JUNE 2017 M. Sc. in Mathematics RODGERS CHISHA

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GREEN'S FUNCTION FOR FINITELY MANY INTERVALS STURM-LIOUVILLE PROBLEMS

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Green’s Function for Finitely Many Intervals Sturm-Liouville Problems

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Assoc. Prof. Dr. Abdullah KABLAN

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Rodgers CHISHA

ABSTRACT

GREEN'S FUNCTION FOR FINITELY MANY INTERVALS STURM-LIOUVILLE PROBLEMS

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The Green's function has been constructed for one-interval and two-interval Sturm-Liouville problems in which the discontinuity of its derivative is not determined beforehand but occurs on its own. This thesis seeks to extend that idea and construct the Green's function for finitely many intervals case. We consider the second order scalar differential equation with its boundary conditions and convert it to its equivalent first order linear system. From this conversion, we formulate the characteristic function whose zeroes are the eigenvalues of the homogeneous system. In addition, we construct the generalized matrix Green's function from which we get the top right component as the Green's function for finitely many intervals Sturm-Liouville problems.

Key Words: Sturm-Liouville problem, Characteristic function, Green's function.

ÖZET

SONLU SAYIDA ARALIĞA SAHİP STURM-LİOUVİLLE PROBLEMLERİ İÇİN GREEN FONKSİYONU

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Türevindeki süreksizliği önceden belirlenmemiş fakat kendiliğinden ortaya çıkan, tek ve iki aralıklı Sturm-Liouville problemleri için Green fonksiyonu inşaa edilmiştir. Bu tezin amacı bu fikri genişletmek ve sonlu sayıdaki aralık durumu için Green fonksiyonunu inşaa etmektir. Bunun için, ikinci mertebeden skaler diferansiyel denklemi sınır şartları ile birlikte ele alacağız ve bu problemi, ona denk olan birinci mertebeden lineer sisteme dönüştüreceğiz. Bu dönüşümden, sıfırları homojen sistemin özdeğerleri olan karekteristik fonksiyonu elde edeceğiz. Daha sonra da genelleştirilmiş matris Green fonksiyonunu inşaa edeceğiz ki bunun sağ üst bileşeni, sonlu sayıda aralığa sahip Sturm-Liouville problemi için Green fonksiyonu olacaktır.

Anahtar Kelimeler: Sturm-Liouville problemi, Karekteristik fonksiyon, Green fonksiyonu

To my parents…..

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LIST OF SYMBOLS/ABBREVIATIONS

I	Interval on real line
$\mathbb{C}$	Set of complex numbers
$\mathbb{Z}^+$	Set of positive integers
$L(I,\mathbb{C})$	Complex-valued Lebesgue integrable functions on I
$AC(I)$	Absolutely continuous functions on I
λ	Eigenvalue
Φ	Primary fundamental matrix
$\prod$	Characteristic function for one interval case
Δ	Characteristic function for finitely many intervals case
IVP	Initial value problem
$M_{n,m}(\mathbb{C})$	$(n \times m)$ matrix with complex entries
ODE	Ordinary differential equation
PDE	Partial differential equation
SLP	Sturm-Liouville problem

CHAPTER 1

INTRODUCTION

The Sturm-Liouville equation

$$-(p(x)y')' + g(x)y = \lambda r(x)y, \qquad x \in I = [a,b],$$

where I is an interval on the real line, has received considerable attention in recent past in a variety of contexts and subject to different conditions such as transmission conditions [1, 2, 3, 4, 5, 6, 7], interface conditions [8, 9, 10], multi-point conditions [9, 11, 12, 13], point interactions [14, 15] and discontinuous conditions [16, 17]. Pokornyi and Borovskikh [14] as well as Pokornyi and Pryadiev [15] give more details on such problems and their applications. To find the solution that must satisfy given conditions, the Green's functions has proved to be a powerful method. As a result, the Green's function has been used very often in physics, engineering and mathematics to solve partial differential equations (PDE) as well as ordinary differential equations (ODE), for example, deflection of a beam with clamped ends [18] and string with fixed ends [19]. In a similar way, we will consider Sturm-Liouville problems (SLP) with some given boundary conditions.

Given the inhomogeneous ODE, one important condition for the Green's function to exist is that the homogeneous equation should have only the trivial solution [19, 20, 21]. If the homogeneous equation has a non-trivial solution, then the inhomogeneous ODE has either no solution or many solutions. It then follows that the Green's function indeed exists and is unique [19, 20, 21]. In most textbooks, the Green's function has been constructed by assigning the discontinuity of its derivative a priori. However, Neuberger's construction [22] is special in that the discontinuity occurs 'naturally'. Such a method has been utilised by Zettl [23] to construct the Green's function for a one-interval SLP and its adjoint form. Wang and Zettl [24], extended this idea and constructed the Green's function for two interval case. We shall build on this work and construct the Green's function for finitely many intervals case.

In chapter Two, we will give a brief fundamental theory on ODE. We will outline important definitions and theorems which are significant in the theory of ODE.

The characteristic function and the Green's function for one-interval SLP with some given boundary conditions will be constructed in Chapter Three. The second order ODE

$$-(p(x)y')' + g(x)y = \lambda r(x)y, \quad x \in I = (a,b), -\infty \leq a < b \leq \infty,$$

is converted to its equivalent first order linear system by writing it as

$$y' = \frac{1}{p(x)}z, \quad z' = (g(x) - \lambda r(x))y,$$

where, as observed, $z = p(x)y'$. Included in this chapter will be the adjoint equivalence of the Green's function and its associated conditions.

We will construct the generalized matrix Green's function from which we get the Green's function for finitely many intervals SLPs in chapter Four. Furthermore, we will make some necessary observation regarding the size of the matrices which form the boundary conditions and the size of the matrix whose determinant gives the characteristic function. Some examples will be given to illustrate the results obtained. Of particular interest are the examples involving disjoint intervals but with common end points and having singularities in the interior which generate self-adjoint boundary conditions in the direct sum of the Hilbert space [25, 26].

In chapter Five, we will give the conclusion based on the results gotten in our research.

CHAPTER 2

PRELIMINARIES

2.1 Notations

We would like to begin by stating the notation to be used in this thesis. Let I be any interval on the real line which can be open, closed, half open, bounded or unbounded. Let $\mathbb{Z}^+$ denote the set of positive integers and $\mathbb{C}$ denote the set of complex numbers. We denote by $M_{n,m}(\mathbb{C})$ the set of $n \times m$ matrices with complex entries, where $n, m \in \mathbb{Z}^+$. The notation $M_n(\mathbb{C})$ indicates $n \times n$ square matrix with complex entries. For any matrix $A \in M_n(\mathbb{C}),$ we denote the transpose of a conjugated complex matrix A by A^*. The determinant of $A \in M_n(\mathbb{C})$ will be designated by $\det A$ while its trace will be symbolised by $Tr(A).$ Let $L(I,\mathbb{C}) = L^1(I,\mathbb{C})$ denote the set of complex-valued Lebesgue integrable functions on a compact interval I for which

$$\int_a^b |\, y(x) \,|\, dx \equiv \int_I |\, y(x) \,|\, dx < \infty$$

and $L_{loc}(I,\mathbb{C})$ signifies the set of complex-valued Lebesgue integrable functions on every compact sub-interval of I. The notation $AC(I)$ is used to denote the collection of absolutely continuous complex-valued functions on I.

2.2 Basic Theory

We now present the basic theory of ODE.

Definition 2.2.1. *A matrix function is absolutely continuous if each of its components is absolutely continuous.*

Definition 2.2.2 (Solution). *Let* I *be any interval and let* $n, m \in \mathbb{Z}^+$. *Let* $P : \mathrm{I} \to M_n(\mathbb{C})$, $F : \mathrm{I} \to M_{n,m}(\mathbb{C})$. *By a solution of the equation*

$$Y' = PY + F \quad \text{on } I \tag{2.1}$$

we mean a function Y *from I into* $M_{n,m}(\mathbb{C})$ *which is absolutely continuous on all compact sub-intervals of I and satisfies the equation a.e on I.*

We now give an important theorem regarding the existence and uniqueness of the solution.

Theorem 2.2.1 (Existence and uniqueness). *Let* I *be any interval and* $n, m \in \mathbb{Z}^+$. *If* $P \in M_n(L(I,\mathbb{C}))$ *and* $F \in M_{n,m}(L(I,\mathbb{C}))$, *then every initial value problem (IVP)*

$$Y' = PY + F, \;\; on\, I \tag{2.2}$$

$$Y(u) = C,\; u \in I,\; C \in M_{n,m}(\mathbb{C}) \tag{2.3}$$

has a unique solution defined on all of I. *Furthermore, if* C, P *and* F *are all real-valued, then there is a unique real-valued solution.*

***Proof*:** The proof is given by using either the fixed point method or the Picard's successive approximation method, (See [20] and [23]). □

Definition 2.2.3 (Fundamental set of solutions). *Consider the homogeneous part of equation* (2.2)

$$Y' = PY \;\; on \;\; I. \tag{2.4}$$

A set of n *solutions of this equation, all defined on the same interval* I, *is called a fundamental set of solutions if the solutions are linearly independent functions on* I.

Definition 2.2.4 (Fundamental matrix solution). A matrix $\mathbb{Y} \in M_n(\mathbb{C})$ *is called a fundamental matrix solution if its columns form a fundamental set of solutions. Note that fundamental matrix solution satisfies*

$$\mathbb{Y}' = P(x)\mathbb{Y}.$$

From Theorem 2.2.1, it follows that for each point $u \in I$, there is exactly one fundamental matrix solution $\mathbb{X}$ of (2.4) satisfying $\mathbb{X}(u) = I_n$, where I_n denotes the $n \times n$ identity matrix.

Definition 2.2.5 (Primary fundamental matrix). *For each fixed* $u \in I$, $\Phi(.,u)$ *is the primary fundamental matrix for the homogeneous linear system (2.4) satisfying*

$$\Phi(u,u) = I_n,$$

and also satisfies

$$\Phi(x,u) = \mathbb{Y}(x)\mathbb{Y}^{-1}(u),$$

for any fundamental matrix solution $\mathbb{Y}$ *of (2.4). Note that* $\Phi(x,u)$ *is invertible and belongs to* $M_n(AC_{loc}(I))$. *Furthermore, if* I *is compact and* $P \in M_n(L(I,\mathbb{C}))$, *then u can be an endpoint of* I.

Proposition 2.2.1. *Primary fundamental matrix has the following properties:*

1. $\Phi(x,x) = I_n$,
2. $\Phi(x,s)\Phi(s,u) = \Phi(x,u)$,
3. $(\Phi(x,u))^{-1} = \Phi(u,x)$, and
4. $\dfrac{\partial}{\partial x}\Phi(x,u) = P(x)\Phi(x,u)$.

