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HANH-BANACH THEOREM, THE SANDWICH THEOREM FOR SUBLINEAR AND SUPERLINEAR FUNCTIONALS

MASTER'S THESIS

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ABSTRACT

HANH-BANACH THEOREM, THE SANDWICH THEOREM FOR SUB-BANACH AND SUPER-BANACH FUNCTIONALS
MSC THESIS
AZAH MUSTAFA MUSTAFA ALMIQASBI
BOLU ABANT IZZET BAYSAL UNIVERSITY
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A linear functional extension that preserves a specified attribute is the ″Hahn-Banach″ Theorem. This means that when we have a linear functional specified on the a subset of a real subspace that is dominated by a sublinear functional over the entire space, we can extend this functional over the entire space that is still controlled by a sublinear functional. In this thesis, we look at the extension of a linear functional given on a subspace of a real linear space that is controlled by a sublinear functional and dominated by a superlinear functional. In addition, we cover various methods for proving the Sandwich Theorem for both sublinear and superlinear functionals.

KEYWORDS: Extension linear functionals, The Hahn - Banach theorem, The Sandwich Theorem, Sublinear functionals, and superlinear functionals.

ÖZET

HANH-BANACH TEOREM, ALT DOĞRUSAL VE SÜPER DOĞRUSAL FONKSİYONELLER İÇİN SANDVİÇ TEOREM
YÜKSEK LISANS TEZI
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Hahn-Banach teoremi, belirli bir özelliği koruyan doğrusal bir işlevsel uzantıdır. Bu, tüm uzayda bir alt-doğrusal fonksiyonelin hâkim olduğu bir gerçek vektör uzayının bir alt uzayında tanımlanmış bir doğrusal fonksiyonelimiz varsa, bu fonksiyoneli, bir alt-doğrusal fonksiyonelin hâkim olduğu tüm uzay boyunca genişletebileceğimizi gösterir. Bu tezde, bir alt lineer fonksiyonelin hâkim olduğu ve bir süper lineer fonksiyonelin hâkim olduğu bir gerçek vektör uzayının bir alt uzayında tanımlanan lineer bir fonksiyonelin uzantısını inceliyoruz. Ayrıca, sandviç teoremini hem lineer hem de süper lineer fonksiyoneller için kanıtlamanın birkaç yolunu tartışıyoruz.

ANAHTAR KELİMELER: Genişleme lineer fonksiyoneller, Hahn - Banach teoremi, Sandviç teoremi, Sublineer fonksiyoneller ve süper lineer fonksiyoneller.

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LIST OF SYMBOLS AND NOTATIONS

$\mathbb{R}$	The set of all real number
$\mathbb{N}$	the set of all natural number
$\mathbb{Q}$	the set of all rational number
$\mathbb{Z}$	the set of all integers number
$\mathbb{R}^n$	Eculidean space
E	Real vector space
F	Subspace of real vector space
I	the identity mapping
$\underline{0}$	the zero vector on E
f	Linear functional on E
p	Sublinear functional on E
q	Superlinear functional on E
$Subl(E)$	The set of all sublinear functionals on E
$Supl(E)$	The set of all sublinear functionals on E
$Lin(E)$	The set of all linear functionals on E

CHAPTER 1

1. INTRODUCTION

The ″Hahn–Banach Theorem″ is a powerful instrument in canalization, which is a major branch of mathematics. Furthermore, there is no technique known without the ″Hahn-Banach Theorem″. It shows that there are "sufficient" continuous linear functionals allowed on every normalization linear space to make studying the dual space "interesting" [1,2]. It enables the extending of limited linear functionals specified on a subset of a subspace to the entire space. The ″Hahn-Banach theorem″ is a linear functional extension theorem with certain features [3,4]. Assume a linear functional is defined on a subset of a real linear space that is dominated by a sublinear functional on the entire area. Then, on the whole space, this functional can be increased to a linear functional, that is nevertheless dominated by a sublinear functional. The proof of this theorem can indeed be obtained in any classic novel on functional analysis. Stefan Banach established the concept of a sublinear function in his explanation of the ″Hahn-Banach theorem″ [4], and the proof of this is also given in this thesis.

The goal of this thesis is to investigate various approaches for extending functionals given on a subspace to the entire space while maintaining control over the extension. Indeed, the ″Hahn-Banach theorem″ is important due to the range of its applications, which also include complicated and functional analysis [2]. The ″Hahn-Banach theorem″ has significant implications for convex sets and serves as the foundation for an appropriate optimization approach. It is also used to address problems in linear algebra, conic duality theory, the minimax theorem, piecewise approximation of convex functionals, and extensions of positive linear functionals [4,5,6].

This dissertation is organized into four major chapters. The first chapter introduces basic definitions and information about functional analysis. The second chapter presents the concept of sublinear and superlinear mapping over linear spaces, which broadens the concept of linear maps. Furthermore, we go over a few of these maps. In chapter three, we demonstrate various applications of the ″Hahn-Banach theorem″. Chapter four offers three distinct approaches for proving the Sandwich Theorem for sublinear and superlinear functionals.

1.1 Basic Concepts

Some of the most frequently used concepts throughout the thesis are introduced in this chapter. To that goal, we began with some fundamental definitions, theorems, and examples required for this thesis. We employ industry-standard vocabulary and notations.

1.1.1 Linear Spaces.

Definition1.1. Assume X is a nonempty set. Any function $*: X \times X \rightarrow X$ referred to as **a binary operation** on X**.** In general, we write

$$*(x, y) = x * y.$$

We start with the meaning of the linear space.

Definition1.2 Let E be a set with binary operation *, which satisfies the requirements:

i. For all $a, d, c \in E$: $a * (d * c) = (a * d) * c$.

ii. There exists $e \in E$ such that $a * e = e * a = e$.

iii. For each $a \in E$ there exists $d \in E$ such that $a * d = d * a = e$.

Then the pair $(E,*)$ is called a **group.■**

Following the above definition, if $(E,*)$ is a set as in the preceding definition demonstrates that element e is unique. It is called as the group's **identity element.** There exists a unique $y \in E$ for each $x \in E$ such that

$$x * y = y * x = e.$$

In this example, y is referred to as the **inverse** of x, and in general, we write

$$y = x^{-1}.$$

If for each $x, y \in E$,

$$x * y = y * x.$$

Then E is a **commutative group.**

Following are some examples of groups [7]**.**

i. Normal multiplication is the composing law for the order two set, which consists of the real numbers 1, -1.

ii. Under multiplication, the set of order four consists of the complicated integers 1, I -1, -i (where i^2 = -i).

iii. $(\mathbb{Z}, +)$ is a group with the fact that + is an associative binary operation on $\mathbb{Z}$. 0 is an identity element, and –x is an inverse of x.

iv. $(\mathbb{R}, +)$. This is a continuing group having 0 as the identity element. The opposite of a number b is its negative - b.

v. $(\mathbb{R} - \{0\}, \cdot)$. The identity item is 1, and the reciprocal of y is 1/y.

vi. Under multiplication, the single point set containing only the unity is a group of order one.

We are dealing with a real linear space, which is defined as a commutative group, in this thesis. The following is the definition:

Definition 1.3 Consider that $(E, +)$ is a commutative group and $m: \mathbb{R} \times E \to E$ be a function, defined by

$$m(r, x) = r\, x,$$

satisfying the following properties:

i. $(r_1 r_2)x = r_1(r_2 x)$.

ii. $r(x+y) = rx + ry$.

iii. $1x = x$.

for all $r_1, r_2, r \in \mathbb{R}\ and\ x, y \in E. And\ 1 \in \mathbb{R}$.

Then the triple $(E, +,\ m)$ is called a **real linear space**.■

In general, when we say "E" is a linear space, we understand the triple $(E, +,\ m)$ as in the above definition. And "+" is called **addition operation,** and "m" is called **scalar multiplication**. The algebraic operation of the linear space is denoted by "+" and "m". Most of the time, the scalar multiplication m is donated by "·" with respect to standard terminology.

Remark 1.1 The identity e of the group $(E, +)$ is called **zero element** of the linear space $(E, +,\ \cdot)$ and it is denoted by O_E. One can show that for each $r \in \mathbb{R}\ a \in E$.

i. $O.a = O_E$.

ii. $r.O_E = O_E$.

iii. $-(r.a) = (-r).a = r.(-a)$.

Some Basic examples of the linear spaces [8,9]:

1. $\mathbb{R}$ is a linear space under the usual addition operation $"+"$ and scalar multiplication "· " .
2. Let $E = \mathbb{R}^n$.Then E is a linear space with the following algebraic operations:

$$(x_1, x_2, \dots, x_n) + (y_1, y_2 \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$r(x_1, x_2, \dots, x_n) = (rx_1, r\, x_2, \dots, rx_n)$$

3. Consider $E = \mathbb{R}^X$. where it is the set of all functions from X into $\mathbb{R}$

$$E = \{f: f: X \to \mathbb{R} \text{ a function}\} = \mathbb{R}^X.$$

Then E is a linear space with the algebraic operations as follows:

$$(f+g)(x) = f(x) + g(x)$$

$$(rf)(x) = rf(x)$$

4. Let I be a nonempty set, let

$$E = \bigoplus_{i\epsilon I} \mathbb{R} = \{ f: f: I \to \mathbb{R} \text{ a function}, i \in I: f(i) \neq 0 \text{ is finite}\}.$$

Then E is a linear space with the following algebraic operations:

$$(f+g)(i) = f(i) + g(i)$$

$$(rf)(i) = rf(i)$$

This linear space E is called a **direct sum of linear space**.

5. Assume that $E = C([0,1])$ be the set of all continuous function from [0,1] into $\mathbb{R}$. Then E is a linear space with the algebraic operations listed below:

$$(f+g)(x) = f(x) + g(x).$$

$$(rf)(x) = rf(x).$$

for every $f, g \in E, r \in \mathbb{R}$ *and* $x \in [0,1]$.

Definition 1.4 Consider M to be a linear subspace and a linear space E is a nonempty subset of E that meets the following requirements

i. $x + y \in M, \ \forall x, y \in M.$

ii. $rx \in M, \ \forall x \in M, \ \ \forall r \in \mathbb{R}. \blacksquare$

Simple Examples of the vector subspace of a linear space E:

i. $\{0\}$ is a subspace of *E*.

ii. *E* is a subspace of itself.

