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SOME APPLICATIONS OF GROEBNER BASES TO NONLINEAR DIFFERENTIAL EQUATIONS

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DECLARATION

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ABSTRACT

SOME APPLICATIONS OF GROEBNER BASES TO NONLINEAR DIFFERENTIAL EQUATIONS
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In this thesis, we study various applications of the Groebner bases method to the problems in nonlinear dynamical systems such as determining the existence and computing symbolically the jerky dynamics of given nonlinear dynamical systems, constructing Lyapunov functions, determining critical levels and investigating robustness problem with structure uncertainty of nonlinear dynamical systems. Numerical examples are given for illustration. All computations are performed by package Mathematica.

KEYWORDS: Groebner base, Jerky dynamics, Nonlinear dynamical system, Lyapunov function, Critical level, Robustness problem with structure uncertainty.

ÖZET

GRÖBNER BAZLARININ LİNEER OLMAYAN DİFERANSİYEL DENKLEMLERE BAZI UYGULAMALARI
YÜKSEK LİSANS TEZİ
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Bu tezde Gröbner bazları yönteminin, verilen lineer olayan dinamik sistemlerin jerk dinamiğinin varlığının belirlenmesi ve var olması durumunda sembolik olarak hesaplanması, Lyapunov fonksiyonlarının kurulması, kritik seviyelerin belirlenmesi ve yapısal belirsizliği olan sağlamlılık problemlerinin araştırılması gibi problemlere uygulamalarını çalıştık. Açıklık getirmek için sayısal örnekler verdik. Bütün hesaplamalar Mathematica programıyla yapıldı.

ANAHTAR KELİMELER: Gröbner bazı, Jerki dinamik, Lineer olmayan dinamik sistem, Lyapunov fonksiyonu, Kritik seviye, Yapısal belirsizliği olan sağlamlılık.

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1. INTRODUCTION

Bruno Buchberger introduced Groebner bases in his PhD thesis in 1965 (Buchberger, 1965). He named them after his PhD supervisor W. Gröbner. The similar concepts were also discovered by F.S.Macaulay and H.Hironaka. (Macaulay, 1927) used some of these ideas at the beginning of the 20th. century to determine certain invariants of ideals in polynomial rings and (Hironaka, 1964) used similar ideas independently to study power series rings in 1964.

The main idea in the Groebner bases theory can be described as a generalization of the theory of polynomials in one variable. In the polynomial ring $k[x]$, where k is a field, any ideal I can be generated by a single element, namely the greatest common divisor of the elements of I. Given any set of generators $\{f_1, ..., f_s\} \subseteq k[x]$ for I, using Euclidean division algorithm one can compute a single polynomial $d = gcd(f_1, ..., f_s)$ such that $I = \langle f_1, ..., f_s \rangle = \langle d \rangle$. Then a polynomial $f \in k[x]$ is in I if and only if the remainder of the division of f by d is zero. Any ideal of $k[x_1, ..., x_n]$ is generated by a Groebner basis. A polynomial $f \in k[x_1, ..., x_n]$ is in I if and only if the remainder of the division of f by the polynomials in the Groebner basis is zero.

The great contribution of Bruno Buchberger to the Groebner basis theory is the Buchberger's algorithm (Buchberger, 1976) which is used to compute Groebner bases. With the development of computer algebra systems Buchberger's algorithm has been improved and refined over forty years (Buchberger, 1979, 1985; Caboara et al., 2004; Faugere et al., 1993; Gebrauer and Möller, 1988; Giovini et al., 1991). Nowadays there are several packages such as CoCoA, Magma, Reduce, Macsyma, Singular, Maple, Mathematica, Drive etc. for implementations of Buchberger's algorithm. The algorithmic algebra of Groebner bases has wide-ranging applications in theoretical physics, applied science and engineering. Groebner bases are certain finite sets of multivariate polynomials. The main reason for the success of Groebner bases is that many problems in mathematics, science and engineering can be represented by multivariable polynomials e.g., ideals, and many problems in polynomial ideal theory can be solved by easy algorithms after transforming the polynomial sets involved in the specification of the problems into Groebner basis form.

In this thesis, we study various applications of Groebner bases method in nonlinear dynamical systems and in some common problems of Lyapunov stability theory by giving concrete examples of 2 and 3-dimensional autonomous systems of nonlinear differential equations. The suggested methods require the nonlinearities of the systems and the Lyapunov functions to be of polynomial type.

In each application of the Groebner bases method, we transform the given system of differential equations into a system of nonlinear algebraic equations in 2 or 3 variables. Then using the Groebner bases method we solve this system. By solving such a system we mean triangulating it, i.e., elimination of variables as far as possible. Elimination consists in finding the elements of an ideal that belong to a certain subring. For example we may wish to find $I \cap k[S]$ for an ideal $I \subseteq k[S,T]$ where S, T are sets of variables. One way of doing elimination is to compute I so called Groebner base of an ideal with respect to the purely lexicographical term-ordering. If the computed ideal is zero dimensional then we get the solution.

In Chapter 2, we present a brief summary of the basic concepts and results of Groebner bases theory.

In Chapter 3, we deal with some applications of the Groebner bases method in nonlinear systems theory. We, first consider the problem of checking the existence and computing the jerky dynamics of the given nonlinear systems. Here, we use the idea introduced in (Linz, 1997; Eichhorn et al., 1998). By applying the suggested method we determine jerky dynamics and also find corresponding jerk functions of some 3-dimensional nonlinear systems.

In the next, we consider the applications of the Groebner bases method to some problems in Lyapunov stability theory. Here, Groebner bases are used to solve following optimization problems arising from the theory of local Lyapunov functions; choosing the parameters in Lyapunov functions for nonlinear dynamical systems in an optimal way (Forsman, 1993), determining the critical levels of the given local Lyapunov functions (Forsman and Glad, 1990; Forsman, 1993), investigating the performance robustness of nonlinear systems (Forsman, 1991). To be illustrative 2- and 3-dimensional nonlinear dynamical systems are given as examples.

All necessary calculations are performed using computer algebra system Mathematica.

2. A BACKGROUND ON GROEBNER BASIS

2.1 Basic Concepts and Some Facts

Definition 2.1.1. *(Adams and Loustaunau, 1994) Let $\mathbb{Z}_{\geq 0}$ denote the nonnegative integers. Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a power vector in $\mathbb{Z}^n_{\geq 0}$, and let $x_1, x_2, ..., x_n$ be any n variables. Then a monomial x^α in $x_1, x_2, ..., x_n$ is defined as the product $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$, $\alpha = mdeg(x^\alpha)$, and $|\alpha| = \sum_{i=1}^n \alpha_i$.*

The mapping from the monomials of $k[x_1, ..., x_n]$ to $\mathbb{Z}^n_{\geq 0}$ is referred as the "multidegree" and $|\alpha|$ as the "total degree".

Definition 2.1.2. *(Adams and Loustaunau, 1994) A multivariate polynomial f in $x_1, ..., x_n$ with coefficients in a field k is a finite linear combination $f(x_1, ..., x_n) = \sum_\alpha c_\alpha x^\alpha$, of monomials x^α and coefficients $c_\alpha \in k$. The total degree of the polynomial f is defined as the maximum of $|\alpha|$ such that $c_\alpha \neq 0$.*

Definition 2.1.3. *(Adams and Loustaunau, 1994) "A ring of polynomials in the variables $x_1, ..., x_n$" is the set of all polynomials in $x_1, ..., x_n$ with coefficients from a ring k together with the usual addition and multiplication of polynomials. This ring is denoted by $k[x_1, ..., x_n]$.*

We notice that a polynomial ring is a commutative ring.

Definition 2.1.4. *A nonempty subset I of $k[x_1, ..., x_n]$ is called a left (right) ideal if for all $f, g \in I$ and for all $h \in k[x_1, ..., x_n]$, $f - g \in I$, $hf \in I$, $(f - g \in I$, $fh \in I)$.*

A nonempty subset I of $k[x_1, ..., x_n]$ is a (two-sided) ideal if I is both a left and a right ideal of the ring k.

Definition 2.1.5. *Let $F = \{f_1, ..., f_s\}$ be a set of multivariate polynomials. Then the ideal generated by F, denoted by $I = \langle F \rangle$, is given by*

$$I = \{\sum_{i=1}^{s} h_i f_i | h_1, ..., h_s \in k[x_1, x_2, ..., x_n]\}$$

The polynomials $f_1, ..., f_s$ are called a basis for the ideal. Since F is finite the ideal is said to be finitely generated.

Definition 2.1.5 tells us that the ideal generated by a family of generators consists of the set of linear combinations of these generators with polynomial coefficients. Furthermore, two polynomials are equivalent with respect to an ideal if their difference belongs to the ideal.

Definition 2.1.6. *(Dummit and Foote, 2004) A commutative ring k with unity is called "Noetherian" if every ideal of k is finitely generated.*

Theorem 2.1.7. *(Hilbert's Basis Theorem) If k is a Noetherian ring then so is the polynomial ring $k[x_1, ..., x_n]$.*

Proof. See (Dummit and Foote, 2004) □

Corollary 2.1.8. *Every ideal in the polynomial ring $k[x_1, ..., x_n]$ with coefficients from a field k is finitely generated.*

Proof. See (Dummit and Foote, 2004) □

If I is an ideal in $k[x_1, ..., x_n]$ generated by a set F of polynomials, Corollary 2.1.8 states that I is generated by a finite number of the polynomials from the set F.

2.2 The Algebraic Geometry Dictionary

Definition 2.2.1. *(Cox et al., 1992) Given a field k and a positive integer n, we define the n-dimensional "affine space" over k to be the set*

$$A^n = \{(a_1, ..., a_n) : a_1, ..., a_n \in k\}$$

For example, $\mathbb{R}$ and $\mathbb{R}^n$ are affine spaces.

Definition 2.2.2. *(Dummit and Foote, 2004) If $x_1, ..., x_n$ are independent variables over k then the polynomials f in $k[x_1, ..., x_n]$ can be viewed as k-valued functions $f : A^n \to k$ on A^n by evaluating f at the points in A^n:*

$$f : (x_1, ..., x_n) \to f(x_1, ..., x_n) \in k$$

This gives a ring of k-valued functions on A^n, denoted by $k[A^n]$ and called the "coordinate ring" of A^n.

Definition 2.2.3. *(Dummit and Foote, 2004) Let S be a subset of functions in the coordinate ring $k[A^n]$ then $Z(S)$, a subset of affine space is defined as*

$$Z(S) = \{(a_1, a_2, ..., a_n) \in A^n |\ f(a_1, a_2, ..., a_n) = 0 \, for \, all \ f \in S\}$$

where $Z(\emptyset) = A^n$.

Definition 2.2.4. *(Dummit and Foote, 2004) A subset V of A^n is called an "affine algebraic set" if V is the set of common zeros of some set S of polynomials, i.e., if $V = Z(S)$ for some $S \subseteq k[A^n]$. It is called the "locus" of S in A^n.*

Example 2.2.5. *(1) The one point subsets of A^n, for any n are affine algebraic sets since $\{(a_1, a_2, ..., a_n)\} = Z(x_1 - a_1, x_2 - a_2, ..., x_n - a_n)$. More generally, any finite subset of A^n is affine algebraic set.*

(2) If f is a nonconstant polynomial then the algebraic set $Z(f)$ is called a "hypersurface" in A^n. Conic sections are familiar algebraic sets in $\mathbb{R}^2$.

Proposition 2.2.6. *(Dummit and Foote, 2004) If S and T are any subsets of $k[A^n]$. Then*

(1) If $S \subseteq T$ then $Z(T) \subseteq Z(S)$.

(2) $Z(S) = Z(I)$, where $I = \langle S \rangle$ is the ideal in $k[A^n]$ generated by the subset S.

(3) $Z(S) \cap Z(T) = Z(S \cup T)$.

More generally, if $\{S_j\}$ is any collection of subsets of $k[A^n]$ then

$$\cap Z(S) = Z(\cup S_j)$$

(4) $Z(I) \cup Z(J) = Z(IJ)$, where I and J are ideals and IJ is their product. In other words, the union of two algebraic sets is again an algebraic set.

(5) $Z(0) = A^n$ and $Z(1) = \emptyset$ (here 0 and 1 denote constant functions).

The second statement of Proposition 2.2.6 says that every affine algebraic set is the algebraic set corresponding to an "ideal" of the coordinate ring. Thus, we may consider

$$Z : \{\text{ideals of } k[A^n]\} \to \{\text{affine algebraic sets in } A^n\}$$

Since every ideal I in the Noetherian ring $k[x_1, ..., x_n]$ is finitely generated, say $I = \langle f_1, ..., f_s \rangle$, the third statement of Proposition 2.2.6 implies that $Z(I) = Z(f_1) \cap Z(f_2) \cap ... \cap Z(f_s)$.

If V is an algebraic set in affine n-affine then there may be many ideals I such that $V = Z(I)$. For instance, in affine 2-space over $\mathbb{R}$ the y-axis is the locus of the ideal $\langle x \rangle$ of $\mathbb{R}[x, y]$, and also for locus of $\langle x^2 \rangle$, $\langle x^3 \rangle$, etc. More generally, the zeros of any polynomial are the same as the zeros of all its positive powers, and so, $Z(I) = Z(I^k)$ for all $k \geq 1$.

