Tropical Hypersurfaces in Groebner Fan

by

Muharrem Barış Ötüş

A Thesis Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Mathematics

Koç University

August 20, 2015

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

Muharrem Barış Ötüş

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee Members:

Prof. Tolga Etgü (Advisor)

Prof. Burak Özbağcı

Prof. Emrah Çakçak

Assoc. Prof. Meral Tosun (Co-Advisor)

Assist. Prof. Serap Gürer

Date:

ABSTRACT

Given an ideal $I \subset k[x_1, \ldots, x_n]$, the Groebner fan of I is a structure in $\mathbb{R}^n$ forming a polyhedral complex. Any ideal I has a tropicalization which is called the tropical variety associated to I. This variety is a piecewise linear object that is directly related to the Groebner fan of that ideal. The aim of this thesis is to have a better understanding of tropical varieties of codimension 1 and 2, and to observe that the tropical variety of a given ideal is in fact a sub-complex of the corresponding Groebner fan.

ÖZETÇE

Verilen bir I idealinin Groebner fanı, $\mathbb{R}^n$'de polihedral kompleks şeklinde gözüken bir yapıdır. Her I idealinin, I ile ilişkili tropik varyete olarak adlandırılan bir tropikalizasyonu vardır. Bu varyete, idealin Groebner fanıyla doğrudan ilişkili olan parçalı doğrusal bir objedir. Tezin amacı, tamamlayıcı boyutu 1 ve 2 olan tropik varyeteleri anlamak ve verilen bir idealin tropik varyetesinin aslında ilgili Groebner fanın bir alt-kompleksi olduğunu gözlemlemektir.

ACKNOWLEDGMENTS

TABLE OF CONTENTS

List of Figures **ix**

Chapter 1: Introduction **1**

Chapter 2: Preliminaries **3**

2.1 Affine Varieties 3

2.2 Term Orders and Groebner Basis 4

2.3 Cones 7

2.4 Newton Polyhedron 11

Chapter 3: Groebner Fan **13**

3.1 w-Order 13

3.2 Groebner Fan 17

Chapter 4: Tropical Algebra **24**

4.1 Tropical Polynomials with one variable 24

4.2 Tropical Polynomials with several variables 29

Chapter 5: Tropical Varieties **33**

5.1 Tropicalization 33

5.2 Initial Ideals 34

5.3 Groebner Fan 37

Chapter 6: Tropical Varieties in $\mathbb{R}^2$ **39**

6.1 Tropical Lines 39

6.2 Tropical Curves 40

6.3 Dual Subdivision and Balancing Condition 42

Chapter 7: Tropical Varieties in $\mathbb{R}^3$ 47

7.1 Tropical Lines 47

7.2 Tropical Hypersurfaces 48

Chapter 8: Conclusion 51

Bibliography 52

LIST OF FIGURES

2.1 The cones σ_1 and σ_2 9
2.2 The dual cones of σ_1 and σ_2 9
2.3 A fan in $\mathbb{R}^2$ 10
2.4 A fan in $\mathbb{R}^3$ 10
2.5 Newton polygon of f 11
2.6 Newton polyhedron of f 12
2.7 Newton polyhedron of f 12

3.1 The convex hull of $(0, 2)$,$(3, 2)$ and $(3, 0)$ 13
3.2 The fan $C(f)$ 14
3.3 The fan $\mathcal{F}(f)$ 17
3.4 The Groebner fan of $x + y + 3$ 18
3.5 The Groebner fan of I 21
3.6 The Groebner fan of I 22
3.7 The Groebner fan of f 23

4.1 The graph of $p(x)$ 25
4.2 The graph of $p(x)$ 26
4.3 The graph of the tropical roots of $p(x, y)$ 30
4.4 The graph of the tropical roots of $p(x, y)$ 31

5.1 The tropical variety of f 34
5.2 The tropical variety of f 35
5.3 The Groebner fan of I 37

6.1 The tropical variety of a line . . . 39
6.2 The tropical variety of $f(x,y)$ . . . 40
6.3 The triangle T_1 . . . 43
6.4 The tropical curve L in the subdivision . . . 43
6.5 The dual subdivision of $p(x,y)$ . . . 44
6.6 The tropical curve L in the subdivision . . . 44

7.1 The tropical variety of $f(x,y,z)$ . . . 49
7.2 The tropical variety of $f(x,y,z)$ . . . 50

Chapter 1

INTRODUCTION

Tropical geometry is a new area of mathematics which is shaped due to the works of R. Bieri, J. Groves and G.M. Bergman in 1980's. Then many remarkable works were done by a number of mathematicians such as B. Sturmfels, A. Viro and G. Mikhalkin. Gaining its shape, tropical geometry began to be a useful tool for algebraic geometers. Together with its benefits in various area of mathematics including algebraic geometry, it has been useful for some other disciplines among which computer science, physics, biology and economy take place.

Tropical geometry is grounded on tropical algebra in which the operations are "$\oplus$"-*the maximum* and "$\odot$"-*usual addition* between terms; that is why it is usually called "max-plus algebra". The geometric objects are piecewise linear graphs. It turns out that these "easy-looking" objects are in a strong relation with affine varieties in usual sense (see Definition 2.1.5); Viro's patchworking method (see [5]) and Mikhalkin's correspondance theorem (see [8]) illustrate the relation.

In this thesis, we try to get a deep understanding of tropical polynomials and more generally tropical ideals (means tropicalization of an ideal) ; which naturally leads to the Tropical Fundamental Theorem of Algebra (see Theorem 4.1.14). It is a well known fact that every affine variety in algebraic geometry defines a tropical variety- *tropicalization*. We examine the tropical varieties of codimension 1 and 2. Although the interesting works in tropical geometry are done in the case of nontrivial valuation, here we use only the trivial valuation.

Some basic notions of algebra like ring, ideal, quotient ring etc. are assumed to be known throughout the text.

In Chapter 2, we recall the definitions of some other important notions that will be helpful for us throughout the text, such as term order, Groebner basis, cone, fan and Newton polyhedron.

In Chapter 3, following [1], we define the Groebner fan of an ideal and give some examples; most of the examples are computed by the software program Sage, which is an interface that uses Singular [4].

In Chapter 4, we introduce tropical algebra by studying tropical polynomials in one variable and giving the statement and the proof of Tropical Fundamental Theorem of Algebra referring to [12]. Then we consider the tropical polynomials in two or three variables and obtain their corresponding tropical varieties. We compare two equivalent definitions of the tropical variety of an ideal. We illustrate via different examples the fact that tropical variety is a sub-complex of Groebner fan.

In Chapter 6 and 7, we aim to understand tropical varieties in $\mathbb{R}^2$ and especialy in $\mathbb{R}^3$.

Chapter 2

PRELIMINARIES

Here we will recall briefly some of basic notions in algebraic geometry that we will need in the sequel.

2.1 Affine Varieties

Let $R := (R, +, \cdot)$ be a ring.

Definition 2.1.1. *An ideal I of R is called prime if $I \neq R$ and whenever $ab \in I$ for with $a, b \in R$ then $a \in I$ or $b \in I$.*

Example 2.1.2. *$2\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$.*

Definition 2.1.3. *Let R be a ring. Consider a chain of prime ideals of the form*

$$I_0 \subsetneq I_1 \subsetneq ... \subsetneq I_l \subsetneq R$$

The number of strict inclusions is called the length l of the chain and the Krull dimension of R is defined to be the supremum of the lenghts of all chains of prime ideals in R.

Example 2.1.4. *Any field k has Krull dimension 0.*

Notation. In the sequel, the field k will denote $\mathbb{R}$ or $\mathbb{C}$ unless stated otherwise.

For us, R will always denote the polynomial ring $k[x_1, \ldots, x_n]$, the the ring of polynomials over k. Let f be an element in $k[x_1, \ldots, x_n]$. The subset

$$V(f) = \{(p_1, \ldots, p_n) \in k^n \mid f(p_1, \ldots, p_n) = 0\}$$

is called an affine hypersurface in k^n. When $n = 2$, the subset $V(f)$ is called a plane algebraic curve.

Definition 2.1.5. *Let $f_1, \ldots, f_k$ be in $k[x_1, \ldots, x_n]$ and I the ideal generated by $f_1, \ldots, f_k$ in $k[x_1, \ldots, x_n]$. The subset*

$$V(I) = \{(p_1, \ldots, p_n) \in k^n \mid f_1(p_1, \ldots, p_n) = \ldots = f_k(p_1, \ldots, p_n) = 0\}$$

is called an affine variety in k^n.

Example 2.1.6. *(i) An element in $\mathbb{R}[x, y]$ of the form $f(x, y) = ax+by+c$ is an affine variety in $\mathbb{R}^2$.*

(ii) The element $f(x, y, z) = z^2 + x^2 + y^2$ in $\mathbb{C}[x, y, z]$ is an affine hypersurface in $\mathbb{C}^3$.

Definition 2.1.7. *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. The dimension of $V(I)$ is defined to be the Krull dimension of the quotient ring $k[x_1, \ldots, x_n]/I$.*

Example 2.1.8. *Let I be the ideal in $k[x, y]$ generated by the polynomial x. Then, $V(I)$ is an affine variety of dimension 2 in k^2 as the longest chain of prime ideals in $k[x, y]/\langle x\rangle \simeq k[y]$ is $\{0\} \subset \langle y\rangle$.*

2.2 Term Orders and Groebner Basis

A monomial in the polynomial ring $R := k[x_1, \ldots, x_n]$ (without its coefficient) is called a term. For a division between two polynomials with several variables we need to define an order between the terms in the polynomials. For simplicity, we will denote a monomial in R by $x^\alpha := x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

Definition 2.2.1. *A relation $\prec$ is called term order on R if:*

1) *We have $1 \prec x^\alpha$ for all $\alpha \in \mathbb{N}^n \backslash \{0\}$ and,*

2) *We have $x^\alpha \prec x^\beta \Longleftrightarrow x^\alpha x^\gamma \prec x^\beta x^\gamma$ for $\alpha, \beta, \gamma \in \mathbb{N}^n$.*

Note that a term order is a total ordering on the monomials.

Definition 2.2.2. *A term order on $k[x_1, \dots, x_n]$ defined as*

$$x^\alpha \prec_l x^\beta \iff \text{ there exists } k \in \{1, ...n\} \text{ s.t. } \alpha_k < \beta_k \text{ and } \alpha_i = \beta_i \ \forall i < k$$

is called the lexicographical order.

Definition 2.2.3. *A term order on $k[x_1, \dots, x_n]$ defined as*

$$x^\alpha \prec_{rl} x^\beta \iff \text{ there exists } k \in \{1, ...n\} \text{ s.t. } \alpha_k < \beta_k \text{ and } \alpha_i = \beta_i \ \forall i > k$$

is called reverse lexicographical order.

Example 2.2.4. *(i) In $k[x, y, z]$, $x^2y^3z \prec_l x^2y^3z^5$ and $x^2y^3z \prec_{rl} x^2y^3z^5$.*

(ii) In $k[x, y, z, w]$, $x^3y^2z^2w^3 \prec_l x^3y^3zw^3$ and $x^3y^3zw^3 \prec_{rl} x^3y^2z^2w^3$

Definition 2.2.5. *Let $w \in (\mathbb{R}_{\geq 0})^n$. A term order $\prec_w$ defined as*

$$x^\alpha \prec_w x^\beta \iff \langle w, \alpha\rangle < \langle w, \beta\rangle \text{ or } (\langle w, \alpha\rangle = \langle w, \beta\rangle \text{ and } x^\alpha \prec_l x^\beta)$$

is called an $\prec_w$-order where $\langle -, -\rangle$ stands for the dot product.

For a term order $\prec$ in R, we define the initial form of an element $f \in k[x_1, \dots, x_n]$ with respect to $\prec$ as the unique maximal term between all the terms of f with respect to $\prec$.