***Proof*:**

1. $\Phi(x,x) = \mathbb{Y}(x)\mathbb{Y}^{-1}(x) = I_n$.
2. LHS: $\Phi(x,s)\Phi(s,u) = \mathbb{Y}(x)\mathbb{Y}^{-1}(s)\mathbb{Y}(s)\mathbb{Y}^{-1}(u)$

$$= \mathbb{Y}(x)\mathbb{Y}^{-1}(u)$$

$$= \Phi(x,u) \text{ : RHS.}$$

3. $(\Phi(x,u))^{-1} = \left(\mathbb{Y}(x)\mathbb{Y}^{-1}(u)\right)^{-1}$

$$= \left(\mathbb{Y}^{-1}(u)\right)^{-1}\left(\mathbb{Y}(x)\right)^{-1}$$

$$= \mathbb{Y}(u)\mathbb{Y}^{-1}(x) = \Phi(u,x).$$

And finally,

4. $\dfrac{\partial}{\partial x}\Phi(x,u) = \dfrac{\partial}{\partial x}\left[\mathbb{Y}(x)\mathbb{Y}^{-1}(u)\right] = \mathbb{Y}'(x)\mathbb{Y}^{-1}(u)$

$= P(x)\Phi(x,u)$, since $\mathbb{Y}$ is a fundamental matrix solution. □

The next theorem provides the relationship between the primary fundamental matrix and a solution of (2.4). Its proof can be found in [20].

Theorem 2.2.2. *Let Φ be a primary fundamental matrix. Then for any constant matrix $C \in M_{n,m}(\mathbb{C})$, ΦC is a solution of* (2.4). *If $C \in M_n(\mathbb{C})$ is invertible, then ΦC is a fundamental matrix solution and every fundamental matrix solution has this form.*

One important property of fundamental matrix solutions is known as Abel's formula. It is presented below as a theorem and its proof is given in [27].

Theorem 2.2.3 (Abel's formula). *Let P(x) be continuous on I and $\mathbb{Y}$ be a fundamental matrix solution. Then $\det \mathbb{Y}$ satisfies the differential equation*

$$\left[\det \mathbb{Y}\right]' = Tr\left[P(x)\right] \det \mathbb{Y} \text{ , on } I. \tag{2.5}$$

In integral form, for $x,t \in I$,

$$\det \mathbb{Y}(x) = \det \mathbb{Y}(t) \left(\exp \int_t^x Tr \ P(s) \ ds \right).$$

We are now ready to unveil the theorem which is fundamental in the theory of linear differential equations as presented in [23]. It is called variation of parameters and its relevance will be realised in the construction of the Green's function.

Theorem 2.2.4 (Variation of parameters formula). *Let I be any interval, let $P \in M_n(L_{loc}(I, \mathbb{C}))$ and let $\Phi(.,.,P)$ be the primary fundamental matrix of (2.4). Let $F \in M_{n,m}(L_{loc}(I, \mathbb{C}))$, $u \in I$ and $C \in M_{n,m}(\mathbb{C})$. Then*

$$Y(x) = \Phi(x,u,P)C + \int_u^x \Phi(x,s,P)F(s)ds, \ on\ I \tag{2.6}$$

is the solution of (2.2), (2.3). Note that if I is compact and $P \in M_n(L(I))$, $F \in M_{n,m}(L(I))$, then $Y \in M_{n,m}(AC(I))$ and u can be an endpoint or an interior point of I.

***Proof*:** Clearly, $Y(u) = C$. We differentiate (2.6) to get,

$$Y'(x) = \frac{\partial}{\partial x}\Big[\Phi(x,u,P\Big]C + \int_u^x \frac{\partial}{\partial x}\Big[\Phi(x,s,P\Big]F(s)ds + \Phi(x,x,P)F(x)$$

$$= \frac{\partial}{\partial x}\Big[\Phi(x,u,P\Big]C + \int_u^x \frac{\partial}{\partial x}\Big[\Phi(x,s,P\Big]F(s)ds + F(x). \tag{2.7}$$

Using (2.2) and the given $Y(x)$, we get

$$Y'(x) = P(x)Y(x) + F(x)$$

$$= P(x) \ \ \Phi(x,u,P)C + \int_u^x \Phi(x,s,P)F(s)ds \ \ + F(x)$$

$$= P(x)\Phi(x,u,P)C + \int_u^x P(x)\Phi(x,s,P)F(s)ds + F(x)$$

$$= \frac{\partial}{\partial x}\Big[\Phi(x,u,P\Big]C + \int_u^x \frac{\partial}{\partial x}\Big[\Phi(x,s,P\Big]F(s)ds + F(x).$$

which is the same as (2.7) (we used property (4) of Proposition 2.2.1 in the last step). This completes the proof. □

Lastly in this Chapter, we state the Adjointness identity whose proof can be found in [23]. It is of paramount importance in determining the adjoint form of the Green's function for one-interval SLP.

Theorem 2.2.5. (Adjointness identity). *Let* $P \in M_n(L_{loc}(I,\mathbb{C}))$, *let* $E \in M_n(\mathbb{C})$. *Assume that*

$$E^{-1}E^{*} = I_n \quad \text{or} \quad E^{-1}E^{*} = -I_n$$

and define

$$P^{+} = -E^{-1}P * E.$$

Then

$$\Phi(x,s,P) = E^{-1}\Phi^{*}(s,x,P^{+})E, \quad s,x \in I.$$

CHAPTER 3

CHARACTERISTIC FUNCTION AND GREEN'S FUNCTION FOR ONE-INTERVAL SLP

In this chapter, we wish to examine Sturm-Liouville equation to determine the characteristic function for one-interval case and the Green's function. We will also consider the adjoint of the Green's function and its associated conditions. This has been necessitated by the Adjointness identity.

3.1 One-Interval SLP

We turn our attention to the SLP consisting of the equation

$$-(p(x)y')' + g(x)y = \lambda r(x)y, \quad x \in I = (a,b), -\infty \leq a < b \leq \infty, \tag{3.1}$$

where

$$s = \frac{1}{p}, g, r \in L(I,\mathbb{C}), \qquad \lambda \in \mathbb{C}, \tag{3.2}$$

together with general homogeneous boundary conditions

$$\begin{aligned} a_{11}y(a) + a_{12}(py')(a) + b_{11}y(b) + b_{12}(py')(b) = 0 \\ a_{21}y(a) + a_{22}(py')(a) + b_{21}y(b) + b_{22}(py')(b) = 0. \end{aligned}$$

When written in matrix form, the boundary conditions become

$$AY(a) + BY(b) = 0, \quad Y = \begin{bmatrix} y \\ py' \end{bmatrix}, \ A, B \in M_2(\mathbb{C}), \tag{3.3}$$

with the assumption that the rows of A and of B are linearly independent. As stated in Stakgold [19], we have only two boundary conditions because having fewer would result in more than one solution and having more would lead to the problem having no solution.

We may choose to omit x when writing (3.1) as it is clear from the context that y and the coefficients depend on x. Equation (3.1) can be converted to the equivalent first order linear system

$$Y' = (P - \lambda R)Y, \tag{3.4}$$

$$AY(a) + BY(b) = 0, \quad Y = \begin{bmatrix} y \\ py' \end{bmatrix}, \; A, B \in M_2(\mathbb{C}), \tag{3.5}$$

where,

$$P = \begin{bmatrix} 0 & s \\ g & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix},$$

by using the substitution

$$y' = \frac{1}{p} z, \qquad z' = (g - \lambda r)y.$$

From this conversion, we note that if y is a solution of (3.1), (3.3), then the vector $Y = \begin{bmatrix} y \\ py' \end{bmatrix}$ is the solution of (3.4), (3.5). Conversely, if the vector Y is the solution of (3.4), (3.5), then its top component is the solution of (3.1), (3.3).

Remark 3.1.1. *Under condition (3.2), the quasi-derivative py' is a continuous function on I and both $p(x)$ and $y'(x)$ may not exist for some $x \in I$. Hence, we write it as (py') to indicate that it cannot generally be separated into $p(x)$ and $y'(x)$ for all $x \in I$.*

3.1.1. One-Interval Characteristic Function

Definition 3.1.1. *By trivial solution of (3.1), (3.3) on some interval I_1 (I_1 may be a sub-interval of I or the whole interval), we mean a solution which is identically zero* *on I_1, i.e. both y and its quasi-derivative py' are both identically zero on I_1. Note that under the assumptions (3.2), y might be identically zero on I_1 while py' might not.*

Definition 3.1.2. *Let (3.2) hold. $\lambda \in \mathbb{C}$ is called an eigenvalue of the Sturm-Liouville boundary value problem consisting of (3.1), (3.3) if (3.1) has a non-trivial solution y satisfying (3.3). Such a solution y is called eigenfunction corresponding to the eigenvalue λ and we say (λ, y) is an eigenpair for the SLP (3.1),(3.3).*

Let Φ be a primary fundamental matrix. Here, since P and R are fixed, we write $\Phi = \Phi(.,.,\lambda)$ to show that the primary fundamental matrix only depends on λ.

Lemma 3.1.1. *Let $P \in M_n(L_{loc}(I))$ and $\Phi(x,u,\lambda)$ be the primary fundamental matrix of P. If $Tr\ \ P(x)\ \ = 0$ for $x \in I$, then*

$$\det\left[\Phi(x,u,\lambda)\right] = 1, \ \ x,u \in I.$$

Proof: By Abel's formula (2.5), we have that

$$\det\ \ \Phi(x,u,\lambda)\quad ' = Tr\left[P(x)\right]\det\ \ \Phi(x,u,\lambda)\ \ = 0.$$

Fixing $u \in I$, we get the desired result (See [28]). $\square$

Definition 3.1.3 (Characteristic function). *The characteristic function Π of the SLP consisting of (3.1), (3.3) in the case of one-interval, is given by*

$$\Pi(\lambda) = \det\left[A + B\Phi(b,a,\lambda)\right].$$

Theorem 3.1.1. *Let assumptions (3.2) hold. $\lambda \in \mathbb{C}$ is an eigenvalue of (3.1), (3.3) if and only if $\Pi(\lambda) = 0$.*

Proof: Suppose λ is an eigenvalue and (λ, y) is an eigenpair of (3.1), (3.3). Then, from the basic theory, there exists a non-trivial constant vector C such that

$$Y(x) = \Phi(x, a, \lambda)C.$$

Substituting into (3.3), we have

$$\left[A + B\Phi(b, a, \lambda)\right]C = 0.$$

Since $C \neq 0$, $\det[A + B\Phi(b, a, \lambda)] = \Pi(\lambda) = 0.$

Conversely, suppose $\Pi(\lambda) = 0$. Then $\left[A + B\Phi(b, a, \lambda)\right]C = 0$ has a non-trivial vector solution for C. Consider the IVP

$$Y' = (P - \lambda R)Y, \qquad Y(a) = C.$$

Then

$$Y(x) = \Phi(x, a, \lambda)C$$

and so

$$Y(b) = \Phi(b, a, \lambda)Y(a).$$

Substituting into (3.3), we have that

$$\left[A + B\Phi(b, a, \lambda)\right]Y(a) = 0.$$

Since $\det[A + B\Phi(b, a, \lambda)] = 0$, it follows that the top component of Y , which is y, is non-trivial meaning λ is an eigenvalue and (λ, y) is an eigenpair. □

3.1.2. One-Interval Green's Function

The Green's function constructed here, as emphasised in [22, 23, 24], is special in that it differs from the usual construction where the jump discontinuity is introduced in advance. Here, we construct the matrix Green's function for the equivalent first order linear system and then extract the top right component as the Green's function for the scalar problem. By so doing, the jump discontinuity appears 'naturally'.