Every subset of a linear space is contained in some linear subspace since every linear space is a linear subspace of itself. We can derive the following theorem from this observation:

Theorem 1.1 *Assume that E be a linear space and A*$\subseteq E$ *be given. Then there exists a linear subspace F of E such that:*

i. *A*$\subseteq F$.

ii. *If M is a linear subspace of E and A*$\subseteq M$ *then F*$\subseteq M$.

Moreover,

$$F = \{ \sum_{i=1}^{n} r_i x_i : n \in \mathbb{N}, r_i \in \mathbb{N}, x_i \in A \}.$$

$$= \cap \{G \mid G \text{ is a linear subspace of } E \text{ and } A \subseteq G\}.$$

And in this case, we write F=<A>.■

Definition 1.5 In theorem 1.1, the linear subspace F is called the **linear subspace generated** by A.■

1.1.2 Basis of Linear Spaces

This section introduces the concept of a linear space basis. Following that, we demonstrate that every linear space is in the direct sum of linear spaces.

Definition 1.6 Consider that E be a subspace and $B \subseteq E$ be a positive real number set and $0 \notin B$. B is said to be **linearly independent** if:

$$r_1 b_1 + r_2 r b_1 + \ldots + r_n b_n = 0 \Longrightarrow r_1 = r_2 = \cdots = r_n = 0 .\ ■$$

Definition 1.7 Consider that E is a linear space and $B \subseteq E$ be linearly independent. If the vector subspace generated by B is equal to E, then B is called a **basis** of E.■

Some examples of basis of linear space.

i. *Assume $n \in \mathbb{N}$ be given. For each $1 \leq i \leq n$, let*

$$e_i = (0,0,\ldots,1,0,0,\ldots) \in \mathbb{R}^n,$$

Then, B= $\{ e_i : 1 \leq i \leq n \}$ is a basis of the linear space $\mathbb{R}^n$.

ii. *Let E= $\bigoplus_I \mathbb{R}$ be a direct sum of linear space. Then*

$$B = \{\chi_i : i \in I\}, \text{ is a basis of } E.$$

Here, $X_i : I \longrightarrow \mathbb{R}$, is the characteristic function i, that is,

$$\chi_i(j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Theorem 1.2 *Every nonzero linear space has a basis.*

It should be mentioned that a linear space might have more than one basis. Wherever, for any vector space has two bases $B_1 \Lambda\ B_2$, there is a one-to-one and onto relation between $B_1 \Lambda\ B_2$. That is, there is a one-to-one relations and an onto function f: $B_1 \rightarrow B_2$. By using this fact, the following definition can be given.

Definition 1.8 Assume E is a positive real number linear space and B is its basis. The dimension of E is then the cardinal number of B.■

Definition 1.9 Consider E and F are two linear spaces. A map $f: E \to F$ is linear is called **linear map** If the following criteria are met:

i. $f(a + b) = f(a)+f(b)$ for every a , $b \in E$ (i.e., f **is additive**),

ii. $f(r\,a) = r\,f(a)$ for every $a \in E$ and $r \in \mathbb{R}$ (i.e., f **is homogeneous**).■

In this case, $F = \mathbb{R}$, f is called a **linear functional**. We note that: $f: E \to F$ is additive for each rational number $r \in \mathbb{Q}$, $a \in E$

$$f(r\,a) = r\,f(a)$$

is satisfied but f is not necessarily homogeneous.

Example:

i. Consider that $E = \bigoplus_I \mathbb{R}$ *is a direct sum of linear space. The map* $T: E \to \mathbb{R}$ *defined by*

$$T(f) = \sum_{i \in I} f(i)$$

is a linear functional.

ii. Consider that $E = \bigoplus_I \mathbb{R}$ *is a direct sum of linear space, F be a linear space, and assume* $b: I \to F$ *be a function. The map* $T: E \to F$ *defined by, for given subset* $J \subseteq I$

$$T_J(f) = \sum_{i \in J} f(i)\, b(i)$$

is a linear map.

Many examples of linear mappings between linear spaces can be defined using the method shown in this example.

Definition1.10 A linear map $T: E \to F$ is called an **isomorphism** if it is one-to-one and onto.■

It should go without saying that the inverse of an isomorphism is also an isomorphism. The concept of "equality" on linear spaces can be modified as following in terms of "isomorphism":

Definition 1.11 Consider E and F to be two linear spaces. If an isomorphism $f: E \to F$ exists, then E and F are **isomorphic spaces.■**

Theorem1.3 *A direct sum linear space is isomorphic to every nonzero linear space.*

Proof. Consider that E is a nonzero linear space. By Theorem (1.2), E has a basis B. Let

$$F = \bigoplus_B \mathbb{R}.$$

Define $T: E \to F$ by

$$T(f) = \sum_{b \in B} f(b)\, b.$$

It is simple to demonstrate that T is one-to-one and on the linear map. As a result, E and F are isomorphic spaces. ■

Definition 1.12 Consider E and E_1 be linear spaces and F be a linear subspace of E. In particular,let g be a linear functional F into E_1. A linear map f from E into E_1 is called a linear **extension** of g if

$$f(a) = g\,(a)$$

for all $a \in F$ This is denoted by $f\,|_F = g$.

CHAPTER 2

2. RELATIONS BETWEEN SUBLINEAR AND SUPERLINEAR FUNCTIONALS DEFINED ON A REAL LINEAR SPACE

2.1 Sublinear and Superlinear Maps.

Basic definitions and functional analysis results for sublinear and superlinear functionals. We consider that E is a real linear space in the following definitions. We introduce the concept of sublinear and superlinear mappings between linear spaces in this chapter, which generalizes the idea of linear maps. In addition, we discuss some of the essential features of these maps.

The following are definitions of sublinear and superlinear linear mappings.

Definition 2.1 Consider that E and F is two linear spaces. A map $f: E \to F$ is called

i. **Subadditive** if $f(a+b) \leq f(a)+f(b)$ for all $a, b \in E$,

ii. **Superadditive** if $f(a+b) \geq f(a)+f(b)$ for all $a, b \in E$,

iii. **Positively homogeneous** if $f(r\,a) = rf(a)$ for all $a \in E$ and $r \geq 0$. ■

Definition 2.2 Consider E and F to be two linear spaces. If a map $f: E \to F$ is subadditive and positively homogenous, it is called **sublinear map**.■

In the case $F = \mathbb{R}$, f is called a **sublinear functional.** The collection of **all sublinear functionals** on a linear space E will be denoted by $Subl(E)$.

Definition 2.3 Assume that E and F be two linear spaces. A map $f: E \to F$ is called **superlinear map** if it is superadditive and positively homogeneous.■

Superlinear map is called **superlinear functional** in the case where $F = \mathbb{R}$. The collection of **all sublinear functionals** on E will be denoted by $Supl(E)$.

Theorem 2.1 *Consider that E is a linear space. Then the followings hold:*

i. A linear functional on E is both sublinear and superlinear.

ii. Lin(E) = Subl(E) ∩ Supl(E), Lin(E) signifies the collection of all linear functionals on E. ■

It should be mentioned that a sublinear map is not necessarily superlinear. Similarly, a superlinear map needs not to be sublinear. An example is here.

Example 2.1 *Consider* $E = \mathbb{R}^n$ *as a linear space, for* $x = (x_1, x_2, \ldots, x_n) \in E$, *define the functional* $p : E \to \mathbb{R}$ *by*

$$p(x) = \sqrt{\sum_{i=1}^{n} {x_i}^2} = \|x\|.$$

P is sublinear but not superlinear. for $x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{E}$ *,If we define* $q: E \to \mathbb{R}$ *by,*

$$P(x) = -\sqrt{\sum_{i=1}^{n-1} {x_i}^2} + x_n,$$

q is superlinear but not sublinear.

The section that follows offers several relations between sublinear and superlinear functional relations on a real linear space *E*.

2.2 Relations Between Sublinear and Superlinear Functionals are Defined in Real Linear Spaces.

This section compiles various relationships between sublinear and superlinear functionals on *E*. Sublinear and superlinear functionals on the real linear space *E* shall be denoted by the symbols *p* and *q* [13].

Lemma 2.1 *Consider that E is a linear space,* $p: E \to \mathbb{R}$ *a sublinear map, and* $q: E \to \mathbb{R}$ *a superlinear map. Then:*

i. $p(\underline{0}) = 0$ *and* $q(\underline{0}) = 0$.

ii. For each $a \in E$, $-p(-a) \leq p(a)$ *and* $-q(-a) \geq q(a)$.

proof:

i. From the definitions of sublınear and superlınear functional we deduce that

$$p(2.\underline{0}) = 2.p(\underline{0}) \text{ i.e., } p(\underline{0}) = p(\underline{0} + \underline{0}) = p(\underline{0}) + p(\underline{0}).$$

Therefore $p(\underline{0}) = 0$.

By the same method, it is easy to show that $q(\underline{0}) = 0$.

ii. Since

$$0 = p(\underline{0}) = p(-a + a) \leq p(-a) + p(a)$$

for every $a \in E$, then

$$-p(-a) \leq p(a).$$

Also,

$$0 = q(\underline{0}) = q(-a + a) \geq q(-a) + q(a)$$

for every $a \in E$, then $-q(-a) \geq q(a)$ for every $a \in E$.■

Lemma 2.2 *Consider that E be a linear space and* $f: E \to \mathbb{R}$ *be a function. Then the following are equivalent:*

i. f is a functional.

ii. f is additive and has a positive homogeneous distribution.

Proof: Assume (ii) is holds. To put it another way, consider f to be an additive positively homogenous functional on E. It is sufficient to demonstrate that f is functional by demonstrating that f is homogeneous. Let $r \in \mathbb{R}$ and $a \in E$ be given if $r \geq 0$ then by the assumption we have

$$f\,(r\,a) = r\,f(a).$$

If $r \leq 0$

$$0 = f\,(0) = f\,a + (-a) = f\,(a) + f\,(-a),$$

We have

$$f\,(r\,a) = f\,((-r)\,(-a)) = -r\,f(-a) = (-r)\,(-f(a)) = r\,f(a).$$

If (i) holds, that is f is linear, then obviously it is additive positively homogenous, by definition. This completes the proof. ■

Corollary 2.1 *If p is a sublınear functional on a real linear space E*

$$p(-a) = -p(a), \text{ for every } a \in E.$$

Then p is linear.

