Harmonic Type Sums and Their Arithmetic Properties

by

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Koç University

Graduate School of Sciences and Engineering

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To my mother and grandparents...

ABSTRACT

Harmonic Type Sums and Their Arithmetic Properties
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Doctor of Philosophy in Mathematics
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The *n-th harmonic number* is the sum of the first n terms of the harmonic series, namely

$$h_n = \sum_{k=1}^{n} \frac{1}{k},$$

for $n \geq 1$. These numbers enjoy several arithmetic properties. For instance, it was first shown by Theisinger in 1915 that the n-th harmonic number is not an integer, when $n \geq 2$. A generalization of these numbers was introduced by Conway and Guy in 1996. They defined *the n-th hyperharmonic number of order r* as

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)},$$

where $h_n^{(1)} = h_n$ and $n, r \in \mathbb{Z}^+$ such that $n \geq 1$ and $r \geq 2$. In 2007, it was conjectured by Mező that there does not exist any hyperharmonic integer except 1. In the same paper, he also proved that $h_n^{(r)}$ is not an integer, for any integer $n > 1$ and $r \leq 3$. In this thesis, we improve the known upper bounds for the order of non-integer hyperharmonic numbers. In particular, we show that for any $r \leq 35\,001$ and $n > 1$, the corresponding hyperharmonic number is not an integer. Also, if n is even or a prime power, or r is odd, then the same property holds for $h_n^{(r)}$. Moreover, using the prime numbers in short intervals, we deduce that almost all (n, r) tuples give us hyperharmonic numbers which are not integers. On the contrary, our final analysis leads to the existence of hyperharmonic integers. More precisely, for $r = 64 \cdot (2^{\alpha} - 1) + 32$, the hyperharmonic number $h_{33}^{(r)}$ is integer for 153 different values of $\alpha \pmod{748\,440}$, where the smallest r is equal to $64 \cdot (2^{2659} - 1) + 32$. This construction yields the infinitude of hyperharmonic integers, and refutes the conjecture of Mező.

ÖZETÇE

Harmonik Tipi Toplamlar ve Aritmetik Özellikleri
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Harmonik serinin ilk n teriminin toplamına $n.$ *harmonik sayı* denir; şöyle ki $n \geq 1$ için,

$$h_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Bu sayılar, bir takım aritmetik özellikleri sağlar. Örneğin, $n \geq 2$ için $n.$ harmonik sayı tam sayı değildir ve bu ilk kez Theisinger tarafından 1915 yılında gösterilmiştir. Bu sayıların bir genelleştirmesi ise, 1996 yılında Conway ve Guy tarafından verilmiştir. Onlar, $h_n^{(1)} = h_n$ olma koşuluyla, $r.$ *dereceden* $n.$ *hiperharmonik sayıyı*

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}$$

şeklinde tanımlamışlardır. Bu tanımda, $n, r \in \mathbb{Z}^+$ için $n \geq 1$ ve $r \geq 2$ şartları da sağlanır. Mező 2007 yılında, bu sayıların 1 dışında tam sayı olmadığını öngören bir sanı ortaya atmıştır. Aynı sanıyı belirttiği makalede, $n > 1$ ve $r \leq 3$ için $h_n^{(r)}$ sayısının tamsayı olmadığını da göstermiştir. Bu tezde, tamsayı olmayan hiperharmonik sayıların dereceleri için bilinen üst sınırlar geliştirilmiştir. Özellikle, tüm $r \leq 35\,001$ ve $n > 1$ değerleri için karşılık gelen hiperharmonik sayıların tamsayı olmadığı kanıtlanmıştır. Ayrıca, n çift bir sayıya veya bir asal kuvvete eşitse, ya da r tek sayı ise, $h_n^{(r)}$ sayısının aynı özelliği sağladığı gösterilmiştir. Üstelik hemen hemen tüm (n, r) ikililerine karşılık gelen hiperharmonik sayıların tam sayı olmadığı, kısa aralıklardaki asal sayılar kullanılarak ispatlanmıştır. Tüm bunların aksine, son analizimiz hiperharmonik tamsayıların varlığını göstermiştir. Özel olarak belirtmek gerekirse, 153 tane farklı $\alpha \pmod{748\,440}$ değeri için $r = 64 \cdot (2^\alpha - 1) + 32$ iken $h_{33}^{(r)}$ sayısının tamsayı olduğu görülmüştür. Bu özelliği sağlayan en küçük r değeri ise $64 \cdot (2^{2659} - 1) + 32$ sayısına eşittir. Belirtilen bu r değerleri, hiperharmonik tamsayıların sonsuzluğuna işaret eder ve Mező'nün sanısını çürütür.

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$\mathrm{dlog}_2(t, n)$	The solution $\beta \pmod{(\mathrm{ord}_n(2))}$ of $2^\beta \equiv t \pmod{n}$, if it exists
$\mathbb{F}_p$	The finite field with p elements, where p is prime
h_n	The n-th harmonic number
$h_n^{(r)}$	The n-th hyperharmonic number of order r
$I(n, r)$	The set of integers from r to $n + r - 1$
$I_a(n, r)$	The set $I(n, r) \cap a\mathbb{Z}$
$\mathcal{M}_p(A)$	The number of integers in A which are divisible by $p^{\mu_p(A)}$
$\mu_p(A)$	The maximum p-adic valuation among the elements of A
$\mathbb{N}$	The set of natural numbers
$(n_\alpha, \ldots, n_0)_b$	The b-ary representation of n where $b \geq 2$ and $b^\alpha \leq n < b^{\alpha+1}$
$\mathcal{O}(\cdot)$	The big-O notation
$o(\cdot)$	The little-o notation
$\mathrm{ord}_m(2)$	The order of 2 in the group $\left(\mathbb{Z}/m\mathbb{Z}\right)^\times$ where m is odd
$\mathbb{P}$	The set of prime numbers
$p^{(n)}$	The greatest prime that is less than or equal to n
$\pi(x)$	The number of primes that is less than or equal to x
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\Re(z)$	The real part of a complex number z
$\mathcal{S}(x)$	The number of (n, r) tuples in $[1, x] \times [1, x]$ where $h_n^{(r)} \notin \mathbb{Z}$
$\nu_p(q)$	The p-adic valuation of a rational number q
$\lfloor x \rfloor$	The floor of x
$\{x\}$	The fractional part of x, i.e. $\{x\} = x - \lfloor x \rfloor$
$\mathbb{Z}$	The set of integers
$\left(\frac{\cdot}{p}\right)$	Legendre symbol modulo p

Chapter 1

INTRODUCTION

Any partial sum of the harmonic series is called a harmonic number. More precisely, the *n-th harmonic number* is the sum of the fractions of the first n positive integers, that is to say

$$h_n := \sum_{k=1}^{n} \frac{1}{k}.$$

These numbers are endowed with many analytic and arithmetical properties. As an arithmetical one, Theisinger [Theisinger, 1915] proved that h_n is never an integer when $n > 1$. Moreover, Kürschák [Kürschák, 1918] showed that for any different positive integers $m, n \geq 1$, the corresponding difference of harmonic numbers $h_m - h_n$ is also a non-integer rational number. These type of differences were also studied in [Erdős and Niven, 1946], where they proved that $h_m - h_n \neq h_k - h_s$, unless $m = k$ and $n = s$. Recently, the same difference was studied in [Pambuccian, 2008], where he observed that $h_m - h_n$ is not an integer under the assumptions of Peano arithmetic, excluding the axiom of induction. Apart from the non-integerness property of harmonic numbers, it was known by the partial summation that

$$h_n = \log n + \gamma + \mathcal{O}\left(\frac{1}{n}\right).$$

A better estimation using Bernoulli numbers can be given as

$$\begin{aligned} h_n &\sim \log n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}} \\ &= \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \cdots, \end{aligned}$$

where n goes to infinity, γ denotes Euler's constant and B_m is the m-th Bernoulli number. Linking the sum of the first n numbers with the sum of their reciprocals,

Ramanujan gave the interesting asymptotic formula

$$h_n \sim \frac{1}{2}\log 2t_n + \gamma + \frac{1}{12t_n} - \frac{1}{120t_n^2} + \frac{1}{630t_n^3} - \frac{1}{1680t_n^4} + \frac{1}{2310t_n^5} - \frac{191}{360360t_n^6} + \frac{29}{30030t_n^7} - \frac{2833}{1166880t_n^8} + \frac{140051}{17459442t_n^9} - \cdots$$

as n tends to infinity, where

$$t_n = \frac{n(n+1)}{2} = 1 + 2 + \cdots + n$$

is the n-th triangular number. However, the proof of this asymptotic formula was missing in Ramanujan's notes and it was later obtained by Berndt in [Berndt, 1998, pg. 531–532]. From the analytical perspective, harmonic numbers have been also considered as the coefficients of Dirichlet series. In particular, it was Euler who first studied the following series (see [Euler, 1776, pg. 164])

$$\zeta_h(s) = \sum_{n=1}^{\infty} \frac{h_n}{n^s},$$

where $\Re(s) > 1$. Such a series is called an Euler sum of harmonic numbers, when $s \geq 2$ is a positive integer, and it satisfies the following striking equation which can be found in [Berndt, 1985, pg. 252]:

$$2\zeta_h(k) = (k+2)\zeta(k+1) - \sum_{\ell=1}^{k-2} \zeta(k-\ell)\zeta(\ell+1). \tag{1.1}$$

In this relation, k can be any integer which is greater than 1, noting that the empty sum is considered to be zero when $k = 2$. Also, $\zeta(s)$ in (1.1) denotes the celebrated Riemann's zeta function which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$. As a consequence of equation (1.1), one can obtain different representations of some irrational numbers, like

$$\mathcal{A} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{h_n}{n^2}, \qquad \pi = \left(\frac{1}{72}\sum_{n=1}^{\infty} \frac{h_n}{n^3}\right)^{\frac{1}{4}},$$

where $\mathcal{A}$ denotes the Apéry's constant and it is equal to

$$\mathcal{A} = \zeta(3) \approx 1.2020569.$$

A recursive generalization of harmonic numbers was introduced by Conway and Guy in [Conway and Guy, 1996]. They defined *the n-th hyperharmonic number of order r* as

$$h_n^{(r)} := \sum_{k=1}^{n} h_k^{(r-1)},$$

for $n \geq 1$ and $r \geq 2$, where the initial case being $h_n^{(1)} = h_n$. In the same book, they also gave the following combinatorial formula for $h_n^{(r)}$, which reveals the relationship between hyperharmonic numbers and the multiplication of a binomial coefficient with a corresponding harmonic difference:

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}). \tag{1.2}$$

For a fixed $r \geq 2$, equation (1.2) yields the following asymptotic

$$h_n^{(r)} \sim \frac{n^{r-1}\log n}{(r-1)!},$$

as n goes to infinity. This fact can also be found in [Dil and Mező, 2010, Lemma 2]. Another combinatorial approach for hyperharmonic numbers was given in [Benjamin et al., 2003]. Using r-Stirling numbers, they gave several identities related to hyperharmonic numbers. Besides these combinatorial results, hyperharmonic numbers have also been studied in an analytical way. Namely, Euler sums of hyperharmonic numbers and their connection with Hurwitz-zeta function were given in [Dil and Mező, 2010, Kamano, 2011, Boyadzhiev and Dil, 2015]. Furthermore, some functional equations regarding the Euler sums of hyperharmonic numbers and the special values of the Riemann zeta function were deduced in [Alkan and Göral, 2018]. Analogous to harmonic numbers, the non-integerness property of hyperharmonic numbers was also investigated in [Mező, 2007, Conjecture 7], where Mező stated the following conjecture:

"None of the hyperharmonic numbers can be integers (for $r, n \geq 2$)."

In the same paper, he also proved this statement for the orders $r = 2, 3$. The extensions of Mező's result were obtained in [Amrane and Belbachir, 2010], [Amrane and Belbachir, 2011], where they showed that $h_n^{(r)}$ is not an integer, for $r \leq 25$. They also verified this conjecture for some certain sets of (n, r) tuples.

In this thesis, our main concern is to find an answer for the open problem of Mező. To do so, we will consider the p-adic valuations of hyperharmonic numbers, for some certain prime numbers p. Unlike Theisinger's idea in [Theisinger, 1915], we will first try to find primes in short intervals, where they may lead to a negative p-adic valuation for $h_n^{(r)}$. More precisely, we consider the intervals of the form $(x - \Phi(x), x]$ which contain prime numbers for a monotonically non-decreasing function $\Phi(x)$ and sufficiently large real number x. An estimation for $\Phi(x)$ can be given by the distribution of prime numbers. Namely, by the well-known Prime Number Theorem, which was independently proved by Hadamard [Hadamard, 1896] and de la Valée Poussin [de La Vallée Poussin, 1896, de La Vallée Poussin, 1899], we know that

$$\pi(x) \sim \frac{x}{\log x},$$

as x tends to infinity. This asymptotic formula yields that $\Phi(x)$ can be taken as $\Phi(x) = \varepsilon x$, for any $\varepsilon > 0$ and sufficiently large $x \in \mathbb{R}$, depending on ε. In order to replace $\Phi(x) = \varepsilon x$ with a relatively slowly increasing function, one can use sieve theoretic methods which may lead to an estimate of the form $\Phi(x) = o(x)$. Applying sieving techniques, Baker, Harman and Pintz [Baker et al., 2001] showed that one can take $\Phi(x) = x^{0.525}$, for all sufficiently large x. To obtain further improvements on the function $\Phi(x)$, one may weaken the condition "for all sufficiently large x", by considering "almost all" short intervals. Here, by saying almost all intervals, we mean that for all integers $x \leq X$, with only $o(X)$ many exceptions, there exists at least one prime in the interval $(x - \Phi(x), x]$, for all sufficiently large $X \in \mathbb{R}$. In that case, the current best known result is due to Jia [Jia, 1996], where he proved that almost all intervals $(x - x^\theta, x]$ contain prime numbers, for any $\theta > \frac{1}{20}$.

Now, we will give our first theorem which states that almost all hyperharmonic numbers are not integers. Using the primes in almost all short intervals, we will obtain a quantitative estimate for the number of tuples $(n,r) \in [1,x] \times [1,x]$, where the corresponding hyperharmonic numbers $h_n^{(r)}$ are not integers.

Theorem 1.1. *Let n, r be positive integers.*

1. *For a fixed $n > 1$, there are infinitely many values of r such that $h_n^{(r)} \notin \mathbb{Z}$.*

2. *Let $d \geq 1$ be a fixed natural number. Then there is an integer n_d, depending on d, such that $n \geq n_d$ together with $r \leq dn$ imply that $h_n^{(r)} \notin \mathbb{Z}$. Moreover, for any constant $C \in \left(0, \frac{1}{2}\right)$, there exists a natural number n_0, depending on C, such that if $n \geq n_0$ and $r \leq Cn^{1.475}$ then $h_n^{(r)}$ is not an integer.*

In particular for a fixed r, there are only finitely many possible values of n where $h_n^{(r)}$ may be an integer. Moreover if

$$\mathcal{S}(x) := \left|\left\{(n,r) \in [1,x] \times [1,x] \,\middle|\, h_n^{(r)} \notin \mathbb{Z}\right\}\right|$$

then for any $A > 0$, we have

$$\mathcal{S}(x) = x^2 + \mathcal{O}_A\left(\frac{x^{\frac{80}{59}}}{(\log x)^A}\right),$$

where the implied constant depends only on A.

Our next theorem depends on combinatorial techniques and eliminates most of n and r values, depending on their parity. As usual, the function $\{x\} = x - \lfloor x \rfloor$ represents the fractional part of a real number x.

Theorem 1.2. *Let $n, r \geq 2$ be given integers. Assume that m is the highest power of a prime number p such that p^m divides at least one of the n consecutive numbers $r, r+1, \ldots, r+n-1$. If p^m divides exactly one of $r, r+1, \ldots, r+n-1$ and the condition*

$$\left\{\frac{r-1}{p^k}\right\} + \left\{\frac{n}{p^k}\right\} < 1$$

holds for some k with $p^k \leq n$, then $h_n^{(r)} \notin \mathbb{Z}$. In particular, if n is even or r is odd, or n is a prime power, then $h_n^{(r)}$ is not an integer.

Our final theorem on non-integer hyperharmonic numbers is of computational flavour and it contains some effective bounds on n and r, which are consequences of our previous theorems.

Theorem 1.3. *If one of the following conditions on n and r is satisfied, then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer:*

1. $r \leq 20n$.
2. $r \leq 35.001$.
3. $1 < n \leq 32$.
4. $n = 3p^\alpha$ *where* $\alpha \geq 1$, p *is a prime and* $p \equiv \pm 5 \pmod{12}$.
5. $n = 5p^\alpha$ *where* $\alpha \geq 1$, p *is a prime and*

$$p \equiv \begin{array}{l} 7, 11, 13, 14, 19, 21, 22, 23, 26, 28, 31, 33, 38, 39, \\ 41, 42, 44, 46, 52, 53, 56, 57, 61, 62, 63, 66, 67, 69, \\ 76, 78, 79, 82, 83, 84, 88, 89, 92, 93, 99, 101, 103, \\ 104, 106, 107, 112, 114, 117, 119, 122, 123, 124, \\ 126, 131, 132, 134, 138 \end{array} \pmod{145},$$

Moreover, the set of integers r such that $h_{33}^{(r)} \notin \mathbb{Z}$ contains a set of density

$$\frac{143}{144} \approx 0.993.$$

According to the previously mentioned results, it seems unlikely to have any hyperharmonic number which is also an integer. However in this thesis, it is proved that there are infinitely many of them. In particular, we prove the following theorem:

Theorem 1.4. *There are infinitely many values of $r \in \mathbb{Z}_{>0}$ such that $h_{33}^{(r)}$ is an integer. More precisely, $h_{33}^{(r)}$ is an integer for*

$$r = 64 \cdot (2^{\alpha + k \cdot 748\,440} - 1) + 32,$$

where $k \geq 0$ *is an integer,* α *takes* 153 *different values in* $\mathbb{Z} \cap [0, 748\,439]$ *and the minimum of these* r *values is equal to* $64 \cdot (2^{2659} - 1) + 32$.

