Hamilton-Jacobi Formulation of the Thermodynamics of Einstein-Born-Infeld-AdS Black Holes

by

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Koç University

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ABSTRACT

A Hamilton-Jacobi formalism for thermodynamics was formulated by Rajeev (Ann. Phys., **323** (2008) 2265) based on the contact structure of the odd dimensional thermodynamic phase space and was applied to chargeless, non-rotating black holes in the same work. In this thesis we apply the formalism to charged, non-rotating black holes using the Born-Infeld theory and a negative cosmological constant. We first review the basics of the causal structure of spacetime, black holes and in particular the Reissner-Nordström black hole. Then we introduce the concepts related to the thermodynamics of black holes such as surface gravity and conserved Komar charges, and discuss the laws of black hole mechanics. We then review the contact geometric formulation of thermodynamics and the formalism developed by Rajeev. Lastly, we give the charged, non-rotating, asymptotically AdS black hole solution in the Einstein-Born-Infeld theory, derive the Hamilton-Jacobi equation and find its most general solution. We show that there exist solutions which are distinct from the equations of state of the Einstein-Born-Infeld-AdS black hole. The meaning of these solutions is discussed.

ÖZETÇE

Termodinamik için, tek boyutlu termodinamik faz uzayının kontakt yapısına dayalı bir Hamilton-Jacobi formalizmi Rajeev tarafından geliştirilmiş (Ann. Phys., **323** (2008) 2265) ve aynı çalışmada yüksüz, dönmeyen kara deliklere uygulanmıştır. Bu tezde aynı formalizm, Born-Infeld kuramını ve negatif bir kozmolojik sabiti kullanarak yüklü, dönmeyen kara deliklere uygulanmaktadır. Öncelikle uzayzamanın nedensel yapısının, kara deliklerin ve özellikle Reissner-Nordström kara deliğinin temelleri gözden geçirilmektedir. Daha sonra yüzey yerçekimi ve korunumlu Komar yükleri gibi kara deliklerin termodinamiğiyle ilgili kavramlar sunulmakta, kara delik mekaniğinin yasaları tartışılmaktadır. Ardından termodinamiğin kontakt geometrik formülasyonu ve Rajeev tarafından geliştirilen formalizm gözden geçirilmektedir. Son olarak, Einstein-Born-Infeld kuramındaki yüklü, dönmeyen, asimptotik olarak AdS kara delik çözümü sunulmakta, Hamilton-Jacobi denklemi türetilmekte ve en genel çözümü bulunmaktadır. Einstein-Born-Infeld-AdS kara deliğinin hal denkleminden farklı olan çözümler olduğu gösterilmekte, bu çözümlerin ne anlama geldiği tartışılmaktadır.

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Chapter 1

INTRODUCTION

It was discovered in the 1970s that general relativity predicts that black holes obey a set of laws which closely resemble the laws of thermodynamics. In 1972, Hawking showed that the horizon area of a black hole does not decrease with time [Hawking, 1972]. This led Bekenstein to argue that a suitable multiple of the horizon area should be interpreted as the entropy of the black hole [Bekenstein, 1973]. Bardeen, Carter and Hawking then formulated the four laws of black hole mechanics which are strikingly similar to the laws of thermodynamics [Bardeen et al., 1973]. These laws show that the mass M and, up to multiplicative constants, the surface gravity κ and the horizon area A of a black hole play roles analogous to the energy, temperature and entropy of a thermodynamical system, respectively. The fact that mass and energy are physically the same quantity suggests that this similarity should have physical significance. Classically, however, this is nothing more than a formal analogy, since a classical black hole is always at absolute zero. One way to see this is by noting that a black hole surrounded by thermal radiation will always absorb some radiation, but never emit back. Hence the black hole can be at equilibrium with its surroundings only if there is no radiation, i.e. at absolute zero. Additionally, the black hole uniqueness theorem implies that a black hole has a single microstate, corresponding to vanishing entropy.

By incorporating quantum mechanical effects, however, Hawking discovered in 1975 [Hawking, 1975] that a black hole emits thermal radiation at a temperature

$$k_{\mathrm{B}}T = \frac{\hbar\kappa}{2\pi}. \tag{1.1}$$

This shows that $\hbar\kappa/2\pi$ is indeed the physical temperature of a black hole and suggests that the laws of black hole mechanics are nothing but the laws of thermodynamics applied to black holes. Since the 1970s, the study of black hole thermodynamics has produced most of our insight about the quantum phenomena occurring in strong gravitational fields [Wald, 2001].

We may expect that in a quantum theory of gravitation, thermodynamic quantities such as mass and area are treated as operators. It is therefore reasonable to attempt to "quantize" thermodynamics. Another possible application of quantized thermodynamics is to systems where the thermal fluctuations are small, but the quantum fluctuations are not.

Thermodynamics can be formulated in terms of contact geometry [Arnold, 1990, Bravetti et al., 2015], where the first law is used to define a contact structure and the equations of state pick out a Legendrian submanifold of the thermodynamic phase space. With the aim of quantizing thermodynamics, a quantization procedure for contact manifolds is established in [Rajeev, 2008b]. Subsequently, a Hamilton-Jacobi formalism for thermodynamics is developed in [Rajeev, 2008a].

In this formalism one extends a thermodynamic system of n degrees of freedom into an n-parameter family; e.g., the ideal gas of a fixed particle number can be extended into the van der Waals family with the parameters a and b. This family can be described as a hypersurface in the phase space, defined by the vanishing of a function which we take to be the Hamiltonian. Given the Hamiltonian function H, one can formulate a Hamilton-Jacobi equation (HJE), the characteristic curves of which correspond to the dynamics generated by H. The equations of state can be obtained in principle by solving the HJE.

In the same work the Hamilton-Jacobi formalism is applied to black holes of one thermodynamical degree of freedom, i.e., to electrically neutral non-rotating black holes. A negative cosmological constant is introduced to extend the Schwarzschild black hole into the Schwarzschild-AdS family. It is also proposed that the Born-

Infeld action be used as a modification of the Einstein-Maxwell equations to describe the family of charged black holes (further modifications can be made to include the rotating ones as well).

The goal of this work is to apply the Hamilton-Jacobi formalism to charged non-rotating black holes using Born-Infeld theory, as suggested by Rajeev. The thesis is organized as follows. We first review the basics of the theory of black holes. We then introduce the concepts related to the thermodynamics of black holes and discuss the laws of black hole mechanics. Afterwards we examine the formulation of thermodynamics in terms of contact geometry and introduce the aforementioned Hamilton-Jacobi formalism for thermodynamics. Lastly, we apply the Hamilton-Jacobi formalism to Einstein-Born-Infeld-AdS black holes, write down the HJE and discuss its solutions. The results of this work were published in Ref. [Dereli and Ünlütürk, 2019]

Throughout this work, we employ units with $c = G = 4\pi\epsilon_0 = 1$ and use the mostly plus metric signature $(- + ++)$.

Chapter 2

BLACK HOLES

2.1 Causal Structure

A spacetime is a pair $(\mathcal{M}, g)$ consisting of a manifold $\mathcal{M}$ and a metric tensor g of Lorentzian signature. At each point $p \in \mathcal{M}$, the tangent space $T_p\mathcal{M}$ is isomorphic to Minkowski spacetime. In particular, we have a well-defined notion of a lightcone passing through the origin of $T_p\mathcal{M}$. Just as in Minkowski spacetime, we can designate half of this lightcone as "future" and the other half as "past". However, it is not possible in general to make this choice continuosly throughout $\mathcal{M}$. If such a continuos designation of future and past can be made, the spacetime is called *time orientable*.

Time orientability is analogous to, but distinct from the notion of orientability of a manifold. For example, one can assign a non-time orientable metric to the orientable band $S^1 \times I$ and a time orientable metric to the non-orientable Möbius band (Fig. 2.1). In the following, we shall only consider time orientable spacetimes, and we shall

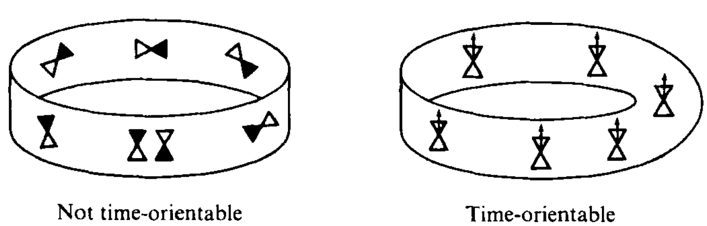

Figure 2.1: Illustrations showing a non-time orientable metric on the orientable manifold $S^1 \times I$ (left) and a time orientable metric on the non-orientable Möbius band (right). Image taken from [O'Neill, 1983].

assume that a particular orientation is chosen.

Let $S \in \mathcal{M}$ be a set. The *chronological future* $I^+(S)$ of S is defined as the set of all points in $\mathcal{M}$ which can be reached from S by a future directed timelike curve (Note that a constant curve, i.e. a single point, is not a timelike curve. Hence $I^+(S)$ may not contain S.). $I^+(S)$ is an open set, since if $p \in \mathcal{M}$ can be reached from S by such a curve, then there is a neighbourhood of p that can be reached so.

The *causal future* $J^+(S)$ of S is defined as the union of S with the set of all points in $\mathcal{M}$ which can be reached from S by a future directed non-spacelike curve. It can be shown [Hawking and Ellis, 1973] that $\overline{I^+}(p) = \overline{J^+}(p)$ and $\dot{I}^+(p) = \dot{J}^+(p)$ where $\bar{A}$ denotes the closure of a set A and

$$\dot{A} = \bar{A} \cap \left(\overline{\mathcal{M} - A}\right) \tag{2.1}$$

denotes the boundary of A.

The definitions for the chronological past $I^-(S)$ and the causal past $J^-(S)$ are analogous.

2.2 Conformal Infinity

The idea of examining the infinity by means of a conformal transformation was first introduced by Penrose [Penrose, 1963, Penrose, 2011], which can be summarized as follows. Consider a spacetime $(\mathcal{M}, g)$. We want to construct an "unphysical" spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ such that $\mathcal{M}$ is conformally isometric to an open region of $\tilde{\mathcal{M}}$ with $\tilde{g} = \omega^2 g$, and the infinity of $\mathcal{M}$ is located at a "finite" surface $\mathscr{I}$. This means that $\omega = 0$ on $\mathscr{I}$; i.e. the metric at $\mathscr{I}$ is stretched by an infinite factor in the passage from $\tilde{\mathcal{M}}$ to $\mathcal{M}$, so $\mathscr{I}$ gets mapped to infinity. Asymptotic properties of $\mathcal{M}$ can now be investigated by studying $\mathscr{I}$, provided that all the relevant concepts can be put into a conformally invariant form.

A conformal transformation

$$g \to \omega^2 g \tag{2.2}$$

preserves angles, which, in the case of a Lorentzian signature, means that the lightcone structure is preserved. Hence we can conveniently apply conformal transformations to investigate the causal structure of a spacetime.

As an example, consider the Minkowski metric in spherical coordinates:

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\Omega^2, \tag{2.3}$$

where $\mathrm{d}\Omega^2$ is a shorthand notation for $\mathrm{d}\Omega^2 = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi$. We introduce the coordinates

$$\tan U = t - r, \quad \tan V = t + r, \tag{2.4}$$

which have the ranges

$$-\frac{\pi}{2} < U < \frac{\pi}{2} \quad \text{and} \quad U \leq V < \frac{\pi}{2}. \tag{2.5}$$

Thus we have brought infinity to finite coordinate values. In these coordinates, the metric (2.3) becomes

$$g = \omega^{-2} \left[-2 \left(\mathrm{d}U \otimes \mathrm{d}V + \mathrm{d}V \otimes \mathrm{d}U \right) + \sin^2(V - U) \mathrm{d}\Omega^2 \right], \tag{2.6}$$

where $\omega = 2 \cos U \cos V$. Hence the Minkowski metric is conformally equivalent to the metric

$$\begin{aligned} \tilde{g} &= \omega^2 g \\ &= -2 \left(\mathrm{d}U \otimes \mathrm{d}V + \mathrm{d}V \otimes \mathrm{d}U \right) + \sin^2(V - U) \mathrm{d}\Omega^2. \end{aligned} \tag{2.7}$$

Transforming to new time and space coordinates

$$T = V + U, \quad R = V - U, \tag{2.8}$$

with ranges

$$0 \leq R < \pi \quad \text{and} \quad |T| + R < \pi, \tag{2.9}$$

this is

$$\tilde{g} = -\mathrm{d}T \otimes \mathrm{d}T + \mathrm{d}R \otimes \mathrm{d}R + \sin^2 R \mathrm{d}\Omega^2. \tag{2.10}$$

The metric (2.10) describes a spacetime with the topology $\mathbb{R} \times S^3$. With the unrestricted coordinates $-\infty < T < \infty$ and $0 \leq R \leq \pi$, it is called the *Einstein static universe*. Hence we have the following result: Minkowski spacetime is conformally isometric to an open region of the Einstein static universe, with the conformal factor ω vanishing at infinity.

One can draw $\mathbb{R} \times S^3$ as a cylinder, where each circle of constant T represents a 3-sphere (Fig. 2.2). Taking the region corresponding to Minkowski space and flattening

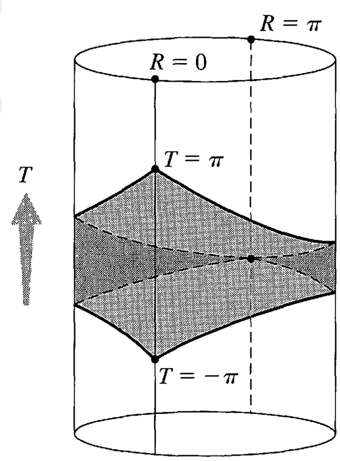

Figure 2.2: The Einstein static universe, $\mathbb{R} \times S^3$. Shaded region is conformally related to Minkowski spacetime. Image taken from [Carroll, 2004].

the cylinder, we get the *conformal (Penrose) diagram* of Minkowski space, illustrated in Fig. 2.3. Minkowski space corresponds to the interior of this diagram (including $R = 0$). The boundary, where $\omega = 0$, is called the *conformal infinity*. The union of the original spacetime with conformal infinity is called the *conformal compactification*.