Let

$$\frac{1}{p}, g, r \in L(I,\mathbb{C}), \quad I=(a,b), -\infty \le a < b \le \infty, \quad \lambda \in \mathbb{C}. \tag{3.6}$$

As done above, the scalar boundary value problem

$$-(py')' + gy = \lambda ry + f, \qquad AY(a) + BY(b) = 0, \tag{3.7}$$

is equivalent to the first order linear system

$$Y' = (P - \lambda R)Y + F, \qquad AY(a) + BY(b) = 0, \tag{3.8}$$

where,

$$Y = \begin{bmatrix} y \\ py' \end{bmatrix}, \ P = \begin{bmatrix} 0 & s \\ g & 0 \end{bmatrix}, \ R = \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}, \ F = \begin{bmatrix} 0 \\ -f \end{bmatrix}, \ A, B \in M_2(\mathbb{C}), \tag{3.9}$$

Let $\Phi(.,.,\lambda)$ be the primary fundamental matrix of the homogeneous system (3.4) . Note that $\Phi(b,a,\lambda)$ exists [23]. We now present a theorem which lays the foundation for the construction of the Green's function.

Theorem 3.1.2. *Let (3.6) to (3.9) hold. Then the following are equivalent:*

1. *When f = 0 on I, the boundary value problem (3.7) (and consequently also (3.8)) has only the trivial solution.*
2. $[A + B\Phi(b,a,\lambda)]^{-1}$ *exists.*
3. *For every* $f \in L(I,\mathbb{C})$, *each of the problems (3.7), (3.8) has a unique solution.*

Proof: To prove that (1) $\Rightarrow$(2), we use (2.6) and the boundary conditions to observe that for every $f \in L(I,\mathbb{C})$, $\det[A + B\Phi(b,a,\lambda)] \neq 0$, i.e. $[A + B\Phi(b,a,\lambda)]^{-1}$ exists.

(2) $\Rightarrow$(3) is clear due to the same fact that $[A + B\Phi(b,a,\lambda)]^{-1}$ exists.

To show that (3) $\Rightarrow$(1), we again use (2.6) and the boundary conditions, and notice that since $[A + B\Phi(b,a,\lambda)]^{-1}$ exists, when $f = 0$, $C = 0$ implying

$$Y(x) = \Phi(x,a,\lambda)C = 0.$$

See Zettl [23] for a detailed proof. $\square$

Theorem 3.1.3. *If* $[A+B\Phi(b,a,\lambda)]^{-1}$ *exists and the matrix Green's function G is defined by*

$$G(x,s,\lambda)=\begin{cases}U(x,s,\lambda), & a\le x<s\le b,\\ V(x,s,\lambda), & a\le s\le x\le b,\end{cases} \tag{3.10}$$

where

$$U(x,s,\lambda)=\Phi(x,a,\lambda)W(\lambda)\Phi(a,s,\lambda),$$

$$V(x,s,\lambda)=\Phi(x,a,\lambda)W(\lambda)\Phi(a,s,\lambda)+\Phi(x,s),$$

$$W(\lambda)=\left[A+B\Phi(b,a,\lambda)\right]^{-1}(-B)\Phi(b,a,\lambda),$$

then for any $f\in L(I,\mathbb{C})$, *the unique solution Y of (3.8) and the unique solution y of (3.7), respectively, are given by*

$$Y(x,\lambda)=\int_a^b G(x,s,\lambda)F(s)ds, \qquad a\le x\le b, \tag{3.11}$$

$$y(x,\lambda)=-\int_a^b g_{12}(x,s,\lambda)f(s)ds, \qquad a\le x\le b, \tag{3.12}$$

with g_{12} *denoting the top right component of G.*

Moreover, the Green's function g_{12} *is continuous on* $[a,b]\times[a,b]$ *and is the unique continuous function on* $[a,b]\times[a,b]$ *with property (3.12), i.e. if (3.12) holds for every* $f\in L(I,\mathbb{C})$ *with* g_{12} *replaced by a continuous* k_{12}, *then* $g_{12}=k_{12}$. *We use the notation* $G=G(.,.,\lambda,A,B,P,R)$ *to indicate the dependence of G on* λ,A,B,P *and* R.

Proof: Assume that $[A+B\Phi(b,a,\lambda)]^{-1}$ exists. Let C be such that

$$C=[A+B\Phi(b,a,\lambda)]^{-1}(-B)\int_a^b \Phi(b,s,\lambda)F(s)ds.$$

Then by (2.6), we have that the boundary conditions are satisfied and

$$Y(x,\lambda) = \Phi(x,a,\lambda)[A + B\Phi(b,a,\lambda)]^{-1}(-B)\int_a^b \Phi(b,s,\lambda)F(s)ds + \int_a^x \Phi(x,s,\lambda)F(s)ds$$

$$= \int_a^b \Big[\Phi(x,a,\lambda)[A + B\Phi(b,a\lambda)]^{-1}(-B)\Phi(b,a,\lambda)\Phi(a,s,\lambda)\Big]F(s)ds + \int_a^x \Phi(x,s,\lambda)F(s)ds$$

$$= \int_a^b G(x,s,\lambda)F(s)ds, \qquad a \le x \le b.$$

Taking the top component of Y, we get (3.12).

The remainder of this Theorem is proved by choosing f which is positive and has compact support (See Zettl [23]). $\square$

Remark 3.1.2. *For this first order linear system, the jump discontinuity for* $x = s$ *is*

$$V_{x=s} - U_{x=s} = \Phi(s,s,\lambda) = I_2.$$

Taking the top right component, we observe that

$$\frac{\partial}{\partial x} g_{12}(s+0,s,\lambda) - \frac{\partial}{\partial x} g_{12}(s-0,s,\lambda) = 0,$$

implying that g_{12} *(even* g_{21}*) is continuous on* $[a,b]\times[a,b]$ *for* $x = s$.

Example 3.1.1. We give a simple example

$$-y'' = y + x, \;\; I = [0,\pi]$$

$$AY(a) + BY(b) = 0,$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & \pi \\ 0 & -1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ -x \end{bmatrix}.$$

It is clear here that $g = 0, \ \lambda = 1 = r$ and

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ F = \begin{bmatrix} 0 \\ -x \end{bmatrix}, \ Y = \begin{bmatrix} y \\ y' \end{bmatrix}.$$

The homogeneous equation

$$-y'' = y$$

has only the trivial solution and the primary fundamental matrix for

$$Y' = (P - \lambda R)Y$$

is

$$\Phi(x,u) = \begin{bmatrix} \cos(x-u) & \sin(x-u) \\ -\sin(x-u) & \cos(x-u) \end{bmatrix}$$

for a fundamental matrix

$$\mathbb{Y}(x) = \begin{bmatrix} -ie^{ix} & e^{-ix} \\ e^{ix} & -ie^{-ix} \end{bmatrix}.$$

It can be checked that $\det \Phi(x,u) = 1$ and we have the following:

$$\Phi(\pi,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ \Phi(x,0) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}, \ \Phi(\pi,s) = \begin{bmatrix} -\cos s & \sin s \\ -\sin s & -\cos s \end{bmatrix},$$
$$\left[A + B\Phi(\pi,0)\right] = \begin{bmatrix} -1 & -\pi \\ 0 & 2 \end{bmatrix}.$$

Letting $\Omega(x,s) = \Phi(x,0)\left[A + B\Phi(\pi,0)\right]^{-1}(-B)\Phi(\pi,s)$, we have that

$$\Omega(x,s) = \begin{bmatrix} (-\frac{\pi}{2}\cos x - \frac{1}{2}\sin x)\sin s & (-\frac{\pi}{2}\cos x - \frac{1}{2}\sin x)\cos s \\ (\frac{\pi}{2}\sin x - \frac{1}{2}\cos x)\sin s & (\frac{\pi}{2}\sin x - \frac{1}{2}\cos x)\cos s \end{bmatrix}$$

and

$$G(x,s)=\begin{cases}\Omega(x,s), & 0\le x<s\le\pi\\ \Omega(x,s)+\Phi(x,s), & 0\le s\le x\le\pi\end{cases}$$

so that

$$g_{12}(x,s)=\begin{cases}(-\frac{\pi}{2}\cos x-\frac{1}{2}\sin x)\cos s, & 0\le s<x\le\pi\\ (-\frac{\pi}{2}\cos x-\frac{1}{2}\sin x)\cos s+\sin(x-s), & 0\le s\le x\le\pi\end{cases}.$$

Using this Green's function, we find

$$y(x)=-\int_x^{\pi}(-\tfrac{\pi}{2}\cos x-\tfrac{1}{2}\sin x)s\cos s ds-\\ \int_0^x s\Big[(-\tfrac{\pi}{2}\cos x-\tfrac{1}{2}\sin x)\cos s+\sin(x-s)\Big]ds,$$

i.e. $y(x)=-\pi\cos x-x.$ △

3.2 Adjoint Green's Function

The adjoint form of the Green's function can easily be found under certain conditions. The Adjoint identity has proved to be very useful in this regard and with the help of two important Lemmas, the required conditions for the adjoint form to exist are stated.