Example 2.2.6. *Let $f(x, y, z) = z^3 + y^3z + x^2y^2 \in \mathbb{R}[x, y, z]$, $w_1 = (1, 1, 1)$ and $w_2 = (1, 2, 3)$*
(i) For $\prec_l$-order, $In_{\prec_l}(f) = x^2y^2$.
(ii) For $\prec_{w_1}$-order, $In_{\prec_{w_1}} = x^2y^2$ and for $\prec_{w_2}$-order, $In_{\prec_{w_2}} = y^3z$

More generally, for an ideal $I \subset k[x_1, \dots, x_n]$, the initial ideal $In_\prec(I)$ with respect to a term order $\prec$ is the ideal generated by the initial forms of all polynomials in I, means

$$In_\prec I := \langle In_\prec f \mid f \in I\rangle$$

We also define the set of vectors whose initial ideal coincide with $In_\prec(I)$;

$$C_\prec(I) := \{w \in \mathbb{R}^n;\ In_w(I) = In_\prec(I)\}$$

Example 2.2.7. *Let* $I = \langle z^2 + y^3 + z^5 \rangle \subset \mathbb{R}[x, y, z]$ *and* $w_1 = (3, 2, 1)$, $w_2 = (1, 0, 1)$. *Then:*

$$In_{<_l}(I) = \langle y^3 \rangle \quad ln_{<_{w_1}}(I) = \langle z^5 \rangle \quad In_{<_{w_2}}(I) = \langle z^5 \rangle$$

If the ideal I is not principal we need to find a good set of generators for computing the initial form of the ideal:

Definition 2.2.8. *For an ideal* $I \subset k[x_1, \ldots, x_n]$ *and a term order* $<$ *on* $k[x_1, \ldots, x_n]$, *if there is a set of generators* $G = \{g_1, \ldots, g_m\}$ *such that*

$$In_< I = \langle In_< g_1, \ldots, In_< g_m \rangle$$

then G is called a Groebner basis for I.

A Groebner basis is computed by By Buchberger's algorithm (see p.90 in [3]). Here we compute the Groebner basis by using the computer programming Sage as an interface to Singular (see [4]).

Example 2.2.9. (i) *Let* $I = \langle xy^2, y^2 - y \rangle \subset \mathbb{Q}[x, y]$. *With respect to the lex order* $<_l$ *on* $\mathbb{Q}[x, y]$, *the set* $G = \{xy, y^2 - y\}$ *is a Groebner basis for I.*

(ii) *Consider the ideal* $I = \langle x^2y - 2, xy^2 - y \rangle \subset \mathbb{R}[x, y]$. *With respect to the lex order* $<_l$ *on* $\mathbb{R}[x, y]$, *the set* $G = \{y - \frac{1}{2}, x - 2\}$ *is a Groebner basis for I.*

(iii) *Consider the ideal* $I = \langle x^2y + xy^2, x^3 + y^5 \rangle \subset \mathbb{Q}[x, y]$.

(a) *With respect to* $<_w$ *with* $w = (1, 1)$ *on* $\mathbb{Q}[x, y]$, *the set* $G = \{x^2y + xy^2, x^3 + y^5, x^5 - xy^4\}$ *is a Groebner basis for I;*

(b) *With respect to* $<_w$ *with* $w = (2, 3)$, *we obtain* $G = \{y^5 + x^3, xy^3 + y^6, y^9 - y^7, x^2y + xy^2\}$.

The Sage commands to be used for computation of the Groebner basis of an ideal are as below:

(1) $R. < x, y >= PolynomialRing(QQ, 2, order = \text{'lex'})$

That command defines the polynomial ring where QQ represents the field of rational numbers, 2 *represents the number of variables in the polynomial ring and order='lex' means that we set the term order to lexicographical order.*

(2) $S = [xy^2, y^2 - y]$

That command simply defines a set of polynomials.

(3) I= R.ideal(S)

That command defines I to be the ideal in the polynomial ring generated by S.

(4) $I.groebner\_basis()$

That command computes the Groebner basis of I. Each of these commands runs after pressing "Enter" and "Shift" together.

Suppose that the initial part of every element $g \in G$ has coefficient 1 and the remaining monomials in G are also monomials in $In_{\prec}(I)$, then G is called a *reduced Groebner basis*. We denote the set of reduced Groebner basis by $R_{\prec}(I)$. In fact, all the Groebner basis in Example 2.2.9 above are reduced Groebner basis.

Remark 2.2.10. *For a principal ideal, the generating set (which is a singleton) is the reduced Groebner basis.*

2.3 Cones

A subset σ in $\mathbb{R}^n$ is called a *cone* if σ consists of a set of vectors $v \in \mathbb{R}^n$ such that we have $\alpha v \in \sigma$ for all $\alpha \in \mathbb{R}$ and $v \in \sigma$. More precisely; let $v_1, \ldots, v_k \in \mathbb{R}^n$. The cone $Cone(v_1, \ldots, v_k)$ generated by the vectors $v_1, \ldots, v_k$ is the set

$$Cone(v_1, \ldots, v_k) := \{\lambda_1 v_1 + \ldots + \lambda_k v_k \mid \lambda_i \in \mathbb{R}_{\geq 0}, i = 1, \ldots, k\} \subset \mathbb{R}^n$$

A subset $\sigma \subset \mathbb{R}^n$ is a polyhedral cone if there exits some vectors $v_1, \ldots, v_k$ such that we have $\sigma = Cone(v_1, \ldots, v_k)$.

Example 2.3.1. (i) *Let* $v_1 = (0,1) \in \mathbb{R}^2$*. The cone* $\sigma := Cone(v_1)$ *is the set* $\{(0,y) \mid y \geq 0\}$.

(ii) *Let* $v_1 = (0,1), v_2 = (1,2) \in \mathbb{R}^2$*. The cone* $\tau := Cone(v_1, v_2)$ *is the set* $\{\lambda_2 v_1 + \lambda_2 v_2 \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$.

The *dimension* of a cone σ is the dimension of the minimal linear subspace containing σ.

Example 2.3.2. *In the Example 2.3.1, the cone* σ *has dimension* 1 *whereas the cone* τ *has dimension* 2.

A cone σ in $\mathbb{R}^n$ is called simplicial if its generating vectors are linearly independent. For a cone $\sigma = Cone(v_1, \ldots, v_k)$ in $\mathbb{R}^n$, we associate an $n \times k$ matrix A_σ such that the columns are the vectors v_i's. The determinant $det(\sigma)$ of A_σ is greatest common divisor of the determinants of all $t \times t$ sub-matrices of A_σ. If $det(\sigma) = \pm 1$ we say that σ is a regular cone.

For example, the cone σ is regular but the cone τ is not regular where σ and τ are the cones given in Example 2.3.1.

The dual cone of a cone σ in $\mathbb{R}^n$ lives in the dual space $(\mathbb{R}^n)^\vee$. Let us first recall the definition of the dual space $V^\vee$ of a vector space $V = span\{v_1, \ldots v_n\}$: it is generated by the vectors $v_1^\vee, v_2^\vee \ldots v_n^\vee$ in $(\mathbb{R}^n)^\vee$ such that:

$$\langle v_i, v_j^\vee \rangle = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and the dual space $V^\vee$ is a vector space of dimension $dimV$. The dual $\sigma^\vee$ of σ (as vector spaces) is the set

$$\sigma^\vee = \{u \in \mathbb{R}^\ltimes \mid (u \cdot v) \geq 0, \forall v \in \sigma\}$$

where $(- \cdot -)$ stands for the inner product.

Proposition 2.3.3. *[3] The dual* $\sigma^\vee$ *is a cone in* $(\mathbb{R}^n)^\vee$.

Example 2.3.4. *Consider the vectors* $v_1 = (1,0)$*,* $v_2 = (1,1)$ *and* $v_3 = (0,1)$ *in* $\mathbb{R}^2$*. The cone* $\sigma_1 := Cone(v_1, v_2)$ *and the cone* $\sigma_2 := Cone(v_2, v_3)$ *is given below respectively:*

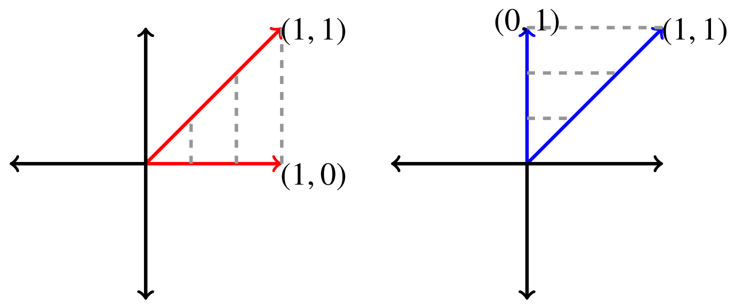


Figure 2.1: The cones σ_1 and σ_2

Example 2.3.5. *The dual cones of the cones given in the Example 2.3.4 are as follows:*

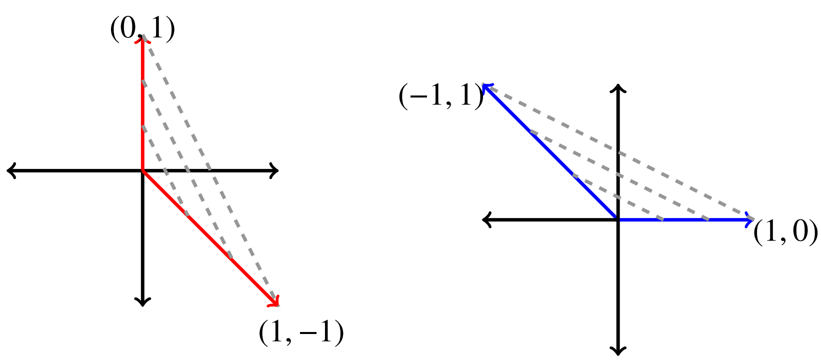


Figure 2.2: The dual cones of σ_1 and σ_2

The *relative interior* of a cone σ in $\mathbb{R}^n$ is the set

$$Int_{rel}(v_1, \dots, v_k) := \{\lambda_1 v_1 + \dots + \lambda_k v_k \mid \lambda_i \in \mathbb{R}_{>0}\}$$

We say that F is a face of σ in $\mathbb{R}^n$ if there is a $v \in \mathbb{R}^n$ and a $\lambda \in \mathbb{R}$ such that

$$F = \{u \in \sigma \mid u \cdot v = \lambda\} \text{ with } \lambda = max\{u \cdot v \mid u \in \sigma\}.$$

The dimension of a face F is defined to be that of the minimal linear space containing F.

Example 2.3.6. *In the Example 2.3.4,* 0*-dimensional face of* σ_1 *is the singleton* $\{(0,0)\}$*,* 1*-dimensional faces of* σ_2 *are the the "semi-lines" generated by* $(1,0)$ *and* $(1,1)$ *respectively, and it has only one* 2*-dimensional face, which is the cone* σ_1 *itself.*

Definition 2.3.7. *Consider the set of cones in* $\mathbb{R}^n$*,* $\mathcal{F}$*. This set is called a fan if the followings are satisfied:*

(i) *Every face of a cone in* $\mathcal{F}$ *is a cone in* $\mathcal{F}$*.*

(ii) *The intersection of any two cones* $\sigma, \tau \in \mathcal{F}$ *is a face of both* σ *and* τ*.*

Example 2.3.8. *The set* S *of the cones* $\sigma_1 = Cone((1,0),(1,1))$ *and* $\sigma_2 = Cone((0,1),(1,1))$ *defines a fan in* $\mathbb{R}^2$ *:*

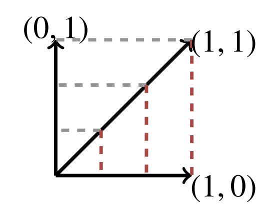


Figure 2.3: A fan in $\mathbb{R}^2$

The set S' *of the cones* $\sigma_1 = Cone((1,0,0),(0,1,0))$*,* $\sigma_2 = Cone((1,0,0),(0,0,1))$ *and* $\sigma_3 = Cone((0,1,0),(0,0,1))$ *forms a fan in* $\mathbb{R}^3$ *:*

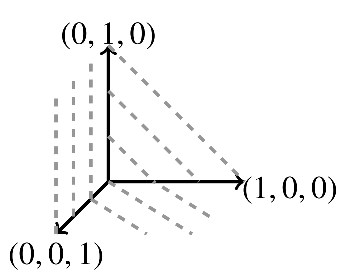


Figure 2.4: A fan in $\mathbb{R}^3$

2.4 Newton Polyhedron

Let $A \subseteq \mathbb{R}^n$. The convex hull of A is the set of all convex combinations of $v_1, \ldots v_k \in A$:

$$Convexhull(A) := \{\sum_i \lambda_i v_i \mid \sum_i \lambda_i = 1, \lambda_i \in \mathbb{R}_{>0} \;\; \forall i\}$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geqslant 0})^n$. Consider the polynomial $f(\mathbf{x}) = \Sigma_{\alpha \in (\mathbb{Z}_{\geqslant 0})^n} a_\alpha \mathbf{x}^\alpha$ in $k[x_1, \ldots, x_n]$. The *support* of f is the set

$$Supp(f) := \{\alpha \in (\mathbb{Z}_{\geqslant 0})^n \mid a_\alpha \neq 0\}$$

Definition 2.4.1. *Let f be a polynomial. The convex hull of the support of f is called the Newton polytope of f.*

.

