

BANACH SPACES WITH THE GROTHENDIECK PROPERTY

by

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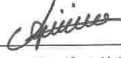
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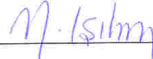
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*To my parents
and
my sister*

ABSTRACT

Let X be an infinite dimensional Banach space and X^* be its dual space. On X^* there are three standard topologies, namely norm topology, weak topology and weak* topology. These topologies are distinct. For certain Banach spaces X , including several familiar ones, weak* convergent sequences in X^* converges weakly. These spaces are said to have the Grothendieck property. A Banach space possessing this property is said to be the Banach space with the Grothendieck property or simply a Grothendieck space. The trivial examples of such spaces are the reflexive Banach spaces. The first nontrivial Grothendieck space ℓ_∞ is presented by A. Grothendieck in 1953 [11, Theorem 2.3]. One can easily find weakly compact operators from a Grothendieck space. In his paper A. Grothendieck proved a criterion for weak compactness in the dual of $C(K)$ the space of continuous complex valued functions on K where K is any compact Hausdorff space. A couple of years later A. Pełczyński gave a measure theoretic proof of Grothendieck's criterion and introduced the notion that is named after him as Pełczyński's property V . We say that a Banach space has the Pełczyński's property V if any unconditionally converging operator is weakly compact. Any dual space having the Pełczyński's property V is a Grothendieck space. In particular, any von-Neumann algebra is a Grothendieck space. The main objective of this Ms. thesis is to learn the basics of the Banach space theory, to study the Banach spaces with the Grothendieck property and make a synthesis of the results obtained so far.

Keywords: Pełczyński's property V , Von-Neumann algebra, Grothendieck property

ÖZET

X sonsuz boyutlu bir Banach uzayı ve X^* da onun dual uzayı olsun. X^* üzerinde norm topoloji, zayıf topoloji ve zayıf-yıldız topoloji olmak üzere üç standart topoloji vardır. Bu topolojilerin hepsi birbirinden farklıdır. Bazı X Banach uzayları için, birkaç bilindik uzay dahil olmak üzere, X^{**} 'daki zayıf-yıldız yakınsak diziler zayıf yakınsar. Bu uzaylar için Grothendieck özelliğine sahiptir denir. Grothendieck özelliğine sahip bir Banach uzayı Grothendieck uzayı olarak adlandırılır. Bu tür uzayların en basit örnekleri refleksif Banach uzaylarıdır. Basit olmayan ilk Grothendieck uzayı örneği olan sınırlı diziler uzayı ℓ_∞ 'yi 1953'te A. Grothendieck[11] vermiştir. Makalesinde A. Grothendieck K tıkız ve Hausdorffken kompleks değerli sürekli fonksiyonların uzayı $C(K)$ uzaylarının duallerinde zayıf tıkızlık için bir kriter ispatlamıştır. Ondan birkaç sene sonra da A. Pełczyński Grothendieck'in kriterini ölçü teorisine dayanan bir yöntemle ispatlamıştır ve kendi adını taşıyan V özelliğini tanımlamıştır. Eğer bir Banach uzayında bütün koşulsuz yakınsayan operatörler zayıf tıkızsa o uzay için Pełczyński'nin V özelliğini taşır diyoruz. Pełczyński'nin V özelliğini taşıyan bir dual uzayı Grothendieck uzayıdır. Her von-Neumann cebiri bir Grothendieck uzayıdır. Bu tezin asıl amacı Banach uzayları teorisini ve Grothendieck özelliğini taşıyan Banach uzaylarını incelemek ve bu konuda günümüze kadar elde edilmiş olan sonuçları sentezlemektir.

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NOMENCLATURE

Blackboard Bold Symbols

$\mathbb{N}$	= the set of all natural numbers.
$\mathbb{R}$	= the set of all real numbers.
$\mathbb{C}$	= the set of all complex numbers.
$\mathbb{F}$	= scalar field ($\mathbb{R}$ or $\mathbb{C}$).

Symbols for Norms

$\ x\ _p$	= $(\sum_{i=1}^{\infty} x_i ^p)^{1/p}$, $1 \leq p < \infty$, where $x = \{x_i\}_{i \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$.
$\ x\ _{\infty}$	= $\sup_i x_i $, where $x = \{x_i\}_{i \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$.
$\ x\ _{c_0}$	= $\max_i x_i $, where $x = \{x_i\}_{i \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$.
$\ f\ _{L_p(\mu)}$	= $(\int_{\Omega} f ^p d\mu)^{1/p}$, where $f \in L_p(\mu)$.
$\ f\ _{L_{\infty}(\mu)}$	= $ess \sup_{x \in \Omega} f(x) $, where $f \in L_{\infty}(\mu)$.
$\ f\ _{C(K)}$	= $\max_{x \in K} f(x) $ if $K \neq \emptyset$, 0 otherwise, where $f \in C(K)$.
$\ f\ _{\infty}$	= $\sup_{x \in K} f(x) $, where $f \in \ell_{\infty}$.
$\ \mu\ $	= $\sup \sum_{i=1}^n \mu(S_i) $, where supremum is taken over all finite sequences $\{S_i\}$ of disjoint sets with $S_i \subset S$.

Symbols for Spaces of Linear Operators

$\mathcal{L}(X, Y)$	= the collection of all bounded linear operators from X into Y .
$\mathcal{L}(X)$	= $\mathcal{L}(X, X)$.
$\mathcal{K}(X, Y)$	= the collection of all compact linear operators from X into Y .
$\mathcal{K}(X)$	= $\mathcal{K}(X, X)$.

Symbols for Special Spaces

$\mathcal{B}(X)$	= the collection of all bounded scalar functions on the set X equipped with norm $\ f\ _\infty$.
$C(K)$	= the space of all continuous real valued functions defined on a compact set K with norm $\ f\ _{C(K)}$.
$C(K, X)$	= the space of all continuous X -valued functions defined on a compact set K with norm $\ f\ _{C(K)}$.
ℓ_p	= the space of all numerical sequences $x = \{x_i\}_{i \in \mathbb{N}}$ satisfying $\sum x_i ^p < \infty$ equipped with the norm $\ \cdot\ _p$, $1 \leq p < \infty$.
ℓ_p^n	= the n -dimensional coordinate space equipped with the norm $\ \cdot\ _p$, $1 \leq p < \infty$.
ℓ_∞	= the space of all bounded numerical sequences equipped with the norm $\ \cdot\ _\infty$.
c_0	= the space of all numerical sequences $x = \{x_i\}_{i \in \mathbb{N}}$ that converge to 0, equipped with the norm $\ \cdot\ _{c_0}$.
$L_p(\mu)$	= $L_p(\Omega, \sigma, \mu)$, $1 \leq p < \infty$, the space of μ measurable scalar functions on (Ω, σ, μ) with the norm $\ f\ _{L_p(\mu)}$.
$L_\infty(\mu)$	= $L_\infty(\Omega, \sigma, \mu)$, the space of bounded measurable functions with the norm $\ f\ _{L_\infty(\mu)}$.
$B(S, \Sigma)$	= uniform limits of all finite linear combinations of characteristic functions of sets in a field Σ of subsets of S with the norm $\ f\ _\infty$.
$ba(S, \Sigma)$	= all bounded finitely additive scalar functions on the field Σ of subsets of S with the total variation norm $\ \mu\ $.
$rca(S)$	= the collection of all regular finite scalar-valued Borel measures on S with the total variation norm $\ \mu\ $.

Symbols Involving Linear Operators

$\mathcal{N}(T)$	$= \{x \in X : Tx = 0\}$, null space of the operator $T : X \rightarrow Y$.
$\mathcal{R}(T)$	$= \{y \in Y : Tx = y \text{ for some } x \in X\}$, range of operator $T : X \rightarrow Y$.
Id_X	$=$ Identity operator on X .
$T _Z$	$=$ the restriction of the operator T on X to the subspace Z of X .
T^*	$=$ the adjoint operator of the operator T .
$T \circ S$	$=$ the composition of operators T and S .

Symbols for Sets Formed from Other Sets

A^+	$=$ nonnegative terms of A , $A \subset \mathbb{R}$.
$\overline{\text{span}}(A)$	$=$ closure of the subspace spanned by A , $A \subset X$.
B_X	$= B_X = \{x \in X : \ x\ \leq 1\}$, the closed unit ball of X .
S_X	$= S_X = \{x \in X : \ x\ = 1\}$, the unit sphere of X .
$\overline{M}$	$=$ norm closure of M , $M \subset X$.
$\overline{M}^w$	$=$ weak closure of M , $M \subset X$.
$\overline{N}^{w*}$	$=$ weak* closure of N , $N \subset X^*$.
X_∞	$=$ one point compactification of a locally compact Hausdorff space X .
$\beta(X)$	$=$ the Stone-Ćech compactification of a completely regular X .

Symbols for C^* algebras

A_{sa}	$=$ set of all self-adjoint elements of the C^* algebra A .
A_1	$=$ the set of all unitary elements of the C^* algebra A .
A_+	$=$ the set of all positive elements of the C^* algebra A .
A'	$=$ the dual space of the C^* algebra A .

Symbols for Convergence and Limits

$x_\alpha \rightarrow x$	= the net $\{x_\alpha\}_{\alpha \in I}$ converges to x in norm.
$x_\alpha \xrightarrow{w} x$	= the net $\{x_\alpha\}_{\alpha \in \mathbb{N}}$ converges to x weakly.
$x_\alpha \xrightarrow{w^*} x$	= the net $\{x_\alpha\}_{\alpha \in \mathbb{N}}$ weak* converges to x .
$\lim_{n \rightarrow \infty} x_n$	= limit of the sequence $\{x_n\}_{n \in \mathbb{N}}$.
$\text{weak} \lim_{n \rightarrow \infty} x_n$	= weak limit of the sequence $\{x_n\}_{n \in \mathbb{N}}$.
$\text{weak}^* \lim_{n \rightarrow \infty} x_n$	= weak* limit of the sequence $\{x_n\}_{n \in \mathbb{N}}$.

Miscellaneous Symbols

X^*	= the dual space of X .
X^{**}	= the second dual space of X .
$\langle x, x^* \rangle$	= the value of the functional $x^* \in X^*$ on $x \in X$.
$\mathcal{P}_\infty(A)$	= the collection of all infinite subsets of A .
$X \setminus M$	= quotient space of X where M is a subspace of X .
e_n	= the unit coordinate vectors of the standard basis.
$\text{Re}(x)$	= real part of the complex number x .
$\text{Im}(x)$	= imaginary part of the complex number x .
$\dim(X)$	= dimension of the vector space X .
$\text{Diam} D$	= diameter of disk D .
$\text{dist}(x, y)$	= distance between x and y .
$\text{dist}(x, A)$	= distance from x to A .
$\inf$	= infimum
$\sup$	= supremum
χ_A	= the characteristic function of A .
δ_{nm}	= Kronecker delta, gives 1 if $n = m$, 0 otherwise.

$\text{sign}(x)$ = the sign of a real number, 1 if x is positive,
-1 if x is negative, 0 otherwise.

wuC = weakly unconditionally Cauchy.

INTRODUCTION

This thesis is divided into three chapters. In the first chapter, we investigate the preliminary results from Banach space theory which will be used throughout the thesis. We start by providing some notations, terminology and principal theorems of linear analysis and Banach space theory. We then introduce two topologies other than the norm topology, namely the weak and weak* topologies, as we usually work on them. We continue by giving the proofs of Alaoglu's theorem, metrizability criterion of the unit ball of the dual space under weak* topology, Eberlein-Šmulian theorem and Rosenthal's ℓ_1 theorem in the given order which we make use of frequently.

Major results on compact and weakly compact operators are given as Grothendieck spaces are the spaces from which weakly compact operators can be easily found. Among them we mostly use Schauder's and Gantmacher's theorem concerning the compactness and weak compactness of an operator and its adjoint.

Then we study the Stone-Čech compactification of a completely regular space which has the extension property, that is, to which one can uniquely extend any continuous function from the completely regular space to a compact Hausdorff space. In particular, we are interested in Stone-Čech compactification of infinite discrete spaces which turns out to be an extremally disconnected space.

In order to understand what a von-Neumann algebra is, we give some basic definitions on C^* algebras and provide some examples of it. Then we introduce von-Neumann algebras.

In the second chapter, we start with basic concepts on series in Banach spaces and investigate the definitions and characterizations of unconditionally convergent and weakly unconditionally Cauchy series in Banach spaces. In this part, some theorems have greater significance in proving the theorems about property V and

Grothendieck's property, namely Bessaga-Pełczyński theorem, Orlicz-Pettis theorem and Schur's lemma.

After that we give the definition for Pełczyński's property V and we characterize the property V by the coincidence of unconditionally converging and weakly compact operators. An alternative definition is related to the existence of a copy of c_0 in the Banach space. We demonstrate the result of Pełczyński in [18] that the space of all continuous linear functionals on a compact Hausdorff space has the property V . In the sequel we also define the properties u and V^* of Pełczyński.

In the last section of this chapter we recall H.Pfzner's result [21] on C^* algebras (in particular von-Neumann algebras) which says that any C^* algebra has the Pełczyński's property V .

In the last chapter we deal with the Grothendieck property which is the main objective of this work. A Banach space is said to have the Grothendieck property if weak and weak* convergence for sequences coincide on the dual space. In this part we first study various characterizations of the Banach spaces having the Grothendieck property and then in the subsequent section by making use of the Dieudonne-Grothendieck theorem, Phillips' lemma, Stone Weierstrass theorem and the result that the space of all continuous linear functionals on a compact Hausdorff space has the property V , we give an example for a Banach space, namely ℓ_∞ , which is a dual space with the Pełczyński's property V and also is a Grothendieck space.

After that in the light of Phillips' lemma, we establish a relation between the Pełczyński's property V and the Grothendieck property which says that any dual space with the property V is a Grothendieck space. Depending on this result we conclude that any Von-Neumann algebra has the Grothendieck property. We finish the chapter by stating the open questions on Grothendieck spaces.

Chapter 1

PRELIMINARY RESULTS**1.1**

In this section we have collected some basic results about Banach spaces. We have also introduced the notation and terminology that we shall use throughout the subsequent sections.

Some properties of normed and Banach spaces:

Let X and Y be normed spaces. $\mathcal{L}(X, Y)$ denotes the collection of all bounded linear mappings of X into Y . The norm on $\mathcal{L}(X, Y)$ defined by

$$\|\Lambda\| = \sup_{\|x\| \leq 1} \|\Lambda x\|, \quad \Lambda \in \mathcal{L}(X, Y)$$

turns $\mathcal{L}(X, Y)$ into a normed space.

Note : Equivalent definitions for $\|\Lambda\|$ are:

$$\|\Lambda\| = \sup_{x \in X} \left\{ \frac{\|Tx\|}{\|x\|} : x \neq 0 \right\} = \inf_{x \in X} \{M : \|Tx\| \leq M\|x\| \}$$

If Y is a Banach space then $\mathcal{L}(X, Y)$ becomes a Banach space, too. In particular, when $Y = \mathbb{F}$, $\mathcal{L}(X, Y) = X^*$ where X^* denotes the space of all bounded linear functionals on X which we call as the dual space of X . For $x^* \in X^*$ and $x \in X$ we shall write $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$ instead of $x^*(x)$ as this notation also represents the action of X on X^* . Thus, the norm on X^* can be written by

$$\|x^*\| = \sup_{x \in B_X} |\langle x, x^* \rangle|, \quad x^* \in X^*.$$

Therefore, $\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{F}$ is a bilinear mapping which is continuous since

$$|\langle x, x^* \rangle| = |x^*(x)| \leq \|x^*\| \|x\|.$$

Dual spaces of some frequently used spaces: $(c_0)^* = \ell_1$, $(\ell_p)^* = \ell_q$ where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $(\ell_1)^* = \ell_\infty$, $(\ell_\infty)^* = ba(\mathbb{N})$, the space of all bounded finitely additive scalar functions on $2^{\mathbb{N}}$ equipped with the total variation norm

$$\|\mu\| = \sup_n \sum_{k=1}^n |\mu(S_k)|$$

where supremum is taken over all finite sequences $\{S_1, S_2, \dots, S_n\}$ of pairwise disjoint sets with $S_k \subset \mathbb{N}$.

For a compact Hausdorff space K , let $rca(K)$ be the space of all the regular Borel measures with the total variation norm where supremum is taken over all finite collections $\{S_1, S_2, \dots, S_n\}$ of pairwise disjoint sets with $S_k \subset K$.

Theorem 1.1.1. (*Riesz Representation Theorem*)

If K is a compact Hausdorff space there is an isometric isomorphism between $C(K)^*$ and $rca(K)$ such that the corresponding elements x^* and μ satisfy the following:

$$\langle f, x^* \rangle = \int_K f(s) d\mu, \quad f \in C(K)$$

□

By this theorem we identify the spaces $C(K)^*$ and $rca(K)$.

The following named theorems, the proofs of which can be found in [9, Chapter II] are known to be the principal theorems of linear analysis and Banach space theory.

Theorem 1.1.2. (*Banach-Steinhaus Uniform Boundedness Principle*)

Let X be a Banach space and Y be a normed space. If $\mathcal{A}$ is a collection of bounded linear mappings of X into Y satisfying

$$\sup_{A \in \mathcal{A}} \|Ax\| < \infty, \quad x \in X,$$

then

$$\sup_{A \in \mathcal{A}} \|A\| < \infty.$$

□

Theorem 1.1.3. *(The Open Mapping Theorem)*

Any bounded linear mapping from a Banach space onto a Banach space is open. $\square$

The following theorem is a consequence of the Open Mapping Theorem.

Theorem 1.1.4. *(The Closed Graph Theorem)*

Let T be a linear operator from a Banach space X into a Banach space Y and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X . If $\{x_n\}_{n \in \mathbb{N}}$ converges to an x in X and $\{T(x_n)\}_{n \in \mathbb{N}}$ converges to a y in Y implies $y = T(x)$ then T is bounded. $\square$

Theorem 1.1.5. *(Hahn-Banach Extension Theorem for normed spaces)*

Let Y be a subspace of a normed space X and $y^* \in Y^*$. Then there exists $x^* \in X^*$ such that

$$x^* = y^* \text{ on } Y, \text{ and } \|x^*\| = \|y^*\|$$

i.e., we can extend any bounded linear functional y^* on Y to a bounded linear functional on X preserving the norm. $\square$

Hahn-Banach theorem answers the question about the existence of continuous linear functionals and gives us the following corollaries which shall be used in most of the proofs.

Corollary 1.1.6. *Let Y be a closed subspace of a normed space X and $x_0 \in X \setminus Y$. Then there exists a bounded linear functional x^* on X such that*

$$\|x^*\| = 1, \langle x, x^* \rangle = 0 \text{ for all } x \in Y, \text{ and } \langle x_0, x^* \rangle = \text{dist}(x_0, Y) = \inf_{y \in Y} \|x_0 - y\|.$$

$\square$

Corollary 1.1.7. *If x is a nonzero element in a normed space X , then there exists a bounded linear functional x^* on X satisfying*

$$\|x^*\| = 1 \text{ and } \langle x, x^* \rangle = \|x\|.$$

$\square$

Corollary 1.1.8. *Let X be a normed space and $x, y \in X$ such that $x \neq y$. Then there exists $x^* \in X^*$ such that $\langle x, x^* \rangle \neq \langle y, x^* \rangle$.* $\square$

As a consequence of the above corollary the dual space X^* of a normed space X is a separating family of functionals. In addition to that if $\langle x, x^* \rangle = 0$ for all $x^* \in X^*$ then $x = 0$.

Definition 1.1.9. *A continuous operator $T : X \rightarrow Y$ satisfying $\|T(x)\| \geq C\|x\|$ for some scalar C and all $x \in X$ is called an isomorphic embedding.* $\square$

X^* is itself a Banach space hence has a normed dual of its own denoted by X^{**} which is actually the space $(X^*)^*$ with the norm

$$\|x^{**}\| = \sup_{x^* \in B_{X^*}} |\langle x^*, x^{**} \rangle|, \quad x^{**} \in X^{**}.$$

We define the canonical embedding $\pi : X \rightarrow X^{**}$ by

$$\langle x, x^* \rangle = \langle x^*, \pi(x) \rangle, \quad x^* \in X^*$$

so that each $x \in X$ defines a unique $\pi(x) \in X^{**}$.

Note that π is a linear isometry. Indeed, for any $x^* \in X^*$, $\alpha, \beta \in \mathbb{F}$,

$$\begin{aligned} \langle x^*, \pi(\alpha x + \beta y) \rangle &= \langle \alpha x + \beta y, x^* \rangle = \alpha \langle x, x^* \rangle + \beta \langle y, x^* \rangle \\ &= \alpha \langle x^*, \pi(x) \rangle + \beta \langle x^*, \pi(y) \rangle = \langle x^*, \alpha \pi(x) + \beta \pi(y) \rangle \end{aligned}$$

Also,

$$\|\pi(x)\| = \sup_{x^* \in B_{X^*}} |\langle x^*, \pi(x) \rangle| = \sup_{x^* \in B_{X^*}} |\langle x, x^* \rangle| \leq \sup_{x^* \in B_{X^*}} (\|x^*\| \cdot \|x\|) \leq \|x\|$$

On the other hand, by 1.1.7 considering $x_0^* \in S_{X^*}$ such that $\langle x, x_0^* \rangle = \|x\|$, we get

$$\|\pi(x)\| = \sup_{x^* \in B_{X^*}} |\langle x, x^* \rangle| \geq \langle x, x_0^* \rangle = \|x\|$$

implying that π is an isomorphic embedding and $\|\pi(x)\| = \|x\|$.

Using this identification, instead of $\pi(x) \in X^{**}$ we write $x \in X^{**}$ i.e., we identify X with $\pi(X) \subset X^{**}$. If X is a Banach space then $\pi(X)$ is closed in X^{**} and thus π is an isometric isomorphism onto a closed subspace of X^{**} .

Definition 1.1.10. A Banach space X is said to be reflexive if $\pi(X) = X^{**}$. $\square$

Note: $L_p(\mu)$, ℓ_p for $1 < p < \infty$ and finite dimensional normed spaces are reflexive whereas c_0 , ℓ_1 , ℓ_∞ , $C(K)$ for compact Hausdorff space K and infinite dimensional $L_1(\mu)$ and $L_\infty(\mu)$ are nonreflexive Banach spaces.

Theorem 1.1.11. Every closed subspace of a reflexive normed space is reflexive. $\square$

On a Banach space, other than the norm topology, we have two weaker topologies related to the linear structure of the space, namely the weak topology and the weak* topology which enables us to use the properties of compactness.

1.1.1 Weak and Weak* Topologies

In this part we will present the basic definitions and properties concerning weak and weak* topologies.

The Weak Topology

Let X be a normed space. Let B be the collection of all finite intersections of the sets of the form

$$V(x^*, \varepsilon) = \{x \in X : |\langle x, x^* \rangle| < \varepsilon\}$$

where $x^* \in X^*$ and $\varepsilon > 0$.

This collection B is a basic neighborhood system of the origin for some locally convex vector space topology. This topology is called the X^* -topology of X or the *weak topology* of X . The weak topology of a normed space X is the smallest topology for the space such that every $x^* \in X^*$ is continuous with respect to that topology. Hence the weak topology on X is weaker than the norm topology on X .

Definition 1.1.12. Let I be any directed set and $\{x_\alpha\}_{\alpha \in I}$ be a net in X . We say that x_α converges to x weakly and denote by $x_\alpha \xrightarrow{w} x$ if and only if $\langle x_\alpha, x^* \rangle \rightarrow \langle x, x^* \rangle$ for every $x^* \in X^*$. The point x is called a weak limit of the net $\{x_\alpha\}_{\alpha \in I}$.

A net $\{x_\alpha\}_{\alpha \in I}$ in X is said to be weakly null if $x_\alpha \xrightarrow{w} 0$. $\square$

Corollary 1.1.13. *If X is a Banach space and $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in X converging weakly to an element x of X , then some sequence of convex combinations of the elements of $\{x_n\}_{n \in \mathbb{N}}$ converges to x in the metric topology.* $\square$

Observe that every norm convergent net converges weakly. Indeed, if $x_\alpha \rightarrow x$ in norm then $x_\alpha - x \rightarrow 0$ and so as x^* is continuous $\langle x_\alpha - x, x^* \rangle \rightarrow 0$ and then linearity of x^* implies $\langle x_\alpha, x^* \rangle \rightarrow \langle x, x^* \rangle$ for every $x^* \in X^*$. Since a convex set is the intersection of the hyperplanes containing it we have the following important result of Mazur.

Theorem 1.1.14. *The weak closure of a convex subset of a normed space equals to its norm closure. A convex subset of a normed space is weakly closed if and only if it is norm closed.* $\square$

The usual definitions and results for boundedness, compactness and completeness for the weak topology are as follows:

Definition 1.1.15. *A subset A of X is said to be weakly bounded if $x^*(A)$ is bounded for each $x^* \in X^*$.* $\square$

Theorem 1.1.16. *A subset of a normed space is bounded if and only if it is weakly bounded.* $\square$

Corollary 1.1.17. *Any weakly convergent sequence of a normed space is bounded.* $\square$

Definition 1.1.18. *Let X be a topological space. A subset A of X is weakly sequentially compact if every sequence $\{x_n\}_{n \in \mathbb{N}}$ in A contains a subsequence which converges weakly to a point in A .* $\square$

Proposition 1.1.19. *Any weakly compact set in a Banach space is norm closed and norm bounded.* $\square$

Definition 1.1.20. *A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is called a weakly Cauchy sequence if $\{\langle x_n, x^* \rangle\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{F}$ for all $x^* \in X^*$.* $\square$

Proposition 1.1.21. *In a normed space weakly Cauchy sequences are bounded.* $\square$

Definition 1.1.22. *A normed space X is said to be weakly sequentially complete if every weakly Cauchy sequence in X converges weakly in X .* $\square$

Example 1.1.23. The space $L_p(\mu)$ and so ℓ_p , $1 \leq p < \infty$, is weakly sequentially complete whereas c_0 , $L_\infty(\mu)$, ℓ_∞ and $C(K)$ for infinite K are not weakly sequentially complete.

Proposition 1.1.24. *Any closed subspace of a weakly sequentially complete space is weakly sequentially complete.* $\square$

Theorem 1.1.25. *Let X and Y be two normed spaces. A linear operator $T : X \rightarrow Y$ is norm continuous if and only if it is weakly continuous.* $\square$

Theorem 1.1.26. *Let X be a Banach space. Then B_X is weakly compact if and only if X is reflexive.* $\square$

As a consequence of theorem 1.1.26,

Theorem 1.1.27. *Let X be a Banach space. Then X is reflexive if and only if any bounded weakly closed subset of X is weakly compact.* $\square$

The Weak-Topology*

On X^* other than the norm topology, we have two new topologies. First of all it may be equipped with the weak topology which is the weakest one such that every $x^{**} \in X^{**}$ is continuous. Observe that, by the canonical mapping, for every $x \in X$, we obtain a bounded linear functional $\pi(x)$ on X^* which maps x^* to $\langle x, x^* \rangle$. Hence by taking the collection of all finite intersections of V of the form

$$V(x, \varepsilon) = \{x^* \in X^* : |\langle x^*, \pi(x) \rangle| < \varepsilon\} = \{x^* \in X^* : |\langle x, x^* \rangle| < \varepsilon\},$$

where $\varepsilon > 0$ and $x \in X$, as a local base we obtain a locally convex topology on X^* which we call as the X topology or the weak* topology of X^* .