Theorem 3.2.1. *Let P, R be as in* (3.6), (3.9). *Let E be defined by*

$$E=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

Then for any $\lambda\in\mathbb{C}$

$$\overline{P}-\overline{\lambda}\overline{R}=E(P-\lambda R)^{*}E,$$

with $\overline{P}$, $\overline{\lambda},\overline{R}$ *denoting conjugate of* P,λ *and* R *respectively.*

Proof: Direct computation gives the result. □

We note that $E^{*}=-E=E^{-1}$ and together with Theorem 3.2.1, we can see that conditions for Adjointness identity (Theorem 2.25) are satisfied such that

$$\Phi(x,s,\lambda,P,R) = -E\Phi^*(s,x,\bar{\lambda},\bar{P},\bar{R})E. \qquad (3.13)$$

Before presenting next lemma, we would like to give the matrix Green's function for $x = s$. Using (3.10), we have that

$$G(x,s,\lambda) = \frac{1}{2}\left[U+V\right] = U + \frac{1}{2}\Phi(x,s,\lambda), \quad a \le x = s \le b, \qquad (3.14)$$

where U and V are as given in Theorem 3.1.3 (See [28]). In the next lemma, it is important that we include all the parameters on which Φ depends.

Lemma 3.2.1. *Let the hypotheses and notation of Theorem 3.1.2 hold. Then for all x, s, with $a \le x, s \le b$, we have that*

$$\begin{aligned} G(x,s,\lambda,A,B,P,R) - EG^*(s,x,\bar{\lambda},C,D,\bar{P},\bar{R})E \\ = \Phi(x,a,\lambda,P,R)Z(\lambda)\Phi(a,s,\lambda,P,R), \end{aligned}$$

where

$$Z(\lambda) = W(\lambda) - ET^*(\lambda)E + I,$$

with $T(\lambda)$ given by

$$T(\lambda) = -\left[C + D\Phi(b,a,\bar{\lambda},\bar{P},\bar{R}\right]^{-1} D\Phi(b,a,\bar{\lambda},\bar{P},\bar{R}).$$

Proof: Using (3.14), we have that

$$G(s,x,\bar{\lambda},\bar{P},\bar{R}) = \Phi(s,a,\bar{\lambda},\bar{P},\bar{R})T(\lambda)\Phi(a,x,\bar{\lambda},\bar{P},\bar{R}) + \frac{1}{2}\Phi(s,x,\bar{\lambda},\bar{P},\bar{R}).$$

Thus,

$$\begin{aligned} &G(x,s,\lambda,A,B,P,R) - EG^*(s,x,\bar{\lambda},C,D,\bar{P},\bar{R})E = \\ &\quad \Phi(x,a,\lambda,P,R)W(\lambda)\Phi(a,s,\lambda,P,R) + \frac{1}{2}\Phi(x,s,\lambda,P,R) - \\ &\quad EE\Phi(x,a,\lambda,P,R)E^{-1}T^*(\lambda)E\Phi(a,s,\lambda,P,R)E^{-1}E + \frac{1}{2}\Phi(x,s,\lambda,P,R) \end{aligned}$$

$$\begin{aligned} &= \Phi(x,a,\lambda,P,R)W(\lambda)\Phi(a,s,\lambda,P,R) - \Phi(x,a,\lambda,P,R)ET^*(\lambda)E\Phi(a,s,\lambda,P,R) + \\ &\quad \Phi(x,s,\lambda,P,R) \end{aligned}$$

$$= \Phi(x,a,\lambda,P,R)Z(\lambda)\Phi(a,s,\lambda,P,R).$$

We have used (3.13) and property (2) of proposition 2.2.1. □

Lemma 3.2.2. *Let the hypotheses and notation of Lemma 3.2.1 hold. Then*

$Z(\lambda) = 0$ *if and only if* $AEC^* = BED^*$.

Proof: $Z(\lambda) = 0 \Leftrightarrow \mathrm{I} = W(\lambda) + ET^*(\lambda)E.$

Multiplying on the left and on the right by

$\left[A + B\Phi(b,a,\lambda,P,R)\right]$ and $E\left[C + D\Phi(b,a,\bar{\lambda},\bar{P},\bar{R})\right]^*$ respectively, we get

LHS: $\left[A + B\Phi(b,a,\lambda,P,R)\right]E\left[C + D\Phi(b,a,\bar{\lambda},\bar{P},\bar{R})\right]^*$

$$\begin{aligned} &= AEC^* + AE\Phi^*(b,a,\bar{\lambda},\bar{P},\bar{R})D^* + B\Phi(b,a,\lambda,P,R)EC^* + \\ &\quad B\Phi(b,a,\lambda,P,R)E\Phi^*(b,a,\bar{\lambda},\bar{P},\bar{R})D^* \end{aligned}$$

$$= AEC^* + AE\Phi^*(b,a,\bar{\lambda},\bar{P},\bar{R})D^* + B\Phi(b,a,\lambda,P,R)EC^* + BED^*.$$

RHS: $B\Phi(b,a,\lambda,P,R)E\left[C + D\Phi(b,a,\bar{\lambda},\bar{P},\bar{R})\right]^* +$

$$\left[A + B\Phi(b,a,\lambda,P,R)\right]E\Phi^*(b,a,\bar{\lambda},\bar{P},\bar{R})D^*$$

$$\begin{aligned} &= B\Phi(b,a,\lambda,P,R)EC^* + 2B\Phi(b,a,\lambda,P,R)E\Phi^*(b,a,\bar{\lambda},\bar{P},\bar{R})D^* + \\ &\quad AE\Phi^*(b,a,\bar{\lambda},\bar{P},\bar{R})D^* \end{aligned}$$

$$= B\Phi(b,a,\lambda,P,R)EC^* + 2BED^* + AE\Phi^*(b,a,\bar{\lambda},\bar{P},\bar{R})D^*.$$

Thus,

$$\begin{aligned} Z(\lambda) = 0 \Leftrightarrow AEC^* + AE\Phi^*(b,a,\bar{\lambda},\bar{P},\bar{R})D^* + B\Phi(b,a,\lambda,P,R)EC^* + BED \\ = B\Phi(b,a,\lambda,P,R)EC^* + 2BED^* + AE\Phi^*(b,a,\bar{\lambda},\bar{P},\bar{R})D^* \end{aligned}$$

$$\Leftrightarrow AEC^* = BED^*.$$

□

Theorem 3.2.2 (Adjoint matrix Green's function). *Let* $C, D \in M_2(\mathbb{C})$, $\lambda \in \mathbb{C}$ *and let (3.6) to (3.8) hold. Assume that both*

$[A + B\Phi(b,a,\lambda)]^{-1}$ *and* $\left[C + D\Phi(b,a,\bar{\lambda},\bar{P},\bar{R}\right]^{-1}$ *exist. Then*

$$G(x,s,\lambda,A,B,P,R) = EG^*(s,x,\bar{\lambda},C,D,\bar{P},\bar{R})E, \quad a \le x,s \le b$$

if and only if $AEC^* = BED^*$*, where E is as defined in Theorem 3.2.1.*

Proof: The proof follows from Lemmas 3.2.1 and 3.2.2. □

Theorem 3.2.3 (Adjoint Green's function). *Let* $C, D \in M_2(\mathbb{C})$, $\lambda \in \mathbb{C}$ *and let (3.6) to (3.8) hold. Assume that both* $[A + B\Phi(b,a,\lambda,P,R)]^{-1}$ *and* $\left[C + D\Phi(b,a,\bar{\lambda},\bar{P},\bar{R}\right]^{-1}$ *exist. Then*

$$g_{12}(x,s,\lambda,P,R) = \bar{g}_{12}(s,x,\lambda,P,R), \quad a \le x,s \le b$$

if and only if $AEC^* = BED^*$*, where E is as defined in Theorem 3.2.1.*

Proof: Taking top right component of G in Theorem 3.2.2, we get the required result (See Zettl [23]). □

CHAPTER 4

GREEN'S FUNCTION FOR FINITELY MANY INTERVALS STURM-LIOUVILLE PROBLEMS

Our main focus in this chapter is to construct the Green's function for finitely many intervals case. We will also formulate the characteristic function. This will be an extension of Wang and Zettl's idea [24] for two intervals case.

We consider SLPs having equations

$$-(p_i y_i')' + g_i y_i = \lambda r_i y_i \quad \text{on } I_i, i = 1, 2, \ldots, n, \lambda \in \mathbb{C} \tag{4.1}$$

with assumptions

$$\frac{1}{p_i}, g_i, r_i \in L(I_i, \mathbb{C}), \; i = 1, 2, \ldots, n \tag{4.2}$$

and boundary conditions

$$A_1 Y_1(a_1) + B_1 Y_1(b_1) + A_2 Y_2(a_2) + B_2 Y_2(b_2) + \ldots + A_n Y_n(a_n) + B_n Y_n(b_n) = 0. \tag{4.3}$$

The intervals I_i, $i = 1, 2, \ldots, n,$ can be identical, overlap or disjoint and the boundary conditions are not necessarily self-adjoint [24]. Equivalently, (4.1), (4.3) can be written as

$$Y_i' = (P_i - \lambda R_i) Y_i, \tag{4.4}$$

$$\sum_{i=1}^{n} \left[A_i Y_i(a_i) + B_i Y_i(b_i) \right] = 0 \tag{4.5}$$

where

$$Y_i = \begin{bmatrix} y_i \\ p_i y_i{}' \end{bmatrix},\ P_i = \begin{bmatrix} 0 & \frac{1}{p_i} \\ g_i & 0 \end{bmatrix},\ R_i = \begin{bmatrix} 0 & 0 \\ r_i & 0 \end{bmatrix},\ i = 1,2,\ldots,n.$$

Let $\Phi_i(.,.,\lambda)$ be the primary fundamental matrix of (4.4). We make the following necessary claim regarding the size of the matrices in the boundary conditions (4.5):

Claim. For $i = 1,2,\ldots,n,\ A_i, B_i \in M_{2n,2}(\mathbb{C})$.

Proof: It is clear that A_i, B_i must have 2 columns in order for multiplication with 2×1 column vectors $Y_i(a_i), Y_i(b_i)$ to occur. We remark that A_i, B_i have either odd or even number of rows. Let R be the number of rows of A_i, B_i. Since $\Phi_i(.,.,\lambda)$ is the primary fundamental matrix, there exists a non-zero constant matrix $C_i \in M_{2,1}(\mathbb{C})$ such that

$$Y_i(x,\lambda) = \Phi_i(x,a_i,\lambda)C_i,$$

is the solution of (4.4). Using (4.5), we have

$$N(\lambda)C = 0,$$

where

$$N(\lambda) = \Big[\big[A_1 + B_1\Phi_1(a_1,b_1,\lambda)\big] \,||\, \big[A_2 + B_2\Phi_2(a_2,b_2,\lambda)\big] \,||\, \ldots \,||\, \big[A_n + B_n\Phi_n(a_n,b_n,\lambda)\big]\Big],$$

is a matrix partitioned in such a way that the first 2 columns are those of $\big[A_1 + B_1\Phi_1(a_1,b_1,\lambda)\big]$, the next 2 columns are those of $\big[A_2 + B_2\Phi_2(a_2,b_2,\lambda)\big]$ and so on up to the last 2 columns which are those of $\big[A_n + B_n\Phi_n(a_n,b_n,\lambda)\big]$, and $C = \big[C_1\ C_2 \ldots C_n\big]^T$ is a block matrix. Clearly, $C \in M_{2n,1}(\mathbb{C})$ and the zero vector $0 \in M_{2n,1}(\mathbb{C})$. If $R = 2n+1,\ n \in \mathbb{Z}^+$, then the order of $N(\lambda)$ is $(2n+1)\times 2n$ and this contradicts the size of the zero vector 0. It follows that $R = 2n,\ n \in \mathbb{Z}^+$. $\square$

4.1 Finitely Many Intervals Green's Function

Our main goal in this section is to construct the Green's function for finitely many intervals case. We begin by constructing the characteristic function whose zeroes give the eigenvalues of the homogeneous system.

