APPLICATIONS GRÖBNER BASES TECHNIQUES TO TANGENT CONE BASES

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ABSTRACT

APPLICATIONS GROBNER BASES TECHNIQUES TO TANGENT CONE BASES

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Tangent cone bases for a polynomial ideal are defined. The properties of tangent cone bases are studied. Using this properties and Gröbner bases techniques, an alternative algorithm to compute a tangent cone basis of an ideal are developed. Furthermore, a lifting problem for tangent cone basis are defined and solved.

Keywords: Polynomial Ideals, Gröbner Bases, Tangent Cone Bases.

ÖZ

GROBNER TABAN TEKNİKLERİNİN

TANJANT KONİK TABANLARINA UYGULAMALARI

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Bir polinom ideali için tanjant konik tabanı tanımlandı. Tanjant konik tabanının özellikleri incelendi. Bu özellikler ve Gröbner Taban Teknikleri kullanılarak, tanjant konik tabanın hesaplanması için bir algoritma geliştirildi. Ayrica, tanjant konik tabanlar için yükseltme problemi tanımlanıp bunun çözümü bulundu.

Anahtar Kelimeler: Polinom İdealleri, Gröbner Tabanları, Tanjant Konik Tabanlar.

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CHAPTER 1

INTRODUCTION

Let $I \subset k[x_1, \ldots, x_n]$ be an ideal of polynomial ring of several variables over a field k and let $V(I)$ the zero locus of I in the affine space k^n. We can define the following ideal

$$\text{in}(I) = \langle \text{in}(f) : f \in I \rangle$$

where $\text{in}(f)$ is the minimal degree part of the polynomial f. The zero locus of $\text{in}(I)$ is called the tangent cone of $V(I)$ at the origin. Tangent cones gives us information about local behavior of ideal at the origin. Hence finding a basis for $\text{in}(I)$ is crucial. There are several approach to find such a basis; Lazard, in [7], use the homogenization of ideal and Gröbner basis with appropriate monomial ordering, in [10] Mora developed a new division algorithm in order to guarantee termination for initial terms and finally Grabe define new monomial orderings which are not total anymore but they are good enough to make necessary calculation in local rings, see [4].

It is not necessary that if ideal I is given a generating set $\{f_1, \ldots, f_s\}$, then $\text{in}(I)$ is generated by $\{\text{in}(f_1), \ldots, \text{in}(f_s)\}$. A generating set with such property is called a tangent cone basis. There is similar object called homogeneous bases, or briefly H-bases, of an ideal which are extensively studied in literature, see [9], [12] and [13] . Let $\text{LF}(I) = \langle \text{LF}(f) : f \in I \rangle$ where $\text{LF}(f)$ is the maximal degree part of the polynomial f. If $\text{LF}(I) = \langle \text{LF}(f_1), \ldots, \text{LF}(f_s) \rangle$, then $\{f_1, \ldots, f_s\}$ is called an H-basis. While tangent cone bases deal with minimal homogeneous part, the H-bases concern about the maximal homogeneous part of the polynomial. In this thesis, we showed that almost all results obtained about H-bases in the literature can also valid for tangent cone bases.

The organization of this thesis is as follows. In Chapter 2 we give basic concepts about computational algebraic geometry. Chapter 3 includes proofs of properties of tangent cone bases. The proofs are modification of the proofs of properties of H-bases. We also developed a new method for finding a tangent cone basis in last chapter. Finally, we modified lifting problem of homogeneous ideals to tangent cones and we completely solve it.

CHAPTER 2

PRELIMINARY CONCEPTS AND DEFINITIONS

In this chapter we give a number of definitions and theorems used in the subsequent parts of the thesis. More detail can be found in [1], [2] and [3].

2.1 Gröbner Basis for Polynomial Ideals

Let $k[x_1, \ldots, x_n]$ be the ring of the polynomials over a field k.

Definition 2.1 *A* **monomial** *in $x_1, \ldots, x_n$ is a product of the form*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

The total degree of this monomial is the sum $\alpha_1 + \cdots + \alpha_n$.

For brevity, let $x = (x_1, \ldots, x_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the monomial can be written as x^n. We also let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ denote the total degree of the monomial x^α.

Definition 2.2 *A* **polynomial** *f is a finite linear combination of monomials with coefficients in a field k. A polynomial, f, will be written in the form*

$$f = \sum_{\alpha} \alpha_n x^n,$$

where the sum is over a finite number of $n-$tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\alpha_\alpha \in k - \{0\}$. The total degree of f, denoted $\deg(f)$, is the maximum $|\alpha| = \alpha_1 + \cdots + \alpha_n$ in the expression of f.

Given a collection of polynomials, $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, we can consider all polynomials which can be written as a linear combination of these polynomials.

Definition 2.3 *Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$. We let $\langle f_1, \ldots, f_s \rangle$ denote the collection*

$$\langle f_1, \ldots, f_s \rangle = \{a_1 f_1 + \cdots + a_s f_s : a_i \in k[x_1, \ldots, x_n], 1 \le i \le s\}.$$

It is easy to show that $\langle f_1, \ldots, f_s \rangle$ is an ideal of $k[x_1, \ldots, x_n]$ and is called ideal generated by $f_1, \ldots, f_s$.

The well-known Hilbert Basis Theorem states that every ideal of $k[x_1, \ldots, x_n]$ is finitely generated.

Definition 2.4 *We will call the set $k^n = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in k\}$ the affine n-dimensional space over k.*

Definition 2.5 *(i) Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal in $k[x_1, \ldots, x_n]$. The set of zero locus of polynomials of I,*

$$V(I) = \{(a_1, \ldots, a_n) : f_i(a_1, \ldots, a_n) = 0, 1 \le i \le s\} \subset k^n$$

is called variety of ideal I.

(ii) Let $V \subset k^n$ be a variety. The collection of polynomials

$$I(V) = \{f \in k[x_1, \ldots, x_n] : f(a_1, \ldots, a_n) = 0, \forall (a_1, \ldots, a_n) \in V\}$$

is called ideal of the variety V.

We always have $V(I(V)) = V$, but $I(V(I)) = \sqrt{I}$ if k is algebraically closed by Strong Nullstellensatz.

Definition 2.6 *A monomial ordering on* $k[x_1, \ldots, x_n]$ *is any relation* $>$ *on* $\mathbb{Z}^n_{\geq 0}$ *or equivalently, any relation on the set of the monomials* x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$ *satisfying:*

(i) $>$ *is a total ordering on* $\mathbb{Z}^n_{\geq 0}$.

(ii) If $\alpha > \beta$ *and* $\gamma \in \mathbb{Z}^n_{\geq 0}$, *then* $\alpha + \gamma > \beta + \gamma$.

(iii) $>$ *is a well-ordering on* $\mathbb{Z}^n_{\geq 0}$. *This means that every nonempty subset of* $\mathbb{Z}^n_{\geq 0}$ *has a smallest element under* $>$.

Definition 2.7 *A monomial ordering* $>$ *in* $k[x_1, x_2, \ldots, x_n]$ *is called a graded monomial ordering if* $x^\alpha > x^\beta$ *whenever* $|\alpha| > |\beta|$.

Now, we will define the most used monomial orderings. Notice that the last two orderings are graded orderings.

Definition 2.8 *Let* x^α *and* x^β *be monomials in* $k[x_1, \ldots, x_n]$

(i) Lexicographic Order: We say $x^\alpha >_{\text{lex}} x^\beta$ *if in the difference* $\alpha - \beta \in \mathbb{Z}^n$, *the left-most nonzero entry is positive.*

(ii) Graded Lexicographic Order: We say $x^\alpha >_{\text{grlex}} x^\beta$ *if* $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$, *or* $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ *and* $x^\alpha >_{\text{lex}} x^\beta$.

(iii) Graded Reverse Lexicographic Order: We say $x^\alpha >_{\text{gevlex}} x^\beta$ *if* $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$, *or in the difference* $\alpha - \beta \in \mathbb{Z}^n$, *the right-most nonzero entry is negative.*

For example; $x^3y^2z^4 >_{\text{lex}} x^3y^2z$, since $(3, 2, 4) - (3, 2, 1) = (0, 0, 3)$ the left-most nonzero entry is positive, and $x^2y^2z^2 >_{\text{grlex}} xy^4z$ since $2+2+2 = 1+1+4$ but $(2, 2, 2) - (1, 4, 1) = (1, -2, 1)$, the left-most nonzero entry is positive. However, $x^2y^2z^2 <_{\text{grevlex}} xy^4z$ since $2+2+2 = 1+1+4$ but $(1, 4, 1) - (2, 2, 2) = (-1, 2, -1)$, the right-most nonzero entry is negative.

Definition 2.9 *Let* $f = \sum_\alpha a_\alpha x^\alpha$ *be a nonzero polynomial in* $k[x_1, x_2, \ldots, x_n]$ *and let* $>$ *be a monomial order.*

(i) The multidegree of f is

$$\text{multideg}(f) = \max(\alpha \in \mathbf{Z}^n_{\geq 0} : a_\alpha \neq 0)$$

the maximum is taken with respect to $>$.

(ii) The leading monomial coefficient of f is

$$\text{LC}(f) = a_{\text{multideg}(f)} \in k.$$

(iii) The leading monomial of f is

$$\text{LM}(f) = x^{\text{multideg}(f)}.$$

(iv) The leading term of f is

$$\text{LT}(f) = \text{LC}(f)\text{LM}(f).$$

To illustrate, let $f = 4xyz + 4z^2 - 5x^3 + 7x^2z^2$. If our order is lexicographic order, $\text{multideg}(f) = (3, 0, 0)$, $\text{LC}(f) = -5$, $\text{LM}(f) = x^3$ and $\text{LT}(f) = -5x^3$. If we choice graded lexicographic order, $\text{multideg}(f) = (2, 0, 2)$, $\text{LC}(f) = 7$, $\text{LM}(f) = x^2z^2$ and $\text{LT}(f) = 7x^2z^2$.

The following is the well-known division algorithm on $k[x_1, \ldots, x_n]$.