The structure of the thesis is as follows: In Chapter 2, we introduce the mathematical tools that we commonly use in this thesis. After that, we consider the non-integer hyperharmonic numbers from three different perspectives, i.e. analytic, combinatorial and algebraic. Together with these approaches, we mention our computational results in Chapter 3. In particular, we prove Theorems 1.1, 1.2 and 1.3 in that chapter. Chapter 4 is dedicated to the existence of hyperharmonic integers, namely the proof of Theorem 1.4. Some concluding remarks related to our results are explained in Chapter 5. Finally, we give the corresponding order of the hyperharmonic integers in Appendix A.

1.1 Notation

Throughout this thesis, n and r always denote positive integers. Usually, we consider the case $n \geq 2$, unless it is stated otherwise, as $h_1^{(r)} = 1$, for any $r \geq 1$. We take $r \geq 2$ in most of our cases, since the case $r = 1$ gives us the usual harmonic numbers. Also, $\mathbb{N}$ is the set of natural numbers (or the set of non-negative integers), and $\mathbb{P}$ is the set of prime numbers. We always denote p as a prime number. Moreover, define

$$p^{(n)} := \max \{p \leq n \,|\, p \in \mathbb{P}\},$$

for any $n \geq 2$. In the entire thesis, $\nu_p(q)$ gives us the p-adic valuation of a given rational number q; that is for any $q \in \mathbb{Z}$,

$$\nu_p(q) := \begin{cases} m, & \text{if } p^m \parallel q \\ \infty, & \text{if } q = 0, \end{cases}$$

and $\nu_p(q) = \nu_p(a) - \nu_p(b)$, where $q = \dfrac{a}{b}$ and $b \neq 0$. The notation $p^m \parallel a$ means that $p^m \mid a$ holds, but $p^{m+1} \nmid a$. For any $b \geq 2$, the b-ary representation of a natural number n will be given as $(n_\alpha, n_{\alpha-1}, \ldots, n_0)_b$, for some non-negative integer α. We also use the big-$\mathcal{O}$ notation in the following sense: suppose that there exists

a constant $C > 0$ and a real number $x_0 > 0$ such that for any $x \geq x_0$, we have $|f(x)| \leq C \cdot g(x)$. Then, we write

$$f(x) = \mathcal{O}\left(g(x)\right) \quad \text{or} \quad f(x) \ll g(x).$$

If the constant $C > 0$ depends on some given parameter $A > 0$, then we say that

$$f(x) = \mathcal{O}_A\left(g(x)\right) \quad \text{or} \quad f(x) \ll_A g(x).$$

Next, we set the notation $f(x) = o(g(x))$, which means that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

In this thesis, we fix $f(x) \sim g(x)$ as

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Furthermore, $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ represent the floor and the ceil functions, respectively. The function $\{x\}$ gives us the fractional part of x, that is $\{x\} = x - \lfloor x \rfloor$. Here, we say that $I(n, r)$ is the set of integers $\{r, \ldots, n + r - 1\}$. Also, we denote $I_a(n, r)$ as the set of multiples of a in $I(n, r)$, i.e. $I_a(n, r) = I(n, r) \cap a\mathbb{Z}$. Define $\mathrm{ord}_m\,(2)$ as the order of 2 in the group $\left(\mathbb{Z}/m\mathbb{Z}\right)^\times$, for a given odd integer $m \geq 3$. Finally, let $\mathrm{dlog}_2\,(t, n)$ represent the solution $\beta \pmod{(\mathrm{ord}_n\,(2))}$ of the equation $2^\beta \equiv t \pmod{n}$, where $t \in \mathbb{Z}$ and $n \geq 3$ is odd. We use this representation, if such a solution β exists.

Chapter 2

PRELIMINARIES

This chapter contains the material that we will use in this thesis. In Section 2.1, we will give some basic facts related to hyperharmonic numbers. Section 2.2 is dedicated to three theorems from the literature that we use to prove Theorems 1.1, 1.3 and 1.4. The next section illustrates some combinatorial facts that will be useful to prove Theorem 1.2. Finally, we give a short survey on primes in short intervals in Section 2.4, which is necessary to show the last part of Theorem 1.1.

2.1 Hyperharmonic Numbers

We will mention two main facts about hyperharmonic numbers. The first one is the fundamental equation (1.2) of hyperharmonic numbers which is given in [Conway and Guy, 1996]. Here we provide two proofs of it:

Proposition 2.1. *Let $n \geq 1$ and $r \geq 2$ be two natural numbers. Then we have*

$$h_n^{(r)} = \binom{n+r-1}{r-1} \left(h_{n+r-1} - h_{r-1}\right). \tag{2.1}$$

Proof. We will prove it by induction on $n + r - 1$. To do this, we first show that equation (2.1) is satisfied for all $n \geq 1$ and $r = 2$. This also includes the initial case, when $n + r - 1 = 2$, as $n \geq 1$ and $r \geq 2$. So, consider

$$\begin{aligned} h_n^{(2)} &= \sum_{k=1}^{n} h_k = \sum_{k\leq n}\sum_{\ell\leq k} \frac{1}{\ell} = \sum_{\ell\leq n}\sum_{k\geq \ell} \frac{1}{\ell} \\ &= \sum_{\ell\leq n} \frac{1}{\ell}(n-\ell+1) = \sum_{\ell\leq n}\left(\frac{n+1}{\ell} - 1\right) = (n+1)\sum_{\ell\leq n}\frac{1}{\ell} - (n+1) + 1 \\ &= (n+1)\left(h_n - 1 + \frac{1}{n+1}\right) = \binom{n+2-1}{2-1}(h_{n+1} - h_1). \end{aligned} \tag{2.2}$$

Therefore, equation (2.1) is satisfied for all $n \geq 1$ and $r = 2$. Now, assume that the same condition also holds for all $k \leq n+r-1$. As we proved the assertion for $r = 2$, it is enough to take $r \geq 3$. Observe that

$$\begin{aligned} h_n^{(r)} &= \sum_{k=1}^{n} h_k^{(r-1)} = h_n^{(r-1)} + \sum_{k=1}^{n-1} h_k^{(r-1)} \\ &= h_n^{(r-1)} + h_{n-1}^{(r)}. \end{aligned}$$

Since $r-1 \geq 2$, we deduce by induction that

$$\begin{aligned} h_n^{(r)} &= \binom{n+r-2}{r-2}(h_{n+r-2} - h_{r-2}) + \binom{n+r-2}{r-1}(h_{n+r-2} - h_{r-1}) \\ &= \frac{(n+1)\cdots(n+r-2)}{(r-2)!} \cdot \left(\frac{1}{r-1} + \frac{1}{r} + \cdots + \frac{1}{n+r-2}\right) \\ &\quad + \frac{n}{r-1} \cdot \frac{(n+1)\cdots(n+r-2)}{(r-2)!} \left(\frac{1}{r} + \cdots + \frac{1}{n+r-2}\right) \\ &= \frac{1}{r-1}\binom{n+r-2}{r-2} + \binom{n+r-2}{r-2}(h_{n+r-2} - h_{r-1})\left(1 + \frac{n}{r-1}\right) \\ &= \frac{1}{r-1}\binom{n+r-2}{r-2}(1 + (n+r-1)h_{n+r-2} - (n+r-1)h_{r-1}) \\ &= \frac{n+r-1}{r-1}\binom{n+r-2}{r-2}\left(h_{n+r-2} - h_{r-1} + \frac{1}{n+r-1}\right) \\ &= \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}), \end{aligned}$$

as desired. □

Using generating functions, one can also deduce Proposition 2.1 as follows:

Another Proof of Proposition 2.1. First of all, we will obtain an equality using the generating function of hyperharmonic numbers. So, fix any $r \geq 2$ and define

$$H_r(z) = \sum_{n\geq 1} h_n^{(r)} z^n.$$

We will show that

$$H_r(z) = -\frac{\log(1-z)}{(1-z)^r}, \tag{2.3}$$

when $|z| < 1$. To do so, first recall that

$$-\log(1-z) = \sum_{n\geq 1} \frac{z^n}{n},$$

for $|z| < 1$. Therefore, we get

$$\begin{aligned} H(z) := \sum_{n\geq 1} h_n z^n &= \sum_{n\geq 1}\left(\sum_{k=1}^{n}\frac{1}{k}\right) z^n = \sum_{n\geq 1}\left(\sum_{k=1}^{n}\frac{1}{k}\cdot 1\right) z^n \\ &= \sum_{n\geq 1}\frac{z^n}{n}\cdot\sum_{n\geq 1}(1\cdot z^n) = -\log(1-z)\cdot\sum_{n\geq 1} z^n = -\frac{\log(1-z)}{1-z} \end{aligned}$$

as $\sum_{n\geq 1} z^n = \dfrac{1}{1-z}$, for any $|z| < 1$. Similarly, we have

$$\begin{aligned} H_2(z) = \sum_{n\geq 1} h_n^{(2)} z^n &= \sum_{n\geq 1}\left(\sum_{k=1}^{n} h_k\right) z^n \\ &= \sum_{n\geq 1} h_n z^n \cdot \sum_{n\geq 1} z^n = H(z)\cdot\frac{1}{1-z} = -\frac{\log(1-z)}{(1-z)^2}, \end{aligned}$$

when $r = 2$ and $|z| < 1$. If we assume that equation (2.3) is true for $r-1$, then we see that

$$\begin{aligned} H_r(z) = \sum_{n\geq 1} h_n^{(r)} z^n &= \sum_{n\geq 1}\left(\sum_{k=1}^{n} h_k^{(r-1)}\right) z^n \\ &= \sum_{n\geq 1} h_n^{(r-1)} z^n \cdot \sum_{n\geq 1} z^n = \frac{H_{r-1}(z)}{1-z}. \end{aligned}$$

Thus, we deduce that

$$\sum_{n\geq 1} h_n^{(r)} z^n = -\frac{\log(1-z)}{(1-z)^r}, \tag{2.4}$$

for $|z| < 1$. Now, taking the derivatives of both sides in equation (2.4) yields that

$$\sum_{n\geq 1} n h_n^{(r)} z^{n-1} = \frac{1}{(1-z)^{r+1}} - r\cdot\frac{\log(1-z)}{(1-z)^{r+1}} = \sum_{n\geq 0}\binom{n+r}{r} z^n + r\sum_{n\geq 1} h_n^{(r+1)} z^n.$$

From these equalities, we see that

$$1 + \sum_{n\geq 1}(n+1)h_{n+1}^{(r)} = 1 + \sum_{n\geq 1}\left(\binom{n+r}{r} + r h_n^{(r+1)}\right) z^n.$$

Comparing the coefficients of both sides, we obtain that

$$r h_n^{(r+1)} = (n+1)h_{n+1}^{(r)} - \binom{n+r}{r}, \tag{2.5}$$

for any integer $n \geq 1$. Now, we will show by induction on $r \geq 2$ that any hyperharmonic number $h_n^{(r)}$ satisfies equation (2.1), for any $n \geq 1$. By (2.2), we know that $h_n^{(2)}$ satisfies equation (2.1). So assume that (2.1) holds for some $r > 2$, and any $n \geq 1$. Then by equation (2.5), we get that

$$rh_n^{(r+1)} = (n+1)\binom{n+r}{r-1}(h_{n+r} - h_{r-1}) - \binom{n+r}{r}.$$

Therefore, we have

$$\begin{aligned} h_n^{(r+1)} &= \frac{n+1}{r} \cdot \frac{(n+2)\cdots(n+r)}{(r-1)!}(h_{n+r} - h_{r-1}) - \frac{1}{r}\binom{n+r}{r} \\ &= \binom{n+r}{r}\left(h_{n+r} - h_{r-1} - \frac{1}{r}\right) = \binom{n+r}{r}(h_{n+r} - h_r), \end{aligned}$$

which concludes the proposition. □

Now, we observe the necessary condition for $h_n^{(r)}$ to be an integer, which will be a main tool for Chapters 3 and 4.

Lemma 2.2. *For any $n, r \geq 1$, $h_n^{(r)} \in \mathbb{Z}$ if and only if $\nu_p\left(h_n^{(r)}\right) \geq 0$ for all primes $p \leq n$.*

Proof. Observe that the statement is trivial when $r = 1$, as $h_n^{(r)} = h_n$ in that case. So take $r \geq 2$. Then, the definition of the p-adic valuation leads to the following fact: $h_n^{(r)}$ is an integer if and only if $\nu_p\left(h_n^{(r)}\right) \geq 0$, for all prime numbers p. By equation (1.2), note that

$$\begin{aligned} h_n^{(r)} &= \frac{r(r+1)\cdots(n+r-1)}{n!} \cdot \left(\frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n+r-1}\right) \\ &= \frac{P}{n!} \cdot \left(\frac{\sum_{i=r}^{n+r-1} \frac{P}{i}}{P}\right) = \frac{P_r + P_{r+1} + \cdots + P_{n+r-1}}{n!}, \end{aligned}$$

where $P = r(r+1)\cdots(n+r-1)$ and $P_i = \dfrac{P}{i}$, for any $i \in \{r, r+1, \ldots, n+r-1\}$. This yields that $\nu_p\left(h_n^{(r)}\right) \geq 0$ for all primes $p > n$. □

So, to show the non-integerness of the corresponding hyperharmonic number $h_n^{(r)}$, it is enough to find a prime number $p \leq n$ with the property $\nu_p\left(h_n^{(r)}\right) < 0$.

2.2 Fundamental Concepts from Number Theory

In order to make this thesis self-contained, we will give some of the basic tools from number theory. To begin with, we first recall the partial summation formula which is also given in [Apostol, 1976, Theorem 4.2].

Abel's Summation Formula. *Let $f : \mathbb{Z}_{>0} \to \mathbb{R}$ be any real valued function. Define*

$$F(x) := \begin{cases} \sum_{n \leq x} f(n), & \text{if } x \geq 1, \\ 0, & \text{if } x < 1. \end{cases}$$

Suppose that g is continuously differentiable on $[y, x]$, for some $y \in (0, x)$. Then, we have

$$\sum_{y < n \leq x} f(n)g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t)g'(t)dt.$$

Denote $\left(\frac{a}{p}\right)$ as the *Legendre symbol*, i.e.

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a square in } \mathbb{F}_p, \\ 0, & \text{if } p \mid a, \\ -1, & \text{otherwise.} \end{cases}$$

In the proof of Theorem 1.3, we will use the very well-known Quadratic Reciprocity Law whose proof can be found in [Koblitz, 1994, Proposition II.2.5].

Quadratic Reciprocity Law. *Let p and q be given two odd primes. Then we have*

$$\left(\frac{p}{q}\right) = (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{q}{p}\right).$$

In Chapter 4, we will try to obtain common solutions in a system of congruences with non-coprime moduli in order to find hyperharmonic integers. In this case, we may use the generalized version of the Chinese Remainder Theorem. For completeness, we will also state this theorem here.

Chinese Remainder Theorem (General Version)**.** *Let $m, n > 1$ be positive integers not necessarily coprime to each other. Then for any $a, b \in \mathbb{Z}$, the system of*

congruences

$$x \equiv a \pmod{m}$$
$$x \equiv b \pmod{n}$$

has a unique solution x *modulo* $\operatorname{lcm}(m,n)$ *if and only if* $d \mid a-b$, *where* $d = \gcd(m,n)$.

In fact, the corresponding common solution is

$$x = \frac{ant + bms}{d}$$

modulo $\operatorname{lcm}(m,n)$, where the integers s and t satisfy the equation $ms + nt = d$. Nevertheless, the existence of such a unique solution will be enough for our purposes.

2.3 Some Combinatorial Facts Related to Divisibility

This section is taken from [Göral and Sertbaş, 2017, Section 3] and contains necessary materials to prove Theorem 1.2. We first define the set $I_a(n,r) := a\mathbb{Z} \cap [r, n+r)$, where a is an arbitrary positive integer. As a start for this section, we give a relationship between the number of elements of $I_a(n,r)$ and $I_a(n,1)$, for any $a \in \mathbb{N}$.

Lemma 2.3. *Let* a *be a given natural number and define* $k = |I_a(n,1)| = \left\lfloor \frac{n}{a} \right\rfloor$. *Then we have,*

$$|I_a(n,r)| \in \{k, k+1\}.$$

Moreover if $a \mid n$, *then* $|I_a(n,r)| = k$.

Proof. Let $b = |I_a(n,r)|$ and $c = \left\lceil \frac{r}{a} \right\rceil$. Since $n < (k+1)a$ and $r \leq ca$, we have $n + r < (k+1+c)a$; or in other words

$$(c+b-1)a \leq n+r-1 < (c+k+1)a.$$

Hence $b - 1 < k + 1$ which is equivalent to $b < k + 2$. Also since $ka \leq n$ and $(c-1)a < r$, we have $(c+k-1)a < n+r$. Therefore,

$$(c+k-1)a \leq n+r-1 < (c+b)a$$

which leads to the fact that $k-1 < b$. Thus, $b \in \{k, k+1\}$. Moreover, if a divides n then $n = ka$. Hence $(n+r-1) = ka+r-1 \leq (k+c)a-1$; and this is equivalent to $(b-k-1)a \leq -1$. So b cannot be equal to $k+1$. Thus, $|I_a(n,r)| = b = k = \frac{n}{a}$. $\square$

Definition 2.4 (Maximal power of a prime in a set)**.** Let I be a finite set and $p \in \mathbb{P}$. We define $\mu_p(I) = \max\{\nu_p(s) \mid s \in I\}$ as *the maximal power of p of I*. The number of integers in $\left(p^{\mu_p(I)}\mathbb{Z}\right) \cap I$ is denoted by $\mathcal{M}_p(I)$, and we say that the maximal power of p of I is *unique* if $\mathcal{M}_p(I) = 1$.