The conformal diagram allows us to depict the whole spacetime (and the infinity) in a finite diagram and the lightcones of the metric (2.10) are at 45° to the horizontal in the Penrose diagram ($\mathrm{d}T = \pm\mathrm{d}R$ for radial null curves). But a conformal transformation preserves the lightcone structure, hence these are precisely the lightcones of Minkowski spacetime. Thus, conformal diagrams provide a simple visualization of the causal structure of a spacetime.

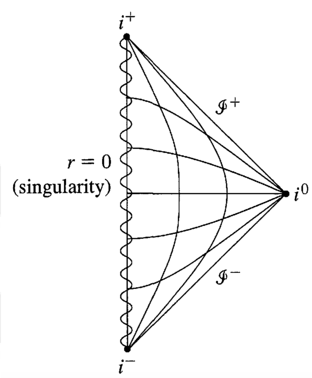

Figure 2.3: Penrose diagram of Minkowski spacetime. Image taken from [Carroll, 2004].

The conformal infinity can be divided into the regions

- i^+: future timelike infinity ($T = \pi$, $R = 0$)
- i^0: spatial infinity ($T = 0$, $R = \pi$)
- i^-: past timelike infinity ($T = -\pi$, $R = 0$)
- $\mathscr{I}^+$: future null infinity ($T = \pi - R$, $0 < R < \pi$)
- $\mathscr{I}^-$: past null infinity ($T = -\pi + R$, $0 < R < \pi$)

Note that unlike the other points in the diagram, $i^\pm$ and i^0 are actually points in $\tilde{\mathcal{M}}$, not 2-spheres. This is due to the usual coordinate singularities at $R = 0, \pi$ which are the north and south poles of S^3.

It can be checked that all timelike geodesics start at i^- and end at i^+, all null geodesics start at $\mathscr{I}^-$ and end at $\mathscr{I}^+$, all spacelike geodesics start and end at i^0.

A spacetime is called *asymptotically flat* if it has the same structure of null infinity as that of the Minkowski spacetime. This definition can be made more precise, see e.g. Ref. [Townsend, 1997], however we shall be content with this intuitive definition for the purposes of this work.

2.3 Black Hole Spacetimes

To discuss black holes, let us start with the simplest example, the Schwarzschild metric, before providing a precise definition. It is given by

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\Omega^2. \tag{2.11}$$

It can be seen that the metric coefficients are singular at $r = 0$ and $r = 2M$. As we shall see, the singularity at $r = 2M$ is simply due to a bad choice of coordinates and can be "lifted" by a coordinate change. This type of singularity is said to be a *coordinate singularity*. The one at $r = 0$, however, is there regardless of the coordinates used, hence $r = 0$ is not a part of the spacetime. This is easily seen by looking at the Kretschmann scalar

$$\begin{aligned} \mathcal{K} &\equiv 2 \star \left(R^a{}_b \wedge \star R^b{}_a\right) \\ &= \frac{48M^2}{r^6}, \end{aligned} \tag{2.12}$$

which diverges as $r \to 0$. Such a singularity is called a *curvature singularity*.

Note that t is a timelike coordinate for $r > 2M$, in the sense that $g_{tt} = g(\partial_t, \partial_t) < 0$. But for $0 < r < 2M$, t becomes spacelike and it is now r that is timelike. We shall indeed see that in the region $0 < r < 2M$, the direction of time is the direction of decreasing r. Therefore, once a particle or a light ray enters $r < 2M$, it cannot go back to $r > 2M$; it has to move in the direction of decreasing r until it "hits" the singularity at $r = 0$, which happens at a finite proper time.

Let us now show that $r = 2M$ is a coordinate singularity by finding coordinates which are well-behaved there. To this end we define the *tortoise coordinate* r^*:

$$\begin{aligned} r^* &\equiv \int \frac{\mathrm{d}r}{1-\frac{2M}{r}} \\ &= r + 2M \ln\left(\frac{r}{2M} - 1\right). \end{aligned} \tag{2.13}$$

With

$$v = t + r^*, \tag{2.14}$$

the metric (2.11) becomes

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}v \otimes \mathrm{d}v + (\mathrm{d}v \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}v) + r^2 \mathrm{d}\Omega^2. \tag{2.15}$$

(v, r, θ, ϕ) are called the Eddington-Finkelstein coordinates. Although $g_{vv} = 0$ at $r = 2M$, this is not a problem since the determinant of metric coefficients;

$$g = -r^4 \sin^2\theta, \tag{2.16}$$

is finite there, hence the metric is well-defined. We see that $r = 2M$ is a perfectly regular part of the spacetime, and the "problem" is merely that the Schwarzschild coordinates (t, r, θ, ϕ) are ill-behaved there.

Let us now examine what happens when we enter the region $r < 2M$. To this end, we look at the radial null curves. These satisfy either $v = \text{const}$ (infalling) or

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2}{1-\frac{2M}{r}} \tag{2.17}$$

(outgoing). The corresponding lightcones are depicted in Fig. 2.4. Thus we see that once a particle or a light ray enters the region $r < 2M$, it has to move in the direction of decreasing r as mentioned. The hypersurface $r = 2M$ is called the *event horizon* and the region $r < 2M$ is said to be the black hole region.

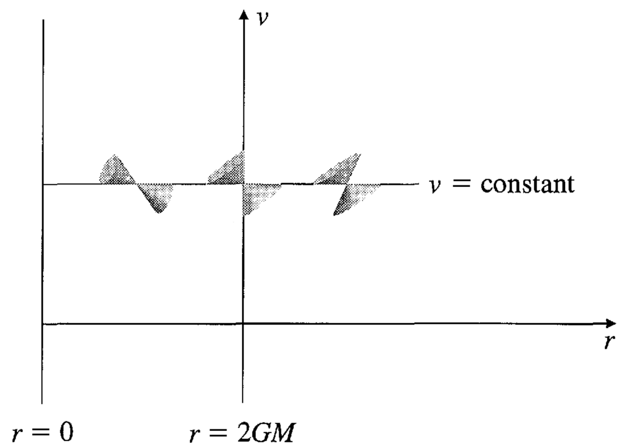

Figure 2.4: Lightcones of Schwarzschild spacetime in Kruskal coordinates. Once inside $r < 2M$, no particle or light ray can escape back to $r > 2M$, since the direction of time here is the direction of decreasing r. Image taken from [Carroll, 2004] with a correction.

As a final remark about the Schwarzschild metric, we note that it can be extended into a larger spacetime that is geodesically complete [Kruskal, 1960] (maximal extension). In the Kruskal coordinates

$$T = \sqrt{\frac{r}{2M} - 1}e^{r/4M} \sinh \frac{t}{4M}, \tag{2.18a}$$

$$R = \sqrt{\frac{r}{2M} - 1}e^{r/4M} \cosh \frac{t}{4M}, \tag{2.18b}$$

the Schwarzschild metric takes the form

$$g = \frac{32M^3}{r}e^{-r/2M} \left(-\mathrm{d}T \otimes \mathrm{d}T + \mathrm{d}R \otimes \mathrm{d}R\right) + r^2 \mathrm{d}\Omega^2, \tag{2.19}$$

where r is now a function of T and R given as the positive solution of

$$T^2 - R^2 = \left(1 - \frac{r}{2M}\right) e^{r/2M}. \tag{2.20}$$

We get the maximal extension if we allow T and R to take every value they can without hitting $r = 0$, i.e.

$$-\infty < R < \infty \quad \text{and} \quad T^2 < R^2 + 1. \tag{2.21}$$

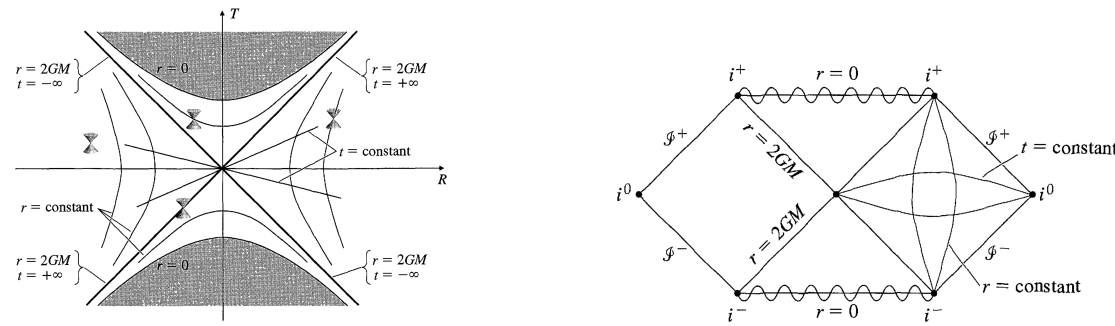

Figure 2.5: (a) Spacetime diagram showing the Kruskal extension of Schwarzschild spacetime. (b) Penrose diagram of the Kruskal extension. Images taken from [Carroll, 2004].

The resulting spacetime diagram is shown in Fig. 2.5a with θ and ϕ supressed. Fig. 2.5b shows the Penrose diagram for this maximally extended Schwarzschild spacetime. The Penrose diagram makes it manifest that nothing in the black hole region can escape to $\mathscr{I}^+$.

The above property of a Schwarzschild black is used as the defining property of a black hole. Select out a specific asymptotically flat region of a spacetime. The selected region has a spacelike infinity i^0 and future/past null infinities $\mathscr{I}^{\pm}$. The surface of black holes (event horizons) separate the external universe, which can send signals to $\mathscr{I}^+$, from the black hole interiors, which cannot. The union of all event horizons is defined as the region

$$\mathcal{H} = \dot{J}^-(\mathscr{I}^+), \tag{2.22}$$

i.e., the boundary of the causal past of $\mathscr{I}^+$ [Misner et al., 1973]. From the definition it is clear that the horizon is a null surface. A spacetime is called a black hole spacetime if it contains an event horizon.

Before proceeding further, we present a simple way to locate the event horizons in certain spacetimes (as described in [Carroll, 2004]), which will be quite useful for our subsequent discussions. Suppose we have a stationary, asymptotically flat spacetime with the time translation Killing vector ∂_t. We can adapt our coordinate system to ∂_t, such that metric components are independent of t. On the surfaces of constant t, we can introduce coordinates (r, θ, ϕ) such that the metric at infinity looks like the Minkowski spacetime in spherical coordinates.

Now suppose that as we decrease r from infinity, the surfaces of contant r remain timelike until some radius r_{H}, for which it is everywhere null. Then the surface $r = r_{\mathrm{H}}$ is clearly an event horizon, since non-spacelike curves that enter the region $r < r_{\mathrm{H}}$ will never be able escape back to infintiy. In such a case, we can simply look at the norm

$$g^{-1}(\mathrm{d}r, \mathrm{d}r) = g^{rr} \tag{2.23}$$

of the normal 1-form $\mathrm{d}r$ and find where it vanishes to locate the horizon, where g^{-1} is the inverse metric. Hence the horizon will be given by $g^{rr} = 0$.

It turns out that all (four dimensional) electrovacuum black hole spacetimes are characterized by a few quantities: their mass, angular momentum and electric charge, the precise definitions of which will be given later. This is called the black hole uniqueness theorem, and a review of the subject can be found in Ref. [Chruściel et al., 2012].

The black hole uniqueness theorem allows a simple characterization of black holes as follows. A black hole with no electric charge and no angular momentum is the already mentioned Schwarzschild black hole. One with electric charge but no angular momentum is called a Reissner-Nordström black hole. If the black hole has angular momentum but no electric charge, it is called a Kerr black hole. And lastly, a black hole with non-zero electric charge and angular momentum is called a Kerr-Newman black hole.

In this work, we shall restrict our attention to non-rotating black holes, hence our primary subject of interest will be the Reissner-Nordström black holes and modifications thereof.

2.4 Reissner-Nordström Black Holes

To derive the Reissner-Nordström metric, we start with the static, spherically symmetric ansatz

$$g = -f(r)\mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}r \otimes \mathrm{d}r}{f(r)} + r^2 \mathrm{d}\Omega^2. \tag{2.24}$$

It is easy to see that the flat space solution

$$F = \frac{Q}{r^2}\mathrm{d}r \wedge \mathrm{d}t \tag{2.25}$$

for the electromagnetic 2-form F of a point charge is a solution of Maxwell's equations $\mathrm{d} \star F = 0$ for the metric in (2.24) as well.