Theorem 2.10 *Fix a monomial order $>$, and let $F = (f_1, \ldots, f_s)$ be an ordered s-tuple of polynomials in $k[x_1, \ldots, x_n]$. Then every $f \in k[x_1, \ldots, x_n]$ can be written as*

$$f = a_1f_1 + \ldots + a_sf_s + r,$$

where $a_i, r \in k[x_1, x_2, \dots, x_n]$, *and either* $r = 0$ *or* r *is a* $k-$*linear combination of monomials, none of which is divisible by any* $\mathrm{LT}(f_1), \dots, \mathrm{LT}(f_s)$. r *is called remainder of* f *on division by F. Furthermore, if* $a_i f_i \neq 0$, *then*

$$\mathrm{multideg}(f) \geq \mathrm{multideg}(a_i f_i)$$

Since we now have a division algorithm in $k[x_1, \dots, x_n]$ that seems to have many of the same features as the one-variable version, it is natural to ask if deciding whether a given $f \in k[x_1, \dots, x_n]$ is a member of a given ideal $I = \langle f_1, \dots, f_s \rangle$ can be done by computing the remainder on division. One direction is easy. Namely, if the remainder $r = 0$ on dividing by $(f_1, \dots, f_s)$, then $f = a_1 f_1 + \cdots + a_s f_s$. By definition then, $f \in \langle f_1, \dots, f_s \rangle$. On the other hand, the following example shows that we are not guaranteed to get remainder $r = 0$ for every $f \in \langle f_1, \dots, f_s \rangle$.

$$p = x^2 + \frac{1}{2}y^2 z - z - 1 = (-\frac{1}{2}z + 1)(x^2 + z^2 - 1) + \frac{1}{2}z(x^2 + y^2 + (z-1)^2 - 4)$$

shows $p \in \langle f_1 = x^2 + z^2 - 1, f_2 = x^2 + y^2 + (z-1)^2 - 4 \rangle$. However the remainder on division of f by (f_1, f_2) is not zero. For instance, using lexicographic order, we get

$$p = 1f_1 + 0f_2 + \frac{1}{2}y^2 z - z^2 - z$$

or

$$p = 0f_1 + 1f_2 + \frac{1}{2}y^2 z - y^2 - z^2 + z + 2$$

The remainder is not zero because it contains terms that cannot be removed by division by this particular generators for I. The leading terms of $f_1 = x^2 + z^2 - 1$

and $f_2 = x^2+y^2+(z-1)^2-4$ the any term of the remainder. In order for division to produce zero remainders for all elements of I, we need to be able to remove all leading terms of elements of I using leading terms of the divisors. That is the motivation for the following definition.

Definition 2.11 *Fix a monomial ordering. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a Gröbner Basis if for every nonzero $f \in I$, $\mathrm{LT}(f)$ is divisible by $\mathrm{LT}(g_i)$ for some i.*

A Gröbner basis for I is indeed a generating set for I. Existence of Gröbner basis for an arbitrary ideal is an easy consequence of Hilbert Basis theorem. Moreover, remainders computed by division with respect to a Gröbner basis are much better behaved than those computed with respect to arbitrary sets of divisors.

Theorem 2.12 *If $G = \{g_1, \ldots, g_s\}$ is a Gröbner basis for I, then for any $f \in I$, the remainder on division f by G is zero.*

More useful for many purposes than the existence proof for Gröbner bases is an algorithm, due to Buchberger, that takes an arbitrary generating set $\{f_1, \ldots, f_s\}$ for I and produce a Gröbner basis G for I from it. This algorithm works by forming new elements of I using expressions guaranteed to cancel leading terms and uncover other possible leading terms.

Definition 2.13 *Let $f, g \in k[x_1, x_2, \ldots, x_n]$ be nonzero polynomials.*

(i) If $\mathrm{multideg}(f) = \alpha$ and $\mathrm{multideg}(g) = \beta$, then let $\gamma = (\gamma_1, \ldots, \gamma_n)$ where $\gamma_i = \max(\alpha_i, \beta_i)$ for each i. x^γ is called the least common multiple of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$, written as $x^\gamma = \mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))$.

(ii) The S-polynomial of f and g is combination

$$S(f,g) = \frac{x^\gamma}{LT(f)}f - \frac{x^{|\gamma}}{LT(g)}g.$$

For example; let $f = x^3y^2 - x^2y^3 + x$ and $g = 3x^4y + y^2$ in $[x, y]$ with the grlex order. Then $\gamma = (4, 2)$ and

$$
\begin{aligned}
S(f, g) &= \frac{x^4y^2}{x^3y^2} f - \frac{x^4y^2}{3x^4y} g \\
&= xf - (1/3)yg \\
&= -x^3y^3 + x^2 - (1/3)y^3.
\end{aligned}
$$

The next theorem gives a criterion for when a basis of an ideal is a Gröbner basis.

Theorem 2.14 *Let I be a polynomial ideal. Then a basis $G = \{g_1, \ldots, g_t\}$ for I Gröbner basis for I if and only if for all pairs $i \neq j$, the remainder on division of $S(g_i, g_j)$ by G (listed in some order) is zero.*

Using this criterion above, Buchberger's Algorithm computes a Gröbner basis for an ideal I from a given basis of that ideal.

Algorithm 2.15 *Let $I = \langle f_1, \ldots, f_s \rangle$ be a polynomial ideal. Then a Gröbner basis for I can be constructed in a finite steps by the following algorithm:*

INPUT: $F = \langle f_1, \ldots, f_s \rangle \subset k[x_1, \ldots, x_n]$ *with* $f_i \neq 0 (1 \leq i \leq s)$

OUTPUT: $G = \{g_1, \ldots, g_t\}$, *a Gröbner basis for* $\langle f_1, \ldots, f_s \rangle$

INITIALIZATION: $G := F, \mathcal{G} := \{(f_i, f_j) | f_i \neq f_j \in G)$

Fix a monomial order.

WHILE $\mathcal{G} \neq \emptyset$ *DO*

  Choose any $(f, g) \in \mathcal{G}$

  $\mathcal{G} := \mathcal{G} - \{(f, g)\}$

  $h := \overline{S(f, g)}^G$

  IF $h \neq 0$ *THEN*

  $\mathcal{G} := \mathcal{G} \cup \{(u, h) | \text{forall} u \in \mathcal{G}\}$

$$G := G \cup \{h\},$$

where $\overline{S(f,g)}^G$ *is remainder on division of* $S(f,g)$ *by* G.

For example; Consider $k[x,y]$ with grlex order, and let $I = \langle f_1, f_2\rangle = \langle x^3 - 2xy, x^2y - 2y^2 + x\rangle$. Recall that $\{f_1, f_2\}$ is not a Gröbner basis for I since $LT(S(f_1, f_2)) = -x^2 \notin \langle LT(f_1), LT(f_2)\rangle$. To produce a Gröbner basis, one natural idea is to try first to extend the original generating set to a Gröbner basis by adding more polynomials in I. We have $S(f_1, f_2) = -x^2 \in I$, and its remainder on division by $F = (f_1, f_2)$ is $-x^2$ which is nonzero. Hence, we should include that remainder in our generating set, as a new generator $f_3 = -x^2$. If we set $F = (f_1, f_2, f_3)$, we can use Theorem 2.14 to test if this new set is a Gröbner basis for I. We compute

$$\begin{aligned}
S(f_1, f_2) &= f_3, \text{ so}\\
\overline{S(f_1, f_2)}^F &= 0,\\
S(f_1, f_3) &= (x^3 - 2xy) - (-x)(-x^2) = -2xy. \text{ but}\\
\overline{S(f_1, f_3)}^F &= -2xy \neq 0.
\end{aligned}$$

Hence, we must add $f_4 = -2xy$ to our generating set. If we let $F = (f_1, f_2, f_3, f_4)$. then

$$\begin{aligned}
\overline{S(f_1, f_2)}^F &= \overline{S(f_1, f_3)}^F = 0,\\
S(f_1, f_4) &= y(x^3 - 2xy) - (-1/2)x^2(-2xy) = -2xy^2 = yf_4, \text{ so}\\
\overline{S(f_1, f_4)}^F &= 0.\\
S(f_2, f_3) &= (x^2y - 2y^2 + x) - (-y)(-x^2) = 2y^2 + x, \text{ but}\\
\overline{S(f_2, f_3)}^F &= -2y^2 + x \neq 0.
\end{aligned}$$

Thus, we must also add $f_5 = -2y^2 + x$ to our generating set. Setting $F = (f_1, f_2, f_3, f_4, f_5)$, one can compute that, for all $1 \leq i < j \leq 5$

$$\overline{S(f_i, f_j)}^F = 0.$$

It follows that a Gröbner basis for I is given by

$$\{f_1, f_2, f_3, f_4, f_5\} = \{x^3 - 2xy, x^2y - 2y^2 + x, -x^2, -2xy, -2y^2 + x\}.$$

2.2 Syzygy Modules

Definition 2.16 *A module over a ring R (or $R-$module) is a set M together with a binary operation, usually written as addition of R on M, and an operation of R on M, called scalar multiplication, satisfying properties.*

a. M is an Abelian group under addition.
b. For all $a \in R$ and all $f, g \in M, a(f + g) = af + ag$.
c. For all $a, b \in R$ and all $f \in M, (a + b)f = af + bf$.
d. For all $a, b \in R$ and all $f \in M, (ab)f = a(bf)$.
e. If is multiplicative identity in R, $1f = f$ for all $f \in M$.

Definition 2.17 *Let M be a module over a ring R. M is said to be a free module if M has a module basis (that is, a generating set which is $R-$linearly independent).*

For instance, the $R-$module $M = R^m$ is a free module. The standard vectors $\mathbf{e_1} = (1, 0, 0, \ldots, 0)^T, \mathbf{e_2} = (0, 1, 0, \ldots, 0)^T, \cdots, \mathbf{e_m} = (0, 0, 0, \ldots, 1)^T$ form one basis for M as $R-$module.

One obtains other examples of $R-$modules by considering submodules of R^m, that is, subsets of R^m which are closed under addition and scalar multiplication by elements of R and which are, therefore, modules in their own right.

We might, for example, choose a finite set of vectors $\mathbf{f_1}, \ldots, \mathbf{f_s}$ and consider the set of all vectors which are can be written as an $R-$linear combination of these vectors:

$$\{a_1\mathbf{f_1} + \cdots + a_s\mathbf{f_s} \in R^m, a_1, \ldots, a_s \in R\}.$$

We denote this set $\langle \mathbf{f_1}, \ldots, \mathbf{f_s} \rangle$, and we can easily show that it is a submodule of R^m. This kind of submodules are called finitely generated submodules.

In this thesis, we consider finitely generated submodules of free module R^m where $R = k[x_1, \ldots, x_n]$ is the polynomial ring over some field k. In particular, we will consider what we call syzygy submodules of R^m.