Note that for any non-empty set I and $p \in \mathbb{P}$, we have $\mathcal{M}_p(I) \geq 1$. Also for any $I = I(n,r)$, observe that $\mathcal{M}_p(I) < p$: if $\mathcal{M}_p(I) \geq p$, then there exists $c \in \mathbb{N}_{\geq 1}$ such that $cp^\theta, (c+1)p^\theta, \ldots (c+p-1)p^\theta \in I$, where θ denotes the $\mu_p(I)$. By the definition of θ, we deduce that $p \nmid c+i$ for all $i \in \{0, \ldots, p-1\}$. However p must divide one of the p consecutive integers. This leads to a contradiction.

Remark 2.5. Note that for any given $n, r \geq 1$, we have $\mathcal{M}_2(I(n,r)) = 1$. To see this, let $\theta = \mu_2(I(n,r))$. By definition, we know that there exists an odd integer c such that $c \cdot 2^\theta \in I(n,r)$. If c is even, then $\mu_2(I(n,r)) \geq \theta + 1$ which contradicts the assumption on θ. Observe that neither $(c-1)2^\theta$ nor $(c+1)2^\theta$ lies in $I(n,r)$, as $(c-1)$ and $(c+1)$ are both even and either of these will lead to $\mu_2(I(n,r)) \geq \theta+1$ again. Hence, we deduce that $\mathcal{M}_2(I(n,r)) = 1$, by the definition of $\mathcal{M}_2(\cdot)$ which is given in Definition 2.4.

Example 2.6. Take $r = 1$ and let $p \in \mathbb{P}$ and $n \in \mathbb{Z}_{\geq 2}$ be given where $p^\alpha \leq n < p^{\alpha+1}$ holds, for some $\alpha \in \mathbb{N}$. In that case, we will have

$$\mu_p(I(n,1)) = \alpha = \lfloor \log_p n \rfloor$$

where $I(n,r) = I(n,1) = \{1, 2, \ldots, n\}$.

Lemma 2.7. *Let $n \geq 2$ and $r \geq 1$ be given. Then for all primes $p \leq n$, we have*

$$\lfloor \log_p n \rfloor \leq \mu_p(I(n,r)) \leq \lfloor \log_p(n+r-1) \rfloor.$$

Proof. Let $\theta = \mu_p(I(n,r))$. By definition, there exists $a \in I(n,r)$ such that $a = cp^\theta$, for some $c \in \mathbb{Z}^+$ where $p \nmid c$. Therefore, $p^\theta \leq a = cp^\theta \leq n+r-1$ which yields that

$\theta \leq \log_p(n+r-1)$. As θ is in $\mathbb{N}$, we have $\theta \leq \lfloor \log_p(n+r-1) \rfloor$.

Let $p^\alpha \leq n < p^{\alpha+1}$ be given. So to show the second inequality, we assume that $\mu_p(I(n,r)) = \theta < \alpha$. That means for all $c \in \mathbb{N}$, we have $cp^\alpha \notin I(n,r)$, by the definition of $\mu_p(\cdot)$. In other words, there exists $c_0 \in \mathbb{N}$ such that $c_0 p^\alpha < r$ and $n+r-1 < (c_0+1)p^\alpha$. This leads to $n < p^\alpha$, which is a contradiction. □

These bounds are sharp indeed. For example, if we take $n = 4$, $r = 25$ and $p = 2$, then $I(n,r) = \{25, 26, 27, 28\}$ and therefore $\mu_p(I(n,r)) = 2 = \lfloor \log_p n \rfloor$.

In order to prove Theorem 1.2, we need to use some combinatorial tools. In particular, we use the following lemma which can be found in [Kummer, 1852, p.116] or [Knuth and Wilf, 1989]:

Lemma 2.8 (Kummer)**.** *Let a, b be given two natural numbers. Then $\nu_p\left(\binom{a+b}{b}\right)$ is equal to the number of carries that occur in the addition of a and b in their p-ary representations.*

The following proposition will be a key step for Theorem 1.2.

Proposition 2.9. *Let p be a given prime number. Assume that*

$$n = (n_\alpha, n_{\alpha-1}, \ldots, n_0)_p \in \mathbb{N}_{\geq 2}$$

and

$$r-1 = (r'_\beta, r'_{\beta-1}, \ldots, r'_0)_p \in \mathbb{N}$$

are the p-ary representations of n and $r-1$, respectively. Also let $\delta = \max\{\alpha, \beta\}$ and $n+r-1 = \sum_{i=0}^{\delta+1} s_i p^i = (s_{\delta+1}, s_\delta, \ldots, s_0)_p$ where $s_{\delta+1} \in \{0,1\}$. Then

$$\mu_p(I(n,r)) = \max\{i \in \{0,1,\ldots,\delta+1\} \mid s_i > r'_i\}. \tag{2.6}$$

Moreover, we have

$$\nu_p\left(\binom{n+r-1}{r-1}\right) \leq \mu_p(I(n,r)), \tag{2.7}$$

and the equality holds if and only if we obtain a carry after the addition of n_i and r'_i, for each digit $i \in \{0,1,\ldots,\mu_p(I(n,r))-1\}$.

Proof. Let $\gamma := \max\{i \in \{0,1,\ldots,\delta+1\} \mid s_i > r'_i\}$ and $\theta := \mu_p(I(n,r))$. We want to prove that $\theta = \gamma$. Note that this γ exists because if not, then for all $i \in \{0,1,\ldots,\delta+1\}$ the inequality $s_i \leq r'_i$ holds. This implies that $n+r-1 = \sum_{i=0}^{\delta+1} s_i p^i \leq \sum_{i=0}^{\delta} r'_i p^i = r-1$, which is a contradiction since $n > 0$. Furthermore, we claim that for all $j > \gamma$, we have $s_j = r'_j$. To show this, suppose contrary, i.e. take the largest $j > \gamma$ such that $s_j < r'_j$, and call it j_0. Then, consider

$$\begin{aligned} 0 > r-1-(n+r-1) &= \sum_{i=0}^{\delta} r'_i p^i - \sum_{i=0}^{\delta+1} s_i p^i = \sum_{i=0}^{j_0} (r'_i - s_i)p^i \\ &= (r'_{j_0} - s_{j_0})p^{j_0} + \sum_{i=0}^{j_0-1}(r'_i - s_i)p^i \geq p^{j_0} - \sum_{i=0}^{j_0-1}(p-1)p^i \\ &= p^{j_0} - (p-1)\cdot\frac{p^{j_0}-1}{p-1} \geq 1. \end{aligned}$$

A contradiction, hence we have the claim. Now, consider

$$b = \sum_{i=\gamma}^{\delta+1} s_i p^i$$

where $s_{\delta+1}$ is possibly zero. We first have to show that $b \in I(n,r)$. To do this, we represent $r-1$ as $\sum_{i=0}^{\delta+1} r'_i p^i$ where $r'_{\beta+1} = \ldots = r'_{\delta+1} = 0$. It is easy to see that

$$b = \sum_{i=\gamma}^{\delta+1} s_i p^i \leq \sum_{i=0}^{\delta+1} s_i p^i = n+r-1.$$

Also by definitions of b and γ, we have $s_i = r'_i$ for all $i \in \{\gamma+1,\ldots,\delta+1\}$ and $s_\gamma > r'_\gamma$; which imply that $b > r-1$. If $\{\gamma+1,\ldots,\delta+1\}$ is empty, then $\gamma = \delta+1$ and by the choice of δ, we have again $b > r-1$; as $s_{\delta+1} > r'_{\delta+1} = 0$. Thus $b \in I(n,r)$; and hence one can easily see that $\gamma = \nu_p(b) \leq \mu_p(I(n,r)) = \theta$. So, it is enough to prove that $\theta \leq \gamma$.

If $\gamma = \delta+1$, then $\gamma = \lfloor \log_p(n+r-1) \rfloor$ and hence by Lemma 2.7 we have $\theta \leq \gamma$. For the second case, suppose that $\gamma < \delta+1$. That means $s_{\delta+1} = r'_{\delta+1}$ and since $\beta \leq \delta$, we see that $s_{\delta+1} = 0$. Also, assume that $\theta > \gamma$. Since $\theta, \gamma \in \mathbb{Z}$, we have $\theta \geq \gamma+1$. By definition of $\mu_p(I(n,r))$, there exists $a \in I(n,r)$ such that $a = c_a p^\theta$ for some $c_a \in \mathbb{N}$ and $p \nmid c_a$. In other words, we have $a = \sum_{i=0}^{\delta} a_i p^i \in I(n,r)$ where

$a_0 = a_1 = \ldots = a_{\theta-1} = 0$ and $a_\theta \neq 0$. If $\delta = \alpha$, then since $n_\alpha > 0$ we have $s_\delta > r_\delta$. This implies that $\gamma \geq \delta$, but since $\gamma < \delta + 1$, this yields that $\gamma = \delta$. However by Lemma 2.7, we have $\theta \leq \delta = \lfloor \log_p(n + r - 1) \rfloor$, which leads to a contradiction. Thus $\delta > \alpha$, as $\delta = \max\{\alpha, \beta\}$. So, assume that $\delta = \beta$, or more precisely $\beta > \alpha$. By the choice of γ, we have $s_i = r'_i$ for all $i \in \{\gamma + 1, \ldots, \delta\}$. Also since $a \in I(n, r)$, observe that $a = \sum_{i=\theta}^{\delta} a_i p^i \leq \sum_{i=\theta}^{\delta} s_i p^i$. Because otherwise the inequality $a - \sum_{i=\theta}^{\delta} s_i p^i = c_0 p^\theta > 0$ holds for some $c_0 \in \mathbb{N}_{\geq 1}$. Therefore,

$$\sum_{i=0}^{\theta-1} s_i p^i \leq \sum_{i=0}^{\theta-1} (p-1)p^i = (p-1) \cdot \frac{p^\theta - 1}{p-1} < p^\theta \leq c_0 p^\theta = a - \sum_{i=\theta}^{\delta} s_i p^i.$$

This implies that $a > \sum_{i=0}^{\delta} s_i p^i = n + r - 1$, which is impossible. So, using this fact and the inequality $\theta \geq \gamma + 1$, we obtain that

$$a = \sum_{i=\theta}^{\delta} a_i p^i \leq \sum_{i=\theta}^{\delta} s_i p^i \leq \sum_{i=\gamma+1}^{\delta} s_i p^i = \sum_{i=\gamma+1}^{\delta} r'_i p^i \leq \sum_{i=0}^{\delta} r'_i p^i = r - 1$$

which is a contradiction. Thus $\theta \leq \gamma$ and hence we have the equality.

To prove the second part, we use Lemma 2.8 and apply the first part of the proposition. Define $n_i = 0$, for all $i \in \{\alpha + 1, \ldots, \delta + 1\}$. Observe that there cannot be any carry for the values of $i = \theta + 1, \ldots, \delta + 1$, due to (2.6). The situation is also same for the case $i = \theta$. Because if a carry occurs for that case, then $s_{\theta+1} \geq r'_{\theta+1} + 1$ which contradicts the fact that $s_i = r'_i$ for all $i \in \{\theta + 1, \ldots, \delta + 1\}$. This in turn implies that the only possible carries in the addition of n and $r - 1$ in their p-ary representations are for the entries $i = 0, \ldots, \theta - 1$. Thus, there are at most θ carries in the addition; and hence $\nu_p\left(\binom{n+r-1}{r-1}\right) \leq \theta = \mu_p\left(I(n, r)\right)$. Note that if there exists an $i \in \{0, \ldots, \theta - 1\}$ such that a carry does not occur after the addition of n_i and r'_i, then the inequality in (2.7) is strict. Otherwise, we conclude that $\nu_p\left(\binom{n+r-1}{r-1}\right) = \theta = \mu_p\left(I(n, r)\right)$. □

2.4 Primes in Short Intervals

To prove non-integerness of hyperharmonic numbers in Chapter 3, we need to have the information about the location of prime numbers. In particular, we will deal

with the intervals

$$(x - \Phi(x), x]$$

which contain prime numbers, for sufficiently large $x \in \mathbb{R}$. Here, the length of the interval $\Phi(x)$ is expected to be small compared to the given real number x. A very first result of this kind is attributed to Joseph Bertrand [Bertrand, 1845] who conjectured the following statement:

Bertrand's Postulate. *For any real number $x \geq 2$, there exists a prime in the interval $\left(\frac{x}{2}, x\right]$.*

In other words, Bertrand suggested to take $\Phi(x) = \dfrac{x}{2}$, when $x \geq 2$. This postulate was first proved by Chebyshev in [Chebyshev, 1852]. Since then, Ramanujan [Ramanujan, 1919] and Erdős [Erdős, 1932] also gave different proofs of this postulate. Using the techniques in [Ramanujan, 1919], an extension of Bertrand's Postulate was given in [Nagura, 1952]. In this thesis, we will use the following consequence of this extension.

Theorem 2.10. *For any integer $n \geq 2$, there is a prime number in $\left(\frac{2n}{3}, n\right]$.*

Proof. By the main theorem of [Nagura, 1952], we know that there exists a prime number in the interval $\left(\frac{2n}{3}, n\right]$, for any $n \geq 12$. A straightforward check shows that the statement is true for all integers $n \geq 2$. □

As it can be noticed from the facts that we mentioned earlier, all of the previous results can be deduced from the distribution of prime numbers. Thanks to Hadamard [Hadamard, 1896] and de la Vallée Poussin [de La Vallée Poussin, 1896, de La Vallée Poussin, 1899], we know the celebrated Prime Number Theorem which can be stated as follows:

Prime Number Theorem. *Let $x \in \mathbb{R}_{>0}$ be given and $\pi(x) := |\{p \leq x \,|\, p \in \mathbb{P}\}|$. Then*

$$\pi(x) \sim \frac{x}{\log x},$$

as x tends to infinity.

One can check [Apostol, 1976] or [Davenport, 1982] to see the details of its proof. To prove Theorem 1.1, we will need the following consequence of the Prime Number Theorem.

Corollary 2.11. *Let $\varepsilon > 0$ be a given real number. Then there is a positive real number $x_0 = x_0(\varepsilon)$ such that for all $x \geq x_0$, we have $\mathbb{P} \cap ((1-\varepsilon)x, x] \neq \emptyset$.*

Remark 2.12. For instance, take $\varepsilon = \frac{1}{9}$. Then we know by [Dusart, 1998] that there exists a prime in the interval

$$\left(x, x + \frac{x}{\log^2 x}\right]$$

for any $x \geq 3275$. So, if we consider our interval in Dusart's setting, then there has to be a prime in the interval $\left(x, \frac{9x}{8}\right]$. Note that

$$\frac{x}{8} \geq \frac{x}{\log^2 x}$$

is satisfied if and only if $\log^2 x \geq 8$. The latter case holds whenever $x \geq e^{2\sqrt{2}} > 16$. When we perform a computer check using SageMath [Sage Developers, 2018], we see that the interval $\left(x, \frac{9x}{8}\right]$ contains a prime, for all $x \in \mathbb{Z}_{\geq 48}$. In other words, we deduced that

$$\mathbb{P} \cap \left(\left(1 - \frac{1}{9}\right) x, x\right] \neq \emptyset,$$

for every integer $x \geq 54$. As $x = 53$ is itself prime, we can conclude that for any natural number $x \geq 53$, the interval $\left(\left(1 - \frac{1}{9}\right) x, x\right]$ contains a prime number. This lower bound on x will be useful in the proof of Theorem 1.1.

By Corollary 2.11, we can take $\Phi(x) = \varepsilon x$, for any $\varepsilon > 0$. Using some sieve theoretic techniques, this function can be replaced by some other slowly increasing function $\Phi(x) = x^\theta$, for some $\theta \in (0, 1)$. The current best known value of θ is 0.525, which is due to Baker, Harman and Pintz [Baker et al., 2001].

Theorem 2.13 ([Baker et al., 2001])**.** *For all sufficiently large x, there is always a prime number in the interval $(x - x^{0.525}, x]$. More precisely, there exists a real number $x_0 > 0$ such that for any $x \geq x_0$, the inequality*

$$\pi\left(x + x^{0.525}\right) - \pi(x) \geq \frac{9}{100} \cdot \frac{x^{0.525}}{\log x}$$

is satisfied, where $\pi(x)$ denotes the number of primes up to x.

This result can be used for all sufficiently large x. However, one can decrease the exponent θ by considering almost all real x's, instead of all sufficiently large ones.

Theorem 2.14 ([Harman, 2007])**.** *Let A be any positive real number and X sufficiently large real number. Then for any integer $x \leq X$ except at most*

$$\mathcal{O}_A\left(\frac{X}{\log^A X}\right)$$

many possible values of x, there exists a prime number in the interval $\left(x - x^{\frac{1}{18}}, x\right]$.

Remark 2.15. An improvement of this theorem can be found in [Jia, 1996]. In that paper, it was shown that the exponent in Theorem 2.14 can be replaced by any θ that is greater than $\frac{1}{20}$. Nevertheless, taking $\Phi(x) = x^{\frac{1}{18}}$ will be enough for our purposes.

Although all of the previous facts were unconditional, one may suppose some unproven statements like Riemann hypothesis to obtain primes in shorter intervals. The following corollary of the Riemann hypothesis is of this type and it was first given by Cramér [Cramér, 1936].

Theorem 2.16. *Conditionally on the Riemann hypothesis, there exists a constant $c > 0$ such that the interval $(x - c\sqrt{x}\log x, x]$ contains a prime number, for all sufficiently large $x \in \mathbb{R}$.*

Another consequence of the Riemann hypothesis is the usage of the facts that are related to difference between consecutive primes. For instance, one can bound the total length of large gaps between prime numbers, which can be obtained by a work of Selberg [Selberg, 1943, Theorem 2] as follows.

Theorem 2.17 ([Selberg, 1943])**.** *If we assume the Riemann hypothesis, then for all $y > 0$ and sufficiently large x, we have*

$$\sum_{\substack{p_n \leq x \\ p_n - p_{n-1} \geq y}} (p_n - p_{n-1}) = \mathcal{O}\left(\frac{x \log^2 x}{y}\right),$$

where p_n denotes the n-th prime number.

This theorem allows us to control both the length of the short interval that contains prime numbers and the size of the set of exceptional integers, simultaneously. In this way, we will find primes in almost all intervals of the form $(x - x^\theta, x]$, for any $\theta \in \left(0, \frac{1}{2}\right]$.