To determine $f(r)$, we look at the Einstein field equations

$$-\frac{1}{2} R^{bc} \wedge \star e_{abc} = 8\pi \tau_a^{(\mathrm{LED})}, \tag{2.26}$$

with the orthonormal coframe

$$e^0 = \sqrt{f}\mathrm{d}t, \quad e^1 = \frac{\mathrm{d}r}{\sqrt{f}}, \quad e^2 = r\mathrm{d}\theta, \quad e^3 = r\sin\theta \mathrm{d}\phi, \tag{2.27}$$

and where $\tau_a^{(\mathrm{LED})}$ are the stress-energy 3-forms of linear (i.e. Maxwell) electrodynamics:

$$\begin{aligned} \tau_a^{(\mathrm{LED})} &= \frac{1}{8\pi}\left(i_a F \wedge \star F - F \wedge i_a \star F\right) \\ &= \begin{cases} -\frac{1}{8\pi}\frac{Q^2}{r^4} \star e_a, & a = 0, 1 \\ +\frac{1}{8\pi}\frac{Q^2}{r^4} \star e_a, & a = 2, 3. \end{cases} \end{aligned} \tag{2.28}$$

To determine the left hand side of Eq. (2.26) in terms of $f(r)$, we first find the connection 1-forms ω^{ab} by solving the equations

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0 \quad \text{and} \quad \omega^{ab} + \omega^{ba} = 0. \tag{2.29}$$

The result is

$$\left(\omega^a{}_b\right) = \begin{pmatrix} 0 & \frac{1}{2}f'(r)\,\mathrm{d}t & 0 & 0 \\ & 0 & -\sqrt{f(r)}\mathrm{d}\theta & -\sqrt{f(r)}\sin\theta\mathrm{d}\phi \\ & & 0 & -\cos\theta\mathrm{d}\phi \\ & & & 0 \end{pmatrix}, \tag{2.30}$$

where the prime denotes the derivative with respect to the argument of f and the blank entries are given by the antisymmetry. From these we get the curvature 2-forms $R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$:

$$\left(R^{ab}\right) = \begin{pmatrix} 0 & -\frac{1}{2}f''(r)e^{01} & -\frac{1}{2r}f'(r)e^{02} & -\frac{1}{2r}f'(r)e^{03} \\ & 0 & -\frac{1}{2r}f'(r)e^{12} & -\frac{1}{2r}f'(r)e^{13} \\ & & 0 & \frac{1-f(r)}{r^2}e^{23} \\ & & & 0 \end{pmatrix}. \tag{2.31}$$

With the curvature 2-forms we can construct the Einstein 3-forms:

$$-\frac{1}{2}R^{bc} \wedge \star e_{abc} = \begin{cases} \frac{1}{r^2}\left[(rf)' - 1\right] \star e_a, & a = 0, 1 \\ \frac{1}{2r}\left(rf\right)'' \star e_a, & a = 2, 3. \end{cases} \tag{2.32}$$

Inserting Eqs. (2.28) and (2.32) into Eq. (2.26), the only independent equation we get is

$$\frac{1}{r^2}\left[(rf)' - 1\right] = -\frac{Q^2}{r^4}, \tag{2.33}$$

and the solution is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{2.34}$$

where M is a constant of integration which gives the total mass (see Sec. 3.2). The resulting metric is called the Reissner-Nordström metric.

Note that we can define the Eddington-Finkelstein coordinates for any metric of the form (2.24). For this, we define the tortoise coordinate r^* through

$$\mathrm{d}r^* = \frac{\mathrm{d}r}{f(r)}, \tag{2.35}$$

where the constant of integration can be chosen arbitrarily. One can again see in the Eddington-Finkelstein coordinates (v, r, θ, ϕ) that the metric is regular everywhere except for $r = 0$, where $v = t + r^*$.

To investigate the horizon structure of the Reissner-Nordström metric, note that the remark at the end of Sec. 2.3 is valid for this metric. Hence the horizon will be given by a zero of $f(r)$. There are three different cases one should consider (Fig. 2.6). If $M^2 < Q^2$, then the metric is everywhere regular in the given coordinates, except for the curvature singularity at $r = 0$. Since there is no horizon, an observer can travel to the singularity and return back. Such a singularity is said to be *naked*. The Penrose diagram of this metric looks like that of the Minkoswki spacetime, but $r = 0$ is now singular (Fig. 2.7a).

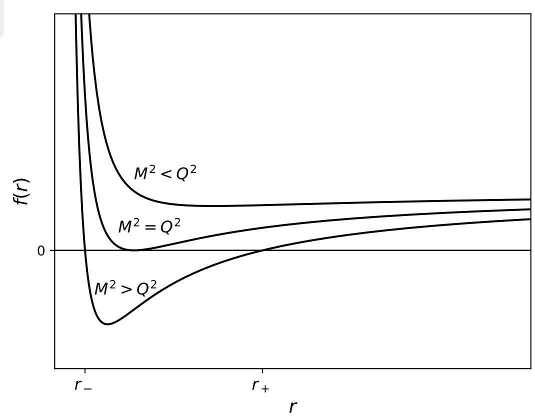

Figure 2.6: Horizon structure of the Reissner-Nordström black hole. In the case $M^2 < Q^2$ there is no horizon; there exists a naked singularity. In the extremal case $M^2 = Q^2$ one has a single horizon. In the case $M^2 > Q^2$ there exist an inner horizon and an outer horizon.

If $M^2 = Q^2$, the metric is called an *extreme* Reissner-Nordström metric. In this case there is a single horizon at $r = M$. The Penrose diagram for this case is shown in Fig. 2.7b.

The third case is $M^2 > Q^2$. This is the case one would expect in the realistic

(a) (b)

Figure 2.7: Penrose diagrams of the Reissner-Nordström black hole for the cases (a) $M^2 < Q^2$ and (b) $M^2 = Q^2$. Images taken from [Carroll, 2004].

gravitational collapse of a star, since we expect stars to be nearly neutral and the gravitational attraction should dominate over the electromagnetic repulsion for it to collapse. In this case there is an inner horizon at r_- and an outer horizon at r_+ given by

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{2.36}$$

The r coordinate is spacelike for $r > r_+$ and $r < r_-$, and timelike for $r_- < r < r_+$. Once a particle or a light ray enters the region $r_- < r < r_+$ from $r > r_+$, it has to move in the direction of decreasing r until it reaches r_-. After it enters the region $r < r_-$, it is again free to move in the direction of increasing r. The Penrose diagram for this case is shown in Fig. 2.8.

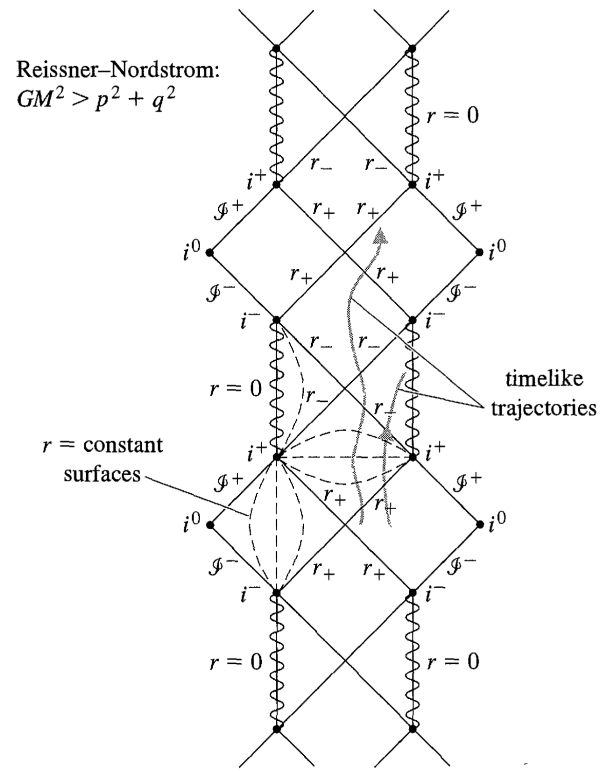

Figure 2.8: Penrose diagram of the Reissner-Nordström black hole for the case $M^2 > Q^2$. Image taken from [Carroll, 2004].

Chapter 3

BLACK HOLE THERMODYNAMICS

3.1 Killing Horizons

A null hypersurface Σ is said to be a Killing horizon of the Killing vector ξ if ξ is normal to Σ. The notion of a Killing horizon is logically independent from an event horizon. However, under reasonable assumptions to be discussed in a moment, we can show that the event horizon of a stationary-axisymmetric black hole is a Killing horizon of

$$\xi = K + \Omega\chi, \tag{3.1}$$

where K is the time translation Killing vector, χ is the rotational Killing vector and Ω is a constant which is called the angular velocity of the horizon. For non-rotating (static) black holes one has $\Omega = 0$, consequently; $\xi = K$. For a justification of the terminology for Ω, see Ref. [Carroll, 2004].

More precisely, Carter has shown that for a static black hole, the Killing vector K must be normal to the horizon, while for a stationary-axisymmetric one with the "$t - \phi$ orthogonality property", a Killing field of the form (3.1) exists, which is normal to the horizon. A stationary-axisymmetric black hole is said to possess the $t - \phi$ orthogonality property, if the 2-planes spanned by K and χ are orthogonal to a family of 2-surfaces. Another result, due to Hawking, shows that in electrovacuum general relativity, the horizon of any stationary black hole is a Killing horizon. See [Wald, 2001] for details.

Note that if a surface is given by the equation $f(x) = 0$ for some function f, then $\mathrm{d}f$ is normal to that surface. In the case of a Killing horizon Σ, we have

$N = \langle \xi, \xi \rangle = 0$, hence $\mathrm{d}N$ is normal to Σ. But since ξ is also a normal, there exists a function κ defined on Σ such that

$$\mathrm{d}N = -2\kappa\tilde{\xi}, \tag{3.2}$$

where we denote the 1-form corresponding to a vector with a tilde over the same symbol; $\tilde{\xi} \equiv g(\xi)$. Since ξ can be scaled arbitrarily, κ is not fixed by itself. However, in a static, asymptotically flat spacetime we shall impose the normalization $\langle K, K \rangle \to -1$ at infinity, which fixes κ up to a sign. To fix the sign, we also require K to be future directed. In the stationary case, fixing K fixes the linear combination (3.1) as well, so that κ is again well-defined.

We can obtain an explicit expression for κ as follows. Using Eqs. (C.21) and (C.24), we have

$$\begin{aligned}\star\left(\mathrm{d}\tilde{\xi} \wedge \star\mathrm{d}\tilde{\xi}\right) \wedge \star\tilde{\xi} &= -i_\xi \left(\mathrm{d}\tilde{\xi} \wedge \star\mathrm{d}\tilde{\xi}\right) \\ &= \mathrm{d}N \wedge \star\mathrm{d}\tilde{\xi} - 2\mathrm{d}\tilde{\xi} \wedge \omega.\end{aligned} \tag{3.3}$$

Since ξ is orthogonal to Σ, we have $\omega = 0$ on Σ by Frobenius's theorem. Hence we get, by repeated application of (3.2) on Σ:

$$\begin{aligned}\star\left(\mathrm{d}\tilde{\xi} \wedge \star\mathrm{d}\tilde{\xi}\right) \wedge \star\tilde{\xi} &= 2\kappa \star i_\xi \mathrm{d}\tilde{\xi} \\ &= -2\kappa \star \mathrm{d}N \\ &= 4\kappa^2 \star \tilde{\xi},\end{aligned} \tag{3.4}$$

which yields

$$\kappa^2 = \frac{1}{4}\left(\mathrm{d}\tilde{\xi} \wedge \star\mathrm{d}\tilde{\xi}\right)\Big|_\Sigma. \tag{3.5}$$

We can now provide a physical interpretation for κ as follows. By Eq. (C.27) and the fact that $\omega = 0$ on Σ, one can write

$$\kappa^2 = -\frac{1}{4N}\left\langle \mathrm{d}N, \mathrm{d}N \right\rangle\big|_\Sigma, \tag{3.6}$$

where the inner product of two p-forms α and β is defined in Eq. (B.3). Eq. (C.23) then yields

$$\kappa^2 = -\frac{1}{N} \left.\langle \nabla_\xi \xi, \nabla_\xi \xi \rangle\right|_\Sigma , \tag{3.7}$$

where we have used the fact that $\langle \tilde{X}, \tilde{Y} \rangle = \langle X, Y \rangle$ for vectors X, Y. Assume now that the spacetime is static so that $\xi = K$. Consider a static observer, i.e., an observer whose four-velocity U is proportional to K:

$$U = \frac{K}{\sqrt{-N}}. \tag{3.8}$$

Its four-acceleration is

$$\begin{aligned} A &= \nabla_U U \\ &= \frac{1}{-N} \nabla_K K, \end{aligned} \tag{3.9}$$

where in the second line we have used $\nabla_K N = 0$. Denote the magnitude of the acceleration by $a = \langle A, A \rangle^{1/2}$. Then from (3.7) we get

$$\kappa = \left. V a \right|_\Sigma . \tag{3.10}$$

Here, $V = (-N)^{1/2}$ is the redshift factor. Hence κ is the acceleration of a static observer on the horizon as seen from infinity. For this reason κ is called the *surface gravity*. This interpretation does not hold for spacetimes that are stationary but not static. Nonetheless, we shall continue to use the term surface gravity.

To conclude this section, we calculate the surface gravity for the static, spherically symmetric metric, Eq. (2.24). The event horizon $r = r_+$, where $f(r_+) = 0$, will be a Killing horizon for $\xi = \partial_t$. Hence we have $N = -f(r)$ and

$$\nabla_\xi \xi = \frac{1}{2} f'(r) f(r) \partial_r, \tag{3.11}$$

which gives

$$\kappa = \frac{1}{2} f'(r_+) \tag{3.12}$$

by Eq. (3.7).

3.2 Komar Charges

For each Killing vector ξ we can define a Komar charge [Komar, 1959] as

$$Q_\xi = \frac{c}{16\pi} \oint_{\partial V} \star \mathrm{d}\tilde{\xi}, \tag{3.13}$$

where c is a constant and ∂V is a topological 2-sphere at infinity. If ∂V is the boundary of a hypersurface V, we can use Stokes's theorem to write

$$Q_\xi = \frac{c}{16\pi} \int_V \mathrm{d} \star \mathrm{d}\tilde{\xi}. \tag{3.14}$$

In this sense, Q_ξ is a conserved charge since it is written as the integral of the conserved current

$$J = \mathrm{d} \star \mathrm{d}\tilde{\xi}, \tag{3.15}$$

i.e. $\mathrm{d}J = 0$.