Definition 2.18 *Let $R = k[x_1, \ldots, x_n]$, and let $(f_1, \ldots, f_t)$ be an ordered $t-$tuple of elements $f_i \in R$. The set of all $(a_1, \ldots, a_t)^T \in R^t$ such that $a_1 f_1 + a_2 f_2 + \cdots + a_t f_t = 0$ is an $R-$submodule of R^t, called the (first) syzygy module of $(f_1, \ldots, f_t)$, and denoted* $\mathrm{syz}(f_1, \ldots, f_t)$.

The syzygy modules are finitely generated $R-$modules, and a set of generators for them can be computed. If the starting point is a Gröbner basis, then finding a set of generators for the syzygy module is accomplished using Buchberger's algorithm. While computing a Gröbner basis $G = \{g_1, \ldots, g_s\}$ for an ideal $I \subset R$ with respect to some fixed monomial ordering, a slight modification of the algorithm would actually compute a set of generators for the syzygy module $\mathrm{syz}(g_1, \ldots, g_s)$ as well as the g_i's themselves.

Let $S(g_i, g_j)$ be the $S-$polynomial of g_i and g_j.

$$S(g_i, g_j) = \frac{x^{\gamma_{ij}}}{LT(g_i)} g_i - \frac{x^{\gamma_{ij}}}{LT(g_j)} g_j,$$

where $x^{\gamma_{ij}}$ is the least common multiple of $\mathrm{LM}(g_i)$ and $\mathrm{LM}(g_j)$. Since G is a Gröbner basis, the remainder of $S(g_i, g_j)$ on division by G is zero, and division algorithm gives an expression

$$S(g_i, g_j) = \sum_{k=1}^{s} a_{ijk} g_k,$$

where $a_{ijk} \in R$, and $\mathrm{LT}(a_{ijk}) \leq \mathrm{LT}(S(g_i, g_j)$ for all i, j, k.

Let $\mathbf{a_{ij}} \in \mathbf{R^s}$ denote the column vector

$$\mathbf{a_{ij}} = a_{ij1}\mathbf{e_1} + a_{ij2}\mathbf{e_2} + \cdots + a_{ijs}\mathbf{e_s} = (a_{ij1}, a_{ij2}, \ldots, a_{ijs})$$

and define $\mathbf{s_{ij}} \in R$ by setting

$$\mathbf{s_{ij}} = \frac{x^{\gamma_{ij}}}{LT(g_i)}\mathbf{e_i} - \frac{x^{\gamma_{ij}}}{LT(g_j)}\mathbf{e_j} + \mathbf{a_{ij}} \qquad (*)$$

in R^s. Then the next theorem shows how to find a basis for the syzygy module.

Theorem 2.19 *Let $G = \{g_1, \ldots, g_s\}$ be a Gröbner basis of an ideal I in R with respect to some fixed monomial order, and let $M = \mathrm{Syz}(g_1, \ldots, g_s)$. The collection $\{\mathbf{s_{ij}}, 1 \leq i, j \leq s\}$ from $(*)$ generates M as an $R-$module.*

The next step is to compute $\mathrm{syz}(f_1, \ldots, f_t)$ for a collection $\{f_1, \ldots, f_t\}$ of nonzero polynomial is R which may not from a Gröbner basis. First compute a Gröbner basis $\{g_1, \ldots, g_s\}$ for $\langle f_1, \ldots, f_t\rangle$. Set $F = (f_1, \ldots, f_t)$ and $G = (g_1, \ldots, g_s)$. Since the elements of F and G generator the same ideal, there are a $t \times s$ matrix A and an $s \times t$ matrix B, both with entries in R, such that $G = FA$ and $F = GB$. The matrix B can be compute by applying the division algorithm with respect to G. The matrix A can be obtained by keeping track of reductions of S-polynomials on the division process during Buchberger's Algorithm.

A generating set $\{\mathbf{s_1}, \ldots, \mathbf{s_p}\}$ for $\mathrm{syz}(G)$ (the $\mathbf{s_i}$'s are column vectors in R^s) can be computed using Theorem 2.19. Let $\mathbf{r_1}, \ldots, \mathbf{r_t}$ be the column of the matrix $I_t - AB$.

Theorem 2.20 *With the notation above,*

$$\{A\mathbf{s_1}, A\mathbf{s_2}, \ldots, A\mathbf{s_p}, \mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_t}\}$$

is a generating set for $\mathrm{Syz}(f_1, \ldots, f_t)$.

for example, let $M = \langle f_1, f_2\rangle$ in $R = k[x, y]$ where

$$f_1 = xy + x, \qquad f_2 = y^2 + 1.$$

Using the lex monomial order, the reduced Gröbner basis for M consist of

$$g_1 = x, \qquad g_2 = y^2 + 1.$$

then it is easy to check that

$$f_1 = (y+1)g_1$$

$$g_1 = -(1/2)(y-1)f_1 + (1/2)xf_2,$$

so that

$$G = (g_1, g_2) = (f_1, f_2)\begin{pmatrix} -(y-1)/2 & 0 \\ x/2 & 1 \end{pmatrix} = FA$$

and

$$F = (f_1, f_2) = (g_1, g_2)\begin{pmatrix} y+1 & 0 \\ 0 & 1 \end{pmatrix} = GB$$

Since

$$S(g_1, g_2) = y^2 g_1 - x g_2 = -x = -g_1,$$

by Theorem 2.19 we have that

$$s_{12} = \begin{pmatrix} y^2+1 \\ -x \end{pmatrix}$$

generates $Syz(g_1, g_2)$. Multiplying By A we get

$$\begin{aligned} As_{12} &= \begin{pmatrix} -(y-1)/2 & 0 \\ x/2 & 1 \end{pmatrix}\begin{pmatrix} y^2+1 \\ -x \end{pmatrix} \\ &= \begin{pmatrix} -(y^3-y^2+y-1)/2 \\ (xy^2-x)/2 \end{pmatrix}. \end{aligned}$$

The columns of $I_2 - AB$ are

$$S_1 = \begin{pmatrix} (y^2+1)/2 \\ -(xy+x)/2 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So by the Theorem 2.20,

$$Syz(f_1, f_2) = \langle As_{12}, S_1 \rangle.$$

2.3 Projective Varieties and Homogeneous Ideals

Definition 2.21 *$n-$dimensional projective space over the field k, denote $P^n(k)$, is the set of equivalence classes of $\sim$ on $k^{n+1} - \{0\}$. Thus,*

$$P^n(k) = (k^{n+1} - \{0\})/\sim .$$

Each nonzero $(n+1)-$tuple $(x_0, ..., x_n) \in k^n + 1$ defines a point p in $P^n(k)$, and we say that $(x_0, ..., x_n)$ are homogeneous coordinates of p.

Proposition 2.22 *Let $f \in k[x_0, ..., x_n]$ be a homogeneous polynomial. If f vanishes on any one set of homogeneous coordinates for a point $p \in P^n$, then f vanishes for all homogeneous coordinates of p. In particular $V(f) = \{p \in P^n(k) : f(p) = 0\}$ is a well-defined subset of $P^n(k)$.*

Definition 2.23 *Let k be a field and let $f_1, ..., f_s \in k[x_0, ..., x_n]$ be homogeneous polynomials. We set*

$$V(f_1, ..., f_s) = \{(a_0, ..., a_n) \in P^n(k) : f_i(a_0, ..., a_n) = 0 forall 1s\}.$$

We call $V(f_1, ..., f_s)$ the projective variety defined by $f_1, ..., f_s$.

Proposition 2.24 *Let $g(x_1, ..., x_n) \in k[x_1, ..., x_n]$ be polynomial of total degree d.*

(i) Let $g = \sum_{i=0}^{d} g_i$ be be the expansion of g as the sum of its homogeneous components, where g_i has total degree i. Then

$$\begin{aligned} g^h(x_0, ..., x_n) &= \textstyle\sum_{i=0}^{d} g_i(x_1, ..., x_n) x_0^{d-i} \\ &= g_d(x_1, ..., x_n) + g_{d-1}(x_1, ..., x_n) x_0 + \cdots + g_0(x_1, ..., x_n) x_0^d \end{aligned}$$

is a homogeneous polynomial of total degree d in $k[x_0, ..., x_n]$. We will call g^h the homogenization of g.

(ii) The homogenization g can be compute using for formula

$$g^h = x_0^d g(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}).$$

(iii) Dehomogenizing g^h yields g. That is, $g^h(1, x_1, ..., x_n) = g(x_1, ..., x_n)$.

(iv) Let $F(x_0, ..., x_n$ be a homogeneous polynomial and let x_0^e to be highest power of x_0 dividing F.If $f = F(1, x_1, ..., x_n)$ is the homogenization of F, then $F = x_0^e f^h$.

Definition 2.25 *An ideal I in $k[x_0, .., x_n]$ is said to be homogeneous if for each $f \in I$, the homogeneous components f_i of f are in I as well.*

Theorem 2.26 *Let $I \subset k[x_0, ..., x_n]$ be an ideal. Then the following are equivalent:*

(i) I is an homogeneous ideal of $k[x_0, ..., x_n]$.

(ii) $I = \langle f_1, ..., f_s \rangle$, where $f_1, ..., f_s$ are homogeneous polynomials.

(iii) A reduced Gröbner basis of I (whith respect to any monomial ordering) consists of homogeneous polynomials.

Proposition 2.27 *Let $I \subset k[x_0, ..., x_n]$ be a homogeneous ideal and suppose that $I = \langle f_1, ..., f_s \rangle$, where $f_1, ..., f_s$ are homogeneous. Then*

$$V(I) = V(f_1, ..., f_s),$$

so that $V(I)$ is a projective variety.

CHAPTER 3
TANGENT CONE BASIS OF AN IDEAL

In this chapter, we define tangent cone basis and proof some of its properties. These properties are modification of that of homogeneous bases. While homogeneous bases deal with maximum degree parts of polynomials, we will concern minimum degree parts of polynomial in tangent cone bases. Homogenous bases are studied in [13], [9] and [12].

