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DEPARTMENT OF MATHEMATICS

ALGORITHMS FOR H-BASES COMPUTATIONS

DOCTOR OF PHILOSOPHY, MATHEMATICS

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THESIS SUPERVISOR

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ACCEPTANCE AND APPROVAL PAGE

The thesis entitled **ALGORITHMS FOR H-BASES COMPUTATIONS** prepared and submitted by **Özlem ALTUNBEZEL** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Department of **MATHEMATICS** with unanimity vote is hereby approved. 7/28/2025

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ABSTRACT

ALGORITHMS FOR H-BASES COMPUTATIONS
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H-basis algorithms that do not rely on Gröbner basis calculations consist of two fundamental stages: the computation of a basis for a syzygy module and the reduction process. This thesis presents enhanced methods that utilize two distinct linear algebraic techniques, namely row reduction and QR Decomposition, for both stages. The efficacy of these methods is demonstrated through examples.

KEYWORDS: Gröbner basis, Syzygy module, H-basis, Homogeneous polynomial, QR decomposition techniques, Extended row reduction, Leading form of polynomial, Extended syzygy.

ÖZET

H-TABAN HESAPLAMALARI İÇİN ALGORİTMALAR
DOKTORA TEZİ
ÖZLEM ALTUNBEZEL
BOLU ABANT İZZET BAYSAL ÜNİVERSİTESİ
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Gröbner taban hesaplamalarına dayanmayan H-taban algoritmaları iki temel aşamadan oluşur: syzygy modülü için bir taban hesaplaması ve indirgeme süreci. Bu tezde, her iki aşama için de satır indirgeme ve QR Ayrıştırma olmak üzere iki ayrı lineer cebirsel tekniği kullanan gelişmiş yöntemler sunulmuştur. Bu yöntemlerin etkinliği örneklerle gösterilmiştir.

ANAHTAR KELİMELER: Gröbner tabanı, Syzygy modül, H-tabanı, Homojen polinom, QR ayrışım teknikleri, Genişletilmiş matris satır indirgemesi, Polinomun öncül formu, Genişletilmiş syzygy.

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LIST OF ABBREVIATIONS AND SYMBOLS

C(A) : Column space of matrix A
deg(f) : Degree of polynomial f
GenGS : General Gram – Schmidt Algorithm
$\mathbf{I_n}$: Identity matrix with dimension nxn
$<_{invlex}$: Inverse lexicographic order
$|J|$: The number of elements of the set J
LCM : Least comman multiple
LF(f) : Leading form of polynomial f
LM(f) : Leading monomial of polynomial f
$M_d(F)$: Set of polynomials with degree d
N(A) : Nullspace of matrix A
P_d : The vector space of homogeneous polynomials of degree d
$\mathbf{P^T}$: Permutation matrix
R(A) : Row space of matrix A
RREF : Row reduced echelon form
Syz(F) : Syzygy module of F
$\mathbf{syz_d(F)}$: Syzygy basis of F for degree d
$\mathbf{V_d(F)}$: The set of homogeneous polynomials of degree d of F

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1. INTRODUCTION

The concept of Gröbner bases, introduced by (Buchberger, 1965), has emerged as a significant component in addressing various issues in computational algebra; for a comprehensive overview, refer to the extensive survey provided in (Bose and Bose, 1995).

Gröbner basis serves as an invaluable tool for the study and resolution of numerous challenges across diverse disciplines, including optimization, coding theory and cryptography, signal and image processing, robotics, statistics, and beyond. One of the primary disadvantages of Gröbner bases is that their structure relies on a monomial ordering, which disrupts the symmetry among the variables. As a result, this approach may also obscure the symmetries inherent in the underlying problem, unless additional measures are taken to preserve them. Furthermore, Gröbner bases typically exhibit a lack of numerical stability, as even minor perturbations in the coefficients of the polynomials that generate an ideal can result in significant alterations to the Gröbner basis of the corresponding ideal (see Cox et al., 2007)). On the other hand, the concept of H-bases, which was introduced by Macaulay (1916) many years ago, relies exclusively on the homogeneous terms of a polynomial. Macaulay computed an H-basis exclusively for a particular example by identifying syzygies among the maximal degree homogeneous components of the polynomials, referred to as leading forms; however, he did not provide a general algorithm for this purpose. The significance of this concept was not fully comprehended, likely due to the absence of facilities for symbolic computations. On the other hand, Buchberger does not only define the concept of Gröbner bases he also gives an algorithm, called Buchberger's algorithm, to compute them. Buchberger's algorithm is implemented to many computer algebra systems. Thus, while Gröbner Bases have been applied to numerous fields, the concept of H-bases did not receive as much attention. However numerous problems in applications that can be addressed using Gröbner techniques can also be effectively resolved with H-bases. Given that H-bases are independent of monomial ordering, numerous application-related problems that arise from the use of Gröbner bases can be effectively addressed using H-bases. An overview of these problems was provided in (Möller and Sauer, 1999) and

(Möller and Sauer, 2000). Therefore, it has become essential to develop algorithms for calculating H-bases.

A Gröbner basis is also an H-basis when the monomial ordering is degree compatible (see Möller and Mora, 1986). Nonetheless, a proper subset of this Gröbner basis may still qualify as an H-basis. Yılmaz and Kılıçarslan (2004) adapt the classical Buchberger's algorithm for the purpose of deriving a minimal H-basis as a subset of a Gröbner basis. However, finding the H-basis in this way requires calculating the Gröbner basis. The structure of H-bases was initially examined in a constructive manner in (Möller and Mora ,1986).

Sauer (2001) provided an algorithm for constructing H-bases without employing any monomial ordering. This algorithm serves as a direct generalization of Buchberger's algorithm and is dependent on a reduction algorithm. This reduction facilitates a characterization of H-bases, which is predicated on the reduction of a generating set of the syzygy module of leading forms. Consequently, determining a basis for the syzygies of finitely many homogeneous forms is essential for constructing the H-basis. However, identifying a basis for the syzygies among homogeneous forms is significantly complex problem. One of the primary methods for constructing a basis for the syzygy module of an ideal involves the computation of Gröbner bases, as described by (Adams and Loustaunau, 1994). However, a Gröbner basis for the given ideal is computed as a byproduct of this method. This Gröbner basis qualifies as an H-basis if the ordering is degree compatible. Consequently, it appears more efficient to compute an H-basis using Buchberger's algorithm with a degree compatible ordering, rather than employing the previously outlined procedure and utilizing Gröbner techniques for the calculation of syzygy modules. On the other hand, the generating set for the syzygy module obtained through this method is generally not minimal. The presence of unnecessary generators within the syzygy module leads to several redundant reductions, thereby rendering the algorithm inefficient. Consequently, identifying a minimal generating set for the module of syzygies associated with a finite number of homogeneous forms is essential for enhancing the calculation of H-bases.

An improved algorithm for computing H-bases of polynomial ideals by leveraging linear algebraic techniques is presented in (Hashemi and Javanbakht, M., 2018). An H-basis is a particular type of generating set whose computation necessitates prior knowledge of a generating set for the associated syzygy module. The proposed method constructs syzygies incrementally, degree by degree, and systematically eliminates redundant generators at each stage, thereby enhancing computational efficiency and structural clarity. Javanbakht and Sauer (2019) present a numerically stable method for computing the H-basis, which is predicated on determining a minimal basis for the module of syzygies through the application of singular value decomposition.

The primary objective of this thesis is to introduce algorithms based on linear algebraic techniques for the computation of H-bases. In Chapter 2, we give the necessary concepts and terminology in linear algebra. In Section 3.1, we use extended row reduced echelon form to obtain the syzygy basis for homogeneous polynomials. We are able to define a reduction procedure depends on the row reduced form of the matrix we used in syzygy computation. Consequently, the algorithm for calculating H-bases that we have developed relies solely on the row reduction of matrices and does not incorporate any monomial ordering. In Section 3.2, H-basis algorithm is studied by using QR Decomposition techniques. In Chapter 4, the linear algebraic H-basis computation methods available in the literature were compared with the methods given in the thesis.

2. MATERIALS and METHOD

In this section, the definitions of linear algebraic tools which we used for our study are given.

2.1 The Fundamentals Subspaces of a Matrix

The following three operations change one matrix into another while keeping its size, and each is known as a row operation.

Definition 2.1.1.

1.Swap Rows ($\boldsymbol{R_i} \leftrightarrow \boldsymbol{R_j}$)**:** Interchange the positions of rows i and j in the matrix.

2.Multiply Row by a Constant ($\boldsymbol{\alpha R_i}$)**:** Multiply every element of a row by a non-zero constant α.

3.Row Addition ($\boldsymbol{\alpha R_i + R_j}$)**:** Multiply each element in row i by a non-zero constant α, and add these values to the corresponding elements in row j. Leave the row i same after this operation, but write the row j with the new values.

Definition 2.1.2. Two same size matrices are called row equivalent if one can be obtained from the other by applying a finite series of row operations.

Definition 2.1.3. A row of a matrix in which all elements are zero is called a zero row.

Definition 2.1.4. The leading entry in a row of a matrix is the first non-zero entry starting from the left.

Definition 2.1.5. A matrix is in row reduced echelon form, RREF for short, if:

1. If there exists a row containing only zero entries, then this row should be positioned below the others.
2. All leading entries are 1.
3. If a column contains a leading entry, every other entry in that column is 0.
4. Any leading entry is strictly to the right of any leading entry in the row above it.

Definition 2.1.6. A column of a matrix in row reduced echelon form containing a leading entry 1 is called a pivot column.

Definition 2.1.7. Let A be an $m \times n$ matrix. The nullspace of A, written as $N(A)$, is the set of all vectors x which satisfies the homogeneous system $Ax = 0$. It is well known that a basis for nullspace of a marix can be easily obtained from its row reduced echelon form. The following theorem gives a method to obtain the basis for nullspace of a matrix.

Theorem 2.1.8. (Beezer, 2015, Theorem SSNS) Let A be an $m \times n$ matrix, and $B = (b_{ij})$ be its row reduced echelon form with r pivot columns. Let $D = \{d_1, d_2, d_3, \dots, d_r\}$ be the set of pivot columns' indices, and $F = \{f_1, f_2, f_3, \dots, f_{n-r}\}$ be the set of non-pivot columns' indices. For $1 \le j \le n - r$ construct the $n - r$ column vectors $\vec{z}_j$ with size n as below:

$$[\vec{z}_j]_i = \begin{cases} 1, & if\ i \in F, i = f_j; \\ 0, & if\ i \in F, i \neq f_j; \\ -b_{k,f_j}, & if\ i \in D, i = d_k. \end{cases}$$

$S = \{\vec{z}_1, \vec{z}_2, \vec{z}_3, \dots, \vec{z}_{n-r}\}$ is a basis for the nullspace of A.

Notice that since the matrix B is in row reduced echelon form $b_{k,f_j} = 0$ if $k > f_j$. Therefore in each basis vector $\vec{z}_j$, $[\vec{z}_j]_{f_j} = 1$ is the last non-zero entry from the top.

Definition 2.1.9. Let A be an $m \times n$ matrix. The column space of A denoted by $C(A)$ is the spanning set of columns of A.

The following is a well known theorem for obtaining a basis for column space of a matrix.

Theorem 2.1.10. Let A be an $m \times n$ matrix with columns $A_1, A_2, A_3, \dots A_n$. Suppose B is the row reduced echelon form of A, with r pivot columns. Define $D = \{d_1, d_2, d_3, \dots, d_r\}$ as the set of pivot columns' indices. Let T be the set of columns corresponding to the pivot columns, i.e., $T = \{A_{d_1}, A_{d_2}, A_{d_3}, \dots, A_{d_r}\}$. Then T is a basis for column space of A.

Definition 2.1.11. Let A be an $m \times n$ matrix. The row space of A is the spanning set of rows of A. Notice that, $R(A) = C(A^T)$.

Theorem 2.1.12. Suppose that A is a matrix, and B is a row equivalent matrix in row reduced echelon form. The non-zero rows of B give a basis for the row space of A.

Example 2.1.13. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ be 6×5 matrix, its row reduced echelon form is

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Pivot elements are in $1^{st}, 2^{nd}, 3^{rd}$ and 5^{th} columns. So, the set of pivot columns' indices is $D = \{1,2,3,5\}$ and the set containing non pivot columns' indices is $F = \{4\}$.

By using Theorem 2.1.8, for $1 \leq j \leq (5-4)$ construct the 1 column vector $\vec{z}_1$ with size 5,

$$\vec{z}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then, $S = \left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the nullspace of A.

T is the set of columns corresponding to the pivot columns, and gives a basis for column space of A,

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The non-zero rows of B give a basis for the row space of A, which are

$$\{[1 \quad 0 \quad 0 \quad 0 \quad 0], [0 \quad 1 \quad 0 \quad 1 \quad 0], [0 \quad 0 \quad 1 \quad 0 \quad 0], [0 \quad 0 \quad 0 \quad 0 \quad 1]\}.$$

In the rest of this section, we consider extended row reduced echelon form of a matrix. Thus we will be able to extend a column space basis of an $m \times n$ matrix A to whole basis of $\mathbb{R}^m$ and we also give an alternative method for obtaining a basis for column space of a matrix. These results will be very useful to us in construction of H-bases of polynomial ideals.

Definition 2.1.14. Consider an $m \times n$ matrix A. Extend A on its right side with the addition of an $m \times m$ identity matrix to form an $m \times (m+n)$ matrix denoted as M. By using row operations, bring M to row reduced echelon form and denote the result as N. N is referred to as the extended row reduced echelon form of A. In this process, five submatrices (B, C, J, K, L) of N are named as follows:

1. B: Denotes the $m \times n$ matrix formed from the first n columns of N.

2. C: Represents the $r \times n$ matrix formed from all the nonzero rows of B (where B has r nonzero rows).

3. J: Denotes the $m \times m$ matrix formed from the last m columns of N.

4. K: Represents the $r \times m$ matrix formed from the first r rows of J.

5. L: Denotes the $(m-r) \times m$ matrix formed from the bottom $(m-r)$ rows of J.

Visually,

$$M = [A \mid I_m] \xrightarrow{RREF} N = [B \mid J] = \begin{bmatrix} C & K \\ 0 & L \end{bmatrix}.$$

Example 2.1.15. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

Then, the extended matrix $M = [A \ | I_m]$ will be

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & \vdots & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The row reduced echelon form of M, called N will be

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Submatrices are

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Theorem 2.1.16. Consider an $m \times n$ matrix A. Let N be extended row reduced echelon form of A. If $\{d_1, d_2, d_3, \dots, d_r\}$ is the set of indices of pivot columns of C and $\{i_1, i_2, i_3, \dots, i_{m-r}\}$ is the set of indices of pivot columns of L, then $\{A_{d_1}, A_{d_2}, A_{d_3}, \dots, A_{d_r}\}$ is a basis for column space of A and $\{e_{i_1}, e_{i_2}, e_{i_3}, \dots, e_{i_{m-r}}\}$ is a basis for the complement of column space of A where e_{i_j} is i_j-th unit vector of $\mathbb{R}^m$. Therefore $ß = \{A_{d_1}, A_{d_2}, A_{d_3}, \dots, A_{d_r}, e_{i_1}, e_{i_2}, e_{i_3}, \dots, e_{i_{m-r}}\}$ is a basis for $\mathbb{R}^m$.