So, the ideal whose locus determines a particular algebraic set V is not unique. On the other hand, there is a unique largest ideal that determines V, given by the set of all polynomials that vanish on V. For any subset A of A^n define

$$\mathbf{I}(A) = \{f \in k[x_1, ..., x_n] | \ f(a_1, a_2, ..., a_n) = 0 \text{ for all } (a_1, a_2, ..., a_n) \in A$$

$\mathbf{I}(A)$ is an "ideal" and is unique largest ideal of functions that are identically zero on A. This defines a correspondence

$$\mathbf{I} : \{\text{subsets in } A^n\} \to \{\text{ideals of } k[A^n]\}$$

Example 2.2.7. *Over any field k, the ideal of functions vanishing at $(a_1, a_2, ..., a_n) \in A^n$ is a maximal ideal since it is the kernel of the surjective ring homomorphism from $k[x_1, ...x_n]$ to the field k given by evaluation at $(a_1, a_2, ..., a_n)$. It follows that*

$$\mathbf{I}((a_1, a_2, ..., a_n)) = (x_1 - a_1, x_2 - a_2, ..., x_n - a_n)$$

Proposition 2.2.8. *(Dummit and Foote, 2004) Let S and T be subsets of A^n. Then*

(1) If $S \subseteq T$ then $\mathbf{I}(T) \subseteq \mathbf{I}(S)$.

(2) $\mathbf{I}(S \cup T) = \mathbf{I}(S) \cap \mathbf{I}(T)$.

(3) $\mathbf{I}(\emptyset) = k[x_1, ..., x_n]$ and if k is infinite, $\mathbf{I}(A^n) = 0$.

(4) If A is any subset of A^n then $A \subseteq Z(\mathbf{I}(A))$, and if I is any ideal $I \subseteq \mathbf{I}(Z(I))$.

(5) If $V = Z(I)$ is an affine algebraic set then $V = Z(\mathbf{I}(V))$, and if $I = \mathbf{I}(A)$ then $\mathbf{I}(Z(I)) = I$, i.e., $Z(\mathbf{I}(Z(I))) = Z(I)$ and $\mathbf{I}(Z(\mathbf{I}(A))) = \mathbf{I}(A)$.

Definition 2.2.9. *If $V \subseteq A^n$ is an affine algebraic set the quotient ring $k[A^n]/\mathbf{I}(V)$ is called the coordinate ring of V, and is denoted by $k[V]$.*

Definition 2.2.10. *An ideal of $k[x_1, ..., x_n]$ is a prime ideal if whenever $f, g \in k[x_1, ..., x_n]$ and $fg \in I$ then either $f \in I$ or $g \in I$.*

Definition 2.2.11. *A polynomial ideal I is a maximal ideal if $I \neq k[x_1, ..., x_n]$ and whenever $I \subseteq J$ for some ideal J, then either $J = I$ or $J = k[x_1, ..., x_n]$.*

Definition 2.2.12. *Let I be an ideal in $k[x_1, ..., x_n]$. The "radical" of I, denoted by $rad\ I$, is the collection of elements in k some powers of which lie in I, i.e.,*

$$rad\ I = \{f \in k[x_1, ..., x_n] | f^m \in I \text{ for some } m \geq 1\}$$

.

Proposition 2.2.13. *Let I be an ideal in $k[x_1, ..., x_n]$. Then $rad\ I$ is an ideal containing I.*

Proof. It follows directly from the definition of $rad\ I$. □

Proposition 2.2.14. *The radical of a proper ideal I is the intersection of all prime ideals containing I.*

Proof. See (Dummit and Foote, 2004) □

Proposition 2.2.15. *An ideal is called a radical ideal if $rad\ I = I$.*

Proof. See (Cox et al., 1992) □

Corollary 2.2.16. *Prime (and also maximal) ideals are radical.*

Proof. If P is a prime ideal then P is the intersection of all the prime ideals containing P, so $P = rad\ P$ by Proposition 2.2.14. □

Theorem 2.2.17. *(Hilbert's Nullstellensatz) Let k be an algebraically closed field. Then $\mathbf{I}(Z(I)) = rad\ I$ for every ideal I of $k[x_1, ..., x_n]$.*

Proof. See (Dummit and Foote, 2004) □

Definition 2.2.18. *let V be a nonempty affine algebraic set. Then V is called "irreducible" if it can not be written as $V = V_1 \cup V_2$, where V_1 and V_2 are proper algebraic sets in V. An irreducible algebraic set is called an "affine variety".*

2.3 Monomial Orderings and Groebner Bases

Groebner bases are special kinds of generating sets for polynomial ideals and have had interesting theoretical as well as computational applications.

2.3.1 Monomial Orderings and The Division Algorithm

The multidegree in Definition 2.1.1 defines a one-to-one mapping between the elements of $\mathbb{Z}^n_{\geq 0}$ and the monomials in $k[x_1, ..., x_n]$. Due to this fact every ordering $\succ$ we establish on $\mathbb{Z}^n_{\geq 0}$ automatically carries over to an ordering of the monomials in $k[x_1, ..., x_n]$. If $\alpha \succ \beta$ we will also say that $x^\alpha \succ x^\beta$.

Definition 2.3.1. *A monomial ordering on $k[x_1, ..., x_n]$ is a relation $\succ$ on $\mathbb{Z}^n_{\geq 0}$, such that:*

(i) The relation $\succ$ is a total ordering.

(ii) If $\alpha \succ \beta$, and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma \succ \beta + \gamma$.

(iii) The relation $\succ$ is a well ordering (every nonempty subset has a least element).

Definition 2.3.2. *The followings are the most commonly used monomial orderings:*

Lexicographic Order: $\alpha \succ_{lex} \beta$ if in $\alpha - \beta$ the leftmost non-zero entry is positive.

Graded Lex Order: $\alpha \succ_{grlex} \beta$ if $|\alpha| > |\beta|$, or if $|\alpha| = |\beta|$ then $\alpha \succ_{lex} \beta$.

Here $|\alpha| = \sum_{i=1}^{n} \alpha_i$, and similarly for $|\beta|$.

Graded Reverse Lex Order: $\alpha \succ_{grevlex} \beta$ if $|\alpha| > |\beta|$, or if $|\alpha| = |\beta|$ then in $\alpha - \beta$ the rightmost non-zero entry is negative.

Example 2.3.3. *For the set*

$$\{xy^2z^3, x^4, y^3, x^3y^2, xy^3z\}.$$

the above orderings are:

lex: $x^4 \succ x^3y^2 \succ xy^3z \succ xy^2z^3 \succ y^3$

grlex: $xy^2z^3 \succ x^3y^2 \succ xy^3z \succ x^4 \succ y^3$

grevlex: $xy^2z^3 \succ xy^3z \succ x^3y^2 \succ x^4 \succ y^3$

Definition 2.3.4. *Let $f = \sum_\alpha c_\alpha x^\alpha \in k[x_1, ..., x_n]$ be a nonzero polynomial and let $\succ$ be a monomial order.*

- *The leading exponent of f is*

$$LE(f) := \max(\alpha \in \mathbb{Z}^n_{\geq 0} : c_\alpha \neq 0).$$

Here the maximum is taken with respect to $\succ$.

- *The leading coefficient of f is*

$$LC(f) := c_{LE(f)} \in k.$$

- *The leading monomial of f is*

$$LM(f) := x^{LE(f)}.$$

(with coefficient 1)

- *The leading term of f is*

$$LT(f) := LC(f).LM(f).$$

Definition 2.3.5. *Fix a monomial ordering on the polynomial ring* $k[x_1, ..., x_n]$. *If I is an ideal in* $k[x_1, ..., x_n]$, *the "ideal of leading terms", denoted by* $LT(I)$, *is the ideal generated by the leading terms of all the elements in the ideal, i.e.,* $LT(I) = \langle LT(f) | f \in I \rangle$.

We note that leading term of a polynomial depends on the choice of the ordering. For instance, $LT(5x^6y^2 + xy^4) = 5x^6y^2$ with multidegree $(6, 2)$ if $x \succ y$ on the other hand, $LT(5x^6y^2 + xy^4) = xy^4$ with multidegree $(1, 4)$ if $y \succ x$. Likewise, the ideal of leading terms $LT(I)$ of an ideal I in general, depends on the ordering used.

By definition, the ideal $LT(I)$ is generated by monomials. Such ideals are called "monomial ideals".

2.3.2 Groebner Bases

In the one variable case, leading terms are used in the Division Algorithm to reduce one polynomial g modulo another polynomial f to get a unique remainder r, and this remainder is zero if and only if g is contained in the ideal $\langle f \rangle$. As we shall see, the situation is more complicated for polynomials in more than one variable.

Definition 2.3.6. *A "Groebner basis" for an ideal I in the polynomial ring* $k[x_1, ..., x_n]$ *is a finite set of generators* $\{g_1, ..., g_t\}$ *for I whose leading terms generate the ideal for all leading terms in I, i.e.,* $I = \langle g_1, ..., g_t \rangle$ *and* $LT(I) = \langle LT(g_1), ..., LT(g_t) \rangle$.

We note that a Groebner basis is a set of generators for I depending on the choice of ordering, i.e., every element in I is a linear combination of the generators. It is not a basis in the sense of vector spaces.

One of the most important properties of a Groebner basis is that every polynomial g can be written "uniquely" as the sum of an element in I and a remainder r obtained by a general polynomial division.

General Polynomial Division

We fix a monomial ordering on $k[x_1, ..., x_n]$, and suppose $g_1, ..., g_t$ is a set of nonzero polynomials in $k[x_1, ..., x_n]$. We let f be a polynomial in $k[x_1, ..., x_n]$ and start with a set of quotients $q_1, ..., q_t$ and a remainder r initially all equal to zero, and successively test whether the leading term of the dividend f is divisible by the leading terms of the divisors $g_1, ..., g_t$ in that order. Then,

(1) If $LT(f)$ is divisible by $LT(g_j)$, say $LT(f) = c_j LT(g_j)$, add c_j to the quotient q_j, replace f by the dividend $f - c_j g_j$, and reiterate the entire process.

(2) If the leading term of dividend f is not divisible by any of the leading terms $LT(g_1), ..., LT(g_t)$, add the leading term of f to the remainder r, replace f by the dividend $f - LT(f)$, and reiterate the entire process.

The process terminates when the dividend is zero and results in a set of quotients $q_1, ..., q_t$ and a remainder r with

$$f = q_1 g_1 + ... + q_t g_t + r$$

Each $q_i g_i$ has multidegree less than or equal to the multidegree of f and the remainder r has the property that no nonzero term in r is divisible by any of the leading terms $LT(g_1), ..., LT(g_t)$.

The next theorem shows that if we use a Groebner basis for the ideal I then we obtain a "unique" remainder, and we can use this to determine whether a polynomial f is an element of the ideal I.

Theorem 2.3.7. *If $\{g_1, g_2, ..., g_t\}$ is a Groebner basis for the nonzero ideal I in $k[x_1, ..., x_n]$ for a fixed monomial ordering. Then*

(a) $\forall f \in k[x_1, ..., x_n]$ can be written uniquely in the form $f = f_I + r$, where $f_I \in I$ and no nonzero monomial term of the remainder r is divisible by the leading terms $LT(g_1), ..., LT(g_t)$.

(b) Both f_I and r can be computed by general polynomial division by $g_1, ..., g_t$ and are independent of the order in which these polynomials are used in the division.

(c) $f \in I$ if and only if $r = 0$.

Proof. See (Dummit and Foote, 2004) □

Proposition 2.3.8. *Fix a monomial ordering on $k[x_1, ..., x_n]$ and let I be a nonzero ideal in $k[x_1, ..., x_n]$. Then*

(1) If $g_1, ..., g_t$ are any elements of I such that $LT(I) = \langle LT(g_1), ..., LT(g_t)\rangle$, then $\{g_1, ..., g_t\}$ is a Groebner basis for I.

(2) The ideal I has a Groebner basis.

Proof. Let $g_1, ..., g_t \in I$ with $LT(I) = \langle LT(g_1), ..., LT(g_t)\rangle$. In order to prove (1) we need to show that $g_1, ..., g_t$ generate the ideal I. If $f \in I$ then using general division algorithm we write $f = \sum_{i=1}^{n} q_i g_i + r$ where no nonzero term in the remainder r is divisible by any $LT(g_i)$. Since $f \in I$, also $r \in I$, we have $LT(r)$ is in $LT(I)$. But, then $LT(r)$ would be divisible by one of $LT(g_1), ..., LT(g_t)$, which is a contradiction unless $r = 0$. Hence $f = \sum_{i=1}^{n} q_i g_i$ and $g_1, ..., g_t$ generate I, so are a Groebner basis for I.

For (2) we note that the ideal $LT(I)$ of leading terms of any ideal I is a monomial ideal generated by all the leading terms of the polynomials in I. We know that a finite number of these leading terms suffice to generate the ideal $LT(I)$, say $LT(I) = \langle LT(h_1), ..., LT(h_m)\rangle$ for some $h_1, ..., h_m \in I$. By (1), the polynomials $h_1, ..., h_m$ are a Groebner basis of I.

□

Proposition 2.3.8 proves that Groebner bases always exist.