3.1 Tangent Cone

Definition 3.1 *Let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. The ideal I gives us the projective variety $V = V(I) \subset \mathbb{P}^n(k)$. On the other hand, we may consider zero set of I in the affine space k^{n+1}. $C_V = V(I) \in k^{n+1}$ is an affine variety which uses same ideal I, but we look solutions in the affine space k^{n+1}. We call C_V the affine cone of V because if we interpret points in $\mathbb{P}^n(k)$ as lines through origin in k^{n+1}, then C_V is the union of the lines determined by the points of V.*

Suppose that $p = (p_1, \ldots, p_n) \in k^n$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, let

$$(x - p)^\alpha = (x_1 - p_1)^{\alpha_1} \cdots (x_n - p_n)^\alpha_n$$

and note that $(x - p)^\alpha$ has total degree $|\alpha| = \alpha_1 + \cdots \alpha_n$. Now, given any polynomial $f \in k[x_1, \ldots, x_n]$of total degree d, we can write f as polynomial in $x_i - p_i$, so that f is a k-linear combination of $(x - p)^\alpha$ for $|\alpha| \leq d$. If we group according to total degree, we can write

$$f = f_{p,0} + f_{p,1} + \cdots + f_{p,d},$$

where $f_{p,j}$ is a k-linear combination of $(x-p)^\alpha$ for $|\alpha| = j$.

Definition 3.2 *Let $V \subset k^n$ be an affine variety and let $p = (p_1, \ldots, p_n) \in V$.*

(i) If $f \in k[x_1, \ldots, x_n]$ is a nonzero polynomial, then $f_{p,min}$ is defined to be $f_{p,j}$ where j is the smallest integer such that $f_{p,j} \neq 0$.

(ii) The tangent cone of V at p, denoted $C_p(V)$, is the variety

$$C_p(V) = V(f_{p,min} : f \in I(V))$$

The tangent cone gets its name from the following proposition.

Proposition 3.3 *Let $p \in V \subset k^n$. Then $C_p(V)$ is a linear transformation of the affine cone o variety in $\mathbb{P}^{n-1}(k)$.*

Proof. Introduce new coordinates on k^n by letting $X_i = x_i - p_i$. Relative to this coordinate system, we can assume that p is the origin. Let $J = \langle f_{0,min} : f \in I(V) \rangle$. Then $C_p(V) = V(J)$. Since J is the homogeneous ideal, we also get a projective variety $W = V(J) \in \mathbb{P}^{n-1}(k)$, and this means $C_p(V)$ is an affine cone $C_W \in k^n$ of W. $\square$

Now, we will define tangent cone of an ideal and tangent cone basis of an ideal. Since tangent cone at an arbitrary point p can be translate to the tangent cone at the origin by a change of coordinates, we will only consider tangent cone at the origin.

Definition 3.4 *Let I be an ideal in $k[x_1, \ldots, x_n]$. The tangent cone of I is the ideal*

$$\text{tancone}(I) = \langle \text{in}(f) : f \in I \rangle$$

where $\text{in}(f)$ *is the nonzero homogenous part of minimum degree of* f*.*

Definition 3.5 *Let* I *be an ideal in* $k[x_1, \ldots, x_n]$*. The set* $\{f_1, \ldots, f_s\}$ *is a tangent cone basis for* I *if* $I = \langle f_1, \ldots, f_s \rangle$ *and* $\text{tancone}(I) = \langle \text{in}(f_1), \ldots, \text{in}(f) \rangle$*.*

Not all bases of an ideal are tangent cone bases. Consider $I = \langle f_1 = xy, f_2 = xz + y^2z - z^3 \rangle$. Clearly, $\text{in}(f_1) = xy$ and $\text{in}(f_2) = xz$. Then $f = zf_1 + yf_2 = y^3z - yz^3 \in I$, but $\text{in}(f) = y^3z - yz^3 \notin \langle xy, xz \rangle$. Hence $\{f_1 = xy, f_2 = xz + y^2z - z^3\}$ is not tangent cone basis for I.

Finding tangent cone of an ideal studied by several mathematicians. Lazard use homogenization of polynomials and Grobner basis techniques and Mora developed a division algorithm which is called Mora's tangent cone algorithm. Then Grabe construct whole new group of monomial orderings which are called local orderings and he showed that the Grobner bases for this orderings can also be computed. In this thesis, we follow Lazard's technique.

Theorem 3.6 *Let* I *be an ideal in* $k[x_1, \ldots, x_n]$*. Let* x_0 *be new variable and pick a monomial order on* $k[x_0, \ldots, x_n]$ *such that among the monomials of same total degree, any monomial involving* x_0 *is grater than any monomial involving only* $x_1, \ldots, x_n$ *and if total degrees are different monomial with bigger total degree is grater than the other.*

Let $I = \langle f_1, \ldots, f_s \rangle$*. Compute the Gröbner Basis* $\{G_1, \ldots, G_t\}$ *of* $\langle F_1, \ldots, F_s \rangle$ *with respect to above order where* F_i *is the homogenization of* f_i*. Then* $\{g_1, \ldots, g_t\}$ *is the tangent cone basis of* I *where* $g_i = G_i(1, x_1, \ldots, x_n)$*.*

Proof. Suppose $g \in I$; we must show that $\text{in}(g)$,s a linear combinations of $\text{in}(g_i)$. Write $g = \sum p_i f_i$. If G, P_i, and F_i are homogenization of g,p_i, af_i, respectively, then for some integers a, b we have

$$x_0^a G = \sum P_i F_i \in \langle F_1, \dots F_s \rangle$$

$$x_0^a G = x_0^b \mathrm{in}(g) + (\mathrm{terms of lower degree in} x_0).$$

Because the G_i's form a Gröbner basis for $\langle F_1, \dots, F_s \rangle$, there is a standart expression

$$x_0^a = \sum Q_i G_i, \qquad \mathrm{LT}(Q_i G_i) \leq \mathrm{LT}(x_0^a G).$$

In particular, the degree in x_0 of $Q_i G_i \leq b$ for each i.

It follows that $x_0^b \mathrm{in}(g)$ is the sum of the products of the terms of highest degree in x_0 in some of the Q_i and G_i. Setting $x_0 = 1$, we see that $\mathrm{in}(g)$ itself is the sum of products of the terms of lowest degree in those $Q_i(1, x_1, \dots, x_n)$ and $G_i(1, x_1, \dots, x_n)$; that is, it is a linear combination of $\mathrm{in}(g_i)$, as claimed. $\square$

Going back to above example; if we compute Gröbner Basis of $I = \langle f_1^h = xy, f_2^h = txz + y^2 z - z^3 \rangle$ with respect to graded lexicographic order with $t > x > y$, then the result is $\{xy, txz + y^2 z - z^3, y^3 z - yz^3\}$. Hence dehomogenization gives the tangent cone basis, $\{xy, xz + y^2 z - z^3, y^3 z - yz^3\}$.

3.2 Minimal Tangent Cone Basis

The following theorem gives us a criterium when a basis is a tangent cone basis.

Theorem 3.7 *A basis $\{f_1, \dots, f_s\}$ is a tangent cone basis for an ideal I if and only if for every $f \in I$ there exists polynomials $a_1, \dots, a_s$ such that $f = a_1 f_1 + \dots + a_s f_s$ where* $\deg(\mathrm{in}(f)) = \min\{\deg(\mathrm{in}(a_i f_i))\}$.

Proof. If for every $f \in I$ there exists polynomials $a_1, \dots, a_s$ such that $f = a_1 f_1 + \dots + a_s f_s$ where $\deg(\mathrm{in}(f)) = \min\{\deg(\mathrm{in}(a_i f_i))\}$, then $\mathrm{in}(f) \in \langle \mathrm{in}(f_1), \dots, \mathrm{in}(f_s) \rangle$.

Hence $\{f_1, \ldots, f_s\}$ is a tangent cone basis.

Assume that $\{f_1, \ldots, f_s\}$ is a tangent cone basis and let $g \in I$. So $\text{in}(g) \in \langle \text{in}(f_1), \ldots, \text{in}(f_s) \rangle$ which implies there exists polynomials $a_1^0, \ldots, a_s^0$ such that $\text{in}(g) = a_1^0 \text{in}(f_1) + \cdots + a_s^0 \text{in}(f_s)$. Consider the polynomial

$$h = a_1^0 f_1 + \ldots + a_s^0 f_s.$$

Notice that $\text{in}(h) = \text{in}(g)$. Let $g_1 = g - h$. Clearly $g_1 \neq g$. Hence if $g_1 \neq 0$, we have the ascending chain $\langle g \rangle \subset \langle g, g_1 \rangle$. If $g_1 = 0$, then $g = h$ which means $g = a_1^0 f_1 + \ldots + a_s^0 f_s$ where $\deg(\text{in}(g)) = \min\{\deg(\text{in}(a_i^0 f_i))\}$ as desired.

Otherwise, since $g_1 \in I$ and $\{f_1, \ldots, f_s\}$ is a tangent cone basis, there exists polynomials $a_1^1, \ldots, a_s^1$ such that $\text{in}(g_1) = a_1^1 \text{in}(f_1) + \cdots + a_s^1 \text{in}(f_s)$. Notice that $\deg(\text{in}(g_1)) > \text{degin}(g))$and therefore $\deg(a_i^1) > \deg(a_i^0)$. Consider the polynomial

$$h_1 = a_1^1 f_1 + \ldots + a_s^1 f_s.$$

Let $g_2 = g_1 - h_1$. If $g_2 \neq$, then we have $\langle g \rangle \subset \langle g, g_1 \rangle \subset \langle g, g_1, g_2 \rangle$. Otherwise,

$$g = h + h_1 = (a_1^0 + a_1^1) f_1 + \ldots + (a_s^0 + a_s^1) f_s$$

where $\deg(\text{in}(g)) = \deg(\text{in}(a_i^0 f_i)) = \min\{\deg(\text{in}((a_i^0 + a_i^1) f_i))\}$ as desired.

Since $k[x_1, \ldots, x_n]$ is Noetherian, ascending chain of ideals must terminate. Hence after finite number of step $g_t = 0$. Then

$$g = h + h_1 + \cdots + h_{t-1} = (a_1^0 + a_1^1 + \cdots + a_1^{t-1}) f_1 + \ldots + (a_s^0 + a_s^1 + \cdots a_s^{t-1}) f_s$$

where $\deg(\text{in}(g)) = \deg(\text{in}(a_i^0 f_i)) = \min\{\deg(\text{in}((a_i^0 + a_i^1 + \cdots + a_i^{t-1}) f_i))\}$ as desired. $\square$

In general, a tangent cone basis contains some unnecessary elements. In other words, a subset of this basis can be still a tangent cone basis.