Furthermore, if $b \in \mathbb{R}^m$, then Jb is the coordinate vector of b with respect to basis ß.

Proof. Since the matrix B is row reduced echelon form of A, Theorem 2.1.10 implies that the set $\{A_{d_1}, A_{d_2}, A_{d_3}, \dots, A_{d_r}\}$ is a basis for column space of A. Similarly, since the matrix N is row reduced echelon form of the matrix M ß $= \{A_{d_1}, A_{d_2}, A_{d_3}, \dots, A_{d_r}, e_{i_1} e_{i_2}, e_{i_3}, \dots, e_{i_{m-r}}\}$ is a basis for $\mathbb{R}^m$. Hence $\{e_{i_1}, e_{i_2}, e_{i_3}, \dots, e_{i_{m-r}}\}$ must be a basis for the complement of column space of A.

Let $b \in \mathbb{R}^m$. To find coordinates of b with respect to ß, we have to obtain row reduced echelon form of augmented matrix $[M|b]$ which will be the matrix $[N|\,Jb]$. Therefore Jb is the coordinate vector of b with respect to basis ß.

Example 2.1.17. Consider the matrix A in the previous example. $d_1 = 1, d_2 = 2$, $d_3 = 3, d_4 = 5$ and $e_1 = 1,\ e_2 = 4$.

Hence

$$C(A) = \left\langle \begin{bmatrix}1\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\\0\end{bmatrix} \right\rangle,$$

$$(C(A))^c = \left\langle \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix} \right\rangle,$$

and

$$Jb = \begin{bmatrix}0 & 1 & 1 & 0 & -1 & -1\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & -1 & 0 & 0 & 1 & 0\\ 1 & -1 & -1 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix} \begin{bmatrix}b_1\\b_2\\b_3\\b_4\\b_5\\b_6\end{bmatrix} = \begin{bmatrix} b_2 + b_3 - b_5 - b_6\\ b_2\\ b_6\\ -b_2 + b_5\\ b_1 - b_2 - b_3 + b_5 + b_6\\ b_4\end{bmatrix}.$$

Then

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = (b_2 + b_3 - b_5 - b_6)\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b_6\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + (-b_2 + b_5)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$+ (b_1 - b_2 - b_3 + b_5 + b_6)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The following theorem gives an alternative method for computing a basis for the column space of a matrix.

Theorem 2.1.18. (Beezer, 2015, Theorem FS) Let A be an $m \times n$ matrix with extended row reduced echelon form N, which is given in Definition 2.1.14. Then, the column space of A is the nullspace of L, $C(A) = N(L)$.

According to the above theorem the alternative basis for the column space of the matrix A in the previous example is given

$$C(A) = N(L) = \left\langle \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\rangle.$$

2.2 QR Decomposition

In this section the primary knowledge and calculation methods in QR Decomposition are given. The detailed information can be found in (Börgers, 2022).

Let $a_1, a_2, a_3, \dots, a_n \in \mathbb{R}^m$ be independent vectors. Then the Gram-Schmidt Procedure finds orthonormal vectors $q_1, q_2, q_3, \dots, q_n$ such that $span(a_1, a_2, a_3, \dots, a_r) = span(q_1, q_2, q_3, \dots, q_r)$ for $r \leq n$.

The following algorithm shows how Gram-Schmidt Procedure works.

Algorithm 2.2.1. (Gram – Schmidt Algorithm)

Input: Vectors $a_1, a_2, \dots, a_n$

Output: Vectors $q_1, q_2, \dots, q_n$

For $i = 1$ **To** n

$\tilde{q}_i = a_i - \sum_{j=1}^{i-1} < q_j, a_i > q_j$

$q_i = \frac{\tilde{q}_i}{\|\tilde{q}_i\|}$

End For

Return $q_1, q_2, \dots, q_n$

Theorem 2.2.2. Let A be an $m \times n$ matrix with linearly independent columns. Then A can be factored as $A = QR$ where Q is an $m \times n$ matrix whose columns are orthonormal and R is an $n \times n$ invertible upper triangular matrix.

The columns of Q can be obtained by applying Gram-Schmidt Procedure to the columns of A.

The entries of R, r_{ij} can be calculated by

$$r_{ij} = \begin{cases} 0, & j < i; \\ < a_j, q_i >, & j \geq i. \end{cases}$$

Where a_j is the j^{th} column of A and q_i is the i^{th} column of Q.

Notice that the columns of Q is an orthonormal basis for the column space of A.

Let A be an $m \times n$ matrix. If the columns of A are linearly dependent, in the given Gram-Schmidt Procedure some of the vectors will be zero vector.

The following General Gram-Schmidt Procedure finds $r \leq n$ orthonormal vectors which form a basis for column space of A.

Algorithm 2.2.3. (General Gram-Schmidt Algorithm (GenGS))

$\boldsymbol{Input}$: $A\ m \times n\ matrix$

$\boldsymbol{Output}$: $(Q, r, C, index)$

$r = 0$

$index = [\,]$

For $i = 1$ **To** n

$\tilde{q}_i = a_i - \sum_{j=1}^{i-1} < q_j, a_i > q_j$

If $\tilde{q}_i \neq 0$ **Then**

$r = r + 1$

$q_r = \frac{\tilde{q}_i}{\|\tilde{q}_i\|}$

$c_r = a_i, append\ i\ to\ index$

End If

End For

$Q = [q_1, q_2, \dots, q_r], C = [c_1, c_2, \dots, c_r]$

Return $(Q, r, C, index)$

Example 2.2.4. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$ be 4×5 matrix.

The columns of A are represented by a_i where $1 \le i \le 5$. Since the number of columns of A is greater than the number of rows, it is clear that the columns of A are linearly dependent. So, we will apply GenGS Algorithm to A.

Now, $r = 0$ and $n = 5$

- $\tilde{q}_1 = a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

  Since $\tilde{q}_1 \neq 0$,

  $q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, then $r = 1$ and $c_1 = a_1, index = [1]$.

- $\tilde{q}_2 = a_2 - < q_1, a_2 > q_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

  Since $\tilde{q}_2 \neq 0$,

  $q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $r = 2, c_2 = a_2,\ index = [1,2]$.

- $\tilde{q}_3 = a_3 - < q_1, a_3 > q_1 - < q_2, a_3 > q_2 = 0$

  Since $\tilde{q}_3 = 0$, r is again 2. So,

  $\tilde{q}_3 = a_4 - < q_1, a_4 > q_1 - < q_2, a_4 > q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

Since $\tilde{q}_3 \neq 0$,

$$q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

Now, $r = 3$, $c_3 = a_4$ and $index = [1,2,4]$

- $\tilde{q}_4 = a_5 - < q_1, a_5 > q_1 - < q_2, a_5 > q_2 - < q_3, a_5 > q_3 = 0.$

By applying the General Gram-Schmidt Procedure we evaluate the matrices Q and R as below:

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Even if $A = QR$, in the above example, the number of columns of Q is less than the number of columns of A and R is not a square matrix. Therefore, we will give a new definition for QR Decomposition.

Definition 2.2.5. Let A be an $m \times n$ matrix and $rank(A) = r < n$, Full QR Decomposition of A is

$$A = [Q_1 \quad Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

where $A = Q_1 R_1$ is the QR Decomposition as in General QR Decomposition, $Q_1$is an $m \times r$, R_1 is an $r \times n$ and Q_2 is an $m \times (m - r)$ matrices.

In the Full QR Decomposition Process, the main idea is to use a matrix which has full rank. In fact, we will get this matrix by adding unit vectors to the columns of A untill getting a full rank matrix.

Let e_j be the j^{th} unit vector in $\mathbb{R}^m$. Then $u_i = e_{m-i}$. We will use this notation in the following algorithm.

Algorithm 2.2.6. (Full QR Algorithm)

Input: A $m \times n$ matrix, such that $rank(A) < m$

Output: $Q_1, Q_2, vectorIndex$

$(Q_1, r, C, index) = GenGS(A)$

vectorIndex=∅

If $r = m$ **Then**

Return Q_1

$i = 1$

$k = 1$

While $k \leq m - r$

$\tilde{p}_k = u_i - \sum_{j=1}^{r} < u_i, q_j > u_i - \sum_{j=1}^{k-1} < u_i, p_k > u_i$

If $\tilde{p}_k \neq 0$ **Then**

$p_k = \frac{\tilde{p}_k}{\|\tilde{p}_k\|}$

$k = k + 1$

Add $m - i$ to vectorIndex

$i = i + 1$

Else

$i = i + 1$

End If-Else

End While

$Q_2 = [p_1 \quad p_2 \quad \cdots \quad \cdots \quad p_{m-r}]$

Return $(Q_1, Q_2, vectorIndex)$

Example 2.2.7. In the previous example, we found that $Q_1 = Q$ and $r = 3$. Now we will find Full QR Decomposition of

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Since $r = 3 < 4$, take $i = 1, k = 1$ and $m - r = 4 - 3 = 1$.

$u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and so,

$$\tilde{p}_1 = u_1 - < u_1, q_1 > u_1 - < u_1, q_2 > u_1 - < u_1, q_3 > u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Since $\tilde{p}_1 \neq 0$,

$$p_1 = \frac{\tilde{p}_1}{\|\tilde{p}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

Then $k = 2, i = 2$. Since $k = 2 \leq 1$ is not true, $Q_2 = [p_1]$.

Thus

$$Q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

and

$$Q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

Note that the columns of Q_1 form an orthonormal basis for the column space of A, and the columns of Q_2 form an orthonormal basis for the complement of column space of A. Hence, they together form an orthonormal basis for $\mathbb{R}^m$.

Definition 2.2.8. A matrix R is in upper staircase form if each of its rows starts a certain number of leading zeros and the number of leading zeros increases as you move down to rows.

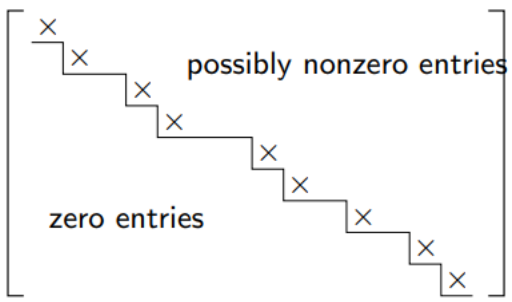

Figure 2.1 Upper Staircase Form

Notice that, in Full QR Decomposition the matrix R_1 is in upper staircase form.

Definition2.2.9. Let $A = [Q_1 \quad Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ be Full QR Decomposition of A. We can write R_1 as

$$[R_{11} \quad R_{12}]$$

where R_{11} is $r \times r$ upper triangular matrix by interchanging columns of R_1. Hence

$$A = [Q_1 \quad Q_2]\begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} P^T$$

where P is an $n \times n$ permutation matrix. This form is called Pivoted QR Decomposition of A.

Example 2.2.10. For the previous example, if we continue by evaluating matrix R, the Full QR Decomposition of the matrix A will be

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & \vdots & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \vdots & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & \vdots & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \vdots & 0 \end{bmatrix} \\ Q_1 \qquad\qquad \vdots \quad Q_2 \end{array} \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} R_1 \\ \cdots \\ 0 \end{array}$$

In this example, interchanging third and fourth columns of R allows us to write R as $\begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}$ where R_{11} is upper triangular. For this purpose, our permutation matrix will be

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$
\begin{array}{c} R_{11} \qquad\qquad R_{12} \\ \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} & \vdots & \sqrt{2} & 0 \\ 0 & \sqrt{2} & \frac{\sqrt{2}}{2} & \vdots & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & \vdots & 0 & \frac{\sqrt{2}}{2} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & 0 \end{bmatrix} \end{array}
\overset{P^T}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}
=
\overset{R}{\begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}
$$

and the Pivoted QR Decomposition of A is

$$A = [Q_1 \quad Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} P^T.$$

Theorem 2.2.11. For an $m \times n$ matrix A with $rank(A) = r < n$ let

$$A = [Q_1 \quad Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} P^T$$

be Pivoted QR Decomposition of the matrix A. Then the columns of

$$P \begin{bmatrix} -R_{11}^{-1} R_{12} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$$

form a basis for the nullspace of A.

Proof.

$$
\begin{aligned}
AP \begin{bmatrix} -R_{11}^{-1} R_{12} \\ I \end{bmatrix} &= [Q_1 \quad Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} P^T P \begin{bmatrix} -R_{11}^{-1} R_{12} \\ I \end{bmatrix} \\
&= [Q_1 \quad Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -R_{11}^{-1} R_{12} \\ I \end{bmatrix}.
\end{aligned}
$$

Since $P^T P = I$

$$
\begin{aligned}
&[Q_1 \quad Q_2] \begin{bmatrix} -R_{11} R_{11}^{-1} R_{12} + R_{12} I \\ 0 \end{bmatrix} \\
&= [Q_1 \quad Q_2] \begin{bmatrix} -R_{12} + R_{12} \\ 0 \end{bmatrix} = [Q_1 \quad Q_2] \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0.
\end{aligned}
$$

By construction of $P \begin{bmatrix} -R_{11}^{-1} R_{12} \\ I \end{bmatrix}$, the columns are linearly independent. So they form a basis for the nullspace of A.

Example 2.2.12. In Example 2.2.10, we found

$$R_{11} = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}, R_{12} = \begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$R_{11}^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\sqrt{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}, -R_{11}R_{12} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$P\begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

Hence $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ is a basis for the nullspace of A.

Theorem 2.2.13. For an $m \times n$ matrix A with $rank(A) = r$ and $r < n$, let

$$A = [Q_1 \quad Q_2]\begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}P^T$$

be Pivoted QR Decomposition of the matrix A. Suppose $AP = [B \quad C]$ where B is $m \times r$ and C is $m \times (n-r)$ matrices. Then

$$Q_1 = BR_{11}{}^{-1}.$$

Proof. Since $P^TP = I$,

$$AP = [B \quad C] = [Q_1 \quad Q_2]\begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} = [Q_1R_{11} \quad Q_2R_{12}].$$

Therefore $B = Q_1R_{11}$ implies $Q_1 = BR_{11}{}^{-1}$.