In the case of the time translation Killing vector K, the Komar charge is the total mass in spacetime with $c = -2$:

$$M = -\frac{1}{8\pi} \oint_{\partial V} \star \mathrm{d}\tilde{K}. \tag{3.16}$$

For the rotational Killing vector χ, the Komar charge is the angular momentum with $c = 1$:

$$J = \frac{1}{16\pi} \oint_{\partial V} \star \mathrm{d}\tilde{\chi}. \tag{3.17}$$

Note that there is a relative factor of -2 between M and J. The minus sign is simply due to the Lorentzian signature of the metric. The idea behind the appearance of the factor 2 is most easily understood in the Newtonian limit. To this end, we write the Komar charge (3.14) as

$$\begin{aligned} Q_\xi &= \frac{c}{16\pi} \int_V \star \mathrm{d}^\dagger \mathrm{d}\tilde{\xi} \\ &= \frac{c}{8\pi} \int_V \xi^a \star \mathrm{Ric}_a, \end{aligned} \tag{3.18}$$

where we have used Eq. (C.28). Using Einstein's equation, this gives

$$Q_\xi = c \int_V \xi^a \left(\tau_a - \frac{1}{2} \mathcal{T} \star e_a \right), \tag{3.19}$$

where τ_a are the stress-energy 3-forms and $\mathcal{T} = i_a \star \tau^a$ is the trace. Let us now go to the Newtonian limit and use an asymptotically Minkowskian coordinate system $\{x^\mu\}$. Then, choosing V to be a hypersurface of $x^0 = \text{const}$ we get

$$Q_\xi = c \int_V \xi^a \tau_a - \frac{c}{2} \int_V \mathcal{T} \xi^0 \star e_0, \tag{3.20}$$

where in the second integral only the $\star e_0$ term contributes since $x^0 = \text{const}$. Hence we see that the second integral is non-zero only if ξ has a component in the time direction x^0. This is the reason behind the factor of 2. To look at it more carefully, we set $\xi = \partial_0$ and take a non-relativistic fluid with mass density ρ;

$$\tau_a \approx -\rho \delta^0{}_a \star e_0. \tag{3.21}$$

Then, $\mathcal{T} \approx -\rho$ and we have

$$Q_\xi = -c \left(\int_V \mathrm{d}^3 x\, \rho - \frac{1}{2} \int_V \mathrm{d}^3 x\, \rho \right), \tag{3.22}$$

with $\mathrm{d}^3 x = \star e_0$. Note that the first integral in the brackets is what we would normally identify as the total mass in V, and the second integral cancels half of the first one. We therefore set $c = -2$.

3.3 Laws of Black Hole Mechanics

As mentioned in the Introduction, black holes obey a set of laws which closely resemble the laws of thermodynamics and Hawking's discovery that black holes emit thermal radiation at a temperature $k_\mathrm{B} T = \hbar \kappa / 2\pi$ suggests that these laws are nothing but the laws of thermodynamics applied to black holes. In this section we shall state these laws and comment on the similarities with the laws of thermodynamics. For more details, see [Wald, 1984, Wald, 2001].

Zeroth law of black hole mechanics: The surface gravity κ of a stationary black hole is constant throughout its horizon. This is in analogy with the zeroth law of thermodynamics which states that the temperature of a thermodynamical system at equilibrium is constant throughout the system.

The constancy of κ is made obvious for static, spherically symmetric black holes by the remark at the end of Sec. 3.1, which states that $\kappa = f'(r_+)/2$ is independent of t, θ and ϕ.

First law of black hole mechanics: Take a spacelike section S of the spacetime outside the black hole, which intersects the horizon in a 2-surface ∂B and let A be the area of ∂B. Then the change $\mathrm{d}M$ in the mass of the black hole when it is perturbed is given by

$$\mathrm{d}M = \frac{\kappa}{8\pi}\mathrm{d}A + \Omega \mathrm{d}J + \Phi \mathrm{d}Q, \tag{3.23}$$

where Ω is the angular velocity, J is the angular momentum, Q is the electric charge of the black hole and Φ is the electrostatic potential at the horizon. Noting that M is the energy of the black hole and comparing Eq. (3.23) with the first law of thermodynamics, we see that $\Omega \mathrm{d}J$ and $\Phi \mathrm{d}Q$ represent the work done on the black hole, whereas κ and A are, up to some multiplicative constants, the temperature and the entropy, respectively. The constants are fixed by Hawking's discovery that the temperature is given by $k_{\mathrm{B}}T = \hbar\kappa/2\pi$, which implies that the entropy of the black hole is given by

$$S = \frac{k_{\mathrm{B}}}{\hbar}\frac{A}{4}. \tag{3.24}$$

For the Reissner-Nordström black hole one can explicitly calculate

$$M = \frac{r_+}{2} + \frac{Q^2}{2r_+}, \quad \kappa = \frac{1}{2r_+} - \frac{Q^2}{2r_+^3} \tag{3.25}$$

and $A = 4\pi r_+^2$, hence

$$\begin{aligned}\frac{\partial M}{\partial Q} &= \frac{Q}{r_+} \\ &= \Phi(r_+) \end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\frac{\partial M}{\partial A} &= \frac{1}{8\pi r_+}\frac{\partial M}{\partial r_+} \\ &= \frac{\kappa}{8\pi}, \end{aligned} \tag{3.27}$$

in accordance with the first law.

The relation between the area A and entropy is strengthened by the

Second law of black hole mechanics: The horizon area A of a black hole does not decrease with time. This is slightly stronger than the second law of thermodynamics: One can transfer entropy from one system to another, provided that the total entropy does not decrease. But one cannot transfer area from one black hole to another, since black holes cannot bifurcate. Hence the second law of black hole mechanics states that the area of each black hole does not decrease [Hawking, 1972, Bardeen et al., 1973].

Third law of black hole mechanics: As argued by Bardeen, Carter and Hawking, this law states that it is impossible by any procedure to reduce κ to zero by a finite sequence of operations. This is in analogy to Nernst's formulation of the third law of thermodynamics, which says that a system cannot be brought to absolute zero in a finite number of operations. A stronger version of this law due to Planck states that as $T \to 0$, the entropy of any system tends to a constant, which may be taken as zero. The analogue of this version is not valid for black holes, since there are black holes with vanishing κ and finite A (extremal black holes). A more precise formulation and a proof of the third law of black hole mechanics was given by Israel [Israel, 1986].

Chapter 4

CONTACT GEOMETRY AND THERMODYNAMICS

4.1 Contact Geometry

Let $\mathcal{M}$ be a smooth manifold and $T\mathcal{M}$ its tangent bundle. A field of hyperplanes on $\mathcal{M}$, i.e. a subbundle $\xi \subset T\mathcal{M}$ of codimension 1 can (at least locally) be written as the kernel of a 1-form: $\xi = \ker\alpha$. Obviously, the 1-form α is not unique and α' defines the same hyperplane field as α precisely if $\alpha' = f\alpha$ for a nowhere-vanishing function f.

If $\mathcal{M}$ is $(2n+1)$ dimensional, a maximally non-integrable hyperplane field ξ is called a *contact structure.* This means that the defining 1-form satisfies

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0 \tag{4.1}$$

everywhere, in which case α is called a *contact form.* Note that the condition (4.1) is independent of the chosen 1-form since

$$(f\alpha) \wedge (\mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha)^n = f^{n+1}\alpha \wedge (\mathrm{d}\alpha)^n \neq 0, \tag{4.2}$$

so that non-integrability is indeed a property of ξ. The pair $(\mathcal{M}, \xi)$ is then called a *contact manifold.*

Proposition 1. *Given a contact form α there exists a unique vector field R_α called the* Reeb vector field *which satisfies*

$$i_{R_\alpha}\mathrm{d}\alpha = 0, \quad i_{R_\alpha}\alpha = 1. \tag{4.3}$$

Proof. At each point $p \in \mathcal{M}$ the 2-form $\mathrm{d}\alpha$ defines a mapping

$$\psi:\ T_p\mathcal{M} \to T_p^*\mathcal{M},\ X \to i_X\mathrm{d}\alpha. \tag{4.4}$$

Expressed in arbitrary bases, the matrix of ψ is odd dimensional and skew-symmetric, hence it has a vanishing determinant. Thus we have shown that $\dim(\ker\psi) \geq 1$.

If $X, Y \in \ker\psi$, then

$$i_X i_Y \left[\alpha \wedge (\mathrm{d}\alpha)^n\right] = 0. \tag{4.5}$$

Since $\alpha \wedge (\mathrm{d}\alpha)^n$ is a volume form, this means that X and Y are linearly dependent, hence $\dim(\ker\psi) = 1$. So the kernel of $\mathrm{d}\alpha$ is a line bundle (which cannot be in the kernel of α), and the normalization condition $i_{R_\alpha}\alpha = 1$ uniquely fixes a section. $\square$

It can be shown (see [Geiges, 2009]) that if $(\mathcal{M}, \xi = \ker\alpha)$ is a contact manifold of dimension $(2n+1)$ and $L \subset \mathcal{M}$ is a submanifold such that $T_pL \subset \xi_p$ for all $p \in L$, then $\dim L \leq n$. Such a submanifold L of *maximal* dimension n is called a *Legendrian submanifold.* Note that the condition $T_pL \subset \xi_p$ can also be written as $\iota^*\alpha = 0$ where $\iota : L \hookrightarrow \mathcal{M}$ is the inclusion of L in $\mathcal{M}$.

4.2 Contact Hamiltonian Dynamics

A diffeomorphism $\phi : (\mathcal{M}, \xi = \ker\alpha) \to (\mathcal{N}, \tilde{\xi} = \ker\tilde{\alpha})$ between two contact manifolds is called a *contactomorphism* if it preserves the contact structure:

$$\phi_*\xi = \tilde{\xi}, \tag{4.6}$$

or equivalently

$$\phi^*\tilde{\alpha} = f\alpha, \quad f \neq 0. \tag{4.7}$$

If a vector field X generates an automorphism of $(\mathcal{M}, \xi = \ker\alpha)$ then we must have

$$\mathcal{L}_X\alpha = \mu\alpha \tag{4.8}$$

for some function μ. Such a vector field X is called a *contact vector field.* It is easy to see that (4.8) is independent of the chosen contact form α since we have

$$\mathcal{L}_X(f\alpha) = (X(f) + f\mu)\,\alpha. \tag{4.9}$$

Proposition 2. *For a fixed contact form α, there is a one-to-one correspondence between the contact vector fields X and the smooth functions $H : \mathcal{M} \to \mathbb{R}$ given by*

- $X \to H_X \equiv i_X\alpha$,
- $H \to X_H$ *defined as the unique solution of* $i_{X_H}\alpha = H$ *and* $i_{X_H}\mathrm{d}\alpha = R_\alpha(H)\alpha - \mathrm{d}H$.

Proof. The fact that a unique solution exists to the equations in the theorem follows from the fact that $\mathrm{d}\alpha$ is non-degenerate on ξ and that $R_\alpha \in \ker[R_\alpha(H)\alpha - \mathrm{d}H]$. This solution is a contact vector field since

$$\begin{aligned} \mathcal{L}_{X_H}\alpha &= i_{X_H}\mathrm{d}\alpha + \mathrm{d}\left(i_{X_H}\alpha\right) \\ &= R_\alpha(H)\alpha, \end{aligned} \tag{4.10}$$

hence X_H is well-defined.

To show that the correspondence is one-to-one, note that if X is a contact vector field with $\mathcal{L}_X\alpha = \mu\alpha$, then $\mu = R_\alpha(i_X\alpha)$. Now observe that

$$\begin{aligned} i_{X_{H_X}}\mathrm{d}\alpha &= R_\alpha(H_X)\alpha - \mathrm{d}H_X \\ &= \mu\alpha - \mathrm{d}\left(i_X\alpha\right) \\ &= i_X\mathrm{d}\alpha. \end{aligned} \tag{4.11}$$

Together with $i_{X_{H_X}}\alpha = H_X = i_X\alpha$ this implies that $X_{H_X} = X$.

Similarly, $H_{X_H} = i_{X_H}\alpha = H$. Therefore the correspondence is one-to-one. □

In this context the function H is called the generating function or the Hamiltonian and X_H the corresponding Hamiltonian vector field. Thus, in close analogy to symplectic geometry, a Hamiltonian function H defines a dynamics on $\mathcal{M}$ by the flow of X_H. However, in contrast to symplectic geometry, a constant added to H does alter the dynamics.

If X_f and X_g are contact vector fields with respective generating functions f and g, then so is $[X_f, X_g]$, as can easily be checked. Hence we can define, analogously to the Poisson bracket, the so-called *Lagrange bracket* as the (negative of the) generating function of $[X_f, X_g]$:

$$\{f, g\} \equiv -i_{[X_f, X_g]}\alpha. \tag{4.12}$$

Using the fact that $\mathcal{L}_{X_f}\alpha = R_\alpha(f)\alpha$ and the identity

$$i_{[X_f, X_g]} = \left[\mathcal{L}_{X_f}, i_{X_g}\right], \tag{4.13}$$

we can put the Lagrange bracket into the equivalent forms

$$\{f, g\} = -\mathcal{L}_{X_f} g + gR_\alpha(f), \tag{4.14a}$$

$$\{f, g\} = i_{X_f} i_{X_g} d\alpha - [fR_\alpha(g) - gR_\alpha(f)]. \tag{4.14b}$$

A *generalized Poisson algebra* is a commutative algebra A with identity together with a bilinear mapping $\{\cdot, \cdot\} : A \times A \to A$ which satisfies

1. $\{f, g\} = -\{g, f\}$ Antisymmetry,
2. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ Jacobi identity,
3. $\{f, gh\} = \{f, g\}h + g\{f, h\} + gh\{1, f\}$ Generalized Leibnitz rule.