Theorem 3.8 *Let $\{f, f_1, \ldots, f_s\}$ be a tangent cone basis for an ideal I. If there exists polynomials $a_1, \ldots, a_s$ such that $f = a_1 f_1 + \cdots + a_s f_s$ where* $\deg(\mathrm{in}(f)) = \min\{\deg(\mathrm{in}(a_i f_i))\}$*, then $\{f_1, \ldots, f_s\}$ is also a tangent cone basis.*

Proof. Let $g \in I$. Since $\{f_1, \ldots, f_s, f\}$ is a tangent cone basis, there exists polynomials $b_1, \ldots, b_{s+1}$ such that

$$g = \sum_{i=1}^{s} b_i f_i + b_{s+1} f$$

and $\deg(\mathrm{in}(g)) = \min\{\deg\{\mathrm{in}(b_i f_i)), \deg(\mathrm{in}(b_{s+1} f))\}$. Substituting the given expression for f into above equation,

$$g = \sum_{i=1}^{s} (b_i + b_{s+1} a_i) f_i.$$

We know that $\deg(\mathrm{in}(g)) \geq \min\{\deg(\mathrm{in}((b_i + b_{s+1} a_i) f_i))\}$.

On the other hand, using the facts that $\deg(\mathrm{in}(fg)) = \deg(\mathrm{in}(f)) + \deg(\mathrm{in}(g))$ and $\deg(\mathrm{in}(f+g)) \geq \min\{\deg(\mathrm{in}(f)), \deg(\mathrm{in}(g))\}$ for any polynomials f and g,

$$\begin{aligned}
\min_{1\leq i\leq s}\{\deg(\mathrm{in}((b_i + b_{s+1} a_i) f_i))\} &\geq \min_{1\leq i\leq s}\{\deg(\mathrm{in}(b_i f_i)), \deg(\mathrm{in}(a_i f_i)) + \deg(\mathrm{in}(b_{s+1}))\} \\
&= \min_{1\leq i\leq s}\{\deg(\mathrm{in}(b_i f_i)), \deg(\mathrm{in}(f)) + \deg(\mathrm{in}(b_{s+1}))\} \\
&= \min_{1\leq i\leq s}\{\deg(\mathrm{in}(b_i f_i)), \deg(\mathrm{in}(b_{s+1} f))\} \\
&= \deg(\mathrm{in}(g)).
\end{aligned}$$

Hence $\deg(\mathrm{in}(g)) = \min_{1\leq i\leq s}\{\deg(\mathrm{in}(b_i + (b_{s+1} a_i) f_i))\}$ Therefore $\{f_1, \ldots, f_s\}$ is a tangent cone basis for I. □

Definition 3.9 *A tangent cone basis $\{f_1, \ldots, f_s\}$ is called minimal if there exists no f_i such that $f_i = \sum_{j \neq i} a_j f_j$ where* $\deg(\mathrm{in}(f_i)) = \min\{\deg(\mathrm{in}(a_j f_j)), j \neq i\}$

For example; let $I = \langle x^2 - y, x^3 - z \rangle$. If we homogenize the elements with respect to new variable and find a Gröbner basis using degree lexicographic order with $t > x > y > z$, the result is $\{x^2 - yt, x^3 - zt^2, x^3y - x^z t, x^3y^2 - x^4 z\}$. Hence $\{x^2 - y, x^3 - z, x^3y - x^2z, x^3y^2 - x^4z\}$ is a tangent cone basis. However, $x^3y - x^2z = -x^3(x^2 - y) + x^2(x^3 - z)$ and

$$\deg(\mathrm{in}(x^3y - x^2z)) = \deg(x^2z) = \min\{\deg(\mathrm{in}(-x^3)\mathrm{in}(x^2 - y)), \deg(\mathrm{in}(x^2)\mathrm{in}(x^3 - z))\}.$$

Furthermore, $x^3y^2 - x^4z = (-x^3y - x^5)(x^2 - y) + x^4(x^3 - z)$ and

$$\deg(\mathrm{in}(x^3y^2 - x^4z)) = \min\{\deg(\mathrm{in}(-x^3y - x^5)\mathrm{in}(x^2 - y)), \deg(\mathrm{in}(x^4)\mathrm{in}(x^3 - z))\}.$$

Hence if we drop last two elements from the tangent cone basis, then we still have a tangent cone basis. Therefore, the original basis is a tangent cone basis, in fact, it is a minimal tangent cone basis since further elimination is impossible.

Definition 3.10 *let $H_1 = \{f_1, ..., f_s, g_1, ..., g_t\}$ be a tangent cone basis for an ideal $I \subset k[x_1..., x_n]$ and $H_2 = \{g_1, ...g_t, h_1, ..., h_p\}$ be a minimal tangent cone basis for the same ideal. Then $p \leq s$.*

Proof. Assume that $p > s$. Since H_2 is a tangent cone basis, there exist polynomials A_{ij} and B_{ir} such that

$$f_i = \sum_{j=1}^{t} A_{ij} g_j + \sum_{r=1}^{p} B_{ir} h_r$$

and

$$\deg(\mathrm{in}(f_i)) = \min_{1 \leq j \leq t} \{\deg(\mathrm{in}(A_{ij} g_j)), \deg(\mathrm{in}(B_{ir} h_r))\}.$$

Similarly since H_1 is a homogeneous basis,

$$h_1 = \sum_{q=1}^{s} D_{1q} f_q + \sum_{j=i}^{t} E_{1j} g_j$$

and

$$\deg(\mathrm{in}(h_1)) = \min_{1 \leq q \leq s} \{\deg(\mathrm{in}(D_{1q} f_q)), \deg(\mathrm{in}(E_{1j} g_j))\}.$$

Clearly $H = \{f_1, ..., f_s, g_1, ..., g_t, h_1\}$ is also tangent cone basis. Now let us substitute f_i's into the equation,

$$h_1 = \sum_{j=1}^{t} \sum_{q=1}^{s} (D_{1q} A_{qj} + E_{1j}) g_j + \sum_{r=1}^{p} (\sum_{q=1}^{s} D_{1q} B_{qr}) h_r.$$

We claim that

$$\deg(\mathrm{in}(h_1)) = \min_{1 \leq j \leq t} \{\deg(\mathrm{in}(\sum_{q=1}^{s} (D_{1q} A_{qj} + E_{1j}) g_j)), \deg(\mathrm{in}((\sum_{q=1}^{s} D_{1q} B_{qr}) h_r))\} = M.$$

Since some cancellations may occur on the right hand side, $\deg(\mathrm{in})(h_1) \geq M$.

$$\begin{aligned} M &\geq \min\{\deg(\mathrm{in}(D_{1q} A_{qj} g_j)), \deg(\mathrm{in}(E_{1j} g_j)), \deg(\mathrm{in}(D_{1q} B_{qr} h_r))\} \\ &\geq \min\{\deg(\mathrm{in}(D_{1q})) + \deg(\mathrm{in}(f_q)), \deg(\mathrm{in}(h_1)), \deg(\mathrm{in}(D_{1q})) + \deg(\mathrm{in}(f_q))\} \\ &\geq \min\{\deg(\mathrm{in}(D_{1q} f_q)), \deg(\mathrm{in}(h_1)\} \\ &\geq \min\{\deg(\mathrm{in}(h_1))\} \end{aligned}$$

Hence

$$\deg(\mathrm{in}(h_1)) = \min_{1 \leq j \leq t} \{\deg(\mathrm{in}(\sum_{q=1}^{s} (D_{1q} A_{qj} + E_{1j}) g_j)), \deg(\mathrm{in}((\sum_{q=1}^{s} D_{1q} B_{qr}) h_r))\} = M.$$

Then

$$h_1 = \sum_{j=1}^{t}\sum_{q=1}^{s}(D_{1q}A_{qj} + E_{1j})g_j + \sum_{r=2}^{p}(\sum_{q=1}^{s} D_{1q}B_{qr})h_r + \sum_{q=1}^{s} D_{1q}B_{q1}h_1.$$

Since deg(in(h_1)) is equal to the .... degree on the right hand side, $\sum_{q=1}^{s} D_{1q}B_{1q}$ will be constant. If $\sum_{q=1}^{s} D_{1q}B_{q1} \neq 1$, then we can write h_1 is a combination of $\{g_1, ..., g_t, h_2, ..., h_p\}$. Therefore $\{g_1, ..., g_t, h_2, ..., h_p\}$ is homogeneous basis. However this contradicts with the fact that H_2 is a minimal basis. Hence $\sum_{q=1}^{s} D_{1q}B_{q1} = 1$.

We know that $\deg(\text{in}(h_1)) \leq \deg(\text{in}(D_{1q}f_q))$ and $\deg(\text{in}(f_q)) \leq \deg(\text{in}(B_{q1}h_1))$ for $q = 1..., s$. Therefore $D_{1q}B_{q1}$ is a constant for $q = 1, ..., s$ and $D_{11}B_{11} + \cdots + D_{1s}B_{1s} = 1$. this implies that there exists at least one integer q_0 such that $D_{1q0}B_{1q0} \neq 0$. Therefore $D_{1q0} \neq 0, B_{q01} \neq 0$ and they are constant. Since

$$h_1 = \sum_{q \neq q0}^{s} D_{1q}f_q + D_{1q0}f_{q0} + \sum_{j=1}^{t} E_{1j}g_j,$$

$$f_{q0} = \frac{1}{D_{1q0}}(h_1 - \sum_{n \neq q0}^{s} D_{1q}f_q - \sum_{j=1}^{t} E_{1j}g_j).$$

Since B_{q01} is constant,

$$\begin{aligned}
\min\{\deg(\text{in}(h_1)), \deg(\text{in}(\sum_{q \neq q0}^{s} D_{1q}f_q)), \deg(\text{in}(\sum_{j=1}^{t} E_{1j}g_j))\} &= \deg(\text{in}(h_1)) \\
&= \deg(\text{in}(B_{q01}h_1)) \\
&\geq \deg(\text{in}(f_q0)).
\end{aligned}$$

Then

$$\deg(\text{in}(f_{q0})) = \min\{\deg(\text{in}(h_1)), \deg(\text{in}(\sum_{q\neq q0}^{s})D_{1q}f_q), \deg(\text{in}(\sum_{j=1}^{t} E_{1j}g_j))\}$$

Since the reverse inequality always holds.Hence

$H_3 = H - \{f_{q0}\} = \{f_1, ..., f_{q0-1}, f_{q0+1}, ..., f_s, g_1, ..., g_t, h_1\}$ is a homogeneous basis by Theorem 3.8 The idea now continue our produce stepwise, and to replace f_i's by $h_1, h_2, ...$ until all the polynomials $f_1, ..., f_s$ are exhausted. So after $s-$ steps, we obtain the homogeneous basis $\{g_1, ..., g_t, h_1, ..., h_{p-s}\}$. It is clear that $\{g_1, ..., g_t, h_1..., h_{p-s}\}$ is a subset of H_2. This is a contradiction since H_2 is a minimal homogeneous basis. Hence $p \leq s$. $\square$

3.3 Tangent Cone Bases and Syzygy Bases

Finally, we give a relation between a tangent cone basis and syzygy basis of initial parts of basis elements. This relation will be used to describe a new method to find a tangent cone basis of an ideal in the last chapter.