Proof. Since p is sublınear then p is positively homogeneous**,** and since

$$p(a) = -p(a),$$

obviously, it is homogenous. Now, It is still necessary to demonstrate that p is additive. Since p is sublınear, for all $x, y \in E$, then

$$p(a) + p(b) = p\,(a + b - b) \leq p(a + b).$$

Hence, it is superlınear. Already it is sublınear, so it is linear. ■

Corollary 2.2 *If q is a superlınear functional on a real linear space E, and*

$$q(-a) = -q(a),$$

for every $a \in E$. Then q is linear.

Proof. The facts follows directly that $-q$ is sublınear and $q(-a) = -q(a)$ for every $a \in E$. ■

Lemma 2.3 *If $p(a) \leq q(a)$ for all $a \in E$, where $p \in Subl(E)$ and $q \in Supl(E)$, then p and q are linear functional on E and moreover $p \equiv q$ on E.*

Proof. Since p is sublınear and q is superlınear, then

$$-p(-a) \leq p(a),$$

and

$$-q(-a) \geq q(a),$$

for all $a \in E$. So

$$p(-a) \leq q(-a) \leq -q(a) \leq -p(a).$$

It means that

$$p(a) \leq -p(-a).$$

Thus

$$p(-a) = -p(a)$$

for all $a \in E$. Then from Corollary 2.1. Then p is linear, and as the same

$$q(a) \geq p(a) \geq -p(-a) \geq -q(-a).$$

it means that

$$q(a) \geq -q(-a).$$

Thus

$$q(-a) = -q(a)$$

for all $a \in E$. Then from Corollary 2.2Then q is linear.

Finally, we show that $q \leq p$ on E,

$$-p(a) \leq p(-a) \leq q(-a) \leq -q(a).$$

Thus

$$q(a) \leq p(a)$$

for all $a \in E$. Therefore, $p \equiv q$ on E, the Lemma follows.■

Corollary 2.3 If f_1 and f_2 are two linear functionals on E such that $f_1 \leq f_2$ on E. Then $f_1 \equiv f_2$ on E.

Proof. The facts directly follows from the fact that every linear functional is sublinear and superlinear and Lemma2.3.

Another Proof.

Let $a \in E$ be given, so

$$f_1(-a) \leq f_2(-a) = -f_1(a) \leq -f_2(a) \Longrightarrow f_2(a) \leq f_1(a),$$

for every $a \in E$.

$$f_1(a) \leq f_2(a), f_2(a) \leq f_1(a)$$

Then $f_1 \equiv f_2$ on E.■

Lemma 2.4 *Assume that E is a linear space and f is a linear functional on E. The following are synonyms:*

i . $f(a) \leq h(a)$.

ii. $g(a) \leq f(a) \leq h(a)$, *where* $g(a) = -h(-a)$.

for every $a \in E$, *where h is an arbitrary functional on E.*

Proof. Consider that (ii) holds. Obviously, if

$$g(a) \leq f(a) \leq h(a)$$

Then

$$f(a) \leq h(a)$$

for every $a \in E$.

Conversely, Consider that (i) holds. That is, $f(a) \leq h(a)$ for all $a \in E$. Then

$$-f(a) = f(-a) \leq h(-a) ,$$

which implies that

$$f(a) \geq -h(-a),$$

Thus,

$$-h(-a) \leq f(a) \leq h(a),$$

which implies that

$$g(a) \leq f(a) \leq h(a)$$

for every $a \in E$.■ [14,15].

Note.

If we put $h(a) = p(a)$ and $q(a) = -p(-a)$ for all $a \in E$. Then,

$$f(a) \leq p(a),$$

which is equivalent to

$$q(a) \leq f(a) \leq p(a)$$

for every $a \in E$.■

Lemma 2.5 *Consider that E is a linear space. And we define the following sets*

$T_1 = \{a \in E\colon q(a) \leq p(a)\}$,

$T_2 = \{a \in E\colon p(a) \leq q(a)\}$,

$T_3 = \{a \in E\colon q(a) = p(a)\}$.

For every pair (q, p), *we establish the following:*

i. $E = T_1 \cup T_2$.

ii. $T_3 = T_1 \cap T_2$.

iii. $-T_2 \subseteq T_1$.

Proof. i. Since $L_1 \subseteq E$ and $L_2 \subseteq E$, then

$$L_1 \cup L_2 \subseteq E, \tag{1}$$

Conversely, let $a \in E$, then $p(a) \in \mathbb{R}$ and $q(a) \in \mathbb{R}$,

but $\mathbb{R}$ *is* ordering. Then,

$$p(a) \leq q(a) \text{ or } q(a) \leq p(a),$$

therefore $a \in L_2$ or $a \in L_1$, and consequently,

$$a \in L_1 \cup L_2,$$

that is,

$$E \subseteq L_1 \cup L, \tag{2}$$

then from (1) and (2), we have,

$$E = L_1 \cup L_{2.}$$

ii. Obviously.

$$T_1 \cap T_2 = \{a \in E: p(a) = q(a)\} = T_{3.}$$

iii. To prove that

$$-T_2 \subseteq T_1$$

it remains to show that if $a \in T_2$ there $-a \in T_1$. So, let $a \in T_2$, then

$$p(a) \leq q(a),$$

for all $a \in E$. From the properties of sublinear and superlinear functional, we get

$$-p(a) \leq p(-a)$$

and

$$-\,q(a) \geq q(-a)$$

For each $a \in E$, Then,

$$-p(-a) \leq p(a) \leq q(a) \leq -q(-a).$$

Which implies that

$$q(-a) \leq p(-a).$$

Then,

$$-a \in T_1,$$

Hence, we have shown,

$$-T_2 \subseteq T_1,$$

the Lemma follows.■

Lemma 2.6 *Consider that E is a linear space, Consider the following sets:*

$L = \{f | f: E \to \mathbb{R}^{linear}\}$ *the set of all linear functionals on E,*

$p = \{p | p: E \to \mathbb{R}^{sublinear}\}$ *the set of all sublinear functionals on E,*

$q = \{q | q: E \to \mathbb{R}^{superlinear}\}$ *the set of all superlinear functionals on E and*

$E^{\#} = \{f | f: E \to \mathbb{R}\}$ *the set of all functional on E.*

Then we have the followings:

i. $Lin(E) = Subl(E) \cap Supl(E)$.

ii. $Supl(E) = -Subl(E)$.

Proof. i. Since every linear functional is sublinear and superlinear, we have

$$Lin(E) \subseteq Subl(E)$$

and

$$Lin(E) \subseteq Supl(E).$$

This means that

$$Lin(E) \subseteq Subl(E) \cap Supl(E).$$

Consequently, let

$$f \in Subl(E) \cap Supl(E).$$

then

$$f \in Subl(E)$$

and

$$f \in Supl(E).$$

it means that f is an additive positively homogeneous functional on E. And from (iii) in Lemma 2.2, f is linear. Hence $f \in Lin(E)$. Therefore

$$Lin(E) = Subl(E) \cap Supl(E).$$

(i) is proved.

ii. The proof is directly because of the reality that if p is sublinear, then $-p$ is superlinear. That is, we have

$$Subl(E) + Subl(E) \subseteq Subl(E)$$

(i.e., $p_1 + p_2 \in Subl(E)$ for every $p_1, p_2 \in E$), $r\,Subl(E) \subseteq Subl(E)$ for every $r \in \mathbb{R}$, (i.e., $r\,p \in Supl(E)$ for every $p \in E$ and for every $r \in \mathbb{R}$).

And

$$Supl(E) + Supl(E) \subseteq Supl(E),$$

and

$$r\,Supl(E) \subseteq Supl(E),$$

for every $r \in \mathbb{R}$. So, the proof is completed.■

Lemma 2.7 *Assume that E is a linear space and F is a linear subspace on E. Consider $f \in Lin(F)$ and $g,h : F \to \mathbb{R}$ is given such that*

$$g(\underline{0}) = h(\underline{0}) = 0,$$

and

$$g(a) \leq h(a)$$

for every $a \in F$. Then the following are interchangeable:

i. $g(a) \leq f(a) \leq h(a)$ for every $a \in F$,

ii. $f(a) \leq h(a+b) - g(b)$ for every $a, b \in F$.

Proof. Consider that (i) holds, and we want to show that

$$f(a) \leq h(a + b) - g(b)$$

for every $a ,b \in F$. Since

$$g(a) \leq f(a) \leq h(a),$$

we have

$$g(a+b) \leq f(a+b) \leq h(a+b)$$

and

$$g(b) \leq f(b) \leq h(b) \Longrightarrow -h(b) \leq -f(b) \leq - gb).$$

Consequently,

$$g(a+b) - h(b) \leq f(a+b) - f(b) \leq h(a+b) - g(b).$$

Since $f \in Lin(F)$, we have

$$g(a+b) - h(b) \leq f(a) \leq h(a+b) - g(b)$$

for every $a,b \in F$.

Consider that *(ii)* holds. We show that

$$g(a) \leq f(a) \leq h(a)$$

for every $a \in F$. Putting

$$b = \underline{0} \text{ and } a= -b$$

in the inequality

$$f(a) \leq h\ (a+b) - g(b)$$

for all $a,b \in M$, we obtain that

$$f(a) \leq h(b)$$

and

$$f(-b) \leq - g(b) \text{ or } g(b) \leq f(b)$$

Therefore,

$$g(a) \leq f(a) \leq h(a) \text{ for every } a \in F. \blacksquare$$

The following example show that if $p \in Subl(E)$ and $q \in Supl(E)$, then the relations

$$q(a) \leq p(a) \text{ or } p(a) \leq q(a),$$

cannot be satisfied for each $a \in E$. In other phrases, there exists a_1, $a_2 \in E$ such that $q(a_1) \leq p(a_1)$ and $p(a_2) \leq q(a_2)$.