S-Polynomials

Let f_1, f_2 be two polynomials in $k[x_1, ..., x_n]$ and x^γ be the least common multiple of the monomial terms $LT(f_1)$ and $LT(f_2)$ then we can cancel the leading terms by taking the difference

$$S(f_1, f_2) = \frac{x^\gamma}{LT(f_1)} f_1 - \frac{x^\gamma}{LT(f_2)} f_2$$

Lemma 2.3.9. *Suppose $f_1, ..., f_m \in k[x_1, ..., x_n]$ are polynomials with the same multidegree β and that their linear combination $l = c_1 f_1 + ... + c_m f_m$ with constants $c_i \in k$ has strictly smaller multidegree. Then*

$$l = \sum_{i=2}^{m} c_i S(f_{i-1}, f_i), \textit{for some constants } c_i \in k$$

Proof. We write, $f_i = u_i f_i'$, where f_i' is a monicpolynomial of degree β and $u_i \in k$. Then

$$\begin{aligned} l &= \sum c_i u_i f_i' = c_1u_1(f_1' - f_2') + (c_1u_1 + c_2u_2)(f_2' - f_3') + ... + \\ &(c_1u_1 + ... + c_{m-1}u_{m-1})(f_{m-1}' - f_m') + (c_1u_1 + ... + c_mu_m)f_m' \end{aligned}$$

We note that $f_{i-1}' - f_i' = S(f_{i-1}, f_i)$. l and each $f_{i-1}' - f_i'$ has multidegree strictly smaller than β, so we get $c_1u_1 + ... + c_mu_m = 0$. Hence, the last term on the right side is zero.

□

Proposition 2.3.10. *(Buchberger's Criterion) Fix a monomial ordering on $k[x_1, ..., x_n]$. If $I = \langle g_1, ..., g_t\rangle$ is a nonzero ideal in $k[x_1, ..., x_n]$, then $G = \{g_1, ..., g_t\}$ is a Groebner basis for I if and only if $S(g_i, g_j) \equiv 0$ mod G for $1 \leq i < j \leq t$.*

Proof. If $\{g_1, ..., g_t\}$ is a Groebner basis for I then since each $S(g_i, g_j)$ is an element of I, we have $S(g_i, g_j) \equiv 0$ mod G by Theorem 2.3.7.

We now assume that $S(g_i, g_j) \equiv 0$ mod G for $1 \leq i < j \leq t$ and take any element $f \in I$. To show that G is a Groebner basis we need to show that $\langle LT(g_1), ..., LT(g_t)\rangle$ contains $LT(f)$. Because $f \in I$ it can be written as $f = \sum_{i=1}^t p_i g_i$ for some polynomials $p_1, ..., p_t$. This representation is not unique. We choose one among all these representations for which the largest multidegree of any summand is minimal, say β. Then multidegree of $f \leq \beta$. So, we write

$$\begin{aligned} f &= \sum_{i=1}^{t} p_i g_i = \sum_{multidegree\ p_ig_i=\beta} p_i g_i + \sum_{multidegree\ p_ig_i<\beta} p_i g_i \\ &= \sum_{multidegree\ p_ig_i=\beta} LT(p_i)g_i + \sum_{multidegree\ p_ig_i=\beta} (p_i - LT(p_i)g_i) \\ &+ \sum_{multidegree\ p_ig_i<\beta} p_i g_i \end{aligned} \tag{2.1}$$

We now suppose that multidegree of $f < \beta$. Then the multidegree of the first sum is strictly smaller than β since the multidegree of the second two sums is strictly smaller than β. Let $b_i \in k$ denote the constant coefficient of the monomial term $LT(p_i)$ then $LT(p_i) = b_i p_i'$ where p_i' is a monomial. Applying Lemma 2.3.9 to $\sum b_i(p_i' g_i)$ we write the first sum above $\sum c_i S(p_{i-1}' g_{i-1}, p_i' g_i)$ with multidegree of $(p_{i-1}' g_{i-1})$ = multidegree of $(p_i' g_i) = \beta$. Let $\gamma_{i-1,i}$ be the multidegree of the monic least common multiple of $LT(g_{i-1})$ and $LT(g_i)$. Then, $S(p_{i-1}' g_{i-1}, p_i' g_i)$ is just $S(g_{i-1}, g_i)$ multiply by the monomial of degree $\beta - \gamma_{i-1,i}$. The polynomial $S(g_{i-1}, g_i)$ has multidegree less than $\gamma_{i-1,i}$ and by assumption, $S(g_{i-1}, g_i) =$ mod G. That is, after general polynomial division of $S(g_{i-1}, g_i)$ by $g_1, ..., g_t$ each $S(g_{i-1}, g_i)$

can be written as a sum $\sum t_i g_i$ with multidegree of $t_i g_i < \gamma_{i-1,i}$. So, each $S(p'_{i-1}g_{i-1}, p'_i g_i)$ is a sum $\sum t'_i g_i$ with multidegree of $t'_i g_i < \beta$. Hence, all the sums on the right hand side of Equation 2.1 can be written as a sum of the terms of the form $q_i g_i$ with polynomials q_i satisfying multidegree of $q_i g_i < \beta$. This contradicts the minimality of β and shows that multidegree of $f = \beta$, i.e., the leading term of f has multidegree β.

When we take the terms in Equation 2.1 of multidegree β we see that

$$LT(f) = \sum_{multidegree\ p_i g_i = \beta} LT(p_i)LT(g_i)$$

. So, indeed $LT(f) \in \langle LT(g_1), ..., LT(g_t)\rangle$. Therefore, $G = \{g_1, ..., g_t\}$ is a Groebner basis for the ideal I.

□

Buchberger's Algorithm

Buchberger's Criterion can be used to provide an algorithm to find a Groebner basis for an ideal I, as follows. If $I = \langle g_1, ..., g_t\rangle$ and each $S(g_i, g_j)$ leaves a remainder of zero when divided by $G = \{g_1, ..., g_t\}$ using general polynomial division then G is a Groebner basis. Otherwise, $S(g_i, g_j)$ has a nonzero remainder r. Increase G by appealing the polynomial $g_{t+1} = r : G' = \{g_1, ..., g_t, g_{t+1}\}$ and begin again. We notice that this is again a set of generators for I since $g_{t+1} \in I$. This procedure terminates after a finite number of steps in a generating set G that satisfies Buchberger's Criterion, hence is a Groebner basis for I. We note that once an $S(g_i, g_j)$ yields a remainder of zero after division by the polynomials in G it also yields a remainder of zero when additional polynomials are appended to G.

If $\{g_1, ..., g_t\}$ is a Groebner basis for the ideal I and $LT(g_j)$ is divisible by $LT(g_i)$ for some $j \neq i$, then $LT(g_j)$ is not needed as a generator for $LT(I)$. By Proposition 2.3.8 we may therefore delete g_j and still retain a Groebner basis for I. Without loss of generality we may also assume that the leading term of each g_i is monic. A Groebner basis $\{g_1, ..., g_t\}$ for I where each $LT(g_i)$ is monic and where $LT(g_j)$ is not divisible by $LT(g_i)$ for $i \neq j$ is called a "minimal Groebner basis". While a minimal Groebner basis is not unique, the number of elements and their leading terms are unique.

Definition 2.3.11. *Fix a monomial ordering on $k[x_1, ..., x_n]$. A Groebner basis $\{g_1, ..., g_t\}$ for the nonzero ideal I in $k[x_1, ..., x_n]$ is called a "reduced Groebner basis" if*

(a) each $LT(g_i)$ is monic, $i = 1, ..., t$, and

(b) no term in g_j is divisible by $LT(g_i)$ for $j \neq i$.

We notice that a reduced Groebner basis is a minimal Groebner basis. The importance of reduced Groebner bases is that they are unique for a given monomial ordering.

Theorem 2.3.12. *There is a unique reduced Groebner basis for every nonzero ideal I in $k[x_1, ..., x_n]$ for a fixed monomial ordering.*

Proof. Let $G = \{g_1, ..., g_t\}$ and $G' = \{g'_1, ..., g'_t\}$ be two reduced bases for the same ideal I. Then after a possible rearrangement we may assume $LT(g_i) = LT(g'_i) = h_i$ for $i = 1, ..., t$. For any fixed i, we consider the polynomial $f_i = g_i - g'_i$. If f_i is nonzero, then since $f_i \in I$, its leading term must be divisible by some h_j. By definition of a reduced basis, h_j for $i \neq j$ does not divide any of the terms in either g_i or g'_i, hence does not divide $LT(f_i)$. But h_i also does not divide $LT(f_i)$, since all the terms in f_i have strictly smaller multidegree. This forces $f_i = 0$ that is, $g_i = g'_i$ for every i, so $G = G'$.

□

Corollary 2.3.13. *Let I and J be two ideals in $k[x_1, ..., x_n]$. Then $I = J$ if and only if I and J have the same reduced Groebner basis with respect to any fixed monomial ordering on $k[x_1, ..., x_n]$.*

2.4 Solving Algebraic Equations Using Groebner Bases:Elimination

One of the important applications of the theory of Groebner bases is explicitly solving systems of algebraic equations. Groebner bases theory is also the basis by which compute algebraic programs attempt to solve systems of equations.

Let $G = \{g_1, ..., g_s\}$ be a collection of polynomials in n-variables $x_1, ..., x_n$ and we want to find the solutions of the system of equations $g_1 = 0, g_2 = 0, ..., g_s = 0$. In other words, we wish to find the common set of zeros of the polynomials in G. If $(a_1, ..., a_n)$ is any solution to this system then every element $f \in I = \langle G \rangle$ also satisfies $f(a_1, ..., a_n = 0$. Furthermore, if $G' = \{h_1, ..., h_m\}$ is any set of generators for the ideal I then the set of solutions to the $h_1 = 0, h_2 = 0, ..., h_m = 0$ is the same as the original solution set.

When $g_1, ..., g_s$ are "linear" polynomials, a solution to the system of equations can be obtained by successively eliminating the variables $x_1, x_2, ...$ The variable x_1 is eliminated by using linear combinations of the original equations, then x_2 is eliminated using these equations, etc. By this way a system of equations is produced that can be easily solved.

Even if the situation for polynomial equations that are nonlinear is naturally more complicated, the basic principle is the same.

If there is a nonzero polynomial in the ideal I involving only one of the variables, say $g(x_n)$, then the last coordinate a_n is a solution of $g(x_n) = 0$. If, now there is a polynomial in I involving only x_{n-1} and x_n, say $h(x_{n-1}, x_n)$, then the coordinate a_{n-1} would be a solution of $h(x_{n-1}, a_n) = 0$, etc. If we can successively find polynomials in I that eliminate the variables $x_1, x_2...$ then we will be able to determine all the solutions $(a_1, ..., a_n)$ to given original system of equations explicitly.

Finding equations that follow from the system of equations in G, in other words., finding elements of the ideal I that do not involve some of the variables, is referred as "elimination theory". The polynomials in I that do not involve the variables $x_1, ..., x_i$, i.e., $I \cap k[x_{i+1}, ..., x_n]$ is an ideal in $k[x_{i+1}, ..., x_n]$.

Definition 2.4.1. *Let I be an ideal in $k[x_1, ..., x_n]$ then $I_i = I \cap k[x_{i+1}, ..., x_n]$ is called the "i^{th} elimination ideal" of I with respect to the ordering $x_1 \succ ... \succ x_n$.*

The success of using elimination to solve a system of equations depends on being able to determine the elimination ideals and, ultimately, on whether these elimination ideals are nonzero.

Proposition 2.4.2. *(Elimination) Let $G = \{g_1, ..., g_t\}$ be a Groebner basis for the nonzero ideal I in $k[x_1, ..., x_n]$ with respect to the lexicographic monomial ordering $x_1 \succ ... \succ x_n$. Then $G \cap k[x_{i+1}, ..., x_n]$ is a Groebner basis for ideal $I_i = I \cap k[x_{i+1}, ..., x_n]$. In particular, $I \cap k[x_{i+1}, ..., x_n] = 0$ if and only if $G \cap k[x_{i+1}, ..., x_n] = \emptyset$.*

Proof. Let $G_i = G \cap k[x_{i+1}, ..., x_n]$. Then $G_i \subseteq I_i$. To see that G_i is a Groebner basis of I_i it enough to show that $LT(G_i)$, the leading terms of the elements G_i, generate the ideal $LT(I_i)$ in $k[x_{i+1}, ..., x_n]$ by Proposition 2.3.8. Surely, $\langle LT(G_i)\rangle \subseteq LT(I_i)$ since they are ideals in $k[x_{i+1}, ..., x_n]$.

To show the converse inclusion, we let f be any element in I_i. Then $f \in I$ and since G is a Groebner basis for I we have

$$LT(f) = b_1(x_1, ..., x_n)LT(g_1) + ... + b_t(x_1, ..., x_n)LT(g_t)$$

for some polynomials $b_1, ..., b_t \in k[x_1, ..., x_n]$. Since each polynomial b_i is written as a sum of monomial terms $LT(f)$ will be a sum of monomial terms of the form $cx_1^{r_1}...x_n^{r_n}LT(g_i)$. Because $LT(f)$ involves only the variables $x_{i+1}, ..., x_n$, the sum of all such terms containing any of the variables $x_1, ..., x_i$ must be zero. Hence, $LT(f)$ is also sum of these monomial terms only involving $x_{i+1}, ..., x_n$. Thus, $LT(f)$ can be written as a $k[x_{i+1}, ..., x_n]$-linear combination of some monomial terms $LT(g_l)$ where $LT(g_l)$ does not involve the variables $x_1, ..., x_i$. On the other hand, by the choice of the ordering, if $LT(g_l)$ does not involve the variables $x_1, ..., x_i$, then $g_l \in G_i$. Therefore, $LT(f)$ can be written as a $k[x_{i+1}, ..., x_n]$-linear combination of elements $LT(G_i)$.