Theorem 3.11 *Let $I = \langle f_1, \ldots, f_t \rangle \subset k[x_1, \ldots, x_n]$. Let the columns of the $t \times l$ matrix $S = (s_{ij})$be a generating set of $\text{syz}\{\text{in}(f_1), \ldots, \text{in}(f_t)\}$. We may assume further that each $s_{ij}\text{in}(f_j)$ is a homogeneous polynomial of same degree for $j = 1, \ldots, t$. Then $F = \{f_1, \ldots, f_t\}$ is a tangent cone basis if and only if*

$$q_i = \sum_{j=1}^{t} s_{ji} f_j = \sum_{j=1}^{t} s_{ji}(f_j - \text{in}(f_j)) = \sum_{j=1}^{t} a_{ji} f_j$$

for some $a_{ij} \in k[x_1, \ldots, x_n]$ such that $\deg(\text{in}(q_i)) = \min\{\deg(\text{in}(a_{ji}f_j)) : j = 1, \ldots, t\}$ for $i = 1, \ldots, l$.

Proof. $\Rightarrow$: Assume F is a tangent cone basis. Since $q_i \in I$, q_i can be expressed in the desired form by Theorem 3.7.

$\Leftarrow$: Given $f \in I$, there are polynomials $g_j \in k[x_1, \ldots, x_n]$ such that

$$f = \sum_{i=1}^{t} g_j f_j. \qquad (*)$$

Since some cancellations may occur on the right-hand side,

$$\deg(\mathrm{in}(f)) \geq \min\{\deg(\mathrm{in}(g_j f_j))\}.$$

Given an expression (*) for f, let $m(j) = \deg(\mathrm{in}(g_j f_j))$, and define $d = \min(m(j))$. Then $\deg(\mathrm{in}(f)) \geq d$. Now, consider all possible ways that f can be expressed the form (*). Let us fix an expression (*) for f such that d is maximal. This is possible because Effective Nullstellenzats gives upper bounds for such expressions. If it can be shown that $\deg(\mathrm{in}(f)) = d$, then $\mathrm{in}(f) \in \langle \mathrm{in}(f_1), \ldots, \mathrm{in}(f_t)\rangle$. This will prove the theorem.

Assume $\deg(\mathrm{in}(f)) > d$. To isolate the forms of degree d let us write f in the following form:

$$f = \sum_{m(j)=d} g_j f_j + \sum_{m(j)>d} g_j f_j = \sum_{m(j)=d} \mathrm{in}(g_j) f_j + \sum_{m(j)=d} (g_j - \mathrm{in}(g_j)) f_j + \sum_{m(j)>d} g_j f_j$$

Since $\deg(\mathrm{in}(f)) > d$, $\deg(\sum \mathrm{in}(g_j) f_j) > d$. Therefore,

$$\sum_{m(j)=d} \mathrm{in}(g_j)\mathrm{in}(f_j) = 0.$$

Define

$$h_j = \begin{cases} \mathrm{in}(g_j), & m(j) = d; \\ 0, & \text{otherwise.} \end{cases}$$

Then $(h_1, \ldots, h_t)$ is a syzygy for $(\mathrm{in}(f_1), \ldots, \mathrm{in}(f_t))$ which implies

$$h_j = \sum_{i=1}^{l} b_i S_{ji},$$

where $b_i \in k[x_1, \ldots x_n]$. Thus,

$$\sum_{m(j)=d} \mathrm{in}(g_j) f_j = \sum_{j=1}^{t} h_j f_j = \sum_{j=1}^{t} (\sum_{i=1}^{l} b_i s_{ji}) f_j = \sum_{i=1}^{l} \sum_{j=1}^{t} b_i s_{ji} f_j.$$

By our assumption,

$$q_i = \sum_{j=1}^{t} s_{ji} f_j = \sum_{j=1}^{t} a_{ji} f_j$$

and $\deg(\mathrm{in}(s_{ji} f_j)) < \deg(\mathrm{in}(q_i)) = \min\{\deg(\mathrm{in}(a_{ji} f_j)) : j = 1, \ldots, t\}$. Therefore, $d = \deg(\mathrm{in}(h_j f_j)) = \deg(\mathrm{in}(b_i s_{ji} f_j)) < \deg(\mathrm{in}(b_i a_{ji} f_j))$, and

$$\sum_{m(j)=d} \mathrm{in}(g_j) f_j = \sum_{j=1}^{t} (\sum_{i=1}^{l} b_i a_{ji}) f_j.$$

If the last expression is substituted into (**), then f can be expressed as linear combination of f_j's, where all the initial terms have degree greater than d. This contradicts the choice of d and completes the proof of the theorem. $\square$

CHAPTER 4

ALGORITHMIC PROBLEMS IN TANGENT CONE BASES

4.1 An Alternative Algorithm for Computation of Tangent Cone Basis

The relation between syzygy basis of initial parts and tangent cone basis given by Theorem 3.11 tends to an alternative way to compute a tangent cone basis of a given ideal. Before explain the method, we need the following definition.

Definition 4.1 *Let* $F = (f_1, \ldots, f_t)$ *be a t-tuples of polynomials and* f *be any polynomial such that* $\mathrm{in}(f) = h_1\mathrm{in}(f_1) + \cdots h_t\mathrm{in}(f_t)$*. We say that* f *reduces to* $\tilde{f}$ *modulo* F *if*

$$\tilde{f} = f - \sum_{i=1}^{t} h_i f_i$$

and we write $f \longrightarrow_F \tilde{f}$*.*

Algorithm 4.2 *The following algorithm computes a tangent cone basis for an ideal from a given basis of that ideal.*

Input: A basis $F := \{f_1, \ldots, f_t\}$ *of a ideal.*

Output: A tangent cone basis $H := \{f_1, \ldots, f_p\}$ *of the same ideal.*

$H := F$

$t := |H|$

$S = \mathrm{syz}(\mathrm{in}(H))$

WHILE $S \neq \{\}$

For $s \in S$, $q := \sum_{i=1}^{t} s_i f_i$

$S := S - \{s\}$

WHILE $\mathrm{in}(q) \in \langle \mathrm{in}(f_1), \ldots, \mathrm{in}(f_t) \rangle$

$q \longrightarrow_F \tilde{q}$.

$q := \tilde{q}$

ENDWHILE

IF $q \neq 0$ *THEN*

$t := t + 1$; $f_t := q$; $H := H \cup \{f_t\}$; $S = \mathrm{syz}(\mathrm{in}(H))$

ENDIF

ENDWHILE

The algorithm suggests the following procedure for computing tangent cone bases. Given a set of polynomials $H = \{f_1, \ldots, f_p\}$ generating an ideal I, first the elements of a basis of

$$\mathrm{syz}(\mathrm{in}(f_1), \ldots, \mathrm{in}(f_p))$$

have to be computed. Then for each basis elements $(s_1, \ldots, s_p)$, the polynomial q as given in the algorithm has to reduced modulo H as far as possible. If reduction procedure terminates with a nonzero polynomial f, the set H is enlarged by f. For the new set H, a basis for the module of syzygies of initial parts is computed etc.

This procedure differs from Buchberger's algorithm only be the module of syzygies. Instead of a syzygy basis of initial parts of polynomials of the generating set, one consider for Gröbner basis computation the S-polynomials which is in fact a syzygy basis of leading terms of generating polynomials. As Buchberger's algorithm terminates correctly with a Gröbner basis, our proposed algorithm terminates with a tangent cone basis.

Example 4.3 *Consider the ideal* $I = \langle f_1 = x^2y - yz, f_2 = y^3 + xz + y2, f_3 = xz^2 + y^2\rangle \in \mathbb{Q}[x, y, z]$. *Then* $g_1 = \mathrm{in}(f_1) = -yz, g_2 = \mathrm{in}(f_2) = xz + y^2$ *and* $g_3 = \mathrm{in}(f_3) = y^2$. *A syzygy basis of* $(g_1, g_2, g - 3)$ *is given by the columns of the following matrix;*

$$\begin{pmatrix} y & x \\ 0 & y \\ z & -y \end{pmatrix}$$

Considering the first column, let $h_1 = yf_1 + zf_3 = x^2y^2 + xz^3$. Then

$$h_1 = yzg_1 + z^2g_2 + x^2g_3.$$

Define

$$h_2 = h_1 - yzf_1 - z^2f_2 - x^2f_3 = -x^3z^2 - y^3z^2 - x^2y^2z.$$

Dividing h_2 by (g_1, g_2, g_3),

$$h_2 = y^2zg_1 - x^2zg_2.$$

Now define

$$h_3 = h_2 - y^2zf_1 + x^2zf_2 = 0.$$

Hence h_1 reduced to zero.

Consider the second column, let

$$h_1 = xf_1 + yf_2 - yf_3 = x^3y + y^4 - xyz^2.$$

Since h_1 can not be reduced in terms of (g_1, g_2, g_3), we define $f_4 = h_1 = x^3y + y^4 - xyz^2$ and $g_4 = \text{in}(f_4) = f_4$. Hence we have to compute syzygy basis for new generating set. The columns of the following matrix is a basis for $\text{syz}(g_1, g_2, g_3, g_4)$;

$$\begin{pmatrix} y & x & 0 & 0 \\ 0 & y & -x^2y + yz^2 & -y^2z \\ z & -y & x^2y - y^2z - yz^2 & x^3 + y^3 + y^2z \\ 0 & 0 & z & -y \end{pmatrix}$$

Since the first two columns are already considered, we start from the third column. Let

$$h_1 = (-x^y + yz^2)f_2 + (x^2y - y^2z - yz^2)f_3 + zf_4 = -x^2y^4 + y^4z^2 + x^3yz^2 - xy^2z^3 - xyz^4.$$

Then

$$h_1 = xyz^2g_1 - x^2y^2g_3 + z^2g_4.$$

Therefore , let

$$h_2 = h_1 - xyz^2f_1 + x^2y^2f_3 - z^2f_4 = 0.$$

Hence $h - 1$ reduced to zero.