It is easy to check that the Lagrange bracket makes the algebra of smooth functions on $\mathcal{M}$ into a generalized Poisson algebra. In contrast to the Poisson bracket, the Lagrange bracket of a constant with a function is not zero in general:

$$\{1, f\} = -R_\alpha(f). \tag{4.15}$$

From (4.14a) we see that the change of a function f along the dynamics of the Hamiltonian H is

$$\mathcal{L}_{X_H} f = \{f, H\} + fR_\alpha(H). \tag{4.16}$$

In particular, we have

$$\mathcal{L}_{X_H} H = H R_\alpha(H), \tag{4.17}$$

so that the Hamiltonian is not conserved in general. But if the initial value of H is zero, it remains zero.

Proposition 3 (Darboux's theorem)**.** *Let α be a contact form on the $(2n+1)$ dimensional manifold $\mathcal{M}$ and p a point on $\mathcal{M}$. Then there are coordinates u, $q^1, \ldots, q^n$, $p_1, \ldots, p_n$ on a neighbourhood $U \subset \mathcal{M}$ of p such that $p = (0, ..., 0)$ and*

$$\alpha = \mathrm{d}u + p_i \mathrm{d}q^i \tag{4.18}$$

on U.

Proof. See [Geiges, 2009]. □

In the Darboux coordinates we have $R_\alpha = \partial/\partial_u$ and the contact vector field X_H of a Hamiltonian H is given by

$$X_H = \left(H - p_i \frac{\partial H}{\partial p_i}\right) \frac{\partial}{\partial u} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} - p_i \frac{\partial H}{\partial u}\right) \frac{\partial}{\partial p_i}, \tag{4.19}$$

which can be seen from Prop. 2 by noting that X_H defined above is a contact vector field and $i_{X_H}\alpha = H$. The integral curves of X_H are determined then by the differential equations

$$\dot{u} = H - p_i \frac{\partial H}{\partial p_i}, \tag{4.20a}$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{4.20b}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial u}. \tag{4.20c}$$

Eq. (4.20b) and (4.20c) are just Hamilton's equations familiar from symplectic geometry, apart from the $\partial/\partial u$ term.

4.3 Contact Geometric Formulation of Thermodynamics

The phase space $\mathcal{M}$ of a thermodynamic system of n degrees of freedom is $(2n+1)$ dimensional. For concreteness, take a gas with a fixed number of particles. The variables U, S, V, T and p can be taken as the coordinates on $\mathcal{M}$. The first law defines a contact form

$$\alpha = -\mathrm{d}U + T\mathrm{d}S - p\mathrm{d}V, \tag{4.21}$$

whose kernel is the set of directions in which the energy is conserved. The equations of state define an n dimensional submanifold, all of whose tangent vectors are annihilated by α, i.e. a Legendrian submanifold:

$$U = U(S, V), \quad T = \left(\frac{\partial U}{\partial S}\right)_V, \quad p = -\left(\frac{\partial U}{\partial V}\right)_S. \tag{4.22}$$

Note that once the fundamental relation $U = U(S, V)$ is given, the remaining n equations of state follow automatically by the vanishing of α.

In general, the 1st law will have the form

$$\alpha = \mathrm{d}u + p_i \mathrm{d}q^i = 0, \tag{4.23}$$

for some thermodynamic potential u and conjugate variables q^i, p_i. These coordinates are by no means the only ones in which the contact form α has the above form. One can find coordinate systems (U, Q^i, P_i) in which α can be written as

$$\alpha = f\left(\mathrm{d}U + P_i \mathrm{d}Q^i\right). \tag{4.24}$$

For instance, the vanishing of (4.21) can be written equivalently as

$$-\mathrm{d}H + T\mathrm{d}S + V\mathrm{d}p = -\mathrm{d}F - p\mathrm{d}V - S\mathrm{d}T = -\mathrm{d}G + V\mathrm{d}p - S\mathrm{d}T = 0, \tag{4.25}$$

where H, F and G are respectively the enthalpy, the Helmholtz free energy and the Gibbs free energy. Coordinate transformations $(u, q^i, p_i) \to (U, Q^i, P_i)$ which preserve the contact form (up to multiplication by a non-zero function) are called Legendre transformations. Geometrically, they correspond precisely to the contactomorphisms of the thermodynamic contact manifold.

4.4 Hamilton-Jacobi Formalism for Thermodynamics

A family of substances can be described by allowing the fundamental relation $u = \Phi(q^1, \ldots, q^n)$ to depend on n parameters[1] $a_1, \ldots, a_n$ as well:

$$u = \Phi(q^1, \ldots, q^n; a_1, \ldots, a_n). \tag{4.26}$$

E.g. for the van der Waals family we have $U = U(S, V; a, b)$, where a and b are the van der Waals parameters.

Given the fundamental relation (4.26) and the other n equations of state

$$p_i = -\frac{\partial \Phi}{\partial q^i}, \tag{4.27}$$

we can eliminate the parameters $a_1, \ldots, a_n$ to get a single relation between the $(2n+1)$ coordinates:

$$H(u, q^1, \ldots, q^n, p_1, \ldots, p_n) = 0. \tag{4.28}$$

Given such a Hamiltonian function H for a family of substances, the equations of state may be recovered by solving the first order PDE

$$H\Big(\Phi, q^1, \ldots, q^n, -\frac{\partial \Phi}{\partial q^1}, \ldots, -\frac{\partial \Phi}{\partial q^n}\Big) = 0. \tag{4.29}$$

A complete integral of this PDE will depend on n parameters $a_1, \ldots, a_n$. The characteristic curves of this PDE are precisely the integral curves of X_H [Courant and Hilbert, 1989, Rajeev, 2008a].

[1] We *choose* the number of parameters to be n since this is going to allow us to define a Hamiltonian function, see [Rajeev, 2008a].

Chapter 5

EINSTEIN-BORN-INFELD-ADS BLACK HOLES

5.1 Born-Infeld Electrodynamics

Born-Infeld theory is a non-linear theory of electrodynamics, which is a modification of Maxwell's linear theory. It was developed with the aim of regularizing the electron's self-energy by introducing an upper bound on the electric field. In this section we shall review Born-Infeld electromagnetism and rederive the point charge solution in a flat background.

The electromagnetic field is described by the Faraday 2-form $F = \mathrm{d}A$, A being the potential 1-form. Given an observer with a four-velocity U, where $\langle U, U \rangle = -1$, one can define the electric field 1-form $\mathbf{e}$ and the magnetic flux density 2-form $\mathbf{b}$ through the equations

$$F = \tilde{U} \wedge \mathbf{e} + \mathbf{b}, \quad i_U \mathbf{e} = i_U \mathbf{b} = 0. \tag{5.1}$$

In the three dimensional notation $\vec{E} = (E_1, E_2, E_3)$, $\vec{B} = (B_1, B_2, B_3)$ and similarly for $\vec{A}$, these can be written as

$$\mathbf{e} = E_1 \mathrm{d}x + E_2 \mathrm{d}y + E_3 \mathrm{d}z \quad \text{and} \quad \mathbf{b} = B_1 \mathrm{d}y \wedge \mathrm{d}z + B_2 \mathrm{d}z \wedge \mathrm{d}x + B_3 \mathrm{d}x \wedge \mathrm{d}y, \tag{5.2}$$

whereas the potential can be written as

$$A = -\Phi \mathrm{d}t + A_1 \mathrm{d}x + A_2 \mathrm{d}y + A_3 \mathrm{d}z. \tag{5.3}$$

We may define the dual field (Maxwell) 2-form $G = -\partial L / \partial F$ where $L = \mathcal{L} \star 1$ is the electromagnetic Lagrangian density 4-form and the derivative with respect to F

is determined by $\delta_F L = \delta F \wedge (\partial L/\partial F)$. Given a total Lagrangian of the form

$$L_{\text{tot}} = L(F) + A \wedge J, \tag{5.4}$$

where J is a fixed current density 3-form, the field equations are simply

$$\mathrm{d}G = J. \tag{5.5}$$

We can similarly define the electric flux density 2-form $\mathbf{d}$ and the magnetic field 1-form $\mathbf{h}$ by

$$G = \mathbf{d} - \tilde{U} \wedge \mathbf{h}, \quad i_U \mathbf{d} = i_U \mathbf{h} = 0. \tag{5.6}$$

E.g., for the Maxwell Lagrangian $L_{\text{Maxwell}} = -(8\pi)^{-1} F \wedge \star F$ we have

$$G_{\text{Maxwell}} = \frac{1}{4\pi} \star F. \tag{5.7}$$

The conserved charge in a spacelike hypersurface V is given by

$$\begin{aligned} Q &= \int_V J = \int_V \mathrm{d}G \\ &= \oint_{\partial V} G. \end{aligned} \tag{5.8}$$

The Lagrangian density 0-form for Born-Infeld electrodynamics is given by [Born and Infeld, 1934]

$$\mathcal{L} = \frac{1}{4\pi\lambda^2} \left(1 - \sqrt{\Delta\,(X, Y)}\right), \tag{5.9}$$

where λ is to be seen as a new fundamental constant of nature and $\Delta(X, Y) = 1 - \lambda^2 X - \lambda^4 Y^2$ with the quadratic invariants of the electromagnetic field

$$X = \star\,(F \wedge \star F) = \vec{E}^2 - \vec{B}^2, \quad Y = \frac{1}{2} \star (F \wedge F) = \vec{E} \cdot \vec{B}. \tag{5.10}$$

In a Minkowskian coordinate system the Lagrangian (5.9) can be written as

$$\mathcal{L} = \frac{1}{4\pi\lambda^2} \left(1 - \sqrt{-\det(\eta_{\mu\nu} + \lambda F_{\mu\nu})}\right), \tag{5.11}$$

with $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$. Note that Maxwell electromagnetism is recovered in the limit $\lambda \to 0$ as $\mathcal{L} \to (8\pi)^{-1}X$.

For the Born-Infeld Lagrangian (5.9) we have the constitutive relation

$$G = \frac{1}{4\pi\sqrt{\Delta}}\left(\star F + \lambda^2 Y F\right). \tag{5.12}$$

Let us determine the point charge solution in the Born-Infeld theory for a flat background. To this end, we go to the rest frame of the point charge. There will be no magnetic field so that the field 2-form is given by $F = \tilde{U} \wedge \mathbf{e}$ or, in terms of the time coordinate t of this frame, $F = \mathbf{e} \wedge \mathrm{d}t$. The electric field must be spherically symmetric, which means that it must be of the form

$$\mathbf{e} = E(r)\mathrm{d}r. \tag{5.13}$$

Thus we get $\star F = Er^2 \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$, $X = E^2$ and $Y = 0$. Putting these into (5.12), the dual field reads

$$G = \frac{1}{4\pi}\frac{E}{\sqrt{1-\lambda^2 E^2}} r^2 \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{5.14}$$

Then the Born-Infeld field equation (5.5) with $J = 0$ becomes, after some algebra:

$$\frac{\mathrm{d}E}{1-\lambda^2 E^2} + \frac{2E}{r}\mathrm{d}r = 0. \tag{5.15}$$

To solve this, we introduce $z = E^{-2}$, in terms of which the field equation is

$$\frac{\mathrm{d}z}{z-\lambda^2} = 4\frac{\mathrm{d}r}{r}. \tag{5.16}$$

The solution is $z = cr^4 + \lambda^2$, or

$$E(r) = \frac{Q}{\sqrt{r^4 + \lambda^2 Q^2}}, \tag{5.17}$$

where $c > 0$ and Q are constants. As the notation suggests, Q is simply the total charge, as given in Eq. (5.8). The electric field can be derived from the electrostatic potential

$$\Phi(r) = \int_r^\infty \mathrm{d}x \frac{Q}{\sqrt{x^4 + \lambda^2 Q^2}}. \tag{5.18}$$

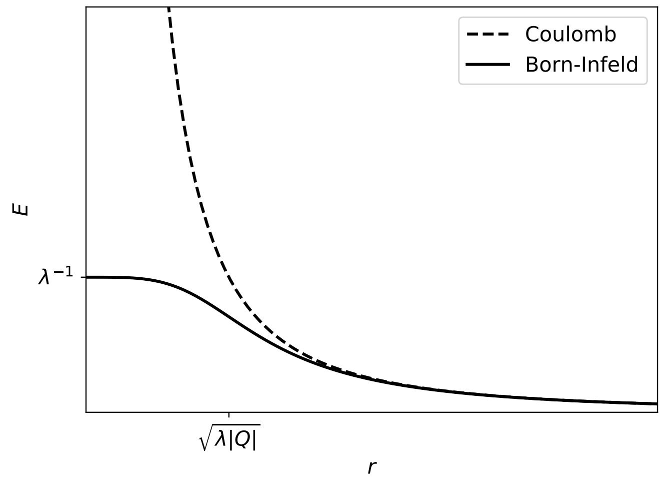

Figure 5.1: Coulomb and Born-Infeld fields around a positive point charge. We see that the Born-Infeld field does not diverge at the origin like the Coulomb field; it assumes its maximal value λ^{-1}.

We can see from (5.17) that in the Born-Infeld theory, the electric field of a point charge is regular at the place of the charge, unlike the Coulomb field (see Fig. 5.1). This is also true for the self-energy of the charge in its rest frame. To see this, note that the energy density 3-form is given by τ^0, where τ_a are the stress-energy 3-forms (see next section for an explicit calculation of τ_a). Integrating this on a constant time hypersurface V yields the energy W of the point charge

$$W = \int_V \tau^0 \approx 1.24 \frac{|Q|^{3/2}}{\sqrt{\lambda}}, \tag{5.19}$$

which is finite for $\lambda > 0$.