The weak* topology of a normed space X is the smallest topology for X^* which makes

the linear functional $x^* \rightarrow \langle x, x^* \rangle$ continuous for each $x \in X$. So it is weaker than the weak topology hence than the norm topology on X^* . Observe that $\{\pi(x): x \in X\}$ is a family of linear functionals which separates points on X^* . Indeed, if $x^*, y^* \in X^*$ where $x^* \neq y^*$, then there exists $x \in X$ such that $\langle x, x^* \rangle \neq \langle x, y^* \rangle$ and so $\langle x^*, \pi(x) \rangle \neq \langle y^*, \pi(x) \rangle$.

Definition 1.1.28. A net x_α^* in X^* is said to weak* converge to x^* in X^* if and only if $\langle x, x_\alpha^* \rangle \rightarrow \langle x, x^* \rangle$ for every $x \in X$ and is denoted by $x_\alpha^* \xrightarrow{w^*} x^*$. $\square$

Clearly, any weakly convergent sequence in X^* is weak* convergent.

Definition 1.1.29. A sequence $\{x_n^*\}_{n \in \mathbb{N}}$ in X^* is called a weak* Cauchy sequence if $\{\langle x, x_n^* \rangle\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{F}$ for all $x \in X$. $\square$

Proposition 1.1.30. Let X be a Banach space. Then weak* Cauchy sequences in X^* are bounded. $\square$

Theorem 1.1.31. (Krein-Smulian Theorem) A convex subset D of X^* is weak* closed if and only if $D \cap tB_{X^*}$ is weak* closed for any $t > 0$. $\square$

The following theorem is due to Amir and Lindenstrauss[2]:

Theorem 1.1.32. If X is a weakly compactly generated Banach space i.e., if there is a relatively weakly compact set K in X such that $\overline{\text{span}}(K) = X$, then bounded sequences in X^* have weak* convergent subsequences. $\square$

Note that weak topology exists in every normed space but the weak* topology exists only in dual spaces. If X is reflexive then the two topologies on X^* coincide.

For further results see [9] and [24].

1.1.2 Alaoglu Theorem

In this section we will give the proof of one of the important results for normed spaces which claims that the closed unit ball of the dual space of any normed space is weak* compact. The following is a classical result of the topology:

Theorem 1.1.33. (*Tychonoff Theorem*)

An arbitrary product of compact spaces is compact in the product topology. $\square$

Theorem 1.1.34. Let $f : X \rightarrow Y$ be a one-to-one and continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism. $\square$

As an application of the Tychonoff Theorem we have the following result which we mentioned above.

Theorem 1.1.35. (*Alaoglu Theorem*)

If X is a normed space, then the closed unit ball B_{X^*} is weak* compact.

Proof. We will prove that B_{X^*} is compact in its weak* topology by identifying B_{X^*} in its weak* topology with a closed subset of a product of compact sets with the product topology. So we will be able to conclude by the Tychonoff theorem that the product is compact and hence B_{X^*} as a closed subset of a compact space will be compact. For each $x \in B_X$, let

$$D_x = \{\alpha \in \mathbb{F} : |\alpha| \leq 1\}$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and define

$$D = \prod_{x \in B_X} D_x$$

As each D_x is compact, Tychonoff's theorem guarantees the compactness of D .

Define $\tau : B_{X^*} \rightarrow D$ by

$$\tau(x^*)(x) = \langle x, x^* \rangle$$

so that the x^{th} -coordinate of $\tau(x^*)$ is $\langle x, x^* \rangle$.

Claim: τ is a homeomorphism from B_{X^*} onto $\tau(B_{X^*})$ and $\tau(B_{X^*})$ is closed in D .

To show that τ is a homeomorphism, by 1.1.34, enough to show that τ is a one-to-one continuous function.

So let $x^*, y^* \in B_{X^*}$ and suppose that $x^* \neq y^*$. Then there exists $x \in X$ such that $\langle x, x^* \rangle \neq \langle x, y^* \rangle$. Hence $\tau(x^*)(x) \neq \tau(y^*)(x)$ for that $x \in X$ implying $\tau(x^*) \neq \tau(y^*)$. Hence τ injective.

Now let $\{x_i^*\}_{i \in I}$ be a net in B_{X^*} such that $x_i^* \xrightarrow{w^*} x^*$.

Then for all $x \in B_X$, $\tau(x_i^*) = \langle x, x_i^* \rangle \rightarrow \langle x, x^* \rangle = \tau(x^*)$ meaning that τ is continuous. Therefore τ is a homeomorphism.

To show that $\tau(B_{X^*})$ is closed in D take any g in $\overline{\tau(B_{X^*})}$ in D so $g = (g_x)_{x \in B_X}$. Then there exists a net $\{x_i^*\}_{i \in I}$ in B_{X^*} such that $\tau(x_i^*) \rightarrow g$ in D . So $\langle x, x_i^* \rangle \rightarrow g_x$ for each $x \in B_X$ and g is linear functional on B_X . Indeed, if $x, y \in B_X$ and $\alpha, \beta \in \mathbb{F}$ such that $\alpha x + \beta y \in B_X$ then

$$g_{\alpha x + \beta y} = \lim_i \langle \alpha x + \beta y, x_i^* \rangle = \alpha \lim_i \langle x, x_i^* \rangle + \beta \lim_i \langle y, x_i^* \rangle = \alpha g_x + \beta g_y$$

The only thing left is to show that $g \in \tau(B_{X^*})$ i.e., there exists $h \in B_{X^*}$ such that $g = \tau(h)$ i.e., $\langle x, h \rangle = g_x$ for all $x \in B_X$.

Now if $x \in X$, choose $\alpha > 0$ such that $\|\alpha x\| \leq 1$ and define $\langle x, h \rangle = \alpha^{-1} g_{\alpha x}$.

If $\beta > 0$ is another scalar different than α satisfying $\|\beta x\| \leq 1$ then

$$\langle x, h \rangle = \alpha^{-1} g_{\alpha x} = \alpha^{-1} \lim_i \langle \alpha x, x_i^* \rangle = \lim_i \langle x, x_i^* \rangle = \beta^{-1} \lim_i \langle \beta x, x_i^* \rangle = \beta^{-1} g_{\beta x}$$

for any $x \in X$ guaranteeing the well-definition of h .

Observe that $|\langle x, h \rangle| \leq 1$ for all $x \in B_X$ because $|\langle x, h \rangle| = |g_x|$ (by taking $\alpha = 1$) and $g_x \in D_x$ for each $x \in B_X$ satisfies $|\langle x, h \rangle| = |g_x| \leq 1$. Hence $h \in B_{X^*}$ and $\tau(h)(x) = \langle x, h \rangle = g_x$ for all $x \in B_X$. Thus $g = \tau(h) \in \tau(B_{X^*})$ and $\tau(B_{X^*})$ is closed in D which in turn implies that B_{X^*} is compact in its weak* topology. $\square$

As a result of Alaoglu Theorem, weak* closed and bounded subsets of X^* are weak* compact from which the next corollary follows:

Corollary 1.1.36. *Let X be a Banach space. A subset of X^* is compact in the weak* topology iff it is closed in the weak* topology and bounded in the metric topology.* $\square$

Theorem 1.1.37. (Goldstine's Theorem)

For any normed linear space X , B_X is weak dense in $B_{X^{**}}$, and so X is weak* dense in X^{**} .* $\square$

As a consequence of Alaoglu's theorem 1.1.35 and Goldstine's theorem 1.1.37, we have the following:

Corollary 1.1.38. Let X be a normed space and π be the natural surjection from X into X^{**} . Then $\pi(X)$ is weak* dense in X^{**} .

1.1.3 Metrizability of the Unit Ball of X^* under Weak* Topology

The weak* topology on an infinite dimensional Banach space is never metrizable.

However the closed unit ball of X^* under the weak* topology is metrizable if X is separable by the following theorem.

Theorem 1.1.39. A Banach space X is separable if and only if B_{X^*} is weak*-metrizable.

Proof. Suppose first that X is separable. So it has a countable dense subset. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in S_X which is dense in S_X . Define

$$d(x^*, y^*) = \sum_{n=1}^{\infty} 2^{-n} | \langle x_n, x^* - y^* \rangle |$$

Obviously d defines a metric on X^* .

Claim: d generates the weak* topology on B_{X^*} .

To prove this we need to show that the identity mapping $I : (B_{X^*}, w^*) \rightarrow (B_{X^*}, d)$ is a homeomorphism.

By Alaoglu's Theorem 1.1.35, B_{X^*} is weak* compact. So by 1.1.34, it will suffice to prove that I is continuous. Take an open ε neighborhood

$$V = \{y^* \in B_{X^*} : d(x^*, y^*) < \varepsilon\}$$

of some element $x^* \in B_{X^*}$. Let i_0 be a natural number satisfying

$$2^{-i_0} = \sum_{n=1}^{\infty} 2^{-i_0-n} < \frac{\varepsilon}{4}$$

Also observe that as $x_n \in S_X$ and $x^*, y^* \in B_{X^*}$,

$$| \langle x_n, x^* - y^* \rangle | \leq \|x_n\| \|x^* - y^*\| \leq \|x^*\| + \|y^*\| \leq 2.$$

Thus for every $y^* \in B_{X^*}$ satisfying $| \langle x_n, x^* - y^* \rangle | < \frac{\varepsilon}{2i_0}$, $n = 1, 2, \dots, i_0$, we have

$$\sum_{n=1}^{\infty} 2^{-n} | \langle x_n, x^* - y^* \rangle | = \sum_{n=1}^{i_0} 2^{-n} | \langle x_n, x^* - y^* \rangle | + \sum_{n=i_0+1}^{\infty} 2^{-n} | \langle x_n, x^* - y^* \rangle |$$

$$< \sum_{n=1}^{i_0} 2^{-n} \frac{\varepsilon}{2^{i_0}} + 2 \sum_{n=i_0+1}^{\infty} 2^{-n} = \frac{\varepsilon}{2^{i_0}} \left[1 - \left(\frac{1}{2} \right)^{i_0} \right] + 2 \sum_{n=1}^{\infty} 2^{-i_0-n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which means $d(x^*, y^*) < \varepsilon$. Therefore V contains a weak* neighborhood of x^* which implies that I is continuous. Hence I is a homeomorphism implying the weak* metrizable-ability of B_{X^*} . Note also that B_{X^*} is compact with the above metric as by Alaoglu's theorem B_{X^*} is weak* compact and the identity map is a homeomorphism.

For the converse assume that B_{X^*} is weak* metrizable. Then it has a countable local base at 0 of B_{X^*} , say $\{V_n^*\}_{n \in \mathbb{N}}$, such that

$$\bigcap_{n=1}^{\infty} V_n^* = \{0\}$$

So every open neighborhood of 0 in the weak topology contains a V_n^* . We may assume that

$$V_n^* = \{x^* \in B_{X^*} : |< x, x^* >| < \varepsilon_n, \quad x \in A_n\}$$

where A_n is a finite subset of X and $\varepsilon_n > 0$. Let

$$A = \bigcup_{n=1}^{\infty} A_n$$

Observe that

$$\{x^* \in B_{X^*} : < x, x^* > = 0, \quad x \in A\} \subset \bigcap_{n=1}^{\infty} V_n^* = \{0\}.$$

Therefore, $x^* = 0$.

Now let $Y = \overline{\text{span}(A)}$. We claim that $Y = X$.

Assume for a contradiction that $Y \neq X$. Then by 1.1.6, there exists a nonzero $x^* \in B_{X^*}$ such that $< x, x^* > = 0$ for all $x \in Y$. So $x^* \in V_n^*$ for each n . Hence $x^* = 0$ which is a contradiction as we assumed x^* to be nonzero. Therefore $Y = X$ and the set of all finite linear combinations of elements of A with rational coefficients is a countable dense subset of X and so X is separable. $\square$

1.1.4 Eberlein-Šmulian Theorem

We have seen that as a result of Alaoglu's Theorem, for any normed space X , weak* closed and bounded sets in X^* are weak* compact. We also know that weakly compact

sets of Banach spaces are norm closed and norm bounded by 1.1.19. The question here is what does a subset K of a Banach Space X need to be weakly compact? The answer lies in the theorem of Eberlein-Šmulian. But before that let us introduce some useful properties.

Definition 1.1.40. *A linear subspace Γ of the space of all linear functionals on X is called total if $x=0$ whenever $f(x)=0$ for all $f \in \Gamma$.* $\square$

Lemma 1.1.41. *If X is a linear topological space and B is a countable subset of X , then $\overline{\text{span}}(B)$ is separable.* $\square$

Theorem 1.1.42. *A compact metric space is complete and separable.* $\square$

Corollary 1.1.43. *Let f be a linear functional on the linear space X and let Γ be a total subspace of the space of all linear functionals on X . Then the following are equivalent:*

- (1) $f \in \Gamma$.
- (2) f is Γ -continuous.
- (3) $\mathcal{N}(f) = \{x \in X : f(x) = 0\}$ is Γ -closed.

$\square$

Now, we are ready to prove the Eberlein-Šmulian theorem.

Theorem 1.1.44. *(Eberlein-Šmulian)*

Let A be a subset of a Banach space X . Then the following are equivalent:

- (1) *A is relatively weakly sequentially compact.*
- (2) *Every countably infinite subset of A has a weak limit point in X .*
- (3) *A is relatively weakly compact (or the closure of A in the X^* topology is weakly compact.)*

Proof. Observe that A is bounded in the metric topology in all (1), (2), (3) by 1.1.2 as $\{ \langle x, x^* \rangle : x \in A \}$ is a bounded set of scalars for all $x^* \in X^*$. We will first prove (3) $\implies$ (2) then (2) $\implies$ (1) and lastly (1) $\implies$ (3).

(3) $\implies$ (2): Let $\{x_n\}_{n \in \mathbb{N}}$ be a countably infinite subset of $A \subseteq \overline{A}^w$. As $\overline{A}^w$ is weakly compact, then $\{x_n\}_{n \in \mathbb{N}}$ has a weak limit point in X .

(2) $\implies$ (1): Take an arbitrary sequence $\{x_n\}_{n \in \mathbb{N}}$ in A . Let $X_0 = \overline{\text{span}}(\{x_n\})$. As a closed subset of X , X_0 is a separable Banach space by 1.1.41. Also theorem 1.1.39 and Alaoglu Theorem 1.1.35 imply the weak* metrizability and weak* compactness of $B_{X_0^*}$. Therefore, by 1.1.42, $B_{X_0^*}$ is separable. Now as

$$X_0^* = \bigcup_{n=1}^{\infty} nB_{X_0^*},$$

X_0^* is also separable in the weak* topology of X_0 .

Let H_0 be a countable dense set in X_0^* . Let $\langle x, f \rangle = 0$ for all $f \in H_0$. For any $f \in X_0^*$ there exists $\{f_n\}_{n \in \mathbb{N}} \in H_0$ such that $f_n \rightarrow f$. Then $\langle x, f_n \rangle \rightarrow \langle x, f \rangle$. Hence $\langle x, f \rangle = 0$ for any $f \in X_0^*$ which implies $x = 0$ by 1.1.8. So H_0 is total on X_0 . Now Hahn-Banach theorem 1.1.5 implies that we can extend every element of H_0 to a bounded linear functional on all of X . Now take only one extension for each element of H_0 to form a countable subset of X^* , say H . Since A is bounded by the observation in the beginning of the proof, after applying a diagonal process we can find a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} \langle x_{n_k}, x^* \rangle_{k \in \mathbb{N}}$ exists for any $x^* \in H$. By (2), $\{x_{n_k}\}_{k \in \mathbb{N}}$ has a weak limit point y_0 in X i.e., there exists $y_0 \in X$ such that every weak neighborhood of y_0 contains an element of $\{x_{n_k}\}_{k \in \mathbb{N}}$. As $\{x_{n_k}\}_{k \in \mathbb{N}}$ is a sequence in X_0 and X_0 is weakly closed, $y_0 \in X_0$. Obviously,

$$\lim_{k \rightarrow \infty} \langle x_{n_k}, x^* \rangle = \langle y_0, x^* \rangle, \quad \forall x^* \in H.$$

Claim: For all $x^* \in X^*$,

$$\lim_{k \rightarrow \infty} \langle x_{n_k}, x^* \rangle = \langle y_0, x^* \rangle$$

Assume otherwise that there exists an $x^* \in X^*$, an $\varepsilon > 0$ and a subsequence $\{x_{n_{k_p}}\}_{p \in \mathbb{N}}$ such that

$$| \langle x_{n_{k_p}} - y_0, x_0^* \rangle | > \varepsilon, \quad p = 1, 2, \dots \quad (1.1.44.1)$$

Again by (2), $\{x_{n_{k_p}}\}_{p \in \mathbb{N}}$ has a weak limit point in X , i.e., there exists a $y'_0 \in X$ such that every weak neighborhood of y'_0 contains an element of $\{x_{n_{k_p}}\}_{p \in \mathbb{N}}$. As $\{x_{n_{k_p}}\}_{p \in \mathbb{N}}$ is a sequence in X_0 and X_0 is weakly closed, $y'_0 \in X_0$ as before and for all x^* in H ,

$$\langle y'_0, x^* \rangle = \lim_{p \rightarrow \infty} \langle x_{n_{k_p}}, x^* \rangle, \quad \forall x^* \in H \quad (1.1.44.2)$$

Hence

$$\langle y_0, x^* \rangle - \langle y'_0, x^* \rangle = \langle y - y'_0, x^* \rangle = 0$$

for all $x^* \in H$. So as H is total (because H_0 is total) on X_0 , $y_0 - y'_0 = 0$ so $y_0 = y'_0$. By 1.1.44.2,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall p \geq N, \quad |\langle x_{n_{k_p}}, x^* \rangle - \langle y_0, x^* \rangle| < \varepsilon$$

which contradicts with 1.1.44.1. Hence the claim is true.

Therefore, as $\{x_n\}_{n \in \mathbb{N}}$ was an arbitrary sequence in A and contains a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ weakly converging to an element of X , A is relatively weakly sequentially compact.

(1) $\implies$ (3): Assume that A is relatively weakly sequentially compact. Consider the canonical embedding $\pi : X \rightarrow X^{**}$ which is a homeomorphism between X and $\pi(X)$ in their weak topologies. As π is a homeomorphism, $\overline{A}^w$ is weakly compact if and only if $\pi(\overline{A}^w)$ is weakly compact. Now as $\overline{A}^w$ is bounded, $\pi(\overline{A}^w)$ is also bounded in the metric topology. So by 1.1.36, to show that $\pi(\overline{A}^w)$ is weakly compact enough to see that $\pi(\overline{A}^w)$ is a weakly closed subset of X^{**} .

Now π is continuous, so $\pi(\overline{A}^w) \subseteq \overline{\pi(A)}^w$. Hence, we are left with showing $\overline{\pi(A)}^w \subseteq \pi(\overline{A}^w)$. Notice again as π is a homeomorphism that

$$\pi(\overline{A}^w) = \overline{\pi(A)}^w \cap \pi(X)$$

Therefore, it will suffice to prove $\overline{\pi(A)}^w \subseteq \pi(X)$.

Let $x^{**} \in \overline{\pi(A)}^w$.

Claim: There exists $x \in X$ such that

$$\langle x^*, x^{**} \rangle = \langle x^*, \pi(x) \rangle = \langle x, x^* \rangle, \quad x^* \in X^*.$$

To prove this we need to show that if $\{x_1^*, x_2^*, \dots, x_n^*\}$ is any finite subset of X^* , then there exists $z \in \overline{A}^w$ such that

$$\langle x_i^*, x^{**} \rangle = \langle z, x_i^* \rangle, \quad i = 1, 2, \dots, n \quad (1.1.44.3)$$

As $x^{**} \in \overline{\pi(A)}^w$, there exists a sequence $\{z_m\}_{m \in \mathbb{N}}$ in A such that $\pi(z_m) \xrightarrow{w} x^{**}$.

Hence, for $i = 1, 2, \dots, n$,

$$|\langle z_m, x_i^* \rangle - \langle x_i^*, x^{**} \rangle| = |\langle x_i^*, \pi(z_m) \rangle - \langle x_i^*, x^{**} \rangle| < \frac{1}{m}$$

As A is relatively weakly sequentially compact there exists a subsequence $\{z_{m_k}\}_{k \in \mathbb{N}}$ of $\{z_m\}_{m \in \mathbb{N}}$ converging weakly to an element of X , say z . Indeed, $z \in \overline{A}^w$ since weak sequential closure of A is in $\overline{A}^w$. Hence, from

$$\begin{aligned} |\langle z, x_i^* \rangle - \langle x_i^*, x^{**} \rangle| &\leq \underbrace{|\langle z, x_i^* \rangle - \langle z_{m_k}, x_i^* \rangle|}_{=0} + \underbrace{|\langle x_i^*, x^{**} \rangle - \langle z_{m_k}, x_i^* \rangle|}_{< \frac{1}{m_k}} \\ &< \frac{1}{m_k} \rightarrow 0, \quad k \rightarrow \infty \end{aligned}$$

it follows that

$$\langle z, x_i^* \rangle = \langle x_i^*, x^{**} \rangle, \quad i = 1, 2, \dots, n.$$

Now to complete the proof we need to see that $x^{**} \in \pi(X)$.

By 1.1.43, enough to show that

$$D = \{x^* \in X^* : \langle x^*, x^{**} \rangle = 0\}$$

is weak* closed. As $D \cap tB_{X^*} = t(D \cap B_{X^*})$ for any $t > 0$, $D \cap B_{X^*}$ is weak* closed if and only if $D \cap tB_{X^*}$ is weak* closed. So by Krein-Smulian theorem 1.1.31, it suffices to show that $D \cap B_{X^*}$ is weak* closed in order to prove the weak* closedness of D .

So let $y_0^* \in \overline{D \cap B_{X^*}}^{w^*}$ and $\varepsilon > 0$.

By 1.1.44.3, there exists $z_1 \in \overline{A}^w$ such that

$$\langle z_1, y_0^* \rangle = \langle y_0^*, x^{**} \rangle$$

As $z_1 \in \overline{A}^w$, we can find an $x_1 \in A$ such that

$$|\langle x_1, y_0^* \rangle - \langle z_1, y_0^* \rangle| < \frac{\varepsilon}{4}$$

Also as $y_0^* \in \overline{D \cap B_{X^*}}^{w^*}$, there exists $y_1^* \in D \cap B_{X^*}$ satisfying

$$| \langle x_1, y_1^* \rangle - \langle x_1, y_0^* \rangle | < \frac{\varepsilon}{4}$$

By induction we select three sequences namely $z_n \in \overline{A}^w$, $x_n \in A$ and $y_n^* \in D \cap B_{X^*}$ satisfying:

1. $\langle z_n, y_m^* \rangle = \langle y_m^*, x^{**} \rangle = 0$, $m = 0, 1, \dots, n-1$;
2. $| \langle x_n, y_m^* \rangle - \langle z_n, y_m^* \rangle | < \frac{\varepsilon}{4}$, $m = 0, 1, \dots, n-1$;
3. $| \langle x_i, y_n^* \rangle - \langle x_i, y_0^* \rangle | < \frac{\varepsilon}{4}$, $i = 1, 2, \dots, n$.

and the sequences we get from this procedure have the following properties:

4. $\|y_m^*\| \leq 1$ (as $y_m^* \in B_{X^*}$) $m = 1, 2, \dots$
5. $\langle z_n, y_m^* \rangle = 0$, $m = 1, 2, \dots, n-1$

Now by letting $m = 0$ in 1 and 2, we get

$$\langle z_n, y_0^* \rangle = \langle y_0^*, x^{**} \rangle = 0 \text{ and } | \langle x_n, y_0^* \rangle - \langle z_n, y_0^* \rangle | < \frac{\varepsilon}{4}.$$

Then with the aid of 3, we get:

$$| \langle y_0^*, x^{**} \rangle - \langle x_i, y_n^* \rangle | \leq | \langle z_n, y_0^* \rangle - \langle x_i, y_0^* \rangle | + | \langle x_i, y_0^* \rangle - \langle x_i, y_n^* \rangle |$$

$$< \underbrace{|\langle z_n, y_0^* \rangle - \langle x_i, y_0^* \rangle|}_{< \frac{\varepsilon}{4} \text{ by 2}} + \underbrace{|\langle x_i, y_0^* \rangle - \langle x_i, y_n^* \rangle|}_{< \frac{\varepsilon}{4} \text{ by 3}} |$$

Hence,

$$6. \quad | \langle y_0^*, x^{**} \rangle - \langle x_i, y_n^* \rangle | < \frac{\varepsilon}{2}, \quad i = 1, 2, \dots, n$$

Since A is relatively weakly sequentially compact and $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in A , there exists a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ converging weakly to some $x \in \overline{A}^w$.

Now by 2 and 5, $| \langle x_{n_k}, y_m^* \rangle | < \frac{\varepsilon}{4}$ where $m = 1, 2, \dots, n_k - 1$. So by letting $k \rightarrow \infty$ we get

$$7. \quad | \langle x, y_m^* \rangle | \leq \frac{\varepsilon}{4} \quad m = 1, 2, \dots$$

By 1.1.13, there exists a convex combination

$$\omega = \sum_{i=1}^N a_i x_{n_i}$$

where $a_i \geq 0$ and $\sum_{i=1}^N a_i = 1$ such that

$$\|x - \omega\| < \frac{\varepsilon}{4} \quad (1.1.44.4)$$

Then, we get;

$$8. \quad | \langle y_0^*, x^{**} \rangle - \langle \omega, y_{n_N}^* \rangle | \leq \sum_{i=1}^N a_i \underbrace{|\langle y_0^*, x^{**} \rangle - \langle x_{n_i}, y_{n_N}^* \rangle|}_{< \frac{\varepsilon}{2} \text{ by 6}} < \sum_{i=1}^N a_i \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

Therefore,

$$| \langle y_0^*, x^{**} \rangle | \leq \underbrace{|\langle y_0^*, x^{**} \rangle - \langle \omega, y_{n_N}^* \rangle|}_{< \frac{\varepsilon}{2} \text{ by 8}} + \underbrace{|\langle \omega, y_{n_N}^* \rangle - \langle x, y_{n_N}^* \rangle|}_{< \frac{\varepsilon}{4} \text{ by 4 and 1.1.44.4}} + \underbrace{|\langle x, y_{n_N}^* \rangle|}_{\leq \frac{\varepsilon}{4} \text{ by 7}}$$

which means $\langle y_0^*, x^{**} \rangle = 0$ as we choose ε arbitrarily. Thus, $y_0^* \in D$ and as B_{X^*} is weak*-closed, $y_0^* \in D \cap B_{X^*}$. Hence $D \cap B_{X^*}$ is weak* closed implying that D is weak* closed. Therefore, $x^{**} \in \pi(X)$ and so $\overline{\pi(A)}^w \subseteq \pi(X)$. Hence, the result follows. $\square$

With theorem 1.1.27, Eberlein-Šmulian theorem says that in order to extract a weakly convergent subsequence from each bounded sequence in X , reflexivity of X is necessary and sufficient.

1.1.5 Rosenthal's ℓ_1 -Theorem

Our concern here is that whether we can extract a weakly Cauchy subsequence from each bounded sequence in a Banach space X or not. Eberlein-Šmulian theorem, the result of the previous section, gives us reflexivity as a sufficient condition to be able to extract a weakly Cauchy subsequence from a bounded one.

Example 1.1.45. ℓ_1 is not reflexive and the sequence of unit vectors $\{e_n\}_{n \in \mathbb{N}}$ in ℓ_1 has no weakly Cauchy subsequence.