Example 2.2 *Consider* $E = \mathbb{R}^3$ *is a linear space. Define* $p: \mathbb{R}^3 \rightarrow \mathbb{R}$ *by*

$$p(x)= \sum_{i=1}^{3} |x_i| ,$$

where $x = (x_1 , x_2 , x_3)$ *and* $q = \mathbb{R}^3 \rightarrow \mathbb{R}$ *defined by*

$$q(x)= 3x_1 - \sum_{i=2}^{3} |x_i| ,$$

where $x = (x_1, x_2, x_3)$. *It is simple to demonstrate that* p *is sublinear and* q *is superlinear. If it is taken* $x = (1,0,0)$ *then,*

$$p(x) = 1, \; q(x) = 3,$$

that means

$$p(x) \leq q(x).$$

Also, if $x = (0,1,1)$, *then*

$$p(x) = 2 \text{ and } q(x) = -2,$$

which implies that

$$q(x) \leq p(x).$$

Therefore, the relations $q(x) \leq p(x)$ *and* $p(x) \leq q(x)$ *cannot be satisfies for every* $x \in E$. ■

CHAPTER 3

3. THE ″HAHN-BANACH THEOREM″

3.1 History

The Hahn-Banach theorem is named after Hans Hahn and Stefan Banach, two mathematicians who independently demonstrated this in the end 1920s. Previously (in 1912), [4,14], Eduard Helly demonstrated a special instance of the theory for continuous functions on an interval in the space C [a,b], and in 1923, Marcel Riesz developed a more comprehensive enhanced version mathematics, the M. Riesz's expansion hypothesis, where the ″Hahn–Banach theorem″ could be deduced [9,15].

In 1921, Eduard Helly discovered the first Hahn–Banach theorem, demonstrating that certain linear functionals generated on a subspace of a specific type of normed space had a norm extension. Helly accomplished this by first performing a one-dimensional extension (which extends the domain of the linear functional by one dimension) and then inducing it. In 1927, Hahn demonstrated general Banach spaces and was using Helly's approach to build a standard version of the ″Hahn–Banach theorem″ for Hilbert space, which states that a limited linear functional on a subspace has a limited linear extension with the same norm throughout the space. In 1929, Banach extended Hahn's theorem by employing the dominant extension version with sublinear functions for the norm-preserving version. Helly's proof relied on mathematical induction, while ″Hahn and Banach″ relied on transfinite installation. [1,2,16].

Attempts to solve infinite systems of linear equations led to the development of the ″Hahn–Banach theorem″. This is essential to deal with issues such as the moment problem [15,16], which requires determining whether a function with all possible moments exists and discovering it in light of the those instances [9, 16, 17]. Another case in point is the Fourier cosine series problem, which necessitates identifying whether or not a function with those coefficients exists and, if so, locating it [17].

Riesz and Helly established that the presence of a fix equates to the availability and continuation of specified linear functionals for certain classes of spaces, such as Lp ([0, 1] and C ([a, b]) [1,16].

The classical ″Hahn-Banach theorem″ is introduced and shown in this chapter. The evidence is divided into two sections. While Zorn's Lemma is not required for demonstration of the first section, it is required for establishing the general case of the ″Hanh Banach theorem″.

3.2 First Step of The ″Hahn Banach Theorem″

The first part of the ″Hahn Banach″ the following is the theorem:

Theorem 3.1 *Consider that:*

i. *E is a linear space.*
ii. *F is a vector subspace of E.*
iii. *E is the linear space generated by $F \cup \{a_0\}$ for some $a_0 \in E \mid F$*
iv. *$p: E \longrightarrow \mathbb{R}$ is sublinear functional.*
v. *$f: F \longrightarrow \mathbb{R}$ is functional such that*

$$f(a) \leq p(a) \text{ for all } a \in F.$$

then there exists a functional $g: E \longrightarrow \mathbb{R}$ such that

$$g(a) = f(a) \text{ for all } a \in F$$

and

$$g(a) \leq p(a) \text{ for all } a \in E.$$

Proof. Clearly,

$$E = \{ a + ra_0 : a \in F \text{ and } r \in \mathbb{R} \}.$$

Consider that such g exists as stated as in the theorem. Then

$$g(a + ra_0) = f(a) + r\, g(a_0)$$

holds for all $a \in F$ and $r \in \mathbb{R}$, now we have to detertmine

$$c = g(a_0),$$

which satisfies the required conditions, that is, the following inequality

$$f(a) + rc \leq p(a + ra_0) \tag{*}$$

must hold for all $a \in F$ and $r \in \mathbb{R}$. For $r > 0$ and $a \in F$, the inequality (*) is equivalent to

$$c \leq p(r^{-1}a + a_0) - f(r^{-1}a)$$

and to

$$-f(r^{-1}a) - p(r^{-1}a + a_0) \leq c$$

for $a \in F$ and $r > 0$. Certainly, these inequalities (and hence, (*)) will be satisfied by a choice of c for which

$$f(a) - p(-a - a_0) \leq c \leq p(a + a_0) - f(a) \tag{**}$$

holds for every $a \in F$. If $a, b \in F$, then the following inequality

$$f(b) - f(a) = -f(b-a) \leq p\, b - a) = p\,(b + a_0 + (-a - a_0)) \leq p\,(b + a_0) + p\,(-a - a_0)$$

shows that

$$-f(a) - p\,(-a - a_0) \leq p\,(b + a_0) - f(b)$$

holds for all a, b ∈ F. Consequently, if

$$p = sup\,\{-f(a) - p\,(-a - a_0) : a \in F\}$$

and

$$t = inf\{\,p\,(b + a_0) - f(b) : b \in F\}$$

then s and t are both real numbers and $s \leq t$. But then any real number c such that $p \leq c \leq t$ satisfies (**), and hence, (*). This completes the proof. ■ [18].

3.3 Hahn-Banach Theorem.

Now we can state one of the classical ″Hahn-Banach theorem″ as seen below

Theorem 3.2 *Let*

i. E is a linear space.
ii. F is a linear subspace of E.
iii. p: E⟶ ℝ is a sublinear functional
iv. f: F⟶ ℝ is functional such that

$$f(a) \leq p(a) \text{ for each } a \in F.$$

Then there exist a functional g: E⟶ ℝ such that

$$g(a) = f(a) \text{ for each } a \in F$$

and

$$g(a) \leq p(a) \text{ for each } a \in E.$$ [10, 15].

The proof of this theorem depends on Zorn's Lemma and Theorem 3.1.1.

Let us present some related definitions before the state of Zorn's Lemma. For this, the terms and definitions are required.

Definition 3.1 A partially order set is a set X with a partial order is a **binary relation** $\leq$ that satisfies the following conditions:

i. $a \leq a$,
ii. *if* $a \leq b$ and $b \leq a$, then $a = b$,
iii. *if* $a \leq b$ and $b \leq c$, then $a \leq c$. ■

For each $a, b \in X$.

Definition 3.2 Assume X is a partially ordered set, and $C \subseteq X$. if for all $a,b \in C$ either $a \leq b$ or $b \leq a$ then C is called a **chain**.■

Definition 3.3 Consider X to be a partially ordered set, and A be a subset of X and $a \in X$, a is said to be **an upper bound** of A if $s \leq a$ for every $s \in A$.■

Definition 3.4 Assume X is a partially ordered set, and **An element** $m \in X$ is called a maximal element if $a = m$ whenever $m \leq a$ in X.■

3.4.1 Now The Zorn's Lemma Can Be Stated As Follows

Zorn's Lemma: *If every chain in the partially ordered set X has an upper-bounded element in X, then X seems to have a maximum element.*

The proof of Theorem 3.1 can be given as follows.

Proof..Let C represent the set of all couples.. (g,Z) where Z is a linear subspace of E containing F and g is a linear functional on Z satisfying

$$g(a) = f(a),$$

for all $a \in F$ and

$$g(a) \leq p(a),$$

for all $a \in Z$.

The collection C is nonempty since $(f, F) \in C$. Define an order relation on C as follows:

$$(g_2, Z_2) \geq (g_1, Z_1) \text{ whenever } Z_1 \subseteq Z_2,$$

and

$$g_2(a) = g_1(a),$$

for all $a \in Z_1$. It is easy to verify that $\geq$ is indeed an order relation on C.

Now, consider a chain in C.

$$\{(g_i, Z_i): i \in I\},$$

(it means that, for all pairs $i, j \in I$ either $g_i, Z_i) \geq (g_j, Z_j)$ or $(g_i, Z_i) \geq (g_j, Z_j)$, holds. Let $Z = \bigcup_{i \in I} Zi$. One can demonstrate that Z is a linear subspace of E. Now, define $g: Z \longrightarrow \mathbb{R}$ by letting

$$g(a) = g_i(a) \text{ if } a \in Z_i.$$

Since $((g_i, Z_i): i \in I)$ is a chain, the value of $g(a)$ is independent of the chosen index i. Clearly, g is a linear functional. In addition,

$$g(a) = f(a),$$

holds for each $a \in F$, and

$$g(a) \leq p(a),$$

for each $a \in Z$. Thus,

$$(g, Z) \in C,$$

and clearly,

$$(g, Z) \geq (g_i, Z_i),$$

holds for each $i \in I$. As a result, for any chain of C, there is a maximum limit. As stated by Zorn's Lemma, C has a maximum component, say (g, Z).

It is sufficient to demonstrate that $Z = X$ to complete the evidence. Furthermore, if $Z \neq X$, then some $a_0 \in E$ with $a_0 \notin Z$ exists. Consider F denote the linear subspace produced by Z and a_0. According to Theorem 3.1.1, there is a linear functional h on F such that

$$h(a) = g(a),$$

for each $a \in Z$ and

$$h(a) \leq p(a),$$

for each $a \in F$. But then $(h, F) \in C$, and

$$(h, F) \geq (g, Z),$$

holds with

$$(h, F) \neq (g, Z).$$

In opposition to the maximality property of (g, Z). As a result, $Z = E$ holds, and the theorem's proof is complete. ■

3.4 Norm Version of ″Hahn-Banach Theorem″

″Hahn-Banach theorem″ can be stated in several ways. Here, we give a norm version of that. For this fist, we define normed space, where a norm is a natural generalization of the absolute value function on the real numbers.

Definition 3.5 Assume E is a linear space. A function $\|\cdot\|: E \longrightarrow \mathbb{R}$ is called a **norm** (we write $\|\cdot\|(a) = \|a\|$) if the following conditions are satisfied.

i. $\|a\| = 0 \text{ if and only if } a = 0.$

ii. $\|ra\| = |r|\|a\|.$

iii. $\|a + b\| \leq \|a\| + \|b\|$. (Triangle inequality)

for each $a,b \in E$, and $r \in \mathbb{R}$.