2.3 H-Bases and Syzygies

In this section we review some definitions and the basic facts related to H-bases. Throughout this thesis, the polynomial ring in variables $x_1, x_2, \dots, x_n$ over a field K will be denoted by $P = K[x_1, x_2, \dots, x_n]$. For a monomial $x^\alpha = x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_n}$, the total degree of x^α is defined as $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The total degree of a polynomial $f \in P$, denoted by $\deg(f)$, is the highest degree of the monomials of f. The vector space of homogeneous polynomials of degree d is denoted by P_d. It is clear that the set of monomials of degree d is a basis for P_d. Therefore the dimension of P_d is

$$\binom{n + (d-1)}{d}.$$

Definition 2.3.1. For a polynomial $f = f^0 + f^1 + \cdots + f^d$ where $d = \deg(f)$, $f^i \in P_i$ with $f^d \neq 0$, the leading form of f is defined as $\mathrm{LF}(f) = f^d$.

Definition 2.3.2. For an ideal $I = \langle f_1, f_2, \dots, f_s \rangle \subset P$ the leading form of I is defined as the set homogeneous polynomials

$$\mathrm{LF}(I) = \{\mathrm{LF}(f) | f \in I\}.$$

Definition 2.3.3. Given $(p_1, p_2, \dots, p_s) \in P^s$, a syzygy basis of polynomials $(p_1, p_2, \dots, p_s)$ is a tuple $(q_1, q_2, \dots, q_s) \in P^s$ such that

$$\sum_{j=1}^{s} q_j \, p_j = 0.$$

The syzygy module of $p_1, p_2, \dots . p_s$, denoted by $Syz(p_1, p_2, \dots . p_s)$, is the set of all syzygies of $p_1, p_2, \dots . p_s$.

Now we ready to give the definition of H-basis.

Definition 2.3.4. The set $\{f_1, f_2, \dots . f_s\} \subset P$ is called an H-basis for the ideal $I = \langle f_1, \dots f_s \rangle$, if one of the following equivalent conditions holds:

1. If $f \in I$, then $\mathrm{LF}(f) \in \langle \mathrm{LF}(f_1), \dots, \mathrm{LF}(f_s) \rangle$.

2. Any $f \in I$ can be witten as

$$f = \sum_{i=1}^{s} h_i \, f_i, \qquad h_i \in P_{\leq \deg(f) - \deg(f_i)}, i = 1, \dots . s.$$

3. If $(h_1, h_2, \dots . h_s) \in Syz\big(\mathrm{LF}(f_1), \dots, \mathrm{LF}(f_s)\big)$ then there exists $g_1, \dots, g_s \in P$ such that

$$\sum_{i=1}^{s} h_i f_i = \sum_{i=1}^{s} g_i f_i, \qquad \deg(g_i f_i) \geq \deg\left(\sum_{i=1}^{s} h_i f_i\right), \; i = 1, \dots, s.$$

To give an algorithm for computation of an H-basis, we first define a reduction procedure which is extension of the division algorithm in polynomial rings.

Algorithm 2.3.5. (Reduction Procedure Algorithm)

Input: $F = \{f_1, f_2, \dots, f_s\}$, $g \in P$

Output: $g = \sum q_{f_i} f_i + r$, $q_{f_i} f_i \in P$

$r = 0, q_{f_1} = 0, q_{f_2} = 0, \dots\ q_{f_s} = 0$

$d = \deg(g)$

For $e = d$ **To** $\min\{\deg(f_i)\}$

$\mathrm{LF}(g) = \sum q_{e,f_i} \mathrm{LF}(f_i) + r_e$

$g \coloneqq g - \sum q_{e,f_i} f_i - r_e$

$r \coloneqq r + r_e$

$q_{f_i} \coloneqq q_{f_i} + q_{e,f_i}$

End For

Return g

We will denote this procedure as $g \xrightarrow{F} r$. By Definition 2.3.4. part (3), $\{f_1, f_2, \dots . f_s\}$ is an H-basis for the ideal $I = \langle f_1, \dots f_s \rangle$ if and only if for each $(h_1, h_2, \dots, h_s) \in Syz\big(\mathrm{LF}(f_1), \dots, \mathrm{LF}(f_s)\big)$,

$$\sum h_i f_i \xrightarrow{F} 0.$$

Hence we can compute an H-basis of an ideal generated by $\{f_1, f_2, \dots, f_s\}$ using the following modified Buchberger's Algorithm.

Algorithm 2.3.6. (H-Basis Algorithm)

Input: $F = \{f_1, f_2, \dots . f_s\} \subset P$

Output: An H-basis for $\langle F \rangle$

$G \coloneqq$ a basis for $Syz\big(\mathrm{LF}(f_1), \dots, \mathrm{LF}(f_s)\big)$

While $G \neq \emptyset$

Select $(h_1, h_2, \dots, h_s) \in G$

$G = G - (h_1, h_2, \dots, h_s)$

Let $\sum h_i f_i \xrightarrow{F} h$

If $h \neq 0$ **then**

$F \coloneqq F \cup \{h\}$

$G \coloneqq$ A basis for $Syz(\mathrm{LF}(F))$

End if

End While

Return F

To find an H-basis for a given ideal, we have to compute a syzygy basis for leading forms of generators of the ideal and we need to apply reduction procedure to polynomials. We will show that we can do these both operations using linear algebraic methods. To do this, we need to define a matrix correspondence for a set of homogeneous polynomials.

Definition 2.3.7. Given a set of homogenous polynomials $F = \{f_1, f_2, \dots, f_s\} \subset P$, define the ideal $I = \langle f_1, f_2, \dots, f_s \rangle$. The set of homogeneous polynomials of degree d of I is

$$V_d(F) = \{f \in I \mid \deg(f) = d\}.$$

It is well known that $V_d(F)$ is a subspace of P_d and

$$M_d(F) = \{x^{\alpha_j} f_i \mid |x^{\alpha_j}| = d - \deg(f_i), i = 1,2, \dots, s\}$$

is a spanning set for $V_d(F)$.

The number of elements of $M_d(F)$ is

$$\sum_{i=1}^{s} \binom{n + d - \deg(f_i) - 1}{d - \deg(f_i)}.$$

Since each polynomial $x^{\alpha_j} f_i$ can be written as a linear combination of monomials of degree d, we can obtain coordinate vector of $x^{\alpha_j} f_i$ with respect to any given order of monomials of degree d. Writing each coordinate vector as a column, $M_d(F)$ can be interpreted as a matrix also. We use $M_d(F)$ to represent both the set of polynomials and their matrix representation. Notice that different orders of polynomials $x^{\alpha_j} f_i$ and monomials of degree d gives different matrix interpretations. We will always use lexicographic order in monomials of degree d, but we will specify which orderings in polynomials $x^{\alpha_j} f_i$ are used when necessary.

Example 2.3.8. Let $F = \{f_1, f_2, f_3\}$, where

$f_1 = x_1 + x_3,\ f_2 = x_1 x_2 + x_2 x_3,\ f_3 = x_1 x_3 + x_2 x_3$ in $\mathbb{R}[x_1, x_2, x_3]$.

Then $M_2(F) = \{x_1 f_1, x_2 f_1, x_3 f_1, f_2, f_3\}$. The matrix represantation of $M_2(F)$ is

$$
\begin{array}{c}
 \\ x_1^2 \\ x_1x_2 \\ x_1x_3 \\ x_2^2 \\ x_2x_3 \\ x_3^2
\end{array}
\begin{array}{c}
\begin{array}{ccccc} x_1f_1 & x_2f_1 & x_3f_1 & f_2 & f_3 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

3. RESULTS

3.1H-Bases Computation via Row Reduction

3.1.1 Syzygy Computation

Calculating a basis for a syzygy module is a fundamental aspect of the methodology involved in constructing H-bases. However, the conventional approach to accomplish this task primarily relies on computations involving Gröbner bases, which renders the H-bases algorithm less than optimal in terms of efficiency. In this section, we present an algorithm designed to compute a basis for the syzygy module associated with a sequence of homogeneous polynomials, utilizing techniques from linear algebra.

Let $F = \{f_1, f_2, \dots, f_s\}$ be a set of homogeneous polynomials, and $d \in \mathbb{N}$.

If $u \in$ nullspace$(M_d(F))$, then

$$\sum_{i=1}^{s} \sum_{|\alpha|=d-\deg(f_i)} u_{i\alpha} x^{\alpha} f_i = 0.$$

Therefore

$$\sum_{i=1}^{s} h_i f_i = 0$$

where

$$h_i = \sum_{|\alpha|=d-\deg(f_i)} u_{i\alpha} x^{\alpha}.$$

Hence an element of nullspace of $M_d(F)$ corresponds to a syzygy in $syz_d(f_1, f_2, \dots, f_s)$. Finding a basis for the nullspace of $M_d(F)$ gives us a basis for the subspace $syz_d(f_1, f_2, \dots, f_s)$.

Example 3.1.1. Let $F = \{f_1, f_2, f_3\}$ where $f_1 = x_1 + x_3$, $f_2 = x_1x_2 + x_2x_3$, $f_3 = x_1x_3 + x_2x_3$ in $R[x_1, x_2, x_3]$. Then $M_2(F) = \{x_1f_1, x_2f_1, x_3f_1, f_2, f_3\}$, and the matrix representation of $M_2(F)$ is

$$
\begin{array}{c|ccccc}
 & x_1 f_1 & x_2 f_1 & x_3 f_1 & f_2 & f_3 \\
\hline
x_1^2 & 1 & 0 & 0 & 0 & 0 \\
x_1 x_2 & 0 & 1 & 0 & 1 & 0 \\
x_1 x_3 & 1 & 0 & 1 & 0 & 1 \\
x_2^2 & 0 & 0 & 0 & 0 & 0 \\
x_2 x_3 & 0 & 1 & 0 & 1 & 1 \\
x_3^2 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

and its row reduced echelon form is

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The basis for the nullspace of $M_2(F)$ is

$$
\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.
$$

The corresponding syzygy is

$$-x_2 f_1 + f_2.$$

As above example, for each degree $, d,$ we can find a basis of $syz_d\big(\mathrm{LF}(f_1), \ldots, \mathrm{LF}(f_s)\big)$; to find the basis of $Syz\big(\mathrm{LF}(f_1), \ldots \mathrm{LF}(f_s)\big)$ we will take the union of these basis for an appropriate choice of d. So, we need to know up to for which values of d we will compute $syz_d\big(\mathrm{LF}(f_1), \ldots \mathrm{LF}(f_s)\big)$. To find such a degree bound we need the following definitions.

Definition 3.1.2. For a polynomial $f \in P$, the leading monomial of f, denoted by $\mathrm{LM}(f)$, is the largest monomial with respect to given monomial order $<$ in $x_1, x_2, \ldots, x_n$.

For a given set $F = \{f_1, f_2, \ldots, f_s\}$ and the ideal $I = \langle f_1, f_2, \ldots, f_s \rangle$, we define

$$\mathrm{LM}(F) = \{\mathrm{LM}(f) | f \in I\}$$

and for a non-negative number d,

$$\mathrm{LM_d}(F) = \{\mathrm{LM}(f) | f \in F \text{ and } \deg(f) = d\}.$$

To find an appropriate bound, we need to find this set. Since each column of $M_d(F)$ corresponds an element of $V_d(F)$, basis for the column space of $M_d(F)$ gives a basis for $V_d(F)$.

In the above example, pivot elements are in $1^{st}, 2^{nd}, 3^{rd}\ and\ 5^{th}$ columns. So, corresponding columns of $M_2(F)$ gives a basis for column space of $V_2(F)$ which are

$$\left\{\begin{bmatrix}1\\0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\1\\0\end{bmatrix}\right\}.$$

If we construct a matrix which takes these vectors as rows, and find its row reduced echelon form, that will be

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0 & -1\\0 & 1 & 0 & 0 & 0 & 1\\0 & 0 & 1 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 1 & -1\end{bmatrix}.$$

The rows of the above matrix form a basis for $V_2(I)$. The corresponding polynomials are

$$\{x_1^2 - x_3^2, x_1x_2 + x_3^2, x_1x_3 + x_3^2, x_2x_3 - x_3^2\}.$$

Hence $\mathrm{LM}_2(F) = \{x_1^2, x_1x_2, x_1x_3, x_2x_3\}$ with respect to lexicographic order.

In the above example, to find a basis for $syz_d(F)$ and the set $\mathrm{LM_d}(F)$, we found row reduced echelon form of two different matrices separately. Instead of this, we can find a basis for $syz_d(F)$ and $\mathrm{LM_d}(F)$ from extended row reduced echelon form of $M_d(F)$.

Definition 3.1.3. Let $x^\alpha,\ x^\beta \in P_d$ where $\alpha = (a_1, a_2, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_n)$. Then we define inverse lexicographic (invlex) order in P_d as $x^\alpha <_{invlex} x^\beta$ if and only if there exists $1 \le i \le n$ such that

$$a_n = b_n, \dots, a_{i+1} = b_{i+1}, a_i < b_i.$$

Theorem 3.1.4. Let $N = \begin{bmatrix}C & K\\0 & L\end{bmatrix}$ be extended row reduced echelon form of $M_d(F)$. The set $\mathrm{LM_d}(F)$ can be found by the basis of nullspace of L.

Proof. Theorem 2.1.17 implies that $N(L) = C(M_d(F))$.

Each column in the basis of $N(L)$ corresponds a polynomial

$$f = x^{\alpha} + \text{lower term}$$

with respect to invlex order. So $\mathrm{LM_d}(f) = x^{\alpha}$. Hence, the assertion easily follows.

Example 3.1.5. Let $F = \{f_1 = x_1 + x_3, f_2 = x_1x_2 + x_2x_3, f_3 = x_1x_3 + x_2x_3\}$ and $I = \langle f_1, f_2, f_3 \rangle$, then

$$[M_2(F) \mid I_6] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & \vdots & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The extended row reduced echelon form of this matrix is

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Here, the nullspace of

$$L = \begin{bmatrix} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

It implies that $V_2(I) = \{x_3^2 - x_1^2, x_3x_2 - x_1^2, x_3x_1 + x_1^2, x_2x_1 + x_1^2\}$.

Hence $\mathrm{LM}_2(F) = \{x_3^2, x_3x_2, x_3x_1, x_2x_1\}$ with respect to invlex order.