So now we will look for a necessary condition. The following theorem shows that example 1.1.45 is essentially the only counterexample.

Theorem 1.1.46. (*Rosenthal-Dor ℓ_1 Theorem*)

Let X be a real or complex Banach space. Any bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ in X have a weakly Cauchy subsequence if and only if X contains no isomorphic copy of ℓ_1 .

To prove the theorem we will show that if $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in X which does not contain any weakly Cauchy subsequences, then it has a subsequence which is the unit vector basis of ℓ_1 . So that we are able to conclude that X contains an isomorphic copy of ℓ_1 .

For the proof of the theorem we need some definitions and results from Ramsey Theory.

First of all what we must understand from “ $\{x_n\}_{n \in \mathbb{N}}$ is the unit vector basis of ℓ_1 ” is that there exists two numbers $a, b > 0$ such that for any scalars $c_1, c_2, \dots, c_n$ and for any positive integer n ,

$$a \sum_{i=1}^n |c_i| \leq \left\| \sum_{i=1}^n c_i x_i \right\| \leq b \sum_{i=1}^n |c_i|$$

Definition 1.1.47. *Let Ω be a set and $\{(E_n, O_n)\}_{n \in \mathbb{N}}$ be a sequence of disjoint pairs of subsets of Ω . Then the sequence $\{(E_n, O_n)\}_{n \in \mathbb{N}}$ is called independent if it satisfies*

$$\bigcap_{n \in N} E_n \cap \bigcap_{n \in P} O_n \neq \emptyset$$

for each pair of infinite subsets N and P of $\mathbb{N}$ with $N \cap P = \emptyset$ □

Proposition 1.1.48. (*due to Leonard Dor*)

Let Ω be a set and $\{(E_n, O_n)\}_{n \in \mathbb{N}}$ be an independent sequence of nonempty disjoint pairs of subsets of Ω , let D_1 and D_2 be disjoint closed disks in $\mathbb{C}$ with centers c_1 and c_2 , respectively, and let $\{b_n\}_{n \in \mathbb{N}}$ be a uniformly bounded sequence of complex-valued functions defined on Ω . Suppose that D_1, D_2 have the same diameter r such that $4r \geq \text{dist}(c_1, c_2)$. Assume that

$$b_n(\omega) \in \begin{cases} D_1 & \text{if } \omega \in E_n, \\ D_2 & \text{if } \omega \in O_n. \end{cases}$$

Then $\{b_n\}_{n \in \mathbb{N}}$ is equivalent to the unit vector basis of ℓ_1 . $\square$

Definition 1.1.49. Let A be a subset of $\mathbb{N}$. We denote the collection of infinite subsets of A by $\mathcal{P}_\infty(A)$ and $\mathcal{P}_\infty(A, B)$ denotes the collection of infinite subsets of $A \cup B$. A collection ζ of $\mathcal{P}_\infty(\mathbb{N})$ is called a Ramsey collection if there exists $S \in \mathcal{P}_\infty(\mathbb{N})$ such that either $\mathcal{P}_\infty(S) \subseteq \zeta$ or $\mathcal{P}_\infty(S) \subseteq \mathcal{P}_\infty(\mathbb{N}) \setminus \zeta$. ζ is called a complete Ramsey collection if for any finite subset A of $\mathbb{N}$ and for any $B \in \mathcal{P}_\infty(\mathbb{N})$ there exists $S \in \mathcal{P}_\infty(B)$ such that either $\mathcal{P}_\infty(A, S) \subseteq \zeta$ or $\mathcal{P}_\infty(A, S) \subseteq \mathcal{P}_\infty(\mathbb{N}) \setminus \zeta$. $\square$

The τ -topology on $\mathcal{P}_\infty(\mathbb{N})$ is formed by taking

$$B = \{\mathcal{P}_\infty(A, B) : A \subset \mathbb{N}, A \text{ finite}, B \in \mathcal{P}_\infty(\mathbb{N})\}$$

as a basis.

Theorem 1.1.50. (Nash-Williams)

Open sets and closed sets of $\mathcal{P}_\infty(\mathbb{N})$ are completely Ramsey in the τ -topology on $\mathcal{P}_\infty(\mathbb{N})$. $\square$

For the proofs of the above results see [7].

The Proof of Rosenthal-Dor ℓ_1 Theorem:

Proof. Let $\{x_n\}_{n \in \mathbb{N}}$ be a bounded sequence in X and assume that it has no weak Cauchy subsequence. We need duality in the proof so we view X as a subspace of continuous functionals on B_{X^*} in its weak* topology i.e, we consider $x \in X$ as a continuous function on B_{X^*} . With this approach, $\{x_n\}_{n \in \mathbb{N}}$ has no weak Cauchy subsequence would mean that $\{x_n\}_{n \in \mathbb{N}}$ has no pointwise convergent subsequence, that is, for any subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ there exists $x^* \in B_{X^*}$ such that the sequence $\{\langle x_{n_k}, x^* \rangle\}_{k \in \mathbb{N}}$ has at least two distinct accumulation points.

Let $\mathcal{D}$ be the countable collection of all pairs (D_1, D_2) of open disks in $\mathbb{C}$, each disk having centers with rational coordinates c_1 and c_2 , respectively and the same rational radius r satisfying $\text{dist}(c_1, c_2) \geq 4r$. Denote the members of $\mathcal{D}$ by (D_1^k, D_2^k) .

Claim: There exists a positive integer K and an infinite subset P of $\mathbb{N}$ such that for any infinite subset M of P , we can find $x_M^* \in B_X^*$ for which $\{ \langle x_k, x_M^* \rangle \}_{k \in M}$ has accumulation points in both D_1^K and D_2^K .

Assume otherwise. Then for any $k \geq 1$ and any infinite subset P of $\mathbb{N}$, there is an infinite subset M of P such that for every $x^* \in B_{X^*}$ the sequence $\{ \langle x_k, x^* \rangle \}_{k \in M}$ would not have accumulation points in each of D_1^k and D_2^k .

So as $\mathbb{N}$ is an infinite subset of itself there exists an infinite subset M_1 of $\mathbb{N}$ such that for every $x^* \in B_{X^*}$, $\{ \langle x_m, x^* \rangle \}_{m \in M_1}$ does not have accumulation points in D_1^1 and D_2^1 . Similarly, there exists an infinite subset M_2 of M_1 such that $\{ \langle x_m, x^* \rangle \}_{m \in M_2}$ does not have accumulation points in D_1^2 and D_2^2 for any $x^* \in B_{X^*}$. Continuing this way, we get a decreasing sequence of infinite subsets $\{M_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ with respect to inclusion such that for each n , $\{ \langle x_m, x^* \rangle \}_{m \in M_n}$ does not have accumulation points in D_1^n and D_2^n , for any $x^* \in B_{X^*}$.

Let P be an infinite subset of $\mathbb{N}$ such that the n^{th} member p_n of P is an element of M_n . As $\{M_n\}_{n \in \mathbb{N}}$ is constructed to be a decreasing sequence of infinite subsets of $\mathbb{N}$, we may assume that $\{p_n\}_{n \in \mathbb{N}}$ is an increasing sequence. Using the approach at the beginning of the proof, no subsequence of $\{x_n\}_{n \in \mathbb{N}}$ converges pointwise on B_{X^*} i.e., we can find an $x_0^* \in B_{X^*}$ such that $\{ \langle x_{p_n}, x_0^* \rangle \}_{n \in \mathbb{N}}$ does not converge. Hence there must be at least two distinct accumulation points d_1 and d_2 of $\{ \langle x_{p_n}, x_0^* \rangle \}_{n \in \mathbb{N}}$ because $\{ \langle x_{p_n}, x_0^* \rangle \}_{n \in \mathbb{N}}$ is a bounded sequence (otherwise the sequence converges.) Note that (d_1, d_2) lie in some $(D_1^k, D_2^k) \in \mathcal{D}$. Then for any $j \geq 1$, both d_1 and d_2 are accumulation points of $\{ \langle x_{p_n}, x_0^* \rangle \}_{n \geq j}$. So as $\{M_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of infinite subsets, for $n \geq j$, $M_n \subseteq M_j$. Thus $\{ \langle x_{p_n}, x_0^* \rangle \}_{n \geq j}$ is a subsequence of $\{ \langle x_m, x_0^* \rangle \}_{m \in M_j}$ implying that d_1 and d_2 are also accumulation points of $\{ \langle x_m, x_0^* \rangle \}_{m \in M_j}$ which contradicts with the definition of M_j 's. Hence our claim holds.

For simplicity, from now on we will assume that $\{x_n\}_{n \in P}$ is in fact $\{x_n\}_{n \in \mathbb{N}}$ and D_1^k and D_2^k are D_1 and D_2 . So to summarize what we have; $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in X with no pointwise convergent subsequence on

B_{X^*} , D_1 and D_2 are disjoint disks of the same rational diameter r having centers with rational coordinates c_1 and c_2 such that $\text{dist}(c_1, c_2) \geq 4r$ and for any infinite subset M of $\mathbb{N}$, there is an $x_M^* \in B_{X^*}$ such that $\{< x_m, x_M^* >\}_{m \in M}$ has accumulation points in both D_1 and D_2 . (★)

Now let

$$O_n = \{x^* \in B_{X^*} : < x_n, x^* > \in D_1\}, \text{ and } E_n = \{x^* \in B_{X^*} : < x_n, x^* > \in D_2\}.$$

Note that for any subsequence n_k of positive integers by (★),

$$\text{neither } \lim_{k \rightarrow \infty} \chi_{E_{n_k}}(x^*) = 0 \text{ nor } \lim_{k \rightarrow \infty} \chi_{O_{n_k}}(x^*) = 0$$

holds for each $x^* \in B_{X^*}$. Denote $O_j = -E_j$. Let

$$\mathcal{P}_k = \{(n_l)_{l \in \mathbb{N}} \in \mathcal{P}_\infty(\mathbb{N}) : \bigcap_{l=1}^k (-1)^l E_{n_l} \neq \emptyset\}, \quad k \in \mathbb{N}$$

Identify each subset of $\mathbb{N}$ with a point in $\{0, 1\}^\mathbb{N}$ i.e., if the subset contains $i \in \mathbb{N}$ then the corresponding point in $\{0, 1\}^\mathbb{N}$ has 1 in the i^{th} coordinate.

Claim: $\bigcap_{k \geq 1} \mathcal{P}_k$ is a closed subset of $\mathcal{P}_\infty(\mathbb{N})$ in the relative topology of $\{0, 1\}^\mathbb{N}$. To show that the claim holds, fix a positive integer k . Then $\mathcal{P}_k$ is relatively closed in $\mathcal{P}_\infty(\mathbb{N})$. Indeed, let $(n_l^0) \in \overline{\mathcal{P}_k} \subseteq \mathcal{P}_\infty(\mathbb{N})$ in the relative topology of $\{0, 1\}^\mathbb{N}$.

Consider the basic neighborhood B of (n_l^0) given by

$$B = \bigcap_{j=1}^{n_k^0} \{(e_p) \in \{0, 1\}^\mathbb{N} : e_j = \chi_{\{n_1^0, n_2^0, \dots, n_k^0\}}(j)\}$$

Then $B \cap \mathcal{P}_k \neq \emptyset$ as (n_l^0) is in the closure of $\mathcal{P}_k$ in the relative topology. So there exists $(n_l^1) \in \mathcal{P}_k$ such that (n_l^1) and (n_l^0) have the same first k entries i.e.,

$$n_1^1 = n_1^0, n_2^1 = n_2^0, \dots, n_k^1 = n_k^0.$$

Hence, as $(n_l^1) \in \mathcal{P}_k$,

$$\bigcap_{l=1}^k (-1)^l E_{n_l^0} = \bigcap_{l=1}^k (-1)^l E_{n_l^1} \neq \emptyset$$

Therefore, $n_l^0 \in P_k$. So, P_k is closed in the relative topology. As arbitrary intersections of closed sets are closed, $\bigcap_{k \geq 1} P_k$ is a relatively closed subset of $\mathcal{P}_\infty(\mathbb{N})$.

So Nash-Williams theorem 1.1.50 implies that $\bigcap_{k \geq 1} P_k$ is completely Ramsey meaning that for any infinite subset K of $\mathbb{N}$ there exists an infinite subset M of K such that either $\mathcal{P}_\infty(M) \subseteq \bigcap_{k \geq 1} P_k$ or $\mathcal{P}_\infty(M) \subseteq (\bigcap_{k \geq 1} P_k)^c$. So we have a sequence $\{m_p\}_{p \in \mathbb{N}}$ of positive integers such that

$$\text{either } \mathcal{P}_\infty(\{m_p\}_{p \in \mathbb{N}}) \subseteq \bigcap_{k \geq 1} P_k \text{ or } \mathcal{P}_\infty(\{m_p\}_{p \in \mathbb{N}}) \subseteq (\bigcap_{k \geq 1} P_k)^c. \quad (1.1.50.1)$$

We may assume that $\{m_p\}_{p \in \mathbb{N}}$ is increasing. By $(\star)$, for $M = \{m_p\}_{p \in \mathbb{N}}$, there is an $x_M^* \in B_{X^*}$ such that $\{ \langle x_m, x_M^* \rangle \}_{m \in M}$ has accumulation points in both D_1 and D_2 . So there exists a subsequence $\{m_{p_q}\}_{q \in \mathbb{N}}$ of $\{m_p\}_{p \in \mathbb{N}}$ such that

$$\begin{cases} \langle x_{m_{p_q}}, x_M^* \rangle \in D_1 & \text{if } q \text{ is odd,} \\ \langle x_{m_{p_q}}, x_M^* \rangle \in D_2 & \text{if } q \text{ is even.} \end{cases}$$

Hence by definition of O_n 's and E_n 's

$$\begin{cases} x_M^* \in O_{m_{p_q}} & \text{if } q \text{ is odd,} \\ x_M^* \in E_{m_{p_q}} & \text{if } q \text{ is even.} \end{cases}$$

i.e., for any $q \in \mathbb{N}$, $x_M^* \in (-1)^q E_{m_{p_q}}$ implying that

$$\bigcap_{q \in \mathbb{N}} (-1)^q E_{m_{p_q}} \neq \emptyset$$

So by definition of P_k , $m_{p_q} \in \bigcap_{k \geq 1} P_k$. Hence $\mathcal{P}_\infty(\{m_p\}_{p \in \mathbb{N}}) \not\subseteq (\bigcap_{k \geq 1} P_k)^c$. So by 1.1.50.1, $\mathcal{P}_\infty(\{m_p\}_{p \in \mathbb{N}}) \subseteq \bigcap_{k \geq 1} P_k$. Therefore we have an increasing sequence $\{m_p\}_{p \in \mathbb{N}}$ of positive integers such that any subsequence $\{m_{p_q}\}_{q \in \mathbb{N}}$ of it is in P_k for any k , because all infinite subsets of $\{m_p\}_{p \in \mathbb{N}}$ is in $\bigcap_{k \geq 1} P_k$. So,

$$\bigcap_{q=1}^k (-1)^q E_{m_{p_q}} \neq \emptyset, \quad \forall k \in \mathbb{N}$$

Now as $O_n = -E_n$,

$$O_{m_{p_1}} \cap E_{m_{p_2}} \cap O_{m_{p_3}} \cap E_{m_{p_4}} \cap \dots \cap E_{m_{p_k}} \neq \emptyset, \quad \forall k \in \mathbb{N}.$$

Now consider the subsequence $\{m_{p_{2q}}\}_{q \in \mathbb{N}}$ and take any subsequence $\{m'_{p_{2q}}\}_{q \in \mathbb{N}}$ of it. Consider the intersection of finitely many E 's and O 's indexed by $\{m'_{p_{2q}}\}_{q \in \mathbb{N}}$ where the successive terms are both E 's, say $E_{m'_{p_{2q}}}$ and $E_{m'_{p_{2(q+1)}}}$. Then

$$E_{m'_{p_{2q}}} \cap E_{m'_{p_{2(q+1)}}} \supseteq E_{m'_{p_{2q}}} \cap O_{m_{p_k}} \cap E_{m'_{p_{2(q+1)}}} \neq \emptyset$$

where k satisfies $m'_{p_{2q}} < m_{p_k} < m'_{p_{2(q+1)}}$. Similarly if the successive terms are both O 's, say $O_{m'_{p_{2q}}}$ and $O_{m'_{p_{2(q+1)}}}$. Then

$$O_{m'_{p_{2q}}} \cap O_{m'_{p_{2(q+1)}}} \supseteq O_{m'_{p_{2q}}} \cap E_{m_{p_k}} \cap O_{m'_{p_{2(q+1)}}} \neq \emptyset$$

where k satisfies $m'_{p_{2q}} < m_{p_k} < m'_{p_{2(q+1)}}$. Now we acquired an independent sequence of disjoint pairs of subsets of B_{X^*} . Also $x_{m_{p_{2q}}}$ is a bounded sequence in X such that

$$\begin{cases} \langle x_{m_{p_{2q}}}, x^* \rangle \in D_1 & \text{if } x^* \in O_{m_{p_{2q}}}, \\ \langle x_{m_{p_{2q}}}, x^* \rangle \in D_2 & \text{if } x^* \in E_{m_{p_{2q}}}. \end{cases}$$

So by 1.1.48, the sequence $\{x_{m_{p_{2q}}}\}_{q \in \mathbb{N}}$ is equivalent to the unit vector basis of ℓ_1 . $\square$

Rosenthal's Theorem says that if $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in a Banach space X then either $\{x_n\}_{n \in \mathbb{N}}$ has a weak Cauchy subsequence or contains a subsequence that is equivalent to the standard unit vector basis in ℓ_1 i.e., there is only one exceptional case where it is impossible to extract a weak Cauchy subsequence from a bounded sequence in a Banach space which happens exactly when the sequence is equivalent to the sequence of unit vectors in ℓ_1 .

1.2 Weakly Compact and Compact Operators

Definitions and Basic Properties

Basic definitions and theorems on weakly compact and compact operators will be given without proof to serve as a tool to our study.

Theorem 1.2.1. *Let X and Y be normed spaces. Then for any bounded linear operator $T : X \rightarrow Y$ there exists a unique bounded linear operator $T^* : Y^* \rightarrow X^*$ satisfying*

$$\langle Tx, y^* \rangle = \langle x, T^*y^* \rangle$$

for all $x \in X$ and for all $y^* \in Y^*$ and $\|T\| = \|T^*\|$. The operator T^* is called the adjoint of T . $\square$

Definition 1.2.2. Let X and Y be two Banach spaces. A linear map $T : X \rightarrow Y$ is said to be (weakly) compact if $T(B)$ is relatively (weakly) compact in Y whenever B is a bounded subset of X . $\square$

From the definition we can say that T is (weakly) compact if $T(B_X)$ is relatively (weakly) compact in Y and obviously any (weakly) compact operator T is bounded, i.e., $T \in \mathcal{L}(X, Y)$.

Definition 1.2.3. A subset E of a metric space X is said to be totally bounded if for all $\varepsilon > 0$, E can be covered by a finite number of open balls with radius ε . $\square$

Theorem 1.2.4. If K is a closed subset of a complete metric space, then the following are equivalent:

- (1) K is compact.
- (2) Every infinite subset of K has a limit point in K .
- (3) K is totally bounded.

$\square$

As a consequence of this theorem, for any two Banach spaces X and Y , since Y is a complete metric space, a linear operator $T : X \rightarrow Y$ is compact if and only if $T(B)$ is totally bounded whenever B is a bounded subset of X . The theorem also implies that T is compact if and only if every bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ contains a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ such that $\{T(x_{n_k})\}_{k \in \mathbb{N}}$ converges to a point of Y . A similar result holds for weakly compact operators as by Eberlein-Šmulian Theorem 1.1.44, T maps bounded sets to relatively weakly sequentially compact sets i.e., T is weakly compact if and only if every bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ contains a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ such that $\{T(x_{n_k})\}_{k \in \mathbb{N}}$ converges weakly to a point of Y .

Theorem 1.2.5. *Any linear combination of (weakly) compact operators is (weakly) compact and the limit of a sequence of (weakly) compact linear operators is a (weakly) compact linear operator if it exists.* $\square$

By the above theorem, $\mathcal{K}(X, Y)$, the collection of all compact operators from X to Y with norm inherited from $\mathcal{L}(X, Y)$, is a closed subspace of $\mathcal{L}(X, Y)$ so is a Banach space.

Theorem 1.2.6. *Let T be a bounded linear operator from a Banach space to a normed space. Then T is an isomorphism from X onto Y if and only if T^* is an isomorphism from Y^* onto X^* .* $\square$

Theorem 1.2.7. *Let U , X , Y and Z be Banach spaces and let $S \in \mathcal{L}(X, Y)$ be (weakly) compact. Let $T \in \mathcal{L}(U, X)$ and $V \in \mathcal{L}(Y, Z)$. Then $S \circ T$ and $V \circ S$ are (weakly) compact.* $\square$

Theorem 1.2.8. *Any bounded linear operator T from a Banach space X to a Banach space Y is compact if and only if T^* sends bounded sequences which converge in the weak* topology of Y^* into sequences which converge in the metric topology of X^* .* $\square$

Theorem 1.2.9. *Let X and Y be Banach spaces. Then any bounded linear operator from X to Y with a finite dimensional range is compact.* $\square$

The compact part of the following theorem which we will be using in the last chapter is due to Schauder, weakly compact part is due to Gantmacher.

Theorem 1.2.10. *Let X and Y be Banach spaces and T is a bounded linear operator from X into Y . Then T is (weakly) compact if and only if T^* is (weakly) compact.* $\square$

Theorem 1.2.11. *Let K be a compact Hausdorff space, Y be a Banach space and $T : C(K) \rightarrow Y$ be a weakly compact linear operator. Then T maps every weakly Cauchy sequence in $C(K)$ into a Cauchy sequence in Y .* $\square$

The proofs for the above theorems may be found in [9].

1.3 Stone-Čech Compactification of Completely Regular Spaces; Extremely Disconnected Spaces

If a topological space is not compact, how close is it to being compact? If the space is locally compact and Hausdorff by adding just one point outside of the space to the space we can turn it into a compact one.

Locally Compact Spaces

Definition 1.3.1. A space X is said to be locally compact if each point of X has a neighborhood which is contained in a compact set. $\square$

So obviously any compact space is locally compact.

We define the one-point compactification of a locally compact Hausdorff space as follows:

Definition 1.3.2. Let X be locally compact noncompact Hausdorff space. Take an object ∞ (point at infinity) outside of X and let $X_\infty = X \cup \{\infty\}$. Then the space X_∞ is called the one-point compactification (or Alexandroff compactification) of X if we put a topology on X_∞ by defining the collection of open sets in X_∞ to be all the sets of the following types:

- (1) U , where U is an open set of X ,
- (2) $X_\infty \setminus C$, where C is a compact subset of X .

$\square$

The following theorem states the properties of the one-point compactification:

Theorem 1.3.3. Let X be a locally compact Hausdorff space X . Then the one-point compactification X_∞ of X is a compact Hausdorff space containing X as a subspace such that $X_\infty \setminus X$ consists of only one point and X is an open dense subset of X_∞ . $\square$

Completely Regular Spaces

In the following we will give some definitions and results for completely regular spaces

because in our forthcoming work we will see that in order to have a compactification a space needs to be completely regular.

Definition 1.3.4. A space X is said to be completely regular if singletons are closed and every nonempty closed set A and every point x outside of it can be separated by a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 1$ and $f(A) = \{0\}$. $\square$

Proposition 1.3.5. Any metric space is completely regular. $\square$

Proposition 1.3.6. Every locally compact Hausdorff space is completely regular. $\square$

Definition 1.3.7. A collection $\{f_\alpha\}_{\alpha \in J}$ of continuous functions $f_\alpha : X \rightarrow \mathbb{R}$ is said to separate points from closed sets in X if for each point x_0 of X and each neighborhood U of x_0 , there is an $\alpha \in J$ such that $f_\alpha(x_0) > 0$ and $f_\alpha(x) = 0$ for each $x \in X \setminus U$. $\square$

Definition 1.3.8. A mapping $f : X \rightarrow Y$ between topological spaces X and Y is an embedding if f yields a homeomorphism between X and $f(X)$ where $f(X)$ has the subspace topology inherited from Y . $\square$

Theorem 1.3.9. (Embedding Theorem)

Let X be a Hausdorff space. Suppose that $f_\alpha : X \rightarrow \mathbb{R}$ separates points from closed sets in X . Then the function $F : X \rightarrow \mathbb{R}^J$ defined by

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is an embedding of X in $\mathbb{R}^J$. $\square$

Theorem 1.3.10. A topological vector space X is completely regular if and only if for some set J , X can be embedded in $[0, 1]^J$. $\square$

Stone-Čech Compactification

In general there are many ways of compactifying a space. We have already seen the one-point compactification of a topological space which is in some sense minimal. In this section, we will introduce the Stone-Čech compactification which we can consider as a maximal compactification.

Definition 1.3.11. A compactification of a space X is a compact Hausdorff space Y containing X such that X is homeomorphic to a dense subspace of Y . So we say that $\overline{X} = Y$ by identifying X with the corresponding dense subspace of Y . Moreover, two compactifications Y_1 and Y_2 of X are said to be equivalent if there exists a homeomorphism $h : Y_1 \rightarrow Y_2$ such that $h(x) = x$ for all $x \in X$. $\square$

However, not all topological spaces have a compactification:

Proposition 1.3.12. A space X has a compactification if and only if X is completely regular. $\square$

We obtain a compactification of a completely regular space X as follows: Let Z be a compact Hausdorff space and $h : X \rightarrow Z$ be an embedding of X in Z . Let $X_0 = h(X) \subset Z$ and $Y_0 = \overline{X_0}$ (closure in Z). Then Y_0 is compact as a closed subspace of the compact space Z and it is Hausdorff as subspaces inherit the property of being Hausdorff. So Y_0 is a compactification of X_0 .

Now let A be a set with $A \cap X = \emptyset$ such that there exists a bijective mapping $k : A \rightarrow Y_0 \setminus X_0$. Let $Y = X \cup A$ and define a bijective map $H : Y \rightarrow Y_0$ as follows:

$$H(x) = \begin{cases} h(x) & \forall x \in X \\ k(x) & \forall x \in A. \end{cases}$$

Now give a topology to Y by defining the open sets as follows:

$$U \text{ is open in } Y \text{ if and only if } H(U) \text{ is open in } Y_0.$$

This topology makes H and H^{-1} continuous. Hence H becomes a homeomorphism as it is bijective. Therefore we get a space Y containing X such that Y is homeomorphic to Y_0 . The space Y is called the compactification of X induced by the embedding h which can be extended to an embedding $H : Y \rightarrow Z$.

At this point, we are interested in finding the conditions under which a continuous real-valued function f defined on X can be continuously extended to a compactification Y of X ? Obviously f must be bounded, because the extension of f will carry the compact space Y into $\mathbb{R}$, and compact subsets of $\mathbb{R}$ are bounded. But being bounded

is not enough. A standard example is $f(x) = \sin(\frac{1}{x})$ defined on $(0,1]$. f is continuous and bounded on $(0,1]$ but we can not extend $f(x)$ continuously to $[0,1]$. This problem of continuously extending any bounded continuous real-valued function defined on X leads us to the Stone-Čech compactification of X , denoted $\beta(X)$.