Example 2.4.2. *Let $f(x) = z^3 + y^3 z + x^2 y^2$. Its support is $Supp(f) = \{(0, 0, 3), (0, 3, 1), (2, 2, 0)\}$. Then its Newton polytope is given below:*

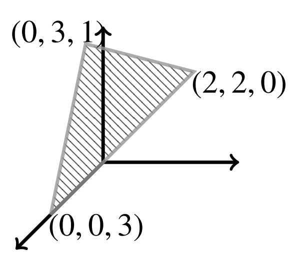


Figure 2.5: Newton polygon of f

Definition 2.4.3. *The Newton polyhedron $NP(f)$ of $f \in k[x_1, ..., x_n]$ is the convex hull of the set*

$$\bigcup \{\alpha + (\mathbb{R}_{\leq 0})^n \;\mid\; \alpha \in \;\; Supp(f)\}$$

Example 2.4.4. *Let us consider* $f(x, y) = 3x - 5xy + 2y + 8y^2 + x^2 \in \mathbb{R}^2$. *The support of* f *is the set:*

$$Supp(f) = \{(1, 0), (1, 1), (0, 1), (0, 2), (2, 0)\}.$$

Then the Newton polyhedron $NP(f)$ *is:*

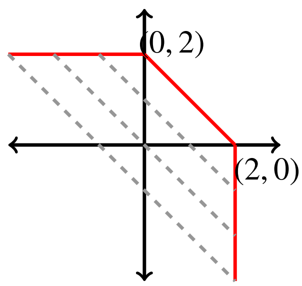


Figure 2.6: Newton polyhedron of f

Example 2.4.5. *Let us consider* $f(x, y, z) = z^2 + y^3 + z^4 \in \mathbb{R}^3$. *The support of* f *is the set:*

$$Supp(f) = \{(0, 0, 2), (0, 3, 0), (0, 0, 4)\}$$

Then we illustrate the Newton polyhedron $NP(f)$ *as:*

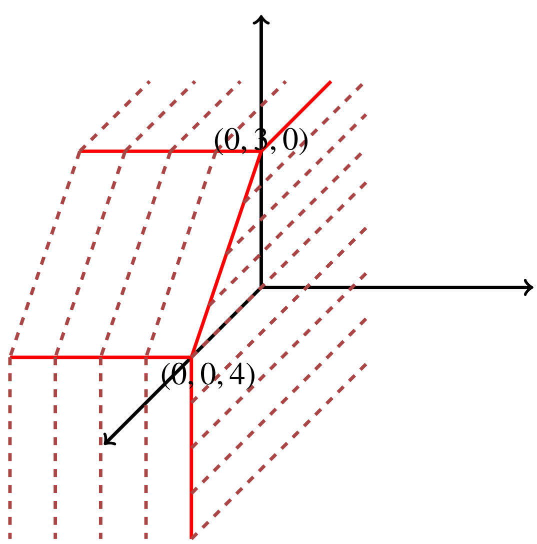


Figure 2.7: Newton polyhedron of f

Chapter 3

GROEBNER FAN

3.1 w-Order

Let us consider all the half spaces H_i in $\mathbb{R}^n$. The intersection of finitely many H_i's is called a convex polyhedron in $\mathbb{R}^n$. When it is bounded it is said to be a convex polytope. For example, the convex hull of the set $\{(3,0),(0,2),(3,2)\}$ defines a convex polytope in $\mathbb{R}^2$:

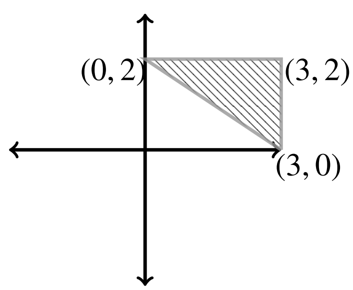


Figure 3.1: The convex hull of $(0,2)$,$(3,2)$ and $(3,0)$

In a convex polyhedron, the faces of dimension 0 are called the vertices; the faces of dimension 1 are called the edges and, the faces of dimension $n-1$ are called the facets. Here we will deal with the vectors in $\mathbb{R}^n$ which are orthogonal to the facets of a given convex polyhedron.

Definition 3.1.1. *The w-order of a polynomial f in $k[x_1,\ldots,x_n]$ where $f \neq 0$ and $w \in (\mathbb{R}_{\geq 0})^n$ is defined as*

$$o_w(f) := max\{w \cdot \alpha \mid \alpha \in Supp(f)\}$$

Example 3.1.2. *Let* $f(x,y) = 3x + 2y + 8xy^2 + x^2y \in \mathbb{R}[x,y]$.

For $w = (2,3)$*, we obtain* $o_w(f) = max\{2,3,8,7\} = 8$.

For $w = (1,1)$*, we obtain* $o_w(f) = max\{1,1,3,3\} = 3$.

Definition 3.1.3. *Let w be a vector in the first quadrant. We define*

$$S_w(f) := \{x \in \mathbb{R}^n; w.x = o_w(f)\}$$

Lemma 3.1.4. *[1]*

i) The intersection $I_w(f) := S_w(f) \cap NP(f)$ *is a face of* $NP(f)$.

ii) Let F *be a face of* $NP(f)$*. Then the set* $C_F := \{w \in (\mathbb{R}_{\geq 0})^n;\ F \subset I_w(f)\}$ *is a cone.*

iii) The collection $C(f) := \{C_F;\ F$ *is a face of NP(f)*$\}$ *forms a fan.*

Example 3.1.5. *Let* $f(x,y) = 3x - 5xy + 2y + 8y^2 + x^2 \in \mathbb{R}[x,y]$ *. The Newton polyhedron of f was given in Example 2.4.4. Clearly* $NP(f)$ *has five faces, namely horizontal edge* F_1*, vertical edge* F_2*, the edge that has slope* -1 F_3*, the point* $\{(2,0)\}$ F_4 *and finally the point* $\{(0,2)\}$ F_5*. Then* C_{F_1} *is the cone generated by* $(0,1)$*,* C_{F_2} *is the cone generated by* $(1,0)$*,* C_{F_3} *is the cone generated by* $(1,1)$*,* C_{F_4} *is the cone generated by* $(1,0)$ *and* $(1,1)$ *and finally* C_{F_5} *is the cone generated by* $(0,1)$ *and* $(1,1)$*. We illustrate the geometric representation of the fan* $C(f)$ *below:*

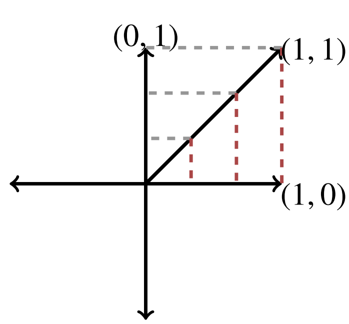


Figure 3.2: The fan $C(f)$

Note that the dual of the Newton polyhedron of a polynomial can simply be found by taking the dual vectors of its boundaries and considering the cones generated by resulting vectors. For example, the dual of the Newton polyhedron given in Example 2.4.4 can be found as taking the dual vectors of the vectors $(-1, 0)$, $(-1, 1)$ and $(0, -1)$, which are $(0, 1)$, $(1, 1)$ and $(1, 0)$ respectively and finding the cones generated by these last three vectors, which in fact gives us exactly the same image that we obtained in Example 3.1.5.

Definition 3.1.6. *The w-initial form of f with $w \in (\mathbb{R}_{\geq 0})^n$ is defined as :*

$$In_w(f) := \sum_{\{\alpha \in Supp(f) | w \cdot \alpha = o_w(f)\}} a_\alpha x^\alpha$$

Example 3.1.7. *Consider the polynomial given in the Example 3.1.2.*

For $w = (2, 3)$, $In_w(f) = 8xy^2$;

For $w = (1, 1)$ we have $In_w(f) = 8xy^2 + x^2y$.

The w-order is not a term order. In fact, for the polynomial f given in the Example 3.1.2, we obtain $In_w(f) = 8xy^2 + x^2y$ where $w = (1, 1)$, so it does not always give a unique monomial. Moreover, for $w = (1, \ldots, 1)$ we obtain the tangent cone of the variety defined by f. We can define an equivalence relation in $\mathbb{R}^n$:

$$w \sim w' \iff In_w(f) = In_{w'}(f)$$

Lemma 3.1.8. [1] *The set*

$$C_w(f) := \overline{\{u \in \mathbb{R}^n \mid In_u(f) = In_w(f)\} \subset \mathbb{R}^n\}}$$

is a cone and the union of all of these cones is the set

$$\mathcal{F}(f) := \{C_w(f) \mid w \in \mathbb{R}^n\}$$

which is a fan.

Example 3.1.9. *Consider the polynomial given in the Example 3.1.2. The vectors $w_1 = (2, 3)$ and $w_2 = (0, 2)$ are equivalent as $In_{w_1}(f) = In_{w_2}(f) = 8y^2$ but $w_3 = (1, 1)$ is not* *equivalent to w_1 as their initial forms with respect to f do not coincide. Calculating by hand, we obtain that there are 9 equivalence classes:*

$C_{(0,0)}(f)$, $C_{(0,-1)}(f)$, $C_{(-1,0)}(f)$, $C_{(1,0)}(f)$, $C_{(2,3)}(f)$, $C_{(-1,1)}(f)$, $C_{(1,1)}(f)$, $C_{(1,-1)}(f)$ *and* $C_{(-1,-1)}(f)$.

In a more general way, we have:

Definition 3.1.10. *For the ideal $I \subset k[x_1, \dots, x_n]$ and $w \in (\mathbb{R}_{\geq 0})^n$ we define the w-initial form of I as:*

$$In_w(I) := \langle In_w f \;\mid\; \text{for all } f \in I\rangle$$

For a vector $w \in \mathbb{R}^n$, the set

$$C_w(I) := \overline{\{v \in \mathbb{R}^n \mid In_v(I) = In_w(I)\}}$$

is a cone in $\mathbb{R}^n$. Moreoever, the set of such equivalence classes forms a fan.

Example 3.1.11. *Let $I = \langle x+y+3\rangle$ be an ideal in $\mathbb{Q}[x, y]$. Consider the vectors $w_0 = (0,0)$, $w_1 = (-1,0)$, $w_2 = (0,1)$, $w_3 = (1,1)$, $w_4 = (1,0)$, $w_5 = (0,-1)$, $w_6 = (-1,-1)$. Note that there are seven different equivalence classes of the form $C_w(I)$ and each w_i (where $0 \leq i \leq 6$) is a representative of these seven different equivalence classes:*

$In_{w_0} = \langle x + y + 3\rangle$, $In_{w_1} = \langle y + 3\rangle$, $In_{w_2} = \langle y\rangle$, $In_{w_3} = \langle x + y\rangle$, $In_{w_4} = \langle x\rangle$, $In_{w_5} = \langle x + 3\rangle$, $In_{w_6} = \langle 3\rangle$.