Definition 4.1.1. *The trivial solution of* (4.1), (4.3) *on some interval* I_σ *(* I_σ *may be a sub-interval of* I_i *or the whole interval), is a solution which is identically zero on* I_σ*, i.e. both* y_i *and its quasi-derivative* $p_i y_i'$ *are identically zero on* I_σ*,* $i = 1,2,...,n$*.*

Definition 4.1.2. *Let* (4.2) *hold.* $\lambda \in \mathbb{C}$ *is called an eigenvalue of the finitely many intervals Sturm-Liouville boundary value problems consisting of* (4.1), (4.3) *if* (4.1) *has a non-trivial solution* $y = \{y_1, y_2, ..., y_n\}$ *satisfying* (4.3)*. Such a solution y is called eigenfunction corresponding to the eigenvalue* λ *and* (λ, y) *is called an eigenpair.*

Remark 4.1.1. *If* y_i *is a solution of* (4.1), (4.3) *then* Y_i *is a solution of* (4.4), (4.5). *Conversely, if* Y_i *is a solution of* (4.4), (4.5)*, then its top component* y_i *is a solution of* (4.1), (4.3).

Theorem 4.1.1. *Let* $\Phi_i(.,.,\lambda)$*,* $i = 1,2,...,n$*, be the primary fundamental matrix for the homogeneous equation (4.4). Define the characteristic function* Δ *for the finitely many intervals SLPs by*

$$\Delta(\lambda) = \det \; N(\lambda) \;,$$

where $N(\lambda) \in M_{2n,2n}(\mathbb{C})$*. Then,* $\lambda \in \mathbb{C}$ *is an eigenvalue of* (4.1), (4.3) *if and only if* $\Delta(\lambda) = 0$.

Proof: Suppose $\lambda \in \mathbb{C}$ is an eigenvalue and $y = \{y_1, y_2, ..., y_n\}$ is an eigenfunction such that (λ, y) is an eigenpair. For the homogeneous equation (4.4), we have that

$$Y_i(x,\lambda) = \Phi_i(x, a_i, \lambda) C_i,$$

where $C_i \in M_{2,1}(\mathbb{C}),\ i = 1,2,\dots,n,$ is non-trivial. Note that $\Phi_i(b_i, a_i, \lambda)$ exists and using (4.5), we have that

$$\sum_{i=1}^{n} \left[A_i C_i + B_i \Phi_i(b_i, a_i, \lambda) C_i \right] = 0$$

which can be written as

$$\left[\left[A_1 + B_1 \Phi_1(a_1, b_1, \lambda) \right] \| \left[A_2 + B_2 \Phi_2(a_2, b_2, \lambda) \right] \| \dots \| \left[A_n + B_n \Phi_n(a_n, b_n, \lambda) \right] \right] C = 0, \tag{4.6}$$

where

$$C = \left[C_1\ C_2 \dots C_n \right]^T \in M_{2n,1}(\mathbb{C}). \tag{4.7}$$

Since $C \neq 0$ and by assumption (λ, y) is an eigenpair, it follows that

$$\det\ N(\lambda)\ = 0,$$

i.e. $\Delta(\lambda) = 0$.

Conversely, suppose $\Delta(\lambda) = 0$. Then (4.6) has a non-zero vector solution $C \in M_{2n,1}(\mathbb{C})$. Consider the IVP consisting of (4.4) and $Y_i(a_i, \lambda) = C_i,\ i = 1,2,\dots,n,$ whose solution is

$$Y_i(x, \lambda) = \Phi_i(x, a_i, \lambda) C_i.$$

It follows that

$$Y_i(b_i, \lambda) = \Phi_i(b_i, a_i, \lambda) C_i.$$

By (4.5),

$$\sum_{i=1}^{n} \left[A_i Y_i(a_i, \lambda) + B_i \Phi_i(b_i, a_i, \lambda) Y_i(a_i, \lambda) \right] = 0$$

which can be written as

$$\left[\left[A_1 + B_1 \Phi_1(a_1, b_1, \lambda) \right] \| \left[A_2 + B_2 \Phi_2(a_2, b_2, \lambda) \right] \| \dots \| \left[A_n + B_n \Phi_n(a_n, b_n, \lambda) \right] \right] \mathbb{Y} = 0,$$

where

$$\mathbb{Y} = \left[Y_1(a_1)\ Y_2(a_2)...Y_n(a_n)\right]^T.$$

Since $\det\ N(\lambda)\ = 0$, taking the top components in each of the (2×1) vectors $Y_i(a_i)$ in $\mathbb{Y}$, we have that $y = \{y_1, y_2, ..., y_n\}$ is non-zero. Therefore, (λ, y) is an eigenpair.

□

Now, let (4.2) hold and consider the scalar boundary value problems

$$-(p_i y_i')' + g_i y_i = \lambda r_i y_i + f_i,\ \sum_{i=1}^{n}\left[A_i Y_i(a_i) + B_i Y_i(b_i)\right] = 0,\ \lambda \in \mathbb{C} \tag{4.8}$$

which is equivalent to

$$Y_i' = (P_i - \lambda R_i)Y_i + F_i,\ \sum_{i=1}^{n}\left[A_i Y_i(a_i) + B_i Y_i(b_i)\right] = 0,\ i = 1,2,...,n,\ \lambda \in \mathbb{C}, \tag{4.9}$$

where

$$Y_i = \begin{bmatrix} y_i \\ p_i y_i{}' \end{bmatrix}, P_i = \begin{bmatrix} 0 & \frac{1}{p_i} \\ g_i & 0 \end{bmatrix}, R_i = \begin{bmatrix} 0 & 0 \\ r_i & 0 \end{bmatrix}, F_i = \begin{bmatrix} 0 \\ -f_i \end{bmatrix}, A_i, B_i \in M_{2n,2}(\mathbb{C}), \tag{4.10}$$

and $f_i \in L(I_i, \mathbb{C}),\ i = 1,2,...,n.$

Theorem 4.1.2. *Let $\Phi_i(.,.,\lambda)$ be the primary fundamental matrix of (4.4) and* let (4.2), *(4.8),* (4.9) and (4.10) *hold. Let $\lambda \in \mathbb{C}$. The following statements are equivalent:*

1. *When $f_i = 0$ (and equivalently $F_i = 0$), the finitely many intervals boundary value problem* (4.8)*, (and consequently* (4.9)*) has only the trivial solution.*
2. *The matrix $N(\lambda)$ has an inverse.*
3. *For every $f_i \in L(I_i, \mathbb{C}),\ i = 1,2,...,n,$ each of the problems* (4.8) *and* (4.9) *has a unique solution.*

Proof: Since Y_i is a solution of (4.9) if and only if y_i is a solution of (4.8), for $C_i \in M_{2,1}(\mathbb{C})$, determine a solution Y_i of (4.9) by the initial condition

$$Y_i(a_i,\lambda) = C_i$$

so that solution y_i of (4.8) is determined by

$$y_i(a_i,\lambda) = c_{i1}, \quad (py')(a_i,\lambda) = c_{i2}$$

where c_{i1} and c_{i2} are the first and second components, respectively, of the column vector C_i and are not both zero. By (2.6),

$$Y_i(x,\lambda) = \Phi_i(x,a_i,\lambda)C_i + \int_{a_i}^{x} \Phi_i(x,s,\lambda)F_i(s)ds, \quad a_i \le x \le b_i. \tag{4.11}$$

Then using (4.11), we get

$$\sum_{i=1}^{n}\left[A_iY_i(a_i) + B_iY_i(b_i)\right] = \sum_{i=1}^{n}\left[A_iC_i + B_i\Phi_i(b_i,a_i,\lambda)C_i\right] + \sum_{r=1}^{n}\left[B_r\int_{a_r}^{b_r}\Phi_r(b_r,s,\lambda)F_r(s)ds\right]$$

$$= N(\lambda)C + \sum_{r=1}^{n}\left[B_r\int_{a_r}^{b_r}\Phi_r(b_r,s,\lambda)F_r(s)ds\right], \tag{4.12}$$

where C is as defined in (4.7). From (4.12), we observe that when $f_i = 0$ on I_i (and consequently $F_i = 0$), there is a non-trivial solution Y_i of (4.9), (and consequently nontrivial solution y_i of (4.8)) if and only if $N(\lambda)$ is singular. It follows that for every $f_i \in L(I_i,\mathbb{C})$ there is the unique solution Y_i of (4.9), (and consequently the unique solution y_i of (4.8)) if and only if $N(\lambda)$ is non-singular. $\square$

Remark 4.1.2. *Under the conditions of Theorem 4.1.2,* $\lambda \in \mathbb{C}$ *is not an eigenvalue of the homogeneous equation due to the fact that when* $f_i = 0$ *(and equivalently* $F_i = 0$*),* (4.8) *(and consequently* (4.9)*) has only the trivial solution. Thus, when constructing the Green's function, we need to take* λ *which is not an eigenvalue.*