□

3. APPLICATIONS OF THE METHOD OF GROEBNER BASES TO DYNAMICAL SYSTEMS

Mathematicians have been used Groebner bases seriously to solve difficult problems in commutative algebra since the mid of 70's. (Buchberger, 1976; Rabbiano, 1989; Beveniste et al., 1989) made significant contributions in the improvement of the Groebner bases algorithms.

The systems of nonlinear algebraic equations in several variables arise in different areas of applied science and engineering, and the problem of finding their solutions are important. There is a general method for solving such systems in the case where the nonlinearities are of polynomial type. The knowledge about this method, based on Groebner bases, seems to be concentrated to mathematics and computer science, though.

By solving such an above system using the method of Groebner bases we mean triangulating it, so that the minimal dimension of the solution becomes apparent, and so that we obtain an equation in one variable only, if the solution is zero dimensional.

When triangulating a system we eliminate variables one by one. A ranking of the variables tells us in what order we should eliminate them. For instance we can have the ranking

$$x_1 \prec x_2 \prec \ldots \prec x_{n-1} \prec x_n \tag{3.1}$$

meaning that x_j is to be eliminated before x_i if $j > i$.

To do the elimination we have able to compare polynomials. This is possible if we have an ordering of the monomials of $k[x_1, \ldots x_n]$. This ordering is achieved if we say that a high ranking monomial should have a high degree in a high ranking variable.

The following theorem gives us the criteria for the existence of the solution of an elimination problem.

Theorem 3.0.1. *(Forsman and Glad, 1990) Let G be a Groebner basis for the ideal I wrt the ranking (3.1). Then the following applies:*

1. $Z(I) = \emptyset \Leftrightarrow G = \{1\}$,

2. *dim* $(Z(I)) = 0 \Leftrightarrow$ *Each variable appears alone in the highest ranking monomial of a polynomial in G,*

3. $I \cap \mathbb{R}[x_1] = \{0\} \Leftrightarrow G \cap \mathbb{R}[x_1] = \emptyset$.

Proof. from (Pauer and Pfeifhofer, 1988). □

In this chapter, we will present same applications of the Groebner bases method to some nonlinear problems of dynamical systems. The studied problems are;

- determining the jerky dynamics of 3-dimensional nonlinear autonomous systems,
- solving some difficult problems in Lyapunov theory by using that method such as,
  - constructing Lyapunov functions,
  - determining critical levels,
  - investigating robustness against structured uncertainty.

3.1 Determination of Jerky Dynamics with aid of Groebner Bases Method

Here, we present a computational method given by (Eichhorn et al., 1998) which is known as Groebner basis technique and based on an algebraic elimination procedure for nonlinear polynomial equations. Using this method we can check the existence and compute jerky dynamics of a given dynamical system.

3.1.1 Jerky Dynamics

Third order explicit autonomous differential equations in one scalar variable are sometimes called jerky dynamics and constitute an interesting subclass of dynamical systems that can exhibit chaotic behavior.

There have been published several articles on jerky dynamics (Eichhorn et al., 1998; Linz, 1997; Malasoma, 2000b,a; Arne'odo et al., 1979, 1981)motivated from the following questions:

(i) Which three-dimensional dynamical systems can be recast into a jerky dynamics?

(ii) Do apparently functionally different dynamical systems obey similar or even identical jerky dynamics?

(iii) If so, can jerky dynamics be used as a tool to classify dynamical systems?

(iv) Can we learn anything about the possible time evolution just by looking at the functional form of a jerky dynamics?

To clarify the subject we now consider a three dimensional autonomous system

$$\dot{x} = \mathbf{f}(\mathbf{x}) \tag{3.2}$$

where $\mathbf{x} = (x, y, z)^T \in \Gamma \subseteq \mathbb{R}^3$, $\mathbf{f}(\mathbf{x})$ in general, nonlinear vector field of the dynamical system and the overdot the derivative with respect to time. $\mathbf{x}(t)$ represent the orbit or trajectory of the dynamical system (3.2) in the phase space when we specify the initial conditions $\mathbf{x}(t = 0) = \mathbf{x}_0$.

As it is shown in (Eichhorn et al., 1998) any autonomous nth order ode can be recast into n-dimensional dynamical system. In particular, third-order explicit ODEs

$$\dddot{x} = J(x, \dot{x}, \ddot{x}) \tag{3.3}$$

can be transformed into a dynamical system (3.2) by introducing, for example, $\dot{x} = u$, $\dot{u} = a$, and $\dot{a} = J(x, u = \dot{x}, a = \ddot{x})$. But the contrary is generally not true and constitutes the starting point of the following investigation.

Our goal here is to see how we can apply Groebner basis method to determine whether a three dimensional autonomous system has an equivalent jerky dynamics.

3.1.2 Groebner Bases Method

For the application of the Groebner bases technique to the problem whether a dynamical system possesses an equivalent jerky dynamics, we have to reformulate it in an algebraic way. Using the equations

$$\dot{x} = f_1(x, y, z), \quad \dot{y} = f_2(x, y, z), \quad \dot{z} = f_3(x, y, z) \tag{3.4}$$

we write the following seven equations

$$p_1(x, y, z) = \dot{x} - f_1(x, y, z) = 0 \tag{3.5}$$

$$p_2(x, y, z) = \dot{y} - f_2(x, y, z) = 0 \tag{3.6}$$

$$p_3(x, y, z) = \dot{z} - f_3(x, y, z) = 0 \tag{3.7}$$

$$p_4(x, y, z) = \ddot{x} - (\dot{x}\partial_x f_1(x, y, z) + \dot{y}\partial_y f_1(x, y, z) + \dot{z}\partial_z f_1(x, y, z)) = 0 \tag{3.8}$$

$$p_5(x, y, z) = \ddot{y} - (\dot{x}\partial_x f_2(x, y, z) + \dot{y}\partial_y f_2(x, y, z) + \dot{z}\partial_z f_2(x, y, z)) = 0 \tag{3.9}$$

$$p_6(x, y, z) = \ddot{z} - (\dot{x}\partial_x f_3(x, y, z) + \dot{y}\partial_y f_3(x, y, z) + \dot{z}\partial_z f_3(x, y, z)) = 0 \tag{3.10}$$

$$\begin{aligned} p_7(x, y, z) \quad = \quad & \dddot{x} - (\dot{x}\partial_x^2 f_1(x, y, z) + \dot{y}\partial_y^2 f_1(x, y, z) + \dot{z}\partial_z^2 f_1(x, y, z) + \\ & \ddot{x}\partial_x f_1(x, y, z) + \ddot{y}\partial_y f_1(x, y, z) + \ddot{z}\partial_z f_1(x, y, z) = 0 \end{aligned} \tag{3.11}$$

We consider $x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, z, \dot{z}, \ddot{z}$ as ten independent variables. By eliminating the six variables $y, \dot{y}, \ddot{y}, z, \dot{z}, \ddot{z}$ from the seven equations we can find a third order differential equations for the variable x, $P(x, \dot{x}, \ddot{x}, \dddot{x})$. The equation $P(x, \dot{x}, \ddot{x}, \dddot{x}) = 0$ may not be explicit. Furthermore, there is no systematic way to find $P(x, \dot{x}, \ddot{x}, \dddot{x}) = 0$ when the vector field **f(x)** has arbitrary nonlinearities. However, for polynomial nonlinearities the Groebner bases method can be applied.

As the solutions of the equation $P(x, \dot{x}, \ddot{x}, \dddot{x}) = 0$ for $\dddot{x}$, jerk functions may contain different types of nonlinearity such as quadratic, cubic, etc. Jerk functions with quadratic nonlinearities are the simplest type displaying deterministic chaos. This type of problems and the corresponding classifications are studied in (Eichhorn et al., 1998; Linz, 1997; Malasoma, 2000b). Even if, the cubic nonlinearities are the simplest among odd algebraically nonlinearities they are rather complicated than the quadratic ones as discussed in (Arne'odo et al., 1979, 1981; Malasoma, 2000b,a).

In the following examples we will see that some three-dimensional nonlinear systems may have equivalent jerky dynamics in x, and/or, y, and/or z with quadratic or cubic nonlinearities or rational forms. While some jerk functions have simple algebraic forms some of them may have rather complex forms.

Example 3.1.1. *(Malasoma, 2000b) The system*

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = -az - x + xy$$

has an equivalent jerky dynamics only in x. For, from the Eq.s (3.5)-(3.11) we obtain the ideal

$$I = \langle \dot{x} - y, \dot{y} - z, \dot{z} + az + x - xy, \ddot{x} - \dot{y}, \ddot{y} - \dot{z}, \ddot{z} + a\dot{z} + \dot{x} - \dot{x}y - x\dot{y}, \dddot{x} - \ddot{y} \rangle$$

Its Groebner basis is computed as

$$a\ddot{x} + \dddot{x} + x - x\dot{x} = 0$$

which is an equivalent jerky dynamics in x for this system. The corresponding jerk function is

$$J(x, \dot{x}, \ddot{x}) = -a\ddot{x} - x + x\dot{x}$$

which contains only one quadratic nonlinearity.

Example 3.1.2. *(Malasoma, 2000a) The system*

$$\dot{x} = y + 1$$

$$\dot{y} = -ay + z$$

$$\dot{z} = xy$$

has equivalent jerky dynamics in x and y. Since the Eq.s (3.5)- (3.10) with (3.11) yield the ideal

$$I_1 = \langle \dot{x} - y - 1, \dot{y} + ay - z, \dot{z} - xy, \ddot{x} - y, \ddot{y} + a\dot{y} - \dot{z}, \ddot{z} - \dot{x}y - x\dot{y}, \dddot{x} - \ddot{y} \rangle$$

On the other hand the equations (3.5)-(3.10) with $p_7 = \dddot{y} + a\ddot{y} - \ddot{z}$ for y result the ideal

$$I_2 = \langle \dot{x} - y - 1, \dot{y} + ay - z, \dot{z} - xy, \ddot{x} - y, \ddot{y} + a\dot{y} - \dot{z}, \ddot{z} - \dot{x}y - x\dot{y}, \dddot{y} + a\ddot{y} - \ddot{z} \rangle$$

Groebner bases for the ideals I_1 and I_2 are obtained as

$$-a\ddot{x} - \dddot{x} - x + x\dot{x} = 0$$

and

$$a\dot{y}^2 + \dot{y}\ddot{y} - ay\ddot{y} - \dddot{y}y + y^2 + y^3 = 0$$

These are the equivalent jerky dynamics in x and y for the given system, respectively. The corresponding jerk functions are

$$J(x, \dot{x}, \ddot{x}) = -x - a\ddot{x} + x\dot{x}$$

and

$$J(y, \dot{y}, \ddot{y})) = y - a\ddot{y} + y^2 + \frac{\dot{y}\ddot{y} + a\dot{y}^2}{y}$$

while the first one has a quadratic nonlinearity, the later is a rational function.

Example 3.1.3. *(Malasoma, 2000a) The dynamical system*

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = -az - x + y^2$$

has equivalent jerky dynamics in x,y and z. Using the Eq.s (3.5)-(3.11), we obtain the ideal

$$I_1 = \langle \dot{x} - y, \dot{y} - z, \dot{z} + az + x - y^2, \ddot{x} - \dot{y}, \ddot{y} - \dot{z}, \ddot{z} + a\dot{z} + \dot{x} - 2y\dot{y}, \dddot{x} - \ddot{y} \rangle$$

with a Groebner basis $-\dot{x}^2 + a\ddot{x} + \dddot{x} + x = 0$, which is an equivalent jerky dynamics in x for the system. The corresponding jerk function

$$J(x, \dot{x}, \ddot{x}) = -x - a\ddot{x} + \dot{x}^2$$

has quadratic nonlinearity.

The Eq.s (3.5)-(3.10) with $p_7 = \dddot{y} - \ddot{z}$ (in terms of y) yield the ideal

$$I_2 = \langle \dot{x} - y, \dot{y} - z, \dot{z} + az + x - y^2, \ddot{x} - \dot{y}, \ddot{y} - \dot{z}, \ddot{z} + a\dot{z} + \dot{x} - 2y\dot{y}, \dddot{y} - \ddot{z} \rangle$$

with a Groebner basis $-a\ddot{y} - \dddot{y} - y + 2y\dot{y} = 0$ which is an equivalent jerky dynamics in y for the system. The corresponding jerk function,

$$J(y, \dot{y}, \ddot{y}) = -y - a\ddot{y} + 2y\dot{y}$$

has quadratic nonlinearity.