A linear space E equipped with a norm p is called a **normed space**. ■

Some standard examples of normed spaces can be given as follows:

Examples:

i. *Consider* $\mathbb{R}^{\boldsymbol{n}}$ *As a linear space. the function* $\|\cdot\|: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$, *defined by*

$$\|\boldsymbol{a}\| = \left(\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} |\boldsymbol{a_i}|^{\boldsymbol{2}}\right)^{\frac{\boldsymbol{1}}{\boldsymbol{2}}},$$

is a norm.

ii. *Let* $p > \boldsymbol{0}$. *let* $\boldsymbol{l_p} = \{(a_n) \mid \sum_{\boldsymbol{i=1}}^{\infty} |\boldsymbol{a_n}|^{\boldsymbol{p}} < \infty\}$. *One can show that* $\boldsymbol{l_p}$ *is a linear space with usual pointwise algebraic operations. Also, it can be shown that,*

$$\|(a_n)\| = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}},$$

define a norm on l_p *Wherever* $p \geq 1$.

iii. *Let E= C [a,b] be the set of all continuous function from [a,b]. It is obvious that E is a linear space with pointwise algebraic operations. The following defined functions on E are norms.*

$$\|x\|_1 = \max_{a \le t \le b} |x(t)|,$$

and

$$\|x\|_2 = \int_a^b |x(t)|\ dt.$$

iv. *Assume that E is a linear space and B is a basis of E. Then the direct sum of linear space* $F = \bigoplus_{\boldsymbol{b \in B}} \mathbb{R}$ *and E are isomorphic, so there exists an isomorphism*

$$f: E \to \bigoplus_{\boldsymbol{b \in B}} \mathbb{R}$$

Then, if $\|.\|$ *is a norm on F, then* $\|a\|_E = \|f(a)\|$ *defines a norm on E.*

Since many norms can be defined on F, the linear space E admits many norms, as much as norms on F.

Definition 3.6 Consider that E is a normed space. A function $f: E \longrightarrow \mathbb{R}$ is a linear functional on N is called **continuous** at a point $a_0 \in N$ if given any $\varepsilon > 0$, there exist positive real number δ *such that*

$$\|a - a_0\| < \delta \implies |f(a) - f(a_0)| < \varepsilon.$$

For each $a \in N$.■

It is possible to demonstrate that a functional on a normalized space is continuous at all points and only then if it is continuous at the origin. Hence, if f is a functional on a normed space E, then f is continuous if for each $\varepsilon > 0$ there exist $\delta > 0$ such that

$$\|a\| < \delta \implies \|f(a)\| < \varepsilon.$$

The continuity of a functional on a normalized space can given in terms of boundedness, the definition is given in the following:

Definition 3.7 Assume that E be a normed space. And a function $f : N \longrightarrow \mathbb{R}$ be a functional on E is called **bounded** if there is such a thing as a real number $r \geq 0$, as well as

$$|f(a)| \leq r\|a\|,$$

For each $a \in E$.■

Following the above definition, on can show that a functional f is bounded if and only if

$$\|\boldsymbol{f}\| = \sup_{\|\boldsymbol{a}\| \leq \mathbf{1}} |\boldsymbol{f}(\boldsymbol{a})| < \infty.$$

In this case, $\|f\|$ is called the norm of f. One might demonstrate that

$$\|f\| = \sup_{\|a\|=1} |f(a)| .$$

Theorem 3.3 *Assume E be a* normed space*, and f:$E \longrightarrow \mathbb{R}$ be a functional. The following are equivalent:*

i. *f is continuous.*

ii. *f is bounded.*

iii. $|f(a)| \leq r\|a\|$, *for all a, for some r*> 0.

It should be obvious that for a continuous functional we have the following inequalities:

$$|\boldsymbol{f}(\boldsymbol{a})| \leq \|\boldsymbol{f}\| \, \|\boldsymbol{a}\| .$$

Now we can stead norm version of the ″Hahn Banach theorem″ as follows.

Theorem 3.4 (″Hahn–Banach Theorem″ -First Version).

Consider that

I. *E is a normed space.*

II. *F is a normed subspace.*

III. *f: $E \longrightarrow \mathbb{R}$ a bounded linear functional.*

Then, there exist a bounded linear functional g: $E \longrightarrow \mathbb{R}$ such that .

IV. *$g(a) = f(a)$ for each $a \in F$.*

V. $\| f\| = \| g\|$.

Proof. let $p: E \to \mathbb{R}$ be defined by

$$p(a) = \|f\| \, \|a\|$$

It is obvious that p is a sub-banach and

$$|f(a)| \leq p(a).$$

for all $a \in F$.

By using Theorem 3.3, f has a linear extension functional $g: E \to \mathbb{R}$ such that

$$g(a) \leq p(a).$$

In particular,

$$-g(a) = g(-a) \leq p(-a) = p(a),$$

so,

$$|g(a)| \leq p(a) = \|f\| \|a\|$$

For each a.

By Theorem 3.3, g is continuous.

Moreover,

$$\| g\| \leq \| f\|.$$

On the other hand, it is obvious that

$$\| f\| \leq \| g\| ,$$

Hence,

$$\| f\| = \| g\|$$

This concludes the evidence.■[19].

3.5 An Application of The ″Hahn Banach Theorem″

As an application of the ″Hahn-Banach theorem″ the following theorem has been proved by some authors (for example Kelly-Naimoka [15], Rudin [11], Koing [2] Simons [6,12] Now, we state this theorem with more simple proof.

Theorem 3.5. *Consider E is a linear space and* $p: E \to \mathbb{R}$ *is a sublinear map. There is then a linear functional* $f: E \to \mathbb{R}$ *such that*

$$f(a) \leq p(a)$$

for every $a \in E$

Proof. If $p = 0$ (*i.e.*, $p(a) = 0$ for every $a \in E$), then the result following immediately by taking

$$f = \underline{0}$$

So we can Consider that

$$p \neq \underline{0}$$

Now let a_0 in E and we construct a one-dimensional subspace F as

$$F = \{r\, a_0 : r \in \mathbb{R}\},$$

and a linear functional is defined g on F as

$$g(a) = g(ra_0) = r\, p\,(a_0)$$

for every $r \in F$. Hence, there are two situations.:

Case 1. $F = \{0\}$**.**Since

$$0 = g(0) = p(\underline{0}),$$

it means that

$$g(a) \leq p(a)$$

for every $a \in F$, then by using the extension form of ″Hahn-Banach Theorem″, a linear functional exists f on E as well as

$$f|_F = g \text{ and } f(a) \leq p(a)$$

for every $a \in E$ as required. [21].

Case 2. $F \neq \{\underline{0}\}$**.** So, we have two subcases:

Subcase 1. $r \geq 0$**.** Let

$$r \geq 0,\ g(a) = g(ra_0) = r\, p(a_0) = p(ra_0) = p(a) \leq p(a)$$

for every $a \in p$.

Subcase 2. $r < 0$**.** Let $r = -t$, where $t > 0$,

$$g(a) = g(ra_0) = r\, p(a_0) = -t\, p(a_0) = -p(ta_0) = -p(-ra_0) = -p(-a) \leq p(a)$$

for every $a \in F$.

Therefore, from the last two subcases, we deduce that

$$g(a) \leq p(a)$$

for every $a \in F$. Then from the extension form of Hahn-Banack theorem, a linear functional exists f on E as well as

$$f|_F = g \text{ and } f(a) \leq p(a)$$

for every $a \in E$.■

This proof has introduced by (A.T.Diad , S.I.Nada, D.L.Fearnley) [3].

In the following Theorems, it can be mentioned a similar fact for a similar theorem to theorem 3.4 can be given for superlinear maps as follows. [10, 20, 21].

Theorem 3.6 *Consider that E is a linear space and q: $E \longrightarrow \mathbb{R}$ is a superlinear map. There is thus a linear functional f: $E \longrightarrow \mathbb{R}$ such that*

$$q(a) \leq f(a)$$

for each $a \in E$.

Proof. It is clear that $-q$ is a sublinear functional. Using Theorem.3.4.1, we can show that there is still a linear functional $f \in E$ such that

$$f(a) \leq -q(a),$$

for every $a \in E$, or $p(a) \leq -f(a)$ for all $a \in E$.

It well known that if $f \in E$, then $-f \in E$. ■

CHAPTER 4

4. THE SANDWICH THEOREM FOR SUB-BANACH AND SUPER-BANACH

The well-known Sandwich Theorem is introduced in the first portion of this chapter. We provide three distinct proofs of this theorem in other parts. These are proofs from the paper [3].

4.1 Statement of The Sandwich Theorem.

We can stated the Sandwich theorem as follows.

Theorem 4.1. *Consider that E is a linear space, and let*

- *F is a vector subspace of E.*
- *p: E⟶ ℝ is a sublinear map.*
- *q: E⟶ ℝ is a superlinear map.*
- *f: F⟶ ℝ is functional. Such that*

$$q(a) \leq f(a \leq p(a),$$

for every a∈ F.Then there exists a functional g: E⟶ ℝ such that

i. *g is an extension of f , that is,*

$$g(a)=f(a), \text{for all } a \in F.$$

ii. *q(a) ≤ g(a) ≤ p(a) for all a ∈ E.*

A function $T: E \to \mathbb{R}$ which is determined by

$$T(a) = inf_{b\in E} \{p(a+b) - q(b)\}$$

for all $a,b \in E$ will be used in the proof the theorem.

4.2 A Proof of The Sandwich Theorem.

In this part, we will proof the theorem 4.1, to do this we still need some lemmas, all lemmas come from [22].

Lemma 4.1 *Let g be a linear functional of a subspace F of E such that*

$$g(a) \leq T(a) \tag{4.2.1}$$

for every a ∈ F. Then the following are satisfied:

i. q(a) ≤ p(a) for every a∈ E. *(4.2.2)*

ii. q(a)≤ $g(a)$ ≤ p(a) for every a ∈ F. *(4.2.3)*

Proof.

i. It may be observed that

$$g(a) \leq p(a + b) - q(b),$$

for every $a \in F$ and $b \in E$. by taking $a = 0$ we have

$$q(b) \leq p(b),$$

on E, so (i) is proved.

ii. In the inequality (4.2.3) by taking $b=0$ we have

$$g(a) \leq p(b),$$

for every $a \in F$. And also by taking $y = -a$ we have

$$g(-b) \leq -q(b),$$

and consequently

$$q(a) \leq g(\mathrm{a}),$$

on F. Hence, we have

$$q(a) \leq g(\mathrm{a}) \leq p(a),$$

on F. Hence (ii) is proved, too.