For example, C_{w_2} gives the cone generated by $(-1,0)$ and $(1,1)$. The geometric representation of $\mathcal{F}(f)$ is as below:

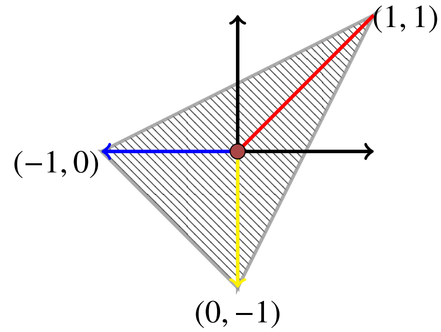


Figure 3.3: The fan $\mathcal{F}(f)$

3.2 Groebner Fan

Following [1], given $w, w' \in Int_{rel}\langle v_1, \ldots, v_k \rangle$, we have

$$In_w(I) = In_{w'}(I) \Longleftrightarrow C_w(I) = C_{w'}(I).$$

Definition 3.2.1. *Consider the set of all equivalence classes $C_w(I)$ intersecting the positive orthant. Then this set is called the Groebner fan of I. The cones in the Groebner fan are called Groebner cones of I.*

Example 3.2.2. *Lef I be the ideal given in the Example 3.1.11. As we already calculated $\mathcal{F}(f)$, the geometric representation of the Groebner fan associated to I is given as :*

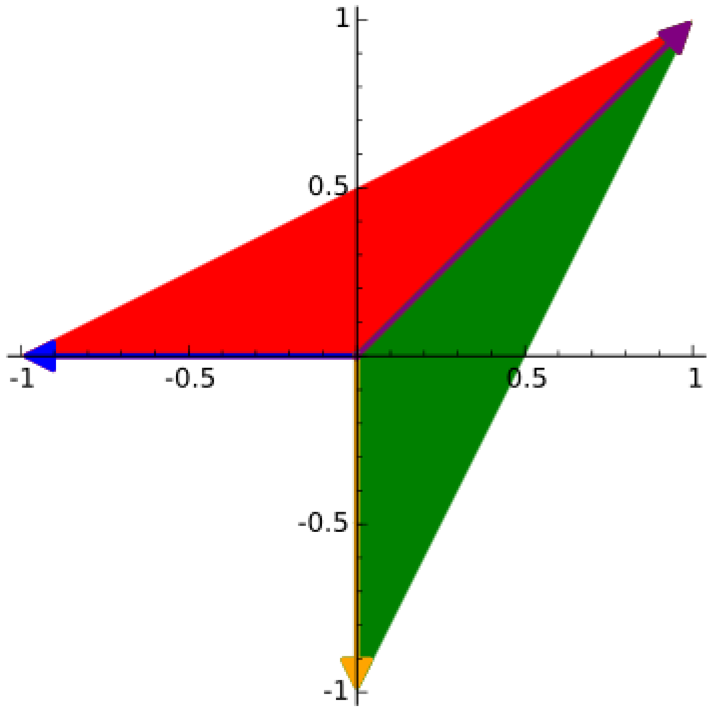


Figure 3.4: The Groebner fan of $x + y + 3$

This computation can also be made by Sage. It can also compute the Groebner fan when the base field is $\mathbb{Q}$. We illustrate the commands used to compute the reduced Groebner basis and Groebner fan of a given ideal via this example :

$R. < x, y >= PolynomialRing(QQ, 2, order = \text{'}lex\text{'})$

$S = [xy^2, y^2 - y]$

$I = R.ideal(S)$

$gf = I.groebner\_fan()$. *(It renames the Groebner fan associated to I as "gf".)*

$gf.reduced\_groebner\_bases()$ *(It computes all possible reduced Groebner basis of I with respect to corresponding term orders. Proposition 3.2.7 will show that the number of reduced Groebner basis of I is equal to the number of maximal cones contained in the Groeb-*

ner fan of I.)

pf=gf.polyhedralfan() (It renames the polyhedral structure that the Groebner fan of I has)

pf.rays() (It computes the rays; in other words it gives a set of generators. For this example, it gives $[-1,1]$*,* $[1,-1]$*,* $[1,1]$ *and it also automatically labels these three vectors as* 0*,* 1 *and* 2 *respectively.)*

pf.cones() (It computes the cones of each dimension. For this example, it gives

$$\{1: \;\; [[0],[1],[2]], \;\;\; 2: \;\; [[0,2],[1,2]]\}.$$

It means there are three 1 *dimensional cones ; the one generated by* $[0]$*,i.e. generated by* $[-1,1]$ *(remember the vector* $[-1,1]$ *was labeled as* 0*), the one generated by* $[1,-1]$ *and the one generated by* $[1,1]$*. There are two* 2*- dimensional cones ; one is generated by* $[0]$ *and* $[2]$ *and the other one is generated by* $[1]$ *and* $[2]$*. Note that this exactly what we illustrated in the above figure.)*

Theorem 3.2.3. [7] *The Groebner fan is a polyhedral complex of cones forming a fan.*

This results from the fact that for $w \in (\mathbb{R}_{\geq 0})^n$

$$w \in C_{\prec}(I) \iff \forall f \in R_{\prec}(I) \mid In_{\prec}(In_w(f)) = In_{\prec}(f)$$

Proposition 3.2.4. *[7] Given a term order* $\prec$ *and* $w \in C_{\prec}(I)$*, we have*

$$In_v(I) = In_w(I) \Longleftrightarrow In_v(f) = In_w(f)$$

for all $f \in R_{\prec}(I)$ *with* v *being an arbitrary vector in* $\mathbb{R}^n$*.*

Example 3.2.5. *Consider the ideal given in part i) of Example 2.2.9. Remember that the set* $\{xy^2, y^2 - y\}$ *is a reduced Groebner basis with respect to the lexicographical order* $\prec_l$*. Then,* $In_{\prec_l}(I) = \langle xy^2, y^2 \rangle = \langle y^2 \rangle$*. Let* $w = (0,1)$ *then we have :*

$$In_w(I) = \langle In_w(xy^2), \;\; In_w(y^2 - y) \rangle = \langle xy^2, y^2 \rangle = \langle y^2 \rangle = In_{\prec_l}(I)$$

so $(0,1) \in C_{<_l}(I)$. *By the Proposition 3.2.4, any vector of the form* $(0,v)$ *with* $v > 0$ *belongs to* $C_{<_l}(I)$ *since*

$$In_{(0,v)}(xy^2) = In_{(0,1)}(xy^2) = xy^2, \ \ In_{(0,v)}(y^2 - y) = In_{(0,1)}(y^2 - y) = y^2$$

Remark 3.2.6. [7] *For an ideal I, the number of the sets* $C_<(I)$ *is finite and these sets cover* $(\mathbb{R}_{\geq 0})^n$. *Also, every initial ideal* $In_<(I)$ *is of the form* $In_w(I)$ *for some* $w \in (\mathbb{R}_{\geq 0})^n$. *Thus, every* $C_<(I)$ *is of the form* $C_w(I)$.

Proposition 3.2.7. *[7] There is a one-to-one correspondance between the reduced Grobner basis of I and the cones with maximum dimension in the Groebner fan. Later, it will be seen that a tropical variety is a spesific subfan of the Grobner fan.*

Example 3.2.8. *Consider the ideal* $I = \langle x{+}y{+}3 \rangle$ *whose Groebner fan was given in Example 3.2.2 . Clearly, there are two maximal dimensional cones, namely the cone* σ_1 *generated by* $(0,-1)$ *and* $(1,1)$ *and the cone* σ_2 *generated by* $(1,1)$ *and* $(-1,0)$. *Note that the reduced Groebner basis* $\{x + y + 3\}$ *with respect to lexicographical term order corresponds to the cone* σ_1 *as* $\sigma_1 = C_{<_l}(I)$. *Similarly, the reduced Groebner basis* $\{x + y + 3\}$ *with respect to reverse lexicographical order corresponds the cone* σ_2 *as* $\sigma_2 = C_{<_{rl}}(I)$.

Example 3.2.9. *i) Let* $I = \langle x^2 + y^3 + z^5 \rangle$ *be an ideal in* $\mathbb{Q}[x,y,z]$.

The Groebner fan has the 1*-dimensional cones given by*

$$In_{w_1}(I) = y^3 + z^5, \ In_{w_2}(I) = x^2 + y^3, \ In_{w_3}(I) = x^2 + z^5$$

where $w_1 = (-68, 75, 45), \ w_2 = (18, 12, -65), \ w_3 = (50, -87, 20)$

and 2*-dimensional cones which are*

$$In_{\sigma_1}(I) = y^3, \ In_{\sigma_2}(I) = z^5, \ In_{\sigma_3}(I) = x^2$$

where $\sigma_1 =< (-68, 75, 45), (18, 12, -65) >$, $\sigma_2 =< (-68, 75, 45), (50, -87, 20) >$, *and* $\sigma_3 =< (18, 12, -65), (50, -87, 20) >$.

Then its Groebner fan is:

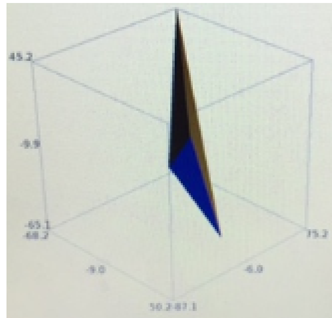


Figure 3.5: The Groebner fan of I

ii) Let $I = \langle z^3 + y^3z + x^2y^2 \rangle$ *be an ideal in* $\mathbb{Q}[x, y, z]$. *The Groebner fan has three* 1-*dimensional cones given by* $w_1 = (-26, 10, 15)$, $w_2 = (2, 17, -13)$, $w_3 = (24, -27, -2)$*:* $In_{w_1}(f) = z^3 + y^3z$, $In_{w_2}(f) = y^3z + x^2y^2$, $In_{w_3}(f) = z^3 + x^2y^2$

It has three 2*-dimensional cones :*

$$\sigma_1 = \langle(-26, 10, 15), (2, 17, -13)\rangle,\ \sigma_2 = \langle(-26, 10, 15), (24, -27, -2)\rangle \ \textit{and}$$

$\sigma_3 = \langle(2, 17, -13), (24, -27, -2)\rangle$ *where* $In_{\sigma_1}(f) = y^3z$, $In_{\sigma_2}(f) = z^3$, $In_{\sigma_3}(f) = x^2y^2$. *Note that* $w_1 = (-26, 10, 15)$*,* $w_2 = (24, -27, -2)$ *and* $w_3 = (2, 17, -13)$*. We give the geometric illustration of its Groebner fan as:*

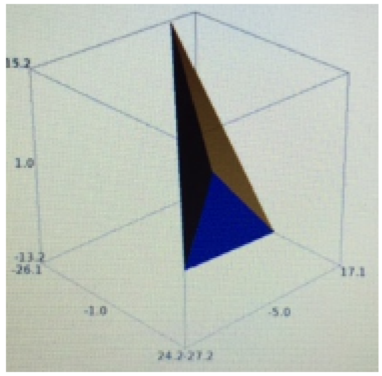


Figure 3.6: The Groebner fan of I

Example 3.2.10. *Consider the polynomial $f(x,y,z) = z^2 + y^3 + z^5$ and let $I = \langle f \rangle$. For the equivalence classes of the Groebner fan associated to I, we obtain:*

Three 1-dimensional cones, whose generatos are $(0,-2,-3)$, $(0,-1,0)$ and $(0,5,3)$ respectively,

Three 2-dimensional cones, namely the cone σ_1 generated by $(0,-2,-3)$, $(0,5,3)$, the cone σ_2 generated by $(0,-2,-3)$, $(0,-1,0)$ and the cone σ_3 generated by $(0,5,3)$, $(0,-1,0)$,

The unique 3-dimensional cone is generated by $(0,-2,-3)$, $(0,5,3)$ and $(0,-1,0)$.

Let $w = (w_1, w_2, w_3)$ be an arbitrary vector in $\mathbb{R}^3$. Note that

$In_w(f) = z^2$ for $w_3 < 0$ and $w_3 > \frac{3w_2}{2}$,

$In_w(f) = y^3$ for $w_2 > \frac{5w_3}{3}$ and $w_2 > 0$,

$In_w(f) = z^5$ for $w_3 > 0$ and $w_3 > \frac{3w_2}{5}$,

$In_w(f) = z^2 + y^3$ for $2w_3 = 3w_2$ and $w_3 < 0$,

$In_w(f) = z^2 + z^5$ for $w_3 = 0$ and $w_2 < 0$,

$In_w(f) = y^3 + z^5$ for $3w_2 = 5w_3$ and $w_3 > 0$,

$In_w(f) = z^2 + y^3 + z^5$ for $w_2 = w_3 = 0$.