On the other hand, the Eq.s (3.5)-(3.10) with $p_7 = \dddot{z} + a\ddot{z} + \ddot{x} - 2\dot{y}^2 - 2y\ddot{y}$ (in terms of z) result the ideal

$$I_3 = \langle \dot{x} - y, \dot{y} - z, \dot{z} + az + x - y^2, \ddot{x} - \dot{y}, \ddot{y} - \dot{z}, \ddot{z} + a\dot{z} + \dot{x} - 2y\dot{y}, \dddot{z} + a\ddot{z} + \ddot{x} - 2\dot{y}^2 - 2y\ddot{y} \rangle$$

with a Groebner basis

$$2a\dot{z}^2 + a\ddot{z} + 2\dot{z}\ddot{z} + \dddot{z} + z - 2az\ddot{z} - 2z\dddot{z} - 4z^2 + 4z^3 = 0$$

which is an equivalent jerky dynamics in z for the system. The corresponding jerk function

$$J(z, \dot{z}, \ddot{z}) = -a\ddot{z} + \frac{z - 4z^2 + 4z^3 + 2\dot{z}\ddot{z} + 2a\dot{z}^2}{2z - 1}$$

is a rational function.

Example 3.1.4. *(Ratschan and She, 2007) Consider the dynamical system*

$$\dot{x} = -y$$

$$\dot{y} = -z$$

$$\dot{z} = -x - 2y - z + x^3$$

The corresponding ideal and the equivalent jerky dynamics in x are,

$$I = \langle \dot{x} + y, \dot{y} + z, \dot{z} + x + 2y + z - x^3, \ddot{x} + \dot{y}, \ddot{y} + \dot{z}, \ddot{z} + \dot{x} + 2\dot{y} + \dot{z} - 3\dot{x}x^2, \dddot{x} + \ddot{y} \rangle$$

and

$$2\dot{x} - \ddot{x} - \dddot{x} - x + x^3 = 0$$

respectively.

The jerk function for the given system

$$J(x, \dot{x}, \ddot{x}) = -x + x^3 + 2\dot{x} - \ddot{x}$$

has cubic nonlinearity.

Example 3.1.5. *The dynamical system*

$$\dot{x} = -x + yz^2$$

$$\dot{y} = -y + xy$$

$$\dot{z} = -z$$

has an equivalent jerky dynamics in x only. The ideal is

$$\begin{aligned} I &= \langle \dot{x} + x - yz^2, \dot{y} + y - xy, \dot{z} + z, \ddot{x} + \dot{x} - \dot{y}z^2 - 2yz\dot{z}, \ddot{y} + \dot{y} - \dot{x}y - x\dot{y}, \\ &\quad \ddot{z} + \dot{z}, \dddot{x} + \ddot{x} - \ddot{y}z^2 - 4z\dot{y}\dot{z} - 2y\dot{z}^2 - 2yz\ddot{z} \rangle \end{aligned}$$

and the equivalent jerky dynamics is

$$\dddot{x} - 9\dot{x} - 9x + 5x\dot{x} + 6x^2 - \dot{x}^2 - \dot{x}x^2 + \ddot{x} = 0$$

The corresponding jerk function,

$$J(x, \dot{x}, \ddot{x}) = 9\dot{x} + 9x - 5x\dot{x} - 6x^2 + \dot{x}^2 + \dot{x}x^2 - \ddot{x}$$

has cubic nonlinearity.

3.2 Groebner Bases Method in Lyapunov Theory

Lyapunov theory is commonly used as a mathematical tool in different areas such as automatic control (Vidyasagar, 1993), power system analysis (Pai, 1981), and mechanics (Hahn, 1967). It is probably the only general tool for determining stability for a system of nonlinear ordinary differential equations.

To see how Groebner bases enter into the study of Lyapunov theory we introduce some important concepts:

Definition 3.2.1. *(Forsman, 1993) A polynomial $p \in k[x_1, ...x_n]$ is positive if $p(0) = 0$ and*

$$\forall x \in \mathbb{R}^n\{0\} : p(x) > 0$$

p is non-negative if $p(0) = 0$ and $\forall x \in \mathbb{R}^n : p(x) \geq 0$. We say that p is locally positive if $p(0) = 0$ and there is a neighborhood $\mathcal{C}$ of the origin such that $\forall x \in \mathcal{C}\{0\} : p(x) > 0$. p is locally non-negative if $p(0) = 0$ and there is a neighborhood $\mathcal{C}$ of the origin such that $\forall x \in \mathcal{C} : p(x) \geq 0$.

We define negative etc. analogously. A dynamical system

$$\dot{x} = f(x), \quad f : \mathbb{R}^n \to \mathbb{R}^n, x \in \mathbb{R}^n \tag{3.12}$$

we suppose that $f(0) = 0$ and that the components of f are polynomials in x:

$$\forall i : f_i \in k[x_1, ..., x_n] \tag{3.13}$$

Since f is polynomial, the solution of (3.12) is uniquely determined by an initial value $\xi \in \mathbb{R}^n$ (Coddington and Levinson, 1985). We write $\pi(t, \xi)$ for the solution starting at ξ, so π is a function $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 3.2.2. *(Forsman, 1993) Given a $V \in k[x_1, ..., x_n]$ the regions $\Omega, \bar{\Omega}$ and B_d are defined by*

$$\Omega = \{x \in \mathbb{R}^n, \quad \dot{V} = V_x \cdot f < 0\} \cup \{0\}$$
$$\bar{\Omega} = \{x \in \mathbb{R}^n, \quad \dot{V} = V_x \cdot f \leq 0\}$$
$$B_d = \{x \in \mathbb{R}^n, \quad V(x) \leq d\}$$

where $\dot{V} = V_x \cdot f = \Sigma f_i \frac{\partial V}{\partial x_i}$ is the time derivative of V along a trajectory of the system (3.12).

This means that the boundary ∂B_d is a level surface of V for each d. From the definition, we can easily see that

$$a \leq b \Rightarrow B_a \subseteq B_b \tag{3.14}$$

for $a, b \in \mathbb{R}$.

Definition 3.2.3. *(Forsman, 1993)*

A function $V \in k[x_1, ..., x_n]$ is a (polynomial) Lyapunov function for the system (3.12) if $\dot{V}$ is non-positive.

Definition 3.2.4. *(Forsman, 1993)*

A function $V \in k[x_1, ..., x_n]$ is a local Lyapunov function for the system (3.12) if $\dot{V}$ is a locally non-positive and $\exists \quad d \in \mathbb{R} : B_d \subseteq \bar{\Omega}$.

If V is a local Lyapunov function for (3.12) and d is such that $B_d \subseteq \bar{\Omega}$ then

$$\xi \in B_d \Rightarrow \forall t \geq 0 : \pi(t, \xi) \in B_d$$

In other words the set B_d is invariant if V is a local Lyapunov function and $B_d \subseteq \bar{\Omega}$. In particular, if V is a Lyapunov function for (3.12). B_d is invariant for all d. B_d is not necessarily connected; make a partition

$$B_d = \bigcup_i B_d^i \tag{3.15}$$

of B_d into disjoint connected sets. Since π is continuous in t we have the following:

Theorem 3.2.5. *(Forsman, 1993) If V is a local Lyapunov function for (3.12) and $B_d^i \subseteq \bar{\Omega}$ then*

$$\xi \in B_d^i \Rightarrow \forall t \geq 0 : \pi(t, \xi) \in B_d^i$$

For a polynomial V the partition (3.15) consists of a finite number of components. The number of its connected components of B_d depends on the value of d.

The most important application of Lyapunov theory is stability theory. In order to establish stability using Lyapunov functions we need one more concept.

Definition 3.2.6. *(Forsman, 1993) A function $p : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded if $p(x) \to \infty$ as $\|x\| \to \infty$.*

Thus, B_d is bounded for all d iff V is radially unbounded.

Theorem 3.2.7. *(Forsman, 1993) If V is positive, radially unbounded, local Lyapunov function for (3.12) and $B_d^0 \subset \Omega$ then*

$$\xi \in B_d^0 \Rightarrow \lim_{t \to \infty} \pi(t, \xi) = 0$$

where $0 \in B_d^0$. Thus the origin is locally asymptotically stable equilibrium of the system (3.12).

3.2.1 Construction of Lyapunov Functions using Groebner Bases

Consider a dynamical system (3.12). Suppose that we have a polynomial parametrized Lyapunov function $V(\eta, x)$ for this system

$$V : \mathbb{R}^n \to \mathbb{R}, V \in k[x_1, x_2, ..., x_n] \tag{3.16}$$

where $V(\eta^0)$ is a local Lyapunov function for a nominal value η^0 of $\eta \in \mathbb{R}^q$. We introduce the notation $Q = V_x \cdot f$, where V_x denote the gradient of V wrt x.

The region $\Omega(\eta)$ and $B_d(\eta)$ are defined by

$$\Omega(\eta) = \{x \in \mathbb{R}^n | Q(x, \eta) \leq 0\}, \qquad B_d(\eta) = \{x \in \mathbb{R}^n | V(x, \eta) \leq d\}$$

with Q as above. It is a classical result in (Hahn, 1967) that the stability of (3.12) is generated within B_d if $B_d(\eta) \subseteq \Omega(\eta)$

In order to find out for which η this is satisfied we can examine the ideal

$$\left\langle V-d, Q, \frac{\partial V}{\partial x_1}-\lambda\frac{\partial Q}{\partial x_1}, ..., \frac{\partial V}{\partial x_n}-\lambda\frac{\partial Q}{\partial x_n}\right\rangle \tag{3.17}$$

Since we are interested in choosing the parameter η it is appropriate to compute a Groebner basis G for the ideal (3.17) wrt a ranking

$$\{\eta_1, ..., \eta_n\} \prec \{x_1, ..., x_n, \lambda\} \tag{3.18}$$

G is typically contains elements that are polynomial in η. In this way we get information about which η are allowed, and then we can apply some criterion for optimality to choose among these.

Example 3.2.8. *(Ratschan and She, 2007) The system*

$$\begin{aligned}\dot{x} &= -4x^3+6x^2-2x\\ \dot{y} &= -2y\end{aligned}$$

has three equilibrium points which are $(0,0)$, $(1,0)$ *and* $(\frac{1}{2},0)$. *While* $(\frac{1}{2},0)$ *is unstable, the points* $(0,0)$ *and* $(1,0)$ *are asymptotically stable.*

To find an estimation to the DOA of the point $(0,0)$, *we let*

$$V(x,y) = ax^4+(a-\frac{1}{2})x^3+(a-\frac{3}{4})x^2+(a+\frac{1}{2})y^2$$

Then the ideal

$$\begin{aligned}I_1 \;=\; &\langle ax^4+ax^3-\frac{x^3}{2}+ax^2-\frac{3x^2}{4}+ay^2+\frac{y^2}{2}-a,\\ &-16ax^6+12ax^5+2ax^4+6ax^3+6x^5-3x^4-6x^3-4ax^2-3x^2-4ay^2-2y^2,\\ &4ax^3+3ax^2-\frac{3x^2}{2}+2ax-\frac{6x}{4}-\lambda(-96ax^5+\\ &x^4(60a+30)+x^3(8a-12)+x^2(18a-18)-x(8a+6),\\ &2ay+y-\lambda(-8ay-4y)\rangle\end{aligned}$$

choosing ranking $\{x,y,\lambda\} > a$ *results the polynomial*

$$
\begin{aligned}
p &= -4829328a - 96147756a^2 - 414076608a^3 + 3162128234a^4 + 35254649548a^5 + \\
&125869478392a^6 + 202507391869a^7 + 161737821216a^8 + 1863101027084a^9 + \\
&16025696039344a^{10} + 46752865477360a^{11} + 42263798224320a^{12} - \\
&42624964664512a^{13} + 30726259919104a^{14} + 781101633494016a^{15} - \\
&789266219830272a^{16} + 138862511087616a^{17}
\end{aligned}
$$

which has the roots

$$-0.198855, \quad -0.16346, \quad -0.109415, \quad 0., \quad 0.132145, \quad 1.35659, \quad 4.39508$$

and 10 non-real roots. Since $a^0 = 1$ *a valid choice for a is* $a \in [0.0132145, 4.39508]$*. When* $a = 4.39508$ *the level curve of the given system is depicted in Figure 3.1.*

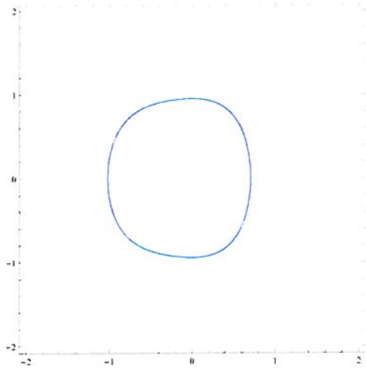

Figure 3.1: $B_a a$ for a=4.39508

To obtain an estimation to the DOA of the point $(1, 0)$*, we transform the given system to*

$$
\begin{aligned}
\dot{u} &= -4u^3 - 6u^2 - 2u \\
\dot{v} &= -2v
\end{aligned}
$$

with the coordinate transformation

$$u = x - 1, \quad v = y$$

So, the point $(1, 0)$ *is translated to the origin, that is* $(0, 0)$ *is an asymptotically stable equilibrium point of the resulting system.*

