (3.38)$$

So we get the function

$$\begin{aligned}\pi_n &= 1-\left\{p_1(1-p\pi_{n-1})^{k_1} + p_2(1-p\pi_{n-1})^{k_2} + ... + p_n(1-p\pi_{n-1})^{k_n}\right\} \\ &= f_{\{p,p_1,..,p_n,k_1,..,k_n\}}(\pi_{n-1})\end{aligned}$$

This function is increasing and concave on the interval $[0,1]$. Besides, it takes the value 0 at the point 0 and less than 1 at the point 1, so it fixes the point x_0 for some $0 < x_0 < 1$ if and only if the derivative of this function is greater that 1; ie;

$$f'_{\{p,p_1,..,p_n,k_1,..,k_n\}}(\pi_{n-1}) = p(p_1k_1 + p_2k_2 + ... + p_nk_n) > 1 \qquad (3.39)$$

Thus, by the same arguments in previous cases, when $p > \frac{1}{(p_1k_1+p_2k_2+...+p_nk_n)}$, we get that $\theta^b_{v_0}(T) > 0$; implying that $p^b_H(T) \leq \frac{1}{(p_1k_1+p_2k_2+...+p_nk_n)}$.

Additionally, when $p < \frac{1}{(p_1k_1+p_2k_2+...+p_nk_n)}$, π_n converges to 0; so $\theta^b_{v_0}(T) = 0$. Therefore, the critical probability $p^b_H(T)$ is equal to $\frac{1}{(p_1k_1+p_2k_2+...+p_nk_n)}$.

For calculating p_T, the conditional expectation is required. Repeatedly, let X_n be the number of sites of T from the root joined to the root by open paths at distance n. Then the sequence $(X_0, X_1, X_2, ...)$ is branching process with $X_0 = 1$. We may simply write the expectation of X_1 as:

$$E(X_1) = p[p_1k_1 + ... + p_nk_n] \qquad (3.40)$$

The other expectations can be found by using the theorem

$$
\begin{aligned}
E(E(X_2|X_1)) = E(X_2) &= p^2[p_1k_1(p_1k_1 + ... + p_nk_n) + ... + p_nk_n(p_1k_1 + ... + p_nk_n)] \\
&= p^2(p_1k_1 + ... + p_nk_n)^2 \\
E(E(X_3|X_2)) = E(X_3) &= p^3[p_1k_1(p_1k_1 + ... + p_nk_n)^2 + ... + p_nk_n(p_1k_1 + ... + p_nk_n)^2] \\
&= p^3(p_1k_1 + ... + p_nk_n)^3 \\
&... \\
E(E(X_n|X_{n-1})) = E(X_n) &= p^n[p_1k_1(p_1k_1 + ... + p_nk_n)^{n-1} + ... \\
&+ p_nk_n(p_1k_1 + ... + p_nk_n)^{n-1}] \\
&= p^n(p_1k_1 + ... + p_nk_n)^n
\end{aligned}
$$

Hence we have

$$
\chi^b_{v_0}(T;p) = \mathbb{E}_p(|C_{v_0}|) = \mathbb{E}[\sum_{i=1}^{\infty} X_i] = \sum_{i=1}^{\infty} \mathbb{E}[X_i] = \sum_{i=1}^{\infty} p^i(p_1k_1 + ... + p_nk_n)^i \quad (3.41)
$$

This equation is finite for $p < 1/(p_1k_1+...+p_nk_n)$ and infinite for $p \geq 1/(p_1k_1+ ... + p_nk_n)$. Hence, the critical probability $p^b_T(T)$ is equal to $1/(p_1k_1 + ... + p_nk_n)$.

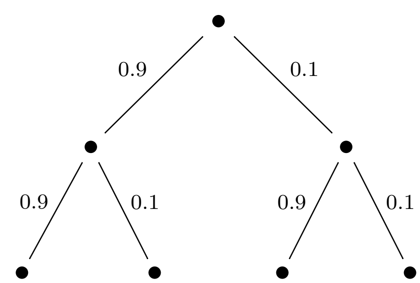
Then, after conditioning on the root x being open, the open clusters containing x in site and bond percolation have exactly same distribution. Thus, we get

$$
p^s_H(T) = p^b_H(T) = p^s_T(T) = p^b_T(T) = \frac{1}{p_1k_1 + p_2k_2 + ... + p_nk_n} \quad (3.42)
$$

□

3.5 The Relation Between The Critical Probabilities p_T and p_H

In all of the examples below, we find out that the events percolation occurs and the expected size of open paths is infinite take place simultaneously. However, by definition, we may have the case in which expected size of open paths is infinite but percolation does not occur. Now we want to show how we can reach this situation. For example, let T be the rooted tree in which each site has two children in such a way that one of them is open with probability $0,9$ and the other one is open with probability $0,1$.