Corollary 2.18. *Let X be a sufficiently large real number and $\theta \in \left(0, \frac{1}{2}\right]$. Suppose that the Riemann hypothesis holds. Then we get a prime in the interval $(x - x^\theta, x]$ for all integers $x \leq X$ with at most $\mathcal{O}_\theta\left(X^{1-\theta}\log^2 X\right)$ many exceptions.*

Proof. As in Theorem 2.16, take X sufficiently large so that the interval

$$(X - c\sqrt{X}\log X, X]$$

contains a prime number, for some constant $c > 0$. We will give an upper bound on the size of the set

$$S(X) := \left\{x \in \mathbb{Z} \;\middle|\; x \leq X \;\wedge\; \mathbb{P} \cap (x - x^\theta, x] = \emptyset\right\}.$$

Let p_N be the greatest prime number that is less than or equal to X. By Theorem 2.16, we know that the interval $(p_N, X] \subseteq (X - c\sqrt{X}\log X, X]$ contains at most

$$\mathcal{O}\left(X^{\frac{1}{2}}\log X\right) = \mathcal{O}\left(X^{1-\theta}\log^2 X\right)$$

many elements, as $\theta \leq \frac{1}{2}$. This yields that

$$S(X) = \left|\left\{x \leq p_N \;\middle|\; \mathbb{P} \cap (x - x^\theta, x] = \emptyset\right\}\right| + \mathcal{O}\left(X^{1-\theta}\log^2 X\right), \tag{2.8}$$

by the definition of $S(X)$. So, it is enough to consider the integers up to p_N. Assume that $x \in (p_{n-1}, p_n]$, for some $n \leq N$. Notice that for each prime number p_n, either the case $p_n - p_{n-1} \geq p_n^\theta$, or $p_n - p_{n-1} < p_n^\theta$ is satisfied. If the latter case holds, then we get

$$x - x^\theta \leq p_n - p_n^\theta < p_{n-1} < x,$$

as $x \in (p_{n-1}, p_n]$ and the function $x - x^\theta$ is increasing, for any $x \geq 1$. This means that $\mathbb{P} \cap (x - x^\theta, x] \neq \emptyset$, for any $x \in (p_{n-1}, p_n]$ when $p_n - p_{n-1} < p_n^\theta$. Therefore, we

have

$$\left|\left\{x \leq p_N \,\middle|\, \mathbb{P} \cap (x - x^\theta, x] = \emptyset\right\}\right| \leq \sum_{\substack{p_n \leq X \\ p_n - p_{n-1} \geq p_n^\theta}} (p_n - p_{n-1}), \tag{2.9}$$

as there are at most $p_n - p_{n-1}$ many integers in the interval $(p_{n-1}, p_n]$. To estimate an upper bound for the sum in (2.9), we first decompose the range $[1, X]$ into dyadic intervals of the form $\left[\frac{X}{2^{k+1}}, \frac{X}{2^k}\right]$, for any $k \geq 0$. Observe that

$$\sum_{\substack{p_n \leq X \\ p_n - p_{n-1} \geq p_n^\theta}} (p_n - p_{n-1}) \leq \sum_{k \geq 0} \sum_{\substack{\frac{X}{2^{k+1}} \leq p_n \leq \frac{X}{2^k} \\ p_n - p_{n-1} \geq p_n^\theta}} (p_n - p_{n-1}). \tag{2.10}$$

Now, we will use Theorem 2.17 to bound the each inner sum on the right hand side of (2.10). This leads to

$$\sum_{\substack{\frac{X}{2^{k+1}} \leq p_n \leq \frac{X}{2^k} \\ p_n - p_{n-1} \geq p_n^\theta}} (p_n - p_{n-1}) \leq \sum_{\substack{\frac{X}{2^{k+1}} \leq p_n \leq \frac{X}{2^k} \\ p_n - p_{n-1} \geq \frac{X^\theta}{2^{\theta(k+1)}}}} (p_n - p_{n-1}) \ll 2^{\theta(k+1)-k} X^{1-\theta} \log^2 X.$$

Putting this bound into inequality (2.10), we deduce that

$$\sum_{\substack{p_n \leq X \\ p_n - p_{n-1} \geq p_n^\theta}} (p_n - p_{n-1}) \ll 2^\theta X^{1-\theta} \log^2 X \sum_{k \geq 0} 2^{k(\theta-1)} \ll_\theta X^{1-\theta} \log^2 X, \tag{2.11}$$

since the series $\sum_{k \geq 0} 2^{k(\theta-1)}$ is convergent, as $\theta < 1$. Combining (2.11) with (2.8) and (2.9), we conclude that there is a prime number in the interval $(x - x^\theta, x]$ for all integers $x \leq X$, except $S(X) \ll_\theta X^{1-\theta} \log^2 X$ many of them. □

Remark 2.19. In [Selberg, 1943, Theorem 1], Selberg already showed that for all integers $x \leq X$, except

$$\mathcal{O}\left(X \left(\frac{\log^2 X}{f(X)}\right)^{\frac{1}{4}}\right)$$

many of them, we have

$$\pi(x + f(x)) - \pi(x) \sim \frac{f(x)}{\log x},$$

for any positive increasing function $f(x)$, where $\dfrac{f(x)}{x}$ is decreasing, $f(x) = o(x)$ and $\log^2 x = o(f(x))$. This leads to an exceptional set of size $\mathcal{O}\left(X^{1-\frac{\theta}{4}} \sqrt{\log X}\right)$, when

we take $f(x) = x^\theta$. However, we will need a smaller set of exceptional integers in our applications. Therefore, we will use Corollary 2.18 instead of this result of Selberg.

Recall by Theorem 2.16 that (assuming the Riemann hypothesis) one can take $\Phi(x) = \mathcal{O}\left(\sqrt{x}\log x\right)$, as it can be deduced from [Cramér, 1936, Theorem 1]. In the same paper, Cramér also constructed a probabilistic model where prime numbers may share some common properties with it. Based on this model, he conjectured that for any $n \in \mathbb{Z}_{>0}$,

$$p_{n+1} - p_n = \mathcal{O}\left(\log^2 p_n\right)$$

holds, where p_n denotes the n-th prime number. This may lead to a decrease in the upper bound on $\Phi(x)$ as follows:

Conjecture 2.20 (Cramér)**.** *For all sufficiently large $x \in \mathbb{R}$, there exists a constant $c > 0$ such that $\mathbb{P} \cap (x - cx\log^2 x, x] \neq \emptyset$.*

By the asymptotic given in the Prime Number Theorem, one may expect to see a prime in the interval $(x - \log x, x]$, for all sufficiently large x. However, this expectation is incorrect, as

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = \infty$$

is satisfied, due to [Westzynthius, 1931, Erdős, 1935, Rankin, 1938, Pintz, 1997, Ford et al., 2018]. So, the given bound $\mathcal{O}\left(\log^2 x\right)$ for $\Phi(x)$ in Conjecture 2.20 may be the best possible one.

Chapter 3

NON-INTEGER HYPERHARMONIC NUMBERS

This chapter is based on the articles [Göral and Sertbaş, 2017] and [Alkan et al., 2018].

3.1 Analytic Approach

In this section, we will prove Theorem 1.1. To do so, first recall that $I(n,r) = \{r, ..., n+r-1\}$. We define $I_p(n,r)$ as the set of integer multiples of p which lie in $I(n,r) = \{r, ..., n+r-1\}$. The following proposition is the beginning step towards the proof of Theorem 1.1.

Proposition 3.1. *Let $n, r \geq 2$ be integers. Assume that there exists a prime number $p \leq n$ such that $|I_p(n,r)| = 1$. Then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Also for a given $r \geq 2$, we have $h_n^{(r)} \notin \mathbb{Z}$, where $n \geq 3r-3$.*

Proof. First of all, observe that if $|I_p(n,r)| = 1$, then $|I_p(n,1)| \in \{0,1\}$ by Lemma 2.3. Since $p \leq n$, we deduce that $|I_p(n,1)| = 1$. This indicates that $p \parallel n!$. By Lemma 2.2, we know that

$$h_n^{(r)} = \frac{P_r + P_{r+1} + \cdots + P_{n+r-1}}{n!}, \tag{3.1}$$

where $P = r(r+1)\cdots(n+r-1)$ and $P_i = \dfrac{P}{i}$, for any $i \in I(n,r)$. Therefore we have $p \nmid P_i$ for $p \mid i$, and $p \mid P_i$ for $p \nmid i$. As $|I_p(n,r)| = 1$, there is only one P_i which is not divisible by p. Hence, we see that $p \nmid P_r + P_{r+1} + \cdots + P_{n+r-1}$. This yields that $\nu_p\left(h_n^{(r)}\right) < 0$, and we conclude that $h_n^{(r)} \notin \mathbb{Z}$.

For the second part of the proposition, assume that $n \geq 3r-3 \geq 3$. This implies that

$$\frac{n+r-1}{2} \leq \frac{2n}{3} < n.$$

Also by Theorem 2.10, we know that there exists a prime number in $\left(\frac{2n}{3}, n\right]$, as $n \geq 2$. Therefore we obtain a prime $p \in \left(\frac{n+r-1}{2}, n\right]$, which indicates that $|I_p(n,1)| = 1$. By Lemma 2.3, we see that $|I_p(n,r)| \in \{1,2\}$. As $n+r-1 < 2p$, the set $I(n,r)$ contains only one multiple of p, which is p itself. Thus, we derive that $|I_p(n,r)| = 1$, and by the first part of the proposition, we conclude that $h_n^{(r)} \notin \mathbb{Z}$. □

This proposition also covers some of the cases, when n is small compared to r. For example, take $n = 2p-1$ and $r = kp+1$ for some prime p and integer $k \geq 3$. Then $I(n,r) = \{kp+1, \ldots, (k+2)p-1\}$. This shows that the only multiple of p in $\{r, \ldots, n+r-1\}$ is $(k+1)p$. Since $\frac{n}{2} < p < n$, we deduce that $h_n^{(r)}$ is not an integer.

Remark 3.2. As a direct consequence of Proposition 3.1, one can deduce that there exist only finitely many n's for a fixed $r \geq 2$ such that $h_n^{(r)}$ may be an integer. This also concludes one of the parts of Theorem 1.1. Moreover, if we take $n = p$ for some $p \in \mathbb{P}$, then by Lemma 2.3, we see that $|I_p(n,r)| = 1$. Thus, we conclude by Proposition 3.1 that $h_n^{(r)} \notin \mathbb{Z}$, which also covers one of the cases in the last part of Theorem 1.2.

Corollary 3.3. *Let $n, r \geq 2$ be given natural numbers. Assume that there exist integers $c, d \geq 1$ and prime numbers p, q, where either the conditions*

$$(c-1)n \leq r < cn, \qquad \frac{n+r}{c+1} < p < n \quad \textit{or,} \tag{3.2}$$

$$\frac{dn}{2} < r \leq dn, \qquad \frac{n+r}{d+2} < q < \frac{r}{d} \quad \textit{hold.} \tag{3.3}$$

Then, any of these two cases implies the non-integerness of $h_n^{(r)}$

Proof. Firstly, assume that both of the conditions given in (3.2) are satisfied. Since the inequalities

$$(c-1)p < (c-1)n \leq r < n+r < (c+1)p$$

hold, we get that $|I_p(n,r)| \leq 1$. As $p < n$, we also have $|I_p(n,1)| \geq 1$. By Lemma 2.3, we deduce that $|I_p(n,r)| = 1$. Hence, we conclude by Proposition 3.1 that $h_n^{(r)} \notin \mathbb{Z}$.

For the second part, if the conditions in (3.3) are satisfied simultaneously, then we obtain the inequalities

$$dq < r < n + r < (d + 2)q.$$

Therefore, we see that $|I_q(n, r)| \leq 1$. Similar to the first case, we also have

$$q < \frac{r}{d} \leq n.$$

This indicates that $|I_q(n, 1)| \geq 1$. Again, Lemma 2.3 yields that $|I_q(n, r)| = 1$. Thus by Proposition 3.1, we once again conclude that $h_n^{(r)}$ is not an integer. □

Using the ideas in Proposition 3.1, we can prove the first part of Theorem 1.1.

Proposition 3.4. *Let $n \geq 2$ be fixed. Then, there exist infinitely many $r \in \mathbb{N}$ such that $h_n^{(r)}$ is not an integer. In particular, define*

$$\mathcal{R}_n(x) := \left|\left\{r \leq x \,\middle|\, h_n^{(r)} \notin \mathbb{Z}\right\}\right|. \tag{3.4}$$

For a fixed natural number $n \geq 2$, we have $\mathcal{R}_n(x) \gg_n x$. Furthermore, if $y = y(n)$ is a function of n, where $n = o(y)$ then $\mathcal{R}_n(y) \sim y$, as n tends to infinity.

Proof. Assume that $n \geq 2$ is a fixed integer and $p^{(n)} \leq n$ is the closest prime to n. We know by Bertrand's Postulate that $p^{(n)} \in \left(\frac{n}{2}, n\right]$. By Proposition 3.1, we also see that if $|I_p(n, r)| = 1$, then $h_n^{(r)}$ is not an integer. Let $r_c = (c+1)\cdot p^{(n)} - n$. We will show that $r \in \bigcup_{c=1}^{\infty} \left((c-1)\cdot p^{(n)}, r_c\right]$ implies $h_n^{(r)} \notin \mathbb{Z}$. Denote I_c as $\left((c-1)\cdot p^{(n)}, r_c\right]$. So, take any natural number $r \in I_c = \left((c-1)\cdot p^{(n)}, r_c\right]$, for some $c \geq 1$. In that case,

$$(c-1)\cdot p^{(n)} < r \leq r_c = (c+1)\cdot p^{(n)} - n = c\cdot p^{(n)} - (n-p) \leq c\cdot p^{(n)}$$

hold. Moreover, we get that $n + (c-1)\cdot p^{(n)} < n + r \leq (c+1)\cdot p^{(n)}$. Both of these inequalities yield that $|I_{p^{(n)}}(n, r)| = 1$, since $cp^{(n)} \in I(n, r)$, but $(c-1)p^{(n)}, (c+1)p^{(n)} \notin I(n, r)$. Hence, we deduce that $h_n^{(r)} \notin \mathbb{Z}$. Observe that the number of integers in the interval I_c is equal to

$$(c+1)\cdot p^{(n)} - n - (c-1)\cdot p^{(n)} = 2p^{(n)} - n \geq 1, \tag{3.5}$$

by Bertrand's Postulate. Now, suppose that $X \in \mathbb{R}$ is positive. Note $I_c \cap I_{c'}$ is empty, whenever $c \neq c'$. By (3.5), we get

$$\left| \mathbb{Z} \cap \bigcup_{c \leq X} I_c \right| = \sum_{c \leq X} \left| \mathbb{Z} \cap \left((c-1) \cdot p^{(n)}, r_c \right] \right| \geq \sum_{c \leq X} 1 \geq X - 1.$$

Thus, we conclude that for a fixed integer $n \geq 2$, there are infinitely many $r \in \mathbb{N}$ where the non-integerness of $h_n^{(r)}$ is satisfied.

Next, take any natural number x, and assume that $n \geq 2$ is fixed. Also, let s be the maximum element of the set $\{c \geq 1 \,|\, r_c \leq x\}$. Then if $r \in I_c$ for some $c \geq 1$, then we know that $h_n^{(r)}$ is not an integer, by the first part. This indicates that

$$\begin{aligned} \mathcal{R}_n(x) &= \left| \left\{ r \leq x \,\middle|\, h_n^{(r)} \notin \mathbb{Z} \right\} \right| \\ &\geq \left| \bigcup_{c=1}^{s} \left(\mathbb{Z} \cap \left((c-1) \cdot p^{(n)}, r_c \right] \right) \right| \\ &= \sum_{c=1}^{s} (r_c - (c-1) \cdot p^{(n)}). \end{aligned}$$

The definition of r_c yields that

$$\begin{aligned} \mathcal{R}_n(x) &\geq \sum_{c=1}^{s} ((c+1) \cdot p^{(n)} - n - (c-1) \cdot p^{(n)}) \\ &= \sum_{c=1}^{s} (2p^{(n)} - n) = s \cdot (2p^{(n)} - n). \end{aligned} \tag{3.6}$$

Moreover, for any $c \geq 1$, we obtain that $r_c = (c+1) \cdot p^{(n)} - n \leq (c+1)n - n = cn$. Therefore, the inequalities $r_s \leq x < r_{s+1} \leq (s+1)n$ are satisfied. Hence, we deduce that

$$s > \frac{x}{n} - 1. \tag{3.7}$$

Equation (3.6) together with (3.7) and Bertrand's Postulate lead to

$$\mathcal{R}_n(x) > \frac{x}{n} - 1, \tag{3.8}$$

as desired. Finally, we will use Theorem 2.13 to obtain the last part. So, assume that n is a sufficiently large fixed natural number such that $(n - n^{0.525}, n]$ contains

a prime number. By the definition of $p^{(n)}$, we have $p^{(n)} \in (n - n^{0.525}, n]$, Therefore, the inequality $2p^{(n)} - n > n - 2n^{0.525}$ holds. Now, suppose that $y = y(n)$ is a function of n where $n = o(y)$. Using (3.6) and (3.7), we deduce that

$$\mathcal{R}_n(y) > \left(\frac{y}{n} - 1\right) \cdot (n - 2n^{0.525}) = (y - n) \cdot \left(1 - \frac{2}{n^{0.475}}\right), \tag{3.9}$$

From (3.9), we see that $\mathcal{R}_n(y) = y + \mathcal{O}\left(n + \frac{y}{n^{0.475}}\right)$ and this yields the asymptotic $\mathcal{R}_n(y) \sim y$, as n goes to infinity. □

Remark 3.5. For any integer $c \geq 1$, if $p^{(n)} < n$ is the closest prime to n and

$$r \in ((c+1)p^{(n)} - n, cp^{(n)}], \tag{3.10}$$

then we cannot say anything about the non-integerness of $h_n^{(r)}$, as $\nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0$ (recall that the case $p^{(n)} = n$ is excluded in Remark 3.2, as $h_{p^{(n)}}^{(r)} \notin \mathbb{Z}$, for all $r \geq 1$). To see this fact, first observe that the given interval is non-empty, as $p^{(n)} < n$ holds. Also, we have

$$r \leq cp^{(n)} < (c+1)p^{(n)} < n + r.$$

Now, the fundamental equation (1.2) yields that

$$\begin{aligned} h_n^{(r)} &= \binom{n+r-1}{r-1} (h_{n+r-1} - h_{r-1}) \\ &= \frac{A}{B} \cdot \frac{cp^{(n)} \cdot (c+1)p^{(n)}}{p^{(n)}} \left(\frac{1}{p^{(n)}} \left(\frac{1}{c} + \frac{1}{c+1}\right) + q\right) \\ &= \frac{A(2c+1)}{B} + qc(c+1)p^{(n)}, \end{aligned} \tag{3.11}$$

for some $A, B \in \mathbb{Z}$ and $q \in \mathbb{Q}$, where $\nu_{p^{(n)}}(A) = \nu_{p^{(n)}}(B) = 0 \leq \nu_{p^{(n)}}(q)$. Thus, by equation (3.11), we conclude that $\nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0$, as $\nu_{p^{(n)}}(B) = 0 \leq \nu_{p^{(n)}}(q)$.