If we let

$$V(u, v) = au^4 + (a - \frac{1}{2})u^3 + (a - \frac{3}{4})u^2 + (a + \frac{1}{2})v^2$$

then the ideal

$$\begin{aligned}
I_2 &= au^4 + au^3 - \frac{u^3}{2} + au^2 - \frac{3u^2}{4} + av^2 + \frac{v^2}{2} - a, \\
&-16au^6 - 36au^5 - 34au^4 - 18au^3 - 4au^2 + 6u^5 + 15u^4 + 12u^3 + 3u^2 - 4av^2 - 2v^2, \\
&4au^3 + 3au^2 - \frac{3u^2}{2} + 2au - \frac{6u}{4} - \lambda(-96au^5 + u^4(-180a + 30) + u^3(-136a + 60) + \\
&u^2(-54a + 36) + u(-8a + 6)), 2av + v - \lambda(-8av - 4v)
\end{aligned}$$

using the ranking $\{u, v, \lambda\} > a$ *contains the polynomial*

$$\begin{aligned}
p &= 168750a + 4471875a^2 + 39915000a^3 + 109838750a^4 - 200021950a^5 - 953238173a^6 + \\
&757535168a^7 + 8199695668a^8 + 9553507216a^9 - 143190703600a^{10} + 44113844544a^{11} - \\
&383408369472a^{12} + 1212137163008a^{13} - 5137406172160a^{14} + 1623919205376a^{15}
\end{aligned}$$

which has the roots

$$-0.153846, \quad -0.110044, \quad 0., \quad 0.337442, \quad 2.93688$$

and 10 non-real roots. Since $a^0 = 1$ *a valid choice for a is* $a \in [0.337442, 2.93688]$. *For* $a = 2.93688$ *the level curve of the above system is plotted in Figure 3.2.*

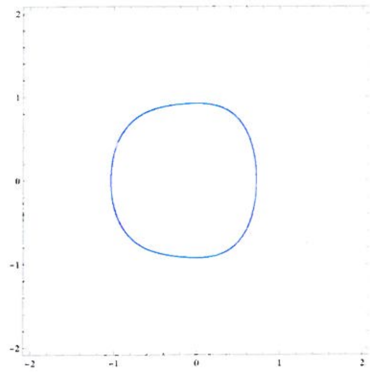

Figure 3.2: $B_a a$ for a=2.93688

Example 3.2.9. *For the system*

$$\begin{aligned}
\dot{x} &= -2x + 2y^4 \\
\dot{y} &= -y
\end{aligned}$$

the origin is the only equilibrium point. Let

$$V(x, y) = x^2 + (2a - 1)y^4$$

Then the ideal

$$\begin{aligned}
I &= \langle x^2 + 2ay^4 - y^4 - a, -4x^2 + 4xy^4 - 8ay^4 + 4y^4, \\
&2x - \lambda(-8x + 4y^4), 8ay^3 - 4y^3 - \lambda(16xy^3 - 32ay^3 + 16y^3)\rangle
\end{aligned}$$

choosing the ranking $\{x, y, \lambda\} > a$ *contains the polynomial*

$$p = 27a - 112a^2 + 108a^3$$

which has the roots

$$0., \quad 0.381182, \quad 0.655856$$

Since $a^0 = 0.6$ *a valid choice for* a *is* $a \in (0, 0.655856]$. *So the desired value of* $a = 0.655856$ *and for this* a *the level curve is sketched in Figure 3.3.*

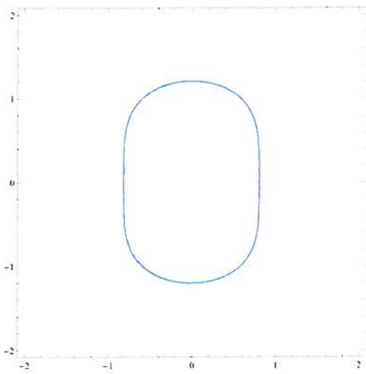

Figure 3.3: $B_a a$ for a=0.655856

Example 3.2.10. *(Ratschan and She, 2007) For the system*

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x - (1 - x^2)y \end{aligned}$$

the origin is the only equilibrium point. Let

$$V(x, y) = ax^2 + (2 - a)y^2$$

then the Groebner basis for the ideal

$$\begin{aligned} I = & \langle ax^2 + 2y^2 - ay^2 - a, -4axy + 4xy - 4y^2 + 4x^2y^2 + 2ay^2 - 2ax^2y^2, \\ & 2ax - \lambda(-4ay + 4y + 8xy^2 - 4axy^2), \\ & 4y - 2ay - \lambda(-4ax + 4x - 8y + 8x^2y + 4ay - 4ax^2y) \rangle \end{aligned}$$

with respect to the ranking $\{x, y, \lambda\} > a$ *contains the polynomial*

$$p = -16a - 22a^2 + 11a^3$$

which has the roots

$$-0.566699, \quad 0., \quad 2.5667$$

Since $a^0 = 1$ *a valid choice for* a *is* $a \in (0., 2.5667]$. *Figure 3.4 shows the situation in the case* $a = 2.5667$.

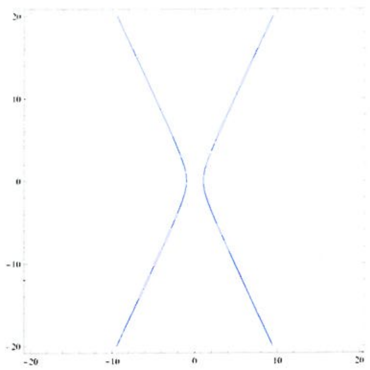

Figure 3.4: $B_a a$ for a=2.5667

Example 3.2.11. *(Ratschan and She, 2007) For the system*

$$\begin{aligned}
\dot{x} &= -x - 3y + 2z + yz \\
\dot{y} &= 3x - y - z + xz \\
\dot{z} &= -2x + y - z + xy
\end{aligned}$$

the origin is the only equilibrium point. Let

$$V(x, y, z) = ax^2 + (2a - 1)y^2 + az^2$$

then the Groebner basis for the ideal

$$\begin{aligned}
I = \; & \langle ax^2 + (2a-1)y^2 + az^2 - a, \\
& -2ax^2 + 6axy + 10axyz - 4ay^2 - 6xy + 2y^2 + 2yz - 2xyz - 2az^2, \\
& 2ax - \lambda(-4ax + 6ay + 10ayz - 6y - 2yz), \\
& 4ay - 2y - \lambda(6ax + 10axz - 8ay - 6x + 4y + 2z - 2xz), \\
& 2az - \lambda(10axy + 2y - 2xy - 4az)\rangle
\end{aligned}$$

using the ranking $\{x, y, z, \lambda\} > a$ contains the minimal polynomial

$$\begin{aligned}
p = \; & 8264970432a - 137519924688a^2 + 1113904458495a^3 - 6272009476027a^4 + \\
& 29690310309860a^5 - 124906381870224a^6 + 448310378616182a^7 - \\
& 1336139001497904a^8 + 3451492759466853a^9 - 8473696488852796a^{10} + \\
& 20559619411734703a^{11} - 45998301530230723a^{12} + 87445235884740725a^{13} - \\
& 139970538666472707a^{14} + 204483382891273147a^{15} - 302256406846123164a^{16} + \\
& 441042807338631371a^{17} - 555234362217762233a^{18} + 538504118805917330a^{19} - \\
& 374114879961222824a^{20} + 173410881313335296a^{21} - 47705577178626000a^{22} + \\
& 5844945576020000a^{23}
\end{aligned}$$

which has the real roots

$$0., \quad 0.632426, \quad 0.712185, \quad 1.0855, \quad 1.877$$

since $a^0 = 1$ *a valid choice for* a *is* $a \in [0.632426, 1.877]$. *When* $a = 1.877$ *Figure 3.5. shows the level surface of the given system.*

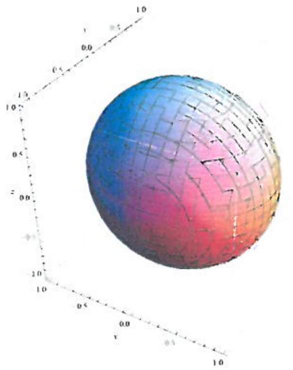

Figure 3.5: $B_a a$ for a=1.877

Example 3.2.12. *(Ratschan and She, 2007) For the system*

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -z - y - x^3 \end{aligned}$$

the origin is the only equilibrium point. Let

$$V(x, y, z) = \frac{1}{2}ax^4 + 2x^3y + (a^2 - 1)y^2 + 2ayz + z^2$$

Then the ideal

$$\begin{aligned} I &= \langle \frac{1}{2}ax^4 + 2x^3y + (a^2 - 1)y^2 + 2ayz + z^2 - a, \\ &\quad 6x^2y^2 + 2a^2yz - 4yz + 2az^2 - 2ayz - 2ay^2 - 2z^2, \\ &\quad 2ax^3 + 6x^2y - \lambda(12xy^2), \\ &\quad 2x^3 + 2a^2y - 2y + 2az - \lambda(12x^2y + 2a^2z - 4z - 2az - 4ay), \\ &\quad 2ay + 2z - \lambda(2a^2y - 4y + 4az - 2ay - 4z)\rangle \end{aligned}$$

with respect to ranking $\{x, y, z, \lambda\} > a$ *contains the polynomial*

$$\begin{aligned} p &= 1679616a - 4945536a^2 + 190138752a^3 - 2857635072a^4 + 17271107280a^5 - \\ &\quad 56965312584a^6 + 115734422808a^7 - 152475765120a^8 + \\ &\quad 131541097680a^9 - 71826431706a^{10} + 21466117254a^{11} + 944804970a^{12} - \\ &\quad 6260953931a^{13} + 4043324518a^{14} - 188512814a^{15} - 1321389302a^{16} + \\ &\quad 988816855a^{17} - 432251042a^{18} + 178270628a^{19} - 77217982a^{20} + \\ &\quad 24143069a^{21} - 2634418a^{22} - 1577032a^{23} + 1160642a^{24} - 551535a^{25} + \\ &\quad 231918a^{26} - 80736a^{27} + 21146a^{28} - 3709a^{29} + \\ &\quad 278a^{30} + 46a^{31} - 14a^{32} + a^{33} \end{aligned}$$

which has the roots

$$-7.34847, \quad -2.62068, \quad 0., \quad 0.296412, \quad 1.22841, \quad 3.01663, \quad 7.34847$$

and 26 non-real roots. Since $a^0 = 1$ *a valid choice for a is* $a \in [0.296412, 7.34847]$. *For* $a = 7.34847$ *the level surface of the system is depicted in Figure 3.6.*

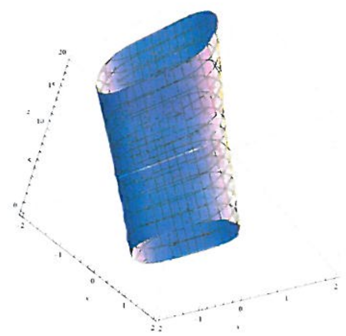

Figure 3.6: $B_a a$ for a=7.34847

3.2.2 Determining Critical Levels

Having defined local Lyapunov function and explained their connection to stability theory above, the following important problem immediately arises:

How do we choose d in order to get B_d as large as possible, while still inside of Ω?

If $B_d \subset \bar{\Omega}$ we can guarantee stability in B_d, according to Theorem 3.2.7. In e.g., (Chiang and Thorp, 1989) an efficient method for solving this problem as given. Here, we shall introduce an algorithm for solving this problem via Groebner bases, in the case of polynomial nonlinearities.

To study the problem we introduce the set $D \subseteq \mathbb{R}$:

$$D = \{d \in \mathbb{R} | B_d \subseteq \bar{\Omega}\} \tag{3.19}$$

This set is obviously the projection of $Z(\langle V - d, V_x \cdot f \rangle)$ on the d-axis (Z(I)) is the set of real zeros of the ideal I in $\mathbb{R}^n$; thus it is semi-algebraic. It is a simple consequence of property (3.14) that D is connected, i.e., consists of a single interval. In view of the property (3.14) we may reformulate the problem as: which is the smallest $d > 0$ such that

$$\partial B_d \cap \partial \Omega \neq \emptyset \tag{3.20}$$

If we view the problem as one of optimization we can formulate it as follows: we wish to minimize V subject to $V_x \cdot f = 0$. This observation is also made in e.g., (Shields and Storey, 1975).

Summing up we get n new equations while introducing one new variable, the Lagrange multiplier λ. So, we get a system of n+2 equations in n+2 variables:

$$V - d = 0, \quad V_x \cdot f = 0, \quad \nabla_x (V_x \cdot f) - \lambda \nabla_x V = 0$$

i.e., we should study the ideal

$$I = \langle V - d, V_x \cdot f, \frac{\partial V}{\partial x_1} - \lambda \frac{\partial}{\partial x_1}(V_x \cdot f), ..., \frac{\partial V}{\partial x_n} - \lambda \frac{\partial}{\partial x_n}(V_x \cdot f) \rangle \tag{3.21}$$

in $k[x_1, ..., x_n, \lambda, d]$. As mentioned we presuppose that we do not have $\nabla V \neq 0 = \nabla(V_x \cdot f)$ at the relevant point. It is thus sufficient that the hypersurface $V - d$ is not singular for the critical d.

A plex Groebner bases for I wrt a term-ordering of the type

$$\{x_1, ..., x_n, \lambda\} > d \tag{3.22}$$

gives us the contraction $I \cap k[d]$. If $I \cap k[d] \neq 0$ we thus obtain a polynomial p in d only, derived from the original equations.