...

By letting π_n be the probability that there is an open path from the root to a leaf of length n, we get the equation

$$1-\pi_n=(1-0,1\pi_{n-1})(1-0,9\pi_{n-1}) \tag{3.43}$$

Then we consider the function

$$\pi_n=1-(1-0,1\pi_{n-1})(1-0,9\pi_{n-1})=f(\pi_{n-1}) \tag{3.44}$$

On the interval $[0,1]$, f is an increasing, concave function. Note that, at the point 0, f takes the value 0 and, at the point 1, it becomes less than 1. So f fixes a point x_0 for some $0<x_0<1$ if and only if its derivative is greater than 1 at the point 0. However $f'(0)=1$. Thus, π_n converges to zero, the unique fixed point of f. This means that percolation does not occur.

On the other hand, we will show that expected size of open paths is infinite. Let X_n be the number of sites of T from the root joined to the root by open paths at distance n. Then the sequence $(X_0, X_1, X_2, ...)$ is branching process with $X_0 = 1$. Then expectations are

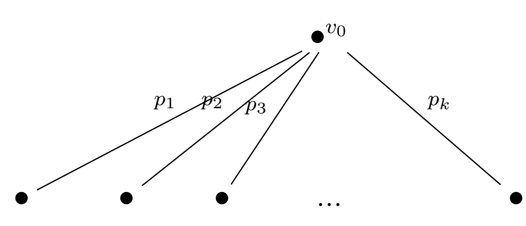
$$E(X_1) = 0,1 + 0,9 = 1$$
$$E(X_2) = 0,1.0,1 + 0,1.0,9 + 0,9.0,1 + 0,9.0,9 = 1$$
$$E(X_3) = 0,1(0,1 + 0,9)^2 + 0,9(0,1 + 0,9)^2 = 1$$
...
$$E(X_n) = 0,1(0,1 + 0,9)^{n-1} + 0,9(0,1 + 0,9)^{n-1} = 1$$

Hence we have

$$\chi^b_{v_0}(T; p) = \mathbb{E}_p(|C_{v_0}|) = \sum_{i=1}^{\infty} \mathbb{E}[X_i] = \sum_{i=1}^{\infty} 1 = \infty \tag{3.45}$$

Thus we get an example in which percolation does not occur but the expected size of open paths is infinite.

This situation can also be generalized for k-branching tree in the following way: Let T be a rooted tree in which each site has k children and each child is open with probabilities in the following order $p_1, p_2, ..., p_k$ such that $p_1+p_2+...+p_k = 1$.



....

Now letting π_n be the probability that there is an open path from the root to a leaf of length n, we attain the below equation

$$1 - \pi_n = (1 - p_1\pi_{n-1})(1 - p_2\pi_{n-1})....(1 - p_k\pi_{n-1}) \tag{3.46}$$

Then we consider the function

$$\pi_n = 1 - \{(1 - p_1\pi_{n-1})(1 - p_2\pi_{n-1})....(1 - p_k\pi_{n-1})\} = f_{p_1,p_2,...,p_k}(\pi_{n-1}) \tag{3.47}$$

On the interval $[0, 1]$, f is an increasing, concave function. Note that, at the point 0, f takes the value 0 and, at the point 1, it becomes less than 1. So f fixes a point x_0 for some $0 < x_0 < 1$ if and only if its derivative is greater than 1 at the point 0. When we take the derivative of f, we get

$$f'_{p_1,p_2,...,p_k}(0) = p_1 + p_2 + ... + p_k \tag{3.48}$$

For the case $p_1 + p_2 + ... + p_k = 1$, $f'(0) = 1$. Thus, π_n converges to zero, the unique fixed point of f. This means that percolation does not occur.

On the other hand, the expected size of open paths is infinite. Let X_n be the number of sites of T from the root joined to the root by open paths at distance n. Then the sequence $(X_0, X_1, X_2, ...)$ is branching process with $X_0 = 1$. Then expectations are

$$\begin{aligned}
E(X_1) &= p_1 + p_2 + ... + p_k = 1 \\
E(X_2) &= p_1(p_1 + p_2 + ... + p_k) + p_2(p_1 + p_2 + ... + p_k) + ... + p_k(p_1 + p_2 + ... + p_k) \\
&= p_1 + p_2 + ... + p_k = 1 \\
E(X_3) &= p_1(p_1 + p_2 + ... + p_k)^2 + p_2(p_1 + p_2 + ... + p_k)^2 + ... + p_k(p_1 + p_2 + ... + p_k)^2 \\
&= p_1 + p_2 + ... + p_k = 1 \\
&... \\
E(X_n) &= p_1(p_1 + p_2 + ... + p_k)^{n-1} + p_2p_k(p_1 + p_2 + ... + p_k)^{n-1} + ... \\
&+ p_k(p_1 + p_2 + ... + p_k)^{n-1} \\
&= p_1 + p_2 + ... + p_k = 1
\end{aligned}$$

Hence we have

$$\chi^b_{v_0}(T;p) = \mathbb{E}_p(|C_{v_0}|) = \sum_{i=1}^{\infty} \mathbb{E}[X_i] = \sum_{i=1}^{\infty} 1 = \infty \tag{3.49}$$

Hence we get a general case in which the expected size of open paths is infinite but percolation does not occur.