Now, consider the last column and define

$$h_1 = -y^2zf_2 + (x^3 + y^3 + y^2z)f_3 - yf_4 = -y^5z + x^4z^2 + xy^3z^2 + xy^2z^3.$$

Dividing h_1 by (g_1, g_2, g_3, g_4),

$$h_1 = x^3zg_2 - yzg_4 + xyz^2g_3.$$

Hence let

$$h_2 = h_1 - x^3zf_2 - xyz^2f_3 + yzf_4 = -x^3y^3z - x^2yz^4$$

Clearly,

$$h_2 = (x^3y^2 + x^2z^3)g_1.$$

Hence

$$h_3 = h_2 - (x^3y^2 + x^2z^3)f_1 = -x^5y^3 - x^4yz^3 = x^4z^2g_1 - x^5yg_3$$

Finally,

$$h_4 = h_3 - x^4z^2f_1 + x^5yf_3 = 0.$$

Since we calculate divisions for all elements of syzygy basis, then $\{f_1, f_2, f_3, f_4\}$ is a tangent cone basis. However, if we compute Gröbner basis of $\{f_1^h = x^2y - yzt, f_2^h = y^3 + xzt + y^2t, f_3^h = xz^2 + y^2t\}$, the result is $\{-y^{12} - y^7\,z^5 + y^3\,z^9, y^{11} + y^7\,z^4 + y^6\,z^5 + x\,z^{10}, y^8 + y^3\,z^5 + x\,y\,z^6, y^7 - x\,y^4\,z^2 + x\,z^6, -x\,y^5 - y^3\,z^3, -y^5 + x\,y^2\,z^2 + x^2\,z^3, x^2\,y^2 + x\,z^3, x^3\,z^2 + y^3\,z^2 - x\,z^4, x^3\,y + y^4 - x\,y\,z^2, x^2\,y - t\,y\,z, t\,y^2 + x\,z^2, y^3 + t\,x\,z - x\,z^2\}$. The dehomogenizations of basis elements gives us a tangent cone basis of 11 elements. However, with new technique, we found a tangent cone basis of 4 elements.

4.2 Lifting A Homogeneous Basis To A Tangent Cone Basis

In polynomial ideals, the following lifting problem for homogeneous ideals is well-known: (see [11] and [8]): Let J be homogeneous ideal in $k[x_1, \ldots, x_n]$. The homogeneous ideal I in $k[x_0, x_1, \ldots, x_n]$ is called a lifting of J if x_0 is not a zero divisor on $k[x_0, x_1, \ldots, x_n]$ and $J = \langle f(0, x_1, \ldots, x_n) : f \in I\rangle$. If a basis for J is given, a basis for the lifting can be found based on the following problem.

Theorem 4.4 *Let $\{f_1, \ldots, f_r\}$ be a set of homogeneous polynomials of $k[x_1, \ldots, x_n]$ which generates a homogeneous ideal J.Let $q_i = f_i + g_i$ $\deg(g_i) < \deg(f_i)$ for $i = 1, \ldots, r$ and $I^* = \langle q_1, \ldots, q_r\rangle$. If $\{q_1, \ldots, q_r\}$ is a homogeneous basis for I^*, then $I = (I^*)^h = \langle q_1^h, \ldots, q_r^h\rangle$ is a lifting of J.*

Now, we will try to modify this problem to the tangent cone bases. Unlike homogeneous bases, if $\{f_1 + g_1, \ldots, f_r + g_r\}$ is a tangent cone basis for some ideal, then $\deg(g_i) > \deg(f_i)$ for $i = 1, \ldots, r$. Hence $\deg(f_i)$ is not an upper bound but a lower bound for the polynomial g_i. Because of this, we propose the following definition for the lifting of a homogeneous ideal to a tangent cone basis.

Definition 4.5 *Let $\{f_1, \ldots, f_r\}$ be a set of homogeneous polynomials of $k[x_1, \ldots, x_n]$ which generates a homogeneous ideal J. Let $q_i = f_i + g_i$ with $d_i = \deg(q_i) = d + \deg(f_i)$ for $i = 1, \ldots, r$. If $\{q_1, \ldots, q_r\}$ is a tangent cone basis for some ideal I_d, then I_d is called a tangent cone lifting of degree d of J.*

The problem is that if a basis $\{f_1, \ldots, f_r\}$ for J is given, then how can we find polynomials $g_1, \ldots, g_r$ with above property. First of all, we will solve this problem assuming that $\{f_1, \ldots, f_r\}$ is a Gröbner basis with respect to an order described in Theorem 3.6.In this case, $\{q_1^h, \ldots, q_r^h\}$ is also a Gröbner basis. Furthermore $\mathrm{LT}(f_i) = \mathrm{LT}(q_i)$ for $i = 1, \ldots, r$. Hence for any s-polynomial $S(f_i, f_j)$, we can define s-polynomial $S(q_i, q_j)$ and division of this s-polynomial by $\{q_1^h, \ldots, q_r^h\}$ must be given a zero remainder. Let us explain this in more detail using following notation. We will use following notation for polynomials g_i's

$$g_i = \sum_{j_1+j_2+\cdots+j_n=d_i} a_{i,(j_1,\ldots,j_n)} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$$

First find a s-polynomial $S(f_i, f_j)$. Since $\{f_1, \ldots, f_r\}$ is a Gröbner basis, division of $S(f_i, f_j)$ by $\{f_1, \ldots, f_r\}$ gives a zero remainder. Hence by Theorem 2.19 , we obtain a syzygy $(s_1, \ldots, s_r)$ of $(f_1, \ldots, f_r)$. Let us define each component of this syzygy as follows;

$$s_i = \sum_{k_1+k_2+\cdots+k_n=e_i} b_{i,(k_1,\ldots,k_n)} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

where $e_i = \deg(s_i)$ for $i = 1, \ldots, r$.

Since $\{q_1^h, \ldots, q_r^h\}$ is also a Gröbner basis and $\mathrm{LT}(f_i) = \mathrm{LT}(q_i)$, the division of $s_1 q_1^h + s_2 q_2^h + \cdots s_r q_r^h = s_1 g_1^h + s_2 g_2^h + \cdots s_r g_r^h$ by $\{q_1^h, \ldots, q_r^h\}$ must be give zero remainder. However, no term of $s_1 g_1^h + s_2 g_2^h + \cdots s_r g_r^h$ is divisible by any $\mathrm{LT}(f_i)$. Therefore $s_1 g_1^h + s_2 g_2^h + \cdots s_r g_r^h = 0$ or by disregarding homogenization $s_1 g_1 + s_2 g_2 + \cdots s_r g_r = 0$. Regarding degree $d_i + e_i$ monomial, this equation gives us linear equations in terms of unknown coefficients of g_i's.

Example 4.6 *Consider the ideal* $I = \langle f_1 = x^2 + y^2, f_2 = y^3 + y^2 z, f_3 = yz^3 + z^4 \rangle$. *By Buchberger's criteria this is a Gröbner basis with respect to degree reverse lexixographic order. By Theorem 2.19, we can find a syzygy basis of* I. *The columns of the following matrix generates the syzygy basis of* I,

$$\begin{pmatrix} 0 & -y^3 - y^2 z & -yz^3 - z^4 \\ -z^3 & x^2 + y^2 & z^3 \\ y^2 & 0 & x^2 \end{pmatrix}$$

Suppose that we would like to find degree 1 *lifting of* I. *Hence we should define*

$$g_1 = a_{1,(3,0,0)} x^3 + a_{1,(2,1,0)} x^2 y + a_{1,(2,0,1)} x^2 z + \cdots + a_{1,(0,1,2)} y z^2 + a_{1,(0,0,3)} z^3.$$

$g_2 = a_{2,(4,0,0)}x^4 + a_{2,(3,1,0)}x^3y + a_{2,(3,0,1)}x^3z + \cdots + a_{2,(0,1,3)}yz^3 + a_{2,(0,0,4)}z^4.$

$g_3 = a_{3,(5,0,0)}x^5 + a_{3,(4,1,0)}x^4y + a_{3,(4,0,1)}x^4z + \cdots + a_{3,(0,1,4)}yz^4 + a_{3,(0,0,5)}z^5.$

Using columns of the above matrix we have the following three equations.

$$-z^3g_2 + y^2g_3 = 0 \qquad (1).$$

$$(-y^3 - y^2z)g_1 + (x^2 + y^2)g_2 = 0 \qquad (2).$$

$$(-yz^3 - z^4)g_1 + z^3g_2 + x^2g_3 = 0 \qquad (3).$$

Equation (1) and equation (3) have degree 7 *while equation (2) has degree* 6. *Considering coefficients of monomials, we have linear equations of unknown coefficients of g_i's. For example coefficient of x^7 in equation (1) gives us $0 = 0$ and the coefficient of z^7 gives $-a_{2,(0,0,4)} = 0$. For equation (2), the coefficient of x^6 gives $a_{2,(4,0,0)} = 0$ and coefficient of y^6 gives $-a_{1,(0,3,0)} + a_{2,(0,4,0)} = 0$. Writing all equations and solving them using linear algebra techniques, we obtain some coefficients of g_i's in terms of others.*

The following remark gives an easy way to write all equations of lifting problem.

Remark 4.7 *Let polynomial g_i's and a syzygy $(s_1, \ldots, s_r)$ as above. The coefficient of $x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n}$ in the expression $s_1g_1 + s_2g_2 + \cdots s_rg_r$ is*

$$\sum_{i=1}^{r} \sum_{k_1+k_2+\cdots+k_n=e_i} a_{i,(l_1-k_1,l_2-k_2,\ldots,l_n-k_n)}b_{i,(k_1,\ldots,k_n)}.$$

Notice that $a_{i,(l_1-k_1,l_2-k_2,\ldots,l_n-k_n)}$ is defined only if $l_1 - k_1 \geq 0, \ldots, l_n - k_n \geq 0$.