Example 3.2.13. *For the system*

$$\begin{aligned} \dot{x} &= -2x(x-1)(2x-1) \\ \dot{y} &= -2y \end{aligned}$$

$(0,0)$, $(1,0)$ *and* $(\frac{1}{2},0)$ *are the equilibrium points. While* $(0,0)$ *and* $(1,0)$ *are asymptotically stable, the point* $(\frac{1}{2},0)$ *is unstable. If we let*

$$V(x,y) = x^2(x-1)^2 + y^2$$

Then the ideal for this problem is computed as

$$\begin{aligned} I &= \langle x^4 - 2x^3 + x^2 + y^2 - d, -16x^6 + 48x^5 - 52x^4 + 24x^3 - 4x^2 - 4y^2, \\ &\quad 4x^3 - 6x^2 + 2x - \lambda(-96x^5 + 240x^4 - 208x^3 + 72x^2 - 8x), 2y - \lambda(-8y)\rangle \end{aligned}$$

A computation of the plex GB for I wrt a term-ordering of the type

$$\{x, y, \lambda\} > d$$

gives us the minimal polynomial for d which is

$$p = -d + 16d^2$$

having

$$0, \quad 0.0625$$

for the real roots.

The level curve is

$$x^2(x-1)^2 + y^2 = 0.0625$$

As it seen from Fig.3.7 the resulting curve gives us an estimations to the domain of the attractions both the points $(0,0)$ *and* $(1,0)$*).*

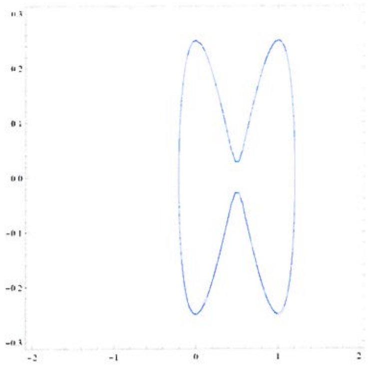

Figure 3.7: Level curve of v=0.0625

Example 3.2.14. *(Tibken and Hachicho, 2000) For the system*

$$\begin{aligned}\dot{x} &= -x + yz^2\\ \dot{y} &= -y + xy\\ \dot{z} &= -z\end{aligned}$$

the origin (0,0,0) is an asymptotically stable equilibrium point.

$$V(x,y,z) = x^2 + y^2 + z^2$$

is a local Lyapunov function for the origin. The ideal for this problem is

$$\begin{aligned}I \;=\; &\langle x^2+y^2+z^2-d, -2x^2+2xyz^2-2y^2+2xy^2-2z^2, 2x-\lambda(-4x+2yz^2+2y^2),\\ &2y-\lambda(2xz^2-4y+4xy), 2z-\lambda(4xyz-4z)\rangle\end{aligned}$$

and a plex GB for I wrt a term-ordering of the type

$$\{x, y, z, \lambda\} > d$$

results the minimal polynomial for d is

$$p = 84375d^2 + 1067500d^3 - 108567808d^4 + 24464976d^5 + 2288640d^6 - 544256d^7 - 24576d^8 + 4096d^9$$

having the approximate roots

$$-0.023341, \quad 0, \quad 0, \quad 0.0333721, \quad 4.91876, \quad 6.75, \quad 10.6723$$

and two non-real roots and the minimum positive real root is 0.0333721 *and the corresponding level surface*

$$x^2 + y^2 + z^2 = 0.0333721$$

is sketched in Fig.3.8

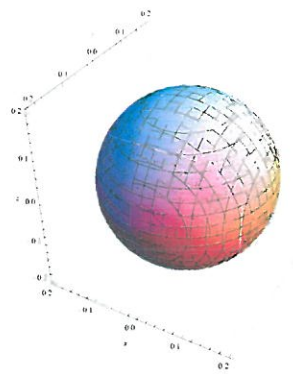

Figure 3.8: Level surface of v=0.0333721

Example 3.2.15. *For the system*

$$\begin{aligned}\dot{x} &= (-x+2y)(z+1)\\ \dot{y} &= (-x-y)(z+1)\\ \dot{z} &= -z^3\end{aligned}$$

the origin (0,0,0) is an asymptotically stable equilibrium point, and

$$V(x,y,z) = x^2 + 2y^2 + z^2$$

is a local Lyapunov function for the origin.

The ideal for this problem

$$\begin{aligned}I &= \langle x^2+2y^2+z^2-d, -2x^2-4y^2-2z^4-2x^2z-4y^2z, 2x-\lambda(-4x-4xz),\\ &\quad 4y-\lambda(-8y-8yz), 2z-\lambda(-8z^3-2x^2-4y^2)\rangle\end{aligned}$$

and a plex GB for I wrt a term-ordering of the type

$$\{x,y,z,\lambda\} > d$$

result the minimal polynomial for d is

$$p = -20d + 196d^2 - 320d^3 + 27d^4$$

having the approximate roots

$$0, \quad 0.128853, \quad 0.512811, \quad 11.2102$$

The minimum positive real root is 0.128853 and the corresponding level surface

$$x^2 + 2y^2 + z^2 = 0.128853$$

is sketched in Fig.3.9.

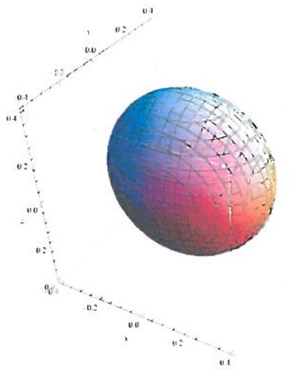

Figure 3.9: Level surface of v=0.128853

Example 3.2.16. *(Sanchez et al., 2010) For the system*

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(1-x^2)x - y \end{aligned}$$

the origin (0,0) is an asymptotically stable equilibrium point.

$$V(x,y) = x^2 + xy + \frac{y^2}{2}$$

is a local Lyapunov function for the origin. The ideal for this problem is

$$I = \langle x^2 + xy + \frac{y^2}{2} - d, x^4 - x^2 + x^3y, 2x + y - \lambda(4x^3 - 2x + 3x^2y), x + y - \lambda(x^3)\rangle$$

and a plex GB for I wrt a term-ordering of the type

$$\{x, y, \lambda\} > d$$

result the minimal polynomial for d is

$$p = -d + d^3$$

having the roots

$$-1, \quad 0, \quad 1.$$

Since the minimum positive real root is 1 *and the corresponding level curve*

$$x^2 + xy + \frac{y^2}{2} = 1$$

is sketched in Fig.3.10.

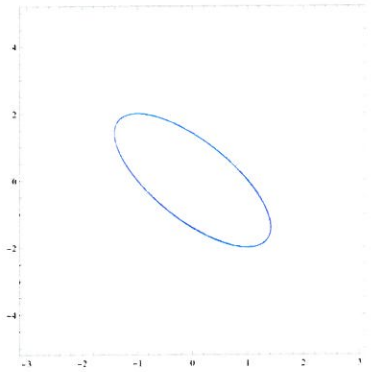

Figure 3.10: Level curve of v=1

Example 3.2.17. *(Sanchez et al., 2010) For the system*

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^3 - y \end{aligned}$$

origin $(0, 0)$ *is an asymptotically stable equilibrium point.*

$$V(x, y) = x^2 - \frac{1}{14}x^4 + xy + \frac{1}{2}y^2 - \frac{1}{7}x^2y^2 - \frac{1}{14}xy^3 - \frac{1}{56}y^4$$

is a local Lyapunov function for the origin.

The ideal for this problem is

$$\begin{aligned} I \;=\; & \langle x^3y - x^2 + x^4 - \frac{2}{7}x^5y + \frac{1}{2}x^2y^2 - \frac{3}{14}x^4y^2 - \frac{1}{14}x^3y^3, \\ & x^2 - \frac{1}{14}x^4 + xy + \frac{1}{2}y^2 - \frac{1}{7}x^2y^2 - \frac{1}{14}xy^3 - \frac{1}{56}y^4 - d, \\ & 2x - \frac{2}{7}x^3 + y - \frac{2}{7}xy^2 - \frac{1}{14}y^3 - \lambda(3x^2y - 2x + 4x^3 - \frac{10}{7}x^4y + xy^2 - \frac{6}{7}x^3y^2 - \frac{3}{14}x^2y^3), \\ & x + y - \frac{2}{7}x^2y - \frac{3}{14}xy^2 - \frac{1}{14}y^3 - \lambda(x^3 - \frac{2}{7}x^5 + x^2y - \frac{3}{7}x^4y - \frac{3}{14}x^3y^2)\rangle \end{aligned}$$

and a plex GB for I wrt a term-ordering of the type

$$\{x, y, \lambda\} > d$$

result the minimal polynomial for d is

$$\begin{aligned} p \;=\; & 316062936d - 2105293848d^2 + 5200653500d^3 - 6244938280d^4 + 3879222914d^5 - \\ & 1193916438d^6 + 139897239d^7 + 13474542d^8 - \\ & 6006779d^9 + 680482d^{10} - 5809d^{11} - 5804d^{12} + 188d^{13} + 32d^{14} \end{aligned}$$

having the approximate roots

$$-6.34707, \quad 0, \quad 0.354249, \quad 1.05405, \quad 3.5, \quad 5.64575$$

and eight non-real roots. The minimum positive real root is 0.354249 and the corresponding level curve

$$x^2 - \frac{1}{14}x^4 + xy + \frac{1}{2}y^2 - \frac{1}{7}x^2y^2 - \frac{1}{14}xy^3 - \frac{1}{56}y^4 = 0.354249$$

is sketched in Fig.3.11.

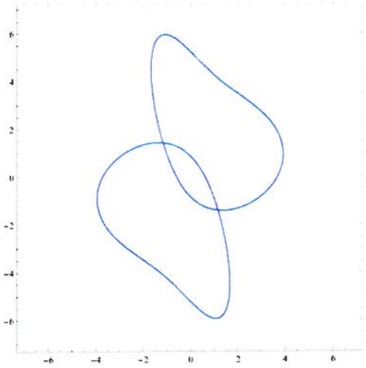

Figure 3.11: Level curve of v=0.354249

3.2.3 Robustness against Structured Uncertainty

In many areas it s very important to ensure that the solutions and designs made for a particular problem do not rely too heavily on the underlying models. This is known as the robustness problem in control theory. Here, we will use Lyapunov theory to deal with a special case of the robustness problem.

An instance of the nonlinear robustness problem is that of a system with structured uncertainty:

$$\dot{x} = f(x, \eta), \quad x \in \mathbb{R}^n \tag{3.23}$$

where $\eta \in \mathbb{R}^q$ are real, time-invariant parameters. The components of f are assumed to be polynomial in both x and η:

$$f_i \in k[x_1, ..., x_n, \eta_1, ..., \eta_q], \quad i = 1, 2, ..., n \tag{3.24}$$

Definition 3.2.18. *(Forsman, 1993) Given V, define the region $H \subseteq \mathbb{R}^q$ to be the set of all η such that V is a local Lyapunov function for the system (3.23).*

We suppose that H is non-empty, and that we have access to a nominal value $\eta^0 \in H$ of the uncertain parameters. The regions Ω and $\bar{\Omega}$ in Definition 3.2.1 now depend on η:

$$\begin{aligned} \Omega(\eta) &= \{x \in \mathbb{R}^n | (V_x \cdot f)(x, \eta) < 0\} \cup \{0\} \\ \bar{\Omega}(\eta) &= \{x \in \mathbb{R}^n | (V_x \cdot f)(x, \eta) \leq 0\} \end{aligned} \tag{3.25}$$

and we want to know if $B_d \subseteq \bar{\Omega}(\eta)$ or not. In the previous section the problem of finding for η^0 was considered. Here, we keep d fixed and vary η.

First we consider an instance of the (local) stability robustness problem:

Given d, for what values of η is $B_d \subseteq \bar{\Omega}(\eta)$?

In words: for what values of the parameters η can we guarantee stability within B_d using the local Lyapunov function V? If we call this set M_d we have

$$M_d = \{\eta \in H | B_d \subset \bar{\Omega}(\eta)\} \tag{3.26}$$

where H is the set of parameters allowed for Definition 3.2.18 and $\bar{\Omega}(\eta)$ is defined in equation (3.25).

We can treated this problem using Groebner bases as follow: The ideal I, $I_{\mathbb{R}}$ now lie in $k[x_1, ..., x_n, \eta_1, ..., \eta_q, \lambda]$. We are interested in finding out for what $\eta \in \mathbb{R}^q$ that I has real zeros. This means that we are looking for η such that

$$\partial B_d \cap \bar{\Omega}(\eta) \neq \emptyset \tag{3.27}$$

Having done this we obtain a hypersurface in parameter space, giving us information on ∂M_d. We need to know in which component of the region bounded by ∂M_d that (3.27) is satisfied. This is easy, since we know that η^0 belongs to this component. Also, we need to know if ∂M_d intersects ∂H or not, i.e., if all the parameters defined by our hypersurface are in the allowed region H or not. Furthermore, we would like to know if the region bounded by the hypersurface is connected or not which is certainly not a simple problem.