Now, we can use the ideas in Corollary 3.3 and Proposition 3.4 to extend our results and obtain the second part of Theorem 1.1.

Theorem 3.6. *Let $d \geq 1$ be a fixed natural number. Then there is an integer n_d, depending on d, such that $n \geq n_d$ together with $r \leq dn$ imply that $h_n^{(r)} \notin \mathbb{Z}$. Moreover, for any constant $C \in \left(0, \frac{1}{2}\right)$, there exists a natural number n_0, depending on C, such that if $n \geq n_0$ and $r \leq Cn^{1.475}$ then $h_n^{(r)}$ is not an integer.*

We will follow the same lines in [Göral and Sertbaş, 2017, Theorem 10].

Proof. By the Prime Number Theorem we know that for a given $d \geq 1$ there exists a real number x_d such that for all $x \geq x_d$ there is a prime in the interval $\left(\left(1-\frac{1}{2d+7}\right)x, x\right]$. Now, we prove the following claim by induction:

Claim: For any given integer $d \geq 1$ and for all $n \geq \frac{9}{7} \cdot x_d$, if $r \leq dn$ then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer.

So first, we deal with the case $d = 1$. Let $x_1 = 53$ and $p^{(n)}$ be the greatest prime that is less than or equal to n. Assume that $n \geq x_1$. By Remark 2.12, we know that $p^{(n)} \in \left(\frac{8n}{9}, n\right]$, as $n \geq 53$ is an integer. Observe by Propositions 3.1 and 3.4 that if $r \leq n$ and $r \notin (2p^{(n)} - n, p^{(n)}]$ then $\nu_{p^{(n)}}\left(h_n^{(r)}\right) = -1$. So it is enough to check the non-integerness property for $r \in (2p^{(n)} - n, p^{(n)}]$. By (3.3) in Corollary 3.3, we can say that if $r \in \left(\frac{n}{2}, n\right]$ and there is a prime in the interval $\left(\frac{n+r}{3}, r\right)$, then the corresponding hyperharmonic number is not an integer. Since $r \in (2p^{(n)} - n, p^{(n)}]$ and $p^{(n)} \in \left(\frac{8n}{9}, n\right]$, we get that

$$n \geq p^{(n)} \geq r > 2p^{(n)} - n > \frac{16n}{9} - n = \frac{7n}{9}.$$

In particular, we have $r \in \left(\frac{n}{2}, n\right]$. Moreover, we obtain that $\frac{n+r}{3} \leq \frac{2n}{3}$. Thus, $I_1 := \left(\frac{2n}{3}, \frac{7n}{9}\right] \subseteq \left(\frac{n+r}{3}, r\right)$. Note that if $n \geq \frac{9}{7} \cdot x_1$, then there is a prime in the interval $\left(\frac{2n}{3}, \frac{7n}{9}\right]$. Thus, we deduce that for all $n \geq 69 \geq \frac{9}{7} \cdot x_1$ and $r \leq n$, we have $h_n^{(r)} \notin \mathbb{N}$.

Assume that there exists a real number x_{d-1} such that there is a prime in $\left(\left(1-\frac{1}{2d+5}\right)x, x\right]$ and the claim holds for $d-1 \geq 1$. By Corollary 2.11, we can guarantee the existence of such an x_{d-1}. Again by the same consequence of the Prime Number Theorem, we may suppose that for any $x \geq x_d$, the interval $\left(\left(1-\frac{1}{2d+7}\right)x, x\right]$ contains a prime number. We can assume that $x_d \geq x_{d-1}$. As we did in the case $d = 1$, let $n \geq x_d$. Then, we have

$$p^{(n)} \in \left(\left(1-\frac{1}{2d+7}\right)n, n\right]. \tag{3.12}$$

By Remark 3.5, we know that the possible values of r that satisfy the property $\nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0$ belong to the set $\left((j+1)p^{(n)} - n, jp^{(n)}\right]$ where $j \in \{1, \ldots, d\}$. Since

$p^{(n)} \in \left(\left(1-\frac{1}{2d+7}\right)n, n\right]$, we see that for all $j \in \{1,\dots,d\}$ we have $jp^{(n)} \leq jn < (j+1)p^{(n)}$. This implies that $\left((j+1)p^{(n)}-n, jp^{(n)}\right] \subseteq ((j-1)n, jn]$. So, it is enough to check the values

$$r \in \left((d+1)p^{(n)}-n, dp^{(n)}\right], \tag{3.13}$$

by the induction hypothesis. We again use (3.3) in Corollary 3.3 to prove the claim for d. Note that as $r > (d-1)n$ and $d \geq 2$, we see that $r > \frac{dn}{2}$. Also

$$r \leq dp^{(n)} \leq dn \tag{3.14}$$

holds. In order to show the claim, we should see that the interval $\left(\frac{n+r}{d+2}, \frac{r}{d}\right)$ contains a prime number. By (3.12), (3.13) and (3.14), observe that

$$I_d := \left(\frac{d+1}{d+2}\cdot n, \frac{2d^2+6d-1}{2d^2+7d}\cdot n\right] \subseteq \left(\frac{n+r}{d+2}, \frac{r}{d}\right).$$

So, finding a prime in the interval I_d leads to the desired result. Note that since

$$\frac{2d^2+6d-1}{2d^2+7d} < \frac{2d+6}{2d+7},$$

the interval I_d does not intersect with the interval $\left(\left(1-\frac{1}{2d+7}\right)n, n\right]$. If we denote $n' = \frac{2d^2+6d-1}{2d^2+7d}\cdot n$, then the lower bound of I_d becomes

$$\frac{d+1}{d+2}\cdot\frac{2d^2+7d}{2d^2+6d-1}\cdot n' = \frac{2d^3+9d^2+7d}{2d^3+10d^2+11d-2}\cdot n'.$$

Computation and simplification indicate that the inequality

$$\frac{2d^3+9d^2+7d}{2d^3+10d^2+11d-2} < \frac{2d+6}{2d+7}$$

holds, as we have $5d^2+13d-12 > 0$ for all $d \geq 1$. So, if $n' \geq x_d$, then there is a prime in I_d. This can only be achieved when $n \geq \frac{2d^2+7d}{2d^2+6d-1}\cdot x_d$. Thus, if

$$n \geq \frac{9}{7}\cdot x_d > \frac{2d^2+7d}{2d^2+6d-1}\cdot x_d,$$

then $I_d \cap \mathbb{P}$ is non-empty by the fact that the interval $\left(\left(1-\frac{1}{2d+7}\right)x, x\right]$ contains a prime number. By using this fact, we deduce that $h_n^{(r)}$ is not an integer, for $n \geq \frac{9}{7}\cdot x_d$ and $r \in ((d-1)n, dn]$. Thus, we have the claim.

On the other hand, the above proof yields that we can choose $d = d(n)$ as a function of n. To see this, first assume that for any $x \geq x_0$ there exists a prime in the interval $(x - \Phi(x), x]$, for some increasing function $\Phi(x)$, where $\Phi(x) = o(x)$. Fix any $n \geq \frac{9}{7} \cdot x_0$ and take $d = \left\lfloor \frac{n - 7\Phi(n)}{2\Phi(n)} \right\rfloor \in \mathbb{Z}$. Observe that

$$\left(1 - \frac{1}{2d+7}\right) n \leq n - \Phi(n)$$

holds and by the claim, we deduce that $h_n^{(r)} \notin \mathbb{Z}$, for any $r \leq dn$. In particular, fix any $C \in \left(0, \frac{1}{2}\right)$ and note that there exists an $n_1 = n_1(C)$ depending on C such that the inequality

$$C \cdot \frac{n}{\Phi(n)} \leq \left\lfloor \frac{n - 7\Phi(n)}{2\Phi(n)} \right\rfloor$$

is satisfied for any $n \geq n_1$. Define $n_0 = \max\left\{\frac{9}{7} \cdot x_0, n_1\right\}$. Then for any $n \geq n_0$ and

$$r \leq C \cdot \frac{n^2}{\Phi(n)}, \tag{3.15}$$

the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Recall by Theorem 2.13 that $\Phi(n)$ can be taken as $n^{0.525}$. Putting this into inequality (3.15) gives the desired result. □

Remark 3.7. We would like to emphasize some remarks regarding the proof of Theorem 3.6.

1. As we said before, we used (3.3) instead of (3.2) in Corollary 3.3, in order to prove the first part of Theorem 3.6. Notice that the same deductions cannot be obtained using (3.2).

2. In order to prove this theorem by induction, we need to fix the denominator with respect to d. Moreover, the choice $2d + 7$ for the denominator is the optimal one, since writing $d + k$ for the denominator is not suitable to show such a general result, for some integer k.

3. The claim that was stated in the proof of Theorem 3.6 can also be proved without using induction:

A Non-inductive Proof of the Claim. Let $d \geq 1$ be a given integer. Assume that there is a prime in the interval $\left(\left(1-\frac{1}{2d+7}\right)x, x\right]$, for all $x \geq x_d$. Take any $n \geq \frac{9}{7}\cdot x_d$ and $r \leq dn$. Define $p^{(n)}$ as the greatest prime that is less than or equal to n, as before. Recall by Remark 3.5 that if $r \in ((c+1)p^{(n)}-n, cp^{(n)}]$ holds for any $c \geq 1$, then $\nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0$; otherwise, $h_n^{(r)} \notin \mathbb{Z}$. So, it is enough to prove that for any $j \in \{1,\ldots,d\}$ and $r \in ((j+1)p^{(n)}-n, jp^{(n)}]$, the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. To see this fact, we will obtain the conditions (3.3) in Corollary 3.3, and deduce that $h_n^{(r)} \notin \mathbb{Z}$. First, observe that $p^{(n)} \in \left(\left(1-\frac{1}{2d+7}\right)x, x\right]$. This yields that $r \leq jp^{(n)} \leq jn$ and

$$(j+1)\,p^{(n)} - n > (j+1)\left(n - \frac{n}{2d+7}\right) - n = n\cdot\frac{2dj+6j-1}{2d+7} \geq \frac{jn}{2},$$

as $d, j \geq 1$. Hence, the condition for r in (3.3) is satisfied. Next, we will find a prime number in the interval

$$\left(\frac{n+jp^{(n)}}{j+2}, \frac{(j+1)p^{(n)}-n}{j}\right) \subseteq \left(\frac{n+r}{j+2}, \frac{r}{j}\right),$$

for any $j \in \{1,\ldots,d\}$. Notice that

$$\left(\frac{j+1}{j+2}\cdot n, \left(\frac{2dj+6j-1}{2dj+7j}\right)\cdot n\right] \subseteq \left(\frac{n+r}{j+2}, \frac{r}{j}\right). \tag{3.16}$$

Let $J = \frac{2dj+6j-1}{2dj+7j}$. Observe that $J \geq \frac{7}{9}$, since $4dj+5j-9 \geq 0$ is satisfied for any $d, j \geq 1$. Therefore, $nJ \geq x_d$, as $n \geq \frac{9}{7}\cdot x_d$. This implies that there is a prime in the interval $\left(\left(1-\frac{1}{2d+7}\right)nJ, nJ\right]$. Note that this prime is not equal to $p^{(n)}$, since

$$J = \frac{2dj+6j-1}{2dj+7j} < \frac{2d+6}{2d+7}.$$

Also, a careful analysis shows that

$$J\cdot\frac{2d+6}{2d+7} > \frac{j+1}{j+2},$$

as $j \in \{1,\ldots,d\}$ and $d \geq 1$. This indicates that

$$\left(\left(1-\frac{1}{2d+7}\right)nJ, nJ\right] \subseteq \left(\frac{j+1}{j+2}\cdot n, \left(\frac{2dj+6j-1}{2dj+7j}\right)\cdot n\right]$$

and the interval on the right hand side contains a prime number. By (3.16), we conclude that the conditions (3.3) in Corollary 3.3 are satisfied, for any $j \in \{1, \dots, d\}$. Thus, we have $h_n^{(r)} \notin \mathbb{Z}$, for any $n \geq \frac{9}{7} \cdot x_d$ and $r \leq dn$. □

4. Similar to the non-inductive proof of the claim, one can also deduce the last part of the theorem with a smaller range for the constant $C > 0$, as follows:

Proof. Take any constant $C \in \left(0, \frac{1}{3}\right)$. Assume that there exists a real number $x_0 > 0$ such that for any $x \geq x_0$, there is a prime in the interval $(x - \Phi(x), x]$ where $\Phi(x)$ is a monotonically increasing function and $\Phi(x) = o(x)$. Also let $n \geq x_0$ be a sufficiently large integer depending on C and

$$r \leq C\frac{n^2}{\Phi(n)}. \tag{3.17}$$

Denote $p^{(n)} = \max\{p \leq n \,|\, p \in \mathbb{P}\}$. We will show that for any sufficiently large $n = n(C)$ and r satisfying (3.17), the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. As we know by Remark 3.5, it is enough to check the non-integerness of $h_n^{(r)}$ for the values of $r \in ((c+1)p^{(n)} - n, cp^{(n)}]$, since otherwise, $h_n^{(r)}$ is not an integer. Define

$$c_0 := \left\lfloor C\frac{n^2}{\Phi(n)p^{(n)}} \right\rfloor + 1. \tag{3.18}$$

We will show that for any $c \in \mathbb{Z} \cap [1, c_0]$, if $r \in ((c+1)p^{(n)} - n, cp^{(n)}]$, then $h_n^{(r)} \notin \mathbb{Z}$. To do so, we will check the conditions (3.3) in Corollary 3.3 again. First of all, observe that $r \leq cp^{(n)} \leq cn$. As $\Phi(x) = o(x)$, we see that

$$\frac{\Phi(n)}{n} \leq \frac{1}{4} \leq \frac{c}{2c+2}$$

is satisfied, for any sufficiently large n and $c \geq 1$. This implies that

$$p^{(n)} > n - \Phi(n) \geq n \cdot \frac{c+2}{2c+2},$$

and hence, $(c+1)p^{(n)} - n > \frac{cn}{2}$. So, the first condition in (3.3) holds for any $c \in \{1, \dots, c_0\}$ and $r \in ((c+1)p^{(n)} - n, cp^{(n)}]$. Now, we will find a prime in

the interval $\left(\frac{n+r}{c+2}, \frac{r}{c}\right)$. Since $n - \Phi(n) < p^{(n)} \leq n$ and $c \geq 1$, we get that

$$\left(\frac{c+1}{c+2} \cdot n, n - 2\Phi(n)\right] \subseteq \left(\frac{n+r}{c+2}, \frac{r}{c}\right). \tag{3.19}$$

Notice that $x - 2\Phi(x)$ is increasing after a while, as $\Phi(x) = o(x)$. Therefore, we may take $n - 2\Phi(n) \geq x_0$. This indicates that there is a prime in the interval $(n - 2\Phi(n) - \Phi(n - 2\Phi(n)), n - 2\Phi(n)]$. Since $\Phi(n)$ is monotonically increasing, we can say that $n - 3\Phi(n) \leq n - 2\Phi(n) - \Phi(n - 2\Phi(n))$, and thus

$$(n - 3\Phi(n), n - 2\Phi(n)] \cap \mathbb{P} \neq \emptyset. \tag{3.20}$$

Observe that $p^{(n)} \notin (n - 3\Phi(n), n - 2\Phi(n)]$. Let

$$C_0 := \frac{3C+1}{6} \in \left(C, \frac{1}{3}\right).$$

Since $\Phi(x) = o(x)$, we have

$$\frac{\Phi(n)}{n} < 1 - 3C_0 = \frac{1-3C}{2},$$

for all sufficiently large n, depending on C. This yields that

$$3C_0 n < n - \Phi(n) < p^{(n)}.$$

Hence, we get

$$C_0 \frac{n^2}{\Phi(n)p^{(n)}} < \frac{n}{3\Phi(n)}. \tag{3.21}$$

Note that the function $\frac{n^2}{\Phi(n)p^{(n)}}$ is increasing, as $\Phi(n) = o(n)$ and $n - \Phi(n) < p^{(n)} \leq n$. Therefore, the inequality

$$C\frac{n^2}{\Phi(n)p^{(n)}} + 3 < C_0 \frac{n^2}{\Phi(n)p^{(n)}} \tag{3.22}$$

is satisfied, as $C_0 \in \left(C, \frac{1}{3}\right)$ and n is sufficiently large in terms of C. Combining (3.18), (3.21) and (3.22) lead to

$$c_0 \leq C\frac{n^2}{\Phi(n)p^{(n)}} + 1 < \frac{n}{3\Phi(n)} - 2.$$

Since $c \in [1, c_0]$, we obtain that

$$\left(1 - \frac{1}{c+2}\right) \cdot n < n - 3\Phi(n).$$

Hence, by (3.19) and (3.20), we deduce that there is a prime in the interval

$$(n - 3\Phi(n), n - 2\Phi(n)] \subseteq \left(\frac{c+1}{c+2} \cdot n, n - 2\Phi(n)\right] \subseteq \left(\frac{n+r}{c+2}, \frac{r}{c}\right).$$

This concludes the last part of the theorem by Corollary 3.3. □

5. With a little effort, one can show that the constant C in inequality (3.15) can be taken as $\frac{1}{2}$ for the case $\Phi(n) = n^{0.525}$. However, the existence of such a constant $C > 0$ will be enough for our purposes.