For each $x^{\alpha} \in LM_d(F)$, there is a polynomial $f \in \langle V_d(F) \rangle$ such that $\mathrm{LM}(f) = x^{\alpha}$. Then for any monomial x^{β}, $x^{\beta} f \in \langle V_e(F) \rangle$ where $e = |x^{\alpha}| + |x^{\beta}|$, $x^{\alpha}x^{\beta} \in \mathrm{LM_e}(F)$. For each $x^{\alpha} \in \mathrm{LM_d}(F)$ and $e > d$ we define the following set

$$A_{\alpha}^{e} = \{x^{\alpha}x^{\beta} \mid |x^{\alpha}| + |x^{\beta}| = e\}.$$

Lemma 3.1.6. For each $x^{\alpha} \in \mathrm{LM_d}(F)$ and $e > d$, the number of elements of A_{α}^{e} is

$$|A_{\alpha}^{e}| = \binom{n + e - d - 1}{e - d}.$$

Proof. For each $x^{\alpha}x^{\beta} \in A_{\alpha}^{e}$, $|x^{\beta}| = e - d$. Since the number of monomial of degree $e - d$ is

$$\binom{n+e-d-1}{e-d},$$

the assertion easily follows.

In the above example, $\mathrm{LM}_2(F) = \{x_3^2, x_3x_2, x_3x_1, x_2x_1\}$. The monomial x_3^2 gives the set

$$A_{\alpha_1}^3 = \{x_3^3, x_3^2x_2, x_3^2x_1\}.$$

On the other hand the monomial x_3x_2 gives the set

$$A_{\alpha_2}^3 = \{x_3^2x_2, x_3x_2^2, x_3x_2x_1\}.$$

Notice that $x_3^2x_2$ is common in both sets. Hence we count this monomial twice when we tried to find $\mathrm{LM}_3(F)$. The Inclusion-Exclusion Principles solves this problem.

Lemma 3.1.7. Let $\mathrm{LM_d}(F) = \{x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_s}\}$ and $e > d$. Then

$$|\mathrm{LM_e}(F)| = \left|\bigcup_{i=1}^{s} A_{\alpha_i}^{e}\right| = \sum_{i=1}^{m} (-1)^{i+1} \left(\sum_{1 \le j_1 < \cdots < j_i \le s} \left| A_{\alpha_{j_1}}^{e} \cap \dots \cap A_{\alpha_{j_s}}^{e} \right| \right)$$

Proof. It is easily follows from Inclusion-Exclusion Principle. Notice that if $x^{\gamma} = \mathrm{LCM}(x^{\alpha_i}, x^{\alpha_j})$

$$A_{\alpha_i}^{e} \cap A_{\alpha_j}^{e} = \left\{ x^{\gamma}x^{\beta} \middle| |x^{\gamma}| + |x^{\beta}| = e \right\}$$

Example 3.1.8. In the above example

$$\mathrm{LM}_2(F) = \{x^{\alpha_1}, x^{\alpha_2}, x^{\alpha_3}, x^{\alpha_4}\} = \{x_3^2, x_3x_2, x_3x_1, x_2x_1\}.$$

For $e > 2$,

$$|A_{\alpha_1}^{e}| = |A_{\alpha_2}^{e}| = |A_{\alpha_3}^{e}| = |A_{\alpha_4}^{e}| = \binom{3+e-2-1}{e-2} = \binom{e}{e-2}.$$

Since

$$\mathrm{LCM}(x^{\alpha_1}, x^{\alpha_2}) = x_3^2x_2, \mathrm{LCM}(x^{\alpha_1}, x^{\alpha_3}) = x_3^2x_1, \mathrm{LCM}(x^{\alpha_1}, x^{\alpha_4}) = x_3^2x_2x_1$$
$$\mathrm{LCM}(x^{\alpha_2}, x^{\alpha_3}) = x_3x_2x_1, \mathrm{LCM}(x^{\alpha_2}, x^{\alpha_4}) = x_3x_2x_1, \mathrm{LCM}(x^{\alpha_3}, x^{\alpha_4}) = x_3x_2x_1,$$

we have

$$\left|A_{\alpha_1}^{e} \cap A_{\alpha_2}^{e}\right| = \left|A_{\alpha_1}^{e} \cap A_{\alpha_3}^{e}\right| = \left|A_{\alpha_2}^{e} \cap A_{\alpha_3}^{e}\right| = \left|A_{\alpha_2}^{e} \cap A_{\alpha_4}^{e}\right| = \left|A_{\alpha_3}^{e} \cap A_{\alpha_4}^{e}\right|$$
$$= \binom{3+e-3-1}{e-3} = \binom{e-1}{e-3}$$

$$\left|A_{\alpha_1}^e \cap A_{\alpha_4}^e\right| = \binom{3+e-4-1}{e-4} = \binom{e-2}{e-4}.$$

For three sets,

$$\mathrm{LCM}(x^{\alpha_1}, x^{\alpha_2}, x^{\alpha_3}) = x_3^2 x_2 x_1, \mathrm{LCM}(x^{\alpha_1}, x^{\alpha_2}, x^{\alpha_4}) = x_3^2 x_2 x_1,$$
$$\mathrm{LCM}(x^{\alpha_2}, x^{\alpha_3}, x^{\alpha_4}) = x_3 x_2 x_1$$

we have

$$\left|A_{\alpha_1}^e \cap A_{\alpha_2}^e \cap A_{\alpha_3}^e\right| = \left|A_{\alpha_1}^e \cap A_{\alpha_2}^e \cap A_{\alpha_4}^e\right| = \binom{e-2}{e-4}$$
$$\left|A_{\alpha_2}^e \cap A_{\alpha_3}^e \cap A_{\alpha_4}^e\right| = \binom{e-1}{e-3}.$$

Finally

$$\mathrm{LCM}(x^{\alpha_1}, x^{\alpha_2}, x^{\alpha_3}, x^{\alpha_4}) = x_3^2 x_2 x_1$$
$$\left|A_{\alpha_1}^e \cap A_{\alpha_2}^e \cap A_{\alpha_3}^e \cap A_{\alpha_4}^e\right| = \binom{e-2}{e-4}.$$

Hence

$$\left|A_{\alpha_1}^e \cup A_{\alpha_2}^e \cup A_{\alpha_3}^e \cup A_{\alpha_4}^e\right| = 4\binom{e}{e-2} - 4\binom{e-1}{e-3}.$$

Notice that we select $\binom{n}{m} = 0$ if $m < 0$ for convenience. For example, if $e = 3$, then $\mathrm{LM}_3(F)$ has a basis containing $4\binom{3}{1} - 4\binom{2}{0} = 8$ elements. If $e = 4$, this number is equal to 12.

Now, let us go back to syzygies.

Definition 3.1.9. Let $x^\alpha f_i,\ x^\beta f_j \in M_d$. Then we define inverse lexicographic (invlex) order in $M_d(F)$ as $x^\alpha f_i < x^\beta f_j$ if $i > j$ or if $i = j, x^\alpha <_{invlex} x^\beta$.

By construction of the nullspace of $M_d(F)$, each syzygy $s \in syz_d(f_1, f_2, \dots, f_s)$ can be written as $s = (x^\alpha f_k + \text{lower terms})$ with respect to the order given in Definition 3.1.9. On the other hand each syzygy $s \in syz_d(f_1, f_2, \dots, f_s)$ produce an equation

$$\sum_{i=1}^{s} h_i f_i = 0.$$

Since for any monomial x^β

$$x^\beta \sum_{i=1}^{s} h_i f_i = 0,$$

$$x^\beta s \in syz_e(f_1, f_2, \dots, f_s) \text{ where } e = d + \left|x^\beta\right|.$$

Now, we group the syzygies in $syz_d(f_1, f_2, \dots, f_s)$ into groups according to the polynomial that their highest term multiplied. Therefore we define the set of monomials.

Definition 3.1.10. Let $F = (f_1, f_2, \dots, f_s)$ be s-tuple of homogeneous polynomials and $d \in \mathbb{Z}^+$. Define

$$J_k^d = \{ x^\alpha \mid x^\alpha f_k \text{ is the highest term of some } s \in syz_d(f_1, f_2, \dots, f_s)\}.$$

For each group of syzygies of degree d, we want to count the total number of syzygies of degree $e > d$.

Definition 3.1.11. Let $e > d$ and $J_k^d = \{x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_m}\}$ for $k = 1,2, \dots, s$. The extension of J_k^d to degree e is defined as

$$J_k^{d,e} = \{x^\beta x^{\alpha_i} \mid |\beta| + |\alpha_i| = e, 1 \le i \le m\} \text{ for } k = 1,2, \dots, s.$$

The number of monomials in $J_k^{d,e}$ can be found by the Lemma 3.1.6. Notice that we may have more than one syzygies whose highest term is $x^\beta x^\alpha f_k$. In this case, suppose we take one of them. Let $\widetilde{N}_{e,k}$ be the matrix such that each row is the coefficient vector of some syzygy whose highest term is $x^\beta x^{\alpha_i} \in J_k^e$. Assuming columns of $\widetilde{N}_{e,k}$ is sorted according the order given Definition 3.1.9, it is easy to order rows of $\widetilde{N}_{e,k}$ so that it is of the row echelon form. Let

$$\widetilde{N}_e = \begin{bmatrix} \widetilde{N}_{e,s} \\ \widetilde{N}_{e,s-1} \\ \vdots \\ \widetilde{N}_{e,1} \end{bmatrix}.$$

Clearly $\widetilde{N}_e$ is also of the row echelon form. Therefore we have at least

$$\sum_{k=1}^{s} |J_k^{d,e}|$$

linearly independent syzygies.

Example 3.1.12. Consider $F = \{f_1 = x_1^2 + x_1x_2, f_2 = x_1^2 - x_2^2, f_3 = x_1 + x_2 \}$

$$M_2(F) = \begin{array}{c} \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & x_1f_3 & x_2f_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

The row reduced echelon form of $M_2(F)$ is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nullspace of $M_2(F)$ is generated by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence

$$N_2 = \begin{array}{c} \begin{array}{cccc} x_2f_3 & x_1f_3 & f_2 & f_1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \end{array}$$

The corresponding syzygies are

$$s_1 = x_2f_3 + f_2 - f_1, s_2 = x_1f_3 - f_1.$$

So, $J_3^2 = \{x_2, x_1\}$. Since $\mathrm{LCM}(x_2, x_1) = x_2x_1$ we have at least

$$|J_3^{2,e}| = \binom{2+e-2-1}{e-2} + \binom{2+e-2-1}{e-2} - \binom{2+e-3-1}{e-3}$$
$$= 2(e-1) - (e-2) = e$$

linearly independent syzygies of degree $e > 2$.

For example for $e = 3, J_3^{2,3} = \{x_2^2, x_2x_1, x_1^2\}$.

$$\tilde{N}_{3,3} = \begin{array}{c} \\ x_2s_1 \\ x_1s_1 \\ x_1s_2 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_2^2f_3 & x_2x_1f_3 & x_1^2f_3 & x_2f_2 & x_1f_2 & x_2f_1 & x_1f_1 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

The matrix is of the row echelon form as we expected. One may construct $\tilde{N}_{3,e}$ for any $e > 3$ to observe the matrix is of the row echelon form.

Notice that we have one more extended syzygy

$$x_2s_2 = x_2x_1f_3 - x_2f_1.$$

We do not take this syzygy in the construction of $\tilde{N}_{3,3}$ because its highest term is same as that of x_1s_1. If we add its coefficients as a new row to the bottom of $\tilde{N}_{3,3}$,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

If we add -1 multiple of row 2 to row 4, and the multiply row 4 by -1, the last row becomes

$$[0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad -1]$$

which corresponds to the syzygy

$$s_3 = x_1 f_2 + x_2 f_1 - x_1 f_1.$$

This is not a new syzygy of degree 3, it is only a linear combination of extended syzygies of s_1 and s_2 of degree 3.

Since

$$s_3 = x_1 f_2 + x_2 f_1 - x_1 f_1, J_2 = \{x_1\} \text{ and } \left|J_2^{3,e}\right| = \binom{2+e-3-1}{e-3} = e-2.$$

Hence we have at least

$$\left|J_3^{2,e}\right| + \left|J_2^{3,e}\right| = e + (e-2) = 2e-2$$

linearly independent syzygies for $e > 3$.

Now consider

$$M_3(F) = \begin{array}{c} \\ x_1^3 \\ x_1^2x_2 \\ x_1x_2^2 \\ x_2^3 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1f_1 & x_2f_1 & x_1f_2 & x_2f_2 & x_1^2f_3 & x_1x_2f_3 & x_2^2f_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and its row reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the nullspace of $M_3(F)$ is

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The corresponding syzygies are

$$x_2^2 f_3 + x_2 f_2 - x_2 f_1 = x_2 s_1,$$

$$x_1 x_2 f_3 - x_2 f_1 = x_2 s_2,$$

$$x_1^2 f_3 - x_1 f_1 = x_1 s_2,$$

$$x_1 f_2 + x_2 f_1 - x_1 f_1 = s_3.$$

All syzygies are extended syzygies. Hence $\{s_1, s_2\}$ is still a basis for $syz_3(f_1, f_2, f_3)$. Recall that, we obtained row echelon form of the matrix of extended syzygies of degree 3. It is easy to obtain row reduced echelon form of this matrix. Notice that during this calculation pivot columns do not change. Furthermore we can write each pivot column as a linear combination of non pivot columns. Hence deleting corresponding columns in $M_3(F)$ do not effect column space of $M_3(F)$.

Therefore let us delete the columns $x_2^2 f_3, x_2 x_1 f_3, x_1^2 f_3, x_1 f_2$ from $M_3(F)$. Call this matrix $\widetilde{M}_3(F)$.

$$\widetilde{M}_3(F) = \begin{array}{c} \begin{array}{ccc} x_1 f_1 & x_2 f_1 & x_2 f_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

and its row reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The nullspace is empty set as we expected.

Before giving general algorithm for finding syzygy basis, we should have the following.