Let X be a completely regular space and $\{f_\alpha\}_{\alpha \in J}$ be the collection of all bounded continuous real-valued functions on X . For each $\alpha \in J$ choose an interval I_α in $\mathbb{R}$ such that

$$I_\alpha = [\inf_{x \in X} f_\alpha(x), \sup_{x \in X} f_\alpha(x)].$$

Then $f_\alpha(X) \subseteq I_\alpha$. Now define

$$h : X \rightarrow \prod_{\alpha \in J} I_\alpha \quad \text{by} \quad h(x) = (f_\alpha(x))_{\alpha \in J}$$

By Tychonoff theorem 1.1.33, $\prod_{\alpha \in J} I_\alpha$ is compact since each interval I_α is compact. As X is completely regular by definition, there exists a continuous bounded function $f : X \rightarrow [0,1]$ separating points from closed sets. So as $f \in \{f_\alpha\}_{\alpha \in J}$, $\{f_\alpha\}_{\alpha \in J}$ separates points from closed sets. By 1.3.9, h is an embedding of X . The compactification of X induced by the embedding h is called the *Stone-Čech compactification* of X and denoted by $\beta(X)$.

Lemma 1.3.13. *Let A be a subset of X , $f : A \rightarrow Y$ be a continuous mapping and Y be a Hausdorff space. Then if f can be extended to a continuous function $g : \overline{A} \rightarrow Y$, then g is uniquely determined by f .* $\square$

Theorem 1.3.14. *Every bounded continuous real-valued function on a completely regular space X can be uniquely extended to a continuous real-valued function on $\beta(X)$.*

Proof. By definition of $\beta(X)$, there is an embedding h inducing $\beta(X)$ such that

$$h : X \rightarrow \prod_{\alpha \in J} I_\alpha, \quad h(x) = (f_\alpha(x))_{\alpha \in J}$$

having the property that h can be extended to an embedding $H : \beta(X) \rightarrow \prod_{\alpha \in J} I_\alpha$. As we represent the collection of all bounded continuous real-valued functions on X by

$\{f_\alpha(x)\}_{\alpha \in J}$, any bounded continuous real-valued function equals f_β for some $\beta \in J$. Now let

$$\pi_\beta : \prod_{\alpha \in J} I_\alpha \rightarrow I_\beta$$

be the projection map onto the β^{th} -coordinate.

Consider the composition of π_β and H , $\pi_\beta \circ H : \beta(X) \rightarrow I_\beta$

Then for $x \in X$,

$$\pi_\beta \circ H(x) = \pi_\beta(H(x)) = \pi_\beta(h(x)) = \pi_\beta((f_\alpha(x))_{\alpha \in J}) = f_\beta(x).$$

Therefore $\pi_\beta \circ H$ is an extension of f_β and by being the composition of two continuous functions it is continuous. As $\beta(X) = \overline{X}$ and $\mathbb{R}$ is Hausdorff, by previous lemma the uniqueness of the extension follows. $\square$

What we have now is every completely regular space X can be embedded as a dense subspace of a compact Hausdorff space $\beta(X)$ such that we can continuously extend bounded continuous real-valued functions on X to $\beta(X)$, uniquely.

Theorem 1.3.15. (*Extension Property*)

Let X be a completely regular space and let Y be a compactification of X such that every bounded continuous real-valued function on X can be uniquely extended to a continuous real-valued function on Y . Then if Z is any compact Hausdorff space and $g : X \rightarrow Z$ is any continuous function, then g can be extended to a continuous function $k : Y \rightarrow Z$. In particular, g extends uniquely to a continuous mapping from $\beta(X)$ to Z .

Proof. As Z is compact it is locally compact. So by 1.3.6, since Z is also Hausdorff, it is completely regular. So by 1.3.10, Z can be embedded in $[0, 1]^J$ for some set J . So we may assume that $Z \subset [0, 1]^J$. Now let $g : X \rightarrow Z \subset [0, 1]^J \subset \mathbb{R}^J$ be any continuous function. So for $x \in X$, $g(x) = (g_\alpha(x))_{\alpha \in J}$ where each component function g_α is bounded continuous real-valued function on X and $g_\alpha(x) \in [0, 1]$ for each $\alpha \in J$. So we can uniquely extend each g_α to a continuous function $k_\alpha : Y \rightarrow \mathbb{R}$ by the assumption.

Let $k : Y \rightarrow \mathbb{R}^J$ be defined by $k(y) = (k_\alpha(y))_{\alpha \in J}$. Then k is continuous as each k_α is continuous because $\mathbb{R}^J$ has the product topology.

The proof will be complete if we show that $k(Y) \subset Z$. Indeed, as $g(X) \subset Z$ and $g(X) = k(X)$, $k(X) \subset Z$. This implies $\overline{k(X)} \subset Z$ since Z is closed in $\mathbb{R}^J$. Hence by continuity of k , $k(\overline{X}) \subset \overline{k(X)} \subset Z$. Therefore as $Y = \overline{X}$, $k(Y) \subset Z$. $\square$

Stone-Čech compactification is essentially unique by the following corollary:

Corollary 1.3.16. *Let Y be a compactification of a completely regular space X . If whenever Z is a compact Hausdorff space and $g : X \rightarrow Z$ is a continuous mapping, g has a unique continuous extension from Y to Z then Y is homeomorphic to $\beta(X)$.*

Proof. Take $Z = \beta(X)$ in 1.3.15. $\square$

Corollary 1.3.17. *A locally compact Hausdorff space X is an open subset of its Stone-Čech compactification $\beta(X)$.* $\square$

Definition 1.3.18. *A topological space X is said to be extremally disconnected (or Stonean) if the closure of any open subset of X is also open.* $\square$

So we may say that the topology of a extremally disconnected space has a base consisting of clopen sets (both closed and open simultaneously.)

Proposition 1.3.19. *Let X be any infinite discrete space. Then if A is a subset of X , then the closure of A in $\beta(X)$ is an open subset of $\beta(X)$. If B is another subset of X disjoint from A , then the closures of A and B in $\beta(X)$ are also disjoint.*

Proof. First observe that any discrete space is metrizable by the discrete metric

$$d : X \rightarrow \{0, 1\} \quad \text{defined by} \quad d(x, y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y. \end{cases}$$

Therefore X is completely regular by 1.3.5 and Hausdorff as any metrizable space is Hausdorff. So $\beta(X)$ exists.

Let A be a subset of X and define $C = X \setminus A$ so that $A \cap C = \emptyset$.

Now define $f : X \rightarrow [0, 1]$ by $f(A) = \{1\}$ and $f(C) = \{0\}$. Then f is continuous

by being a function on a discrete space. So by 1.3.14, as f is a continuous bounded real-valued function it extends uniquely to a continuous function $F : \beta(X) \rightarrow [0, 1]$.

Now as $\overline{X} = \beta(X)$ and $X = A \cup C$,

$$\overline{A \cup C} = \overline{A \cup C} = \overline{X} = \beta(X)$$

Therefore, $\overline{A} = F^{-1}(\{1\})$ and $\overline{C} = F^{-1}(\{0\})$.

Thus, $\overline{A} \cap \overline{C} = \emptyset$ which implies that $\overline{A}$ is open.

Now let B be any subset of X such that $A \cap B = \emptyset$. Then $B \subseteq C$, so that

$$\overline{A} \cap \overline{B} \subset \overline{A} \cap \overline{C} = \emptyset.$$

Hence $\overline{A} \cap \overline{B} = \emptyset$ as desired. $\square$

Theorem 1.3.20. *For any infinite discrete space X , $\beta(X)$ is extremally disconnected.*

Proof. Let V be an open subset of $\beta(X)$. To prove that $\overline{V}$ is open in $\beta(X)$, let $x \in \overline{V}$. Then for any open neighborhood U of x , $V \cap U \neq \emptyset$. Since X is dense in $\beta(X)$ it has nonempty intersection with open sets. So as $V \cap U$ is open and nonempty, $V \cap U \cap X \neq \emptyset$, i.e., $x \in \overline{V \cap X}$. Hence $\overline{V} = \overline{V \cap X}$. By previous proposition $\overline{V \cap X}$ is open in $\beta(X)$ and therefore $\overline{V}$ is open. $\square$

1.4 C^* and Von-Neumann Algebras

Basic Definitions

Before giving the definition of a C^* algebra, we have to know what an algebra is and what turns it to a $*$ -algebra. So we proceed with the following:

Definition 1.4.1. *An algebra A is a vector space over a field $\mathbb{F}$ together with an $\mathbb{F}$ -bilinear mapping $A \times A \rightarrow A$ defined by $(x, y) \mapsto xy$ and called the multiplication of x and y . When we say $\mathbb{F}$ -bilinearity we require the following properties:*

$$(1) \quad (\alpha x + \beta y)z = \alpha(xz) + \beta(yz) \quad \forall \alpha, \beta \in \mathbb{F}, \quad \forall x, y, z \in A.$$

$$(2) \quad x(\alpha y + \beta z) = \alpha(xy) + \beta(xz) \quad \forall \alpha, \beta \in \mathbb{F}, \quad \forall x, y, z \in A.$$

□

Definition 1.4.2. Let A be an algebra. A subalgebra of A is a vector subspace B such that whenever $x, y \in B$, $xy \in B$. □

Definition 1.4.3. A mapping $x \mapsto x^*$ of a complex algebra A into A is said to be an involution on A if the following properties hold for all $x, y \in A$ and $\alpha \in \mathbb{C}$:

$$(1) \quad (x + y)^* = x^* + y^*.$$

$$(2) \quad (\alpha x)^* = \bar{\alpha} x^*.$$

$$(3) \quad (xy)^* = y^* x^*.$$

$$(4) \quad (x^*)^* = x.$$

The pair $(A, *)$ is called an involutive algebra or a $*$ -algebra. If A also admits a unit 1 ($x1 = 1x = x$, $x \in A$) then we say that A is a unital $*$ -algebra. □

Definition 1.4.4. A Banach $*$ -algebra A is a $*$ -algebra A together with a complete submultiplicative norm that is a complete norm such that for any $x, y \in A$,

$$\|xy\| \leq \|x\|\|y\|$$

If A is unital and $\|1\| = 1$ then we say that A is a unital Banach $*$ -algebra. □

If A is unital then $1^* = 1$ as $1^* = (11^*)^* = 1$. We say that $a \in A$ is invertible if there exists $b \in A$ such that $ab = ba = 1$. In this case we denote b as a^{-1} .

Definition 1.4.5. A C^* -algebra is a Banach $*$ -algebra satisfying

$$\|x^*x\| = \|x\|^2, \quad x \in A$$

□

Proposition 1.4.6. Let A be a C^* -algebra then for any $x \in A$, $\|x^*\| = \|x\|$. □

Example 1.4.7. All of the following are C^* algebras with the indicated involutions:

- (1) $L_\infty(\mu)$, $f \mapsto \bar{f}$, where $\bar{f}$ is the pointwise complex conjugate of f .
- (2) ℓ_∞ , $x = \{x_n\}_{n \in \mathbb{N}} \mapsto \bar{x} = \{\bar{x}_n\}_{n \in \mathbb{N}}$ (follows from 1).
- (3) $\mathcal{L}(H)$, where H is a Hilbert space, $T \mapsto T^*$, where T^* is the adjoint of T .
- (4) $\mathcal{K}(H)$, where H is a Hilbert space, is a norm closed subalgebra of $\mathcal{L}(H)$ which is closed under involution $T \mapsto T^*$, where T^* is the adjoint of T .
- (5) $\mathcal{B}(S, \Sigma)$ bounded scalar valued Σ measurable functions defined on S where S is a set and Σ is a σ -algebra of subsets of S , $f \mapsto \bar{f}$, $\bar{f}$ as above.

We also have $C(K)$, where K is a compact Hausdorff space as an example of a C^* -algebra.

Theorem 1.4.8. *Let K be a compact (locally compact) Hausdorff space. Then $C(K)$ the space of all complex valued continuous functions on K is a commutative C^* -algebra.*

Proof. Observe that $C(K)$ is an algebra under pointwise multiplication of all complex valued functions on K . With the norm $\|f\| = \sup_{x \in K} |f(x)|$ and the involution $f^*(x) = \overline{f(x)}$ for $f \in C(K)$, $C(K)$ becomes a commutative C^* -algebra. $\square$

Definition 1.4.9. *An element x of a C^* -algebra A is said to be self-adjoint if $x^* = x$ and unitary if $x^*x = 1 = xx^*$ where 1 is the identity element with respect to the multiplication.* $\square$

Notation: For any C^* -algebra A , we denote the set of all self-adjoint elements by A_{sa} ; the set of all unitary elements of A (or the unit ball of A) when A is unital by A_1 .

Note: Not to cause confusion, we denote the dual space of a C^* algebra A by A' rather than our normal notation A^* . And we denote the elements of the dual space A' by a' rather than a^* .

Definition 1.4.10. Let H be a Hilbert space. A C^* -algebra of $\mathcal{L}(H)$ which contains the identity operator and which is closed in weak operator topology is called a Von-Neumann (or a W^*) algebra. $\square$

Theorem 1.4.11. A C^* algebra A is a W^* algebra if and only if A is the dual space of some Banach space. $\square$

Examples 1.4.12. The following are some examples of von-Neumann algebras with the indicated involutions in 1.4.7:

- (1) ℓ_∞ .
- (2) $L_\infty(\mu)$.
- (3) $\mathcal{L}(H)$ where H is a Hilbert space.
- (4) $\mathcal{B}(S, \Sigma)$ where Σ is the σ -algebra of all subsets of S .

All these and further can be found in any book on C^* algebras, especially in [27].

Chapter 2

PELCZYŃSKI'S V PROPERTY

2.1 Unconditional and Weakly Unconditional Series in Banach Spaces

Before giving the basics of unconditional and weakly unconditional series in Banach spaces, let us first give some background definitions and results about series.

Definition 2.1.1. Let X be a normed space. An infinite sum $\sum_{k=1}^{\infty} x_k$ is said to be a series in X . If the sequence $S_n = \sum_{k=1}^n x_k$ of the partial sums of the series converges then we say that the series $\sum_{k=1}^{\infty} x_k$ converges and write $\sum_{k=1}^{\infty} x_k = \lim_{n \rightarrow \infty} S_n$. Otherwise we say that the series $\sum_{k=1}^{\infty} x_k$ diverges.

A subseries of a series $\sum_{n=1}^{\infty} x_n$ is a series $\sum_{i=1}^{\infty} x_{n_i}$ obtained from a subsequence $\{x_{n_i}\}_{i \in \mathbb{N}}$ of the sequence $\{x_n\}_{n \in \mathbb{N}}$. $\square$

Proposition 2.1.2. Let X and Y be two normed spaces. Let $\sum_{k=1}^{\infty} x_k$ be a convergent series in X . If $T : X \rightarrow Y$ is a continuous linear operator then $\sum_{k=1}^{\infty} T(x_k)$ also converges and

$$\sum_{k=1}^{\infty} T(x_k) = T\left(\sum_{k=1}^{\infty} x_k\right).$$

Proof. Assume that $T : X \rightarrow Y$ is a continuous linear operator. Then

$$\sum_{k=1}^n T(x_k) = T\left(\sum_{k=1}^n x_k\right) \rightarrow T\left(\sum_{k=1}^{\infty} x_k\right) \text{ as } n \rightarrow \infty$$

so the series converges and by the uniqueness of the limit

$$\sum_{k=1}^{\infty} T(x_k) = T\left(\sum_{k=1}^{\infty} x_k\right).$$

$\square$

Definition 2.1.3. Let $\sum_{k=1}^{\infty} x_k$ be a series in a normed space X .

- (a) $\sum_{k=1}^{\infty} x_k$ is said to be absolutely convergent if $\sum_{k=1}^{\infty} \|x_k\| < \infty$.
- (b) $\sum_{k=1}^{\infty} x_k$ is said to be conditionally convergent if it converges but not absolutely.
- (c) A series $\sum_{k=1}^{\infty} x_k$ is said to be unconditionally convergent if it converges for any rearrangement of its terms, i.e., for any bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$ the series $\sum_{k=1}^{\infty} x_{\pi(k)}$ converges.

□

We recall the Riemann's theorem:

Theorem 2.1.4. (*Riemann's Theorem*)

Let $\sum_{k=1}^{\infty} x_k$ be a series in $\mathbb{R}$. Then $\sum_{k=1}^{\infty} x_k$ is absolutely convergent if and only if it is unconditionally convergent.

□

Theorem 2.1.5. Let X be a Banach space. Then any absolutely convergent series $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent.

Proof. Let $\sum_{k=1}^{\infty} x_k$ be an absolutely convergent series in X . Then $\sum_{k=1}^{\infty} \|x_k\| < \infty$, say $\sum_{k=1}^{\infty} \|x_k\| = M$, for some constant M . Let $\pi : \mathbb{N} \rightarrow \mathbb{N}$ be any bijection. Then for any positive integer n , $\sum_{k=1}^n \|x_{\pi(k)}\| \leq M$. So as the sequence of the partial sums of this series is bounded and the terms of the series is positive, $\sum_{k=1}^{\infty} \|x_{\pi(k)}\|$ converges. Hence $\sum_{k=1}^n x_{\pi(k)}$ is convergent. Therefore as π was arbitrary, $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent.

□

The converse which is the analogue of Riemann's result in infinite dimensional Banach spaces is not true as the following famous theorem implies that in every infinite dimensional Banach space there exists an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ such that $\sum_{n=1}^{\infty} \|x_n\|$ diverges:

Theorem 2.1.6. (*Dvoretzky-Rogers Theorem*)

If each unconditionally convergent series in a Banach space X is absolutely convergent, then X is finite dimensional.

Now we are ready to study unconditional and weakly unconditional series.

Proposition 2.1.7. *Let $\sum_{k=1}^{\infty} x_k$ be an unconditionally convergent series in a Banach space X . If Y is a Banach space then for any continuous linear operator $T : X \rightarrow Y$, $\sum_{k=1}^{\infty} T(x_k)$ is unconditionally convergent.*

Proof. Let $\pi : \mathbb{N} \rightarrow \mathbb{N}$ be any bijection, then as $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent, $\sum_{k=1}^{\infty} x_{\pi(k)}$ converges. Then $T(\sum_{k=1}^{\infty} x_{\pi(k)}) = \sum_{k=1}^{\infty} T(x_{\pi(k)})$ also converges by 2.1.2 which means that $\sum_{k=1}^{\infty} T(x_k)$ is unconditionally convergent. $\square$

Definition 2.1.8. *A series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is said to be subseries convergent if any subseries of it converges.* $\square$

Theorem 2.1.9. *A series $\sum_{n=1}^{\infty} x_n$ in a Banach space is unconditionally convergent if and only if $\sum_{n=1}^{\infty} x_n$ is subseries convergent.*

Proof. Let $\sum_{n=1}^{\infty} x_n$ be an unconditionally convergent series in X and assume for a contradiction that there exist a divergent subseries $\sum_{k=1}^{\infty} x_{n_k}$ of $\sum_{n=1}^{\infty} x_n$. Then by Cauchy criterion, there is an $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ there exist $p_k, q_k > k$ satisfying

$$\left\| \sum_{i=p_k}^{q_k} x_{n_i} \right\| \geq \varepsilon$$

We may assume that $p_1 \leq q_1 < p_2 \leq q_2 < p_3 \leq q_3 < \dots$

We will denote the sequence of positive integers which do not appear in any of the $\{n_j : p_k < j < q_k\}$ by $\{y_k\}_{k \in \mathbb{N}}$. Let π be the following rearrangement of positive integers:

$$n_{p_1}, \dots, n_{q_1}, y_1, n_{p_2}, \dots, n_{q_2}, y_2, n_{p_3}, \dots, n_{q_3}, y_3, \dots$$

So again by Cauchy condition, $\sum_{n=1}^{\infty} x_{\pi(n)}$ diverges. But then the series $\sum_{n=1}^{\infty} x_n$ is not unconditionally convergent which is a contradiction.

For the converse suppose that $\sum_{n=1}^{\infty} x_n$ is not unconditionally convergent. So we can assume that there exists a bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{n=1}^{\infty} x_{\pi(n)}$ diverges. By Cauchy condition, there exists an $\varepsilon > 0$ and two sequences $\{r_n\}_{n \in \mathbb{N}}, \{s_n\}_{n \in \mathbb{N}}$ of

positive integers satisfying $r_1 \leq s_1 < r_2 \leq s_2 < r_3 \leq s_3 < \dots$ such that for any $k \in \mathbb{N}$,

$$\left\| \sum_{n=r_k}^{s_k} x_{\pi(n)} \right\| \geq \varepsilon$$

We may also assume that for each positive integer k ,

$$\max_{r_k \leq n \leq s_k} \{\pi(n)\} < \min_{r_{k+1} \leq n \leq s_{k+1}} \{\pi(n)\}$$

Then sort the elements of $\pi(r_1), \dots, \pi(s_1), \dots, \pi(r_2), \dots, \pi(s_2), \dots$ in the ascending order and rename them as $\{n_j\}_{j \in \mathbb{N}}$. Then the subseries $\sum_{j=1}^{\infty} x_{n_j}$ does not converge. $\square$

Theorem 2.1.10. *Let $\sum_{k=1}^{\infty} x_k$ be an unconditionally convergent series in a Banach space X . Then all its rearrangements have the same sum, that is, for any bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$,*

$$\sum_{k=1}^{\infty} x_{\pi(k)} = \sum_{k=1}^{\infty} x_k$$

Proof. Let $\sum_{k=1}^{\infty} x_k = s$. Assume for a contradiction that there exists a bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{k=1}^{\infty} x_{\pi(k)} = r$ where $r \neq s$. We know that X^* is a separating family of functionals on X . So there exists an $x^* \in X^*$ such that $\langle s, x^* \rangle \neq \langle r, x^* \rangle$.

Then

$$\sum_{k=1}^{\infty} \langle x_k, x^* \rangle = \langle \sum_{k=1}^{\infty} x_k, x^* \rangle = \langle s, x^* \rangle$$

Similarly,

$$\sum_{k=1}^{\infty} \langle x_{\pi(k)}, x^* \rangle = \langle \sum_{k=1}^{\infty} x_{\pi(k)}, x^* \rangle = \langle r, x^* \rangle$$

So the series $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ does not converge absolutely. So By Riemann's theorem 2.1.4, we can find a bijection $\tau : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{k=1}^{\infty} \langle x_{\tau(k)}, x^* \rangle$ diverges. Hence $\sum_{k=1}^{\infty} x_{\tau(k)}$ diverges by continuity of x^* . This leads to a contradiction as the series $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent. Therefore all rearrangements have the same sum with the original series. $\square$

Theorem 2.1.11. *A series $\sum_{n=1}^{\infty} x_n$ in a Banach space is unconditionally convergent if and only if the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges for any choice of sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of ∓ 1 's.*

Proof. Assume that $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent. Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be any sequence of ∓ 1 's. Define $A = \{i \in \mathbb{N} : \alpha_i = 1\}$ and $B = \{i \in \mathbb{N} : \alpha_i = -1\}$. Rename A by $\{n_i\}_{i \in \mathbb{N}}$ and B by $\{m_i\}_{i \in \mathbb{N}}$. Then the subseries $\sum_{k=1}^{\infty} x_{n_k}$ and $\sum_{k=1}^{\infty} x_{m_k}$ converges by 2.1.9. Observe that $A \cup B = \mathbb{N}$. Therefore as

$$\sum_{n=1}^{\infty} \alpha_n x_n = \sum_{k=1}^{\infty} x_{n_k} - \sum_{k=1}^{\infty} x_{m_k}$$

$\sum_{n=1}^{\infty} \alpha_n x_n$ also converges for any choice of $\{\alpha_n\}_{n \in \mathbb{N}}$.

For the converse assume that $\sum_{n=1}^{\infty} \alpha_n x_n$ converges for any choice of sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of ∓ 1 's. Let $n_1 < n_2 < n_3 < \dots$ be a sequence of positive integers. Let $A = \{n_1, n_2, \dots\}$ and $B = \mathbb{N} \setminus A$. Define two sequences $\{\alpha_i\}_{i \in \mathbb{N}}$ and $\{\beta_i\}_{i \in \mathbb{N}}$ as follows:

$$\alpha_i = 1, \quad \forall i \in \mathbb{N} \quad \text{and} \quad \beta_i = \begin{cases} 1, & \text{if } i \in A; \\ -1, & \text{if } i \in B. \end{cases}$$

Then the series $\sum_{n=1}^{\infty} \alpha_n x_n$ and $\sum_{n=1}^{\infty} \beta_n x_n$ converges by the assumption. So as

$$\sum_{n=1}^{\infty} \frac{1}{2} (\alpha_n x_n + \beta_n x_n) = \frac{1}{2} \left[\sum_{n \in A} \underbrace{(\alpha_n x_n + \beta_n x_n)}_{2x_n} + \sum_{n \in B} \underbrace{(\alpha_n x_n + \beta_n x_n)}_0 \right] = \sum_{n \in A} x_n = \sum_{i=1}^{\infty} x_{n_i}$$

the subseries $\sum_{i=1}^{\infty} x_{n_i}$ converges. As $n_1 < n_2 < \dots < \infty$ was arbitrary, by 2.1.9, the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally. $\square$

The following two lemmas will be necessary in order to find another characterization for the unconditional convergence of a series in a Banach space.

Lemma 2.1.12. *Let Y be a closed subspace of ℓ_{∞} containing all sequences taking values from $\{0, 1\}$. Then $Y = \ell_{\infty}$.*

Proof. To prove $Y = \ell_{\infty}$ it is enough to show that $S_{\ell_{\infty}}^+ \subset Y$. So let $\{\alpha_n\}_{n \in \mathbb{N}} \in S_{\ell_{\infty}}^+$. Then for each positive integer n , $\alpha_n \in [0, 1]$. So let $\alpha_n = 0.\alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\dots$ be the binary expansion of α_n i.e., $\alpha_n = \sum_{j=1}^{\infty} 2^{-j} \alpha_{n,j}$, for all n . But as $\{\alpha_{n,j}\}_{j \in \mathbb{N}}$ is a sequence in Y for any n and Y is a subspace, $\{\alpha_n\}_{n \in \mathbb{N}}$ being a linear combination of elements of Y is in Y . Therefore $S_{\ell_{\infty}}^+ \subset Y$. $\square$

Lemma 2.1.13. *Let $\sum_{k=1}^{\infty} x_k$ be a series in a normed space X . If for every $x^* \in X^*$, $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ is absolutely convergent then there exists a constant $M > 0$ such that for any $\{\alpha_k\}_{k \in \mathbb{N}} \in \ell_{\infty}$,*

$$\sup_n \left\| \sum_{k=1}^n \alpha_k x_k \right\| \leq M \|\{\alpha_k\}\|_{\infty}.$$

Proof. Assume that $\sum_{k=1}^{\infty} |\langle x_k, x^* \rangle| < \infty$ for any $x^* \in X^*$. Then

$\{\langle x_k, x^* \rangle\}_{k \in \mathbb{N}} \in \ell_1$ for any $x^* \in X^*$. So we can define an operator $T : X^* \rightarrow \ell_1$ by

$$T(x^*) = \{\langle x_k, x^* \rangle\}_{k \in \mathbb{N}}.$$

Obviously T is linear and, by the closed graph theorem 1.1.4, T is bounded.