As a result of these investigations, by letting T be a rooted tree in which each site has k children and each child is open with probabilities in the following order $p_1, p_2, ..., p_k$, we conclude that when $p_1+p_2+...+p_k < 1$, the expected size of open paths is finite and percolation does not occur. Moreover, if $p_1 + p_2 + ... + p_k = 1$, then the expected size of open paths is infinite, in spite of the fact that percolation does not occur. And if $p_1 + p_2 + ... + p_k > 1$, then both happens, ie, the expected size of open paths is infinite and percolation also occurs.

3.6 Conclusion

Percolation is a model which is easy to define and yet it exhibits a variety of fascinating phenomena. It is therefore a source of many deep and beautiful mathematical problems. Indeed, percolation theory has many profound and rather difficult theorems, with a host of open problems remaining. We start the thesis with classical approaches in percolation theory. We calculate the critical probabilities for some lattice-like graphs in 2-dimension. The key challenge in percolation is to uncover the relation between the percolation critical behavior and the properties of the underlying graph from which we obtain percolation by removing edges. Especially with the advent of new tools in probabilistic combinatorics many simple proofs have been founded.

Throughout the thesis, we discuss the basics of percolation and work on the critical probabilities for some graphs. In particular, we are interested in the percolation on rooted trees and calculate the critical probabilities at which the percolation occurs. Moreover, we show the relation between site percolation and bond percolation in the sense that site percolation is more general. We importantly reach the fact that the probability that there is an infinite open cluster is either 0 or 1. We also make some changes to a graph and investigate the effects of these changes on the critical probabilities and, by calculating this effect, we explore the critical probabilities for a family of graphs. Furthermore, we illustrate the critical probabilities for trees in which we have additional randomness. Lastly, we get an example in which percolation does not occur but the expected size of open paths is infinite.

The fields that this contribution covers, percolation and random graph theory, have attracted tremendous attention in the past decades, and enormous progress has been made. Over the last several decades, a tremendous amount of work has gone into finding exact and approximate values of the critical probabilities for a variety of systems. Exact critical probabilities are only known for certain two-dimensional lattices that can be broken up into a self-dual array, such that under a triangle-triangle transformation, the system remains the same. Studies using numerical methods have led to numerous improvements in algorithms and several theoretical discoveries.

Bibliography

[1] S. R. Broadbent and J. Hammersley, "Percolation processes i. crystals and mazes," *Math. Proc. Cambridge Philos. Soc.*, vol. 53, pp. 629–641, 1957.

[2] B. Bollobás, "Modern graph theory," *Springer Verlag*, 1998.

[3] H. Kesten, "The critical probability of bond percolation on the square lattice equals 1/2," *Commun. Math. Phys.*, vol. 74, pp. 41–59, 1980.

[4] J. M. Hammersley, "Percolation process. lower bounds for the critical probability," *Ann. Math. Stat.*, vol. 28, pp. 790–795, 1957.

[5] T. E. Harris, "A lower bound for the critical probability in a certain percolation process," *Proc. Camb. Phil. Soc.*, vol. 56, pp. 13–20, 1960.

[6] J. M. Hammersley, "Comparison of atom and bond percolation," *J. Math. Phys.*, vol. 2, pp. 728–733, 1961.

[7] Y. Higuchi, "Coexistence of the infinite (*) cluster: a remark on the square lattice site percolation," *Z. W.*, vol. 61, pp. 75–81, 1982.

[8] B. Tóth, "A lower bond for the critical probability of the square lattice site percolation," *Z. W.*, vol. 69, pp. 19–22, 1985.

[9] S. A. Zuev, "A lower bond for percolation threshold for the square lattice," *Math. Bull.*, vol. 43(5), pp. 59–61(66–69 in translation), 1985.

[10] M. V. Menshikov and K. D. Pelikh, "[engl. transl.: Percolation with several defect types. an estimate of critical probability for a square lattice math.

notes acad. sci. ussr, 46(4), 778-785 (1990)]," *Mat. Zametki,*, vol. 46, pp. 38–47, 1985.

[11] B. Bollobás and O. Riordan, "Percolation," *Cambridge Univ. Pres.*, vol. 36, 2006.

[12] A. N. Shiryaev, "Probability," *Springer*, 1984.