We can finally proceed to the construction of the Green's function. We start by first assuming that $N^{-1}(\lambda)$ exists. Let $N_1(\lambda)$ denote the $(2\times 2n)$ matrix formed from the first two rows of $N^{-1}(\lambda)$, $N_2(\lambda)$ be the $(2\times 2n)$ matrix obtained from the next two rows of $N^{-1}(\lambda)$ and so on up to $N_n(\lambda)$ the $(2\times 2n)$ matrix formed from the last two rows of $N^{-1}(\lambda)$. Define $M_{ir}(\lambda),\ 1\le i,r\le n,$ by

$$M_{ir}(\lambda)=N_i(\lambda)(-B_r).$$

Theorem 4.1.3. *Assume* $N^{-1}(\lambda)$ *exists. Define the matrix Green's function* G *by* $G(x,s,\lambda)=\{G_i(x,s,\lambda)\},\ 1\le i\le n$ *with*

$$G_i(x,s,\lambda)=\begin{cases} G_{i1}(x,s,\lambda),\ a_1\le s\le b_1 \\ G_{i2}(x,s,\lambda),\ a_2\le s\le b_2 \\ \vdots \\ G_{in}(x,s,\lambda),\ a_n\le s\le b_n \end{cases},\quad a_i\le x\le b_i,$$

where, for $r\ne i,\ r=1,2,\dots,n,$

$$G_{ir}(x,s,\lambda)=\Phi_i(x,a_i,\lambda)M_{ir}(\lambda)\Phi_r(b_r,s,\lambda),\ \ a_i\le x\le b_i,\ \ a_r\le s\le b_r,$$

and when $i=r$

$$G_{ii}(x,s,\lambda)=\begin{cases} \Phi_i(x,a_i,\lambda)M_{ii}(\lambda)\Phi_i(b_i,s,\lambda), & a_i\le x<s\le b_i \\ \Phi_i(x,a_i,\lambda)M_{ii}(\lambda)\Phi_i(b_i,s,\lambda)+\Phi_i(x,s,\lambda), & a_i\le s\le x\le b_i \end{cases}.$$

Then for any $f_i\in L(I_i,\mathbb{C})$, *the unique solution* Y_i *of* (4.9) *and the unique solution* y_i *of (4.8), respectively, are given by*

$$Y_i(x,\lambda)=\sum_{r=1}^{n}\left(\int_{a_r}^{b_r}G_{ir}(x,s,\lambda)F_r(s)ds\right),\ \ a_i\le x\le b_i,\ \ i=1,2,\dots,n, \tag{4.13}$$

$$y_i(x,\lambda)=-\sum_{r=1}^{n}\left(\int_{a_r}^{b_r}G_{(ir),(12)}(x,s,\lambda)f_r(s)ds\right),\ \ a_i\le x\le b_i,\ i=1,2,\dots,n, \tag{4.14}$$

where $G_{(ir),(12)}$ denotes the top right component of G_{ir}. To show dependence of G on the variables of the problem, we write

$$G_{ir}(x,s,\lambda) = G_{ir}(x,s,\lambda,A_i,B_i,P_i,R_i).$$

We call $G_{(ir),(12)}$ the Green's function for finitely many intervals SLPs.

***Proof*:** Let $C \in M_{2n,1}(\mathbb{C})$ be given by

$$C = N^{-1}(\lambda)\left[\sum_{r=1}^{n}(-B_r)\int_{a_r}^{b_r}\Phi_r(b_r,s,\lambda)F_r(s)ds\right].$$

Then, C comprises of $C_i \in M_{2,1}(\mathbb{C})$ given by

$$C_i = N_i(\lambda)\left[\sum_{r=1}^{n}(-B_r)\int_{a_r}^{b_r}\Phi_r(b_r,s,\lambda)F_r(s)ds\right],\ \ i = 1,2,\ldots,n.$$

This choice of C satisfies (4.12), i.e. boundary conditions of (4.8) are satisfied. By variation of parameters formula (2.6), we have

$$\begin{aligned} Y_i(x,\lambda) &= \Phi_i(x,a_i,\lambda)\left\{N_i(\lambda)\left[\sum_{r=1}^{n}(-B_r)\int_{a_r}^{b_r}\Phi_r(b_r,s,\lambda)F_r(s)ds\right]\right\} + \\ &\qquad \int_{a_i}^{x}\Phi_i(x,s,\lambda)F_i(s)ds \\ &= \sum_{r=1}^{n}\int_{a_r}^{b_r}\left[\Phi_i(x,a_i,\lambda)N_i(\lambda)(-B_r)\Phi_r(b_r,s,\lambda)\right]F_r(s)ds \ + \\ &\qquad \int_{a_i}^{x}\Phi_i(x,s,\lambda)F_i(s)ds \\ &= \sum_{r=1}^{n}\left[\int_{a_r}^{b_r}G_{ir}(x,s,\lambda)F_r(s)ds\right],\ \ a_i \le x \le b_i,\ \ i = 1,2,\ldots,n. \end{aligned}$$

Note that taking the top component of $Y_i(x,\lambda)$, we get (4.14). □

Remark 4.1.3. *The jump discontinuity when $x = s$ occurs from G_{ii} and since it is an identity, we conclude that the Green's function $G_{(ir),(12)}$ is continuous on*

$[a_i, b_i] \times [a_i, b_i]$. *Furthermore, as stated in [13], this method of constructing the Green's function does not assume symmetry or self-adjointness of the problems.*

Suppose the coefficients in the Sturm-Liouville problem satisfy

$$\tfrac{1}{p_i}, g_i, r_i \in L(I_i, \mathbb{R}) \text{ and } r_i > 0$$

and consider the Hilbert space $H_i = L^2(I_i, r_i)$ with usual inner product

$$\left(f, l\right)_i = \int_{I_i} f(x)\overline{l(x)}r(x)dx, \; i = 1, 2, ..., n,$$

and norm $\| f \| = \left(f, f\right)_i^{1/2}$. Then, the direct sum H of the Hilbert space can be denoted by

$$H = \oplus_{i=1}^{n} H_i = \oplus_{i=1}^{n} L^2(I_i, r_i).$$

As stated in [14, 27] for the two intervals case, this direct sum can be replaced by the one with direct sum inner product

$$\left\langle \mathbb{f}, \mathbb{l} \right\rangle = \oplus_{i=1}^{n} h_i \left(f_i, l_i\right),$$

where $\mathbb{f} = \{f_1, f_2, ..., f_n\}$ and $\mathbb{l} = \{l_1, l_2, ..., l_n\}$ are the elements of H and $h_i > 0$ [5]. We wish consider the boundary conditions which generate self-adjoint finitely many intervals SLP in the direct sum of the Hilbert space H of the 'outer' interval $I = (a_1, b_n)$ having singularities in the interior.

It is well known that self-adjoint boundary conditions can mainly be categorized into separated and coupled. Their canonical forms are also stated in [23]. According to [5, 24, 26, 29, 30] one important condition for the boundary conditions to be self-adjoint is that the determinant of the coupling matrix must be different from zero. We will use this fact in the next two examples to show that the boundary conditions under our consideration are self-adjoint.

Example 4.1.1. Consider transmission conditions at all end points of the intervals given by

$$\begin{aligned}
&a_{11}y_1(a_1)+a_{12}(p_1y_1')(a_1)=0\\
&b_{21}y_1(b_1)+b_{22}(p_1y_1')(b_1)=0\\
&a_{31}y_2(a_2)+a_{32}(p_2y_2')(a_2)=0\\
&b_{41}y_2(b_2)+b_{42}(p_2y_2')(b_2)=0\\
&\vdots\\
&a_{(2n-1),1}y_n(a_n)+a_{(2n-1),2}(p_ny_n')(a_n)=0\\
&b_{2n,1}y_n(b_n)+b_{2n,2}(p_ny_n')(b_n)=0,
\end{aligned}$$

having

$$a_{(2i-1),1}, a_{(2i-1),2}, b_{2i,1}, b_{2i,2}\in\mathbb{R},\ a_{(2i-1),1}\neq 0, a_{(2i-1),2}\neq 0,\ b_{2i,1}\neq 0, b_{2i,2}\neq 0,\ i=1,2,...,n.$$

From these conditions, we can define

$$A_i=(a_{(2i-1),s}),\ 1\le s\le 2$$

in such a way that only the $(2i-1)^{th}$ row is non-zero, and also define

$$B_i=(b_{2i,s}),\ 1\le s\le 2$$

in such a way that only the $(2i)^{th}$ row is non-zero. Then,

$$Rank\left[A_i, B_i\right]=2n$$

and

$$0=A_iEA_i^*=B_iEB_i^*,$$

where E is as defined in Theorem 3.4.1, which satisfies the conditions for self-adjointness. $\triangle$

Example 4.1.2. Let $h_i \in \mathbb{R},\ h_i \neq 0,\ i = 1,2,...,n.$ Suppose we have transmission conditions at 'outer' end points a_1, b_n

$$a_{11}y_1(a_1) + a_{12}(p_1y_1{}')(a_1) = 0, \quad a_{11}, a_{12} \in \mathbb{R},\ a_{11} \neq 0 \neq a_{12},$$
$$b_{2n,1}y_n(b_n) + b_{2n,2}(p_ny_n{}')(b_n) = 0,\ b_{2n,1}, b_{2n,2} \in \mathbb{R},\ b_{2n,1} \neq 0 \neq b_{2n,2},$$

and coupled conditions at interior points with $b_1 = a_2,\ b_2 = a_3, ..., b_{n-1} = a_n,$

$$Y_2(a_2) = e^{i\gamma_1}K_1Y_1(b_1),\ Y_3(a_3) = e^{i\gamma_2}K_2Y_2(b_2), ..., Y_n(a_n) = e^{i\gamma_{n-1}}K_{n-1}Y_{n-1}(b_{n-1}),$$

where $K_j \in M_2(\mathbb{R}),\ -\pi < \gamma_j \leq \pi,\ \ j = 1,2,...,n-1.$ As in example 4.1.1, define

$A_1 = (a_{1s}),\ 1 \leq s \leq 2$ and $B_n = (b_{2n,s}),\ 1 \leq s \leq 2.$ For the first coupled condition, define

$$A_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 0 \\ k_{1,(11)} & k_{1,(12)} \\ k_{1,(21)} & k_{1,(22)} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

For the second coupled condition, define

$$A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ k_{2,(11)} & k_{2,(12)} \\ k_{2,(21)} & k_{2,(22)} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

and so on up to the last coupled condition which will have

$$A_n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B_{n-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ k_{(n-1),(11)} & k_{(n-1),(12)} \\ k_{(n-1),(21)} & k_{(n-1),(22)} \\ 0 & 0 \end{bmatrix}.$$

Since $A_1 E A_1^* = B_n E B_n^* = 0,$ it follows that $h_1 A_1 E A_{1*} = h_n B_n E B_n^*.$ By direct computation, we have that if $h_2 A_2 E A_2^* = h_1 B_1 E B_1^*,$ then $h_2 = h_1 \det K_1$ implying that $\det K_1 \neq 0.$ Similarly, $h_3 A_3 E A_3^* = h_2 B_2 E B_2^*$ if $h_3 = h_2 \det K_2$ implying that $\det K_2 \neq 0.$ Continuing in this manner, we get that $h_n A_n E A_n^* = h_{n-1} B_{n-1} E B_{n-1}^*$ which is equivalent to $h_n = h_{n-1} \det K_{n-1}$ meaning that $\det K_{n-1} \neq 0.$ Therefore, the boundary conditions are self-adjoint for any $h_i \in \mathbb{R},\ h_i \neq 0$ and for any $K_j \in M_2(\mathbb{R}),\ j = 1,2,...n-1$ with $\det K_j \neq 0.$ It can be shown in a similar way that even coupled boundary conditions at both end points a_1, b_n and at other interior points are self adjoint. More examples of similar form can be found in [26] and can easily be adapted to finitely many intervals case.