Geometrically the Groebner fan of f looks like:

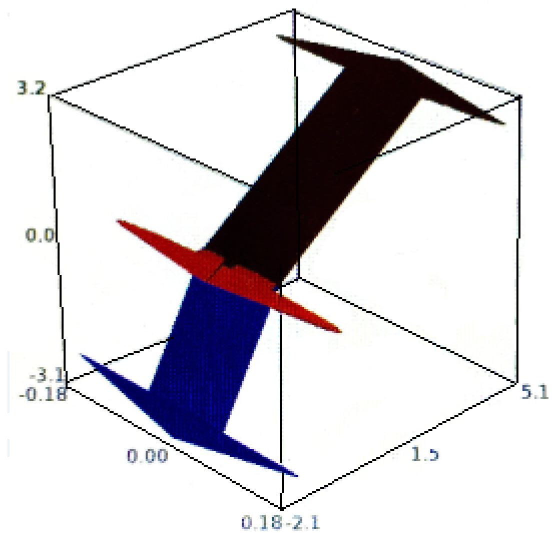


Figure 3.7: The Groebner fan of f

Example 3.2.11. *Let $I = \langle f_1, f_2, f_3 \rangle$ where $f_1 = z^2 - yw$, $f_2 = zw - yx^2 - y^4$, $f_3 = w^2 - zx^2 - zy^3$. Its Groebner fan is much more complicated: It has eleven 1-dimensional cones, thirty 2-dimensional cones and twenty 3-dimensional cones. This computation has been made by using Sage by using the same commands illustrated in Example 3.2.2.*

Chapter 4

TROPICAL ALGEBRA

Let us replace the usual addition + and multiplication $\times$ in $\mathbb{R}$ by two new operations $\oplus$ and $\odot$ respectively, defined as

$$x \oplus y = max\{x, y\} \text{ and } x \odot y = x + y$$

for all $x, y \in \mathbb{R}$. For example, we have $10 \oplus 1 = 10$ and $10 \odot 1 = 11$.

The operations $\oplus$ and $\odot$ are called *tropical addition* and *tropical multiplication* respectively. It is easy to see that $\oplus$ and $\odot$ are commutative and $\odot$ is distributive onto $\oplus$. The essential difference between the classical case is that the tropical addition does not have an identity element in $\mathbb{R}$. However, we can extend these two operations to $-\infty$ by defining

$$x \oplus (-\infty) = x \text{ and } x \odot (-\infty) = -\infty$$

Clearly, in this case $-\infty$ is the identity element with respect to the tropical addition. Then $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ is a semi-field, called tropical semi-field. Remark that $(\mathbb{T}, \oplus, \odot)$ has all the axioms of being a field except that the axiom "every element has an additive inverse".

4.1 Tropical Polynomials with one variable

Let $\mathbb{T}$ be the semi-field defined in the previous section. Consider the polynomial ring $\mathbb{T}[x]$ with one variable over $\mathbb{T}$, called the tropical polynomial ring. It is the set of all finite sums of the form

$$\bigoplus_i \alpha_i \odot x^i$$

Such a sum is called a tropical polynomial. Choosing the minimum or maximum values of $\bigoplus_i \alpha_i \odot x^i$, we may define the roots of a tropical polynomial. So let us consider the tropical polynomial

$$p(x) = max_{i=1}^{d}(\alpha_i + ix)$$

Definition 4.1.1. *If there are two distinct indices i and j where*

$$p(x_0) = \alpha_i + ix_0 = \alpha_j + jx_0$$

is satisfied for some $x_0 \in \mathbb{T}$, then x_0 is called a tropical root of the polynomial $p(x)$.

Example 4.1.2. *Let p(x) = $0 \oplus x \oplus (-1) \odot x^2$ be a tropical polynomial in $\mathbb{T}[x]$. We have:*

$$p(x) = max\{0, x, 2x - 1\}$$

Clearly, 0 and 1 are the only roots of $p(x)$ as $p(x)$ attains its maximum at 0 and 1 by two monomials. The graph of $p(x)$ is given below:

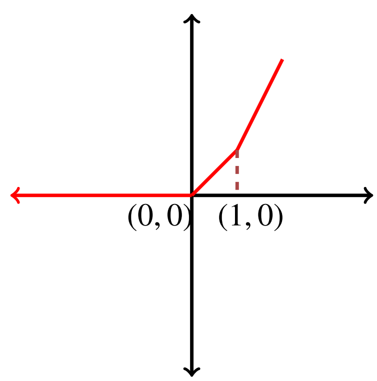


Figure 4.1: The graph of $p(x)$

Note that the roots of $P(x)$ are the points where the linearity changes.

Example 4.1.3. *Let $p(x) = 0 \oplus x \oplus (-2) \odot x^3$, in classical sense $p(x) = max\{0,\ x,\ 3x - 2\}$. Hence, the graph of $p(x)$ is given as :*

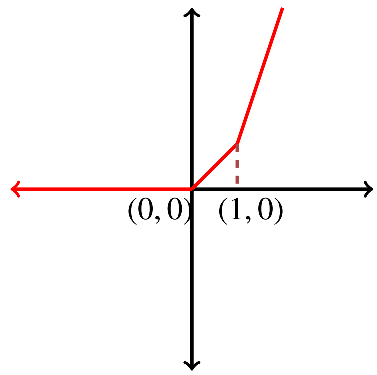


Figure 4.2: The graph of $p(x)$

As in the classical case, we have the fundamental theorem of tropical algebra: every tropical polynomial can be factorised into a product of linear tropical polynomials. We will introduce all the necessary definitions, lemmas and theorems, which will help us understand the proof of the theorem. Along the way, the proofs of some lemmas will be skipped as they are dispensable for our purposes.

Definition 4.1.4. *[12] If two tropical polynomials $p(x)$ and $q(x)$ give the same value for all $x \in \mathbb{T}$, then they are called functionally equivalent.*

Note that unlike in classical geometry, functionally equivalent polynomials need not to be the same.

Example 4.1.5. $p(x) = max\{2x, x + 2, 4\}$ *and* $q(x) = max\{2x, x + 1, 4\}$ *are different polynomials which are functionally equivalent since the maximum of two sets coincide for all $x \in \mathbb{T}$. However, they are not the same polynomials.*

Definition 4.1.6. [12] *Let $p(x)$ be a tropical polynomial*

$$p(x) = \alpha_n \odot x^n \oplus \alpha_{n-1} \odot x^{n-1} \oplus \ldots \oplus \alpha_r \odot x^r$$

If there is no coefficient $\beta_i > \alpha_i$ such that the polynomial obtained by substituting β_i for α_i in $p(x)$ is functionally equivalent to $p(x)$, then α_i is called a largest coefficient of $p(x)$.

Example 4.1.7. *Every coefficient of the polynomial* $p(x) = x^2 \oplus 2 \odot x \oplus 3$ *is a largest coefficient. However,* 2 *is not a largest coefficient of the polynomial* $p_1(x) = x^2 \oplus 2 \odot x \oplus 6$ *since* $p_2(x) = x^2 \oplus 3 \odot x \oplus 6$ *is functionally equivalent to* $p_1(x)$.

Remark that, if α_n and α_r are not equal to $-\infty$, they are the largest coefficients of $p(x)$ since any coefficient being equal to $-\infty$ is not a largest coefficient.

Definition 4.1.8. *[12] A tropical polynomial* $p(x) = \alpha_n \odot x^n \oplus \alpha_{n-1} \odot x^{n-1} \oplus \ldots \oplus \alpha_r \odot x^r$ *is said to be largest if* α_i *is a largest coefficient of* $f(x)$ *for all* $1 \leq i \leq n$.

Example 4.1.9. *The polynomial* $p(x) = x^2 \oplus 2 \odot x \oplus 3$ *is a largest polynomial since any replacement of its coefficients will lead a polynomial which is not functionally equivalent to* $p(x)$*. This fact will be more clear after the Lemma 4.1.12.*

Lemma 4.1.10. [12] *Two largest tropical polynomials are functionally equivalent if they are the same polynomials.*

Lemma 4.1.11. [12] *Given a tropical polynomial* $p(x) = \alpha_n \odot x^n \oplus \alpha_{n-1} \odot x^{n-1} \oplus \ldots \oplus \alpha_r \odot x^r$. *Let* $q(x) = \beta_n \odot x^n \oplus \beta_{n-1} \odot x^{n-1} \oplus \ldots \oplus \beta_r \odot x^r$ *be a largest coefficient polynomial which is functionally equivalent to* $p(x)$*. Then*

$$\beta_i = max\{\{\alpha_i\} \cup \{\frac{\alpha_j(k-i) + \alpha_k(i-j)}{k-j} \mid r \leq j < i < k \leq n\}\}.$$

Lemma 4.1.12. [12] *Let* $p(x) = \alpha_n \odot x^n \oplus \alpha_{n-1} \odot x^{n-1} \oplus \ldots \oplus \alpha_r \odot x^r$ *be a polynomial in* $\mathbb{T}[x]$ *with* $\alpha_i \neq -\infty,\ i \in \{r, r+1, \ldots, n\}$*. The polynomial* $p(x)$ *is a largest tropical polynomial if and only if* $d_n \geq d_{n-1} \geq ... \geq d_{r+1}$ *where* $d_i = \alpha_{i-1} - \alpha_i$.

Example 4.1.13. *The polynomial* $p(x)$ *given in Example 4.1.9 is a largest polynomial as we indicated before since* $2 - 0 = 2 > 1 = 3 - 2$.

Finally we are ready to announce the Fundamental theorem:

Theorem 4.1.14. [12] *(Fundamental Theorem of Tropical Algebra) A largest tropical polynomial* $p(x) = \alpha_n \odot x^n \oplus \alpha_{n-1} \odot x^{n-1} \oplus \ldots \oplus \alpha_r \odot x^r$ *has a unique factorization*

$$p(x) = \alpha_n \odot x^r \odot (x \oplus d_n) \odot (x \oplus d_{n-1}) \ldots \odot (x \oplus d_{r+1})$$

where $d_i = \alpha_{i-1} - \alpha_i$ and $i \in \{r+1, \ldots, n\}$.

Proof. By Lemma 4.1.12 the differences between consecutive coefficients of $p(x)$ are non-increasing. Notice that

$\alpha_n \odot x^r \odot (x \oplus d_n) \odot (x \oplus d_{n-1}) ... \odot (x \oplus d_{r+1}) = \alpha_n \odot x^n \oplus \alpha_n \odot d_n \odot x^{n-1} \oplus ... \oplus \alpha_n \odot d_n \odot d_{n-1} ... d_{r+1} \odot x^r$

which shows that the coefficient of term x^i is

$$\alpha_n \odot d_n \odot d_{n-1} ... \odot d_{i+1} = \alpha_i.$$

Hence, we conclude that $f(x)$ and $\alpha_n \odot x^n \oplus \alpha_n \odot d_n \odot x^{n-1} \oplus ... \oplus \alpha_n \odot d_n \odot d_{n-1} ... d_{r+1} \odot x^r$ are the same.

It remains to prove that the factorisation is unique. Let $q(x) = b_n \odot x^r \odot (x \oplus c_n) \odot (x \oplus c_{n-1}) ... \odot (x \oplus c_{r+1})$ be another factorisation of $f(x)$, where $c_n \geq c_{n-1} \geq ... \geq c_{r+1}$. Note that when $q(x)$ is expanded the coefficients of x^{i-1}, x^i and x^{i+1} appear to be $p_{i-1} = b_n + c_n + ... + c_{i+1} + c_i$, $q_i = b_n + c_n + ... + c_{i+1}$ and $r_{i+1} = b_n + c_n + ... + c_{i+2}$ respectively . By Lemma 4.1.12, we conclude that $q(x)$ is a largest tropical polynomial as $p_{i-1} - q_i = c_i \geq q_i - r_{i+1} = c_{i+1}$. By Lemma 4.1.10, $p(x)$ and $q(x)$ are the same polynomials as they are functionally equivalent to each other and as they both are largest polynomials. □

Proposition 4.1.15. [12] *$x \oplus d$ is a factor of a largest polynomial $p(x)$ if and only if d is a tropical root of $p(x)$.*

Proof. Let $x \oplus d$ be a factor of $p(x)$. Then according to Theorem 4.1.14, d must be equal to $\alpha_{i-1} - \alpha_i$ for some i. If $j < i$, since $d_i \geq d_{i-1} \geq ... \geq d_{j+1}$ we have

$$a_i + id_i = a_i + (i-j)d_i + jd_i \geq a_i + d_i ... + d_{j+1} + jd_i = a_j + jd_i$$

Otherwise if $j > i$, as $-d_i \geq -d_{i+1} \geq ... \geq -d_j$, we have

$$a_i + id_i = a_i - (j-i)d_i + jd_i \geq a_i - d_{i+1} - d_{i+2} + ... - d_j + jd_i = a_j + jd_i.$$

Hence, $f(d_i) = a_i \odot d_i^i$. Note that another expression of the term $a_i d_i^i$ can be given as

$$a_i \odot d_i^i = a_n \odot d_n ... d_{i+1} \odot d_i^i = a_n \odot d_n \odot d_{n-1} ... d_{i+1} \odot d_i \odot d_i^{i-1} = a_{i-1} \odot d_i^{i-1}.$$

which is enough to conclude that d is a tropical root of $p(x)$ as the maximum of $p(x)$ is obtained by at least two monomials at d.