The following example demonstrates that the inverse of lemma 4.1. is not always true.■

Example 4.1 *Consider* $E = \mathbb{R}^2$ *as a linear space. Define* $p, q: E \rightarrow R$ *by*

$$q(a) = -|a_1| + a_2,$$

and

$$p(a) = \sqrt{a_1^2 + a_2^2},$$

where $a = (a_1,a_2)$. *Let* $F = \{(a_1,0) : a_1 \in \mathbb{R}\}$. *F is a linear subspace of E.*

Consider that $g : F \longrightarrow \mathbb{R}$ *is defined as*

$$g(a) = a_1,$$

for each $a = (a_1,0) \in F$.

First we show that for p, q and f{0} satisfies the condition of the Lemma:

It effortless to verify that q is a superlinear functional on E, p is a sublinear functional

on E and g is a linear functional on F.

Now, let $a = (a_1,a_2) \in E$, *then*

$$p(a) = \sqrt{a_1^2 + a_2^2} \geq a_2 \geq -|a_1| + a_2 = q(a),$$

it means that

$$q(a) \leq p(a) \text{ on } E.$$

Also, let a = $(a_1,0) \in F$, then

$$q(a) = -|a_1| \le a_1 = f_0(a) \le |a_1| \le \sqrt{a_1^2 + a_2^2} = p(a),$$

so, $q(a) \le g(a) \le p(a)$ on F, as required.

We show that the following statement

$$g(a) \le T(a)$$

for each $a \in F$, is not true, in general: Consider that (4.2.1) is true for every $a \in F$ and for every $b \in E$. so, let $a = (a_1,0) \in F$ where $a_1 > 0$. Then

$$a_1 \le \sqrt{(a_1 + b_1)^2 + b_2^2} + |b_1| - b_2$$

for every $b = (b_1,b_2) \in E$. now taking $b=(0,b_2)$, where $b> 0$. Hence and consequently $a_1 \le 0$,

$$a_1 \le \sqrt{a^2{}_1 + b^2{}_2} - b_2 = \frac{\left(\sqrt{a^2{}_1 + b^2{}_2} - b_2\right)\left(\sqrt{a^2{}_1 + b^2{}_2} + b_2\right)}{\left(\sqrt{a^2{}_1 + b^2{}_2} + b_2\right)}$$

$$= \frac{a^2{}_1}{\left(\sqrt{a^2{}_1 + b^2{}_2} + b_2\right)} \to 0 \text{ as } b_2 \to \infty,$$

which gives a contradiction with assumption that $a_1 > 0$. Then g cannot be satisfied (4.1). ■

The following Lemma gives the condition under which the converse can be satisfied.

Lemma 4.2 *If $q(b) = -p(-b)$ for every $b \in E$, then conditions (4.2.2), (4.2.3) are satisfied, if and only if condition (4.2.1) is true.*

Proof. Since $q(b) = -p(-b)$ for every $b \in E$, and condition *(4.2.2), (4.2.3)* are satisfies, then

$$g(a) \le p(a + b - b) \le p(a + b) + p(-b)$$

Then

$$g(a) \le p(a + b) - q(b).$$

Taking the infimum over $b \in E$, we get the desired. ■

Theorem 4.2 *Consider that $q(a) \le p(a)$ on E and f is a linear functional on E such that*

$$q(a) \le f(a) \le p(a) \text{ on } E.$$

Then $f(a) \le T(a)$ on E, where $T: E \to \mathbb{R}$ is sublinear defined as

$$T(a) = inf_{b\in E} \{p(a + b) - q(b)\}$$

for all $a \in E$ and

$$q(a) \le T(a) \le p(a) \text{ on } E.$$

Proof. Since $q(a) \leq L(a)$ on E, then

$$q(u) \leq f(u)$$

for all $u \in E$ and consequently

$$q(-u) \leq f(-u) = -f(u) - q(-u) \leq f(u).$$

Also, from $f(a) \leq p(a)$ on E, we reach the result that $f(v) \leq p(v)$ for each $v \in E$ and then

$$f(u + v) = f(u) + f(v) \leq p(v) - q(-u) \text{ for every } u,v \in E.$$

Letting $u + v = a$ and $u = -b$, then

$$f(x) \leq p(a + b) - q(b).$$

Taking the infimum over $b \in E$ gives $f(a) \leq T(a)$ for every $a \in E$, It means that $f \leq T$ on E as required.

We now prove that $T: E \to \mathbb{R}$ is sublinear. If $b \in E$, then

$$q(b) \leq p(b) \leq p(a + b) + p(-a).$$

So,

$$-p(-a) \leq p(a + b) - q(b).$$

Taking the infimum over $b \in E$, we get

$$T(a) \geq -p(-a) > -\infty.$$

As a result $T: E \to \mathbb{R}$. Now, it simple to demonstrate that T *is positively symmetric.* all that remains to show that T is subadditive. [23]. In fact, let $u, v \in E$,

$$T(u + v) = inf_{b+z\in E} \{p(u+ v+ y+ z) - q(b+ z)\} \leq inf_{b\in E} \{p(u+ z) - q(b)\} + inf_{z\in E} \{p(v + z) - q(z)\} = T(u)+T(v).$$

So,

$$T(u + v) \leq T(u) + T(v).$$

Thus T is sublinear.

Since

$$T(a) \leq p(a + b) - q(b),$$

then

$$T(a) \leq p(a + 0) - q(0),$$

It means

$$T(a) \leq p(a)$$

for all $a \in E$ and

$$T(-b) \leq -q(b)$$

for every $b \in E$. So

$$-T(b) \leq T(-b) \leq -q(b)$$

and hence $q(b) \leq T(b)$ for every $b \in E$. Therefore

$$q(a) \leq T(a) \leq p(a)$$

on E, and the proof is ready.■

Now we are ready to give a different proof of the first method to prove Theorem 4.1.

Proof of Theorem 4.1 Let that the functional g extends a linear functional f on E, such that

$$q(a) \leq f(a) \leq p(a)$$

on E, then from Theorem 4.2,

$$F(a) = g(a) \leq T(a)$$

for every $a \in F$, where

$$T(a) = inf_{b \in E} \{p(a + b) - q(b)\}$$

For all $b \in E$, as required.

In contrast, Consider that

$$g(a) \leq T(a)$$

for every $a \in F$ and from Theorem 4.2. T is sublinear. Therefore, from the extension from of ″Hahn-Banach theorem″, there exists a linear functional f such that

$$f|_F = g \text{ and } f \leq T$$

on E. From the definition of T, we deduce that

$$q\ (a) \leq f(a) \leq p(a) \text{ on } E. \ ■$$

The following corollary show that if we take $q(a) = -p(-a)$ for every $a \in E$, then we obtain immediately the extension form of the ″Hahn-Banach theorem″ as a special case from our work.

Corollary 4.1 *If* $q(a) = -p(-a)$ for every $a \in E$ and g is a linear functional on F such that

$$g(a) \leq p(a) \text{ on } F.$$

Then there exists a linear functional f on E such that

$$f|_F = g \text{ and } f \leq p \text{ on } E.$$

Proof.

The proof is inextricably linked to Lemma 4.2 and Theorem 4.1.

4.3 Another Proof of The Sandwich Theorem

Here's a second proof, another proof of the Sandwich Theorem for superlinear and sublinear functionals. For this we need some lemmas.

Lemma 4.3. *Let* $x_0 \notin F$,

$a = sup_{y \in F} \{ g(y) - p(y - x_0)\}$,

$b = inf_{y \in F} \{-g(y) + p(y + x_0)\}$,

$c = inf_{y \in F} \{ g(y) - q(y - x_0)\}$,

and $d = sup_{y \in F} \{- g(y) + q(y + x_0)\}$ *such that* $a \le b$ *and* $d \le c$.

The following assertions are therefore equivalent:

i. $g(x) \le p(x+ y) - q(y)$ *for all* $x \in F$ *and for all* $y \in E$.

ii. $a \le c$ *and* $d \le b$.

Proof. Consider that $g(x) \le p(x + y) - q(y)$ for every $x \in F$ and $y \in E$. Then

$$g(w-z) \le P(w-x_0)-q(z-x_0)$$

for every $w, z \in F$, where $x = w - z$ and $y = z - x_0$.

Consequently, from the linearity of g we obtain that

$$g(w)-p(w-x_0) \le g(z)-q(z-x_0)$$

for every $w,z \in F$. Consider now that z is held fixed while w is allowed to vary through all of M. This indicates that the collection of all real numbers is complete.

$$\{ g(w) - P(w - x_0) : w \in F\}$$

has an upper bounded and hence has a least upper bounded, say

$$a = sup_{w \in F} \{ g(w) - P(w - x_0)\},$$

In a similar way, we are also assured of the existence of

$$c = inf_{z \in F} \{ g(z) - q(z - x_0)\}.$$

Then we have

$$a = sup_{w \in F} \{ g(w) - P(w - x_0)\} \le c = inf_{z \in F} \{ g(z) - P(z - x_0)\}.$$

Similarly Consider that

$$x = w-z \text{ and } y = z+x_0$$

in the inequality $g(x) \le P(x+ y)-q(y)$, we conclude that

$$- g(z) + q(z - x_0) \le - g(w) + P(w + x_0)$$

for every $w, z \in F$ and hence we obtain that $d \le b$, where

$$d = sup_{z \in F} \{- g(z) + q(z+ x_0)\} \text{ and } b = inf_{z \in F} \{- g(w) + P(w + x_0)\}.$$

Conversely, Consider that $a \le c$ and $d \le b$, where a, b, c and d are defined above. Since $a \le c$, then

$$sup_{w\in F}\{\ g(w)-p(w-x_0)\}\leq\ inf_{z\in F}\{\ g(z){-}q(z{-}x_0)\}$$

and hence

$$g(w)-p(w-x_0)\leq\ g(z){-}q(z{-}\ x_0)$$

for each $w, z \in F$, and consequently

$$g(w{-}z)\leq p(w{-}x_0){-}\ q(z-x_0),$$

It means

$$g(x)\leq p(x+y)-q(y)$$

for all $x = w - z \in F$ and $y = z - x_0 \in E$.