Example 4.1.3. We consider the problem

$$-(p_i y_i')' + g_i y_i = \lambda r_i y_i + f_i,\ \sum_{i=1}^{n} \left[A_i Y_i(a_i) + B_i Y_i(b_i) \right] = 0 \text{ on } I_i, \qquad (4.15)$$

For $i = 1,2,3$ and the primary fundamental matrices $\Phi_i(x, a_i, \lambda),$ we have

$$G_{11}(x,s,\lambda) = \begin{cases} \Phi_1(x,a_1,\lambda) M_{11}(\lambda) \Phi_1(b_1,s,\lambda), & a_1 \le x < s \le b_1 \\ \Phi_1(x,a_1,\lambda) M_{11}(\lambda) \Phi_1(b_1,s,\lambda) + \Phi_1(x,s,\lambda), & a_1 \le s \le x \le b_1 \end{cases},$$

$$G_{12}(x,s,\lambda) = \Phi_1(x,a_1,\lambda) M_{12}(\lambda) \Phi_2(b_2,s,\lambda),\ \ a_1 \le x \le b_1,\ \ a_2 \le s \le b_2,$$

$$G_{13}(x,s,\lambda) = \Phi_1(x,a_1,\lambda) M_{13}(\lambda) \Phi_3(b_3,s,\lambda),\ \ a_1 \le x \le b_1,\ \ a_3 \le s \le b_3,$$

$$G_{21}(x,s,\lambda) = \Phi_2(x,a_2,\lambda) M_{21}(\lambda) \Phi_1(b_1,s,\lambda),\ \ a_2 \le x \le b_2,\ \ a_1 \le s \le b_1,$$

$$G_{22}(x,s,\lambda)=\begin{cases}\Phi_2(x,a_2,\lambda)M_{22}(\lambda)\Phi_2(b_2,s,\lambda), & a_2\le x<s\le b_2\\ \Phi_2(x,a_2,\lambda)M_{22}(\lambda)\Phi_2(b_2,s,\lambda)+\Phi_2(x,s,\lambda), & a_2\le s\le x\le b_2\end{cases},$$

$$G_{23}(x,s,\lambda)=\Phi_2(x,a_2,\lambda)M_{23}(\lambda)\Phi_3(b_3,s,\lambda),\ \ a_2\le x\le b_2,\ \ a_3\le s\le b_3,$$

$$G_{31}(x,s,\lambda)=\Phi_3(x,a_3,\lambda)M_{31}(\lambda)\Phi_1(b_1,s,\lambda),\ \ a_3\le x\le b_3,\ \ a_1\le s\le b_1,$$

$$G_{32}(x,s,\lambda)=\Phi_3(x,a_3,\lambda)M_{32}(\lambda)\Phi_2(b_2,s,\lambda),\ \ a_3\le x\le b_3,\ \ a_2\le s\le b_2,$$

$$G_{33}(x,s,\lambda)=\begin{cases}\Phi_3(x,a_3,\lambda)M_{33}(\lambda)\Phi_3(b_3,s,\lambda), & a_3\le x<s\le b_3\\ \Phi_3(x,a_3,\lambda)M_{33}(\lambda)\Phi_3(b_3,s,\lambda)+\Phi_3(x,s,\lambda), & a_3\le s\le x\le b_3\end{cases}.$$

Therefore, $G(x,s,\lambda)=\{G_1,G_2,G_3\}$ with $G_i(x,s,\lambda)=\begin{cases}G_{i1}(x,s,\lambda)\\ G_{i2}(x,s,\lambda),\\ G_{i3}(x,s,\lambda)\end{cases}$ so that

$$Y_1(x,\lambda)=\sum_{r=1}^{3}\left[\int_{a_r}^{b_r}G_{1r}(x,s,\lambda)F_r(s)ds\right]$$

$$=\int_{a_1}^{b_1}G_{11}(x,s,\lambda)F_1(s)ds+\int_{a_2}^{b_2}G_{12}(x,s,\lambda)F_2(s)ds+$$
$$\int_{a_3}^{b_3}G_{13}(x,s,\lambda)F_3(s)ds,\qquad a_1\le x\le b_1,$$

$$Y_2(x,\lambda)=\sum_{r=1}^{3}\left[\int_{a_r}^{b_r}G_{2r}(x,s,\lambda)F_r(s)ds\right]$$

$$=\int_{a_1}^{b_1}G_{21}(x,s,\lambda)F_1(s)ds+\int_{a_2}^{b_2}G_{22}(x,s,\lambda)F_2(s)ds+$$
$$\int_{a_3}^{b_3}G_{23}(x,s,\lambda)F_3(s)ds,\qquad a_2\le x\le b_2,$$

and

$$Y_3(x,\lambda)=\sum_{r=1}^{3}\left[\int_{a_r}^{b_r}G_{3r}(x,s,\lambda)F_r(s)ds\right]$$

$$= \int_{a_1}^{b_1} G_{31}(x,s,\lambda)F_1(s)ds + \int_{a_2}^{b_2} G_{32}(x,s,\lambda)F_2(s)ds +$$
$$\int_{a_3}^{b_3} G_{33}(x,s,\lambda)F_3(s)ds, \quad a_3 \le x \le b_3.$$

If $\lambda = 1, p_1(x) = p_2(x) = 1, p_3(x) = x^{-2}, g_i(x) = 0, r_1(x) = 1, r_2(x) = 4,$

$r_3(x) = 2x^{-4}, f_i(x) = x, I_1 = [0,\pi], I_2 = [0,\frac{\pi}{2}], I_3 = [1,2],$

and

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & \pi \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Then (4.15) can equivalently be written as

$$Y_i' = (P_i - \lambda R_i)Y_i + F_i, \sum_{i=1}^{n} \left[A_i Y_i(a_i) + B_i Y_i(b_i)\right] = 0 \qquad \textit{(4.16)}$$

where

$$Y_1 = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}, Y_2 = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}, Y_3 = \begin{bmatrix} y_3 \\ x^{-2}y_3' \end{bmatrix}, P_1 - \lambda R_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, P_2 - \lambda R_2 = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix},$$
$$P_3 - \lambda R_3 = \begin{bmatrix} 0 & x^2 \\ -2x^{-4} & 0 \end{bmatrix}, F_i = \begin{bmatrix} 0 \\ -x \end{bmatrix}.$$

It can be checked that when $f_i = 0$ the resulting homogeneous equation has only the trivial solution and that

$$\Phi_1(x,u) = \begin{bmatrix} \cos(x-u) & \sin(x-u) \\ -\sin(x-u) & \cos(x-u) \end{bmatrix}, \ \Phi_2(x,u) = \begin{bmatrix} \cos\left[2(x-u)\right] & \frac{1}{2}\sin\left[2(x-u)\right] \\ -2\sin\left[2(x-u)\right] & \cos\left[2(x-u)\right] \end{bmatrix}$$

and

$$\Phi_3(x,u)=\begin{bmatrix} 2xu^{-1}-x^2u^{-2} & -xu^2+x^2u \\ 2x^{-2}u^{-1}-2x^{-1}u^{-2} & -x^{-2}u^2+2x^{-1}u \end{bmatrix}$$

are the respective primary fundamental matrices. Clearly, $\det\Phi_i(x,u)=1,\ i=1,2,3$ and we have the following:

$$A_1+B_1\Phi_1(\pi,0)=\begin{bmatrix} -1 & -\pi \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ A_2+B_2\Phi_2\left(\frac{\pi}{2},0\right)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ A_3+B_3\Phi_3(2,1)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix},$$

$$\Phi_1(x,0)=\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix},\ \Phi_1(\pi,s)=\begin{bmatrix} -\cos s & \sin s \\ -\sin s & -\cos s \end{bmatrix},$$

$$\Phi_2(x,0)=\begin{bmatrix} \cos 2x & \frac{1}{2}\sin 2x \\ -2\sin 2x & \cos 2x \end{bmatrix},\ \Phi_2\left(\frac{\pi}{2},s\right)=\begin{bmatrix} -\cos 2s & \frac{1}{2}\sin 2s \\ -2\sin 2s & -\cos 2s \end{bmatrix},$$

$$\Phi_3(x,1)=\begin{bmatrix} 2x-x^2 & -x+x^2 \\ 2x^{-2}-2x^{-1} & -x^{-2}+2x^{-1} \end{bmatrix},\ \Phi_3(2,s)=\begin{bmatrix} 4s^{-1}-4s^{-2} & -2s^2+4s \\ \frac{1}{2}s^{-1}-s^{-2} & -\frac{1}{4}s^2+s \end{bmatrix}.$$

Then, we have that

$$N=\begin{bmatrix} -1 & -\pi & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix},\ N^{-1}=\begin{bmatrix} -1 & -\frac{\pi}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

so that

$$N_1=\begin{bmatrix} -1 & -\frac{\pi}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix},\ N_2=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Thus,

$$M_{11} = N_1(-B_1) = \begin{bmatrix} 0 & \frac{\pi}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$M_{22} = N_2(-B_2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M_{33} = N_3(-B_3) = \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}.$$

and

$$0 = M_{12} = M_{13} = M_{21} = M_{23} = M_{31} = M_{32},$$

implying that

$$0 = G_{12} = G_{13} = G_{21} = G_{23} = G_{31} = G_{32}.$$

It follows that

$$\Phi_1(x,0)M_{11}\Phi_1(\pi,s) = \begin{bmatrix} -\left(\frac{\pi}{2}\cos x + \frac{1}{2}\sin x\right)\sin s & -\left(\frac{\pi}{2}\cos x + \frac{1}{2}\sin x\right)\cos s \\ -\left(\frac{-\pi}{2}\sin x + \frac{1}{2}\cos x\right)\sin s & -\left(\frac{-\pi}{2}\sin x + \frac{1}{2}\cos x\right)\cos s \end{bmatrix}$$

$$\Phi_2(x,0)M_{22}\Phi_2\left(\tfrac{\pi}{2}, s\right) = \begin{bmatrix} -\sin 2x \sin 2s & -\frac{1}{2}\sin 2x \cos 2s \\ -2\cos 2x \sin 2s & -\cos 2x \cos 2s \end{bmatrix} \text{ and}$$

$$\Phi_3(x,1)M_{33}\Phi_3(2,s) = \begin{bmatrix} 2x(s^{-2} - s^{-1}) & x(s^2 - 2s) \\ 2x^{-2}(s^{-2} - s^{-1}) & x^{-2}(s^2 - 2s) \end{bmatrix}.$$