Now let $\{\alpha_k\}_{k \in \mathbb{N}} \in \ell_{\infty}$. Then for any $x^* \in B_{X^*}$ and any $n \in \mathbb{N}$,

$$\left| \left\langle \sum_{k=1}^n \alpha_k x_k, x^* \right\rangle \right| = \left| \sum_{k=1}^n \alpha_k \langle x_k, x^* \rangle \right| \leq \sup_k |\alpha_k| \|T(x^*)\|_1 \leq \|\{\alpha_k\}\|_{\infty} \|T\|$$

Then,

$$\left\| \sum_{k=1}^n \alpha_k x_k \right\| = \sup_{x^* \in B_{X^*}} \left| \left\langle \sum_{k=1}^n \alpha_k x_k, x^* \right\rangle \right| \leq \|\{\alpha_k\}\|_{\infty} \|T\|$$

Now let $M = \|T\|$. As n was arbitrary,

$$\sup_n \left\| \sum_{k=1}^n \alpha_k x_k \right\| \leq M \|\{\alpha_k\}\|_{\infty}$$

□

Next theorem gives us a characterization for unconditionally convergent series in Banach spaces.

Theorem 2.1.14. (*Bounded Multiplier Test*)

Let X be a Banach space. The series $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent if and only if $\sum_{k=1}^{\infty} \alpha_k x_k$ converges for any $\{\alpha_k\}_{k \in \mathbb{N}} \in \ell_{\infty}$.

Proof. Let $\{\alpha_k\}_{k \in \mathbb{N}} \in \ell_{\infty}$ and assume that $\sum_{k=1}^{\infty} \alpha_k x_k$ converges. Let $\{\beta_k\}_{k \in \mathbb{N}}$ be any sequence having only ∓ 1 as elements. Then $\{\beta_k\}_{k \in \mathbb{N}} \in \ell_{\infty}$ so $\sum_{k=1}^{\infty} \beta_k x_k$ converges

by the assumption. Hence by 2.1.11, $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent.

Now assume that $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent and let

$$Y = \{ \{ \alpha_k \}_{k \in \mathbb{N}} \in \ell_{\infty} : \sum_{k=1}^{\infty} \alpha_k x_k \text{ converges} \}$$

We need to show that $Y = \ell_{\infty}$ in order to prove that $\sum_{k=1}^{\infty} \alpha_k x_k$ converges for any $\{ \alpha_k \}_{k \in \mathbb{N}} \in \ell_{\infty}$.

Indeed, as $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent, by 2.1.9, any subseries of it also converges. So every sequence whose terms come from $\{0, 1\}$ is an element of Y . By 2.1.12, it is enough to show that Y is a closed subspace of ℓ_{∞} . Y is obviously a subspace of ℓ_{∞} . So let $\{ \alpha_k^j \}_{j \in \mathbb{N}}$ be a sequence in Y where $\alpha_k^j = (\alpha_1^j, \alpha_2^j, \dots)$ with

$$\lim_{j \rightarrow \infty} \alpha_k^j = \alpha_k$$

where $\alpha_k = (\alpha_1, \alpha_2, \dots) \in \ell_{\infty}$. We need to show that $\sum_{k=1}^{\infty} \alpha_k x_k$ converges. By 2.1.7, for any $x^* \in X^*$, $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ is unconditionally convergent. Since for numerical series unconditional convergence implies absolute convergence, $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ is absolutely convergent. 2.1.13 implies that there exists a constant M such that for every $\{ \alpha_k \}_{k \in \mathbb{N}} \in \ell_{\infty}$,

$$\sup_n \left\| \sum_{k=1}^n \alpha_k x_k \right\| \leq M \| \{ \alpha_k \} \|_{\infty}$$

Let $\varepsilon > 0$. Then we can find a positive integer J satisfying

$$\| \{ \alpha_k^J \} - \{ \alpha_k \} \|_{\infty} < \frac{\varepsilon}{1 + M}$$

As we fixed J , $\{ \alpha_k^J \}_{k \in \mathbb{N}}$ is in Y . Therefore, $\sum_{k=1}^n \alpha_k^J x_k$ converges so it is Cauchy.

Thus for some positive integer N and for any $q, p \geq N$ and $q \geq p$

$$\left\| \sum_{k=p+1}^q \alpha_k^J x_k \right\| < \frac{\varepsilon}{1 + M}$$

All the above observations leads us to

$$\left\| \sum_{k=p+1}^q \alpha_k x_k \right\| \leq \underbrace{\left\| \sum_{k=p+1}^q \alpha_k^J x_k \right\|}_{< \frac{\varepsilon}{1+M}} + \underbrace{\left\| \sum_{k=p+1}^q (\alpha_k^J - \alpha_k) x_k \right\|}_{\leq M \| \{ \alpha_k^J \} - \{ \alpha_k \} \|_{\infty}} < \frac{\varepsilon}{1 + M} + M \underbrace{\| \{ \alpha_k^J \} - \{ \alpha_k \} \|_{\infty}}_{< \frac{\varepsilon}{1+M}} < \varepsilon$$

Hence the sequence of the partial sums $\sum_{k=1}^n \alpha_k x_k$ is a Cauchy sequence in a Banach space. Thus $\sum_{k=1}^{\infty} \alpha_k x_k$ converges and so Y is closed in ℓ_{∞} . $\square$

Definition 2.1.15. A series $\sum_{k=1}^{\infty} x_k$ is weakly unconditionally Cauchy (or wuC) if for any bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$, the series $\sum_{k=1}^{\infty} x_{\pi(k)}$ is weakly Cauchy. $\square$

From the definition we can reach to an alternative definition that is $\sum_{k=1}^{\infty} x_k$ is wuC if and only if $\sum_{k=1}^{\infty} |\langle x_k, x^* \rangle| < \infty$ for every $x^* \in X^*$ (or $\sum_{k=1}^{\infty} x_k$ is weakly absolutely convergent.) Indeed, $\sum_{k=1}^{\infty} x_k$ is wuC if and only if for any bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$, $\sum_{k=1}^{\infty} \langle x_{\pi(k)}, x^* \rangle$ converges for any $x^* \in X^*$. So it is necessary and sufficient that for any π and for every $x^* \in X^*$ the numerical series $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ is unconditionally convergent. But this holds if and only if for any $x^* \in X^*$, $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ is absolutely convergent.

Theorem 2.1.16. Let $\sum_{k=1}^{\infty} x_k$ be a series in a Banach space X . Then the following are equivalent:

- (1) $\sum_{k=1}^{\infty} x_k$ is wuC .
- (2) For any $\{\alpha_k\}_{k \in \mathbb{N}}$ in c_0 , $\sum_{k=1}^{\infty} \alpha_k x_k$ converges.
- (3) There exists a constant M such that for any finite subset F of $\mathbb{N}$ and any sequence $\{\alpha_k\}_{k \in \mathbb{N}}$ of signs $\mp$,

$$\left\| \sum_{k \in F} \alpha_k x_k \right\| \leq M$$

Proof. (1) $\implies$ (2): Assume that $\sum_{k=1}^{\infty} x_k$ is wuC . Then for every $x^* \in X^*$, $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ is absolutely convergent. Then by 2.1.13, when $\{\alpha_k\}_{k \in \mathbb{N}} \in c_0 \subset \ell_{\infty}$, for some scalar M ,

$$\sup_n \left\| \sum_{k=1}^n \alpha_k x_k \right\| \leq M \sup_k |\alpha_k|.$$

Thus,

$$\left\| \sum_{k=m+1}^n \alpha_k x_k \right\| \leq M \max_{m+1 \leq k \leq n} |\alpha_k| \rightarrow 0$$

as $m, n \rightarrow \infty$. Therefore $\sum_{k=1}^{\infty} \alpha_k x_k$ is Cauchy in the Banach space X . Hence converges.

(2) $\implies$ (3): Define $T : c_0 \rightarrow X$ by

$$T(\{\alpha_k\}) = \sum_{k=1}^{\infty} \alpha_k x_k.$$

By (2), T is well-defined. T is obviously linear and by the closed graph theorem 1.1.4, T is bounded. As T maps bounded sets to bounded sets, for some constant M and for every $\{\beta_k\}_{k \in \mathbb{N}} \in B_{c_0}$, $\|\sum_{k=1}^{\infty} \beta_k x_k\| \leq M$. Let F be a finite subset of $\mathbb{N}$. Then the sequences having ∓ 1 as k^{th} elements if $k \in F$ and 0 otherwise, are elements of B_{c_0} .

Thus,

$$\left\| \sum_{k \in F} \mp x_k \right\| \leq M.$$

(3) $\implies$ (1): Let $x^* \in B_{X^*}$. Define a sequence of scalars $\{\alpha_k\}_{k \in \mathbb{N}}$ by

$$\alpha_k = \begin{cases} \text{sign}(\text{Re}(\langle x_k, x^* \rangle)), & \text{if } \text{Re} \langle x_k, x^* \rangle \neq 0; \\ 1, & \text{if } \text{Re} \langle x_k, x^* \rangle = 0. \end{cases}$$

Hence $\{\alpha_k\}_{k \in \mathbb{N}}$ is a sequence of signs $\mp$. So by the assumption in (3), we can find a constant $M > 0$ such that

$$\begin{aligned} \sum_{k \in F} |\text{Re}(\langle x_k, x^* \rangle)| &= \sum_{k \in F} \alpha_k \text{Re} \langle x_k, x^* \rangle = \text{Re} \left(\sum_{k \in F} \alpha_k x_k, x^* \right) \\ &\leq \left| \sum_{k \in F} \alpha_k x_k, x^* \right| = \left| \langle \sum_{k \in F} \alpha_k x_k, x^* \rangle \right| \leq \left\| \sum_{k \in F} \alpha_k x_k \right\| \|x^*\| \leq M \end{aligned}$$

holds for any finite subset F of $\mathbb{N}$. In a similar way,

$$\sum_{k \in F} |\text{Im}(\langle x_k, x^* \rangle)| \leq N$$

for some $N > 0$. So,

$$\sum_{k=1}^{\infty} |\langle x_k, x^* \rangle| \leq \sum_{k \in F} |\text{Re}(\langle x_k, x^* \rangle)| + \sum_{k \in F} |\text{Im}(\langle x_k, x^* \rangle)| < \infty$$

Therefore, $\sum_{k=1}^{\infty} x_k$ is wuC. □

In a Banach space if $\sum_{k=1}^{\infty} x_k$ is wuC then any subseries $\sum_{j=1}^{\infty} x_{k_j}$ is weakly Cauchy and vice versa. Indeed, $\sum_{k=1}^{\infty} x_k$ is wuC if and only if $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ is absolutely convergent for any $x^* \in X^*$. This is true if and only if for any subseries $\sum_{j=1}^{\infty} x_{k_j}$, $\sum_{j=1}^{\infty} \langle x_{k_j}, x^* \rangle$ converges for each $x^* \in X^*$. Then it is necessary and sufficient that each subseries $\sum_{j=1}^{\infty} x_{k_j}$ is weakly Cauchy.

In our proceeding work, we need the following facts about bases in Banach spaces.

Definition 2.1.17. Let X be a Banach space. Two bases $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ are equivalent if for any sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of scalars, the series $\sum_{k=1}^{\infty} \alpha_k x_k$ converges if and only if $\sum_{k=1}^{\infty} \alpha_k y_k$ converges. $\square$

If $\{x_n\}_{n \in \mathbb{N}}$ is a basis for the Banach space X , define $x_n^* : X \rightarrow \mathbb{F}$ by

$$\langle x, x_n^* \rangle = \alpha_n$$

where $\sum_{n=1}^{\infty} \alpha_n x_n$ is the unique expansion for x . Obviously, $x = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle x_n$ and x_n^* 's are continuous linear functionals on X . These x_n^* 's are said to be the coefficient functionals on X and the basis $\{x_n\}_{n \in \mathbb{N}}$ uniquely determines them.

Theorem 2.1.18. Let X and Y be Banach spaces. The bases $\{x_n\}_{n \in \mathbb{N}}$, $\{y_n\}_{n \in \mathbb{N}}$ for X and Y , respectively are equivalent if and only if there exists an isomorphism from X to Y carrying each x_n to y_n .

Proof. Let $S : X \rightarrow Y$ be an isomorphism such that $S(x_n) = y_n$ for each n . As S is continuous, convergence of $\sum_{n=1}^{\infty} \alpha_n x_n$ for any sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of scalars implies the convergence of $\sum_{n=1}^{\infty} \alpha_n S(x_n) = \sum_{n=1}^{\infty} \alpha_n y_n$. Similarly as S^{-1} is continuous, convergence of $\sum_{n=1}^{\infty} \alpha_n y_n$ for any sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of scalars implies the convergence of $\sum_{n=1}^{\infty} \alpha_n S^{-1}(y_n) = \sum_{n=1}^{\infty} \alpha_n x_n$. So $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ are equivalent.

Now let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be two equivalent bases and let $\{x_n^*\}_{n \in \mathbb{N}}$ and $\{y_n^*\}_{n \in \mathbb{N}}$ be the coefficient functionals of them respectively. Let $x \in X$. Then there exists a unique expansion corresponding to x such that

$$x = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle x_n.$$

Equivalence of the bases implies that, $\sum_{n=1}^{\infty} \langle x, x_n^* \rangle x_n$ converges in X if and only if $\sum_{n=1}^{\infty} \langle x, x_n^* \rangle y_n$ converges in Y . So define $S : X \rightarrow Y$ by

$$S(x) = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle y_n.$$

S is clearly linear. S is well-defined as the expansion of any $x \in X$ is unique. Now assume that $S(x) = 0$. Then

$$\sum_{n=1}^{\infty} 0y_n = 0 = S(x) = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle y_n.$$

Hence $\langle x, x_n^* \rangle = 0$ for each n as $\{y_n\}_{n \in \mathbb{N}}$ is a basis. So $x = 0$ which implies that S is one-to-one. To check onto, let $y \in Y$. Then $y = \sum_{n=1}^{\infty} \langle y, y_n^* \rangle y_n$. As above, $\sum_{n=1}^{\infty} \langle y, y_n^* \rangle x_n$ converges in X , say $x = \sum_{n=1}^{\infty} \langle y, y_n^* \rangle x_n$. As $x = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle x_n$ also, $\langle x, x_n^* \rangle = \langle y, y_n^* \rangle$ for each n as $\{x_n\}_{n \in \mathbb{N}}$ is a basis. Therefore $S(x) = y$ implying the onto of S . Lastly we need to check that S is continuous. Define a sequence of mappings $T_k : X \rightarrow Y$ by

$$T_k(x) = \sum_{n=1}^k \langle x, x_n^* \rangle y_n.$$

As each x_n^* is continuous,

$$\|T_k(x)\| = \left\| \sum_{n=1}^k \langle x, x_n^* \rangle y_n \right\| \leq \sum_{n=1}^k |\langle x, x_n^* \rangle| \|y_n\| \leq \|x\| \sum_{n=1}^k \|x_n^*\| \|y_n\|.$$

So as $\lim_{k \rightarrow \infty} T_k(x) = S(x)$, for each $x \in X$,

$$\|S(x)\| \leq \sup_k \|T_k(x)\| < \infty.$$

By the Uniform Boundedness principle 1.1.2, $\sup_k \|T_k\| < \infty$. So as $\|S\| < \sup_k \|T_k\|$, S is continuous. Therefore S is an isomorphism and obviously $S(x_n) = y_n$ for each n . □

Definition 2.1.19. Let X be a Banach space. A basis $\{x_n\}_{n \in \mathbb{N}}$ for X is said to be an unconditional basis if for every $x \in X$, there exists a unique representation of the form

$$x = \sum_{n=1}^{\infty} \alpha_n x_n$$

where the sum is unconditionally convergent. □

Example 2.1.20. For c_0 or ℓ_p , $1 \leq p < \infty$, the sequence $\{e_n\}_{n \in \mathbb{N}}$ where

$$e_n = \delta_{nm} = \begin{cases} 1, & n = m; \\ 0, & n \neq m. \end{cases}$$

is an unconditional basis called the standard unit vector basis. $\{e_n\}_{n \in \mathbb{N}}$ is not a basis for ℓ_∞ . Indeed as ℓ_∞ is not separable it has no Schauder basis. $\square$

Definition 2.1.21. A sequence $\{x_n\}_{n \in \mathbb{N}}$ of a Banach space X is said to be a basic sequence if it is a basis of $\overline{\text{span}}(\{x_n\})$ of X . $\square$

Theorem 2.1.22. (Bessaga-Pelczyński Selection Principle)

Let X be a Banach space and $\{x_n\}_{n \in \mathbb{N}}$ be a weakly null, normalized sequence in X . Then $\{x_n\}_{n \in \mathbb{N}}$ has a subsequence which is a basic sequence in itself.

Proof. [7, p.42] $\square$

Theorem 2.1.23. Let X be a closed subspace of a Banach space with an unconditional basis. Then the following are true:

- (1) X is weakly sequentially complete if and only if no subspace of X is isomorphic to c_0 .
- (2) X contains no subspace isomorphic to ℓ_1 if and only if X^* is weakly sequentially complete.
- (3) X is reflexive if and only if neither c_0 nor ℓ_1 is embedded in X .

Proof. [20],[14, p.410] $\square$

Theorem 2.1.24. Let X be a Banach space. Then the following are equivalent:

- (1) X contains a complemented copy of ℓ_1 .
- (2) X^* contains an isomorphic copy of c_0 .
- (3) X^* contains a subspace isomorphic to ℓ_∞ .

Proof. [7, p.48],[3] □

Now we proceed with the question of whether unconditional convergence and being wuC are equivalent in Banach spaces or not and if not under which additional conditions it is achieved? We already know that unconditionally convergent series are wuC . Since if $\sum_{k=1}^{\infty} x_k$ is any unconditionally convergent series in a Banach space X , then for any $x^* \in X^*$, $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ is an unconditionally convergent numerical series by 2.1.7. So as a numerical series, it converges absolutely which is exactly the definition of being a wuC series. But the converse may not hold as we can see in the following example.

Example 2.1.25. A simple counterexample is the series formed from the sequence $\{e_n\}_{n \in \mathbb{N}}$ of the standard unit vectors in c_0 . Observe that $\sum_{n=1}^{\infty} e_n$ is wuC . Indeed, for any $x^* \in c_0^* = \ell_1$ where $x^* = \{x_1, x_2, \dots\}$,

$$\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle| = \sum_{n=1}^{\infty} |x_n| = \|x^*\|_1 < \infty$$

However as $\inf_n \|\{e_n\}\|_{c_0} > 0$, $\{e_n\}_{n \in \mathbb{N}}$ does not converge to 0. So the series $\sum_{n=1}^{\infty} e_n$ diverges. Hence $\sum_{n=1}^{\infty} e_n$ cannot be unconditionally convergent. □

Corollary 2.1.26. *Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a Banach space X . If $\inf_n \|x_n\| > 0$ and $\sum_{n=1}^{\infty} x_n$ is wuC (so that $\{x_n\}_{n \in \mathbb{N}}$ is weakly null), then $\{x_n\}_{n \in \mathbb{N}}$ is equivalent to the unit vector basis of c_0 .*

Proof. Suppose that $\inf_n \|x_n\| > 0$ and $\sum_{n=1}^{\infty} x_n$ is wuC . Let $\{e_n\}_{n \in \mathbb{N}}$ be the standard unit vector basis of c_0 and $\{\alpha_n\}_{n \in \mathbb{N}}$ be any sequence of scalars. Now let $\sum_{n=1}^{\infty} \alpha_n x_n$ be convergent, then $\|\alpha_n x_n\| = |\alpha_n| \|x_n\| \rightarrow 0$. By assumption since $\inf_n \|x_n\| > 0$, $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$ i.e., $\{\alpha_n\}_{n \in \mathbb{N}} \in c_0$. Hence as $\sum_{n=1}^{\infty} e_n$ is wuC , $\sum_{n=1}^{\infty} \alpha_n e_n$ also converges by 2.1.16.

On the other hand if $\{\alpha_n\}_{n \in \mathbb{N}} \in c_0$ then by 2.1.16, $\sum_{n=1}^{\infty} \alpha_n x_n$ converges. The result follows by the definition 2.1.17. □

The following theorem is one of the most important theorems that we need for the last chapter which actually claims that the sequence in example 2.1.25 is essentially the only counterexample.

Theorem 2.1.27. (*Bessaga-Pelczyński*)

Let X be a Banach space. Then each wuC series $\sum_{n=1}^{\infty} x_n$ in X is unconditionally convergent if and only if X does not contain an isomorphic copy of c_0 .

Proof. If X contains an isomorphic copy of c_0 , there exists a series $\sum_{n=1}^{\infty} x_n$ in X corresponding to $\sum_{n=1}^{\infty} e_n$. Thus by example 2.1.25 $\sum_{n=1}^{\infty} x_n$ is a wuC series in X which is not unconditionally convergent.

For the converse suppose that $\sum_{k=1}^{\infty} x_k$ is a wuC series which is not unconditionally convergent. Then there exists a subseries $\sum_{j=1}^{\infty} x_{k_j}$ which diverges i.e., there exists two sequences $\{p_n\}_{n \in \mathbb{N}}, \{q_n\}_{n \in \mathbb{N}}$ of positive integers with $p_1 \leq q_1 \leq p_2 \leq q_2 \leq \dots$ satisfying

$$\inf_n \left\| \sum_{j=p_n}^{q_n} x_{k_j} \right\| > 0$$

Now let $y_n = \sum_{j=p_n}^{q_n} x_{k_j}$. Then for each $x^* \in X^*$,

$$| \langle y_n, x^* \rangle | = \left| \langle \sum_{k=p_n}^{q_n} x_{k_j}, x^* \rangle \right| \leq \sum_{j=p_n}^{q_n} | \langle x_{k_j}, x^* \rangle | \leq \sum_{j=1}^{\infty} | \langle x_{k_j}, x^* \rangle | < \infty$$

Hence $\{y_n\}_{n \in \mathbb{N}}$ is weakly null and $\inf_n \|y_n\| > 0$. Define $z_n = \frac{y_n}{\|y_n\|}$ for each positive integer n . Then $\{z_n\}_{n \in \mathbb{N}}$ is a normalized weakly null sequence. Hence by Bessaga-Pelczyński selection principle 2.1.22, $\{z_n\}_{n \in \mathbb{N}}$ has a subsequence which is a basic sequence of itself. Also $\{z_n\}_{n \in \mathbb{N}}$ inherits other properties of $\{y_n\}_{n \in \mathbb{N}}$ i.e., $\inf_n \|z_n\| > 0$ and $\sum_{n=1}^{\infty} | \langle z_n, x^* \rangle | < \infty$ for each $x^* \in X^*$. Therefore by 2.1.26, $\{z_n\}_{n \in \mathbb{N}}$ is equivalent to the unit vector basis of c_0 . Then theorem 2.1.18 implies that there is an isomorphism between c_0 and X carrying each e_n of the standard unit vector basis of c_0 to z_n . Hence an isomorphic copy of c_0 is contained in X . $\square$

The following is a result similar to the bounded multiplier test but for the wuC series.

Theorem 2.1.28. *Let X be a Banach space. The series $\sum_{k=1}^{\infty} x_k$ is wuC if and only if $\sum_{k=1}^{\infty} \alpha_k x_k$ converges for any $\{\alpha_k\}_{k \in \mathbb{N}} \in c_0$.*

Proof. Follows directly from 2.1.16. □

The next theorem of Schur says that it is impossible to find a sequence in ℓ_1 converging weakly but not converging in norm so that weak and norm convergence of sequences are equivalent in ℓ_1 .

Theorem 2.1.29. (*Schur's Lemma*)

In ℓ_1 every weakly Cauchy sequence converges.

Proof. Let $\{x^k\}_{k \in \mathbb{N}}$ be a weakly Cauchy sequence in ℓ_1 where $x^k = (x_1^k, x_2^k, x_3^k, \dots)$ $k = 1, 2, \dots$. Then for any $x^* \in \ell_1^*$,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall k, l \geq N \quad | \langle x^k, x^* \rangle - \langle x^l, x^* \rangle | < \varepsilon \quad (2.1.29.1)$$

Now observe that for any two positive integers k and p , the map $x^k \mapsto x_p^k$ is a bounded linear functional on ℓ_1 since $|x_p^k| \leq \sum_{k=1}^{\infty} |x_p^k| = \|x^k\|_1 < \infty$. Hence by 2.1.29.1, $|x_p^k - x_p^l| < \varepsilon$ where $k, l \geq N$ for any positive integer p . Therefore for any p , $\{x_p^k\}_{k \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$ hence converges to say x_p as $k \rightarrow \infty$.

Claim: $x = \{x_p\}_{p \in \mathbb{N}} \in \ell_1$ and $\{x^k\}_{k \in \mathbb{N}}$ converges to $\{x_p\}_{p \in \mathbb{N}}$ i.e., $\|x^k - x\|_1 \rightarrow 0$ as $k \rightarrow \infty$.

Indeed as $x^k \in \ell_1$ for any k , there exists a constant $M > 0$ such that

$$\sum_{p=1}^m |x_p^k| \leq \sum_{p=1}^{\infty} |x_p^k| = \|x^k\|_1 \leq M$$

Letting $k \rightarrow \infty$ we get, $\sum_{p=1}^m |x_p| \leq M$ for all $m \in \mathbb{N}$. This time by letting $m \rightarrow \infty$, we get $\sum_{p=1}^{\infty} |x_p| \leq M$ which means $\{x_p\}_{p \in \mathbb{N}} \in \ell_1$.

Now assume for a contradiction that $\|x^k - x\|_1$ does not converge to 0. So there exists $\varepsilon > 0$ such that for any positive integer j there exists $k_j > j$ satisfying

$$\|x^{k_j} - x\|_1 \geq \varepsilon$$

We may assume that $k_1 < k_2 < \dots$

Define $y^j = (y_1^j, y_2^j, \dots)$ where $y_n^j = x_n^{k_j} - x_n$ for each positive integers j and n . So by 2.1.29.1, as for any $j, r > N$,

$$| \langle y^j - y^r, x^* \rangle | = | \langle x^{k_j}, x^* \rangle - \langle x^{k_r}, x^* \rangle | < \varepsilon$$

Thus, the sequence $\{y^j\}_{j \in \mathbb{N}}$ is weakly Cauchy in ℓ_1 with $\|y^j\|_1 = \sum_{n=1}^{\infty} |y_n^j| \geq \varepsilon$ for each j . Also $y_n^j \rightarrow 0$ as $j \rightarrow \infty$ for each positive integer n because for any n , $\lim_{k \rightarrow \infty} x_n^k \rightarrow x_n$.

Now we may assume that after passing to an appropriate subsequence there exists a sequence satisfying $0 = m_1 < m_2 < \dots$ such that for every positive integer j ,

$$\sum_{n=m_j+1}^{m_{j+1}} |y_n^j| > \frac{3}{4} \|y^j\|_1$$

so that

$$\|y^j\|_1 = \sum_{n=m_j+1}^{m_{j+1}} |y_n^j| + \sum_{\substack{n < m_j+1, \\ n > m_{j+1}}} |y_n^j| > \frac{3}{4} \|y^j\|_1 + \sum_{\substack{n < m_j+1, \\ n > m_{j+1}}} |y_n^j|$$

and hence

$$\sum_{\substack{n < m_j+1, \\ n > m_{j+1}}} |y_n^j| < \frac{1}{4} \|y^j\|_1$$

Now let $\alpha_n = (-1)^j \frac{|y_n^j|}{y_n^j}$ for $m_j + 1 < n < m_{j+1}$ so that $|\alpha_n| = 1$.