To finalize our analytic section, we will show that almost all tuples $(n, r) \in [1, x] \times [1, x]$ lead to non-integer hyperharmonic numbers. Note that this is the last part of Theorem 1.1 and in order to prove this fact, we will use prime numbers in short intervals. To obtain better estimations on the number of (n, r) tuples, we will apply the well-known consequences of the Riemann hypothesis and the Cramér's conjecture, which are given in Section 2.4.

Theorem 3.8. *If $\mathcal{S}(x)$ is the number of pairs $(n, r) \in [1, x] \times [1, x]$ such that $h_n^{(r)}$ is not an integer, then for any $A > 0$, we have*

$$\mathcal{S}(x) = x^2 + \mathcal{O}_A\left(\frac{x^{\frac{80}{59}}}{(\log x)^A}\right),$$

where the implied constant depends only on A. Assuming the Riemann hypothesis, we have

$$\mathcal{S}(x) = x^2 + \mathcal{O}\left(x^{\frac{7}{6}} (\log x)^{\frac{19}{6}}\right).$$

Finally, assuming Cramér's conjecture, we have

$$\mathcal{S}(x) = x^2 + \mathcal{O}\left(x \log^3 x\right).$$

Proof. Consider a pair $(n, r) \in [1, x] \times [1, x]$, where $n, r \in \mathbb{Z}$ and x is sufficiently large. Fix any $C \in \left(0, \frac{1}{2}\right)$ (for instance, choose $C = \frac{1}{4}$) and assume that the interval

$(x - \Phi(x), x]$ contains a prime number, for some monotonically increasing function $\Phi(x)$ with $\Phi(x) = o(x)$. Recall by the last part of Theorem 3.6 that there exists an $n_0 = n_0(C)$, depending on C, such that for any $n \geq n_0$ and any

$$r \leq C \cdot \frac{n^2}{\Phi(n)}, \tag{3.23}$$

we have $h_n^{(r)} \notin \mathbb{Z}$. As we did in the last part of Theorem 3.6, it follows from (3.23) that if $n \geq n_0$ and $r \leq Cn^{1.475}$, then $h_n^{(r)} \notin \mathbb{Z}$. We may suppose that $n \geq n_0$, as the number of pairs $(n, r) \in [1, x] \times [1, x]$ with $n < n_0$ is

$$(n_0 - 1) \cdot \lfloor x \rfloor = \mathcal{O}(x) \tag{3.24}$$

which will be a negligible error in our calculations. Next, we will put an upper bound on the size of n in terms of x. As $r \leq x$, it is clear from $r \leq Cn^{1.475}$ that we can take $n \leq cx^{\frac{1}{1.475}}$ for some constant $c > 0$, depending on C, since otherwise $h_n^{(r)}$ is guaranteed to be non-integer as it can be seen from Figure 3.1.

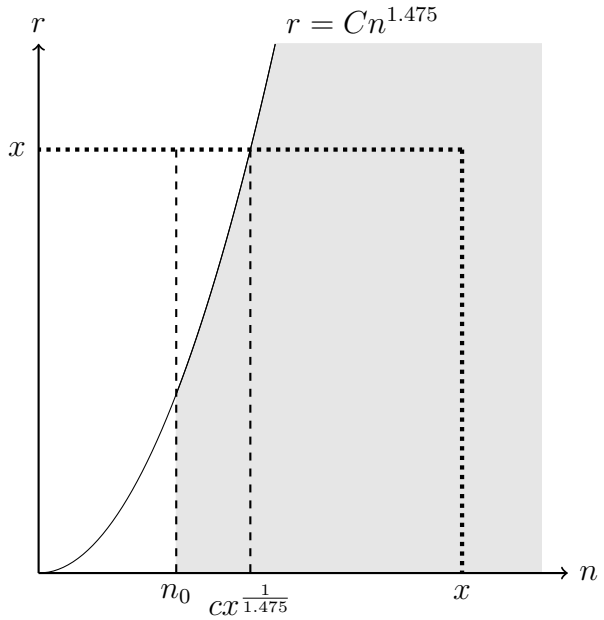

Figure 3.1: The gray area contains the integral points (n, r) which give non-integer hyperharmonics.

Now, for any given natural number $n \leq x$, define

$$E_n(x) := \left|\left\{r \leq x \,\middle|\, h_n^{(r)} \in \mathbb{Z}\right\}\right|. \tag{3.25}$$

Notice that if n is a prime number, then $E_n(x) = 0$ by Remark 3.2. Using the definition given in (3.25), we see that

$$E_n(x) \leq \left|\left\{r \leq x \,\middle|\, \nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0\right\}\right|, \tag{3.26}$$

where $p^{(n)}$ denotes the greatest prime that is less than n. Recall by Remark 3.5 that if $r \in ((c+1)p^{(n)} - n, cp^{(n)}]$ for some $c \geq 1$, then $\nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0$. Observe that there are at most $\left\lfloor \dfrac{x}{p^{(n)}} \right\rfloor$ many values of c, as $r \leq x$. Also note that the number of integers in the interval $((c+1)p^{(n)} - n, cp^{(n)}]$ is at most

$$cp^{(n)} - ((c+1)p^{(n)} - n) = n - p^{(n)} < \Phi(n), \tag{3.27}$$

as $p^{(n)} \in (n - \Phi(n), n]$ when $n \geq n_0$. Thus, we get

$$E_n(x) = \mathcal{O}\left(\frac{x}{p^{(n)}} \cdot \Phi(n)\right) = \mathcal{O}\left(\frac{x}{n} \cdot \Phi(n)\right), \tag{3.28}$$

for all $n \geq n_0$, as $p^{(n)} \in (n - \Phi(n), n] \subseteq \left(\frac{n}{2}, n\right]$. Again by (3.25), we infer that

$$\mathcal{S}(x) + \sum_{n \leq x} E_n(x) = \lfloor x \rfloor^2 = x^2 + \mathcal{O}(x). \tag{3.29}$$

Also by (3.24), we see that

$$\sum_{n \leq x} E_n(x) = \sum_{n_0 \leq n \leq cx^{\frac{1}{1.475}}} E_n(x) + \mathcal{O}(x) \tag{3.30}$$

is satisfied, as for all $n > cx^{\frac{1}{1.475}}$, we have $h_n^{(r)} \notin \mathbb{Z}$. Since (3.28) holds for all $n \geq n_0$, we obtain that

$$\sum_{n \leq x} E_n(x) = \mathcal{O}\left(\sum_{n \leq cx^{\frac{1}{1.475}}} \frac{x}{n} \cdot \Phi(n)\right). \tag{3.31}$$

Combining this with equation (3.29), we deduce that

$$\mathcal{S}(x) = x^2 + \mathcal{O}\left(x \sum_{n \leq cx^{\frac{1}{1.475}}} \frac{\Phi(n)}{n}\right), \tag{3.32}$$

where $\Phi(n)$ is subject to the only condition that $(n-\Phi(n), n]$ contains a prime number. Put $X = cx^{\frac{1}{1.475}}$. By Theorem 2.14, we may take $\Phi(n) = n^{\frac{1}{18}}$, for all $n \leq X$ with at most

$$\mathcal{O}_A\left(\frac{X}{(\log X)^A}\right)$$

exceptions, where A is any positive number and the implied constant depends only on A. Let us define a set of good values of n as follows:

$$\mathfrak{G} := \left\{ n \leq X \,\middle|\, \Phi(n) = n^{\frac{1}{18}} \right\}. \tag{3.33}$$

According to (3.33), the set of bad values of n is defined as follows:

$$\mathfrak{B} := \{ n \leq X \,|\, n \notin \mathfrak{G} \}. \tag{3.34}$$

From (3.33) and (3.34), we have

$$\sum_{n\leq X} \frac{\Phi(n)}{n} = \sum_{n\in\mathfrak{G}} \frac{\Phi(n)}{n} + \sum_{n\in\mathfrak{B}} \frac{\Phi(n)}{n}. \tag{3.35}$$

Again using (3.33), one obtains

$$\sum_{n\in\mathfrak{G}} \frac{\Phi(n)}{n} \leq \sum_{n\leq X} n^{\frac{1}{18}-1} \leq X^{\frac{1}{18}}. \tag{3.36}$$

We also know that

$$\mathcal{B}(X) := \sum_{n\in\mathfrak{B}} 1 \ll_A \frac{X}{(\log X)^A}. \tag{3.37}$$

As we know from Theorem 2.13, we can take $\Phi(n) = n^{0.525}$ for the sum over $n \in \mathfrak{B}$. Hence, we can apply partial summation by taking $\Phi(n) = n^{0.525}$ and we get

$$\sum_{n\in\mathfrak{B}} \frac{\Phi(n)}{n} = \sum_{n\in\mathfrak{B}} \frac{1}{n^{0.475}} = \frac{\mathcal{B}(X)}{X^{0.475}} + 0.475 \int_2^X \frac{\mathcal{B}(t)}{t^{1.475}}\, dt + \mathcal{O}(1). \tag{3.38}$$

Using (3.37), we see that

$$\frac{\mathcal{B}(X)}{X^{0.475}} \ll_A \frac{X^{0.525}}{(\log X)^A} \tag{3.39}$$

and

$$\int_2^X \frac{\mathcal{B}(t)}{t^{1.475}}\, dt \ll_A \int_2^X \frac{1}{t^{0.475}(\log t)^A}\, dt. \tag{3.40}$$

By partial integration, one obtains for $2 \leq Y \leq X$ that

$$\int_Y^X \frac{1}{t^{0.475}(\log t)^A}\, dt = \frac{X^{0.525}}{0.525(\log X)^A} + \frac{A}{0.525}\int_Y^X \frac{1}{t^{0.475}(\log t)^{A+1}}\, dt + \mathcal{O}_Y(1). \tag{3.41}$$

If Y is large enough only in terms of A, then

$$\frac{A}{(0.525)\log t} \leq \frac{1}{2} \tag{3.42}$$

holds, for any $t \geq Y$. It is clear from (3.42) by the monotonicity of integrals that

$$\frac{A}{0.525}\int_Y^X \frac{1}{t^{0.475}(\log t)^{A+1}}\, dt \leq \frac{1}{2}\int_Y^X \frac{1}{t^{0.475}(\log t)^A}\, dt, \tag{3.43}$$

when Y is large enough only in terms of A. Assembling (3.41) and (3.43), we obtain

$$\int_2^X \frac{1}{t^{0.475}(\log t)^A}\, dt = \int_2^Y \frac{1}{t^{0.475}(\log t)^A}\, dt + \int_Y^X \frac{1}{t^{0.475}(\log t)^A}\, dt \ll_A \frac{X^{0.525}}{(\log X)^A}. \tag{3.44}$$

Therefore, (3.40) and (3.44) give

$$\int_2^X \frac{\mathcal{B}(t)}{t^{1.475}}\, dt \ll_A \frac{X^{0.525}}{(\log X)^A}. \tag{3.45}$$

Gathering (3.38), (3.39) and (3.45), we have

$$\sum_{n\in\mathfrak{B}} \frac{\Phi(n)}{n} \ll_A \frac{X^{0.525}}{(\log X)^A}. \tag{3.46}$$

Next (3.35), (3.36) and (3.46) yield that

$$\sum_{n\leq X} \frac{\Phi(n)}{n} \ll_A \frac{X^{0.525}}{(\log X)^A} \ll_A \frac{x^{\frac{0.525}{1.475}}}{(\log x)^A}. \tag{3.47}$$

Feeding (3.47) into (3.32),

$$\mathcal{S}(x) = x^2 + \mathcal{O}_A\left(\frac{x^{\frac{2}{1.475}}}{(\log x)^A}\right)$$

follows. This completes the proof of the first part of Theorem 3.8.

For the second part, assume that the Riemann hypothesis holds. By Theorem 2.16, we may take $\Phi(n) = c_0\sqrt{n}\log n$, for some $c_0 > 0$. It follows from (3.23) that if

$$r \leq \frac{C}{c_0} \cdot \frac{n^{\frac{3}{2}}}{\log n} = C_1 \frac{n^{\frac{3}{2}}}{\log n}, \tag{3.48}$$

for some $C \in \left(0, \frac{1}{2}\right)$, then $h_n^{(r)}$ is guaranteed to be non-integer. As $r \leq x$, we see from (3.48) that there exists a constant $c_1 > 0$ depending on $C_1 = \frac{C}{c_0}$ such that

$$n \leq X := c_1 x^{2/3} (\log x)^{2/3}, \tag{3.49}$$

as otherwise $h_n^{(r)}$ is not an integer. Using (3.49) and arguing as above, we arrive at the formula

$$\mathcal{S}(x) = x^2 + \mathcal{O}\left(x \sum_{n \leq X} \frac{\Phi(n)}{n}\right), \tag{3.50}$$

where $\Phi(n)$ is subject to the only condition that $(n - \Phi(n), n]$ contains a prime number. By Corollary 2.18, we know that $\Phi(n)$ can be taken as n^θ, for any $\theta \in \left(0, \frac{1}{2}\right]$ and for all $n \leq X$ except for $\mathcal{O}_\theta\left(X^{1-\theta}\log^2 X\right)$ many values. So our definition of a set of good n's is as follows:

$$\mathfrak{G} := \left\{n \leq X \,\middle|\, \Phi(n) = n^\theta\right\}, \tag{3.51}$$

where $\theta \in \left(0, \frac{1}{2}\right]$ is to be determined later. As a result of (3.51), the set of bad n's becomes

$$\mathfrak{B} := \{n \leq X \,|\, n \notin \mathfrak{G}\}. \tag{3.52}$$

Using (3.50), (3.51) and (3.52), we may now express

$$\mathcal{S}(x) = x^2 + \mathcal{O}\left(x \sum_{n \in \mathfrak{G}} \frac{\Phi(n)}{n} + x \sum_{n \in \mathfrak{B}} \frac{\Phi(n)}{n}\right). \tag{3.53}$$

It follows that

$$\sum_{n \in \mathfrak{G}} \frac{\Phi(n)}{n} \leq \sum_{n \leq X} n^{\theta - 1} \leq X^\theta. \tag{3.54}$$

We also know that

$$\mathcal{B}(X) := \sum_{n\in\mathfrak{B}} 1 \ll_\theta X^{1-\theta}\log^2 X. \tag{3.55}$$

Of course for $n \in \mathfrak{B}$, we may take $\Phi(n) = c_0\sqrt{n}\log n$ and this gives

$$\sum_{n\in\mathfrak{B}} \frac{\Phi\left(n\right)}{n} \ll \sum_{n\in\mathfrak{B}} \frac{\log n}{n^{1/2}}. \tag{3.56}$$

Using (3.55) and applying partial summation, one obtains that

$$\sum_{n\in\mathfrak{B}} \frac{\log n}{n^{1/2}} \ll_\theta X^{\frac{1}{2}-\theta}\log^3 X + \int_2^X t^{-\frac{1}{2}-\theta}\log^3 t\ dt. \tag{3.57}$$

Clearly, we have

$$\int_2^X t^{-\frac{1}{2}-\theta}\log^3 t\ dt \ll_\theta X^{\frac{1}{2}-\theta}\log^3 X. \tag{3.58}$$

Assembling (3.56)–(3.58), one gets the estimate

$$\sum_{n\in\mathfrak{B}} \frac{\Phi\left(n\right)}{n} \ll_\theta X^{\frac{1}{2}-\theta}\log^3 X. \tag{3.59}$$

To optimize the exponent, take $\theta = \frac{1}{4}$ so that from (3.49), (3.54) and (3.59),

$$\sum_{n\in\mathfrak{G}} \frac{\Phi\left(n\right)}{n} + \sum_{n\in\mathfrak{B}} \frac{\Phi\left(n\right)}{n} \ll X^{\frac{1}{4}}\log^3 X \ll x^{\frac{1}{6}}(\log x)^{\frac{19}{6}} \tag{3.60}$$

follows. Lastly, feeding (3.60) into (3.53), we complete the proof of the second part of Theorem 3.8, under the assumption of the Riemann hypothesis.

Let us now assume Conjecture 2.20. This shows that it is possible to take $\Phi(n) = c_2\log^2 n$ for some constant $c_2 > 0$. Similarly as above, we get from (3.23) that if we take any $C \in \left(0, \frac{1}{2}\right)$ and

$$r \le \frac{C}{c_2}\cdot\frac{n^2}{\log^2 n} = C_2\frac{n^2}{\log^2 n}, \tag{3.61}$$

then $h_n^{(r)}$ is not an integer. As $r \le x$, from (3.61),

$$n \le X := c_3\sqrt{x}\log x \tag{3.62}$$

follows for some constant $c_3 > 0$, depending on C_2. If $n > X$, then we know by Theorem 3.6 that $h_n^{(r)}$ is not an integer, as (3.61) is satisfied for all $n > X$. Using (3.62), we have

$$\mathcal{S}(x) = x^2 + \mathcal{O}\left(x \sum_{n \leq X} \frac{\Phi(n)}{n}\right). \tag{3.63}$$

Just noting that

$$\sum_{n \leq X} \frac{\Phi(n)}{n} \ll \sum_{n \leq X} \frac{\log^2 n}{n} \ll \log^3 X \ll \log^3 x \tag{3.64}$$

and combining (3.63) and (3.64), one completes the proof of Theorem 3.8. □

Note that the best error term that we can obtain for $\mathcal{S}(x)$ is $\mathcal{O}(x)$, since there are $x^2 + \mathcal{O}(x)$ many lattice points in $[1, x] \times [1, x]$. So, the error term that we obtained under Cramér's conjecture is very close to get the best possible bound for $\mathcal{S}(x) - x^2$.