Algorithm 3.1.13. (Extended Syzygy Algorithm)

Input: $syz_d(f_1, f_2, \dots, f_s)$

Output: $J_1, J_2, \dots, J_s$

For k=1 **To** s

Find $J_k^{d,d+1}$

End For

Obtain the matrix $\widetilde{N}_{d+1}$

Obtain the matrix $\widetilde{\widetilde{N}}_{d+1}$, coefficient matrix of other extended syzygies

Define $N_{d+1} = \begin{bmatrix} \widetilde{N}_{d+1} \\ \widetilde{\widetilde{N}}_{d+1} \end{bmatrix}$

Obtain R_{d+1}, the row echelon form of N_{d+1}

For k=1 **To** s

Return $J_k = \{x^\alpha \mid x^\alpha f_k$ is pivot column in $R_{d+1}\}$

End For

We know that the number of columns of $M_e(F)$ is

$$r = \sum_{i=1}^{s} \binom{n + e - \deg(f_i) - 1}{e - \deg(f_i)}.$$

The well known theorem of Linear Algebra indicates that rank of a matrix plus nullity of that matrix is equal to number of the columns of the matrix. After computation of extended row reduced echelon form of $M_d(F)$, we know the rank of $M_e(F)$ is at least $|\mathrm{LM_e}(F)|$ and nullity of $M_e(F)$ is at least $\sum_{k=1}^{s}\left|J_k^{d,e}\right|$ for every $e > d$. Therefore if $r = |\mathrm{LM_e}(F)| + \sum_{k=1}^{s}\left|J_k^{d,e}\right|$, then there is no new syzygies to add $Syz(F)$.

After this observation, we are ready to give an algorithm to find a basis for $Syz(f_1, f_2, \dots, f_s)$.

Algorithm 3.1.14. (Syzygy Algorithm)

Input: $F = \{f_1, f_2, \dots, f_s\} \subseteq k[x_1, x_2, \dots, x_n]$, a set of homogeneous polynomials

Output: S, a minimal basis for $Syz(f_1, f_2, \dots, f_s)$

$S = \emptyset$

$r = \sum_{i=1}^{s} \binom{n+e-\deg(f_i)-1}{e-\deg(f_i)}$

For k=1 **To** s

$\quad J_k = \emptyset$

End For

Do

$\quad$ Obtain $M_d(F)$

$\quad$ **For** k=1 **To** s

$\quad\quad$ Delete the columns $\{x^\alpha f_k \mid x^\alpha \in J_k\}$ from $M_d(F)$

$\quad$ **End For**

$\quad$ Find row reduced echelon form of $[M_d(F)|I]$

$\quad$ Find $\mathrm{LM_d}(F)$

$\quad$ Find $syz_d(F)$

$\quad$ $S = S \cup syz_d(F)$

For k=1 **To** s

$J_k = J_k^d$

End For

If $|\mathrm{LM_e}(F)| + \sum_{k=1}^{s} |J_i^{d,e}| = r$ **Then Exit**

$\{J_1, J_2, \dots, J_k\} = \text{Extended Syzygy}(S)$

d=d+1

While $|\mathrm{LM_e}(F)| + \sum_{i=1}^{s} |J_i^{d,e}| < r$

Example 3.1.15. Let $F = \{f_1 = x_2^3, f_2 = x_1x_2 - x_2^2, f_3 = x_1^2 + x_2^2, f_4 = x_1 + x_2\}$. Then

$$r = \binom{2+e-3-1}{e-3} + 2\binom{2+e-2-1}{e-2} + \binom{2+e-1-1}{e-1} = 4e-4,$$

$d = min\{\deg(f_i), i = 1,2,3,4\} = 1$ and

$$[M_1(F) : I_2] = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} f_4 \\ \left[\begin{array}{c} 1 \\ 1 \end{array}\right. \end{array} \left.\begin{array}{ccc} \vdots & 1 & 0 \\ \vdots & 0 & 1 \end{array}\right]$$

The row reduced echelon form is

$$\begin{bmatrix} 1 & \vdots & 0 & 1 \\ 0 & \vdots & 1 & -1 \end{bmatrix}.$$

So $L = [1 \quad -1]$. The nullspace of L is generated by $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. So $\mathrm{LM}_1(F) = \{x_2\}$. Therefore

$$|\mathrm{LM_e}(F)| = \binom{2+e-1-1}{e-1} = e.$$

The nullspace of $M_1(F)$ is empty set.

Since $e < 4e-4$, $d = 2$.

$$[M_2(F) : I_3] = \begin{array}{c|cccc|ccc} & f_2 & f_3 & x_1f_4 & x_2f_4 & & & \\ \hline x_1^2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ x_1x_2 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ x_2^2 & -1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}$$

and its row reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \vdots & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The basis for the nullspace of $M_2(F)$ is $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$. The corresponding syzygy is

$$s_1 = x_1 f_4 - f_3 - f_2 \text{ and } S = \{s_1\}.$$

Therefore

$$J_4 = \{x_1\}, |J_4^e| = \binom{2+e-2-1}{e-2} = e-1.$$

Since the matrix L did not occur at this step, $\mathrm{LM}_2(F) = \{x_1^2, x_1x_2, x_2^2\}$ and

$$|\mathrm{LM_e}(F)| = e+1.$$

Since $(e+1)+(e-1) = 2e < 4e-4$, we apply extended syzygy algorithm.

$$N_3 = \begin{array}{c} \\ x_2s_1 \\ x_1s_1 \end{array} \begin{array}{c} \begin{array}{cccccccc} x_2^2f_4 & x_2x_1f_4 & x_1^2f_4 & x_2f_3 & x_1f_3 & x_2f_2 & x_1f_2 & f_1 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

It is already in row echelon form. Then

$$J_4 = \{x_2x_1, x_1^2\} \text{ and } \left|J_4^{3,e}\right| = e-1.$$

Now $d = 3$. We drop columns $x_2x_1f_4, x_1^2f_4$ from $M_3(F)$. So we have the matrix

$$[M_3(F) : I_4] = \begin{array}{c} \\ x_1^3 \\ x_1^2x_2 \\ x_1x_2^2 \\ x_2^3 \end{array} \begin{array}{c} \begin{array}{cccccc} f_1 & x_1f_2 & x_2f_2 & x_1f_3 & x_2f_3 & x_2^2f_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 1 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and its row reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 2 & \vdots & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \vdots & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the nullspace is

$$\left\{\begin{bmatrix}-2\\0\\-1\\0\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-1\\-1\\0\\1\\0\end{bmatrix}\right\}.$$

The corresponding syzygies are $s_2 = {x_2}^2 f_4 - x_2 f_2 - 2f_1$ and $s_3 = x_2 f_3 - x_2 f_2 - x_1 f_2 - 2f_1$. Hence $S = \{s_1, s_2, s_3\}$. Then $J_3 = \{x_2\}$, $J_4 = \{x_1^2, x_1x_2, x_2^2\}$ and so $|J_3| = e - 2$, $|J_4| = e$.

On the other hand, $|\mathrm{LM_e}(F)| = e + 1$ remains the same. Since

$$(e+1) + (e-2) + e = 3e - 1 < r,$$

we apply extended syzygy algorithm

$$\begin{array}{c} \\ x_2s_2 \\ x_1s_2 \\ x_2x_1s_1 \\ x_1^2s_1 \\ x_2s_3 \\ x_1s_3 \\ x_2^2s_1 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} x_2^3f_4 & x_2^2x_1f_4 & x_2x_1^2f_4 & x_1^3f_4 & x_2^2f_3 & x_2x_1f_3 & x_1^2f_3 & x_2^2f_2 & x_2x_1f_2 & x_1^2f_2 & x_2f_1 & x_1f_1 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -2 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

Except the last row, the matrix is in row echelon form. After some row operations, the last row becomes

$$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad -1]$$

The corresponding syzygy is $x_2^2 f_2 + x_2 f_1 - x_1 f_1$.

Then $J_2 = \{x_2^2\}$ so $|J_2| = e - 3$. The sets J_3 and J_4 remain same. Since

$$|J_2| + |J_3| + |J_4| + |\mathrm{LT_e}(F)| = (e-3) + (e-2) + e + (e+1) = 4e - 4 = r,$$

the algorithm terminates. Hence

$$Syz(F) = \{s_1, s_2, s_3\}.$$

3.1.2 Reduction Procedure

The second component essential to the H-basis construction is reduction, which serves as a multivariate analogue of the Euclidean division process. In this section we will describe a reduction algorithm based on linear algebraic techniques.

During this subsection, let $H = \{f_1, f_2, \dots, f_s\} \subset P$ and $g \in P$ with $\deg(g) = d$. Define $F = \{\mathrm{LF}(f_1), \mathrm{LF}(f_2), \dots, \mathrm{LF}(f_s)\}$.

Let $N = [B \mid J]$ be the extended row reduced echelon form of the matrix $M_d(F)$. Furthermore let $\{x^{\alpha_{i_1}}\mathrm{LF}(f_{i_1}), x^{\alpha_{i_2}}\mathrm{LF}(f_{i_2}), \dots, x^{\alpha_{i_m}}\mathrm{LF}(f_{i_m})\}$ be the pivot columns of B and $\{x^{\beta_{j_1}}, x^{\beta_{j_2}}, \dots, x^{\beta_{j_n}}\}$ be the pivot columns of J.

The following lemma is a consequence of Theorem 2.1.16.

Lemma 3.1.16. If v is the coefficient vector of $\mathrm{LF}(g)$, then

$$\mathrm{LF}(g) = h + r = \sum_{k=1}^{m} c_{i_k}\left(x^{\alpha_{i_k}} f_{i_k}\right) + \sum_{k=1}^{n} d_{j_k} x^{\beta_{j_k}}$$

where $h \in V_d(\langle F\rangle)$, $r \in V_d(\langle F\rangle)^C$ and

$$\left(c_{i_1}, \dots c_{i_m}, d_{j_1}, \dots, d_{j_n}\right) = J.v$$

The *r* is called remainder of $\mathrm{LF}(g)$ with respect to F.

Example 3.1.17. Let

$H = \{f_1 = x_1 + x_3, f_2 = x_1x_2 + x_2x_3 + x_1 + 2x_2, f_3 = x_1x_3 + x_2x_3 + x_3 + 1\}$

and $g = x_1^2 + x_1x_3 + x_2^2 + x_2 + x_3 + 1$. Therefore

$F = \{\mathrm{LF}(f_1) = x_1 + x_3, \mathrm{LF}(f_2) = x_1x_2 + x_2x_3, \mathrm{LF}(f_3) = x_1x_3 + x_2x_3\}$,

$\deg(g) = 2$ and

$$\mathrm{LF}(g) = x_1^2 + x_1x_3 + x_2^2.$$

Then

$$[M_2(F) \mid I_6] = \begin{array}{cc} & \begin{array}{cccccccccccc} x_1LF(f_1) & x_2LF(f_1) & x_3LF(f_1) & f_2 & f_3 & & x_1^2 & x_1x_2 & x_1x_3 & x_2^2 & x_2x_3 & x_3^2 \end{array} \\ \begin{array}{c} x_1^2 \\ x_1x_2 \\ x_1x_3 \\ x_2^2 \\ x_2x_3 \\ x_3^2 \end{array} & \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & \vdots & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

and the row reduced echelon form is

$$N = [B \mid J] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Pivot columns of B are $(x_1\mathrm{LF}(f_1), x_2\mathrm{LF}(f_1), x_3\mathrm{LF}(f_1), f_3)$ and the pivot columns of J are (x_1^2, x_2^2).

$\mathrm{LF}(g) = x_1^2 + x_1x_3 + x_2^2$ can be represented as $v = (1,0,1,1,0,0)^T$.

Then $Jv = (1,0,0,0,0,1)^T$. Therefore

$\mathrm{LF}(g) = x_1\mathrm{LF}(f_1) + x_2^2$ where $r = x_2^2 \in \left(M_2(F)\right)^C$.

The next algorithm gives the polynomials $a_1, a_2, \dots, a_s$ and r such that

$$g = \sum_{i=1}^{s} a_i f_i + r \qquad (3.1)$$

where $\deg(a_i f_i) \leq \deg(g)$ and $r = 0$ or $\deg(r) \leq \deg(g)$. In this case we say g is reduced to the remainder r modulo H and write $g \overset{H}{\rightarrow} r$. The following algorithm outlines the steps of the reduction process.

Algorithm 3.1.18. (Reduction Algorithm)

Input: $H = \{f_1, f_2, \dots, f_s\} \subseteq P_i, g \in P$

Output: $\{a_1, a_2, \dots, a_s, r\} \subseteq P$ satisfying (3.1)

$r = 0, a_1 = 0, a_2 = 0, \dots, a_s = 0,$

$F = \{\mathrm{LF}(f_1), \mathrm{LF}(f_2), \dots, \mathrm{LF}(f_s)\}$

$d = \deg(g)$

While $d \geq 0$

  Find $N = [B \mid J]$ (the extended row reduced echelon form of $M_d(F)$)

  Write $\mathrm{LF}(g) = \sum_{i=1}^{s} b_i \mathrm{LF}(f_i) + \tilde{r}$ using N

  For $i = 1$ **To** s

    $a_i = a_i + b_i$

  End For

  $r = r + \tilde{r}$

  $g = g - (\sum_{i=1}^{s} b_i f_i + r)$

  $d = \deg(g)$

End While

One of the output of the algorithm is the remainder r. We denote this process by $g \overset{H}{\rightarrow} r$.

Example 3.1.19. Continuing from previous example, $a_1 = x_1, r = x_2^2$.

$$g = g - (x_1 f_1 + x_2^2) = x_2 + x_3 + 1,$$

$$d = deg(g) = 1$$

$$\mathrm{LF}(g) = x_2 + x_3$$

$$[M_1(F) : I_3] = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} LF(f_1) \\ \left[\begin{array}{ccccc} 1 & \vdots & 1 & 0 & 0 \\ 0 & \vdots & 0 & 1 & 0 \\ 1 & \vdots & 0 & 0 & 1 \end{array}\right] \end{array}$$

Row reduced echelon form is

$$N = [B \mid J] = \begin{bmatrix} 1 & \vdots & 0 & 0 & 1 \\ 0 & \vdots & 1 & 0 & -1 \\ 0 & \vdots & 0 & 1 & 0 \end{bmatrix}.$$

The pivot column of B is $\{\mathrm{LF}(f_1)\}$ and the pivot columns of J are $\{x_1, x_2\}$. Then

$\mathrm{LF}(g) = x_2 + x_3$ can be represented as $v = (0,1,1)^T$ and $Jv = (1,-1,1)^T$. Therefore

$$\mathrm{LF}(g) = x_2 + x_3 = \mathrm{LF}(f_1) - x_1 + x_2.$$

So, $a_1 = x_1 + 1, r = x_2^2 - x_1 + x_2$,

$$g = g - (f_1 - x_1 + x_2) = x_2 + x_3 + 1 - (x_1 + x_3 - x_1 + x_2) = 1$$

$$d = \deg(g) = 0.$$

$M_0(F) = \emptyset$ implies $g = 0 + r = 0 + 1$. So $r = x_2^2 - x_1 + x_2 + 1$.