The ideal I is generated by n+2 polynomials in n+q+1 variables (the x, the η and λ): it typically contains an element in η only. The Groebner basis for I wrt a term ordering of the type

$$\{x_1, ..., x_n, \lambda\} > \{\eta_1, ..., \eta_q\} \tag{3.28}$$

gives us a generator for the wanted subideal.

The following examples indicate how these ideas work in practice.

Example 3.2.19. *For the system*

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y + y^2 \end{aligned}$$

the origin is an equilibrium point,and

$$V(x, y) = x^2 + y^2$$

is a local Lyapunov function for the origin (0,0) and $d = 2$, *then the corresponding ideal*

$$I = \langle x^2 + y^2 - 2, 2ax^2 - 2y^2 + 2y^3, 2x - \lambda(4ax), 2y - \lambda(-4y + 6y^2) \rangle$$

using the ranking $\{x, y, \lambda\} > a$ *yields the polynomial*

$$p = 2 - 2a + 6a^2 + 2a^3$$

in terms of a which has roots

$$-5.09808, \quad 0.0980762, \quad 2.$$

Fig.3.12 shows the set $\{x \in \mathbb{R}^2\,,\, V(x) \leq 2\}$ *which is contained in the DOA of the origin for* $a \in (-5.09808, 0)$.

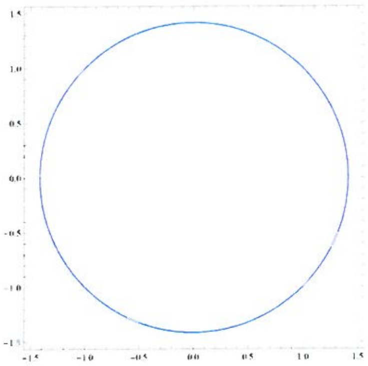

Figure 3.12: Statespace: The curve of v=2

Example 3.2.20. *For the system*

$$\begin{aligned}\dot{x} &= -x + y + x(x^2 + y^2)\\ \dot{y} &= \beta x - y + y(x^2 + y^2)\end{aligned}$$

If we take

$$V(x, y) = x^2 + y^2$$

be a local Lyapunov function for the origin (0,0) and $d = 2$, *then the corresponding ideal*

$$\begin{aligned}I &= \langle x^2 + y^2 - 2, -2x^2 - 2y^2 + 2xy + 2\beta xy + 2x^4 + 2y^4 + 4x^2y^2,\\ &\quad 2x - \lambda(-4x + 2y + 2\beta y + 8x^3 + 8xy^2), 2y - \lambda(-4y + 2x + 2\beta x + 8y^3 + 8x^2y)\rangle\end{aligned}$$

using the ranking $\{x, y, \lambda\} \geq \beta$ *yields the polynomial*

$$p = -3 + 2\beta + \beta^2$$

which has roots

$$-3, \quad 1$$

Fig.3.13 shows the set $\{x \in \mathbb{R}^2\,,\, V(x) \leq 2\}$ *which is contained in the DOA of the origin for* $\beta \in (-3, 1)$

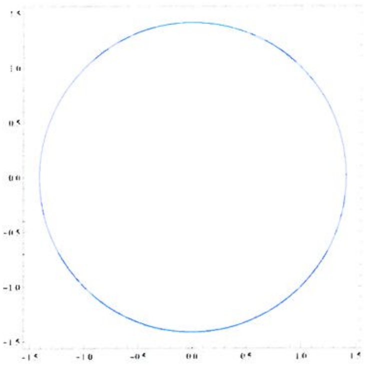

Figure 3.13: Statespace: The curve of v=2

Example 3.2.21. *For the system*

$$\begin{aligned}\dot{x} &= -x + yz^2\\ \dot{y} &= -2y + xy\\ \dot{z} &= -az\end{aligned}$$

if we take

$$V(x, y, z) = x^2 + y^2 + z^2$$

be a local Lyapunov function for the origin $(0, 0, 0)$ *and* $d = 1$*, then the ideal I*

$$\begin{aligned}I &= \langle x^2 + y^2 + z^2 - 1, -2x^2 + 2xyz^2 - 4y^2 + 2xy^2 - 2az^2, 2x - \lambda(-4x + 2yz^2 + 2y^2),\\ &\quad 2y - \lambda(2xz^2 - 8y + 4xy), 2z - \lambda(4xyz - 4az)\rangle\end{aligned}$$

using the ranking $\{x, y, z, \lambda\} \geq a$ *results the polynomial*

$$\begin{aligned}p &= 6779052 - 40673610a - 32335947a^2 + 548278676a^3 - 109373644a^4 - 2196522144a^5 +\\ &\quad 4793364288a^6 - 5550925440a^7 + 4294516880a^8 - 2369291392a^9 +\\ &\quad 940965248a^{10} - 262912320a^{11} + 48843072a^{12} - 5373952a^{13} + 262144a^{14}\end{aligned}$$

in terms of a which has roots

$$2.97447, \quad 4.91984$$

and 12 non-real roots. Fig.3.14 shows the set $\{x \in \mathbb{R}^2\ ,\ V(x) \leq 1\}$ *which is contained in the DOA of the origin for* $a \in (2.97447, 4.91984)$.

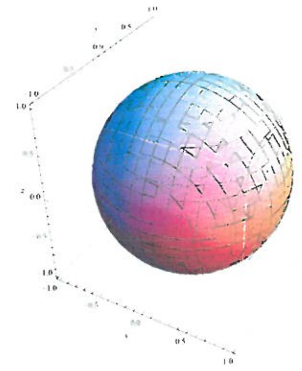

Figure 3.14: Statespace: The surface of v=1

Example 3.2.22. *For the system*

$$\begin{aligned}\dot{x} &= 10(y-x)\\ \dot{y} &= \beta x - y - xz\\ \dot{z} &= xy - 2.67z\end{aligned}$$

if we take

$$V = \frac{1}{2}x^2 + 10y^2 + 10(z-1)^2$$

a local Lyapunov function for the origin $(0,0,0)$ *and* $d = 1$, *then the ideal I choosing the ranking results the polynomial*

$$\begin{aligned} I &= \langle \frac{1}{2}x^2 + 10y^2 + 10(z-1)^2 - 1, 10xy - 10x^2 + 20\beta xy - 20y^2 - 53.4z^2 - 20,\\ &\quad x - \lambda(10y - 20x + 20\beta y), 20y - \lambda(10x - 20\beta x - 40y), 20z - 20 - \lambda(-106.8z)\rangle\end{aligned}$$

using the ranking $x, y, z, \lambda \geq \beta$. *The polynomial of this Groebner basis is*

$$\begin{aligned} p &= -668127 + 396796\beta + 421156\beta^2 - 899576\beta^3 + 512707\beta^4 - 969644\beta^5 -\\ &\quad 487908\beta^6 + \beta^7 + \beta^8\end{aligned}$$

in terms of a which has roots

$$-0.87935, \quad 0.146311$$

and 6 non-real roots. Fig.3.15 shows the set $\{x \in \mathbb{R}^3, V(x) \leq 1\}$ *which is contained in the DOA of the origin for* $\beta \in (-0.87935, 0.146311)$.

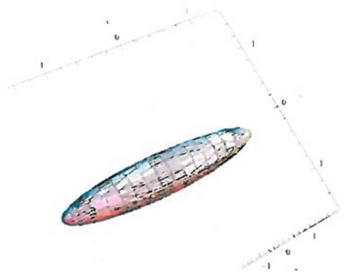

Figure 3.15: Statespace: The surface of v=1

Example 3.2.23. *For the system*

$$\begin{aligned}\dot{x} &= -2x + \beta y^4\\ \dot{y} &= -y\end{aligned}$$

if we take

$$V(x,y) = 6x^2 + 12y^2 + 4xy^4 + y^3$$

be a local Lyapunov function for the origin $(0,0)$ *and* $d = 1$ *, then the ideal I with ranking yields the polynomial of the Groebner basis*

$$\begin{aligned} I &= \langle 6x^2 + 12y^2 + 4xy^4 + y^3 - 1,\\ &\quad -24x^2 + 12\beta xy^4 - 24y^2 - 24xy^4 + 4\beta y^8 - 3y^3,\\ &\quad 12x + 4y^4 - \lambda(-48x + 12\beta y^4 - 24y^4),\\ &\quad 24y + 16xy^3 + 3y^2 - \lambda(48\beta xy^3 - 48y - 96xy^3 + 32\beta y^7 - 9y^2)\rangle \end{aligned}$$

using the ranking $\{x, y, \lambda\} \geq \beta$*. The polynomial of this Groebner basis is*

$$
\begin{aligned}
p \;=\;& 659798454836431404815625 - 13748983737043204175459507309628\beta + \\
& 73864635111735921579287041829325\beta^2 - 218928071958336979498199152827504\beta^3 + \\
& 444172274719484478962617419397434\beta^4 - 679281740172266748768467925247080\beta^5 + \\
& 818858637689595260468957767139130\beta^6 - 798899128809478254307412556864624\beta^7 + \\
& 639650677673075769112842352200141\beta^8 - 423098013527578962831399682372668\beta^9 + \\
& 231455256370878611600640642091785\beta^{10} - 104292080770502144500854778789888\beta^{11} + \\
& 38454290874386023486086162493440\beta^{12} - 11467127910649891538340586819584\beta^{13} + \\
& 2749149882761850789923949474816\beta^{14} - 519773630278734654499231432704\beta^{15} + \\
& 78543289367637919217713428480\beta^{16} - 7888999493592955352241930240\beta^{17} + \\
& 656134568955351504449961984\beta^{18} + 303609176146625943306240\beta^{19} - \\
& 22757356931577324503040\beta^{20} - 2856702732775981056\beta^{21} + 194775186325635072\beta^{22}
\end{aligned}
$$

in terms of a which has roots

$$-253.505, \quad -228.222, \quad 4.79889x10^{-8}, \quad 1.33333, \quad 229.555, \quad 254.838$$

and 16 non-real roots. Fig.3.16 shows the set $\{x \in \mathbb{R}^2, V(x) \leq 1\}$ *which is contained in the DOA of the origin for* $\beta \in (-253.505, 1.33333)$.

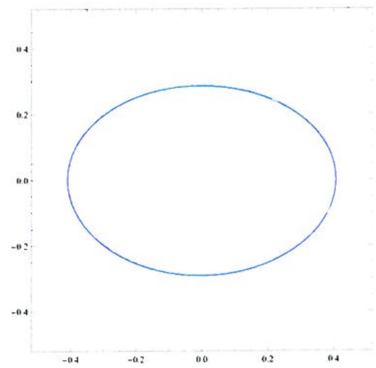

Figure 3.16: Statespace: The curve of v=1

4. RESULTS AND DISCUSSIONS

The theoretical background for checking the existence and determining the jerky dynamics of a given nonlinear autonomous systems is presented in (Linz, 1997; Eichhorn et al., 1998). However, the offered most practical method is the Groebner bases method in the literature. To see how Groebner bases method is efficient in the determination of jerky dynamics we computed all jerky dynamics analytically of the examples given in the section 3.1.2. and we obtained exactly the same results except Example 3.1.5. In this example while Groebner basis method yields $J(x, \dot{x}, \ddot{x}) = 9\ddot{x} + 9x - 5x\ddot{x} - 6x^2 + \dot{x}^2 + \dot{x}x^2 - \dot{x}$, the analytical computation gives $J(x, \dot{x}, \ddot{x}) = 9\ddot{x} + 9x - 5x\ddot{x} - 6x^2 + \dot{x}^2 + \dot{x}x^2 - \ddot{x} + x^3$.

There exist a large number of articles in the literature on how to construct Lyapunov functions, how to determine the critical levels and how to investigate the robustness problem for nonlinear dynamical systems. But, in this thesis we only consider the applications of the Groebner bases method to solve the problems using suggested techniques. In all illustrations we choose 2-and 3-dimensional examples which can be found in any books dynamical systems. We use a few of these systems and corresponding Lyapunov functions directly. Depending on our purpose we introduce a single parameter either in systems or Lyapunov functions. We also compute some Lyapunov functions by ourselves. Such examples are not available in other articles.

The ideals in all worked examples are zero dimensional since the resulting minimal polynomials have finite number of solutions.

5. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, we show how Groebner bases can be used to solve some problems in nonlinear dynamical systems. These problems consists of checking the existence and determining jerky dynamics of given nonlinear systems, constructing Lyapunov functions, finding critical levels of local Lyapunov functions and investigating robustness problem of nonlinear systems. In the applications, it is assumed that the nonlinearities of the systems and the Lyapunov functions to be of polynomial types and the solution quality depends on the chosen Lyapunov function. In each problem, 2-and 3-dimensional nonlinear systems are studied as concrete examples. Since all these systems results zero dimensional ideals after triangulating process Groebner bases method applications yield desired results.

To improve the obtained results in this thesis we may solve the above problems using different techniques suggested in the literature. By this way, we can compare our results and see the efficiency of the Groebner bases method.

Here, we only use Groebner bases method to solve discussed problems we do not dealt with the computational complexity of the method. The main disadvantage with Groebner bases is that their computational complexity is in general high (Forsman, 1993). In such case, in order to reduce computational complexity the alternative methods, like Boege, Gebauer and Kredel algorithm (Boege et al., 1986), resultants (Geddes et al., 1992) or characteristic sets (Chou, 1988) can be used.

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