5.2 Black Hole Solution

The Einstein-Born-Infeld field equations with a cosmological constant Λ can be derived from the action

$$S[e, A] = \frac{1}{16\pi} \int (\mathcal{R} - 2\Lambda) \star 1 + \int \mathcal{L} \star 1, \tag{5.20}$$

where $\mathcal{R}$ is the scalar curvature and e^a are the orthonormal coframe 1-forms. The variational field equations for the case $\Lambda = 0$ are derived in Ref. [Dereli and Tucker, 2010] using the invariant tensor notation. Below we note the analogous derivation for $\Lambda \neq 0$.

To vary the action with respect to e^a we make use of the identity

$$(\delta \star - \star \delta)\, \phi = \delta e^a \wedge i_a \star \phi - \star \left(\delta e^a \wedge i_a \phi\right) \tag{5.21}$$

for an arbitrary differential form ϕ [Muench et al., 1998]. Then the variation with respect to e^a yields the equations

$$-\frac{1}{2} R^{bc} \wedge \star e_{abc} + \Lambda \star e_a = 8\pi \tau_a, \tag{5.22}$$

with the stress-energy 3-forms

$$\begin{aligned} \tau_a &= \frac{\partial\left(\mathcal{L} \star 1\right)}{\partial e^a} \\ &= M \star e_a + N \tau_a^{(\mathrm{LED})}, \end{aligned} \tag{5.23}$$

where $M = \mathcal{L} - X\partial_X \mathcal{L} - Y\partial_Y \mathcal{L}$, $N = 8\pi\partial_X \mathcal{L}$ and $\tau_a^{(\mathrm{LED})}$ are the stress-energy 3-forms of linear (i.e. Maxwell) electrodynamics:

$$\tau_a^{(\mathrm{LED})} = \frac{1}{8\pi}\left(i_a F \wedge \star F - F \wedge i_a \star F\right). \tag{5.24}$$

The variation with respect to A yields the equation $\mathrm{d}G = 0$, where G is given as in (5.12), which is to be solved together with (5.22).

The static, spherically symmetric solution without the cosmological constant was first found by Hoffmann [Hoffmann, 1935] and then rediscovered by Demianski [Demianski, 1986]. The solution in the presence of a cosmological constant was noted in Ref. [Fernando and Krug, 2003]. Note that the action used in Ref. [Fernando and Krug, 2003] differs from ours by the absence of the Y^2 term. However, we shall see below that in the static case one has $Y = 0$, hence our results agree.

With the static, spherically symmetric ansatz, Eq. (2.24), for the metric, it is straightforward to see that

$$F = \frac{Q}{\sqrt{r^4 + a^4}} \mathrm{d}r \wedge \mathrm{d}t \tag{5.25}$$

solves the electromagnetic field equation $\mathrm{d}G = 0$ where Q is the total electric charge and $a = (\lambda |Q|)^{1/2}$.

To find $f(r)$, we first calculate the stress-energy 3-forms τ_a. Using

$$e^0 = \sqrt{f}\mathrm{d}t, \quad e^1 = \frac{\mathrm{d}r}{\sqrt{f}}, \quad e^2 = r\mathrm{d}\theta, \quad e^3 = r\sin\theta\mathrm{d}\phi \tag{5.26}$$

and $F = \sqrt{X}e^{10}$, we have

$$\begin{aligned} \tau_a^{(\mathrm{LED})} &= -\frac{1}{8\pi} X \left(\delta_a^0 \star e_0 + \delta_a^1 \star e_1 - \delta_a^2 \star e_2 - \delta_a^3 \star e_3\right) \\ &= \begin{cases} -\frac{1}{8\pi} X \star e_a, & a = 0, 1 \\ +\frac{1}{8\pi} X \star e_a, & a = 2, 3. \end{cases} \end{aligned} \tag{5.27}$$

So:

$$\tau_a = \begin{cases} \frac{1}{4\pi\lambda^2}\left(1 - \frac{1}{\sqrt{1-\lambda^2 X}}\right) \star e_a, & a = 0, 1 \\ \frac{1}{4\pi\lambda^2}\left(1 - \sqrt{1-\lambda^2 X}\right) \star e_a, & a = 2, 3. \end{cases} \tag{5.28}$$

The Einstein 3-forms for the metric (2.24) have been calculated in Eq. (2.32). Inserting Eqs. (2.32) and (5.28) into (5.22), the only independent equation we get is

$$\frac{1}{r^2}\left[(rf)' - 1\right] + \Lambda = \frac{2}{\lambda^2}\left(1 - \frac{\sqrt{r^4 + a^4}}{r^2}\right), \tag{5.29}$$

and the solution is given by

$$f(r) = -\frac{\Lambda}{3} r^2 + 1 - \frac{2M}{r} + \frac{2Q^2}{ar} h\Big(\frac{r}{a}\Big), \quad h(x) \equiv \int_x^\infty \mathrm{d}y \left(\sqrt{y^4 + 1} - y^2\right), \tag{5.30}$$

where M is a constant of integration which is equal to the total mass as given in Eq. (D.10).

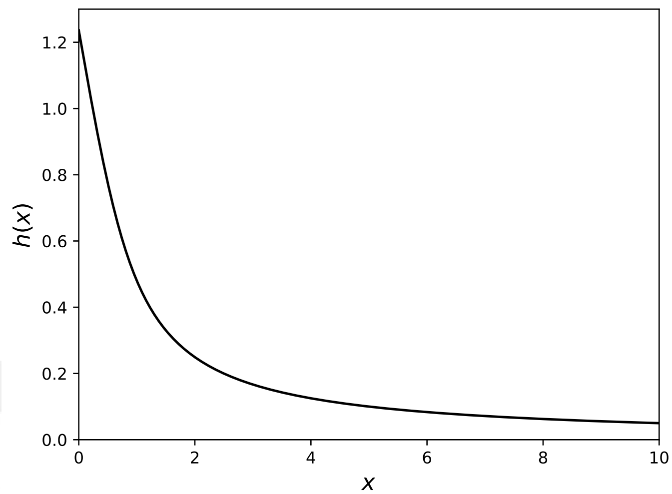

Figure 5.2: Plot of the function $h(x)$ defined in Eq. (5.30). It satisfies $h(0) \approx 1.24$ and $\lim_{x\to\infty} h(x) = 0$.

For simplicity we shall work with a negative cosmological constant $\Lambda = -3l^{-2}$ so that there is no cosmological horizon. In that case we can expand $f(r)$ for $r \gg a$ as

$$f(r) = \frac{r^2}{l^2} + 1 - \frac{2M}{r} + \frac{Q^2}{r^2}\left[1 - \frac{a^4}{20r^4} + \mathcal{O}\left(\frac{a^8}{r^8}\right)\right]. \tag{5.31}$$

In particular the spacetime is asymptotically AdS and in the limit $\lambda \to 0$ we recover the Reissner-Nordström-AdS black hole as expected. We shall call this solution *the Einstein-Born-Infeld-AdS black hole*.

To investigate the horizon structure we first note that

$$\lim_{r\to\infty} rf(r) = \infty \quad \text{and} \quad rf(r)|_{r=0} = -2M + \frac{2Q^2}{a}h(0) \tag{5.32}$$

($h(0) \approx 1.24$ is finite, see Fig. 5.2.). We shall limit our attention to the case

$$2M < \frac{2Q^2}{a}h(0) \quad \text{and} \quad \lambda < 2Q, \tag{5.33}$$

which includes the Reissner-Nordström-AdS black hole. With the assumptions in (5.33), the function $rf(r)$ has exactly one minimum whose position is independent of

M:

$$r_0^2 = \frac{2 - \lambda^2\Lambda}{4\Lambda - \lambda^2\Lambda^2} + \sqrt{\left(\frac{2 - \lambda^2\Lambda}{4\Lambda - \lambda^2\Lambda^2}\right)^2 - \frac{4Q^2 - \lambda^2}{4\Lambda - \lambda^2\Lambda^2}}. \tag{5.34}$$

Since $rf(r)$ is positive at $r = 0$ and $r = \infty$, it will have no zeros, a double zero or two zeros if $r_0 f(r_0)$ is positive, zero or negative, respectively. Hence from (5.30) we see that there is a critical mass

$$2M_{\rm c} = \frac{r_0^3}{l^2} + r_0 + \frac{2Q^2}{a} h\left(\frac{r_0}{a}\right), \tag{5.35}$$

such that (see Fig. 5.3) if

1. $M < M_{\rm c}$, there is no horizon. In that case there is a naked singularity at $r = 0$ where the Kretschmann scalar $\mathcal{K} = 2 \star ({R^a}_b \wedge \star {R^b}_a)$ diverges.

2. $M = M_{\rm c}$, we have an extremal black hole with one horizon.

3. $M > M_{\rm c}$, there are two horizons.

Thus the situation is completely analogous to the Reissner-Nordström case.

If l is small, it may not be possible to satisfy the condition $M > M_{\rm c}$ simultaneously with (5.33), but it is possible if l is sufficiently large. In the subsequent discussion we shall assume that $M > M_{\rm c}$, which should cover the physically relevant cases. We denote the outer horizon radius by r_+.

5.3 Thermodynamics

The thermodynamical equations of state were given in Ref. [Rasheed, 1997] for the case with $\Lambda = 0$ and then in Ref. [Dey, 2004] for a non-zero Λ. In the following we rederive these.

Using the fact that $f(r_+) = 0$ we can write the mass of the black hole as

$$2M = \frac{r_+^3}{l^2} + r_+ + \frac{2Q^2}{a} h\left(\frac{r_+}{a}\right). \tag{5.36}$$

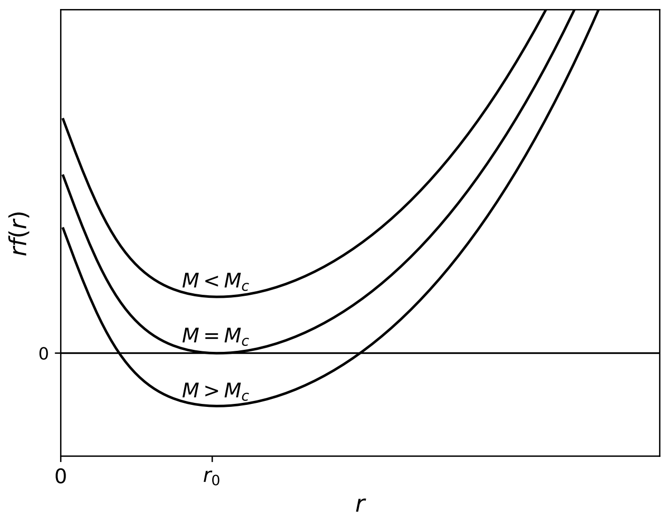

Figure 5.3: Typical plots of $rf(r)$ for the three cases $M < M_{\rm c}$, $M = M_{\rm c}$ and $M > M_{\rm c}$. The number of horizons is different in each case, analogous to the Reissner-Nordström black hole.

The surface gravity $\kappa = (1/2)\,\mathrm{d}f/\mathrm{d}r|_{r=r_+}$ is given by

$$\kappa = \frac{r_+}{l^2} + \frac{M}{r_+^2} - \frac{Q^2}{ar_+^2}h\Big(\frac{r_+}{a}\Big) - \frac{Q^2}{a^4 r_+}\left(\sqrt{r_+^4 + a^4} - r_+^2\right), \tag{5.37}$$

and the electrostatic potential on the horizon is

$$\Phi = \int_{r_+}^{\infty} \mathrm{d}x \frac{Q}{\sqrt{x^4 + a^4}}. \tag{5.38}$$

Introducing the surface area $A = 4\pi r_+^2$, it is straightforward to verify the first law

$$\mathrm{d}M = \frac{\kappa}{8\pi}\mathrm{d}A + \Phi \mathrm{d}Q. \tag{5.39}$$

Here we have used the identity

$$h(x) = \frac{2}{3}\int_x^{\infty} \frac{\mathrm{d}y}{\sqrt{y^4+1}} - \frac{x}{3}\left(\sqrt{x^4+1} - x^2\right), \tag{5.40}$$

which can be seen by noting that both sides of the equation converge to zero as $x \to \infty$ and that they have the same derivative. The first law (5.39) defines our contact structure.

We can use Eq. (5.36) in (5.37) to eliminate l:

$$\kappa r_+^2 = 3M - r_+ - \frac{3Q^2}{a} h\left(\frac{r_+}{a}\right) - \frac{Q^2}{a^4} r_+ \left(\sqrt{r_+^4 + a^4} - r_+^2\right). \tag{5.41}$$

Next we use Eq. (5.38) in (5.41) to eliminate a with the help of the identity (5.40):

$$\frac{\kappa A}{4\pi} = 3M - \sqrt{\frac{A}{4\pi}} - 2Q\Phi. \tag{5.42}$$

Comparing this with the first law, Eq. (5.39), we see that the Einstein-Born-Infeld-AdS family is described by the hypersurface $F(M, A, Q, p_A, p_Q) = 0$ with the Hamiltonian

$$F(M, A, Q, p_A, p_Q) = 3M - 2p_A A - 2p_Q Q - \sqrt{\frac{A}{4\pi}}. \tag{5.43}$$

The HJE we get from this Hamiltonian is

$$3M - 2A\frac{\partial M}{\partial A} - 2Q\frac{\partial M}{\partial Q} = \sqrt{\frac{A}{4\pi}}. \tag{5.44}$$

This is a first order linear PDE, and the most general solution can be found easily as follows. Let us first look at the homogenous equation

$$3\tilde{M} - 2A\frac{\partial \tilde{M}}{\partial A} - 2Q\frac{\partial \tilde{M}}{\partial Q} = 0. \tag{5.45}$$

Consider the coordinate transformation

$$x = A, \tag{5.46a}$$

$$y = Q/A. \tag{5.46b}$$

Eq. (5.45) then becomes

$$3\tilde{M} - 2x\frac{\partial \tilde{M}}{\partial x} = 0. \tag{5.47}$$

For a fixed y, this is a separable ODE and integrating along a line of constant y gives

$$\tilde{M} = c\left(\frac{Q}{A}\right) A^{3/2}, \tag{5.48}$$

where the constant of integration c will depend on $y = Q/A$ in general. It can be checked that a particular solution of the inhomogenous equation (5.44) is given by

$$M_{\mathrm{p}} = \frac{1}{2}\sqrt{\frac{A}{4\pi}}. \tag{5.49}$$

Hence the most general solution of (5.44) is

$$M = \frac{1}{2}\sqrt{\frac{A}{4\pi}} + \frac{1}{2}\left(\frac{A}{4\pi}\right)^{3/2} u\Big(\frac{Q}{A/4\pi}\Big), \tag{5.50}$$

where u is an arbitrary function of a single variable that must be fixed by a boundary condition. The complete integral corresponding to the actual equation of state (5.36) is given by the choice

$$u(x) = \frac{1}{l^2} + 2\sqrt{\frac{x^3}{\lambda}}h\bigg(\frac{1}{\sqrt{\lambda x}}\bigg). \tag{5.51}$$

It should be noted, however, that (5.36) is not the only complete integral of the PDE (5.44) as, e.g., the choice $u(x) = l^{-2} + 2x^{3/2}\lambda^{-1/2}$ also yields a complete integral. It is not exactly clear what this non-uniqueness means, but at the very least it shows that one must be careful to use the HJE to get the equations of state of a family of substances.