In a similar way, we can easily show that from $d \leq b$, we conclude that

$$g(x)\leq p(x+y)-q(y)$$

for each $x = w - z \in F$ and $y = z + x_0 \in E$.■

Remark 4.1 It is easy to confirm that, if the condition (i) exists in the above Lemma, this is equivalent to

$$g(x)\leq T(x)=inf_{y\in E}\ \{p(x+y)-q(y)\},$$

for each $x \in F$. Also, the condition (ii) in the above lemma is equivalent to $[a,b]\cap[d,c]\neq\emptyset$.

In the following lemma, we Consider that $x_0 \notin F$ and we construct a subspace F_1 of E generated by F and x_0.

Lemma 4.4 *Let g be a linea r functional on F with the property that*

$$g(x)\leq p(x+y)-q(y)$$

for each $x \in F$ and for each $y \in F_1$. Then there exists a functional $f \in F_1$ such that $f|_F = g$ and $q(x) \leq f(x) \leq p(x)$ for each $x \in F_1$.

Proof. From theorem 4.1. we have

$$g(x)\leq p(x)$$

for each $x \in F$. Therefore for arbitrary $y_1,y_2 \in F$ we have

$$g(y_1+y_2)\leq p(y_1+y_2).$$

So, from the linearity of g and the sublinear it y of p we get

$$g(y_1)+g(y_2)=g(y_1{+}y_2)\leq p(y_1{+}y_2)\leq p(y_1{-}x_0){+}p(y_2{+}x_0).$$

Thus, by grouping the terms y_1 on one side and that of y_2 on the other side [24, 25] we have

$$g(y_1)-p(y_1-x_0)\leq -g(y_2)+p(y_2+x_0) \qquad (4.1.1).$$

Consider now y_2 is held fixed and y_1 is allowed to vary over F, then from (4.1.1) we get that the set of real numbers

$$\{\, g(y_1) - p(y_1 - x_0) : y_1 \in F\},$$

has upper bounds and hence has a least upper bound. Let

$$a = sup_{y1 \in F}\{\, g(y_1) - p(y_1 - x_0)\}.$$

If we keep y_1 fixed and y_2 is allowed to vary over F, then from (4.1.1) we get that the set of real numbers

$$\{- g(y_2) + p(y_2 + x_0) : y_2 \in F\},$$

has lower bounds and hence a greatest lower bound. Let

$$b = inf_{y2 \in E}\{\text{-}\, g(y_2) + p(y_2 - x_0)\}.$$

It is clear from (4.1.1) that $a \leq b$. Similarly from theorem 4.1, we have

$$q(x) \leq g(x),$$

for every $x \in F$. Therefore for arbitrary $y_1, y_2 \in F$ we have

$$q(y_1 + y_2) \leq g(y_1 + y_2).$$

So, from the linearity of f_0 and from the superlinear it y of q, we get

$$q(y_1 + x_0) + q(y_2 - x_0) \leq q(y_1 + y_2) \leq g(y_1) + g(y_2).$$

Thus, by grouping the terms y_1 on one side and that of y_2 on the other side, we have

$$- g(y_1) + q(y_1 + x_0) \leq g(y_2) - q(y_2 - x_0) \qquad (4.1.2).$$

Consider now y_2 is held fixed and y_1 is allowed to vary over F, then from (4.1.2) we get that the set of real numbers

$$\{- g(y_1) + q(y_1 + x_0) : y_1 \in F\}$$

has upper bounds and hence a least upper bound. Let

$$d = sup_{y1 \in F}\{- g(y_1) + q(y_1 + x_0)\}.$$

If y_1 is held fixed and y_2 is allowed to vary over F, then from (4.1.2) we get that the set of real numbers

$$\{\, g(y_2) - q(y_2 - x_0) : y_2 \in F\}$$

has lower bounds and hence a greatest lower bound. Let

$$c = \inf_{y2 \in F}\{\, g(y_2) - q(y_2 - x_0)\}.$$

It is clear from (4.1.2) that $d \leq c$. By Remark 4.1, we obtain that $[a,b] \cap [d,c] \neq \emptyset$ and then there exists

$$k \in [a,b] \cap [d,c],$$

it means

$$a \leq k \leq b \text{ and } d \leq k \leq c.$$

Now let $x \in F_1$, then there exists $y \in F$ and $r \in \mathbb{R}$ such that

$$x = y + r\, x_0 \qquad (4.1.3).$$

Now we define a real valued functional f on F_1 as follows:

$$f(x) = f(y + r\, x_0) = g(y) + r\, k \qquad (4.1.4)$$

where $x \in F_1$, $y \in F$, $r \in \mathbb{R}$ and $k \in [a,b] \cap [d,c]$.

Now we verify that the well-defined functional f on F_1 satisfies the desired conditions:

1. f is linear.
2. $f(x) = g(x)$ for every $x \in F$.
3. $q(x) \leq f(x) \leq p(x)$ for every $x \in F_1$.

First, we show f is linear. Let $z_1, z_2 \in F_1$, then there exists $y_1, y_2 \in F$ and $r_1, r_2 \in \mathbb{R}$, such that

$$z_1 = y_1 + r_1 x_0 \text{ and } z_2 = y_2 + r_2 x_0.$$

Hence for every $r, n \in \mathbb{R}$, we get

$$f(rz_1 + nz_2) = f((ry_1 + ny_2) + (rr_1 + nr_2)\, x_0) = g(ry_1 + ny_2) + (rr_1 + nr_2)\, k = r\,(g(y_1) + r_1 k) + n\,(g(y_2) + r_2 k) = f(z_1) + f(z_2),$$

is proved.

1. If $x \in F$, then r must be zero in (4.1.3), and (4.1.4) gives $f(x) = g(x)$ for every $x \in F$, it means $f|_F = g$ is proved.

2. Here we consider three cases:

Case 1. $r = 0$.

We have seen that $f(x) = g(x)$, and we get $q(x) \leq f(x) \leq p(x)$ for every $x \in F$.

Case 2. $r > 0$. Since $k \in [a, b] \cap [d, c]$, that is

$$d = \sup_{y \in F} \{-g(y) + q(y + x_0)\} \leq k \leq b = \inf_{y \in F} \{-g(y) + p(y + x_0)\},$$

where $d \leq k \leq b$. Thus,

$$-g(y) + q(y + x_0) \leq k \leq -g(y) + p(y + x_0)$$

for every $y \in F$. Replacing y by $\frac{y}{r}$, where $r > 0$ and multiplying both sides by $r > 0$, we get

$$q(y + rx_0) \leq g(y) + rk \leq p(y + rx_0)$$

for every $y \in F$. Therefore $q(x) \leq f(x) \leq p(x)$ for every $x \in F_1$, where $r > 0$.

Cases 3. $r < 0$.

Let $r = -t$, where $t > 0$ and since $k \in [a,b] \cap [d,c]$, then $a \leq k \leq b$ and $dk \leq c$. Therefore $a \leq k \leq c$ and Consequently ,

$$a = sup_{y \in F} \{ g(y) - p(y - x_0) \} \leq k \leq c = inf_{y \in F} \{ g(y) - q(y - x_0) \},$$

or

$$g(y) - p(y - x_0) \leq k \leq g(y) - q(y - x_0),$$

for every $y \in F$ or

$$-p(y - x_0) \leq - g(y) + k \leq -q(y - x_0),$$

for every $y \in F$.

Replacing y by $\frac{y}{t}$, where $t > 0$ and multiplying both sides by $t > 0$, we get

$$-p(y - tx_0) \leq - g(y) + t\,k \leq -q(y - tx_0),$$

for every $y \in F$ or

$$-p(y + rx_0) \leq - g(y) - r\,k \leq -q(y + rx_0),$$

for every $y \in F$. This means that

$$q(y+rx_0) \leq g(y) + rk \leq p(y+rx_0),$$

for every $y \in F$. Therefore

$$q(x) \leq f(x) \leq p(x),$$

for every $x \in F_1$, where $r < 0$, as required. This proves the lemma completely.■

Theorem 4.3 *Suppose that F and E_1 be two linear subspaces of E such that $F \subseteq E_1 \subseteq E$. Assume that g is a linear functional on F such that*

$$g(x) \leq p(x + y) - q(y)$$

for all $x \in F$ and for each $x \in E_1$. Then there's the linear functional f on E_1 such that $f|_F = g$ and $q(x) \leq f(x) \leq p(x)$ for each $x \in E_1$.

Proof. Lemma 4.4. and Zorn's Lemma provide direct support for the proof (See Kolmogorov [1], p. 134 for this argument).■

Remark 4.2 It is worth noting that such an extension of a linear functional g on F does not always exist on the whole space E.

In other words, if the inequality 4.1.1 is not valid on the whole space E but for some linear subspace E_1 containing F, then by using the above theorem, we can extend g on the space E_1 which preserves conditions in Theorem 4.1, i.e., there exists a linear functional f on E_1 such that

$$f|_F = g \text{ and } q(x) \leq g(x) \leq p(x)$$

for every $x \in E_1$, and we cannot extend g to the whole space E. However, it is valid on the whole space E if the inequality (4.2.1) is valid on E, it means that

$$g(x) \leq p(x+y) - q(y)$$

for every $x \in F$ and for each $y \in E$.■

In the following, we give a simple example to explain the above discussion

Example 4.2 *Let* $E = \mathbb{R}^4$

$$, p(x) = \sqrt{\sum_{i=1}^{4} x_i^2}$$

and

$$q(x) = -\sqrt{\sum_{i=1}^{4} x_i^2} + x_4$$

every $x \in \mathbb{R}^4$, *Also, let* $F = \{(x_1,0,0,0): x_1 \in \mathbb{R}\}$ *be a subspace of* $\mathbb{R}^4$ *and we define a functional*

$$g \in F \text{ as } g(x) = x_1 \text{ for each } x = (x_1,0,0,0) \in F.$$

Then we show that there exists a functional f *on a subspace* $E_1 = \mathbb{R}^3$ *as in*

$$f|_F = g \text{ and } q(x) \leq f(x) \leq p(x)$$

for each $x \in \mathbb{R}^3$. *Moreover, there does not exists a linear functional* f *on* $\mathbb{R}^4$ *such that* $f|_F = g$ *and* $q(x) \leq f(x) \leq p(x)$ *for each* $x \in \mathbb{R}^4$.