Then $\alpha = \{\alpha_n\}_{n \in \mathbb{N}} \in \ell_\infty = (\ell_1)^*$. If j is even,

$$\begin{aligned} \operatorname{Re}(\langle y^j, \alpha \rangle) &= \operatorname{Re} \left(\sum_{n=1}^{\infty} \alpha_n y_n^j \right) = \operatorname{Re} \left(\sum_{n=m_j+1}^{m_{j+1}} \alpha_n y_n^j \right) + \operatorname{Re} \left(\sum_{n < m_j+1, n > m_{j+1}} \alpha_n y_n^j \right) \\ &= \sum_{n=m_j+1}^{m_{j+1}} |y_n^j| + \operatorname{Re} \left(\sum_{n < m_j+1, n > m_{j+1}} \alpha_n y_n^j \right) \geq \sum_{n=m_j+1}^{m_{j+1}} |y_n^j| - \sum_{n < m_j+1, n > m_{j+1}} |y_n^j| \\ &\geq \frac{3}{4} \|\{y^j\}\|_1 - \frac{1}{4} \|y^j\|_1 = \frac{1}{2} \|y^j\|_1 \geq \frac{1}{2} \varepsilon \end{aligned}$$

Similarly we get for any odd j ,

$$\operatorname{Re}(\langle y^j, \{\alpha_n\} \rangle) < -\frac{1}{2}\varepsilon$$

which is a contradiction as $\{y^j\}_{j \in \mathbb{N}}$ is weakly Cauchy. Therefore our claim is true. Thus every weakly Cauchy sequence in ℓ_1 actually converges. $\square$

By Schur's lemma, ℓ_1 is weakly sequentially complete. In addition as any weakly convergent sequence is weakly Cauchy, weakly convergent sequences in ℓ_1 are actually norm convergent so that weak and norm convergence of sequences coincide on ℓ_1 .

Remark: The property stated in the theorem that ℓ_1 have is called the Schur property. The spaces c_0 and ℓ_p for $1 < p < \infty$ fail to have the Schur property. Indeed, the sequence of the standard unit vector basis $\{e_n\}_{n \in \mathbb{N}}$ of c_0 is weakly Cauchy but not convergent as $\inf_n \|e_n\| > 0$. The same reasoning holds for ℓ_p .

Definition 2.1.30. A series $\sum_{k=1}^{\infty} x_k$ is said to be weakly subseries convergent if each subseries of $\sum_{k=1}^{\infty} x_k$ is weakly convergent. $\square$

Theorem 2.1.31. (The Orlicz-Pettis Theorem)

Let X be a Banach space. Then a series in X is weakly subseries convergent if and only if it is unconditionally convergent.

Proof. Let X be a Banach space and take any unconditionally convergent series in X . Then by 2.1.9, every subseries of the series is norm convergent. Thus as norm convergence implies weak convergence, each subseries is weakly convergent which is exactly the definition of weak subseries convergence.

Now assume for a contradiction that there exists a weakly subseries convergent series $\sum_{k=1}^{\infty} x_k$ in X which fails to be unconditionally convergent. Then there exists a subseries of $\sum_{k=1}^{\infty} x_k$ which diverges. By the construction in the proof of 2.1.27, we can find a subsequence $\{x_{k_j}\}_{j \in \mathbb{N}}$ of $\{x_k\}_{k \in \mathbb{N}}$ and two sequences $\{p_n\}_{n \in \mathbb{N}}$, $\{q_n\}_{n \in \mathbb{N}}$ of positive integers with $p_1 \leq q_1 \leq p_2 \leq q_2 \leq \dots$ such that by letting $y_n = \sum_{j=p_n}^{q_n} x_{k_j}$ and normalizing it we get a subsequence of $\{x_k\}_{k \in \mathbb{N}}$ which is equivalent to the unit vector basis of c_0 . For simplicity we continue to denote that subsequence by $\{y_n\}_{n \in \mathbb{N}}$.

By 2.1.17, there exists an isomorphism T from the unit vector basis $\{e_n\}_{n \in \mathbb{N}}$ of c_0 to $\{y_n\}_{n \in \mathbb{N}}$ such that $T(e_n) = y_n$ for any n . Also being a subseries of the weakly subseries convergent $\sum_{k=1}^{\infty} x_k$, $\sum_{n=1}^{\infty} y_n$ is weakly convergent. So weak continuity of T^{-1} because of 1.1.25 and the equality $T(\sum_{n=1}^{\infty} e_n) = \sum_{n=1}^{\infty} T(e_n) = \sum_{n=1}^{\infty} y_n$ implies that $\sum_{n=1}^{\infty} e_n$ is weakly convergent which is a contradiction. Thus any weakly subseries convergent series is unconditionally convergent. $\square$

The theorem stated differently says that in any Banach space a series is weakly subseries convergent if and only if it is subseries convergent. As a consequence of the theorem, in a Banach space any weakly subseries convergent series is wuC. Actually, Orlicz proved that the theorem holds for any weakly sequentially complete space. Indeed, let X be a weakly sequentially complete space. Assume for a contradiction that there exists a weakly subseries convergent series which is not unconditionally convergent. Then by 2.1.27, X contains an isomorphic copy of c_0 . But c_0 is not weakly sequentially complete which violates the weak sequential completeness of X by 1.1.24.

Example 2.1.32. Let $\{e_n\}_{n \in \mathbb{N}}$ be the standard unit vector basis of c_0 and let $\{x_n\}_{n \in \mathbb{N}}$ be the sequence defined by:

$$x_1 = e_1, \quad x_n = e_n - e_{n-1}, \quad n \geq 2$$

Then the series $\sum_{k=1}^{\infty} x_k$ is wuc (actually weakly converges to 0 but not convergent) but not weakly subseries convergent. Indeed, $S_n = \sum_{k=1}^n x_k = e_n$, so for any bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$, for all $x^* = (x_1, x_2, \dots) \in c_0^* = \ell_1$

$$\lim_{n \rightarrow \infty} \langle S_{\pi(n)}, x^* \rangle = \lim_{n \rightarrow \infty} \langle e_{\pi(n)}, x^* \rangle = \lim_{n \rightarrow \infty} x_{\pi(n)} = 0,$$

meaning that $\sum_{k=1}^{\infty} x_k$ is wuC. But as $\inf \|x_k\|_{c_0} = \|e_k - e_{k-1}\|_{c_0} > 0$, $\sum_{k=1}^{\infty} x_k$ diverges hence not unconditionally convergent. We conclude by Orlicz-Pettis theorem that $\sum_{k=1}^{\infty} x_k$ is not weakly subseries convergent. $\square$

Indeed, in Banach spaces containing an isomorphic copy of c_0 being weakly subseries convergent and being wuC are not the same:

Theorem 2.1.33. *Being weakly unconditional Cauchy and being weakly subseries convergent in a Banach space X are equivalent if and only if the space X does not have an isomorphic copy of c_0 .*

Proof. Follows readily from the Orlicz-Pettis theorem and the example following it. $\square$

Corollary 2.1.34. *A subset M of a separable Banach space is relatively compact if and only if every weak* convergent sequence of functionals in X^* converges uniformly on M i.e.,*

$$\lim_{n \rightarrow \infty} \sup_{x \in M} | \langle x, x_n^* \rangle | = 0 \text{ whenever } \lim_{n \rightarrow \infty} \langle x, x_n^* \rangle = 0 \text{ for any } x \in X.$$

Proof. [22, Corollary 3.2] $\square$

Theorem 2.1.35. *If a series $\sum_{n=1}^{\infty} x_n$ is weakly subseries convergent in a Banach space X then for any sequence $\{x_m^*\}_{m \in \mathbb{N}}$ in X^* satisfying $\lim_{m \rightarrow \infty} \langle x, x_m^* \rangle = 0$ for each $x \in \overline{\text{span}}(\{x_n\})$,*

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} | \langle x_n, x_m^* \rangle | = 0$$

Proof. Let $\sum_{n=1}^{\infty} x_n$ be weakly subseries convergent and $\{x_m^*\}_{m \in \mathbb{N}}$ be a sequence in X^* satisfying the given property. Then $\{\langle x_n, x_m^* \rangle\}_{n \in \mathbb{N}} \in \ell_1$ for any m . We want to show that

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} | \langle x_n, x_m^* \rangle | = \lim_{m \rightarrow \infty} \| \{ \langle x_n, x_m^* \rangle \}_n \|_1 = 0$$

In view of Schur's theorem 2.1.29 as weak convergence and norm convergence coincide on ℓ_1 , it is enough to show that

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} \alpha_n \langle x_n, x_m^* \rangle = 0$$

for any $\{\alpha_n\}_{n \in \mathbb{N}}$ where $\alpha_n \in \{-1, 0, 1\}$ for each positive integer n . So take any such $\{\alpha_n\}_{n \in \mathbb{N}}$. Form two sets A and B containing the nonnegative and negative terms of the sequence $\{\alpha_n\}_{n \in \mathbb{N}}$, respectively i.e.,

$$A = \{n : \alpha_n \geq 0\}, \quad B = \{n : \alpha_n < 0\}$$

As the series $\sum_{n=1}^{\infty} x_n$ is weakly subseries convergent, the two subseries $\sum_{n \in A} x_n$ and $\sum_{n \in B} x_n$ weakly converges to say $x_A, x_B \in X$, respectively. We claim that both x_A and x_B are in $\overline{\text{span}}(\{x_n\})$. Assume otherwise that $x_A \notin \overline{\text{span}}(\{x_n\})$ or $x_B \notin \overline{\text{span}}(\{x_n\})$. Say $x_A \notin \overline{\text{span}}(\{x_n\})$. Then by 1.1.6, there exists $x^* \in X^*$ such that $\langle x, x^* \rangle = 0$ for all $x \in \overline{\text{span}}(\{x_n\})$ but $\langle x_A, x^* \rangle \neq 0$. This contradicts with the fact that $\langle x_A, x^* \rangle = \sum_{n \in A} \langle x_n, x^* \rangle = 0$. Hence our claim is true, that is, both x_A and x_B are elements of $\overline{\text{span}}(\{x_n\})$. Thus,

$$\sum_{n=1}^{\infty} \alpha_n \langle x_n, x_m^* \rangle = \sum_{n \in A} \langle x_n, x_m^* \rangle - \sum_{n \in B} \langle x_n, x_m^* \rangle = \langle x_A, x_m^* \rangle - \langle x_B, x_m^* \rangle$$

As we assumed that $\lim_{m \rightarrow \infty} \langle x, x_m^* \rangle = 0$ for each $x \in \overline{\text{span}}(\{x_n\})$,

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} \alpha_n \langle x_n, x_m^* \rangle = 0$$

□

From the theorem we can deduce the following:

If a series $\sum_{n=1}^{\infty} x_n$ is weakly subseries convergent in a Banach space X , then for any sequence $\{x_m^*\}_{m \in \mathbb{N}}$ in X^* satisfying $\lim_{m \rightarrow \infty} \langle x, x_m^* \rangle = 0$ for each $x \in \text{span}(\{x_m\})$ and for any bounded $K \subset \ell_{\infty}$,

$\lim_{m \rightarrow \infty} \langle x, x_m^* \rangle = 0$ uniformly for $x \in S(K)$ where

$$S(K) = \left\{ \sum_{n \in F} \alpha_n x_n : \{\alpha_n\} \in K \text{ and } F \subset \mathbb{N} \text{ is finite} \right\}.$$

Corollary 2.1.36. *In ℓ_p , $1 \leq p < \infty$, a set X is compact if and only if X is bounded and*

$$\lim_{n \rightarrow \infty} \sum_{j=n}^{\infty} |x_j|^p = 0$$

uniformly with respect to $\{x_j\}_{j \in \mathbb{N}}$ in X .

Proof. [4, p.691]

□

Theorem 2.1.37. *Let $\sum_{k=1}^{\infty} x_k$ be a series in a Banach space X . Then $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent if and only if*

$$\lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} |\langle x_k, x^* \rangle| = 0 \text{ uniformly for } x^* \in X^* \text{ where } \|x^*\| \leq 1$$

Proof. Let $\sum_{k=1}^{\infty} x_k$ be an unconditionally convergent series. By Orlicz-Pettis theorem 2.1.31, $\sum_{k=1}^{\infty} x_k$ is weakly subseries convergent. Also by 2.1.14, for any $\alpha = \{\alpha_k\}_{k \in \mathbb{N}} \in \ell_{\infty}$, $\sum_{k=1}^{\infty} \alpha_k x_k$ converges. Then the operator defined by $T(\alpha) = \sum_{k=1}^{\infty} \alpha_k x_k$ for all $\{\alpha_k\}_{k \in \mathbb{N}} \in \ell_{\infty}$ is clearly well-defined and linear. Observe that

$$S(K) = \left\{ \sum_{k \in F} \alpha_k x_k : \{\alpha_k\}_{k \in \mathbb{N}} \in K \text{ and } F \text{ is finite} \right\}$$

where K is any bounded subset of ℓ_{∞} is relatively compact by 2.1.34. So $\overline{S(B_{\ell_{\infty}})}$ is compact and $T(B_{\ell_{\infty}}) \subset \overline{S(B_{\ell_{\infty}})}$. Hence T is compact. As $\sum_{k=1}^{\infty} x_k$ is weakly subseries convergent, $\sum_{k=1}^{\infty} \langle x_k, x^* \rangle$ is subseries convergent so unconditionally convergent for each $x^* \in X^*$ implying that $\sum_{k=1}^{\infty} |\langle x_k, x^* \rangle| < \infty$. Hence $\{\langle x_k, x^* \rangle\}_{k \in \mathbb{N}} \in \ell_1$. Now define $S : X^* \rightarrow \ell_1$ by

$$S(x^*) = \{\langle x_k, x^* \rangle\}_{k \in \mathbb{N}}, \quad x^* \in X^*.$$

Clearly S is linear and $T = S^*$. Therefore as T is compact, by Gantmacher's theorem 1.2.10, $T = S^*$ is also compact. Hence by the corollary 2.1.36 for $p = 1$ applied to the set

$$\{\{\langle x_k, x^* \rangle\} : \|x^*\| \leq 1\},$$

the result follows.

For the converse implication let $\varepsilon > 0$. Then there exists a positive integer N such that

$$\sum_{k=N}^{\infty} |\langle x_k, x^* \rangle| < \varepsilon, \text{ for } \|x^*\| \leq 1.$$

Then for any finite subset F of $\mathbb{N}$ which only contains elements greater than N ,

$$\left\| \sum_{k \in F} x_k \right\| = \sup_{x^* \in B_{X^*}} \left| \sum_{k \in F} \langle x_k, x^* \rangle \right| \leq \sum_{k=N}^{\infty} |\langle x_k, x^* \rangle| < \varepsilon.$$

Therefore any subseries of $\sum_{k=1}^{\infty} x_k$ is Cauchy in the complete space X . Hence $\sum_{k=1}^{\infty} x_k$ is subseries convergent implying the unconditional convergence. $\square$

2.2 Pelczyński's V Property

In this subsection, we deal with how the convergence of certain types of series which we studied in our previous work is affected by linear operators. As we will see it will be important for us to know that whether a Banach space contains a copy of c_0 or not.

Definition 2.2.1. *Let X and Y be two Banach spaces. A bounded linear operator $T : X \rightarrow Y$ is said to be unconditionally converging if T takes every wuC series to an unconditionally convergent series in Y .* $\square$

By the above definition and as a consequence of Orlicz-Pettis theorem 2.1.31, every weakly compact operator from a Banach space X to a Banach space Y is unconditionally converging.

Pelczyński in [18] studied the converse of Orlicz's result and he introduced what he called property V as follows:

Definition 2.2.2. *A Banach space X has the Pelczyński's property V (or just the property V) whenever given any Banach space Y , every unconditionally converging operator $T : X \rightarrow Y$ is weakly compact.* $\square$

For an equivalent definition of the property V of Pelczyński, we need the following theorem:

Theorem 2.2.3. *Let X be a Banach space. Then the following are equivalent:*

- (1) *X has the Pelczyński's property V .*
- (2) *Let E be a subset of X^* such that for any wuC series $\sum_{k=1}^{\infty} x_k$ in X ,*

$$\lim_{k \rightarrow \infty} \sup_{x^* \in E} | \langle x_k, x^* \rangle | = 0.$$

Then E is relatively weakly compact.

Proof. Assume that X has the property V and let E be a subset of X^* such that for any wuC series $\sum_{k=1}^{\infty} x_k$ in X ,

$$\lim_{k \rightarrow \infty} \sup_{x^* \in E} | \langle x_k, x^* \rangle | = 0$$

Obviously E is a bounded set. Think X as acting on E . Let $\mathcal{B}(E)$ be the space of all bounded real valued functions on E with the norm $\|f\| = \sup_{x^* \in E} |f(x^*)|$. Define an operator $T : X \rightarrow \mathcal{B}(E)$ by $Tx(x^*) = \langle x, x^* \rangle$ for every $x \in X$ and $x^* \in E$. Then by our assumption on E , T is bounded, linear and unconditionally converging. So by (1), T is weakly compact. Hence using Gantmacher's theorem 1.2.10, T^* is also weakly compact. Now take any sequence $\{x_n^*\}_{n \in \mathbb{N}}$ in E . Define a sequence of functionals $f_n^* : \mathcal{B}(E) \rightarrow \mathbb{R}$ by

$$f_n^*(f) = f(x_n^*), \quad n = 1, 2, \dots$$

i.e., the evaluation on the n^{th} coordinate for any $f \in \mathcal{B}(E)$. Then $\|f_n^*\| = 1$ for any n . So the sequence $\{f_n^*\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{B}(E)^*$. Hence as T^* is weakly compact, $\{T^*(f_n^*)\}_{n \in \mathbb{N}} = \{x_n^*\}_{n \in \mathbb{N}}$ has a weakly convergent subsequence in X^* . Therefore as $\{x_n^*\}_{n \in \mathbb{N}}$ was any sequence in E and has a weakly convergent subsequence, E is relatively weakly compact.

For the other direction, assume (2) holds and let $T : X \rightarrow Y$ be an unconditionally converging operator. Take any bounded sequence $\{y_n^*\}_{n \in \mathbb{N}}$ in Y^* . Let $\sum_{n=1}^{\infty} x_n$ be a wuC series in X . Then T , by being unconditionally converging, maps $\sum_{n=1}^{\infty} x_n$ to the unconditionally convergent series $\sum_{n=1}^{\infty} T(x_n)$ in Y . So $\|T(x_n)\| \rightarrow 0$ and that $T(x_n)$ tends to zero uniformly on bounded subsets of Y^* . By 2.1.37,

$$\lim_{n \rightarrow \infty} \sup_m | \langle T(x_n), y_m^* \rangle | = \lim_{n \rightarrow \infty} \sup_m | \langle x_n, T^*(y_m^*) \rangle | = 0$$

By the assumption in (2), $\{T^*(y_m^*)\}_{m \in \mathbb{N}}$ is relatively weakly compact. So has a weakly convergent subsequence by Eberlein-Šmulian theorem 1.1.44. Hence as $\{y_m^*\}_{m \in \mathbb{N}}$ was an arbitrary bounded sequence in Y^* , the adjoint T^* of T is weakly compact. As a result by Gantmacher's theorem 1.2.10, T is weakly compact, that is, X has the property V . $\square$

The above theorem gives us a tool to check whether a Banach space has the property V or not. To do so it is enough to see for any given non relatively weakly compact subset E of X^* , there exists a wuC series $\sum_{k=1}^{\infty} x_k$ in X such that for some $\delta > 0$,

$$\lim_{k \rightarrow \infty} \sup_{x^* \in E} | \langle x_k, x^* \rangle | > \delta.$$

Or alternatively, we can use the following theorem as a characterization for the Banach spaces having the property V .

Theorem 2.2.4. *Let X be a Banach space with the property V . If an operator $T : X \rightarrow Y$, where Y is any Banach space, fails to be weakly compact then there exists a subspace M of X such that M is isomorphic to c_0 and the restriction $T|_M$ is an isomorphic embedding (or T fixes a copy of c_0).*

Proof. Assume that $T : X \rightarrow Y$, where Y is any Banach space, is not weakly compact. Then by Gantmacher's theorem 1.2.10, the adjoint T^* is not weakly compact. Let $E = T^*(B_{Y^*})$. Then E is not relatively weakly compact. So as X has the property V , we can find a wuC series $\sum_{n=1}^{\infty} x_n$ in X and a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ such that for some $\delta > 0$,

$$\sup_{x^* \in E} | \langle x_{n_k}, x^* \rangle | > \delta \text{ where } k = 1, 2, \dots$$

Now as $\sum_{k=1}^{\infty} x_{n_k}$ is wuC, for any $\alpha = \{\alpha_k\}_{k \in \mathbb{N}}$ in c_0 , $\sum_{k=1}^{\infty} \alpha_k x_{n_k}$ converges by 2.1.16. Define an operator $S : c_0 \rightarrow X$ by

$$S(\alpha) = \sum_{k=1}^{\infty} \alpha_k x_{n_k}$$

Let $U = T \circ S : c_0 \rightarrow Y$. Then for all n ,

$$U(\{e_n\}) = (T \circ S)(\{e_n\}) = T(S(\{e_n\})) = T(x_{n_k})$$

where $\{e_n\}_{n \in \mathbb{N}}$ is the standard unit vector of c_0 . So, for any $n = 1, 2, \dots$,

$$\|U(\{e_n\})\| = \|T(x_{n_k})\| = \sup_{y^* \in B_{Y^*}} | \langle T(x_{n_k}), y^* \rangle | = \sup_{T^*(y^*) \in E} | \langle x_{n_k}, T^*(y^*) \rangle | > \delta$$

which means that T fixes a copy of c_0 by 2.1.26. $\square$

We now know that saying X has the property V is equivalent to saying that any bounded linear operator T from X to any Banach space Y is either weakly compact or X contains an isomorphic copy S of c_0 such that the restriction $T|_S$ is an isomorphism. Also by the theorem in any Banach space X with the property V , as ℓ_∞ is a Banach space, every non-weakly compact bounded linear operator $T : X \rightarrow \ell_\infty$ fixes a copy of c_0 . The converse also holds:

Theorem 2.2.5. *Let X be a Banach space. If every non-weakly compact operator $T : X \rightarrow \ell_\infty$ fixes a copy of c_0 , then X has the property V .*

Proof. Let $E \subset X^*$ be such that for any wuC series $\sum_{n=1}^\infty x_n$ in X ,

$$\lim_{n \rightarrow \infty} \sup_{x^* \in E} | \langle x_n, x^* \rangle | = 0.$$

Assume for a contradiction that there exists a sequence $\{x_n^*\}_{n \in \mathbb{N}}$ in E which has no weakly convergent subsequence. Define an operator $S : X \rightarrow \ell_\infty$ by

$$S(x) = \{ \langle x, x_n^* \rangle \}_{n \in \mathbb{N}}, \quad x \in X.$$

As $\sum_{n=1}^\infty x_n$ is wuC, S is well-defined. Observe that $\{e_n\}_{n \in \mathbb{N}} \subset \ell_1 \subset \ell_1^{**} = \ell_\infty^*$ where $\{e_n\}_{n \in \mathbb{N}}$ is the sequence of the standard unit vectors of ℓ_1 . Then the adjoint $S^* : \ell_\infty^* \rightarrow X^*$ satisfies $S^*(\{e_n\}) = \{x_n^*\}_{n \in \mathbb{N}}$. Therefore S^* is not weakly compact since $\{e_n\}_{n \in \mathbb{N}}$ is a bounded sequence but we assumed that $\{x_n^*\}_{n \in \mathbb{N}}$ has no weakly convergent subsequence. By Gantmacher's theorem 1.2.10, S is not weakly compact, either. So by the assumption of the theorem, S fixes a copy of c_0 . Let M be the subspace of X which is isomorphic to c_0 and $\{u_n\}_{n \in \mathbb{N}}$ be the corresponding sequence in M for the standard unit vectors $\{e_n\}_{n \in \mathbb{N}}$ of c_0 . Then $\sum_{n=1}^\infty u_n$ is wuC since $\sum_{n=1}^\infty e_n$ is wuC as we saw in 2.1.25. But also, for some $\delta > 0$,

$$\sup_{x^* \in E} | \langle u_n, x^* \rangle | \geq \sup_m | \langle u_n, x_m^* \rangle | = \|S(u_n)\|_\infty \geq \delta \text{ for sufficiently large } n$$

which is a contradiction. Thus E is relatively weakly compact. Hence X has the property V . $\square$

Theorem 2.2.6. *A Banach space X has the property V if and only if every non weakly compact linear operator from X to any Banach space Y fixes a copy of c_0 .*

Proof. Follows by combining the results of 2.2.4 and 2.2.5. $\square$

The following corollaries stating that the property V of a Banach space passes to complemented closed subspaces, quotients and cartesian products and is stable under isomorphisms follows readily from the above results.

Corollary 2.2.7. *Let X and Y be two Banach spaces. Then the following holds:*

- (1) *If X has the property V , then any quotient space of X has the property V .*
- (2) *The Cartesian product $X \times Y$ of X and Y has the property V if and only if both X and Y have the property V .*
- (3) *Let $T : X \rightarrow Y$ be an isomorphism. If X has the property V then Y also has the property V .*
- (4) *If X has the property V , then any complemented closed subspace of X has the property V .*

$\square$

The property u of Pełczyński is defined in the following proposition:

Proposition 2.2.8. *Let X be a Banach space satisfying the following two properties:*

- (1) *For any weakly Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in X there exists a wuC series $\sum_{k=1}^{\infty} u_k$ such that the sequence $\{x_n - \sum_{k=1}^n u_k\}_{n \in \mathbb{N}}$ converges weakly to 0 (or X has the property u .)*
- (2) *X does not contain an isomorphic copy of ℓ_1 .*

Then X has the property V .

Proof. Let Y be any Banach space and $T : X \rightarrow Y$ be an unconditionally converging operator. Take any bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ in X . Then by (2) in the light of Rosenthal's ℓ_1 theorem 1.1.46, $\{x_n\}_{n \in \mathbb{N}}$ has a weakly Cauchy subsequence, say $\{x_{n_k}\}_{k \in \mathbb{N}}$. Using (1), we can find a wuC series $\sum_{k=1}^{\infty} u_k$ such that $(x_{n_k} - \sum_{k=1}^{n_k} u_k) \xrightarrow{w} 0$. Now as T is an unconditionally converging operator, $\sum_{k=1}^{\infty} T(u_k)$ is an unconditionally convergent series in Y and $T(x_{n_k}) \xrightarrow{w} \sum_{k=1}^{\infty} T(u_k)$ as T is weakly continuous by 1.1.25. Therefore, T sends any bounded sequence to a sequence which has a weakly convergent subsequence. Hence T is weakly compact i.e., X has the property V . $\square$

Obviously, any weakly sequentially complete Banach space has the property u . So in particular, every reflexive space has the property u . This implies that any reflexive space has the property V since it does not contain a copy of ℓ_1 .

Corollary 2.2.9. *Any Banach space with an unconditional basis has the property u . Any subspace of a space with an unconditional basis has the property u .*

Proof. [19] [15, p.327] $\square$

Corollary 2.2.10. *Let X be a Banach space with the following properties:*

- (1) *X does not contain an isomorphic copy of ℓ_1 , and*
- (2) *X is isomorphic to a subspace of a space Y with an unconditional basis.*

Then X has the property V .