3.2 Combinatorial Techniques

In this section, we will show some consequences of our results that we obtained in Section 2.3. For instance, we will use Proposition 2.9 to prove the last part of Theorem 1.2.

Theorem 3.9. *Suppose that $n \geq 2$ and $r \geq 2$ are positive integers. If there is a $p \in \mathbb{P}$ such that $\mathcal{M}_p(I(n,r)) = 1$ and $\mu_p(I(n,r)) > \nu_p\left(\binom{n+r-1}{r-1}\right)$, then $h_n^{(r)}$ is not an integer. In particular, for all $n, r \geq 2$, if n is even or r is odd then $h_n^{(r)} \notin \mathbb{Z}$. Moreover for all $\alpha \geq 1$ and prime p, we have $h_{p^\alpha}^{(r)} \notin \mathbb{Z}$.*

Proof. Assume that there is a prime number p such that $\mathcal{M}_p(I(n,r)) = 1$ and $\mu_p(I(n,r)) > \nu_p\left(\binom{n+r-1}{r-1}\right)$. Denote θ as the highest power of p in $I(n,r)$, that is $\mu_p(I(n,r)) = \theta$. By (1.2), we can say that $h_n^{(r)}$ is not an integer if and only if there is a prime p such that

$$\nu_p\left(\binom{n+r-1}{r-1}\right) + \nu_p(h_{n+r-1} - h_{r-1}) < 0. \tag{3.65}$$

We will show that the condition on p implies (3.65). So, consider the harmonic difference

$$h_{n+r-1} - h_{r-1} = \sum_{i=r}^{n+r-1} \frac{1}{i} = \frac{1}{cp^{\theta}} + \sum_{\substack{i=r \\ i \neq cp^{\theta}}}^{n+r-1} \frac{1}{i}, \tag{3.66}$$

where $p \nmid c$. As $\mathcal{M}_p\left(I(n,r)\right) = 1$, we have $\mu_p\left(I(n,r) \setminus \left\{cp^{\theta}\right\}\right) \leq \theta - 1$. Therefore, we get that

$$\nu_p\left(\sum_{\substack{i=r \\ i \neq cp^{\theta}}}^{n+r-1} \frac{1}{i}\right) > -\theta, \tag{3.67}$$

by the properties of the p-adic valuation; and thus $\nu_p\left(h_{n+r-1} - h_{r-1}\right) = -\theta$. By the assumption, we know that $\theta = \mu_p\left(I(n,r)\right) > \nu_p\left(\binom{n+r-1}{r-1}\right)$. This indicates that $\nu_p\left(h_n^{(r)}\right) < 0$; and hence we conclude by (3.65) that $h_n^{(r)}$ is not an integer.

To consider the second part, take $p = 2$. In that case, we know by Remark 2.5 that $\mathcal{M}_2\left(I(n,r)\right) = 1$. So if $n \geq 2$ is even, then we see that $n = (n_{\alpha}, \ldots, n_1, 0)_2$. This indicates that a carry cannot occur after the addition of $n_0 = 0$ and r'_0, where $r - 1 = (r'_{\beta}, r'_{\beta-1}, \ldots, r'_0)_2$. By Proposition 2.9, we get that $\nu_2\left(\binom{n+r-1}{r-1}\right) < \mu_2\left(I(n,r)\right)$. Thus, we deduce that $h_n^{(r)}$ is not an integer. Note that the same situation also holds, when $r \geq 2$ is odd.

In the last case, assume that $n = p^{\alpha}$, for some $\alpha \geq 1$ and $p \in \mathbb{P}$. Define $\theta_p = \mu_p\left(I(n,r)\right)$. As n is a positive power of p, we see that $\mathcal{M}_p\left(I(n,r)\right) = 1$, by Lemma 2.3. Moreover, we have

$$n = (1, \underbrace{0, 0, \ldots, 0}_{\alpha \text{ many}})_p,$$

where $\alpha \geq 1$. Similar to $p = 2$ case, we cannot obtain any carry after the addition of the first entries of n and $r - 1$. This indicates that $\nu_p\left(\binom{n+r-1}{r-1}\right) < \theta_p$, by Proposition 2.9 again. Hence, we conclude by the first part of the theorem that $h_n^{(r)}$ is not an integer. □

Remark 3.10. Note that the same result cannot be obtained, when we take $p > 2$. For example, choose any $n > 2$ and $p = 3$. In that case, we see that $\mathcal{M}_3(I(n,r))$ is either 1 or 2, for any integer $r \geq 1$. If we assume the conditions $\mathcal{M}_3(I(n,r)) = 1$ and $\nu_3\left(\binom{n+r-1}{r-1}\right) < \mu_3(I(n,r))$, then we get by Theorem 3.9 that $h_n^{(r)}$ is not an integer. But we may not deduce the same result, when $\mathcal{M}_3(I(n,r)) = 2$, even if the condition $\nu_3\left(\binom{n+r-1}{r-1}\right) < \mu_3(I(n,r))$ is satisfied: let $\mathcal{M}_3(I(n,r)) = 2$ and $\theta_3 = \mu_3(I(n,r))$. Then, there is an integer $c \geq 1$, where $c3^{\theta_3}, (c+1)3^{\theta_3} \in I(n,r)$ and $c \equiv 1 \pmod 3$. Observe that if $c \equiv 0, 2 \pmod 3$, then either c or $c+1$ is divisible by 3 which leads to the fact that $\mathcal{M}_3(I(n,r)) = 1$. So, we have

$$h_{n+r-1} - h_{r-1} = \frac{1}{c3^{\theta_3}} + \frac{1}{(c+1)3^{\theta_3}} + \sum_{\substack{i=r \\ 3^{\theta_3} \nmid i}}^{n+r-1} \frac{1}{i} = \frac{2c+1}{c(c+1)3^{\theta_3}} + q,$$

for some q, where the case $\nu_3(q) \geq 1 - \theta_3$ holds, which is similar to (3.67). Since $c \equiv 1 \pmod 3$, we have $\nu_3(c) = \nu_3(c+1) = 0$. Moreover, we obtain that $2c+1 \equiv 0 \pmod 3$. Hence, we get that $\nu_3\left(\frac{2c+1}{c(c+1)3^{\theta_3}}\right) = \nu_3(2c+1) - \theta_3 \geq 1 - \theta_3$. This indicates that

$$\nu_3(h_{n+r-1} - h_{r-1}) \geq \min\left\{\nu_3\left(\frac{2c+1}{c(c+1)3^{\theta_3}}\right), \nu_3(q)\right\} \geq 1 - \theta_3.$$

Therefore, if we assume that $\nu_3\left(\binom{n+r-1}{r-1}\right) = \theta_3 - 1$ which is obviously less than $\mu_3(I(n,r)) = \theta_3$, then equation (1.2) implies that

$$\nu_3\left(h_n^{(r)}\right) = \nu_3\left(\binom{n+r-1}{r-1}\right) + \nu_3(h_{n+r-1} - h_{r-1}) \geq 0.$$

Thus, we can say that $\nu_3\left(h_n^{(r)}\right) \geq 0$ may hold, for some special value of $r \in \mathbb{Z}_{>1}$.

It is possible to use Theorem 3.9 to obtain a huge class of (n,r) pairs which lead to non-integer $h_n^{(r)}$'s. Instead of giving all such possible tuples here, we will particularly mention the following consequence.

Corollary 3.11. *Let $a \in \mathbb{N}$ and $m, k, b \in \mathbb{Z}_{>0}$. Take any $p \in \mathbb{P}$, where $a + b \leq p$ holds. Assume that $n = mp + a = (n_\alpha, \ldots, n_0)_p$ and $r = kp + b = (r_\beta, \ldots, r_0)_p$. If, either*

1. *$r \in (cp^{\kappa} - n, cp^{\kappa}]$, for some integers $c \geq 1$ and $\kappa > \alpha$, or*

2. *p is a sufficiently large prime, depending on n and r,*

then we have $h_n^{(r)} \notin \mathbb{Z}$.

Proof. First of all, define θ as $\mu_p\left(I(n,r)\right)$. Note that $r_0 = b > 0$ and $n_0 = a \geq 0$. This implies that

$$0 \leq n_0 + (r_0 - 1) = a + b - 1 < p.$$

In other words, we cannot obtain any carry after the addition of the first entries of $n = (n_\alpha, \ldots, n_0)_p$ and $r - 1 = (r_\beta, \ldots, r_0 - 1)_p$, as $r_0 = b \geq 1$. Thus, we get that

$$\nu_p\left(\binom{n+r-1}{r-1}\right) < \theta, \tag{3.68}$$

by Proposition 2.9. Additionally, if we assume the first condition $r \in (cp^{\kappa} - n, cp^{\kappa}]$, then we have $r \leq cp^{\kappa} < n + r$, for some $c \geq 1$ and $\kappa > \alpha$. Therefore, we obtain that $\theta \geq \nu_p(c) + \kappa > \alpha$. From Lemma 2.3, we see that

$$\mathcal{M}_p\left(I(n,r)\right) = \left|I_{p^\theta}(n,r)\right| \in \left\{\left\lfloor \frac{n}{p^\theta} \right\rfloor, \left\lfloor \frac{n}{p^\theta} \right\rfloor + 1\right\} = \{0,1\},$$

as $\theta > \alpha$. Hence, we get that

$$\mathcal{M}_p\left(I(n,r)\right) = 1, \tag{3.69}$$

as $I(n,r) \neq \emptyset$. Since the conditions of Theorem 3.9 are satisfied, we conclude that $h_n^{(r)}$ is not an integer, as $n = mp + a \geq 2$ and $r = kp + b \geq 3$.

For the last part, we can take $\theta = \alpha \geq 1$ which is the lowest possible value for θ by Lemma 2.7. Notice that taking $\theta > \alpha$ yields $\mathcal{M}_p\left(I(n,r)\right) = 1$, as it is given in (3.69). Since (3.68) is satisfied, we deduce by Theorem 3.9 that $h_n^{(r)}$ is not an integer, when $\theta > \alpha$. So choose $\theta = \alpha$, as it is mentioned earlier. Let $d = \left\lceil \frac{r}{p^\alpha} \right\rceil$ and

$$s = \mathcal{M}_p\left(I(n,r)\right) = \left|I_{p^\alpha}(n,r)\right| \geq 1. \tag{3.70}$$

Obviously, d depends on r and s depends on both n and r. Now, we observe that

$$h_{n+r-1} - h_{r-1} = \frac{1}{p^{\alpha}} \cdot \left(\frac{1}{d} + \cdots + \frac{1}{d+s-1} \right) + \sum_{\substack{i=r \\ p^{\alpha} \nmid i}}^{n+r-1} \frac{1}{i}. \tag{3.71}$$

Assume that p is greater than the numerator of

$$z = \frac{N_z}{D_z} = \frac{1}{d} + \cdots + \frac{1}{d+s-1}. \tag{3.72}$$

Note that p cannot divide any of $j \in \{d, d+1, \ldots, d+s-1\}$, since otherwise we have $jp^{\alpha} \in I(n,r)$ and $\nu_p\left(jp^{\alpha}\right) \geq \alpha + 1 = \theta + 1$, which leads to a contradiction by the definition of $\theta = \mu_p\left(I(n,r)\right)$. Also, $p > N_z$ implies that $\nu_p(z) = 0$. Moreover, we have

$$\nu_p \left(\sum_{\substack{i=r \\ p^{\alpha} \nmid i}}^{n+r-1} \frac{1}{i} \right) \geq 1 - \alpha. \tag{3.73}$$

Applying the non-Archimedean property to equation (3.71), we get that

$$\nu_p\left(h_{n+r-1} - h_{r-1}\right) = -\alpha,$$

as $\nu_p(z) = 0$ and (3.73) hold. Combining this fact with (3.68), we conclude that $h_n^{(r)}$ is not an integer, by (3.65) in Theorem 3.9. □

Example 3.12. In order to provide an example for the first part, let $n = 33$, $p = 31$ and

$$r = 11\,082\,235 = (357\,491 \cdot 31 + 14) \in (12 \cdot 31^4 - 33, 12 \cdot 31^4].$$

By the first part of Corollary 3.11, we see that $h_n^{(r)} \notin \mathbb{Z}$. Note that we cannot obtain the non-integerness of $h_n^{(r)}$ for this value of r, when we use the methods given in Section 3.1. More precisely, we need to take

$$d = \left\lceil \frac{12 \cdot 31^4}{33} \right\rceil = 335\,826,$$

in order to use Theorem 3.6. After that, we will get the lower bound n_d in Theorem 3.6, where the lattice point

$$(n,r) \in \left\{ (n,r) \in \mathbb{N}^2 \;\middle|\; n \geq n_d \wedge r \leq dn \right\}$$

gives us a non-integer $h_n^{(r)}$. Unfortunately, this lower bound n_d may be so large to show the non-integerness of $h_n^{(r)}$, for $n = 33$ and $r = 11\,082\,235$. Therefore, combinatorial methods can surpass the analytic techniques in such cases.

To provide an example for the last part of Corollary 3.11, choose any prime number $p > 3$ and $n = 3p^\alpha$, for some $\alpha \geq 1$. Since $n = 3p^\alpha = (3, 0, \ldots, 0)_p$, observe that a carry cannot be obtained after the addition of the first entries of n and $r-1$, for any natural number $r \geq 2$. Also, if we let $s = \mathcal{M}_p\left(I(n,r)\right)$, then s is either equal to 1 or 3, by Lemma 2.3. As $I(n,r)$ is not empty, the case $s = 1$ occurs if and only if $\mu_p\left(I(n,r)\right) > \alpha$, by the same lemma again. Observe that this fact yields the non-integerness of $h_n^{(r)}$ by Theorem 3.9. So, assume that $s = 3$. In that case, we have

$$z = \frac{1}{d} + \frac{1}{d+1} + \frac{1}{d+2} = \frac{3d^2 + 6d + 2}{d(d+1)(d+2)},$$

by equation (3.72). If $p > 3d^2 + 6d + 2$, we know by Corollary 3.11 that $h_n^{(r)} \notin \mathbb{Z}$. Notice that the condition

$$\frac{3r^2}{p^{2\alpha}} + \frac{12r}{p^\alpha} + (11 - p) < 0$$

implies $3d^2 + 6d + (2 - p) < 0$. Computations give us the following bounds for r:

$$-2p^\alpha - p^\alpha\sqrt{\frac{p+1}{3}} < r < -2p^\alpha + p^\alpha\sqrt{\frac{p+1}{3}}.$$

Hence, we conclude that $h_{3p^\alpha}^{(r)}$ is not an integer, if $r < p^\alpha\left(\sqrt{\frac{p+1}{3}} - 2\right)$. Note that the latter inequality is valid when $p > 11$, as $r \geq 1$.

Remark 3.13. To compare the consequences of analytic and combinatorial techniques, take $n = sp^\alpha$ where s and α are positive integers and p is a prime. Recall by Theorem 3.9 that $h_n^{(r)} \notin \mathbb{Z}$, if $s \in 2\mathbb{Z} \cup \{1\}$. Moreover, by Example 3.12, we know that r can be taken as $r = \mathcal{O}\left(n \cdot n^{\frac{1}{2\alpha}}\right)$, when $s = 3$. Observe that choosing $\alpha = 1$ leads to an upper bound $\mathcal{O}\left(n\sqrt{n}\right)$ for r. Unfortunately, analytic methods do not yield such a bound for r, even if we assume Riemann hypothesis. Note by equation (3.48) that r can be at most of order $n^{\frac{3}{2}-\varepsilon}$, for any $\varepsilon > 0$. Thus, applying

combinatorial techniques to $n = sp^{\alpha}$ (for $s \leq 4$ and $\alpha > 0$) may lead to better bounds for r in some cases, when it is compared to analytic methods.

Unfortunately, this situation cannot hold for the other values of s. To see this, first pick $s = 2t + 1$, where $t \geq 2$. Assume that $d = \left\lceil \frac{r}{p^{\alpha}} \right\rceil$ and $\mathcal{M}_p\left(I(n,r)\right) = s \geq 5$ (the case $\mathcal{M}_p\left(I(n,r)\right) = 1$ is already covered in Corollary 3.11 and Example 3.12). Also, let H be the harmonic mean of the integers $d, d+1, \ldots, d+s-1$ and $P = d(d+1)\cdots(d+s-1)$. By equation (3.72), we have

$$z = \frac{1}{d} + \cdots + \frac{1}{d+s-1} = \frac{s}{H} = \frac{\frac{sP}{H}}{d(d+1)\cdots(d+s-1)}.$$

Obviously, the inequality $d \leq H$ is satisfied, as

$$\frac{1}{d} + \cdots + \frac{1}{d+s-1} \leq \frac{s}{d}.$$

Therefore, we have

$$\frac{sP}{H} \leq \frac{sP}{d} = s(d+1)\cdots(d+s-1).$$

Hence, if $s(d+1)\cdots(d+s-1) < p$, we see by Corollary 3.11 that $h_n^{(r)}$ is not an integer. This indicates that we may have a bound of the form $\mathcal{O}\left(p^{\frac{1}{s-1}}\right)$ for d, when s is fixed. As $d = \left\lceil \frac{r}{p^{\alpha}} \right\rceil$ and $n = sp^{\alpha}$, we get that

$$r = \mathcal{O}\left(p^{\alpha + \frac{1}{s-1}}\right) = \mathcal{O}\left(n^{1 + \frac{1}{\alpha(s-1)}}\right). \tag{3.74}$$

Notice that this bound for r is small when it is compared to the one that we obtained using analytic methods. In particular, we know that r can be taken of order $n^{1.475}$, by Theorem 3.6. However, the best possible bound that we can obtain from (3.74) is $\mathcal{O}\left(n^{1.25}\right)$, since

$$1 + \frac{1}{\alpha(s-1)} \leq 1.25,$$

as $\alpha \geq 1$ and $s \geq 5$. Thus, we need to apply different techniques in different situations to get more general results on the non-integerness problem.

Now, we can prove the first part of Theorem 1.2 as a corollary of Theorem 3.9.