Let d be a tropical root of f. Then by definition, there exists at least two terms such that $p(d) = a_i \odot d^i = a_k \odot d^k$. Suppose that $|i - k| > 1$. For the monomial $a_j \odot d^j$ if $x \geq d$, we have that

$$a_k + kx = a_k + (k-j)x + jx \geq a_k + jx + (k-j)d = a_k + kd + j(x-d) \geq a_j + jd + j(x-d) = a_j + jx$$

where $i < j < k$. In the same way, for $x \leq d$ the inequality $a_i + ix < a_j + jx$ holds, which suffices to conclude that a_j is not a largest coefficient as there is no x satisfying $p(x) = a_j \odot x^j$. Therefore we get a contradiction as we assumed that $p(x)$ is a largest polynomial, which means that there exist some t such that $a_t \odot d^t = a_{t-1} \odot d^{t-1}$ is satisfied. Thus,

$$0 = a_{t-1} + (t-1)d - (a_t + td) = a_{t-1} - a_t - d$$

which gives us $d = a_{t-1} - a_t$, meaning that d is the difference between consecutive coefficients of $f(x)$. Hence, $x \oplus d$ is a factor of $p(x)$. □

In algebraic geometrical point of view, the important result is:

Corollary 4.1.16. *The semi-field $\mathbb{T}$ is algebraically closed.*

It results from the fact that every tropical polynomial is functionally equivalent to a largest coefficient polynomial.

4.2 Tropical Polynomials with several variables

A tropical polynomial in the tropical polynomial ring $\mathbb{T}[x, y]$ in two variables can be written as:

$$p(x, y) = \bigoplus_{i,j} \alpha_{i,j} \odot x^i \odot y^j$$

Hence we have:

$$p(x, y) = max_{i,j}(\alpha_{i,j} + ix + jy)$$

Definition 4.2.1. *Let $p(x, y)$ be a tropical polynomial and $P = (x_0, y_0)$ be in $\mathbb{R}^2$. P is a tropical root of $p(x, y)$ if the maximum of p is attained by at least two monomials of p at P.*

Definition 4.2.2. *[5] Let T denote the set of tropical roots of a given tropical polynomial $p(x, y)$ and e be an edge of T. Consider all the pairs (i, j) and (k, l) which correspond to the edge e. The largest value of all greatest common divisors of $|i - k|$ and $|j - l|$ for these pairs is called the weight of e.*

Example 4.2.3. *Let $p(x, y) = x \oplus y \oplus 1$ be a tropical polynomial. The roots of $p(x, y)$ are the points (x_0, y_0) where the maximum of $p(x, y)$ is attained by at least two monomials of $p(x, y)$ at (x_0, y_0). Then we must find the points (x_0, y_0) satisfying one of the following three systems of equations:*

$$x_0 = y_0 \geq 1, \quad x_0 = 1 \geq y_0 \quad y_0 = 1 \geq x_0.$$

Hence the intersection point of $x_0 = y_0 = 1$ is the point $(1, 1)$. We illustrate the graph of the tropical roots of $P(x, y)$ as below:

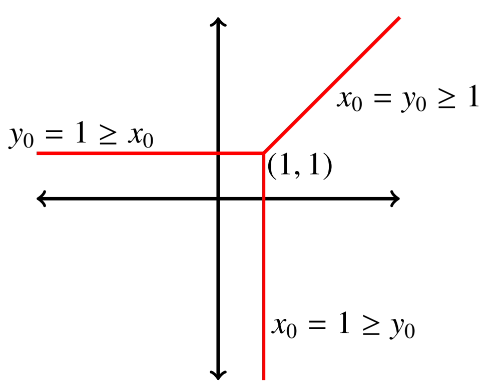


Figure 4.3: The graph of the tropical roots of $p(x, y)$

Note that all edges are of weight 1.

Example 4.2.4. *Let* $p(x, y) = 5 \oplus 5 \odot x \oplus 5 \odot y \oplus 4 \odot x \odot y \oplus 1 \odot y^2 \oplus x^2$ *be a tropical polynomial. By solving systems of equations as in the case of above example, we illustrate the geometric representation of the roots of* $p(x, y)$ *below:*

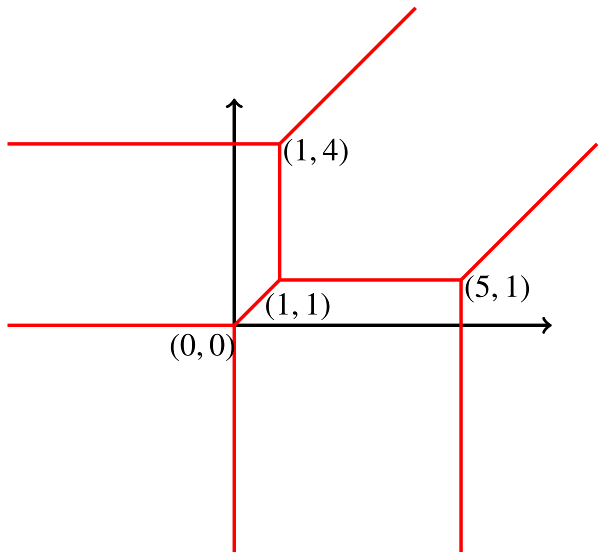


Figure 4.4: The graph of the tropical roots of $p(x, y)$

Note once again that each edge is of weight 1.

We have seen that a tropical polynomial in one variable can be uniquely factorised into linear terms. However, this is not always true in the case of a tropical polynomial in two or more variables. (see [12] for more details about the factorization of tropical polynomials in several variables.)

When a tropical polynomial has more than two variables the general representation as in the first paragraph above comes cumbersome. Instead of giving the formal definition, we illustrate the definition of the root of a tropical polynomial in three variables by an example :

Example 4.2.5. *Let* $p(x, y, z) = 2 \odot x \oplus 3 \odot y \oplus 5 \odot z$ *be a tropical polynomial. As in the* *cases of tropical polynomials in one or two variables, we calculate the tropical roots of* f *as the points* (x_0, y_0, z_0) *which satisfy one of the following three inequalities:*

$$x_0 + 2 = y_0 + 3 \geq z_0 + 5, \quad x_0 + 2 = 5 + z_0 \geq 3 + y_0, \quad y_0 + 3 = 5 + z_0 \geq x_0 + 2$$

Chapter 5

TROPICAL VARIETIES

5.1 Tropicalization

In a polynomial $f \in k[x_1, \ldots, x_n]$, let us replace the usual sum $+$ by tropical sum $\oplus$, the usual multiplication $\times$ by the tropical multiplication $\odot$ and set all coefficients of f to zero. The new polynomial obtained will be called the tropicalization of f and denoted by $trop(f)$. Hence $trop(f) \in \mathbb{T}[x_1, \ldots, x_n]$.

Example 5.1.1. *Consider* $f(x, y) = 2x^2y + 3x^6y \in \mathbb{R}[x, y]$. *Then*

$$trop(2x^2y + 3x^6y) = 0 \odot x^2 \odot y \oplus 0 \odot x^6 \odot y$$

Definition 5.1.2. *Let* $f \in \mathbb{T}[x_1, \ldots, x_n]$ *be a tropical polynomial. The set of roots of* f *is defined as the set*

$R(f) := \{w \in \mathbb{R}^n \mid$ *the maximum of* f *is attained by at least two monomials of* f *at* $w\ \}$

Definition 5.1.3. *The tropical variety* $TV(f)$ *of a polynomial* f *is defined as the set of tropical roots of* $trop(f)$, *i.e.* $R(trop(f))$.

Example 5.1.4. *Let* $f(x, y) = x + y + 1$. *We have* $trop(f) = x \oplus y \oplus 0$ *and calculated its tropical roots above in Example 4.2.3. The tropical variety* $TV(f)$ *is given by the red line below:*

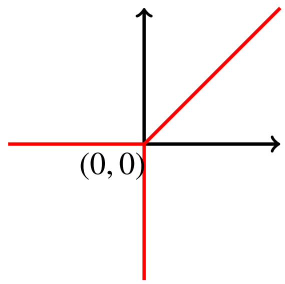


Figure 5.1: The tropical variety of f

Hence the tropical variety $TV(f)$ is a tropical hypersurface in $\mathbb{R}^n$ defined as the set of points in $\mathbb{R}^n$ where $trop(f) : \mathbb{R}^n \to \mathbb{R}$ is not linear. This means that the tropical varieties of codimension 1 are simply the tropicalization of the algebraic hypersurfaces defined by the polynomials in $k[x_1, \ldots, x_n]$. A tropical variety defined as the tropicalizations of a finite number of polynomials $f_1, \ldots, f_k \in k[x_1, \ldots, x_n]$ is not straightforward. For this, we have:

Proposition 5.1.5. *Let k be an algebraically closed field. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. The tropical variety defined by I is:*

$$TV(I) = \bigcap_{f \in I} TV(f)$$

The tropicalization process for an ideal (which is not principal) becomes easier if we find a suitable set of generators for the ideal I. Let us examine that situation in the following section.

5.2 Initial Ideals

The tropical variety $TV(f)$ of the polynomial $f \in k[x_1, \ldots, x_n]$ is the set

$$TV(f) := \{w \in \mathbb{R}^n \mid In_w f \text{ is monomial-free}\}.$$

This definition is equivalent to the definition in the preceding section.

Example 5.2.1. *Consider $f(x, y) = x+y+1$. We look for the vectors $w = (w_1, w_2)$ satisfying one of these three equalities*

$$In_w(f) = x + y,\ \ In_w(f) = x + 1,\ \ In_w(f) = y + 1, In_w(f) = x + y + 1$$

which is to say that

$$w_1 = w_2 \geq 0,\ \ w_1 = 0 \geq w_2,\ \ w_2 = 0 \geq w_1.$$

It is nothing but $R(trop(f)$, so the tropical variety is the same as in Example 5.1.4:

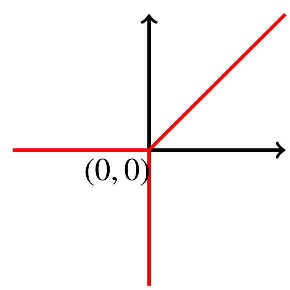


Figure 5.2: The tropical variety of f

Definition 5.2.2. *Let k be algebraically closed and $I \subset k[x_1, \ldots, x_n]$ be an ideal.*

$$TV(I) := \{w \in \mathbb{R}^n \mid In_w I \text{ is monomial-free}\}$$

is called the tropical varitety associated with the ideal I.

By this last definition, we clearly have the equality given in Proposition 5.1.5. Hence, any tropical variety in $\mathbb{T}^n$ is the intersection of some tropical hypersurfaces in $\mathbb{R}^n$.

Definition 5.2.3. *Let $I \subset k[x_1, \ldots, x_n]$ and $\mathcal{T} \subseteq I$. If $TV(I) = \bigcap_{f \in \mathcal{T}} TV(f)$ then the subset $\mathcal{T}$ is called a tropical basis for I.*

The tropical geometric version of Hilbert Nullstellensatz:

Theorem 5.2.4. [2] *Every tropical ideal in $\mathbb{T}[x_1, \ldots, x_n]$ has a finitely generated tropical basis.*

Moreover:

Proposition 5.2.5. [6] *Let I be a prime ideal in $k[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_k$. Let $< g_1, \ldots, g_r >$ be a Groebner basis of I. Then the set*

$$\mathcal{T} = \{f_1, \ldots, f_k, g_1, \ldots, g_r\}$$

is a tropical basis for I of cardinality $k + r$.