Chapter 6

CONCLUSION

In this thesis, we have reviewed the basics of the theory of black holes by first introducing the concept of conformal infinity and then providing the general definition of black holes in light of the example of the Schwarzschild black hole. We have then examined the charged, non-rotating (i.e. Reissner-Nordström) black holes in more detail. Afterwards, we introduced the concepts related to the thermodynamics of black holes, such as the surface gravity, horizon area and the total mass and angular momentum in a spacetime. With these at hand, we have discussed the thermodynamics of black holes.

Following this, we examined contact geometry and how one can construct Hamiltonian dynamics and similar structures in contact geometry, analogous to the ones in symplectic geometry. We have discussed how thermodynamics can be desribed naturally in the language of contact geometry, and introduced the Hamilton-Jacobi formalism for thermodynamics developed by Rajeev.

To study the thermodynamic HJE for charged black holes we have made the Reissner-Nordström black hole into a two-parameter family by introducing a (negative) cosmological constant and replacing the Maxwell Lagrangian by the Born-Infeld one. Under the assumption that the Born-Infeld parameter λ is sufficiently small, the horizon structure of the resulting static black hole is quite similar to that of the Reissner-Nordström one. By this we mean that there is a critical mass below which there is no horizon (Fig. 5.3).

An element of the Einstein-Born-Infeld-AdS family has two thermodynamical degrees of freedom. Hence its phase space is five dimensional, which can be coordinatized by M, κ, A, Φ and Q, and the two parameters Λ and λ specify the particular element. Using the three equations of state (5.36)-(5.38), we were able to eliminate the parameters Λ and λ to get a single relation between the phase space variables. This relation defines a hypersurface by the vanishing of a Hamiltonian function and therefore yields a HJE.

The HJE (5.44) we get for the Einstein-Born-Infeld-AdS family is linear and we can find an analytic expression for the most general solution. It is interesting to note that the solution is far more general than the equation of state of the Einstein-Born-Infeld-AdS black hole. A function must be specified by a boundary condition to get the actual equation of state, not just two constants of integration. As we mentioned, the precise meaning of the existence of these solutions that do not correspond to the actual equation of state is not clear. This may get clearer if it is understood whether and how the HJE is related to the quantization of black holes. In any case it provides a caveat against the use of the Hamilton-Jacobi formalism to determine the equations of state of a family of substances.

It should also be remarked that the above is not the only way of extending the Reissner-Nordström black hole into a two-parameter family; one may think of parameters other than the cosmological constant and the Born-Infeld parameter. In fact, we can introduce n parameters to a system of n degrees of freedom in a completely arbitrary manner. It is possible that there is another choice of extension which is free of this ambiguity. Moreover, the question of finding a HJE for rotating black holes is still open.

Appendix A

AFFINE CONNECTIONS

Vector fields on a manifold $\mathcal{M}$ provides the notion of a directional derivative of functions. However, on a general manifold there is no natural definition of the directional derivative of an arbitrary tensor along a vector. This requires an additional structure on the manifold, called a connection. In the following, we shall introduce affine connections and the related curvature forms. For more details, see e.g. Ref. [Nakahara, 2003].

An affine connection ∇ is a map that takes two vector fields and produces a vector field

$$\nabla : \ (X, Y) \to \nabla_X Y, \tag{A.1}$$

such that for all vector fields X, Y, Z and functions f and g:

1. $\nabla_X Y$ is linear in X:

$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z, \tag{A.2}$$

2. $\nabla_X Y$ is additive in Y:

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z, \tag{A.3}$$

3. ∇_X obeys the Leibnitz rule:

$$\nabla_X (fY) = X(f)Y + f\nabla_X Y. \tag{A.4}$$

The vector $\nabla_X Y$ is called the *covariant derivative* of Y in the direction of X. The notion of covariant derivative can be extended to tensors of arbitrary type as follows. Applied on a function f, we require that the covariant derivative $\nabla_X f$ coincides with the usual directional derivative:

$$\nabla_X f = X(f). \tag{A.5}$$

The covariant derivative of a 1-form is defined in a way that is compatible with the Leibnitz rule and tensor contraction. Namely, for all vector fields X, Y and 1-forms ω,

$$\nabla_X \omega(Y) = \nabla_X(i_Y \omega) - \omega(\nabla_X Y). \tag{A.6}$$

The covariant derivative can then be defined for a tensor of arbitrary type by demanding that for all tensors A and B:

$$\nabla_X(A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B, \tag{A.7}$$

and for all tensors A and B of the same type:

$$\nabla_X(A + B) = \nabla_X A + \nabla_X B. \tag{A.8}$$

One can also define the total covariant derivative of a tensor T of type (m, n) as the $(m, n+1)$ tensor

$$\nabla T(\omega_1, \ldots, \omega_m, X_1, \ldots, X_n, X) \equiv \nabla_X T(\omega_1, \ldots, \omega_m, X_1, \ldots, X_n). \tag{A.9}$$

In a coordinate chart, the components of ∇T are written equivalently as

$$(\nabla T)^{\alpha_1 \cdots \alpha_m}{}_{\beta_1 \cdots \beta_n \mu} = T^{\alpha_1 \cdots \alpha_m}{}_{\beta_1 \cdots \beta_n ; \mu}, \tag{A.10}$$

or

$$(\nabla T)^{\alpha_1 \cdots \alpha_m}{}_{\beta_1 \cdots \beta_n \mu} = \nabla_\mu T^{\alpha_1 \cdots \alpha_m}{}_{\beta_1 \cdots \beta_n}. \tag{A.11}$$

One can define the torsion tensor T and the Riemann curvature tensor R of a connection ∇ by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{A.12}$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{A.13}$$

Although T and R may seem to be differential operators, it is not difficult to show that they are multilinear objects. Hence T and R are indeed tensors of type $(1,2)$ and $(1,3)$, respectively. If the torsion tensor of a connection vanishes, it is called a *symmetric* or *torsion-free* connection.

If the manifold in question is endowed with a metric g, a connection with

$$\nabla_X g = 0 \tag{A.14}$$

for all vectors X is called a *metric compatible* or simply a *metric* connection.

Proposition 4. *On a (pseudo-)Riemannian manifold* $(\mathcal{M}, g)$*, there exists a unique symmetric connection which is compatible with the metric* g*. This connection is called the* Levi-Civita connection.

Proof. See [Nakahara, 2003]. □

Let $\{x_a\}$ be a frame and let $\{e^a\}$ be the dual coframe. The connection forms of this frame are the 1-forms $\omega^a{}_b$ such that for all vectors X,

$$\omega^a{}_b(X) = e^a(\nabla_X x_b). \tag{A.15}$$

The curvature 2-forms $R^a{}_b$ are given by

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{A.16}$$

whereas the Ricci 1-forms Ric_a and the Ricci scalar $\mathcal{R}$ are given by

$$\mathrm{Ric}_a = i_b R^b{}_a \tag{A.17}$$

and

$$\mathcal{R} = i_a \mathrm{Ric}^a. \tag{A.18}$$

For a vector $X = X^a x_a$ let

$$R(X)^a \equiv X^b R^a{}_b. \tag{A.19}$$

Then the curvature 2-forms and the Riemann curvature tensor are related by

$$i_Z i_Y R(X)^a = e^a[R(X,Y)Z]. \tag{A.20}$$

Appendix B

HODGE DUAL

On an n dimensional manifold $\mathcal{M}$, the space of p-forms $\Omega^p(\mathcal{M})$ is isomorphic to the space of $(n-p)$-forms $\Omega^{n-p}(\mathcal{M})$, both being $n!/p!(n-p)!$ dimensional. In the presence of a metric g and with a chosen orientation, one can define a natural isomorphism between them called the *Hodge star* $\star$. Let $\varepsilon_{a_1\cdots a_n}$ be the Levi-Civita symbol with $\varepsilon_{01\cdots(n-1)} = 1$. Then $\star : \Omega^p(\mathcal{M}) \to \Omega^{n-p}(\mathcal{M})$ is the linear mapping whose action on a basis p-form is given by

$$\star\,(e^{a_1} \wedge \cdots \wedge e^{a_p}) = \frac{1}{(n-p)!}\varepsilon^{a_1\cdots a_p}{}_{b_{p+1}\cdots b_n} e^{b_{p+1}} \wedge \cdots \wedge e^{b_n}. \tag{B.1}$$

One can show that on a Lorentzian manifold

$$\star\star\,\omega = -(-1)^{p(n-p)}\omega, \tag{B.2}$$

where ω is a p-form.

The inner product $\langle\omega,\eta\rangle$ of two p-forms ω and η is defined by the equation

$$\omega \wedge \star\eta = \langle\omega,\eta\rangle \star 1. \tag{B.3}$$

It can be shown that this inner product is symmetric; $\langle\omega,\eta\rangle = \langle\eta,\omega\rangle$. One can similarly define the L^2 inner product of p-forms by

$$(\omega,\eta) = \int_{\mathcal{M}} \omega \wedge \star\eta. \tag{B.4}$$

The adjoint exterior derivative operator $\mathrm{d}^\dagger : \Omega^p(\mathcal{M}) \to \Omega^{p-1}(\mathcal{M})$ is defined, on a Lorentzian spacetime, as

$$\mathrm{d}^\dagger = (-1)^{n+np} \star \mathrm{d} \star. \tag{B.5}$$

The name “adjoint” is due to the fact that if $\mathcal{M}$ is a manifold without a boundary, then

$$(\mathrm{d}\omega, \eta) = (\omega, \mathrm{d}^\dagger \eta). \tag{B.6}$$

In a local coordinate system, the action of $\mathrm{d}^\dagger$ on a p-form ω can be written as

$$\left(\mathrm{d}^\dagger \omega\right)_{\mu_1 \cdots \mu_{p-1}} = -\nabla_\nu \omega^\nu{}_{\mu_1 \cdots \mu_{p-1}}. \tag{B.7}$$

Appendix C

KILLING VECTORS AND SYMMETRIES

Let $(\mathcal{M}, g)$ be a (pseudo-)Riemannian manifold. A vector field ξ is called a Killing vector field if

$$\mathcal{L}_\xi g = 0. \tag{C.1}$$

This means that ξ is the generator of an isometry. Hence if $\mathcal{M}$ is a spacetime, the existence of a Killing vector implies the existence of a physical symmetry.

Note that $\forall \xi, X, Y$ we have

$$\begin{aligned}\mathcal{L}_\xi g(X,Y) &= \mathcal{L}_\xi \left[g(X,Y)\right] - g(\mathcal{L}_\xi X, Y) - g(X, \mathcal{L}_\xi Y) \\ &= \nabla_\xi \langle X, Y\rangle - \langle \nabla_\xi X, Y\rangle + \langle \nabla_X \xi, Y\rangle - \langle X, \nabla_\xi Y\rangle + \langle X, \nabla_Y \xi\rangle \\ &= \langle \nabla_X \xi, Y\rangle + \langle X, \nabla_Y \xi\rangle ,\end{aligned} \tag{C.2}$$

where ∇ is the Levi-Civita connection and we have used the fact that

$$\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X. \tag{C.3}$$

Therefore, an equivalent definition of a Killing vector ξ is that for all X, Y the *Killing equation*

$$\langle \nabla_X \xi, Y\rangle + \langle \nabla_Y \xi, X\rangle = 0 \tag{C.4}$$

is satisfied. Killing equation can be written in a coordinate chart as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{C.5}$$

with $\xi_\mu = g_{\mu\nu}\xi^\nu$. Also note that a linear combination of two Killing vectors with constant coefficients is again a Killing vector.

Proposition 5. *Let $g_{\mu\nu}$ be the metric coefficients in a coordinate system $\{x^\mu\}$, i.e., $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$. If all the metric coefficients are independent of one of the coordinates x^α, i.e. $\forall \mu, \nu$:*

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha} = 0, \tag{C.6}$$

then ∂_α is a Killing vector.