Hence $g \xrightarrow{H} x_2^2 - x_1 + x_2 + 1$.

3.1.3 H-Basis Algorithm

In this section, we detailed Algorithm 2.3.6 using degree by degree approach. Basically, given $H = \{f_1, f_2, \dots, f_s\} \subseteq P$, we will find a basis for $syz_d(F) = syz_d\big(\mathrm{LF}(f_1), \mathrm{LF}(f_2), \dots, \mathrm{LF}(f_s)\big)$ degree by degree starting from

$$d = \min\{\deg(f_i); i = 1,2,\dots,s\}$$

For each syzygy $(h_1, h_2, \dots, h_s) \in syz_d(F)$, we apply reduction procedure to

$$h = h_1 f_1 + h_2 f_2 + \cdots + h_s f_s$$

to obtain $h \xrightarrow{H} r$. If $r \neq 0$, we extend H by r and F by $\mathrm{LF}(r)$. If $r = 0$, we do nothing and pass the next syzygies. We repeat this process for each basis elements of $Syz(F)$ and then

$$h = h_1 f_1 + h_2 f_2 + \cdots + h_s f_s \xrightarrow{H} 0 \text{ for all } (f_1, f_2, \dots, f_s) \in Syz(F).$$

Notice that if the set $F = \{\mathrm{LF}(f_1), \mathrm{LF}(f_2), \ldots, \mathrm{LF}(f_s)\}$ is extended by $f = \mathrm{LF}(r)$, then we will not compute the extended row reduced echelon form of new F from scratch.

If $N_d = [B_d \mid J_d]$ is row reduced echelon form of old F and K is the matrix representation of $\{x^\alpha f : |x^\alpha| + \deg(f) = d\}$, then row reduced echelon form of $[(B_d|J_k K) \mid J_d]$ will be the extended row reduced echelon form of (F, f). This observation is easily followed by Theorem 2.1.16.

Algorithm 3.1.20.

Input: $d, F = (f_1, f_2, \ldots, f_s), f$

Output: $N_d = [B_d \mid J_d]$, the extended row reduced echelon form of $M_d(F)$

If $f = 0$ **Then**

$N_d = [B_d \mid J_d] =$ The row reduced echelon form of $[M_d(F) \mid I]$

Else

K is the matrix representation of $\{x^\alpha f : |x^\alpha| + \deg(f) = d\}$

$L = J_D K$

$M = [B_d L \mid J_d]$

$N_d =$ The row reduced echelon form of M

End If-Else

Algorithm 3.1.21. (H-Basis Algorithm)

Input: $H = \{f_1, f_2, \ldots, f_s\} \subset P$

Output: an H-basis for the ideal $\langle H \rangle$

$F = \{\mathrm{LF}(f_1), \mathrm{LF}(f_2), \ldots, \mathrm{LF}(f_s)\}$

$S = \emptyset, \bar{S} = \emptyset$

$dim = \sum_{i=1}^{s} \binom{n+e-\deg(f_i)-1}{e-\deg(f_i)}$

For k=1 **To** s

$J_k = \emptyset$

End For

$LT_e(F) = \emptyset$

$f = 0$

$r = 0$

While $|\mathrm{LT_e}(F)| + \sum_{k=1}^{s} |J_k^e| < dim$

$d = \min\{\deg(f_i); i = 1,2, \dots, s\}$

Obtain $M_d(F)$

$f = \mathrm{LF}(r)$

For k=1 **To** s

Remove the columns $\{x^\alpha f_k \mid x^\alpha \in J_k; \; k = 1,2, \dots, s\}$ from $M_d(F)$

End For

(*) $N_d = [B_d \mid J_d] = Reduction(d, F, f)$

Find $\mathrm{LT_d}(F)$

Find $Syz_d(F)$

$S = S \cup Syz_d(F), \bar{S} = \bar{S} \cup Syz_d(F)$

For k=1 **To** s

$J_k = J_k^d$

End For

If $r \neq 0$, **Then** $H = (H, r)$

If $f \neq 0$, **Then** $F = (F, s), s = s + 1$

While $\bar{S} \neq \emptyset$

Select $(h_1, h_2, \dots, h_s) \in S$

$h = \sum_{i=1}^{s} h_i f_i$

$h \to r$

If $r \neq 0$ **Then**

$f = \mathrm{LF}(r)$

$d = \min\{\deg(f_1), \deg(f_2), \dots, \deg(f_s), \deg(f)\}$

Go To Line (*)

$\bar{S} = \bar{S} - \{(h_1, h_2, \dots, h_s)\}$

End While

If $|\mathrm{LT_e}(F)| + \sum_{k=1}^{s} |J_k^e| = dim$ **Then Exit**

$\{J_1, J_2, \dots, J_s\} = Extended(S)$

$d = d + 1$

End While

Return H

Example 3.1.22. Let

$$H = \{h_1 = x_1^3 x_2^2 + 2x_1 x_2^4, h_2 = x_1 x_2, h_3 = x_1 x_2^3 + x_1 x_2 + 3x_2^2\}.$$

Then

$$\mathrm{LF}(H) = F = \{f_1 = x_1^3x_2^2 + 2x_1x_2^4, f_2 = x_1x_2, f_3 = x_1x_2^3\}.$$

We start with the minimum degree, $d = 2$

$$[M_2(F) \mid I_3] = \begin{array}{c} \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} \begin{array}{c} f_2 \\ \left[\begin{array}{c:ccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

the row reduced echelon form of $[M_2(F) \mid I_3]$ is

$$\left[\begin{array}{c:ccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

So, $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

hence $\mathrm{nullspace}(L) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\mathrm{LM}_2(F) = \{x_1x_2\}$.

Continue with $d = 3$

$$[M_3(F) \mid I_4] = \begin{array}{c} \\ x_1^3 \\ x_1^2x_2 \\ x_1x_2^2 \\ x_2^3 \end{array} \begin{array}{c} \quad x_1f_2 \quad x_2f_2 \qquad\qquad\qquad \\ \left[\begin{array}{cc:cccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

The row reduced echelon form of $[M_3(F) \mid I_4]$ is

$$\left[\begin{array}{cc:cccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

So $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Hence $\mathrm{nullspace}(L) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. Then $\mathrm{LM}_3(F) = \{x_1x_2^2, x_1^2x_2\}$.

For $d = 4$,

$$[M_4(F)\mid I_5] = \begin{array}{c} \\ x_1^4 \\ x_1^3x_2 \\ x_1^2x_2^2 \\ x_1x_2^3 \\ x_2^4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1^2f_2 & x_1x_2f_2 & x_2^2f_2 & f_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and row reduced echelon form of $[M_4(F)\ \mid I_5]$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$,

and $\text{nullspace}(L) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Then $\text{LM}_4(F) = \{x_1x_2^3, x_1^2x_2^2, x_1^3x_2\}$ and

$\text{nullspace}\big(M_4(F)\big) = \left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$,

The corresponding syzygy is

$$f_3 - x_2^2f_2$$

$$h_3 - x_2^2h_2 = x_1x_2^3 + x_1x_2 + 3x_2^2 - x_1x_2^3$$

$$= x_1x_2 + 3x_2^2 \xrightarrow{F} 3x_2^2.$$

Then

$$h_4 = 3x_2^2, \text{and so } \ f_4 = 3x_2^2.$$

Thus $F = F \cup f_4$ and

$$r = (e-4) + (e-1) + (e-3) + (e-1) = 4e - 9.$$

Since $\deg(f_4) = 2$, we turn back to second degree. Now $d = 2$. Therefore

$$\left[M_2\widetilde{(F)}\mid I_3\right] = \begin{array}{c} x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} \begin{bmatrix} 1 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 3 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

and the row reduced echelon form of $\left[M_2\widetilde{(F)}\mid I_3\right]$ is

$$\begin{bmatrix} 1 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 1 & \vdots & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \vdots & 1 & 0 & 0 \end{bmatrix}.$$

So $L = [\,1 \quad 0 \quad 0]$ and

$\text{nullspace}(L) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Then $\text{LM}_2(F) = \{x_2^2, x_1x_2\}$.

Now $d = 3$.

$$[\tilde{M_3}(F) \mid I_4] = \begin{array}{c} \begin{array}{cccc} x_1f_2 & x_2f_2 & x_1f_4 & x_2f_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and the row reduced echelon form of $\left[M_3\widetilde{(F)} \mid I_4\right]$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \end{bmatrix}.$$

So $L = [\,1 \quad 0 \quad 0 \quad 0]$. Then

$\text{nullspace}(L) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\text{LM}_3(F) = \{x_1^2x_2, x_1x_2^2, x_2^3\}$. Then

$$\text{nullspace}\big(M_3(F)\big) = \left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\},$$

the corresponding syzygy is

$$x_1f_4 - 3x_2f_2$$

and

$$x_1h_4 - 3x_2h_2 = 3x_1x_2^2 - 3x_1x_2^2 = 0.$$

Let $d = 4$.

Extended syzygies are

$$x_1^2h_4 - 3x_1x_2h_2,$$

$$x_1x_2h_4 - 3x_2^2h_2.$$

We can delete the columns corresponding to $x_1^2h_4$ and $x_1x_2h_4$.

$$\left[M_4\widetilde{(F)}\mid I_5\right]=\begin{bmatrix}1&0&0&0&\vdots&0&1&0&0&0\\0&1&0&0&\vdots&0&0&1&0&0\\0&0&1&0&\vdots&0&0&0&1&0\\0&0&0&0&\vdots&1&0&0&0&0\\0&0&0&3&\vdots&0&0&0&0&1\end{bmatrix}.$$

The row reduced echelon form of $[M_4\widetilde{(F)}\mid I_5]$ is

$$\begin{bmatrix}1&0&0&0&\vdots&0&1&0&0&0\\0&1&0&0&\vdots&0&0&1&0&0\\0&0&1&0&\vdots&0&0&0&1&0\\0&0&0&1&\vdots&0&0&0&0&\frac{1}{3}\\0&0&0&0&\vdots&1&0&0&0&0\end{bmatrix}.$$

So $L=[1\quad 0\quad 0\quad 0\quad 0]$

and

$$\text{nullspace}(L)=\left\{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}.$$

Therefore $\mathrm{LM}_4(F)=\{x_1^3x_2,x_1^2x_2^2,x_1x_2^3,x_2^4\}$. The extended syzygies are

$$x_1f_3-x_1x_2^2f_2,$$
$$x_2f_3-x_2^3f_2,$$
$$x_1^3f_4-3x_1^2x_2f_2,$$
$$x_1^2x_2f_4-3x_1x_2^2f_2,$$
$$x_1x_2^2f_4-3x_2^4f_2.$$

So, we can delete the columns $x_1f_3,x_2f_3,x_1^3f_4,x_1^2x_2f_4,x_1x_2^2f_4$ in $[M_5(F)\mid I_6]$.

For $d=5$

$$[M_5(F)\mid I_6]=\begin{array}{c}\\ x_1^5\\ x_1^4x_2\\ x_1^3x_2^2\\ x_1^2x_2^3\\ x_1x_2^4\\ x_2^5\end{array}\begin{array}{c}\begin{array}{cccccc}f_1&x_1^3f_2&x_1^2x_2f_2&x_1x_2^2f_2&x_2^3f_2&x_2^3f_4\end{array}\\ \begin{bmatrix}0&0&0&0&0&0&\vdots&1&0&0&0&0&0\\0&1&0&0&0&0&\vdots&0&1&0&0&0&0\\1&0&1&0&0&0&\vdots&0&0&1&0&0&0\\0&0&0&1&0&0&\vdots&0&0&0&1&0&0\\2&0&0&0&1&0&\vdots&0&0&0&0&1&0\\0&0&0&0&0&3&\vdots&0&0&0&0&0&1\end{bmatrix}\end{array}$$

and the row reduced echelon form of $[M_5(F)\mid I_6]$ is

$$\left[\begin{array}{cccccc:cccccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore $L = [\,1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$

and $\text{nullspace}(L) = \left\{\begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\right\}$. So

$$\text{LM}_5(F) = \{x_1^4x_2, x_1^3x_2^2, x_1^2x_2^3, x_1x_2^4, x_2^5\}.$$

Then

$$\text{nullspace}\big(M_5(F)\big) = \left\{\begin{bmatrix}-\frac{1}{2}\\0\\\frac{1}{2}\\0\\1\\0\end{bmatrix}\right\}.$$

Corresponding syzygy is

$$x_3^2f_2 + \frac{1}{2}x_1^2x_2f_2 - \frac{1}{2}f_1.$$

Therefore

$J_2 = \{x_3^2\}, J_3 = \{1\}$ and $J_4 = \{x_1\}$. This set produce

$$\left|J_1^{5,e}\right| = \binom{2+e-5-1}{e-5} = e-4,$$

$$\left|J_2^{4,e}\right| = \binom{2+e-4-1}{e-4} = e-3,$$

and

$$\left|J_4^{3,e}\right| = \binom{2+e-3-1}{e-3} = e-2.$$

Since $|\text{LM}_5(F)| = e$,

$$|\text{LM}_5(F)| + \left|J_1^{5,e}\right| + \left|J_2^{4,e}\right| + \left|J_4^{3,e}\right| = e + (e-4) + (e-3) + (e-2) = 4e-9.$$

As a result, the equation $|\mathrm{LM_e}(F)| + \sum_{k=1}^{s}|J_i^e| = r$ is satisfied at degree 5, so our calculation terminates here. We have an H-basis as below

$$H = \{h_1 = x_1^3x_2^2 + 2x_1x_2^4, h_2 = x_1x_2, h_3 = x_1x_2^3 + x_1x_2 + 3x_2^2, h_4 = 3x_2^2\}.$$

3.2 H-Bases Computation via Full QR Decomposition

3.2.1 Syzygy Computation

Given a set of homogeneous polynomials $F = \{f_1, \dots, f_s\}$, we will use QR Decomposition techniques to find a basis for $Syz(F)$. In this section, Algorithm 3.1.15 also constitutes the primary foundation for the $Syz(F)$ computation. The Algorithm 3.1.15 uses row reduction techniques for the calculations of $syz_d(F)$ and $\mathrm{LM_d}(F)$ but in this section QR Decomposition will be used for these calculations.

We intend to develop a recursive method. Because of this, we sort elements of $M_d(F)$ in a different way by using the following definition.

Definition 3.2.1. Let $x^\alpha f_i, x^\beta f_j \in M_d(F)$. Then $x^\alpha f_i < x^\beta f_j$ if $x^\alpha >_{\mathrm{lex}} x^\beta$ or if $x^\alpha =_{\mathrm{lex}} x^\beta$ and $i < j$.