Corollary 5.2.6. *If $I \subset k[x_1, \ldots, x_n]$ is a prime ideal containing no monomials where k is algebraically closed, the tropical variety $TV(I)$ is a finite union of polyhedras where the dimension of each polyhedra is $dim V(I)$.*

Example 5.2.7. (1) *If $I = \langle f \rangle$ is a principal ideal, then $\{f\}$ is a tropical basis.*

(2) *Let $I = \langle x + z + 1, y^2 + 3 \rangle$ be an ideal in $\mathbb{Q}[x, y, z]$. By Sage, we compute that the tropical basis and Groebner basis for I coincide and they are sam as $\langle x + z + 1, y^2 + 3 \rangle$.*

(3) *Let $I = \langle x + z + 1, x + y + 3z \rangle$ be an ideal in $\mathbb{Q}[x, y, z]$. A Groebner basis for I is $\{x + z + 1, y + 2z - 1\}$ and a tropical basis is $\{x + z + 1, x + y + 3z, x - \frac{1}{2}y + \frac{3}{2}\}$.*

Proposition 5.2.8. *[1] An ideal $I \subset k[x_1, \ldots, x_n]$ is monomial-free if and only if*

$$V(I) \cap (k^*)^n \neq \emptyset$$

Proof. [1] Assume that $V(I) \cap (k^*)^n = \emptyset$, then we have $V(I) \subset \cup_{i=1}^n \{x_i = 0\}$. For the monomial $x_1 ... x_n$, $V(x_1 ... x_n) \supset V(I)$ is satisfied. Then, considering Hilbert's Nullstellensatz theorem there is a number k such that $x_1^k ... x_n^k \in I$, i.e. I has no monomials (The theorem states that if $f(x)$ is a polynomial in $k[x_1, ..., x_n]$ such that $f(x) = 0$ for all x in the algebraic variety $V(I)$ for an ideal I then there exists a positive integer n with $f^n(x)$ is in I).
Assume that $\alpha x^\alpha \in I$ and let z be in V(I). Then $\alpha_1 z_1^{\alpha_1} ... z_n^{\alpha_n} = 0$, which means that there exists some i such that $z_i = 0$. Therefore, $z \notin (k^*)^n$. □

Now let us see the close relation between a tropical variety which are polyhedral objects and the corresponding algebraic variety.

5.3 Groebner Fan

We saw that the Groebner fan of a given ideal in $k[x_1, \ldots, x_n]$ consists of a suitable subdivision of $(\mathbb{R}_{\geq 0})^n$.

i) Consider the ideal given in Example 3.1.11 again. In Example 3.2.2 and Example 5.1.4, we already illustrated the Groebner fan and the tropical variety associated to I. It is clearly seen that the tropical variety is a sub-variety of the Groebner fan when considered geometrically.

ii) Consider the ideal and its Groebner fan given in part i) of Example 3.2.9. Computing its tropical variety via Sage, we obtain the figure below :

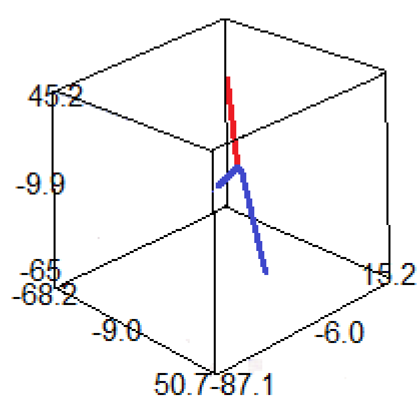


Figure 5.3: The Groebner fan of I

The commands to compute a tropical variety by Sage are:

$R. < x, y >= PolynomialRing(QQ, 2, order = \text{'lex'})$

$S = [xy^2, y^2 - y]$

$I = R.ideal(S)$

$gf = I.groebner\_fan()$

$gf.tropical\_basis()$ (computes the tropical basis of I)

$tv = gf.tropical\_intersection()$ (computes the tropical variety and rename it as ”tv”)

$tv.rays()$ (describes the rays (1-dimensional cones) in the tropical variety)

$tv.cones()$ (describes the cones in the tropical variety in accordance with the dimension)

Remark 5.3.1. *$In_w(f)$ is a monomial if and only if w belongs to the relative interior of a maximal dimensional cone in the corresponding Groebner fan.*

Chapter 6

TROPICAL VARIETIES IN $\mathbb{R}^2$

6.1 Tropical Lines

Consider the line equation $f(x, y) = ax+by+c = 0$ in $\mathbb{R}^2$ where $a, b, c \in \mathbb{R}$. Its tropical roots are the points (x_0, y_0) where f attains its maximum at (x_0, y_0) by at least two of $x+a,\ y+b,\ c$. Hence a tropical line in $\mathbb{R}^2$ is a tropical hypersurface $TV(f)$ consisting of a single vertex and 3 edges, one in vertical direction, one in horizontal direction and one in the direction of the line $x = y$. More precisely, the centre of the line is $(c-a, c-b)$ and three unbounded rays come out from the centre which are in the directions $(-1, 0)$, $(0, -1)$ and $(1, 1)$.

Example 6.1.1. *Let* $f(x, y) = x + 3y + 4$. *Tropicalisation of* f *is* $x \oplus y \oplus 0$. *Then,* $TV(f)$ *can be given as :*

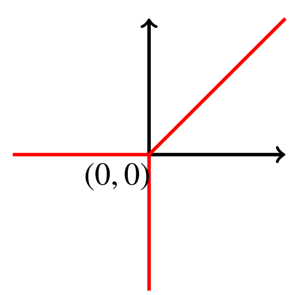


Figure 6.1: The tropical variety of a line

Example 6.1.2. *For* $f(x, y) = 2x + y + 3$, *we obtain the same image as above. In fact, any tropical line has clearly the unique form as in Example 6.1.1.*

Remark 6.1.3. *We may obtain the same tropical variety from different tropical polynomials. For example, a tropical polynomial g which is a monomial multiple of another tropical polynomial define the same tropical variety pushed on some other points of the ambient space.*

Example 6.1.4. *Let* $f(x, y) = x(x + 3y + 4)$*. As the remark above indicates, we obtain the same tropical line given in Example 6.1.1 as TV(f).*

6.2 Tropical Curves

It is natural that, for a polynomial $f(x, y)$ of degree d, ($d \geq 2$) in $k[x, y]$, the tropical variety $TV(f)$ defines a tropical curve in $\mathbb{R}^2$.

Remark 6.2.1. *If the ideal I is generated by a unique monomial then the tropical variety* $TV(f)$ *is empty, so has dimension zero.*

Example 6.2.2. *Let us consider the polynomial* $f(x, y) = x^2y^3 + x + y^4$ *in* $\mathbb{Q}[x, y]$*. Its tropicalization* $TV(f)$ *is:*

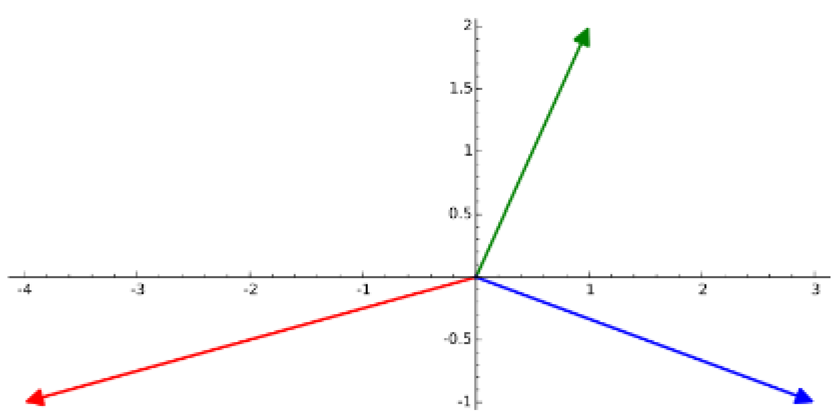


Figure 6.2: The tropical variety of $f(x, y)$

Remark 6.2.3. *The polynomial $g(x, y) = x^3(x^2y^3 + x + y^4)$ in $k[x, y]$ has the same figure above as its tropical variety.*

Let T_d denote the triangle in $\mathbb{R}^2$ with the vertices $(0, 0)$, $(d, 0)$ and $(0, d)$. If there exists some d such that T_d contains the Newton polygon of f and Newton polygon of f is not contained by T_{d_0} for any $d_0 \leq d$ then the tropical curve $R(f)$ is said to be of degree d.

Example 6.2.4. *Consider the tropical polynomial given in Example 4.2.4. The degree of the tropical curve associated to the polynomial is 2 as T_2 contains the Newton polygon of f, which is just the triangle with vertices $(0, 0)$, $(0, 2)$ and $(2, 0)$ and there is no $d < 2$ such that T_d contains the Newton polygon of f.*

Remark 6.2.5. *A tropical curve of degree 1 in $\mathbb{R}^2$ is a tropical line.*

A vertex N in a tropical curve $R(f)$ is called a node if there exists at least three edges in $R(f)$ attached to N. Let $\mathbf{v}_i = (u_1^{(i)}$ and $u_2^{(i)})$, $\mathbf{v}_j = (u_1^{(j)}, u_2^{(j)})$ be two primitive direction vectors attached to N with integer coordinates. The number

$$a_i a_j. \det\begin{pmatrix} u_1^{(i)} & u_2^{(i)} \\ u_1^{(j)} & u_2^{(j)} \end{pmatrix}$$

associated to each node N in a tropical curve is called the multiplicity of N with a_i and a_j are the weights of the edges that v_i and v_j correspond respectively. A tropical curve C is said to be smooth if all its nodes have exactly three adjacent edges and each node has multiplicity 1. For example, a tropical line is a smooth tropical variety.

Tropical curves in $\mathbb{R}^2$ associated with a polynomial of degree d satisfy the following three properties:

(i) All of the edges have a rational slope,

(ii) The balancing condition is satisfied at each vertex,

(iii) The sum of the multiplicities of unbounded edges in the direction of $(1, 1)$ is equal to the degree of the tropical polynomial. This statement is also valid for the unbounded edges in the direction of $(-1, 0)$ and the unbounded edges in the direction of $(0, -1)$.

Theorem 6.2.6. *[9] Tropical polynomial can be reconstructed from its curve up to an additive (tropically multiplicative) constant. Moreover, any weighted graph in the plane having the conditions* $(i) - (iii)$ *above is a tropical curve associated to some polynomial of degree* d.

6.3 Dual Subdivision and Balancing Condition

Let $f(x, y)$ be a tropical polynomial in $\mathbb{T}[x, y]$. Let us plot the points in $Supp(f)$ in $\mathbb{Z}^2$ and denote the Newton polygon of $f(x, y)$ by $\Delta(f)$. Remember that tropical roots of $f(x, y)$ are the points where the maximum is attained by at least two monomials at these points, so every tropical root corresponds to at least two monomials. Put an edge between the points $(i, j) \in Supp(f)$ which belong to monomials that tropical roots of $f(x, y)$ correspond. This gives a subdivision of $\Delta(f)$, called the dual subdivision of f. The new smaller Newton polygons in the subdivision of $\Delta(f)$ are called the cells of $\Delta(f)$. Note that every node N corresponds to a polygon as there is at least three monomials which give the maximum of f at N. We denote this polygon by $\Delta(N)$. The word "duality" comes from the fact that the faces of $\Delta(p)$ corresponds to the edges of the tropical curve $p(x, y)$.

Example 6.3.1. *Let us consider the tropical line given in the Example 4.2.3. The point* $(1, 1)$ *is the vertex of the line* L. *It is the point where the monomials* $1 = 1 \odot x^0 \odot y^0$, $x = x^1 \odot y^0$ *and* $y = x^0 \odot y^1$ *take the same value. Consider the triangle* T_1 *with vertices* $(0, 0), (1, 0)$ *and* $(0, 1)$*):*

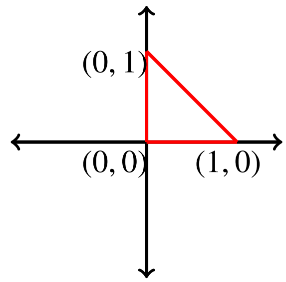


Figure 6.3: The triangle T_1

The monomials with exponents $(0,0)$ *and* $(0,1)$ *give the the horizontal edge of L, the monomials with exponents* $(0,0)$ *and* $(1,0)$ *give the vertical edge of L and monomials with exponents* $(1,0)$ *and* $(0,1)$ *give the edge of L that has slope* 1*. Note that the node N in L is associated with* T_1 *and each edge v of L is associated with an edge* v_i *of* T_1*, whose direction is perpendicular to that of v. We have:*

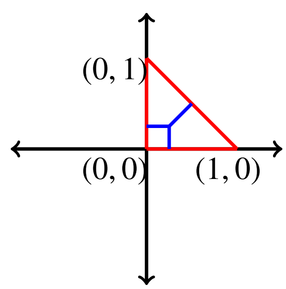


Figure 6.4: The tropical curve L in the subdivision

Example 6.3.2. *The dual subdivision of the tropical polynomial given in Example 4.2.4 is given as:*

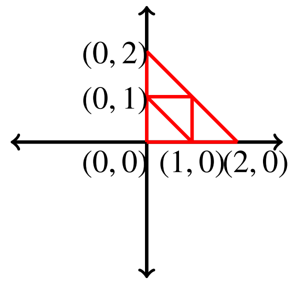


Figure 6.5: The dual subdivision of $p(x, y)$

From this subdivision, we actually can deduce our tropical variety using the same method explained above:

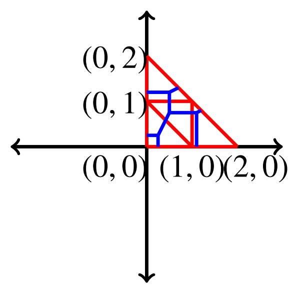


Figure 6.6: The tropical curve L in the subdivision

We may obtain almost all geometric properties of the tropical curve C defined by f from a given subdivision $\Delta(f)$. Note that the number of sub-polygons in a subdivision equals to the number of nodes in the corresponding tropical curve and the weight of an edge of the tropical curve is one minus the number of lattice points of corresponding edge in the subdivision. Conversely, we may obtain the subdivision or Newton polygon from the tropical curve itself.