The equation $-z^3g_2 + y^2g_3 = 0$ gives the following system;

$$
\begin{aligned}
&-a_{2,\{0,0,4\}} = 0 && -a_{2,\{0,1,3\}} = 0 \\
&-a_{2,\{1,0,3\}} = 0 && -a_{2,\{1,1,2\}} = 0 \\
&-a_{2,\{2,0,2\}} = 0 && -a_{2,\{2,1,1\}} = 0 \\
&-a_{2,\{3,0,1\}} = 0 && -a_{2,\{3,1,0\}} = 0 \\
&-a_{2,\{4,0,0\}} = 0 && a_{3,\{0,0,5\}} - a_{2,\{0,2,2\}} = 0 \\
&a_{3,\{0,1,4\}} - a_{2,\{0,3,1\}} = 0 && a_{3,\{0,2,3\}} - a_{2,\{0,4,0\}} = 0 \\
&a_{3,\{0,3,2\}} = 0 && a_{3,\{0,4,1\}} = 0 \\
&a_{3,\{0,5,0\}} = 0 && a_{3,\{1,0,4\}} - a_{2,\{1,2,1\}} = 0 \\
&a_{3,\{1,1,3\}} - a_{2,\{1,3,0\}} = 0 && a_{3,\{1,2,2\}} = 0 \\
&a_{3,\{1,3,1\}} = 0 && a_{3,\{1,4,0\}} = 0 \\
&a_{3,\{2,0,3\}} - a_{2,\{2,2,0\}} = 0 && a_{3,\{2,1,2\}} = 0 \\
&a_{3,\{2,2,1\}} = 0 && a_{3,\{2,3,0\}} = 0 \\
&a_{3,\{3,0,2\}} = 0 && a_{3,\{3,1,1\}} = 0 \\
&a_{3,\{3,2,0\}} = 0 && a_{3,\{4,0,1\}} = 0 \\
&a_{3,\{4,1,0\}} = 0 && a_{3,\{5,0,0\}} = 0
\end{aligned}
$$

The equation $(-y^3 - y^2z)g_1 + (x^2 + y^2)g_2 = 0$ gives the following system;

$a_{2,\{0,0,4\}} = 0$

$a_{2,\{0,1,3\}} = 0$

$-a_{1,\{0,1,2\}} - a_{1,\{0,2,1\}} + a_{2,\{0,2,2\}} = 0$

$a_{2,\{0,4,0\}} - a_{1,\{0,3,0\}} = 0$

$a_{2,\{1,0,3\}} - a_{1,\{1,0,2\}} = 0$

$-a_{1,\{1,0,2\}} - a_{1,\{1,1,1\}} + a_{2,\{1,1,2\}} = 0$

$a_{2,\{1,3,0\}} - a_{1,\{1,2,0\}} = 0$

$-a_{1,\{2,0,1\}} + a_{2,\{0,2,2\}} + a_{2,\{2,0,2\}} = 0$

$-a_{1,\{2,0,1\}} - a_{1,\{2,1,0\}} + a_{2,\{0,3,1\}} + a_{2,\{2,1,1\}} = 0$

$a_{2,\{3,0,1\}} = 0$

$a_{2,\{3,1,0\}} = 0$

$a_{2,\{4,0,0\}} = 0$

$a_{2,\{0,0,4\}} - a_{1,\{0,0,3\}} = 0$

$-a_{1,\{0,0,3\}} - a_{1,\{0,1,2\}} + a_{2,\{0,1,3\}} = 0$

$-a_{1,\{0,2,1\}} - a_{1,\{0,3,0\}} + a_{2,\{0,3,1\}} = 0$

$a_{2,\{1,0,3\}} = 0$

$a_{2,\{1,1,2\}} = 0$

$-a_{1,\{1,1,1\}} - a_{1,\{1,2,0\}} + a_{2,\{1,2,1\}} = 0$

$a_{2,\{2,0,2\}} = 0$

$a_{2,\{2,1,1\}} = 0$

$-a_{1,\{2,1,0\}} + a_{2,\{0,4,0\}} + a_{2,\{2,2,0\}} = 0$

$-a_{1,\{3,0,0\}} + a_{2,\{1,2,1\}} + a_{2,\{3,0,1\}} = 0$

$-a_{1,\{3,0,0\}} + a_{2,\{1,3,0\}} + a_{2,\{3,1,0\}} = 0$

$a_{2,\{2,2,0\}} + a_{2,\{4,0,0\}} = 0$

The equation $(-yz^3 - z^4)g_1 + z^3g_2 + x^2g_3 = 0$ gives the following system;

$a_{2,\{0,0,4\}} - a_{1,\{0,0,3\}} = 0$

$-a_{1,\{0,0,3\}} - a_{1,\{0,1,2\}} + a_{2,\{0,1,3\}} = 0$

$-a_{1,\{0,1,2\}} - a_{1,\{0,2,1\}} + a_{2,\{0,2,2\}} = 0$

$-a_{1,\{0,2,1\}} - a_{1,\{0,3,0\}} + a_{2,\{0,3,1\}} = 0$

$a_{2,\{0,4,0\}} - a_{1,\{0,3,0\}} = 0$

$a_{2,\{1,0,3\}} - a_{1,\{1,0,2\}} = 0$

$-a_{1,\{1,0,2\}} - a_{1,\{1,1,1\}} + a_{2,\{1,1,2\}} = 0$

$-a_{1,\{1,1,1\}} - a_{1,\{1,2,0\}} + a_{2,\{1,2,1\}} = 0$

$a_{2,\{1,3,0\}} - a_{1,\{1,2,0\}} = 0$

$-a_{1,\{2,0,1\}} + a_{2,\{2,0,2\}} + a_{3,\{0,0,5\}} = 0$

$-a_{1,\{2,0,1\}} - a_{1,\{2,1,0\}} + a_{2,\{2,1,1\}} + a_{3,\{0,1,4\}} = 0$

$-a_{1,\{2,1,0\}} + a_{2,\{2,2,0\}} + a_{3,\{0,2,3\}} = 0$

$a_{3,\{0,3,2\}} = 0$

$a_{3,\{0,4,1\}} = 0$

$a_{3,\{0,5,0\}} = 0$

$-a_{1,\{3,0,0\}} + a_{2,\{3,0,1\}} + a_{3,\{1,0,4\}} = 0$

$-a_{1,\{3,0,0\}} + a_{2,\{3,1,0\}} + a_{3,\{1,1,3\}} = 0$

$a_{3,\{1,2,2\}} = 0$

$a_{3,\{1,3,1\}} = 0$

$a_{3,\{1,4,0\}} = 0$

$a_{2,\{4,0,0\}} + a_{3,\{2,0,3\}} = 0$

$a_{3,\{2,1,2\}} = 0$

$a_{3,\{2,2,1\}} = 0$

$a_{3,\{2,3,0\}} = 0$

$a_{3,\{3,0,2\}} = 0$

$a_{3,\{3,1,1\}} = 0$

$a_{3,\{3,2,0\}} = 0$

$a_{3,\{4,0,1\}} = 0$

$a_{3,\{4,1,0\}} = 0$

$a_{3,\{5,0,0\}} = 0$

Solving above systems of linear equations, we obtain;

$$
\begin{array}{ll}
a_{1,\{3,0,0\}} = a_{3,\{1,0,4\}} & a_{1,\{2,1,0\}} = a_{3,\{0,1,4\}} - a_{3,\{0,0,5\}} \\
a_{1,\{2,0,1\}} = a_{3,\{0,0,5\}} & a_{1,\{1,2,0\}} = a_{3,\{1,0,4\}} \\
a_{1,\{1,1,1\}} = 0 & a_{1,\{1,0,2\}} = 0 \\
a_{1,\{0,3,0\}} = a_{3,\{0,1,4\}} - a_{3,\{0,0,5\}} & a_{1,\{0,2,1\}} = a_{3,\{0,0,5\}} \\
a_{1,\{0,1,2\}} = 0 & a_{1,\{0,0,3\}} = 0 \\
a_{2,\{4,0,0\}} = 0 & a_{2,\{3,1,0\}} = 0 \\
a_{2,\{3,0,1\}} = 0 & a_{2,\{2,2,0\}} = 0 \\
a_{2,\{2,1,1\}} = 0 & a_{2,\{2,0,2\}} = 0 \\
a_{2,\{1,3,0\}} = a_{3,\{1,0,4\}} & a_{2,\{1,2,1\}} = a_{3,\{1,0,4\}} \\
a_{2,\{1,1,2\}} = 0 & a_{2,\{1,0,3\}} = 0 \\
a_{2,\{0,4,0\}} = a_{3,\{0,1,4\}} - a_{3,\{0,0,5\}} & a_{2,\{0,3,1\}} = a_{3,\{0,1,4\}} \\
a_{2,\{0,2,2\}} = a_{3,\{0,0,5\}} & a_{2,\{0,1,3\}} = 0 \\
a_{2,\{0,0,4\}} = 0 & a_{3,\{5,0,0\}} = 0 \\
a_{3,\{4,1,0\}} = 0 & a_{3,\{4,0,1\}} = 0 \\
a_{3,\{3,2,0\}} = 0 & a_{3,\{3,1,1\}} = 0 \\
a_{3,\{3,0,2\}} = 0 & a_{3,\{2,3,0\}} = 0 \\
a_{3,\{2,2,1\}} = 0 & a_{3,\{2,1,2\}} = 0 \\
a_{3,\{2,0,3\}} = 0 & a_{3,\{1,4,0\}} = 0 \\
a_{3,\{1,3,1\}} = 0 & a_{3,\{1,2,2\}} = 0 \\
a_{3,\{1,1,3\}} = a_{3,\{1,0,4\}} & a_{3,\{0,5,0\}} = 0 \\
a_{3,\{0,4,1\}} = 0 & a_{3,\{0,3,2\}} = 0 \\
a_{3,\{0,2,3\}} = a_{3,\{0,1,4\}} - a_{3,\{0,0,5\}} &
\end{array}.
$$

Let $a_{3,\{1,0,4\}} = a_1, a_{3,\{0,0,5\}} = a_2$ and $a_{3,\{0,1,4\}} = a_3$. Then
$g_1 = a_1x^3 + (a_3 - a_2)x^2y + a_2x^2z + a_1xz^4 + (a_3 - a_2)y^3 + a_2y^2z.$

$$g_2 = a_1xy^3 + a_1xy^2z + (a_3 - a_2)y^4 + a_3yz^4 + a_2y^2z^2$$

$$g_3 = a_1xyz^3 + (a_2 - a_3)y^2z^3.$$

Hence $\{f_1 + g_1, f_2 + g_2, f_3 + g_3\}$ is a tangent cone lifting of degree 1 of I.

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