Notice that using Pivoted QR Decomposition of $M_d(F)$, Theorem 2.2.11 gives the matrix N such that columns of N form a basis for the nullspace of $M_d(F)$. Then the corresponding polynomials of nullspace gives a generating set for $syz_d(f_1, \dots, f_s)$.

Let C_d^n be the set of monomials of degree d. For a polynomial f of degree d, $\mathrm{LM}(f)$ is the largest monomial $x^\alpha \in C_d^n$ with respect to lex order.

Let $F = (f_1, f_2, \dots, f_s)$

$$\mathrm{LM_d}(F) = \{x^\alpha \in C_d^n \mid x^\alpha = \mathrm{LM}(f), f \in \langle F \rangle\}$$

then the set $B_d = C_d^n \backslash \mathrm{LM_d}(F)$ is called standart monomials.

Theorem 3.2.2. Let $I = \{i_1, i_2, \dots, i_t\}$ be the index set obtained in Full QR Decomposition of $M_d(F)$. Suppose that the monomials in C_d^n are ordered from the smaller to larger with respect to lex order. Then,

$$B_d = \{C_d^n[i_1], C_d^n[i_2], \dots, C_d^n[i_t]\}.$$

Proof. Let $J = \{C_d^n[i_1], C_d^n[i_2], \dots, C_d^n[i_t]\}$. Let $x^\alpha \notin J$. Suppose

$$C_d^{(n)}(i_j) < x^\alpha < C_d^{(n)}(i_{j+1}).$$

By Gram-Schmidt procedure,

$$x^\alpha \in span\left(M_d(F), C_d^{(n)}(i_1), \dots, C_d^{(n)}(i_j)\right).$$

So, there is a polynomial

$$f = x^\alpha + a_1 C_d^{(n)}(i_1) + \cdots + a_j C_d^{(n)}(i_j) \in span(M_d(F)).$$

So $\mathrm{LM}(f) = x^\alpha \in \mathrm{LM_d}(F)$. Then $x^\alpha \notin B_d$.

Conversely, suppose $x^\alpha \notin B_d$. This implies $x^\alpha \in \mathrm{LM_d}(F)$. Therefore there is a polynomial $f \in \langle F \rangle$ such that $f = x^\alpha + \text{lower degree terms}$. Therefore during the Gram-Schmidt Procedure corresponding x^α gives zero vector. Hence $x^\alpha \notin J$. Thus, we may conclude that $B_d = \{C_d^n[i_1], C_d^n[i_2], \dots, C_d^n[i_t]\}$.

Example 3.2.3. Let $F = \{x_1 + x_3\ , f_2 = x_1x_2 + x_2x_3\ , f_3 = x_1x_3 + x_2x_3\}$ we will find a basis for the vector space $Syz(F)$.

$$r = \binom{3 + e - 1 - 1}{e - 1} + \binom{3 + e - 2 - 1}{e - 2} + \binom{3 + e - 2 - 1}{e - 2}$$

$$= \binom{e + 1}{e - 1} + 2\binom{e}{e - 2}.$$

$$d = \min\{\deg(f_i)\,, i = 1,2,3\} = 1$$

Start with $d = 1$. $M_1(F) = \{f_1\}$. Then the corresponding matrix is

$$M_1(F) = \begin{array}{c} f_1 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

and the Full QR Decomposition of $M_1(F)$ is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \vdots & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \vdots & 0 & 1 \\ \frac{1}{\sqrt{2}} & \vdots & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} = [Q_1 \quad \vdots \quad Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}.$$

The Full QR Algorithm returns the vectorIndex={3,2}. Hence

$$B_1 = \{V_1(3), V_1(2)\} = \{x_3, x_2\}, \text{and}$$

$$\mathrm{LM}_1(F) = V_1 - B_1 = \{x_1\}.$$

Therefore for $e > 1$,

$$|\mathrm{LM}_1(F)| = \binom{e+1}{e-1}.$$

The nullspace of $M_1(F)$ is empty set. Thus there is no syzygy in degree 1. Since $|\mathrm{LM}_1(F)| = \binom{e+1}{e-1} < r$, we will continue with $d = 2$.

Let $d = 2$.

$$M_2(F) = \{x_1f_1, x_2f_1, x_3f_1, f_2, f_3\}.$$

Then the corresponding matrix is

$$M_2(F) = \begin{array}{c} \begin{array}{ccccc} x_1f_1 & x_2f_1 & x_3f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

now we need to find the Full QR Decomposition of $M_2(F)$. Therefore we find

$$Q = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & & \end{array} \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{15}} & \vdots & \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{3}}{\sqrt{10}} & \vdots & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{15}} & \vdots & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{\sqrt{10}} & \vdots & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{15}} & \vdots & \frac{1}{\sqrt{5}} & 0 \end{bmatrix} \end{array} = [Q_1 : Q_2]$$

$$R = \begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{\sqrt{6}}{2} & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{30}}{6} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ \cdots \\ 0 \end{bmatrix}.$$

The monomials in degree 2 with respect to the lex order

$$V_2 = \{x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\}.$$

The Full QR Algorithm returns the vectorIndex={6,4}. Hence

$$B_2 = \{V_2(6), V_2(4)\} = \{x_3^2, x_2^2\}, \text{and}$$

$$\mathrm{LM}_2(F) = V_2 - B_2 = \{x_2x_3, x_1x_3, x_1x_2, x_1^2\}.$$

Therefore for $e > 2$,

$$|\mathrm{LM}_2(F)| = \binom{e+1}{e-1} + \binom{e}{e-2} - \binom{e-1}{e-3}$$

To find $syz_2(F)$, we need to obtain the nullspace of $M_2(F)$. Let us find Pivoted QR Decomposition of $M_2(F)$.

The matrix R in Full QR Decomposition of $M_2(F)$ is not an upper triangular matrix. So, we need to use

$$A = [Q_1 \quad Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} P^T.$$

We need to change 4^{th} and 5^{th} columns of R.

So,

$$\begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} P^T = \begin{array}{c} \begin{array}{cccccc} & & R_{11} & & & R_{12} \end{array} \\ \begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \vdots & 0 \\ 0 & \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & \vdots & \sqrt{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} & \frac{1}{\sqrt{6}} & \vdots & 0 \\ 0 & 0 & 0 & \frac{\sqrt{30}}{6} & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \end{array}$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then

$$-R_{11}^{-1}R_{12} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -R_{11}^{-1}R_{12} \\ I_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Theorem 2.2.11 implies

$$P\begin{bmatrix} -R_{11}^{-1}R_{12} \\ I_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

forms a basis for the nullspace of $M_2(F)$. Then

$$M_2(F).\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \{x_1f_1, x_2f_1, x_3f_1, f_2, f_3\}.\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = f_2 - x_2f_1.$$

Thus

$$syz_2(F) = \{f_2 - x_2f_1\}.$$

So, $J_2 = \{1\}$ and $|J_2| = \binom{e}{e-2}$. Then

$$|J_2| + |\mathrm{LM}_2(F)| = \binom{e+1}{e-1} + 2\binom{e}{e-2} - \binom{e-1}{e-3} < r.$$

Now, we give a recursive method for computing the Full QR Decomposition of $M_{d+1}(F)$. That means instead of computing the Full QR Decomposition of the $M_{d+1}(F)$ from scratch, we will compute it using the QR Decomposition of the $M_d(F)$.

Notice that $M_{d+1}(F)$ can be written as

$$M_{d+1}(F) = \begin{bmatrix} M_d(F) & A \\ 0 & B \end{bmatrix}.$$

Let $M_d(F) = [Q_1 \quad Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ be Full QR Decomposition. Since Q_1 is an orthonormal basis for column space of $M_d(F)$, we can substitute Q_1 for $M_d(F)$ in the above matrix. It is enough to find Full QR Decomposition of

$$\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix}$$

for our purposes.

Theorem 3.2.4. Let $M_d(F), Q_1, Q_2, R_1, M_{d+1}, A, B$ be matrices as defined above, and r be the number of column of Q_1.

If

$$\begin{bmatrix} Q_2^T A \\ B \end{bmatrix} = [\bar{Q}_1 \quad | \quad \bar{Q}_2] \begin{bmatrix} \bar{R}_1 \\ 0 \end{bmatrix}$$

is the Full QR Decomposition and

$$[\bar{Q}_1 \quad | \quad \bar{Q}_2] = \begin{bmatrix} S_1 & | & S_2 \\ T_1 & | & T_2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 S_1 & Q_2 S_2 \\ 0 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} I_r & Q_1^T A \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix}$$

is the Full QR Decomposition of $\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix}$.

Proof.

Notice that

$$\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I_r & Q_1^T A \\ 0 & Q_2^T A \\ 0 & B \end{bmatrix} \quad (*)$$

On the other hand

$$\begin{bmatrix} Q_2^T A \\ B \end{bmatrix} = \begin{bmatrix} S_1 & | & S_2 \\ T_1 & | & T_2 \end{bmatrix} \begin{bmatrix} \bar{R}_1 \\ 0 \end{bmatrix}$$

implies

$$\begin{bmatrix} I_r & Q_1^T A \\ 0 & Q_2^T A \\ 0 & B \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & S_1 & S_2 \\ 0 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} I_r & Q_1^T A \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I_r & 0 & 0 \\ 0 & S_1 & S_2 \\ 0 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} I_r & Q_1^T A \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} Q_1 & Q_2 S_1 & Q_2 S_2 \\ 0 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} I_r & Q_1^T A \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix}.$$

Corollary 3.2.5. Let

$$\begin{bmatrix} Q_2^T A \\ B \end{bmatrix} = [\bar{Q}_1 \quad | \quad \bar{Q}_2] \begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} \\ 0 & 0 \end{bmatrix} P^T$$

be the Pivoted QR Decomposition and

$$\begin{bmatrix} Q_1^T A \\ \bar{R} \end{bmatrix} = \begin{bmatrix} C & | & D \\ \bar{R}_{11} & | & \bar{R}_{12} \end{bmatrix} P^T.$$

Then

$$-[I_r \quad | \quad P] \begin{bmatrix} D - C\bar{R}_{11}^{-1}\bar{R}_{12} \\ \bar{R}_{11}^{-1}\bar{R}_{12} \\ I \end{bmatrix}$$

gives a nullspace for the matrix

$$\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix}.$$

Proof.

$$\begin{bmatrix} I_r & Q_1^T A \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & C & \vdots & D \\ 0 & \bar{R}_{11} & \vdots & \bar{R}_{12} \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & 0 \end{bmatrix} [I_r \quad | \quad P^T].$$

The nullspace can be obtained as

$$-[I_r \quad | \quad P] \begin{bmatrix} I_r & C \\ 0 & \bar{R}_{11} \end{bmatrix}^{-1} \begin{bmatrix} D \\ \bar{R}_{12} \end{bmatrix} = -[I_r \quad | \quad P] \begin{bmatrix} I_r & -C\bar{R}_{11}^{-1} \\ 0 & \bar{R}_{11}^{-1} \end{bmatrix} \begin{bmatrix} D \\ \bar{R}_{12} \end{bmatrix}$$

$$= -[I_r \quad | \quad P] \begin{bmatrix} D - C\bar{R}_{11}^{-1}\bar{R}_{12} \\ \bar{R}_{11}^{-1}\bar{R}_{12} \end{bmatrix}$$

Example 3.2.6. If we continue Example 3.2.3. Take $d = 3$. To find $M_3(F)$ we use recursive method given in Theorem 3.2.4. Since $x_2 f_2 - {x_2}^2 f_1$ is an extended syzygy, we can delete the column $x_2 f_2$. Then the matrices A and B are

$$\begin{array}{c} \\ x_1^3 \\ x_1^2x_2 \\ x_1^2x_3 \\ x_1x_2^2 \\ x_1x_2x_3 \\ x_1x_3^2 \\ \\ x_2^3 \\ x_2^2x_3 \\ x_2x_3^2 \\ x_3^3 \end{array}
\begin{array}{c} \begin{array}{ccccc} x_2^2f_1 & x_2x_3f_1 & x_2f_3 & x_3^2f_1 & x_3f_3 \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} A \\ \cdots \\ B \end{bmatrix}$$

Hence

$$\widetilde{M}_3(F) = \begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{15}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{3}}{\sqrt{10}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{15}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{\sqrt{10}} & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{15}} & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

As a next step we have to find Full QR Decomposition of

$$X = \begin{bmatrix} Q_2^T A \\ B \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 1 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\bar{Q} = \begin{bmatrix} 0 & \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{2\sqrt{6}} & \frac{\sqrt{5}}{6\sqrt{2}} & \vdots & -\frac{\sqrt{5}}{3} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{6\sqrt{2}} & \vdots & -\frac{1}{3} & 0 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{6\sqrt{2}} & \vdots & \frac{1}{3} & 0 \\ 0 & \frac{\sqrt{5}}{\sqrt{6}} & \frac{-1}{2\sqrt{6}} & \frac{-1}{6\sqrt{2}} & \vdots & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{2}}{3} & \vdots & \frac{1}{3} & 0 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ T_1 & T_2 \end{bmatrix}$$

$$\bar{R} = \begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & 0 & \vdots & 0 \\ 0 & \frac{\sqrt{6}}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \vdots & \frac{\sqrt{6}}{\sqrt{5}} \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{2\sqrt{6}} & \vdots & 0 \\ 0 & 0 & 0 & \frac{3}{2\sqrt{2}} & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} \bar{R}_{11} & \vdots & \bar{R}_{12} \\ 0 & \vdots & 0 \end{bmatrix}.$$

The set of monomials in degree 3 with respect to the lex order is

$$V_3 = \{x_1^3, x_1^2x_2, x_1^2x_3, x_1\, x_2^2, x_1\, x_2x_3, x_1x_3^2, x_2^3, x_2^2x_3, x_2x_3^2, x_3^3\}.$$

The Full QR Algorithm returns the vectorIndex={10,7}. Hence

$$B_3 = \{V_3(10), V_3(7)\} = \{x_3^3, x_2^3\}, \text{and}$$

$$\mathrm{LM}_3(F) = V_3 - B_3 = \{x_1^3, x_1^2x_2, x_1^2x_3, x_1\, x_2^2, x_1\, x_2x_3, x_1x_3^2, x_2^2x_3, x_2x_3^2\}.$$

Then

$$|\mathrm{LM}_3(F)| = \binom{e+1}{e-1} + \binom{e}{e-2} - \binom{e-1}{e-3}.$$

$$Q_1{}^T A = \begin{bmatrix} 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \vdots & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \vdots & \sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{3}{10}} & \sqrt{\frac{3}{10}} & -\sqrt{\frac{2}{15}} & \vdots & -\sqrt{\frac{2}{15}} \end{bmatrix} = [C \;\; \vdots \;\; D].$$