It is simple to demonstrate that p *is a sub-banach on* $\mathbb{R}^4$ *and* q *is a super-banach on* $\mathbb{R}^4$.

First, we demonstrate that the following criteria are fulfilled:

1) $g(x) \leq p(x+y) - q(y)$ *for each* $x,y \in F$.
2) *for all* $y \in F_1 = \mathbb{R}^2$, *where*

$$F_1 = \mathbb{R}^2 = \{(x_1,x_2,0,0) : x_1,x_2 \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^4$.

3) *There exists a functional* $f_1 \in F_1$ *such that*

$$f_1|_F = g \text{ and } q(x) \leq f_1(x) \leq p(x)$$

for every $x \in F_1$.

In fact, Let $x = (x_1,0,0,0) \in F$ *and* $y = (y_1,0,0,0) \in F$, *then*

$$g(x) = x_1,\ p(x+y) = |x_1+y_1| \text{ and } q(y) = -|y_1|.$$

Therefore

$$g(x) = x_1 \leq |x_1| \leq |x_1| - |y_1| + |y_1| \leq |x_1 + y_1| + |y_1| \leq \sqrt{|x_1 + y_1|^2} + |y_1| = p(x+y) - q(y),$$

Thus. (1) is proved.

Let $x = (x_1,0,0,0) \in F$ *and* $y = (y_1,y_2,0,0) \in F_1$,

then

$$g(x_1) = x_1, p(x+y) = \sqrt{(x_1+y_1)^2 + y^2{}_2} \text{ and } q(y) = -\sqrt{y^2{}_1 + y^2{}_2}$$

Hence,

$$g(x) = x_1 \leq |x_1| \leq |x_1| - |y_1| + |y_1| \leq |x_1 + y_1| + |y_1| \leq \sqrt{(x_1+y_1)^2 + y^2{}_2} + \sqrt{y^2{}_1 + y^2{}_2} \leq p(x+y) - q(y),$$

Thus, (2) is proved.

Now from Lemma 2.4, we conclude that

$$q(x) \leq p(x)$$

for each $x \in E$ *and*

$$q(x) \leq g(x) \leq p(x)$$

for each $x \in F$. *By using Theorem 4.3, then there exists a functional* $f_1 \in F_1$ *such that*

$$f_1|_F = g \text{ and } q(x) \leq f_1(x) \leq p(x)$$

for all $x \in F_1$, *now (3) is proved.*

Second, it easy to verify that the linear functional f_1 *on* F_1, *which satisfies the above conditions is defined as*

$$f_1(x) = x_1$$

for each $x = (x_1,x_2,0,0) \in F_1$. *Now we show that*

$$g(x) \leq p(x+y) - q(y)$$

for each $x \in F$ *and for every* $y \in F_2$, *where* $F_2 = \{(x_1,x_2,x_3,0) : x_1,x_2,x_3 \in \mathbb{R}\}$ *is a subspace of* $\mathbb{R}^4$ *as follows. let*

$$x = (x_1,0,0,0) \in F \text{ and } y = (y_1,y_2,y_3,0) \in F_2,$$

then

$$g(x) = x_1, p(x+y) = \sqrt{(x_1+y_1)^2 + y^2{}_2 + y^2{}_3} \text{ and } q(y) = -\sqrt{y^2{}_1 + y^2{}_2 + y^2{}_3},$$

Therefore

$$g(x) = x_1 \leq |x_1| \leq |x_1| - |y_1| + |y_1| \leq |x_1 + y_1| + |y_1| \leq \sqrt{(x_1 + y_1)^2 + y^2{}_2 + y^2{}_3} + \sqrt{y^2{}_1 + y^2{}_2 + y^2{}_3} \leq p(x+y) - q(y).$$

Again, by using Theorem 4.3, there exists a functional $f_2 \in F_2$ such that

$$f_2|_F = g$$

and

$$q(x) \leq f_2(x) \leq p(x)$$

for all $x \in F_2$.

Third, it easy to verify that the linear functional f_2 on F_2, which satisfies the above conditions is defined as

$$f_2(x) = x_1$$

for each $x = (x_1,x_2,x_3,0) \in F_2$.

We now show that the linear functional f_2 on F_2 does not satisfy the inequality

$$f_0(x) \leq p(x+y) - q(y)$$

on the whole space $E = \mathbb{R}^4$. Consider that the above inequality is valid for every $x \in F$ and for each $y \in E = \mathbb{R}^4$. Then we can take

$$x = (10,0,0,0) \in F \text{ and } y = (0,0,0,1) \in E.$$

So,

$$f_0(x) = 10,\ p(x+y)\ \sqrt{101, q(y)} = 1$$

$$\textit{and consequently}$$

$$p(x+y) - q(y) \sim= 9.489 < 10 = f_0(x),$$

which gives a contradiction with the assumption that

$$g(x) \leq p(x+y) - q(y)$$

for each $x \in F$ and for every $y \in E = \mathbb{R}^4$. Finally, by using Theorem 4.1 and Remark 4.1, There is no linear functional extension of f_2 on the space E, which preserves the required conditions. ■

Theorem 4.4 *Let $g \in Lin(F)$ such that*

$$g(x) \leq p(x+y) - q(y),$$

for each $x \in F$ and for each $x \in E$. Then there is a linear functional f on E as in

$$f|_F = g \text{ and } q(x) \leq f(x) \leq p(x),$$

for each $x \in E$.

Proof. The proof follows directly from Theorem 4.3 with $E = E_1$. ■

4.4 One Other Proof of The Sandwich Theorem.

In the following, we introduce third proof for the Theorem 4.1. Define the two non-empty sets

$$\Gamma_1 = \{f \in Lin(E) : f|_F = g \text{ and } f(x) \leq p(x) \text{ for each } x \in E\}$$

and

$$\Gamma_2 = \{f \in Lin(E) : f|_F = g \text{ and } q(x) \leq f(x) \text{ for each } x \in E\}.$$

Remark 4.3 If $g \in F$ is a functional and

$$q(x) \leq g(x) \leq p(x)$$

for each $x \in F$, as in

$$q(x) \leq p(x)$$

for each $x \in E$. Then, using the extended format of the ″Hahn-Banach theorem″ for sublinear functionals (refer Theorem 3.2) and the extended version of the ″Hahn-Banach theorem″ for superlinear functionals (see Theorem 3.6) we conclude that $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$.■

Now we establish the following theorems.

Theorem 4.5 *A condition that would be both necessary and sufficient for $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ is that*

$$g(x) \leq T(x)$$

for each $x \in F$, where

$$T(x) = inf_{y \in F} \{p(x + y) - q(y)\}.$$

Proof. Consider that $\Gamma_1 \cap \Gamma_2 \neq \emptyset$, then there exists a functional $f \in \Gamma_1$ and $f \in \Gamma_2$. This means that

$$f|_F = g \text{ , } f(x) \leq p(x)$$

and

$$q(x) \leq f(x)$$

for each $x \in E$. So, $q(u) \leq f(u)$ for every $u \in E$ and consequently

$$q(-u) \leq f(-u) = -f(u), \text{ or } f(u) \leq -q(-u)$$

for each $u \in E$. Also,

$$f(v) \leq p(v)$$

for each $v \in E$. Therefore from the linearity of f, we have

$$f(u+v) = f(u)+f(v) \leq p(v)-q(-u).$$

This means that

$$f(x) \leq p(x-u)-q(-u), \text{ or } f(x) \leq p(x+y)- q(y),$$

where $x = u + v \in E$ and $y = -u \in E$. Since $f|_F = g$, then

$$g(x) \leq p(x + y) - q(y)$$

for each $x \in F$ and $y \in E$. Taking the infimum over $y \in E$, $g(x) \leq T(x)$ for each $x \in F$. Conversely, let

$$T(x) = inf_{y \in E} \{p(x + y) - q(y)\}$$

for each $x \in E$ and $g(x) \leq T(x)$ for each $x \in F$.

It is simple to prove that T is a sublinear functional on E. Using the extension form of the ″Hahn-Banach theorem″ for sublinear functionals, There is a linear functional that exists f on E such that

$$f/_F = g \text{ and } f(x) \leq T(x)$$

for each $x \in E$. Hence from Lemma 4.1 we conclude that

$$q(x) \leq f(x) \leq p(x)$$

for each $x \in E$. Hence $L \in \Gamma_1$ and $L \in \Gamma_2$, it means that

$$\Gamma_1 \cap \Gamma_2 \neq \emptyset.$$

the theorem follows. ■

Theorem 4.6 *If $f_0 \in Lin(F)$ where F is a linear subspace of E as in*

$$q(x) \leq g(x) \leq p(x)$$

for every $x \in F$. Then the a necessary requirement that a functional $f \in Lin(E)$ exists as in

$$f/_F = g \text{ and } q(x) \leq f(x) \leq p(x)$$

for every $x \in E$ is that $g(x) \leq T(x)$ for every $x \in F$, where

$$T(x) = inf_{y \in E} \{p(x+y)-q(y)\}$$

for each $x \in E$.

Proof. Since $g \in Lin(F)$ and

$$q(x) \leq g(x) \leq p(x)$$

for every $x \in F$, then from Remark 4.3, we obtain that $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$. So let $f \in Lin(E)$ as in

$$f|_F = g \text{ and } q(x) \leq f(x) \leq p(x),$$

for each $x \in E$, then $f \in \Gamma_1$ and $f \in \Gamma_2$, it means $f \in \Gamma_1$ and $f \in \Gamma_2$.

Consequently, $f \in \Gamma_1 \cap \Gamma_2$ and then $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ and from Theorem 4.5,

$$g(x) \leq T(x),$$

for each $x \in F$, where

$$T(x) = \inf_{y \in E} \{p(x+y) - q(y)\}$$

for every $x \in E$. Conversely, let

$$g(x) \leq T(x)$$

for each $x \in F$, where

$$T(x) = \inf_{y \in E} \{p(x + y) - q(y)\}$$

for every $x \in E$, then from Theorem 4.4.1, $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ and consequently There is a functional $f \in Lin(E)$ as in

$$f/_F = g \text{ and } q(x) \leq f(x) \text{ and } f(x) \leq p(x)$$

for every $x \in E$. This implies that a functional system exists $f \in Lin(E)$ as in

$$f/_F = g \text{ and } q(x) \leq f(x) \leq p(x)$$

for each $x \in E$.■

5. REFERENCES

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