Proof. Assume that Y contains an unconditional basis and let U be a subspace of Y that is isomorphic to X . By 2.2.9, U has the property V . Let $\{x_n\}_{n \in \mathbb{N}}$ be a weakly Cauchy sequence in X . Assume that $\{y_n\}_{n \in \mathbb{N}}$ is the corresponding sequence for $\{x_n\}_{n \in \mathbb{N}}$ in U . So it is also weakly Cauchy. Then as U has the property V , there exists a wuC series $\sum_{k=1}^{\infty} u_k$ such that $\{y_n - \sum_{k=1}^n u_k\} \xrightarrow{w} 0$. So the corresponding series for $\sum_{k=1}^{\infty} u_k$ in X satisfy the hypothesis in (1) of 2.2.8. Hence as the second condition of 2.2.8 is given already, X has the property V . $\square$

The following results will be necessary in our investigation of property V in Banach spaces.

Recall that $C(K)^*$ is identified with $rca(K)$ defined on the class Σ of all Borel subsets of K by 1.1.1, that is, if x^* is a bounded linear functional on $C(K)$ and μ is the corresponding measure in $rca(K)$ then for any $f \in C(K)$,

$$\langle f, x^* \rangle = \int_K f d\mu.$$

Lemma 2.2.11. *Let K be a compact Hausdorff space. Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $C(K)^*$ such that there exists $\delta > 0$ and a sequence $\{O_n\}_{n \in \mathbb{N}}$ of disjoint sets in Σ satisfying $\mu_n(O_n) > \delta$ for each positive integer n . Then there exists a subsequence $\{\nu_n\}_{n \in \mathbb{N}}$ of $\{\mu_n\}_{n \in \mathbb{N}}$ and a sequence of disjoint open sets $\{E_n\}_{n \in \mathbb{N}}$ in Σ such that $\nu_n(E_n) > \frac{\delta}{2}$ for any positive integer n .*

Proof. [18, Lemma 1] □

Theorem 2.2.12. (Dieudonne-Grothendieck)

Let K be a compact Hausdorff space and Σ be the collection of all Borel subsets of K . Suppose $\mathcal{H}$ is a bounded subset of $rca(K, \Sigma)$. Then $\mathcal{H}$ is relatively weakly compact if and only if given any sequence $\{O_n\}_{n \in \mathbb{N}}$ of disjoint open subsets of K ,

$$\lim_{n \rightarrow \infty} \sup_{\mu \in \mathcal{H}} \mu(O_n) = 0.$$

Proof. [7, p.98 Theorem 14] □

Now we can proceed with the important result of Pełczyński that the space of all the scalar valued continuous functions on a compact Hausdorff space has the property V .

Theorem 2.2.13. *Let K be any compact Hausdorff space. Then $C(K)$ has property V .*

Proof. Let $\mathcal{H}$ be a non relatively weakly compact bounded subset of $C(K)^* = rca(K)$. Then Dieudonne-Grothendieck theorem 2.2.12 says that there exists a sequence $\{O_n\}_{n \in \mathbb{N}}$

of disjoint open subsets of K and a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ such that $\mu_n(O_n) > \delta$ for some $\delta > 0$, for any positive integer n . So by 2.2.11, there exists a subsequence $\{\nu_n\}_{n \in \mathbb{N}}$ of $\{\mu_n\}_{n \in \mathbb{N}}$ and sequence of disjoint open sets $\{E_n\}_{n \in \mathbb{N}}$ such that $\nu_n(E_n) > \frac{\delta}{2}$ for all n . As $K \setminus E_n$ is closed for any n , by 1.1.6, there exists $f_n \in C(K)$ satisfying $\|f_n\| = 1$ and $f_n(x) = 0$ if $x \in K \setminus E_n$. So

$$\int_K f_n(x) d\nu_n(x) > \frac{\delta}{2}, \quad n = 1, 2, \dots$$

The series $\sum_{n=1}^{\infty} f_n$ is wuC because E_n 's are disjoint, f_n vanishes outside of E_n and $\|f_n\| \leq 1$ for $n = 1, 2, \dots$

Clearly,

$$\lim_{n \rightarrow \infty} \sup_{\mu \in \mathcal{H}} \int_K f_n(x) d\mu(x) > \frac{\delta}{2} > 0$$

Therefore by proposition 2.2.3, $C(K)$ has the property V . $\square$

We can deduce from this result due to Pelczyński that every linear operator $T : C(K) \rightarrow Y$, where Y is any Banach space, is unconditionally convergent if and only if it is weakly compact. Also from the theorem 1.2.11, T is unconditionally converging if and only if it maps every weak Cauchy sequence in $C(K)$ into a Cauchy sequence in Y .

Example 2.2.14. In particular $L_{\infty}(\mu)$ has the property V . Since $L_{\infty}(\mu)$ is isomorphic to $C(K)$ for some compact Hausdorff K .

We can generalize the theorem 2.2.13 as follows:

Theorem 2.2.15. *For any reflexive space X and any compact Hausdorff space K , $C(K, X)$ has the property V .*

Proof. We can represent $C(K, X)$ by vector valued measures on the Borel subsets Σ of K with values in X^* . To prove the theorem we need to do the following. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $C(K, X)$. Let R be the relation on K defined by

$$k R \tilde{k} \Leftrightarrow f_n(k) = f_n(\tilde{k}), \quad n = 1, 2, \dots$$

Let K' be the set of all the equivalence classes of K under R . K' is actually a metric space with the distance defined by

$$d(k, l) = \sum_{n=1}^{\infty} 2^{-n} |f_n(k) - f_n(l)|, \quad k, l \in K'.$$

Let $\phi : K \rightarrow K'$ be the canonical map. Obviously ϕ is continuous because each f_n is continuous. So as K is compact, K' is a compact metric space. Thus enough to consider the case when X is a separable, reflexive space and K is a compact metric space. The rest is an analogue of the proof of 2.2.13. $\square$

Corollary 2.2.16. *Let K be a compact Hausdorff space and Y be a Banach space which does not contain any isomorphic copy of c_0 . Then any linear operator $T : C(K) \rightarrow Y$ is weakly compact.*

Proof. This follows from 2.2.13 and the comments after 2.2.4 because for any compact Hausdorff space K , $C(K)$ has the property V and in turn if Y satisfies the property stated, then $T : C(K) \rightarrow Y$ is weakly compact. $\square$

Other than property V and u , Pełczyński has defined a dual companion to V , namely V^* .

Definition 2.2.17. *Let X be a Banach space. Then X is said to have the property V^* whenever any subset K of X with*

$$\lim_{n \rightarrow \infty} \sup_{x \in K} | \langle x, x_n^* \rangle | = 0 \text{ for any } wuC \text{ series } \sum_{n=1}^{\infty} x_n^* \text{ in } X^*$$

is relatively weakly compact. $\square$

The next proposition follows readily from the definition.

Proposition 2.2.18. *Any closed subspace of a space having the property V^* has the property V^* .* $\square$

There is a one way relation between the property V and the property V^* .

Proposition 2.2.19. *Let X be a Banach space. Then the following are true:*

(1) If X has the property V then X^* has the property V^* .

(2) If X^* has the property V then X has the property V^* .

Proof. (1) follows from the definitions 2.2.3 and 2.2.17.

(2) If X^* has the property V then X^{**} has the property V^* thus X has property V^* by 2.2.18. $\square$

Example 2.2.20. c_0 does not have the property V^* as in $c_0^* = \ell_1$, every wuC series converges in norm by Schur's lemma 2.1.29. Thus by (2) of proposition 2.2.19, ℓ_1 does not have the property V .

Note: The converses are proven to be false. See [25, Example 5].

Proposition 2.2.21. Any Banach space X with the property V^* is weakly sequentially complete.

Proof. Take any weakly Cauchy sequence $\{x_m\}_{m \in \mathbb{N}}$ in X . Then $\{< x_m, x^* >\}_{m \in \mathbb{N}}$ is a Cauchy sequence for any $x^* \in X^*$. So $\{< x_m, x^* >\}_{m \in \mathbb{N}}$ converges for any $x^* \in X^*$.

Claim: $\{x_m\}_{m \in \mathbb{N}}$ has a weak limit point.

To see this define a mapping $f_\alpha : X \rightarrow \mathbb{F}$ by

$$f_\alpha(x) = \sum_{n=1}^{\infty} \alpha_n < x, x_n^* >$$

where $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ is a bounded real sequence and $\sum_{n=1}^{\infty} x_n^*$ is any wuC series. f_α is obviously linear. By 2.1.13, for some constant M ,

$$\sup_m \left\| \sum_{n=1}^m \alpha_n x_n^* \right\| \leq M \|\{\alpha_n\}\|_\infty$$

that is

$$\sup_m \sup_{x \in B_X} \left| \sum_{n=1}^m \alpha_n < x, x_n^* > \right| = \sup_m \sup_{x \in B_X} \left| < x, \sum_{n=1}^m \alpha_n x_n^* > \right| \leq M \sup_n |\alpha_n|$$

Hence f_α is a well-defined linear functional on X . f_α is also bounded because

$$|f_\alpha(x)| = \left| \sum_{n=1}^{\infty} \alpha_n < x, x_n^* > \right| \leq \sup_n |\alpha_n| \sum_{n=1}^{\infty} |< x, x_n^* >| \leq \sup_n |\alpha_n| \left(\sum_{n=1}^{\infty} \|x_n^*\| \right) \|x\|$$

as α_n is bounded and $\sum_{n=1}^{\infty} x_n^*$ is wuC. Thus $f_\alpha \in X^*$.

Now as $\{\langle x_m, x^* \rangle\}_{m \in \mathbb{N}}$ converges for any $x^* \in X^*$,

$$\lim_{m \rightarrow \infty} f_\alpha(x_m) = \lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} \alpha_n \langle x_m, x_n^* \rangle$$

exists for any $\{\alpha_n\}_{n \in \mathbb{N}} \in \ell_\infty$. Also since $\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle| < \infty$, $\{\langle x, x_n^* \rangle\}_{n \in \mathbb{N}} \in \ell_1$.

Thus,

$$\lim_{m \rightarrow \infty} \langle \{\langle x_m, x_n^* \rangle\}, \{\alpha_n\} \rangle = \lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} \alpha_n \langle x_m, x_n^* \rangle$$

exists for any $\{\alpha_n\}_{n \in \mathbb{N}} \in \ell_\infty = \ell_1^*$ which means that $\{\langle x_m, x_n^* \rangle\}_{m \in \mathbb{N}}$ is a weakly Cauchy sequence in ℓ_1 . So by Schur's lemma 2.1.29, $\{\langle x_m, x_n^* \rangle\}_{m \in \mathbb{N}}$ converges in norm i.e.,

$$\lim_{m \rightarrow \infty} \langle x_m, x_n^* \rangle = \{s_n\}_{n \in \mathbb{N}} \text{ for some } \{s_n\}_{n \in \mathbb{N}} \in \ell_1.$$

Therefore,

$$\|\{\langle x_m, x_n^* \rangle\}_m - \{s_n\}\|_1 \rightarrow 0 \text{ and } \sum_{n=1}^{\infty} |s_n| < \infty$$

implying

$$\lim_{n \rightarrow \infty} \sup_m \langle x_m, x_n^* \rangle = 0$$

for every wuC series $\sum_{n=1}^{\infty} x_n^*$ in X^* . So since X has the property V^* , $\{x_m\}_{m \in \mathbb{N}}$ is relatively weakly compact i.e., $\{x_m\}_{m \in \mathbb{N}}$ contains a weakly convergent subsequence, say $\{x_{m_k}\}_{k \in \mathbb{N}}$ with weak limit point $x \in X$. From this, and the fact that $\{x_m\}_{m \in \mathbb{N}}$ is weakly Cauchy we get, for all $x^* \in X^*$,

$$|\langle x_m, x^* \rangle - \langle x, x^* \rangle| \leq |\langle x_m, x^* \rangle - \langle x_{m_k}, x^* \rangle| + |\langle x_{m_k}, x^* \rangle - \langle x, x^* \rangle|$$

converges to 0 as $k, m \rightarrow \infty$ which says that $\{x_m\}_{m \in \mathbb{N}}$ converges weakly to $x \in X$.

Thus X is weakly sequentially complete. $\square$

Theorem 2.2.22. *If X is any Banach space with the property V , then X^* is weakly sequentially complete.*

Proof. By the proposition 2.2.19, if X has property V then X^* has the property V^* .

As a result the proposition 2.2.21, X^* is weakly sequentially complete. $\square$

Example 2.2.23. The space ℓ_1 does not have the property V as its dual space ℓ_∞ is not weakly sequentially complete.

The space ℓ_1 does not have the property V and is not reflexive. So we can make a guess that spaces which contain ℓ_1 do not have the property V . However this is not true. Actually $C(K)$ contains a copy of ℓ_1 but as we proved in 2.2.13, it also has the property V . So a space which contains ℓ_1 must have an extra condition on it to fail to have the property V .

Proposition 2.2.24. *Let X be a closed subspace of a Banach space with an unconditional basis. Then X has the property V if and only if no subspace of X is isomorphic to ℓ_1 .*

Proof. Suppose that X has the property V . Then by 2.2.22, X^* is weakly sequentially complete. So 2.1.23 says that X contains no subspace isomorphic to ℓ_1 .

Now assume that no subspace of X is isomorphic to ℓ_1 , then by 2.2.9 as X has the property u , 2.2.8 guarantees that X has the property V . $\square$

We obtain a relation between reflexivity and the properties V and V^* with the aid of the following propositions.

Proposition 2.2.25. *Let X be a Banach space. Then X is reflexive if and only if X both has the properties V and V^* .*

Proof. Let A be any bounded subset of X . Then by 1.1.27, A is relatively weakly compact. Similarly any bounded set in X^* is relatively weakly compact as reflexivity of X implies reflexivity of X^* . So X has both the properties V and V^* . For the converse suppose that X has both the properties V and V^* . Then by 2.2.22, X^* is weakly sequentially complete. Orlicz-Pettis theorem 2.1.31 for weakly sequentially complete spaces implies that every wuC series in X^* is unconditionally convergent so if A is any bounded set in X , for any wuC series $\sum_{n=1}^{\infty} x_n^*$ in X^* ,

$$\lim_{n \rightarrow \infty} \sup_{x \in A} \langle x, x_n^* \rangle = 0$$

Hence as X has the property V^* , A is relatively weakly compact. So X is reflexive by 1.1.27. $\square$

Observe that dual spaces of nonreflexive Banach spaces do not respect the property V . Indeed let X be a Banach space with the property V and assume that the dual space X^* has the property V . Then by 2.2.19, X has the property V^* . This leads us to the reflexivity of X by 2.2.25.

Example 2.2.26. We know that $C(K)$ has the property V for any compact Hausdorff space K . Also $C(K)$ is the dual space of an infinite dimensional $L_1(\mu)$ for some compact Hausdorff K . But infinite dimensional $L_1(\mu)$ is a nonreflexive space so by the above observation, infinite dimensional $L_1(\mu)$ does not have the property V . As a conclusion $L_1(\mu)$ has the property V if and only if it is finite dimensional if and only if it is reflexive.

2.3 Pfitzner's Theorem: Every Von-Neumann Algebra Has Pełczyński's V Property

H. Pfitzner in [21] has proven the following from which he deduced that C^* algebras have Pełczyński's property V as a consequence.

Theorem 2.3.1. *If A is a C^* algebra and $\mathcal{H} \subset A'$ is bounded then $\mathcal{H}$ is not relatively weakly compact if and only if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in A_1 of pairwise orthogonal self-adjoint elements such that*

$$\sup_{a' \in \mathcal{H}} |\langle x_n, a' \rangle| \not\rightarrow 0.$$

$\square$

Note: A' denotes the dual space of A .

Theorem 2.3.2. Any C^* algebra has property V .

Proof. Let A be a C^* -algebra, $\mathcal{H}$ be a bounded subset of A' and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of pairwise orthogonal self-adjoint elements in A_1 satisfying

$$\sup_{a' \in \mathcal{H}} | \langle x_n, a' \rangle | \not\rightarrow 0 \quad (2.3.2.1)$$

as in the theorem. Then $\mathcal{H}$ is not relatively weakly compact. But in a C^* algebra if $\{x_n\}_{n \in \mathbb{N}}$ is any bounded sequence of pairwise orthogonal self-adjoint nonzero elements, then the series $\sum_{n=1}^{\infty} x_n$ is wuC. Since $\{\langle x_n, a' \rangle\}_{n \in \mathbb{N}}$ does not converge uniformly to 0 by 2.3.2.1, A satisfies the property V . $\square$

This result leads us to the following as every von-Neumann algebra is actually a C^* algebra.

Corollary 2.3.3. *Every von-Neumann algebra has Pelczyński's property V .* $\square$

By our previous work we can conclude that any bounded linear operator from a von-Neumann algebra to a Banach space which does not contain any isomorphic copy of c_0 is weakly compact.

Observe also that from 1.4.8 and 2.3.2 we can easily deduce the result that $C(K)$ has the property V which we have proven in the previous section.

Chapter 3

BANACH SPACES WITH GROTHENDIECK PROPERTY

This last chapter focuses on the main issue of this thesis, namely Banach spaces with the Grothendieck property from which weakly compact operators can be easily found. First we will study what a Grothendieck space is and what characterizes a Banach space with the Grothendieck property. In this context property V introduced by Pełczyński is very useful.

3.1 Various Characterizations of the Banach Spaces Having the Grothendieck Property

Definition 3.1.1. *A Banach space X has the Grothendieck property if weak* convergent sequences in X^* are weakly convergent. A Banach space is said to be a Grothendieck space if it has the Grothendieck property.* $\square$

Clearly any reflexive space has the Grothendieck property. Indeed, if $\{x_n\}_{n \in \mathbb{N}}$ is a weak* convergent sequence in the dual space of a reflexive space, then since $\pi(X) = X^{**}$,

$$\langle x_n^*, \pi(x) \rangle = \langle x, x_n^* \rangle \text{ converges for all } \pi(x) \in \pi(X) = X^{**}.$$

Actually, the class of reflexive spaces contains the trivial examples of Grothendieck spaces.

An alternative definition for the Grothendieck property is that a Banach space X is Grothendieck if weak* null sequences in X^* are weakly null. By the definition it is obvious that we need information about the dual space of a Banach space to understand whether the space is Grothendieck or not.

It readily follows from the definition that dual space of a Grothendieck space X is

weakly sequentially complete. Indeed, if X^* is the dual space and $\{x_n^*\}_{n \in \mathbb{N}}$ is a weakly Cauchy sequence in X^* , then it is weak* Cauchy. So it is a bounded sequence. By Alaoglu's theorem, there exists a weak* convergent subsequence of $\{x_n^*\}_{n \in \mathbb{N}}$. Hence as the sequence is weak* Cauchy and has a weak* convergent subsequence, it converges weak* to the same limit with its subsequence. Now since X is a Grothendieck space the sequence converges weakly.

In addition, the definition implies that linear isomorphisms respect the Grothendieck property. Indeed, let $T : X \rightarrow Y$ be a linear isomorphism onto a Banach space Y and $\{y_n^*\}_{n \in \mathbb{N}}$ is weak* null in Y^* . So, $\lim_{n \rightarrow \infty} \langle y, y_n^* \rangle = 0$ for any $y \in Y$. As T is a bijection, for any $y \in Y$, there exists a unique $x \in X$ such that $Tx = y$. Hence $\lim_{n \rightarrow \infty} \langle x, T^*y_n^* \rangle = \lim_{n \rightarrow \infty} \langle Tx, y_n^* \rangle = 0$ for all $x \in X$ where T^* is the adjoint of T . Thus $\{T^*y_n^*\}_{n \in \mathbb{N}}$ is a weak* null sequence in X^* . As X is Grothendieck, $\{T^*y_n^*\}_{n \in \mathbb{N}}$ is weakly null, i.e., $\lim_{n \rightarrow \infty} \langle T^*y_n^*, x^{**} \rangle = 0$ for all $x^{**} \in X^{**}$. But the second adjoint $T^{**} : X^{**} \rightarrow Y^{**}$ defined by $T^{**}x^{**} = y^{**}$ is an isomorphism by 1.2.6 as T is, so $\lim_{n \rightarrow \infty} \langle y_n^*, T^{**}x^{**} \rangle = \lim_{n \rightarrow \infty} \langle y_n^*, y^{**} \rangle = 0$ for all $y^{**} \in Y^{**}$. Therefore, $\{y_n^*\}_{n \in \mathbb{N}}$ is weakly null implying that Y is a Grothendieck space.

Also we can easily show that quotient spaces of Grothendieck spaces have the Grothendieck property. First of all, we identify $(X/M)^*$ with $(M)^\perp = \{x^* \in X^* : \langle x, x^* \rangle = 0 \text{ for all } x \in M\}$ by the map $T : (X/M)^* \rightarrow (M)^\perp$, $Ty^* = x^*$, $y^* \in (X/M)^*$ satisfying $\langle x + M, y^* \rangle = \langle x, Ty^* \rangle = \langle x, x^* \rangle$ where $x + M \in X/M$. So if $\{y_n^*\}_{n \in \mathbb{N}}$ is a weak* null sequence in $(X/M)^* = M^\perp$, then $\langle x + M, y_n^* \rangle \rightarrow 0$ for any $x \in X$. Also as the elements of the sequence belongs to $M^\perp$, we conclude that $\langle x, y_n^* \rangle \rightarrow 0$ for any $x \in X$. Thus $\{y_n^*\}_{n \in \mathbb{N}}$ is weak* null in X^* so is weakly null as X is Grothendieck. Hence, $\langle y_n^*, x^{**} \rangle$ converges to 0 for any $x^{**} \in X^{**}$. But as $M^\perp \subset X^*$, $X^{**} \subset M^{\perp*}$ $\langle y_n^*, x^{**} \rangle$ converges to 0 for any $x^{**} \in (M^\perp)^* = (X/M)^{**}$ implying that $\{y_n^*\}_{n \in \mathbb{N}}$ is weakly null in $(X/M)^*$.

Now let us characterize the Grothendieck property in terms of weakly compact operators.

Proposition 3.1.2. *Let X be a Banach space. All the following implies the other*

two.

- (1) X is a Grothendieck space.
- (2) If Y is a separable Banach space, then every bounded linear operator $T : X \rightarrow Y$ is weakly compact.
- (3) Every bounded linear operator $T : X \rightarrow c_0$ is weakly compact.

Proof. (1) $\implies$ (2): Let Y be a separable Banach space and take a bounded linear operator $T : X \rightarrow Y$. Then by 1.1.39 and 1.1.35, B_{Y^*} is weak* metrizable and weak* compact. So for any sequence $\{y_n^*\}_{n \in \mathbb{N}}$ in B_{Y^*} , we can find a weak* convergent subsequence $\{y_{n_k}^*\}_{k \in \mathbb{N}}$ of $\{y_n^*\}_{n \in \mathbb{N}}$ after a Cantor's diagonalization process. Consider the adjoint $T^* : Y^* \rightarrow X^*$ of T . As T^* is weak* continuous, $\{T^*(y_{n_k}^*)\}_{k \in \mathbb{N}}$ is also weak* convergent in X^* . Hence as the space X is Grothendieck, the weak* convergent sequence $\{T^*(y_{n_k}^*)\}_{k \in \mathbb{N}}$ is actually weakly convergent. Therefore, T^* is weakly compact. It follows that T is weakly compact by Gantmacher's theorem 1.2.10.

(2) $\implies$ (3): Obvious as c_0 is a separable Banach space.

(3) $\implies$ (1): We will use the correspondence of the weak* convergent sequences in X^* with the bounded operators from X to c_0 to give the proof. Let $\{x_n^*\}_{n \in \mathbb{N}}$ in X^* be a weak* convergent sequence and say $x_n^* \xrightarrow{w^*} x^* \in X^*$. We may assume that $\|x_n^*\| \leq 1$ for any $n = 1, 2, \dots$

Now define an operator $T : X \rightarrow c_0$ by

$$T(x) = \{\langle x, x_n^* - x^* \rangle\}_{n \in \mathbb{N}}$$

Clearly, T is a bounded linear operator so by assumption in (3), T is weakly compact. Using Gantmacher's theorem 1.2.10, we get T^* is weakly compact which implies that the set $\{x_n^* - x^* : n \in \mathbb{N}\}$ is relatively weakly compact.

Claim: Any relatively weakly compact weak* null sequence in X^* converges weakly to 0.

Indeed if $\{y_n^*\}_{n \in \mathbb{N}}$ is such a sequence in X^* , then by Eberlein-Šmulian theorem

1.1.44, we can extract a weakly convergent subsequence $\{y_{n_k}^*\}_{k \in \mathbb{N}}$ of $\{y_n^*\}_{n \in \mathbb{N}}$ which converges weakly to, say y^* in X^* , i.e., for any $\varepsilon > 0$ for sufficiently large k ,

$$| \langle y_{n_k}^*, y^{**} \rangle - \langle y^*, y^{**} \rangle | < \varepsilon/3$$

for all $y^{**} \in X^{**}$. Also as a subsequence of a weak* null sequence, $\{y_{n_k}^*\}_{k \in \mathbb{N}}$ is itself weak* null, i.e., for any $\varepsilon > 0$, for sufficiently large k ,

$$| \langle y_{n_k}^*, \pi(y) \rangle | = | \langle y, y_{n_k}^* \rangle | < \varepsilon/3$$

for all $y \in X$. We also know by the corollary 1.1.38 to Goldstine's theorem that $\pi(X)$ is weak* dense in X^{**} . So for any $y^{**} \in X^{**}$, there exists $y \in X$ such that

$$| \langle y^*, \pi(y) \rangle - \langle y^*, y^{**} \rangle | < \varepsilon/3$$

for all $y^* \in X^*$. Combining all these we get,

$$\begin{aligned} | \langle y_n^*, y^{**} \rangle | &\leq | \langle y_n^*, y^{**} \rangle - \langle y_{n_k}^*, y^{**} \rangle | + | \langle y_{n_k}^*, \pi(y) \rangle - \langle y_{n_k}^*, y^{**} \rangle | \\ &\quad + | \langle y_{n_k}^*, \pi(y) \rangle | < \varepsilon \end{aligned}$$

So the claim is true. Now letting $y_n = x_n^* - x^*$ for all n , we can conclude that $x_n^* - x^* \xrightarrow{w} 0$. Hence $\{x_n^*\}_{n \in \mathbb{N}}$ converges weakly to x^* i.e., X is a Grothendieck space. $\square$

Here is another characterization:

Theorem 3.1.3. *The following are equivalent for any Banach space X :*

- (1) X is a Grothendieck space.
- (2) For a weakly compactly generated Banach space Y every bounded linear operator $T : X \rightarrow Y$ is weakly compact.
- (3) For any Banach space Y , the weakly compact linear operators from X to Y are closed under the process of taking pointwise weak sequential limits.

- (4) For any Banach space Y , the weakly compact linear operators from X to Y are closed under the process of taking pointwise norm sequential limits.

Proof. (1) $\implies$ (2): Let X be a Grothendieck space and take a continuous linear operator $T : X \rightarrow Y$ where Y is a weakly compactly generated Banach space. Consider the adjoint operator $T^* : Y^* \rightarrow X^*$. As any bounded sequence $\{y_n^*\}_{n \in \mathbb{N}}$ in Y^* has a weak* convergent subsequence $\{y_{n_k}^*\}_{k \in \mathbb{N}}$ by 1.1.32 and T^* is weak* continuous, $\{T^*(y_{n_k})\}_{k \in \mathbb{N}}$ is weak* convergent. Hence $\{T^*(y_{n_k})\}_{k \in \mathbb{N}}$ is weakly convergent since X is a Grothendieck space. So T^* is weakly compact from which we can conclude that T is weakly compact by Gantmacher's theorem 1.2.10.

(2) $\implies$ (3): Let $T_n : X \rightarrow Y$ be a sequence of weakly compact linear operators.