Taking the top right components, we find the following Green's functions:

$$G_{1,(12)}(x,s) = G_{(11),(12)}(x,s) = \begin{cases} -\left(\frac{\pi}{2}\cos x + \frac{1}{2}\sin x\right)\cos s, & 0 \le x < s \le \pi \\ -\left(\frac{\pi}{2}\cos x + \frac{1}{2}\sin x\right)\cos s + \sin(x-s), & 0 \le s \le x \le \pi \end{cases},$$

$$G_{2,(12)}(x,s)=G_{(22),(12)}(x,s)=\begin{cases}-\tfrac{1}{2}\sin 2x\cos 2s, & 0\le x<s\le\tfrac{\pi}{2}\\ -\tfrac{1}{2}\sin 2x\cos 2s+\dfrac{1}{2}\sin\left[2(x-s)\right], & 0\le s\le x\le\tfrac{\pi}{2}\end{cases}$$

$$=\begin{cases}-\tfrac{1}{2}\sin 2x\cos 2s, & 0\le x<s\le\tfrac{\pi}{2}\\ -\tfrac{1}{2}\cos 2x\sin 2s, & 0\le x\le s\le\tfrac{\pi}{2}\end{cases},$$

$$G_{3,(12)}(x,s)=G_{(33),(12)}(x,s)=\begin{cases}x(s^2-2s), & 1\le x<s\le 2\\ x(s^2-2s)-xs^2+x^2s, & 1\le s\le x\le 2\end{cases}$$

$$=\begin{cases}x(s^2-2s), & 1\le x<s\le 2\\ (x^2-2x)s, & 1\le s\le x\le 2\end{cases}.$$

Hence,

$$y_1(x)=-\sum_{r=1}^{3}\left[\int_{a_r}^{b_r}G_{(1r),(12)}(x,s)f_r(s)ds\right]$$

$$=-\int_0^{\pi}G_{(11),(12)}(x,s)f_1(s)ds-\int_0^{\frac{\pi}{2}}G_{(12),(12)}(x,s)f_2(s)ds-\int_1^2 G_{(13),(12)}(x,s)f_3(s)ds$$

$$=-\int_0^{\pi}G_{(11),(12)}(x,s)f_1(s)ds$$

$$=-\int_x^{\pi}(-\tfrac{\pi}{2}\cos x-\tfrac{1}{2}\sin x)s\cos s ds-\int_0^x s\left[(-\tfrac{\pi}{2}\cos x-\tfrac{1}{2}\sin x)\cos s+\sin(x-s)\right]ds$$

i.e. $y_1(x)=-\pi\cos x-x.$

$$y_2(x)=-\sum_{r=1}^{3}\left[\int_{a_r}^{b_r}G_{(2r),(12)}(x,s)f_r(s)ds\right]$$

$$=-\int_0^{\pi}G_{(21),(12)}(x,s)f_1(s)ds-\int_0^{\frac{\pi}{2}}G_{(22),(12)}(x,s)f_2(s)ds-\int_1^2 G_{(23),(12)}(x,s)f_3(s)ds$$

$$=-\int_0^{\frac{\pi}{2}}G_{(22),(12)}(x,s)f_2(s)ds$$

$$=-\int_x^{\frac{\pi}{2}}(-\tfrac{1}{2}\sin 2x)s\cos 2s ds-\int_0^x(-\tfrac{1}{2}\cos 2x)s\sin 2s ds,$$

i.e. $y_2(x) = -\frac{1}{8}\sin 2x - \frac{1}{4}x.$

$$y_3(x) = -\sum_{r=1}^{3}\left[\int_{a_r}^{b_r} G_{(3r),(12)}(x,s)f_r(s)ds\right]$$

$$= -\int_0^{\pi} G_{(31),(12)}(x,s)f_1(s)ds - \int_0^{\frac{\pi}{2}} G_{(32),(12)}(x,s)f_2(s)ds - \int_1^2 G_{(33),(12)}(x,s)f_3(s)ds$$

$$= -\int_1^2 G_{(33),(12)}(x,s)f_3(s)ds$$

$$= -\int_x^2 x(s^3 - 2s^2)ds - \int_1^x (x^2 - 2x)s^2 ds$$

i.e. $y_3(x) = -\frac{1}{12}x^5 + \frac{1}{3}x^2 + \frac{2}{3}x.$ $\triangle$

CHAPTER 5

CONCLUSION

Motivated by the application of the Green's function in partial differential equations and ordinary differential equations and also by the works of other scholars, this research sought to construct the Green's function for finitely many intervals SLPs. The second order scalar problem was converted to its equivalent first order linear system. The size of the matrices in the boundary conditions was found to be $(2n \times 2)$, $n \in \mathbb{Z}^{+}$. We constructed the characteristic function for the homogeneous SLP whose zeroes give the eigenvalues. Using the converted first order linear system, the generalized matrix Green's function has been constructed from which we got the top right component as the Green's function for finitely many intervals SLPs. This Green's function is not symmetric and in general, the whole problem is not self-adjoint. Interesting examples were given which create self-adjoint boundary conditions in the direct sum of the Hilbert space on the 'outer' interval with singularities in the interior.

REFERENCES

[1] Akdoğan, Z., Demirci, M., Mukhtarov, O. S. (2007). Green function of discontinuous boundary value problem with transmission conditions, *Math. Meth. Appl. Sci.*, **30**, 1719-1738.

[2] Ao, J. J., Sun, J., Zhang, M. Z. (2011). The finite spectrum of Sturm-liouville problems with transmission conditions, *Appl. Math. Comp., **218***, 1166-1173.

[3] Mukhtarov, O. S., Kadakal, M., Muhtarov, F. S. (2004). Eigenvalues and normalized eigenfunctions of discontinuous Sturm-Liouville problems with transmission conditions, *Report on Math. Physics, **54***, 41-58.

[4] Mukhtarov, O. S., Kadakal, M., Muhtarov, F. S. (2004). On discontinuous Sturm-Liouville probroblems with transmission conditions, *J. Math. Kyoto Univ. (JMKYAZ), **44 (4)***, 779-798.

[5] Mukhtarov, O. S., Yakubov, S. (2002). Problems for differential equations with transmission conditions, *Applicable Analysis, **81***, 1033-1064.

[6] Tunc, E., Mukhtarov, O. S. (2004). Fundamental Solutions and Eigenvalues of one Boundary Value Problem with Transmission Conditions, *Appl. Math. Comp., **157***, 347-355.

[7] Wang, A., Sun, J., Hao, X., Yao, S. (2009). Completeness of Eigenfunctions of Sturm-Liouville Problems with Transmission Conditions, *Methods and Applications of Analysis, **16***, 299-312.

[8] Kong, Q., Wang, Q. R. (2013). Using time scales to study multi-interval Sturm-Liouville problems with interface conditions, *Results. Math.*, **63**, 451-465.

[9] Sun, J., Wang, A. (2008). Sturm-Liouville operators with interface conditions, *The progress of research for Math, Mech, Phys. and High Tech. Science Press, Beijing **12***, 513-516.

[10] Zettl, A. (1968). Adjoint and self-adjoint problems with interface conditions, *SIAM J. Applied Math*, **16**, 851-859.

[11] Kong, L., Kong, Q., Kwong, M. K., Wong, J. S. W. (2013). Linear Sturm-Liouville problems with multi-point boundary conditions, *Math. Nachr.*, **286**, 1167-1179.

[12] Loud, W. S. (1968). Self-Adjoint multi-point boundary value problems, *Pacific J. Math., **24***, 303-317.

[13] Zettl, A. (1966). The lack of self-adjointness in three point boundary value problems, *Proc. Amer.Math. Soc., **17***, 368-371.

[14] Pokornyi, Y. V., Borovskikh, A. V. (2004). Differential Equations on Networks (Geometric Graphs), *Journal of Mathematical Sciences, **119 (6)**, 691-718.*

[15] Pokornyi, Y. V., Pryadiev, V. L. (2004). Sturm-Liouville Theory on Spacial Networks, *Journal of Mathematical Sciences, **119 (6)**, 788-834.*

[16] Demir, O., Akdoğan, Z. (2010). Green function of one discontinuous Sturm-Liouville problem, *Bulletin of Pure and Applied Sciences, **29E (2)***, 435-452.

[17] Kadakal, M., Muktarov, O. S. (2007). Sturm-Liouville Problems with discontinuities at two points, *Comput. Math. Appl., **54***, 1367-1379.

[18] Bateman, H. (1959). *Partial Differential Equations of Mathematical Physics,* Cambridge: University Press.

[19] Stakgold, I. (2000). *Boundary Value Problems of Mathematical Physics, I,II,* Philadelphia: Society for Industrial and Applied Mathematics.

[20] Coddington, E. A., Levinson, N. (1955). *Theory of ordinary differential equations,* New York: McGraw-Hill.

[21] Naimark, M. A. (1967). *Linear Differential Operators-Part I: Elementary Theory of Linear Differential Operators,* New York: Frederick Ungar Publishing Co., Inc.

[22] Neuberger, J. W. (1960). Concerning Boundary value problems, *Pacific Journal of Mathematics, **10 (4)***, 1385-1392.

[23] Zettl, A. (2005). Sturm-Liouville Theory, *American Mathematical Society, Mathematical Surveys and Monographs.*

[24] Wang, A., Zettl, A. (2013). Green's function for two-interval Sturm-Liouville problems, *Electronic Journal of Differential Equations,* ***76***, 1-13.

[25] Ibrahim, S. E. (1999). On boundary conditions for Sturm-Liouville differential operators in the direct sum spaces, *Rocky Mountains Journal of Mathematics,* ***29***, 873-892.

[26] Wang, P. A., Zettl, A. (2005). Self-Adjoint Sturm-Liouville Problems with discontinuous boundary conditions, *Methods and Applications of Analysis,* ***22***, 37-66.

[27] Chicone, C. (1999). *Ordinary Differential Equations with Applications*, New York: Spring Science & Business Media.

[28] Wang, A., Ridenhour, J., Zettl, A. (2012). Construction of reqular and singular Green's function, *Proceedings of the Royal Society of Edinburg Section A Mathematics*, 171-198.

[29] Sun, J., Wang, A. P., Zettl, A. (2007). Two-Interval Sturm-Liouville Operators in Direct Sum Spaces with inner product multiples, *Results in Mathematics,* ***50***, 155-168.

[30] Wang, A. P., Sun, J., Zettl, A. (2007). Two-Interval Sturm-Liouville in Modified Hilbert Spaces, *Journal of Mathematical Analysis and Applications,* ***328***, 390-399.