Therefore

$$-\begin{bmatrix} D + C\bar{R}_{11}{}^{-1}\bar{R}_{12} \\ \bar{R}_{11}{}^{-1}\bar{R}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{5}{6}} \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{Y} = P\begin{bmatrix} -\tilde{R}_{11}^{-1}\tilde{R}_{12} \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{5}}{\sqrt{6}} \\ \cdots \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{Y}_1 \\ \cdots \\ \tilde{Y}_2 \end{bmatrix}.$$

The columns of above matrix form a basis for the nullspace of $\widetilde{M}_3(F)$. The corresponding polynomials are given by

$$\{q_1, q_2, q_3, q_4, x_2^2 f_1, x_2 x_3 f_1, x_2 f_2, x_2 f_3, x_3^2 f_1, x_3 f_2, x_3 f_3\}\begin{bmatrix} \tilde{Y}_1 \\ \cdots \\ \tilde{Y}_2 \end{bmatrix}.$$

Using the equation

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = [x_1{}^2 f_1 \quad x_1 x_2 f_1 \quad x_1 x_3 f_1 \quad x_1 f_3] R_{11}^{-1},$$

we find the syzygy

$$\{x_1^2 f_1, x_1 x_2 f_1, x_1 x_3 f_1, x_1 f_3, x_2^2 f_1, x_2 x_3 f_1, x_2 f_2, x_2 f_3, x_3^2 f_1, x_3 f_2, x_3 f_3\}\begin{bmatrix} R_{11}^{-1}\tilde{Y}_1 \\ \cdots \\ \tilde{Y}_2 \end{bmatrix}$$

$$= \{x_1^2f_1, x_1x_2f_1, x_1x_3f_1, x_1f_3, x_2^2f_1, x_2x_3f_1, x_2f_2, x_2f_3, x_3^2f_1, x_3f_2, x_3f_3\}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ \cdots \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_3f_3 - x_2x_3f_1 + x_1f_3 - x_1x_3f_1.$$

Hence

$syz_3(F) = \{x_3f_3 - x_2x_3f_1 + x_1f_3 - x_1x_3f_1\}$. Then $J_3 = \{x_3\}$ and $|J_3| = \binom{e-1}{e-3}$. Therefore

$$|J_2| + |J_3| + |\mathrm{LM}_2(F)| = \binom{e+1}{e-1} + 2\binom{e}{e-2} = r.$$

Hence the algorithm terminates here, and the syzygy basis is found as

$$Syz(F) = \{f_2 - x_2f_1, x_3f_3 - x_2x_3f_1 + x_1f_3 - x_1x_3f_1,\}.$$

3.2.2 Reduction Procedure

Using the same terminology as in the previous section, we denote the orthogonal complement of $M_d(F)$ with $W_d(F)$ so that $P_d = M_d(F) \oplus W_d(F)$. These orthogonal vector spaces allow for the decomposition of every homogeneous polynomials $g \in P_d$ into two distinct components, one component is in $M_d(F)$ and the other component is in $W_d(F)$.

Let $Q = [Q_1|Q_2]$ be the orthogonal matrix obtained in the Full QR Decomposition of $M_d(F)$. Suppose that the rank of $A = M_d(F)$ is k. If v is the coefficient vector of the polynomial $g \in P_d$, then

$$v = \sum_{i=1}^{k}\langle v, (Q_1)_i\rangle (Q_1)_i + \sum_{i=1}^{m-k}\langle v, (Q_2)_i\rangle (Q_2)_i \qquad (3.2)$$

where $m = \binom{n+d-1}{d}$.

Let $\{s_1, s_2, \cdots, s_k\}$ be the set of index set return in the Full QR Decomposition of $M_d(F)$ and $\{e_{t_1}, e_{t_2}, \cdots, e_{t_{m-k}}\}$ be the set of unit vectors used in the Full QR Decomposition of $M_d(F)$. Define the matrix $\tilde{R}$ with

$$\tilde{r}_{ij} = \begin{cases} 0, & \text{if } i < j; \\ \langle A_{s_j}, (Q_1)_i \rangle, & \text{if } i \geq j, 1 \leq j \leq k; \\ \langle e_{t_{k-j}}, (Q_2)_i \rangle, & \text{if } i \geq j, k+1 \leq j \leq m. \end{cases}$$

Therefore

$$Q = [Q_1|Q_2] = \left[A_{s_1}, \cdots, A_{s_k}, e_{t_1}, \cdots, e_{t_{m-k}}\right]\tilde{R}^{-1}.$$

Replacing this matrix equation into equation (3.2) and going back to polynomials gives

$$g = \sum_{i=1}^{s} h_i f_i + r^d$$

where $h_i \in P_{d-\deg(f_i)}$ and $r^d \in W_d(F)$. This is the key computational process of the reduction algorithm. Now let $p \in P$ with $\deg(p) = d$. Therefore we can write

$$\mathrm{LF}(p) = \sum_{i=1}^{s} g_i f_i + r^d$$

where $g_i \in P_{d-\deg(f_i)}$ and $r^d \in W_d(F)$. Then we can apply same process to the polynomial

$$p - \sum_{i=1}^{s} g_i h_i - r^d.$$

After at most d steps, this process provides an expression for each polynomial $p \in P$ as

$$p = \sum_{i=1}^{s} q_i h_i + r \qquad (3.3)$$

where $\deg(q_i h_i) \leq d$ and $r \in \bigcup_{k=0}^{n} W_k(F)$. In this case we say p is reduced to the remainder r modulo H and write $p \xrightarrow{H} r$. The following algorithm outlines the steps of the reduction process.

Algorithm 3.2.7. (Reduction QR Algorithm)

Input: $H = \{h_1, \ldots, h_s\} \subseteq P, p \in P$

Output: $q_1, \ldots, q_s, r \in P$ satisfying (3.3).

$F = \{f_1 = \mathrm{LF}(h_1), \ldots, f_s = \mathrm{LF}(h_s)\}$

$r = 0,\ q_i = 0, 1 \leq i \leq s.$

While $p \neq 0$

$g = \mathrm{LF}(p), d = \deg(g)$

Construct $M_d(F)$

If $M_d(F) \neq 0$ **Then**

Find Full QR Decomposition of $M_d(F)$

Compute the expression $g = \sum_{i=1}^{s} h_i f_i + r^d$

$r = r + r^d$

$p = p - \sum_{i=1}^{s} g_i h_i - r^d$

$q_i = q_i + g_i, 1 \leq i \leq s$

Else

$r = r + g$

End If-Else

End While

Example 3.2.8. Let $H = \{h_1 = x_1 + x_3\ , h_2 = x_1x_2 + x_2x_3 + x_2\ , h_3 = x_1x_3 + x_2x_3 + x_1 + 1\}$, $F = \{f_1 = \mathrm{LF}(h_1) = x_1 + x_3, f_2 = \mathrm{LF}(h_2) = x_1x_2 + x_2x_3, f_3 = \mathrm{LF}(h_3) = x_1x_3 + x_2x_3\}$ and $f = x_1^2 + x_1x_3 + x_3^2 + x_1 + x_3 + 1$. So $\mathrm{LF}(f) = x_1^2 + x_1x_3 + x_3^2$. Then

$$M_2(F) = \begin{array}{c} \begin{array}{ccccc} x_1f_1 & x_2f_1 & x_3f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

If we apply the Full QR Decomposition to $A = M_2(F)$, then it returns index $=$ $\{1,2,3,5\}$, vectorIndex $= \{6,4\}$. So $B = [x_1f_1 \quad x_2f_1 \quad x_3f_1 \quad f_3 \quad x_3^2 \quad x_2^2]$,

$$Q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{15}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{3}}{\sqrt{10}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{15}} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{\sqrt{10}} \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{15}} \end{bmatrix}, Q_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & 0 \end{bmatrix},$$

and

$$\tilde{R} = \begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{5}}{\sqrt{6}} & \frac{-\sqrt{2}}{\sqrt{15}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The vector form of $\mathrm{LF}(f)$ is $v = (1,0,1,0,0,1)$. The coordinate vectors of v can be computed as follows:

$$\langle v, (Q_1)_1 \rangle = \sqrt{2}, \langle v, (Q_1)_2 \rangle = 0, \langle v, (Q_1)_3 \rangle = \frac{\sqrt{2}}{\sqrt{3}},$$

$$\langle v, (Q_1)_4 \rangle = \frac{-\sqrt{2}}{\sqrt{15}}, \langle v, (Q_2)_1 \rangle = \frac{1}{\sqrt{5}}, \langle v, (Q_2)_2 \rangle = 0.$$

$$u = \left(\sqrt{2}, 0, \frac{\sqrt{2}}{\sqrt{3}}, \frac{-\sqrt{2}}{\sqrt{15}}, \frac{1}{\sqrt{5}}, 0\right).$$

Then the corresponding polynomial is

$$u[x_1 f_1 \quad x_2 f_1 \quad x_3 f_1 \quad f_3 \quad x_3^2 \quad x_2^2]\tilde{R}^{-1} = x_1 f_1 + x_3^2.$$

So $a_1 = x_1, r_1 = x_3^2$. Then

$$f = f - (x_1 f_1 + x_3^2) = x_1 + x_3 + 1, \mathrm{LF}(f) = x_1 + x_3.$$

Since $\deg(\mathrm{LF}(f)) = 1$, we need to use

$$M_1(F) = \begin{array}{c} f_1 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

To find the Full QR Decomposition of $M_1(F)$, we find the QR Decomposition of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then we find

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \vdots & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \vdots & 0 & 1 \\ \frac{1}{\sqrt{2}} & \vdots & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} = [Q_1 \quad \vdots \quad Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

index $= \{1\}$, vectorIndex $= \{3,2\}$, $B = [\mathrm{LF}(f_1) \quad x_3 \quad x_2]$.

Extended QR Decomposition of $M_1(F)$ is $[Q_1 \quad \vdots \quad Q_2]\tilde{R}$ where

$$\tilde{R} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The vector form of $\mathrm{LF}(f)$ is $v = (1,0,1)$. The coordinate vectors of v can be computed as follows:

$$\langle v, \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\rangle = \sqrt{2}, \qquad \langle v, \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\rangle = 0, \qquad \langle v, (0,1,0)\rangle = 0.$$

Then the corresponding polynomial is

$$\left(\sqrt{2}, 0,0\right)^T [\mathrm{LF}(f_1), x_3, x_2]\tilde{R}^{-1} = \left(\sqrt{2}, 0,0\right)^T \left(\frac{f_1}{\sqrt{2}}, \frac{-f_1}{\sqrt{2}} + \sqrt{2}x_3, x_2\right) = f_1.$$

Then $f = f - h_1 = x_1 + x_3 + 1 - h_1 = 1$. So $\mathrm{LF}(f) = 1$. Since $M_0(F) = \emptyset$, $1 = 0 + 1$ and $f = f - 1 = 0$. So

$$f = (x_1 + 1)f_1 + x_3^2 + 1.$$

Hence

$$f \xrightarrow{H} x_3^2 + 1.$$

3.2.3 H-Basis Algorithm

Upon executing the $syz_d(F)$ and $\mathrm{LM}_{\mathrm{d}}(F)$ computations as outlined in Algorithm 3.1.21, using the methodologies described in Subsection 3.2.1, and subsequently applying the reduction operation as specified in Algorithm 3.2.7, the algorithm will be transformed into one that conducts the H-basis calculation exclusively through QR Decomposition techniques.

4. DISCUSSION

Hashemi and Javanbakht, M. (2018) propose a method for generating a basis for the syzygies of a finitely many homogeneous polynomials, organized degree by degree. The fundamental aspect of this approach involves the generation of a matrix composed of coefficient vectors corresponding to the leading forms of polynomials. Then using the row reduced echelon forms of this matrix and its transpose, they obtain a basis for syzygy module. We use extended row reduced echelon form to obtain the syzygy basis for homogeneous polynomials. In contrast to approach given in (Hashemi and Javanbakht, M., 2018), we conduct all necessary computations through a single matrix reduction. On the other hand, the primary factor in determining the termination degree of the algorithm presented in (Hashemi and Javanbakht, M., 2018) is the presence of a Gröbner basis within a specific matrix, alongside the application of Schreyer's theorem (see Schreyer 1980). In the algorithm we propose, the upper bound is determined independently of the Gröbner basis. Moreover, the reduction procedure described in (Hashemi and Javanbakht, M., 2018) is based on a very complicated orthogonalization process through an inner product (see Pena and Sauer, 2007). However, we are able to define a reduction procedure depends on the row reduced form of the matrix we used in syzygy computation. Consequently, the algorithm for calculating H-bases that we have developed relies solely on the row reduction of matrices and does not incorporate any monomial ordering.

Javanbakht and Sauer (2019) present a method for computing the H-basis, which relies on determining a basis for the module of syzygies through the application of singular value decomposition. It is established that the singular value decomposition is the most reliable method for determining the rank and identifying orthogonal bases. Nevertheless, a more computationally efficient alternative is the QR Decomposition. Because of this, we thought that QR Decomposition would serve as a viable alternative for the calculation of H-bases. We give a recursive method for syzygy module computation using QR Decomposition techniques. We also define reduction process with QR Decomposition. Thus, we developed an H-bases computation algorithm that relies exclusively on QR Decomposition techniques. Another advantage of our proposed algorithm in comparison to the algorithm given in (Javanbakht and Sauer, 2019) is the distinct upper bound utilized for the algorithm's termination. Consequently, there is no necessity for us to conduct any supplementary matrix reduction operations.

5. CONCLUSION

In this thesis, we present a discussion of various linear algebraic techniques aimed at proposing enhanced algorithms for the computation of H-bases. One potential future research area is the development of a signature-based algorithm for H-bases through the adaptation of F4-type algorithms designed for Gröbner bases (Faugere,1999).

In one step of the proposed H-basis algorithms, a syzygy is selected from a group of syzygies. This selection does not compromise the correctness of the algorithm; however, it can significantly influence performance. Machine learning techniques may be employed to assist in the development of a selection strategy within the algorithms.

If the methods established for identifying the syzygy bases of a finite set of homogeneous polynomials can be applied to modules of homogeneous polynomials, then the syzygy chain of these homogeneous polynomials can also be determined, which is called free resolution. This is an essential tool in commutative and computational algebra that yield significant invariants, including Betti numbers, Hilbert regularity, and the Krull dimension of the corresponding ideal.

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