Assume that pointwise weak limit of $\{T_n\}_{n \in \mathbb{N}}$ exists i.e., for each $x \in X$,

$$T_0(x) = \text{weak} \lim_{n \rightarrow \infty} T_n(x)$$

exists. Let $Z_n = T_n(X)$ for each positive integer n and set

$$Z = \overline{\text{span}}\left(\bigcup_{n \in \mathbb{N}} Z_n\right)$$

Then by the theorem of Mazur 1.1.14, as T_0 is the pointwise weak limit of $T_n : X \rightarrow Z$, $T_0(X) \subset \overline{\bigcup_{n \in \mathbb{N}} Z_n}^w \subset Z$. Now let $B_n = T_n(B_X)$. As each T_n is weakly compact, B_n is relatively weakly compact for any n . Hence

$$M_n = \sup_{y \in B_n} \|y\|$$

exists for each n . We may assume that $M_n > 0$. Set

$$B = \bigcup_{n \geq 1} \left(\frac{1}{nM_n}\right) B_n$$

Obviously, $\overline{\text{span}}(B) = Z$.

Claim: B is relatively weakly compact.

To show this, take a sequence $\{y_n\}_{n \in \mathbb{N}}$ in B . We must consider two cases:

- (i) Assume that $\{y_n\}_{n \in \mathbb{N}}$ visits one of the $(\frac{1}{nM_n})B_n$'s infinitely many times. Observe that $(\frac{1}{nM_n})B_n$ is relatively weakly compact as B_n is relatively weakly compact. So

there exists a weakly convergent subsequence of $\{y_n\}_{n \in \mathbb{N}}$.

(ii) Assume on the contrary that $\{y_n\}_{n \in \mathbb{N}}$ visits none of the $(\frac{1}{nM_n})B_n$'s infinitely many times. Then we can find two strictly increasing sequences $\{m_k\}_{k \in \mathbb{N}}$ and $\{n_k\}_{k \in \mathbb{N}}$ of positive integers satisfying

$$y_{m_k} \in (\frac{1}{n_k M_{n_k}})B_{n_k}, \quad k = 1, 2, \dots$$

Hence

$$\|y_{m_k}\| \leq \frac{1}{n_k} \rightarrow 0, \quad k \rightarrow \infty$$

Hence $\{y_n\}_{n \in \mathbb{N}}$ has a norm convergent subsequence.

So in both cases, B turns out to be relatively weakly compact.

Therefore as $\overline{\text{span}}(B) = Z$, Z is weakly compactly generated. By Banach-Steinhaus uniform boundedness principle 1.1.2, T_0 is bounded. So by assumption, T_0 is weakly compact.

(3) $\implies$ (4): As norm convergence implies weak convergence the result is obvious.

(4) $\implies$ (1): By 3.1.2, enough to show that any bounded linear operator from X to a separable Banach space is weakly compact. So let $T : X \rightarrow Y$ be a bounded linear operator where Y is separable. If Y has a Schauder basis $\{y_n\}_{n \in \mathbb{N}}$ then

$$T(x) = \lim_{n \rightarrow \infty} \pi_n(T(x))$$

where π_n is the n^{th} projection associated with the basis of Y i.e.,

$$\pi_n(y) = \pi_n\left(\sum_{k=1}^{\infty} \alpha_k y_k\right) = \sum_{k=1}^n \alpha_k y_k$$

and $\sum_{k=1}^{\infty} \alpha_k y_k$ is the unique representation of the element y of Y . So as

$$\dim(\text{Im}(\pi_n)) < \infty,$$

by 1.2.9, π_n is a compact operator for each n . Therefore by assumption, T , by being the norm pointwise limit of compact linear operators, is weakly compact. If Y does not have a Schauder basis, by Banach-Mazur theorem¹ Y is isometrically isomorphic

¹Banach-Mazur Theorem: Every separable Banach space can be isometrically embedded into $C([0, 1])$.

to a subspace of $C([0, 1])$. As $C([0, 1])$ has a Schauder basis (see [14, p.352]) we are in the same situation with the case when Y has a Schauder basis. Hence by 3.1.2, X is a Grothendieck space. $\square$

Observe that separable spaces are weakly compactly generated as they are the closed linear span of a compact set. Indeed if $X \neq \emptyset$ is separable, then there is a sequence $\{x_n\}_{n \in \mathbb{N}}$ of nonzero elements of X such that

$$X = \overline{\{x_n : n \in \mathbb{N}\}} = \overline{\text{span}}(\{x_n\}).$$

So,

$$X = \overline{\text{span}}(\{\|nx_n\|^{-1}x_n : n \in \mathbb{N}\} \cup \{0\})$$

and $\{\|nx_n\|^{-1}x_n : n \in \mathbb{N}\} \cup \{0\}$ is compact since it consists of the elements and the limit of a convergent sequence in X . So every bounded linear operator from a Grothendieck space to a separable space is weakly compact which is the result that we have proven before.

Theorem 3.1.4. *A Banach space X is a Grothendieck space if and only if every bounded linear operator $T : X \rightarrow Y$ is weakly compact whenever Y is any Banach space such that Y^* has a weak* sequentially compact unit ball.*

Proof. Let X be a Grothendieck space and Y be a Banach space such that Y^* has a weak* sequentially compact unit ball B_{Y^*} . Take a bounded linear operator $T : X \rightarrow Y$. Let $\{y_n^*\}_{n \in \mathbb{N}}$ be a sequence in B_{Y^*} . As B_{Y^*} is weak* sequentially compact, there exists a subsequence $\{y_{n_k}^*\}_{k \in \mathbb{N}}$ which converges weak*ly. Consider the adjoint $T^* : Y^* \rightarrow X^*$. As T^* is weak* continuous it takes $\{y_{n_k}^*\}_{k \in \mathbb{N}}$ to a weak* convergent sequence $\{T^*(y_{n_k}^*)\}_{k \in \mathbb{N}}$ in X^* . Now the Grothendieck property of X implies that $\{T^*(y_{n_k}^*)\}_{k \in \mathbb{N}}$ is actually weakly convergent. Thus T^* is weakly compact and with Gantmacher's theorem 1.2.10, T is weakly compact.

Conversely, to show that X is a Grothendieck space, using 3.1.2 it is enough to prove that any bounded linear operator T from X to a separable Banach space Y is weakly compact. So let Y be a separable Banach space. By 1.1.39 and Alaoglu

theorem 1.1.35, B_{Y^*} is metrizable and weak* compact. So B_{Y^*} is weak* sequentially compact. Thus by assumption any bounded linear $T : X \rightarrow Y$ is weakly compact. $\square$

Notice that we can deduce the above theorem from 3.1.3 as weakly compactly generated Banach spaces have weak* sequentially compact unit balls by 1.1.32. Also from the above results, we saw that if Y is in some sense a small Banach space (meaning separable, weakly compactly generated, failure to contain a copy of ℓ_1 or to have a weak* sequentially compact unit ball) then every bounded linear operator from a Grothendieck space X to Y is weakly compact.

It is easy to see that separable quotients of Grothendieck spaces are reflexive. For that purpose let X/M be a separable space where X is a Grothendieck space. By 3.1.2 the quotient map $\phi : X \rightarrow X/M$, $x \mapsto x + M$ is weakly compact since it is bounded and linear. But $\phi(X) = X/M$ is closed implying the reflexivity of X/M . In particular, no complemented subspace of a Grothendieck space is isomorphic to c_0 and c_0 or any other nonreflexive separable space cannot be a quotient space of a Grothendieck space.

Theorem 3.1.5. *For any Banach space X the following are equivalent:*

- (1) X has the Grothendieck property.
- (2) X^* is weakly sequentially complete and no quotient of X is isomorphic to c_0 .

Proof. (1) $\implies$ (2): Follows from the comment after the definition 3.1.1 and the above one.

(2) $\implies$ (1): Assume for a contradiction that X fails to have the Grothendieck property. Then we can find a weak* null sequence in X^* which is not weakly null, say $\{x_n^*\}_{n \in \mathbb{N}}$. Then as X^* is weakly sequentially complete, $\{x_n^*\}_{n \in \mathbb{N}}$ has no weakly Cauchy subsequence. Otherwise it would be a weakly convergent subsequence which forces the sequence to be weakly null. Then by 1.1.46, X^* contains an isomorphic copy of ℓ_1 and that as in the proof of 1.1.46 we can find a subsequence $\{x_{n_k}^*\}_{k \in \mathbb{N}}$ which is

equivalent to the standard unit vector basis of ℓ_1 . By being a subsequence of a weak* null sequence, $\{x_{n_k}^*\}_{k \in \mathbb{N}}$ is weak* null. So we define $T : X \rightarrow c_0$ by

$$T(x) = \{\langle x, x_{n_k}^* \rangle\}_{k \in \mathbb{N}}, \quad x \in X$$

Then $T^* : \ell_1 \rightarrow X^*$, where T^* maps the standard unit vector basis vectors of ℓ_1 to $\{x_{n_k}^*\}_{k \in \mathbb{N}}$, is an isomorphism. So T is surjective by 1.2.6 which leads to a contradiction as X has no quotient space isomorphic to c_0 . Hence the result follows. $\square$

Lemma 3.1.6. *If X is a separable Banach space containing an isomorphic copy U of c_0 , then there exists a subspace V of U isomorphic to c_0 which is complemented in X .*

Proof. Let $\{e_n\}_{n \in \mathbb{N}}$ be the basis of U equivalent to the standard unit vector basis of c_0 . Let $\{u_m^*\}_{m \in \mathbb{N}}$ be a bounded sequence in X^* satisfying

$$\langle e_n, u_m^* \rangle = \delta_{mn} = \begin{cases} 1, & n = m; \\ 0, & n \neq m. \end{cases}$$

As X is separable, B_X is separable so there exists a countable dense subset A of B_X . For any $x \in A$, $\{\langle u_n^*, x \rangle\}_{n \in \mathbb{N}}$ is bounded. So if we apply a diagonalization process to $\{\langle u_n^*, x \rangle\}_{n \in \mathbb{N}}$, we can find a convergent subsequence $\{\langle u_{n_k}^*, x \rangle\}_{k \in \mathbb{N}}$ for all $x \in A$. Since A is dense in X , actually $\{\langle u_{n_k}^*, x \rangle\}_{k \in \mathbb{N}}$ converges for all $x \in X$. Denote the limit of $\{\langle u_{n_k}^*, x \rangle\}_{k \in \mathbb{N}}$ by u_x . Let $V = \overline{\text{span}}(\{e_{n_k}\})$. Then V is isomorphic to c_0 . Now define $P : X \rightarrow V$ by

$$P(x) = \sum_{k=1}^{\infty} (\langle x, u_{n_k}^* \rangle - u_x) e_{n_k}$$

P is a bounded linear projection. Indeed,

$$\begin{aligned} P^2(x) &= P \left(\sum_{k=1}^{\infty} (\langle x, u_{n_k}^* \rangle - u_x) e_{n_k} \right) = \sum_{j=1}^{\infty} \left(\left\langle \sum_{k=1}^{\infty} (\langle x, u_{n_k}^* \rangle - u_x) e_{n_k}, u_{n_j}^* \right\rangle - u_{P(x)} \right) e_{n_j} \\ &= \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} (\langle x, u_{n_k}^* \rangle - u_x) \langle e_{n_k}, u_{n_j}^* \rangle - u_{P(x)} \right) e_{n_j} \\ &= \sum_{j=1}^{\infty} (\langle x, u_{n_j}^* \rangle - u_x) e_{n_j} - \underbrace{u_{P(x)}}_{=0} e_{n_j} = P(x) \end{aligned}$$

So P is a projection. Also as $\sum_{k=1}^{\infty} e_{n_k}$ is wuC and $\{ \langle x, u_{n_k}^* \rangle - u_x \}_{k \in \mathbb{N}}$ is a sequence in c_0 ,

$$\|P(x)\| = \left\| \sum_{k=1}^{\infty} (\langle x, u_{n_k}^* \rangle - u_x) e_{n_k} \right\| < \infty.$$

So P is bounded. Hence the result follows. $\square$

Proposition 3.1.7. *X is a Grothendieck space if and only if X contains no complemented subspace isomorphic to c_0 and every non-weakly compact operator $T : X \rightarrow c_0$ fixes a copy of c_0 .*

Proof. Assume for a contradiction that X is not Grothendieck. Then by 3.1.2, there exists a bounded linear $T : X \rightarrow c_0$ which fails to be weakly compact. By hypothesis T fixes a copy of c_0 , that is, there exists a subspace U of X isomorphic to c_0 on which T is an isomorphism. Then by 3.1.6, we can find a subspace V of $T(U)$ isomorphic to c_0 which is complemented in $T(U)$. Suppose that $P : c_0 \rightarrow V$ be the projection. Define a projection by

$$Q = T|_U^{-1} \circ P \circ T : X \rightarrow T|_U^{-1}(V)$$

onto $T|_U^{-1}(V)$. But $T|_U^{-1}(V)$ is isomorphic to c_0 which leads to a contradiction as we assumed that there is no complemented subspace isomorphic to c_0 in X .

The converse follows by 3.1.2 $\square$

3.1.7 with 2.2.4 asserts that a Banach space with the property V is a Grothendieck space if it has no complemented copy of c_0 . Also we can deduce from 3.1.7 and 3.1.6 that a Grothendieck space having an isomorphic copy of c_0 is not separable.

Examples 3.1.8. We have reflexive spaces such as $L_p(\mu)$ and ℓ_p for $1 < p < \infty$, finite dimensional $L_1(\mu)$, and finite dimensional $L_\infty(\mu)$, as the trivial examples of Grothendieck spaces. There are also non-reflexive Grothendieck spaces the examples of which are below:

- (1) $C(K)$, where K is Stonean (so in particular $L_\infty(\mu)$ since $L_\infty(\mu)$ is isometrically isomorphic to $C(K)$ for a compact Hausdorff space K .)

- (2) $C(F)$, where F is an F -space, i.e., disjoint open F_σ (union of countable number of closed sets) subsets of F have disjoint closures.
- (3) $C(K, X)$, K is finite and X is a Grothendieck space.
- (4) $C(K, X)$, X is finite dimensional and $C(X)$ is a Grothendieck space.
- (5) $B(S, \Sigma)$ bounded scalar valued Σ measurable functions defined on S where S is a set and Σ is a σ -algebra of subsets of S .
- (6) $\mathcal{L}(H)$ is a Grothendieck space when H is a Hilbert space.

(1) is the first non-trivial example of a Grothendieck space which is given by A. Grothendieck in [11, 168]. In his paper A. Grothendieck proved that for Stonean K , weak* null sequences in $rca(K)$ are weakly null. Example (2) is the improved version of A. Grothendieck's result given by Seever[26, Theorem B]. He introduced a notion called G -space for compact Hausdorff spaces K which makes $C(K)$ a Grothendieck space and proved that F -spaces are indeed G -spaces. (3) and (4) follows by the work of S. S. Khurana in [13] where he investigated the conditions under which $C(K, X)$ is a Grothendieck space. Indeed he proved that if $C(K, X)$ is a Grothendieck space either the conditions in (3) or in (4) are satisfied. Examples (5) and (6) follows by the oncoming result that any von-Neumann algebra has the Grothendieck property. Other than the above examples we also have ℓ_∞ as a non-reflexive Grothendieck space.

3.2 The space ℓ_∞ has the Grothendieck Property

Before giving the proof, first notice that ℓ_∞ is isometrically isomorphic to the space of all continuous scalar valued functions on the Stone-Ćech compactification of the natural numbers i.e., $\ell_\infty = C(\beta(\mathbb{N}))$. To show this let us introduce some notations. We denote by $B(S, \Sigma)$ the uniform limits of all finite linear combinations of characteristic functions of sets in a field Σ of subsets of S with the norm $\|f\|_\infty$. $\mathcal{B}(S, \Sigma) = \mathcal{B}(S)$ if Σ is the family of all subsets of S . In this case, $\mathcal{B}(\mathbb{N})$ is the space

of all bounded scalar functions on $\mathbb{N}$ which is obviously ℓ_∞ when we consider ℓ_∞ as $\ell_\infty = \{f : \mathbb{N} \rightarrow \mathbb{C} : f \text{ is bounded}\}$.

Hence to show that $\ell_\infty = C(\beta(\mathbb{N}))$, enough to see that there is a linear isometry from $\mathcal{B}(\mathbb{N})$ to $C(\beta(\mathbb{N}))$. For this purpose define an operator T on the collection of finite linear combinations of characteristic functions of subsets of $\mathbb{N}$, that is, on the simple functions $\sum_{k=1}^n \alpha_k \chi_{A_k}$ in $\mathcal{B}(\mathbb{N})$ by

$$\sum_{k=1}^n \alpha_k \chi_{A_k} \mapsto \sum_{k=1}^n \alpha_k \chi_{\hat{A}_k}$$

where $\hat{A}_k$ is the counterpart of A_k in the Stone-Ćech compactification $\beta(\mathbb{N})$ of $\mathbb{N}$. As $\mathbb{N}$ is discrete, $\beta(\mathbb{N})$ is a extremally disconnected compact Hausdorff space so we may assume that $\hat{A}_k$'s are clopen. Then T is a well-defined linear isometry. The simple functions $\sum_{k=1}^n \alpha_k \chi_{A_k}$'s are dense in $\mathcal{B}(\mathbb{N})$. The closure of the simple functions $\sum_{k=1}^n \alpha_k \chi_{\hat{A}_k}$ forms a closed subalgebra of $C(\beta(\mathbb{N}))$ which contains the unit and separates the points of $\beta(\mathbb{N})$. Thus by Stone Representation Theorem² and Stone-Weierstrass theorem³, they are dense in $C(\beta(\mathbb{N}))$.

So the isometry we define extends to a linear isometry of $\mathcal{B}(\mathbb{N}) = \ell_\infty$ onto $C(\beta(\mathbb{N}))$. So as the Stone-Ćech compactification of a discrete space is Stonean, $C(\beta(\mathbb{N}))$ is Grothendieck from which the result follows by 3.1.8 (1). But let us give the proof directly. To do this let us first state the following classical convergence theorem related to ℓ_∞ , namely the Phillips' Lemma:

Theorem 3.2.1. (*Phillips' Lemma*)

Let $\mu_n \in ba(\mathbb{N})$ such that $\lim_{n \rightarrow \infty} \mu_n(F) = 0$ for each subset F of $\mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} |\mu_n(\{j\})| = 0.$$

Proof. [7, p.83] □

²Stone Representation Theorem: Every Boolean algebra with a unit is isomorphic to the Boolean algebra of all clopen subsets of a extremally disconnected compact Hausdorff space.

³Stone-Weierstrass Theorem: Let S be a compact Hausdorff space and $C(S)$ be the algebra of all real continuous functions on S . Let $\mathcal{H}$ be a closed subalgebra of $C(S)$ which contains the unit 1. Then $\mathcal{H} = C(S)$ if and only if $\mathcal{H}$ separates the points of S .

Now we proceed with the result that gave the name of this subsection.

Theorem 3.2.2. ℓ_∞ has the Grothendieck property.

Proof. First of all as $\beta(\mathbb{N})$ is a compact Hausdorff space, by Riesz representation theorem 1.1.1, $C(\beta(\mathbb{N}))^* = rca(\beta(\mathbb{N}))$. Take a weak* null sequence $\{\mu_n\}_{n \in \mathbb{N}}$ in $\ell_\infty^* = C(\beta(\mathbb{N}))^*$. We want to show that $\{\mu_n\}_{n \in \mathbb{N}}$ is relatively weakly compact. So assume for a contradiction that this is not the case. Then by Dieudonne-Grothendieck theorem 2.2.12, we can find a sequence $\{O_n\}_{n \in \mathbb{N}}$ of open disjoint subsets of $\beta(\mathbb{N})$ and a subsequence $\{\nu_n\}_{n \in \mathbb{N}}$ of $\{\mu_n\}_{n \in \mathbb{N}}$ such that

$$|\nu_n(O_n)| \geq \delta$$

for some $\delta > 0$. As $\beta(\mathbb{N})$ is extremally disconnected and ν_n 's are regular, we may assume that O_n 's to be clopen. As every countable collection of elements in $2^{\mathbb{N}}$ has a supremum, the Boolean algebra of $2^{\mathbb{N}}$ is σ -complete. $\beta(\mathbb{N})$ is also σ -complete. Now define $\hat{\nu}_n \in ba(\mathbb{N})$ by

$$\hat{\nu}_n(F) = \nu_n(\sup_{k \in F} O_k), \quad F \subset \mathbb{N}$$

Now as the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ is weak* null, by being a subsequence $\{\nu_n\}_{n \in \mathbb{N}}$ is also weak* null. Also for any subset F of positive integers, $\sup_{k \in F} O_k$ is clopen as each O_k is clopen in $\beta(\mathbb{N})$. Hence the characteristic function $\chi_{\sup_{k \in F} O_k}$ belongs to ℓ_∞ and so

$$\lim_{n \rightarrow \infty} \hat{\nu}_n(F) = \lim_{n \rightarrow \infty} \nu_n(\chi_{\sup_{k \in F} O_k}) = 0$$

Thus by Phillips' lemma,

$$0 = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} |\hat{\nu}_n(\{j\})| = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} |\nu_n(\sup_{k \in \{j\}} O_k)| = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} |\nu_n(O_j)|$$

which contradicts with our assumption that $|\nu_n(O_n)| \geq \delta$. Hence $\{\mu_n\}_{n \in \mathbb{N}}$ is relatively weakly compact so that $\{\mu_n\}_{n \in \mathbb{N}}$ is weakly null. Therefore ℓ_∞ is a Grothendieck space. $\square$

What this result means is that in $(\ell_\infty)^*$ any weak* convergent sequence is actually weakly convergent. It is known that for compact and Hausdorff K or more generally

for K an F space, $C(K)$ spaces contain isomorphic copies of ℓ_∞ . A natural question is whether every non-reflexive Grothendieck space has a subspace isomorphic to ℓ_∞ . In 1981, R. Haydon[10] has given an answer to this question by constructing an infinite compact space K such that in the dual space of $C(K)$ weak and weak* convergence of sequences coincide but $C(K)$ does not contain an isomorphic copy of ℓ_∞ .

Observe that as $\ell_\infty = C(\beta(\mathbb{N}))$ and $\beta(\mathbb{N})$ is a compact Hausdorff space, so ℓ_∞ has the property V by 2.2.13. ℓ_∞ is also a dual space, i.e., $\ell_\infty = \ell_1^*$. We may guess that any dual space with the property V is a Grothendieck space which is actually true.

3.3 Any Dual Space with Pełczyński's V Property Has the Grothendieck Property

Theorem 3.3.1. *Any dual space with the Pełczyński's Property V is a Grothendieck space.*

We will prove this result by making use of the Phillips' lemma 2.1.34.

Proof. Assume that X is a Banach space with the dual X^* having the Pełczyński's property V . To show that X^* has the Grothendieck property, take any weak* null sequence $\{x_n^{**}\}_{n \in \mathbb{N}}$ in X^{**} . We need to show that this sequence is actually weakly null. So it is enough to show that $E = \{x_n^{**} : n \in \mathbb{N}\}$ in X^{**} is relatively weakly compact by the same argument as in the proof of the claim in 3.1.2. If we can show that

$$\lim_{k \rightarrow \infty} \sup_n | \langle x_k^*, x_n^{**} \rangle | = 0$$

for every wuC series $\sum_{n=1}^{\infty} x_n^*$ in X^* then the result will follow immediately as X^* has the property V .

So take a wuC series $\sum_{k=1}^{\infty} x_k^*$ in X^* . If F is any subset of $\mathbb{N}$, for any positive integer n ,

$$\langle x, \sum_{k \in F, k \leq n} x_k^* \rangle = \sum_{k \in F, k \leq n} \langle x, x_k^* \rangle \leq \sum_{k \in F, k \leq n} | \langle x, x_k^* \rangle | \leq \sum_{k=1}^n | \langle x, x_k^* \rangle |$$

As n was arbitrary, for any $x \in X$,

$$\lim_{n \rightarrow \infty} \langle x, \sum_{k \in F, k \leq n} x_k^* \rangle \leq \sum_{n=1}^{\infty} |\langle x, x_k^* \rangle| < \infty$$

Hence the sequence $\{\sum_{k \in F, k \leq n} x_k^*\}_{n \in \mathbb{N}}$ is weak* Cauchy in X^* , so bounded by 1.1.30. Thus as Alaoglu's theorem 1.1.35 implies the relative weak* compactness of bounded subsets of X^* , it has a weak* convergent subsequence. Therefore the sequence also converges to the same limit of that subsequence since it is weak* Cauchy. Hence

$$\text{weak}^* \lim_{n \rightarrow \infty} \sum_{i \in F, i \leq n} x_i^*$$

exists, say x_F^* . Now define $\mu_n : 2^{\mathbb{N}} \rightarrow \mathbb{F}$ by

$$\mu_n(F) = \langle x_F^*, x_n^{**} \rangle, \quad F \subset \mathbb{N}.$$

Then μ_n is a bounded finitely additive scalar function on $2^{\mathbb{N}}$, i.e., $\mu_n \in ba(\mathbb{N})$. As $\{x_n^{**}\}_{n \in \mathbb{N}}$ is weak* null,

$$\lim_{n \rightarrow \infty} \mu_n(F) = \lim_{n \rightarrow \infty} \langle x_F^*, x_n^{**} \rangle = 0 \text{ for any } F \subset \mathbb{N}.$$

So by applying Phillips' lemma, we get

$$0 = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |\mu_n(\{k\})| = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |\langle x_k^*, x_n^{**} \rangle|$$

which implies

$$\lim_{k \rightarrow \infty} \sup_n |\langle x_k^*, x_n^{**} \rangle| = 0$$

Therefore E is relatively weakly compact implying $\{x_n^{**}\}_{n \in \mathbb{N}}$ in X^{**} is weakly null. Thus X^* has the Grothendieck property. $\square$

3.4 Every Von-Neumann Algebra Has the Grothendieck Property

By combining the results of the sections 2.3 and 3.3, we can deduce the following as any Von-Neumann algebra is a dual space which has the Pełczyński's V property.

Theorem 3.4.1. *Any von-Neumann algebra has the Grothendieck property.* $\square$

3.5 Some Open Questions Concerning Grothendieck Spaces

There are still unsolved problems concerning the Grothendieck spaces:

- (1) If X is a reflexive space need $\mathcal{L}(X)$ be a Grothendieck space? Notice that 3.1.8 example (6) constitutes a special case when X is Hilbert.
- (2) If X is a Grothendieck space is X^{**} necessarily a Grothendieck space?
- (3) For X reflexive obviously both X and X^* has the Grothendieck property. An answer to the converse which can be stated as “if both X and X^* are Grothendieck spaces need X be reflexive?” is not known.
- (4) If Y contains no isomorphic copy of ℓ_∞ and X is a Grothendieck space need every bounded linear operator from X to Y be weakly compact?
- (5) For which class of Banach spaces Y and given any Grothendieck space X any bounded linear operator from X to Y is weakly compact? We may think by (2) of 3.1.3 that an answer to this question can be the class of weakly compactly generated spaces which includes the separable spaces. But if Y has the property that every bounded linear operator from X to Y is weakly compact where X is a Grothendieck space Y need not be weakly compactly generated. Indeed $\ell_1(\Omega)$, Ω is uncountable is a non weakly compactly generated space but satisfies the given property.
- (6) We proved in 3.2 that any dual space having the V property is a Grothendieck space. From there we may reach to the question that whether all Grothendieck spaces have the property V or not. This also remains open.

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