Proof. Since $\{\partial_\mu\}$ is a coordinate basis, we have $\forall \mu, \nu$:

$$[\partial_\mu, \partial_\nu] = 0. \tag{C.7}$$

Hence $\forall \mu, \nu$:

$$\begin{aligned} \mathcal{L}_{\partial_\alpha} g\big(\partial_\mu, \partial_\nu\big) &= \mathcal{L}_{\partial_\alpha} g_{\mu\nu} - g\big(\underbrace{\mathcal{L}_{\partial_\alpha}\partial_\mu}_{=0}, \partial_\nu\big) - g\big(\partial_\mu, \underbrace{\mathcal{L}_{\partial_\alpha}\partial_\nu}_{=0}\big) \\ &= \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \\ &= 0. \end{aligned} \tag{C.8}$$

□

An asymptotically flat spacetime is said to be *stationary* if it possesses a Killing vector K which is timelike near infinity. In such a spacetime, we can find a coordinate system (t, x^1, x^2, x^3) adapted to K, such that $\partial_t = K$. Then the metric takes the form

$$g = g_{00}(\vec{x})\mathrm{d}t \otimes \mathrm{d}t + g_{0i}(\vec{x})\left(\mathrm{d}t \otimes \mathrm{d}x^i + \mathrm{d}x^i \otimes \mathrm{d}t\right) + g_{ij}(\vec{x})\mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{C.9}$$

with $\vec{x} = (x^1, x^2, x^3)$ and $g_{00} < 0$ near infinity. Conversely if there exists a coordinate system in which all the metric coefficients are independent of the (asymptotically) timelike coordinate, then the spacetime is stationary. Intuitively, a stationary spacetime is one that is symmetric under time translations $t \to t + t_0$.

A stationary spacetime is said to be *static* if, additionally, the Killing vector K is orthogonal to a family of hypersurfaces. By Frobenius's theorem this implies that

$$\tilde{K} \wedge \mathrm{d}\tilde{K} = 0, \tag{C.10}$$

which can be written in the adapted coordinate system as

$$(g_{00}\partial_i g_{0j} + g_{0i}\partial_j g_{00})\,\mathrm{d}t \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j + g_{0i}\partial_j g_{0k}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k = 0. \tag{C.11}$$

In particular, from the $\mathrm{d}t \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j$ terms we get

$$g_{00}\partial_i g_{0j} - g_{0j}\partial_i g_{00} = g_{00}\partial_j g_{0i} - g_{0i}\partial_j g_{00}, \tag{C.12}$$

or, after dividing by $(g_{00})^2$:

$$\partial_i\left(\frac{g_{0j}}{g_{00}}\right) = \partial_j\left(\frac{g_{0i}}{g_{00}}\right). \tag{C.13}$$

This means that there exists a function $h(\vec{x})$ such that

$$\frac{g_{0i}}{g_{00}} = \partial_i h. \tag{C.14}$$

Consider now the coordinate transformation

$$t' = t + h(\vec{x}), \quad x'^i = x^i, \tag{C.15}$$

which is also adapted to K in the sense that $\partial_{t'} = K$. It is straightforward to show that in the new system the metric (C.9) takes the form

$$g = g_{00}(\vec{x})\mathrm{d}t' \otimes \mathrm{d}t' + \left(g_{ij}(\vec{x}) - \frac{g_{0i}(\vec{x})g_{0j}(\vec{x})}{g_{00}(\vec{x})}\right)\mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{C.16}$$

Hence in a static spacetime one can find a coordinate system adapted to K, in which the time and space components of the metric separate and all the metric coefficients are independent of the time. Conversely, if the metric can be put into the form (C.16) where all the metric coefficients are independent of the timelike coordinate and there are no time-space cross terms, then the spacetime is static, with the Killing vector $\partial_{t'}$ being orthogonal to the surfaces of constant t'. Intuitively, a static spacetime is one which has time reversal symmetry $t \to -t$ in addition to time translation symmetry.

A spacetime is called spherically symmetric if it possesses the symmetries of a 2-sphere. More precisely, it is a spacetime that has three independent spacelike Killing vectors χ_1, χ_2, χ_3 whose orbits are closed and which satisfy the Lie algebra of $SO(3)$:

$$[\chi_1, \chi_2] = \chi_3, \quad [\chi_2, \chi_3] = \chi_1, \quad [\chi_3, \chi_1] = \chi_2. \tag{C.17}$$

If there is only one spacelike Killing vector with closed orbits, the spacetime is said to be axially symmetric.

Example. *The Schwarzschild metric is given in Schwarzschild coordinates by*

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - \frac{2M}{r}} + r^2 \mathrm{d}\Omega^2. \tag{C.18}$$

Hence the two Killing vectors ∂_t and ∂_ϕ are manifest. Physically, ∂_t is associated with the time translation invariance and ∂_ϕ with the axial symmetry of the Schwarzschild metric. In fact, it has two more Killing vectors which, together with ∂_ϕ, form the Lie algebra of SO(3)*. So the Schwarzschild spacetime does not only have an axial symmetry, but a spherical symmetry.*

It should be remarked that Killing vectors are quite important for particle mechanics in general relativity. To see how, note that if $V = \mathrm{d}/\mathrm{d}\lambda$ is tangent to a geodesic ($\nabla_V V = 0$) parametrized by λ and ξ is a Killing vector, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} \langle \xi, V \rangle &= \nabla_V \langle \xi, V \rangle \\ &= \langle \nabla_V \xi, V \rangle + \langle \xi, \nabla_V V \rangle \\ &= 0. \end{aligned} \tag{C.19}$$

Here in the second line, the first term vanishes due to the Killing equation. Hence the quantity $\langle \xi, V \rangle$ is conserved along the geodesic.

Example. *If p is the four-momentum of a free particle ($\nabla_p p = 0$) and the spacetime has time translation invariance with Killing vector ∂_t, then the quantity*

$$E = -\langle \partial_t, p \rangle \tag{C.20}$$

is conserved. We call E the energy of the particle.

We shall now write down some useful identities concerning Killing vectors, closely following Ref. [Heusler, 1996].

The norm N and the twist ω of ξ are defined as

$$N = \langle \xi, \xi \rangle , \quad \omega = \frac{1}{2} \star \left(\tilde{\xi} \wedge \mathrm{d}\tilde{\xi} \right) , \tag{C.21}$$

By Killing's equation (C.4) we have for any vector X

$$\begin{aligned} i_X \mathrm{d}N &= \nabla_X N \\ &= 2 \langle \nabla_X \xi, \xi \rangle \\ &= -2 \langle \nabla_\xi \xi, X \rangle , \end{aligned} \tag{C.22}$$

hence

$$\mathrm{d}N = -2\widetilde{\nabla_\xi \xi}. \tag{C.23}$$

Since ξ is a Killing vector we have $\mathcal{L}_\xi \tilde{\xi} = 0$, which implies, using Cartan's identity,

$$i_\xi \mathrm{d}\tilde{\xi} = -\mathrm{d}N. \tag{C.24}$$

With the identity $i_X \star \alpha = \star \left(\alpha \wedge \tilde{X} \right)$ and Eq. (C.24), we get

$$\begin{aligned} i_\xi \left(\tilde{\xi} \wedge \mathrm{d}\tilde{\xi} \right) &= 2 i_\xi \star \omega \\ &= -2 \star \left(\tilde{\xi} \wedge \omega \right) . \end{aligned} \tag{C.25}$$

Hence one obtains

$$-\mathrm{d}\tilde{\xi} = \frac{1}{N} \left[2 \star \left(\tilde{\xi} \wedge \omega \right) + \tilde{\xi} \wedge \mathrm{d}N \right] . \tag{C.26}$$

Taking the inner product of Eq. (C.26) with itself and using the facts $i_\xi \mathrm{d}N = \mathcal{L}_\xi N = 0$ and $i_\xi \omega = 0$ we can also show that

$$N \mathrm{d}\tilde{\xi} \wedge \star \mathrm{d}\tilde{\xi} = \mathrm{d}N \wedge \star \mathrm{d}N - 4\omega \wedge \star\omega. \tag{C.27}$$

Lastly, another useful identity concerning a Killing vector $\xi = \xi^a x_a$ is

$$\mathrm{d}^\dagger \mathrm{d}\tilde{\xi} = 2\xi^a \mathrm{Ric}_a. \tag{C.28}$$

Appendix D

ASYMPTOTICALLY ADS SPACETIMES

The four dimensional anti-de Sitter (AdS) spacetime can be represented as the hyperboloid

$$-u^2 + x^2 + y^2 + z^2 - v^2 = -l^2 \tag{D.1}$$

in $\mathbb{R}^5$ with the metric

$$g = -\mathrm{d}u \otimes \mathrm{d}u + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z - \mathrm{d}v \otimes \mathrm{d}v, \tag{D.2}$$

where $l > 0$ is a constant. It has the topology $S^1 \times \mathbb{R}^3$.

There are closed timelike curves in this spacetime, such as the intersection with $x = y = z = 0$. One can get rid of this by unwrapping the circle S^1 (to obtain its covering space $\mathbb{R}$) and thereby passing to a covering space of AdS. This covering space has the topology $\mathbb{R}^4$ and contains no closed timelike curves. It can be coordinatized as follows. Define the usual spherical coordinates (r, θ, ϕ) on the (x, y, z) space. Define the coordinate t by

$$u = \sqrt{l^2 + r^2} \cos \frac{t}{l}, \quad v = \sqrt{l^2 + r^2} \sin \frac{t}{l}, \tag{D.3}$$

which is allowed to take any value $-\infty < t < \infty$. Then the metric induced from (D.2) takes the form

$$g = -\left(1 + \frac{r^2}{l^2}\right) \mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}r \otimes \mathrm{d}r}{1 + \frac{r^2}{l^2}} + r^2 \mathrm{d}\Omega^2. \tag{D.4}$$

In the following, we shall refer to this covering space when we say “AdS”.

The AdS spacetime is a solution of the Einstein field equations with a negative cosmological constant $\Lambda = -3/l^2$, and has a constant negative scalar curvature

$$\mathcal{R} = -\frac{12}{l^2}. \tag{D.5}$$

To draw the conformal diagram of AdS, consider the coordinate transformation

$$T = t/l, \quad \tan R = r/l, \tag{D.6}$$

in which the metric takes the form

$$g = \frac{l^2}{\cos^2 R}\left[-\mathrm{d}T \otimes \mathrm{d}T + \mathrm{d}R \otimes \mathrm{d}R + \sin^2 R \mathrm{d}\Omega^2\right]. \tag{D.7}$$

In this form, it is manifest that the AdS spacetime is conformally isometric to the region $0 \leq R < \pi/2$, i.e. half, of the Einstein static universe. The Penrose diagram of AdS is shown in Fig. D.1. Note that the timelike infinities $i^{\pm}$ are still infinitely far away; one cannot find a transformation which makes timelike infinity finite without pinching off the Einstein static universe to a point [Hawking and Ellis, 1973]. In addition to this, the null and spacelike infinities are given by the timelike surface $\mathscr{I}$, unlike Minkowski spacetime which has the two null surfaces $\mathscr{I}^{\pm}$ and the point i^0. A spacetime is called *asymptotically AdS* if its infinity has the same structure as that of AdS.

An important example of an asymptotically AdS spacetime is the Reissner-Nordström-AdS black hole. This is the static charged black hole solution of the Einstein field equations with a negative cosmological constant. It has the same form as (2.24), but with

$$f(r) = \frac{r^2}{l^2} + 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{D.8}$$

The Penrose diagram for this spacetime is shown in Fig. D.2.

A crucial point here is that the Komar mass integral (3.16) is actually divergent for an asymptotically AdS spacetime, hence the M in Eq. (D.8) is not equal to the

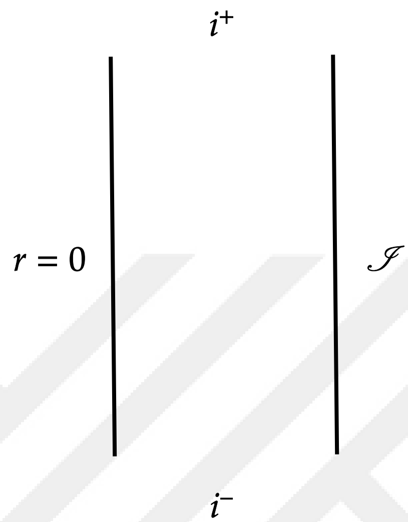

Figure D.1: Penrose diagram of the AdS spacetime.

Komar mass for an asymptotically flat spacetime, but to a suitable generalization of it. To generalize the definition of the Komar charge, observe that Eq. (3.18) can be written using the Einstein field equations with cosmological constant Λ as

$$Q_\xi = c \int_V \xi^a \left(\tau_a - \frac{1}{2} \mathcal{T} \star e_a \right) + \frac{c}{8\pi} \int_V \Lambda \star \tilde{\xi}. \tag{D.9}$$

Hence we can define a finite Komar charge by "subtracting" the divergent term:

$$Q_\xi^{(\mathrm{AdS})} = \frac{c}{16\pi} \oint_{\partial V} \star \mathrm{d}\tilde{\xi} - \frac{c}{8\pi} \int_V \Lambda \star \tilde{\xi}. \tag{D.10}$$

For a more detailed discussion of conserved charges in an asymptotically AdS spacetime, see e.g. [Abbott and Deser, 1982, Magnon, 1985].

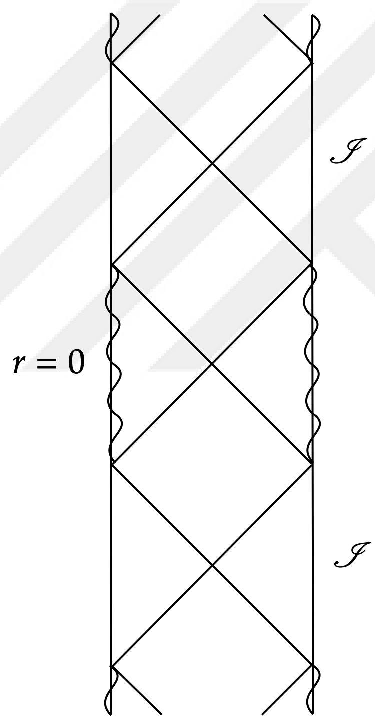

Figure D.2: Penrose diagram of the Reissner-Nordström-AdS spacetime.

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