Remark 6.3.3. *If we want to add the informations on the coefficients in $p(x, y)$ to $\Delta(f)$ either we add the coefficients at each point on the Newton polygon or we can consider the points (i, j, c_{ij}) in $\mathbb{R}^3$ by taking the coefficients as the height of the points $(i, j) \in Supp(p)$.*

Determining factors of a tropical curve is its dual subdivision up to a translation and lengths of its edges.

Proposition 6.3.4. *[5] Let C be tropical curve and $v_1, \ldots, v_k$ denote the edges. There exists a unique integer vector $m_i = (a_i, b_i)$ in the direction of v_i such that $gcd(a_i, b_i) = 1$.*

This comes from the fact that every edge v_i is contained in a line defined by an equation with integer coefficients.

Example 6.3.5. *Consider the Example 6.2.2. By a simple calculation, one can find that for the edge in the first quadrant, the unique vector is $(1, 2)$, for the edge in the third quadrant , the unique vector is $(-1, -2)$ and for the edge in the fourth quadrant, it is $(3, -1)$.*

One of the important geometric information about the tropical curve coming from the dual subdivision is:

Proposition 6.3.6. *[5] At every node N of the tropical curve C, the weighted sum of the primitive integral vectors of the edges around N is zero. This property is known as the balancing condition in C.*

Proof. Let N be a node in C attached to the edges $e_1, \ldots, e_k$ with the weights $w_1, \ldots, w_k$ respectively. We aim to see that

$$\sum_{i=1}^{k} w_i v_i = 0$$

where v_i is the corresponding primitive integral vectors in the direction of e_i. We use that the polygon $\triangle_v$'s being closed.

Renumber the vertices of $\triangle_v$ in clockwise order as $p_1, \ldots, p_k$ in order to make $p_{i+1} - p_i$ orthogonal to v_i for $i = 1, \ldots, k-1$ and $p_1 - p_k$ orthogonal to v_k. Notice that $M(p_{i+1} - p_i) = w_i v_i$ where $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The reason for this is that $w_i v_i$ is a vector orthogonal to $p_{i+1} - p_i$ which is a w_i multiple of the primitive vector v_i. Since $\sum_{i=1}^{k} w_i v_i$ is equal to $\sum_{i=1}^{k-1} M(p_{i+1} - p_i) + M(p_1 - p_k)$ and M is linear, the equality is satisfied. $\square$

For more details about dual subdivision, see [5].

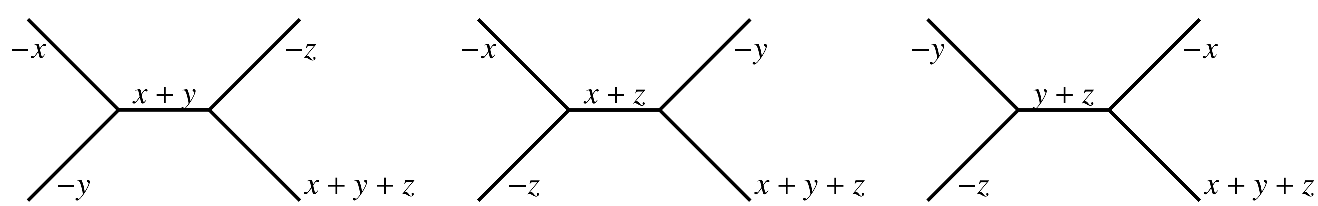
Chapter 7

TROPICAL VARIETIES IN $\mathbb{R}^3$

We will examine the tropical varieties in $\mathbb{R}^3$ in two parts as tropical lines and tropical hypersurfaces in $\mathbb{R}^3$.

7.1 Tropical Lines

A tropical line in three dimension is the intersection of two tropical planes, i.e. tropical roots of a polynomial of the form $ax + by + cz + d$. There are three types of tropical line in three dimension :



In an abstract way:

Definition 7.1.1. *A tropical line in $\mathbb{R}^3$ is an unrooted tree such that:*

(i) it has five edges one of which is a bounded edge and the others are unbounded half rays,

(ii) The directions of the edges are spanned by integer vectors,

(iii) For the primitive integer vectors v_1, v_2, v_3, v_4 in the directions of edges at a node we have $v_1 + v_2 + v_3 + v_4 = 0$.

This definition is generalized to the case of a tropical linear subspace in $\mathbb{R}^n$ (see [10]). Note that it can also be resumed to the lines in $\mathbb{R}^2$.

Proposition 7.1.2. [11] *Let A, B be two points in $\mathbb{R}^3$ and $B - A = (a_1, a_2, a_3)$. There are infinitely many tropical lines which pass through A and B if and only if at least one of a_i's is zero or $a_i = a_j$ for $i \neq j$. Otherwise, there exists a unique tropical line passing through A and B.*

Proof. See [10] for the detailed proof. □

7.2 Tropical Hypersurfaces

Consider a polynomial in $k[x, y, z]$ is of the form

$$f(x, y, z) = \sum_{(a,b,c)\in Supp(f)} \lambda_{abc} x^a y^b z^c$$

where λ_{abc} are real numbers. Its tropicalization $trop(f)$ defines a tropical hypersurface in $\mathbb{R}^3$. In other words, the nonlinear locus of a convex, piecewise linear function with rational slopes $f : \mathbb{R}^3 \to \mathbb{R}$ is called a tropical hypersurface defined by f (see [9]). It is a connected polyhedral complex of dimension 2 contained in a union of finite number of hyperplanes in $\mathbb{R}^n$.

Example 7.2.1. *Consider $f(x, y, z) = x^2 + y^3 + z^5$. We illustrate the geometric representation of its tropical variety as below :*

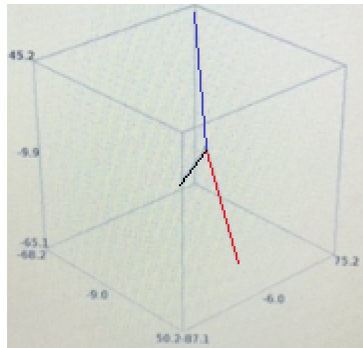


Figure 7.1: The tropical variety of $f(x, y, z)$

$TV(f)$ consists of three 1-dimensional cones where the rays are $g_1 = (-68, 75, 45)$, $g_2 = (18, 12, -65)$ and $g_3 = (50, -87, 20)$.

We already saw that different tropical polynomials may define the same tropical hypersurface. In particular, if $f = g \odot u$ with $u = a \odot x$, then $TV(f) = TV(g)$.

The degree d of f is defined as follows: consider the lattice points contained in the simplex

$$\Gamma_d := conv((0,0,0),(d,0,0),(0,d,0),(0,0,d))$$

The number of lattice points in Γ_d is $\binom{d+3}{3}$. The polynomial is of degree of d if and only if $Supp(f)$ is contained in Γ_d such that there is no $d' < d$ with $\Gamma'_d \subset \Gamma_d$, $d' \in \mathbb{N}$ and $Supp(f) \subset \Gamma_{d'}$ as simplex.

Example 7.2.2. *Let f be the polynomial given in Example 7.2.1. As the corresponding simplex in the definition of degree is Γ_5 in this case, f is of degree 5.*

Definition 7.2.3. *A tropical hypersurface $TV(f)$ in $\mathbb{R}^3$ is said to be smooth if each of the maximal dimensional cones in the dual subdivision has volume $\frac{1}{3!}$.*

Proposition 7.2.4. *[11] Assume that $TV(f)$ is a smooth tropical hypersurface in $\mathbb{R}^3$. Let F_1, F_2, F_3 be among the faces of dimension 2 in $TV(f)$ which are attached to the face of dimension 1, F. Then F_1, F_2, F_3 span different planes in $\mathbb{R}^3$.*

Example 7.2.5. $f(x, y, z) = z^3 + y^3z + x^2y^2$ *We obtain its tropical variety geometrically as below :*

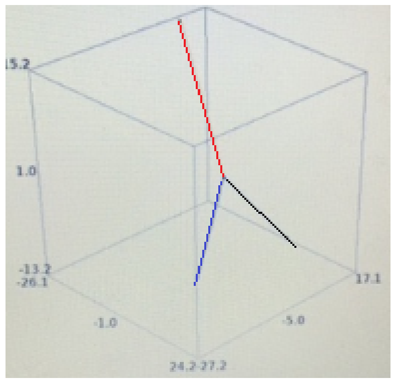


Figure 7.2: The tropical variety of $f(x, y, z)$

Remark 7.2.6. *The hypersurface* $TV(f)$ *is balanced: For any* 1*-dimensional face* F *of* $TV(f)$ *, let* $F_1, \ldots, F_t$ *be the* 2*-dimensional faces attached to* F*. Let* $p : \mathbb{R}^3 \to \mathbb{R}^2$ *be the projection along* F*. Put* $C := p(\cup F_i)$*. It is a one- dimensional polyhedral complex in* $\mathbb{R}^2$ *and balanced by the weights coming form* $TV(f)$*.*

Chapter 8

CONCLUSION

In this thesis, we studied the tropical varieties in $\mathbb{R}^2$ and $\mathbb{R}^3$. We considered only the tropical varieties of codimension 1, called tropical hypersurfaces. In this case, we tried to understand the relation between the tropical variety, Groebner fan and the dual Newton polyhedron corresponding to an algebraic variety defined by f. Using this relation, we aimed to understand tropical varieties of codimension two in $\mathbb{R}^3$- which is one of the actual research problems in tropical geometry. Although those tropical varieties have not been computed by using gfan yet (neither by using Sage), there are few examples of the varieties in the literature. Tropical lines are studied in many works of mathematicians, [10] and [8] are two of the examples from the literature. We left that topic for a future work as time didn't permit to arrive at that point.

BIBLIOGRAPHY

[1] Aroca, F., Gomez-Morales, M., Shabbir, K, *Torical modifcation of Newton non-degenerate ideals*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Mat. V.107, (1), pp 221-239, 2013.

[2] Bogart, T., Jensen, A., Speyer, A., Sturmfels, B., Thomas, R, *Computing Tropical Varieties*, J. Symb. Comput. 42, 2007.

[3] Cox, D.,Little, J.,O'Shea, D, *Ideals, Varieties and Algorithms*, 4th Ed., Undergraduate Texts in Math, Springer, New York, 2010.

[4] Decker, W., Greuel, G.-M., Pfister, G., Schönemann, H., *SINGULAR 3-1-6- A computer algebra system for polynomial computations* http://www.singular.uni-kl.de.

[5] Erwan, B., Shaw, K., *A bit of tropical geometry*, Amer. Math. Monthly 121, no. 7, 563-589, 2014.

[6] Hept, K., *Projections of Tropical Varieties and an Application to Small Tropical Bases*, PhD Dissertation, University Goethe, 2009.

[7] Jensen, A., *Algorithmic Aspects of Groebner Fans and Tropical Varieties*, PhD Dissertation, University of Aarhus 2007.

[8] Mikhalkin, G., *Enumerative tropical geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. 18 (2005), 313–377.

[9] Richter-Gebert J., Sturmfels, B., Theobald T., *First steps in tropical geometry*, Contemp. Math., pp. 289-317, 2005.

[10] Speyer, D., Sturmfels, B., *The tropical Grassmannian*, Adv. Geom. 4, 389-411, 2004.

[11] Vigeland, M.D., *Tropical lines on smooth tropical surfaces*, arXiv:0708.3847

[12] 林如苹, *Largest-coefficient Tropical Polynomials and Their Applications, PhD thesis, National Chengchi University, 2009.*