Corollary 3.14. *Let $n, r \geq 2$ be given integers. Assume that $\mathcal{M}_p\left(I(n,r)\right) = 1$ and*

$$\left\{\frac{r-1}{p^k}\right\} + \left\{\frac{n}{p^k}\right\} < 1 \tag{3.75}$$

holds for some k with $p^k \leq n$. Then $h_n^{(r)} \notin \mathbb{Z}$. In particular, if

$$\left\{\frac{r-1}{2^k}\right\} + \left\{\frac{n}{2^k}\right\} < 1 \tag{3.76}$$

holds for some k with $2^k \leq n$, then $h_n^{(r)} \notin \mathbb{Z}$.

Proof. Let $n = (n_\alpha, n_{\alpha-1}, \ldots, n_0)_p$ and $r-1 = (r'_\beta, r'_{\beta-1}, \ldots, r'_0)_p$ be p-ary representations of n and $r-1$, respectively. We will show that condition (3.75) implies that a carry does not occur after the addition of n_{k-1} and r'_{k-1}. Suppose not. Consider the sum $s = \left\{\frac{n}{p^k}\right\} + \left\{\frac{r-1}{p^k}\right\}$. Note that

$$\begin{aligned}
\frac{n}{p^k} &= n_\alpha p^{\alpha-k} + \cdots + n_k + \frac{n_{k-1}p^{k-1} + \cdots + n_0}{p^k} \\
\frac{r-1}{p^k} &= r'_\beta p^{\beta-k} + \cdots + r'_k + \frac{r'_{k-1}p^{k-1} + \cdots + r'_0}{p^k}.
\end{aligned}$$

Therefore, we get that

$$\begin{aligned}
\left\{\frac{n}{p^k}\right\} &= \frac{n_{k-1}p^{k-1} + \cdots + n_0}{p^k} = (0; n_{k-1}, \ldots, n_0)_p, \\
\left\{\frac{r-1}{p^k}\right\} &= \frac{r'_{k-1}p^{k-1} + \cdots + r'_0}{p^k} = (0; r'_{k-1}, \ldots, r'_0)_p.
\end{aligned}$$

If a carry occurs after the addition of n_{k-1} and r'_{k-1}, then $s = (1; s_{-1}, s_{-2}, \ldots, s_{-k})_p$, for some $s_{-1}, s_{-2}, \ldots, s_{-k} \in \{0, \ldots, p-1\}$. In that case, we have

$$s = 1 + \frac{s_{-1}p^{k-1} + \cdots + s_{-k}}{p^k} \geq 1,$$

which is a contradiction. Thus, for the digit $i = k-1$, a carry does not occur after the addition of n_i and r'_i. Hence, we get that $\nu_p\left(\binom{n+r-1}{r-1}\right) < \mu_p\left(I(n,r)\right)$ by Proposition 2.9. By the assumption $\mathcal{M}_p\left(I(n,r)\right) = 1$, we conclude that $h_n^{(r)} \notin \mathbb{Z}$, since the conditions of Theorem 3.9 are satisfied. In particular, when $p = 2$, $\mathcal{M}_2\left(I(n,r)\right) = 1$ comes for free from Remark 2.5. So condition (3.76) is sufficient to say that $h_n^{(r)} \notin \mathbb{Z}$. □

Let us see that if $n \geq 2$ is even, then taking $k = 1$ in (3.76), we see that $h_n^{(r)} \notin \mathbb{Z}$. If r is odd, then we see again from (3.76) and [Theisinger, 1915] that $h_n^{(r)} \notin \mathbb{Z}$ for $n \geq 2$. Lastly, if $n = p^s \geq 2$ is a prime power, where we may assume that p is odd, then taking $k = s$ in (3.75) and observing that p^m with $m \geq s$ divides exactly one of $r, r+1, \ldots, r+n-1$, we have $h_n^{(r)} \notin \mathbb{Z}$. Therefore, the rest of Theorem 1.2 can also be deduced from Corollary 3.14.

Remark 3.15. Let n, a be non-negative integers and $k \geq 1$. Assume that $n \equiv a \pmod{2^k}$, $a \in \left\{0, 1, \ldots, 2^k - 1\right\}$ and $2^k \leq n$. It follows that

$$\left\{\frac{n}{2^k}\right\} = \frac{a}{2^k}. \tag{3.77}$$

Also, if $r \equiv 1, 2, \ldots 2^k - a \pmod{2^k}$, then

$$\left\{\frac{r-1}{2^k}\right\} \leq \frac{2^k - a - 1}{2^k}. \tag{3.78}$$

Combining (3.77) and (3.78), one obtains that

$$\left\{\frac{r-1}{2^k}\right\} + \left\{\frac{n}{2^k}\right\} \leq \frac{2^k - 1}{2^k}. \tag{3.79}$$

Since the conditions of the last part of Corollary 3.14 are satisfied, we conclude that $h_n^{(r)} \notin \mathbb{Z}$, even if $r = 1$, as $n \geq 2^k \geq 2$. This fact will be useful to obtain density results for a fixed $n \in \mathbb{Z}_{\geq 2}$.

As a final result of this section, we will show the last part of Theorem 1.3.

Corollary 3.16. *The set of integers r such that $h_{33}^{(r)} \notin \mathbb{Z}$ contains a set of density*

$$\frac{143}{144} \approx 0.993.$$

Proof. We will try to find the set of r values such that $h_{33}^{(r)}$ may be an integer. So first of all, we may use Remark 3.15 to see that $h_n^{(r)} \notin \mathbb{Z}$ unless $r \equiv 0 \pmod{2^5}$, as $33 \equiv 1 \pmod{2^5}$. Now for any $r \geq 2$, assume that

$$\left\{\frac{r-1}{3^3}\right\} + \left\{\frac{33}{3^3}\right\} < 1, \tag{3.80}$$

and let $r - 1 \equiv a \pmod{27}$, for some $a \in \{0, \ldots, 26\}$. Note that the condition (3.80) is equivalent to $a + 6 < 27$, and this inequality is satisfied if and only if

$r \equiv 1, 2, \ldots, 21 \pmod{27}$. If we let $r = b + 27\ell$, for some $\ell \in \mathbb{N}$ and $b \in \{1, \ldots, 21\}$, then we will show that $r + 27 - b$ is the only multiple of 27 in $I(33, r)$. Obviously the inequalities

$$r - b < r \leq r + 27 - b \leq r + 32$$

hold, as $b \in \{1, \ldots, 21\}$. Moreover, since $b < 22$, we see that $r + 32 < r + 54 - b$, where the latter one is a multiple of 27. Hence, we deduce that $|I(33, r) \cap 27\mathbb{Z}| = 1$. Recall by Lemma 2.7, we have $\mu_3(I(33, r)) \geq 3$, which indicates that at least one of the integers in $I(33, r)$ is divisible by $3^3 = 27$. Hence by Definition 2.4, we deduce that $\mathcal{M}_3(I(n, r)) = 1$. By Corollary 3.14, we observe that the condition (3.80) yields that $h_{33}^{(r)} \notin \mathbb{Z}$, as $3^3 \leq 33$. As a side remark, note that $h_{33}^{(r)} \notin \mathbb{Z}$ is also satisfied, when $r = 1$. So $h_{33}^{(r)}$ is not an integer, unless $r \equiv 0, 22, 23, 24, 25, 26 \pmod{27}$ and $r \equiv 0 \pmod{32}$. By the well-known Chinese Remainder Theorem, it can be observed that there are 6 common solutions for r modulo $2^5 \cdot 3^3 = 864$, where $h_{33}^{(r)}$ may be an integer. Thus, the set $\left\{ r \in \mathbb{N} \,\middle|\, h_{33}^{(r)} \notin \mathbb{Z} \right\}$ contains a set of density

$$1 - \frac{6}{864} = \frac{143}{144},$$

as desired. □

3.3 Algebraic Methods & Computations

This section contains the necessary materials which were also given in [Göral and Sertbaş, 2017, Section 4], in order to prove Theorem 1.3 except its last part. To do so, we need to prove a general result first.

Theorem 3.17. *Assume that $n = kp^\alpha$ is an odd integer for some prime number p, integer $\alpha \geq 1$ and $r \geq 2$. Let $a = \frac{k-1}{2}$, $c = \left\lceil \frac{r}{p^\alpha} \right\rceil$ and*

$$F_k(x) := \sum_{i=-a}^{a} \left[\prod_{j=-a}^{a} (x - j) \right] \frac{1}{x - i}.$$

If $\nu_p(F_k(c + a)) \leq \nu_p(k!)$, then $h_n^{(r)} \notin \mathbb{Z}$. In particular, if $c + a$ is not a root of $F_k(x)$ in modulo p, then $h_n^{(r)}$ is not an integer.

Proof. First, observe that $|I_{p^\alpha}(n,r)| = \frac{n}{p^\alpha} = k$, by Lemma 2.3. Now, consider

$$\begin{aligned} h_n^{(r)} &= \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}) \\ &= \frac{\mathrm{N}}{\mathrm{D}} \cdot \frac{cp^\alpha \cdot (c+1)p^\alpha \cdots (c+k-1)p^\alpha}{p^\alpha \cdot 2p^\alpha \cdots kp^\alpha} \cdot \left(\sum_{i=0}^{k-1} \frac{1}{(c+i)p^\alpha} + \sum_{\substack{\ell=r \\ p^\alpha \nmid \ell}}^{n+r-1} \frac{1}{\ell} \right), \end{aligned}$$

for some $\mathrm{N}, \mathrm{D} \in \mathbb{Z}$, where $\nu_p(\mathrm{N}) = \nu_p(\mathrm{D})$. To observe the latter fact, first notice that N is the multiplication of the elements in $I(n,r)$ which are not divisible by p^α, whereas D is the multiplication of the elements in $I(n,1)$ which are not divisible by p^α. Again by Lemma 2.3, we see that for any $1 \leq \beta < \alpha$ we have $\left|I_{p^\beta}(n,r)\right| = \frac{n}{p^\beta} = kp^{\alpha-\beta}$. Therefore, the cardinality

$$\begin{aligned} T_{p^\beta}(n,r) &:= \left|\left\{a \in I(n,r) \,\middle|\, p^\beta \parallel a\right\}\right| \\ &= \left|\left\{a \in I(n,r) \,\middle|\, p^\beta \mid a\right\} \setminus \left\{a \in I(n,r) \,\middle|\, p^{\beta+1} \mid a\right\}\right| \\ &= kp^{\alpha-\beta} - kp^{\alpha-\beta-1}. \end{aligned}$$

Similarly, we get that $T_{p^\beta}(n,1) = kp^{\alpha-\beta} - kp^{\alpha-\beta-1}$, as $\left|I_{p^\beta}(n,1)\right| = \left\lfloor \frac{n}{p^\beta} \right\rfloor = kp^{\alpha-\beta}$. Hence, for each $1 \leq \beta < \alpha$, the number of terms which are exactly divisible by p^β in N and D are the same. Canceling each of these terms yields that $\nu_p(\mathrm{N}) = \nu_p(\mathrm{D})$. Next, let $q_1 = \frac{\mathrm{N}}{\mathrm{D}}$,

$$q_2 = \sum_{\substack{\ell=r \\ p^\alpha \nmid \ell}}^{n+r-1} \frac{1}{\ell}$$

and $\mathrm{C} = c(c+1)\cdots(c+k-1)$. Notice that $\nu_p(q_1) = 0$. Also, observe that $\nu_p(\mathrm{C}) \geq 0$, as C is an integer. Moreover, since the lowest power of p which is contained in the sum q_2 is equal to $1-\alpha$, we see that $\nu_p(q_2) > -\alpha$, by the non-Archimedean property of the p-adic valuation. Note that

$$h_n^{(r)} = \frac{q_1}{p^\alpha \cdot k!} \cdot \sum_{i=0}^{k-1} \left[\left(\prod_{j=0}^{k-1} (c+j) \right) \cdot \frac{1}{c+i} \right] + \frac{\mathrm{C} \cdot q_1 q_2}{k!}.$$

Definitions of $F_k(x)$ and a yield that

$$F_k(x+a) = \sum_{i=0}^{k-1} \left[\left(\prod_{j=0}^{k-1} (x+j) \right) \cdot \frac{1}{x+i} \right].$$

Again by the non-Archimedean property of the p-adic valuation, we have that

$$\nu_p\left(h_n^{(r)}\right) \geq \min\left\{\nu_p\left(\frac{q_1 \cdot F_k(c+a)}{p^\alpha \cdot k!}\right), \nu_p\left(\frac{\mathrm{C} \cdot q_1 q_2}{k!}\right)\right\}. \tag{3.81}$$

By the assumption, we take $\nu_p\left(F_k(c+a)\right) \leq \nu_p\left(k!\right)$. Then the p-adic valuation of the first term is

$$\nu_p\left(\frac{q_1 \cdot F_k(c+a)}{p^\alpha \cdot k!}\right) = \nu_p\left(q_1\right) + \nu_p\left(F_k(c+a)\right) - \nu_p\left(k!\right) - \nu_p\left(p^\alpha\right) \leq -\alpha, \tag{3.82}$$

as $\nu_p\left(q_1\right) = 0$. Also, we have

$$\nu_p\left(\frac{\mathrm{C} \cdot q_1 q_2}{k!}\right) = \nu_p\left(\mathrm{C}\right) + \nu_p\left(q_2\right) - \nu_p\left(k!\right).$$

Note that

$$\frac{\mathrm{C}}{k!} = \frac{c \cdot (c+1) \cdots (c+k-1)}{1 \cdot 2 \cdots k} = \binom{c+k-1}{k}$$

is an integer. Hence, we see that $\nu_p\left(\mathrm{C}\right) \geq \nu_p\left(k!\right)$. This shows that

$$\begin{aligned}\nu_p\left(\frac{\mathrm{C} \cdot q_1 q_2}{k!}\right) &= \nu_p\left(\mathrm{C}\right) + \nu_p\left(q_2\right) - \nu_p\left(k!\right) \geq \nu_p\left(q_2\right) \\ &> -\alpha \geq \nu_p\left(\frac{q_1 \cdot F_k(c+a)}{p^\alpha \cdot k!}\right),\end{aligned} \tag{3.83}$$

by (3.82). Combining the properties of the p-adic valuation with (3.81) and (3.83), we conclude that

$$\nu_p\left(h_n^{(r)}\right) = \nu_p\left(\frac{q_1 \cdot F_k(c+a)}{p^\alpha \cdot k!}\right) \leq -\alpha \leq -1,$$

which indicates the non-integerness of $h_n^{(r)}$.

As a special case, if $c+a$ is not a root of $F_k(x)$ in $\mathbb{F}_p$, then $F_k(c+a) \not\equiv 0 \pmod{p}$, which implies that $\nu_p\left(F_k(c+a)\right) = 0$. Note that $\nu_p\left(k!\right) \geq 0 = \nu_p\left(F_k(c+a)\right)$, as $k! \in \mathbb{Z}$. Hence, we deduce that $h_n^{(r)} \notin \mathbb{Z}$, by the first part of the theorem. □

Remark 3.18. As we obtained in Theorem 3.9, one can also apply Theorem 3.17 to show that $h_n^{(r)}$ is not an integer, when n is a prime power. To see this fact, let $k = 1$. Then, we have $a = 0$ and $F_1(x) = 1$, for any real number x. Therefore, $\nu_p\left(F_1(c)\right) = 0$, for any $c \in \mathbb{Z}$ and $p \in \mathbb{P}$. Also, we know that $\nu_p\left(k!\right) = \nu_p\left(1\right) = 0$. This

indicates that $\nu_p\left(F_k(c)\right) \leq \nu_p\left(k!\right)$. Using Theorem 3.17, we deduce that $h_{p^\alpha}^{(r)} \notin \mathbb{Z}$, for any integer $r \geq 1$. Note that the initial case $r = 1$ also satisfies this property, as for all $p \in \mathbb{P}$ and $\alpha \geq 1$, we have $p^\alpha \geq 2$.

We would like to point out that for a given $r \geq 2$ and a sufficiently large prime $p > F_k(c+a)$, we see that $h_n^{(r)} \notin \mathbb{Z}$, as $c+a$ cannot be a root of $F_k(x)$. Similar situations and its generalizations were also considered in Example 3.12 and Remark 3.13. Apart from that, Theorem 3.17 can be used to classify the values of n, for which $h_n^{(r)} \notin \mathbb{Z}$. In particular, we can obtain the fourth and the fifth parts of Theorem 1.3 as a consequence of Theorem 3.17.

Corollary 3.19. *Suppose that $\alpha \geq 1$ and $r \geq 2$ are two integers. For any prime number p, if $p \equiv \pm 5 \pmod{12}$, then $h_{3p^\alpha}^{(r)}$ is not an integer. Moreover, if*

$$p \equiv \begin{array}{l} 7, 11, 13, 14, 19, 21, 22, 23, 26, 28, 31, 33, 38, 39, \\ 41, 42, 44, 46, 52, 53, 56, 57, 61, 62, 63, 66, 67, 69, \\ 76, 78, 79, 82, 83, 84, 88, 89, 92, 93, 99, 101, 103, \\ 104, 106, 107, 112, 114, 117, 119, 122, 123, 124, \\ 126, 131, 132, 134, 138 \end{array} \pmod{145}, \tag{3.84}$$

then $h_{5p^\alpha}^{(r)} \notin \mathbb{Z}$.

Proof. To prove such a corollary of Theorem 3.17, we obtain the necessary form of the prime p, so that the polynomial $F_k(x)$ has no roots in $\mathbb{F}_p$, for $k = 3, 5$. So first, observe that $F_3(x) = 3x^2 - 1$. Note that $F_3(x)$ does not have any root in $\mathbb{F}_p$ if and only if

$$3x^2 - 1 \not\equiv 0 \pmod{p} \;\Leftrightarrow\; x^2 \not\equiv 3^{-1} \pmod{p}. \tag{3.85}$$

Here a^{-1} represents the inverse of $a \in \mathbb{F}_p^\times$. By (3.85), we see that $h_{3p^\alpha}^{(r)} \notin \mathbb{Z}$, unless 3 is a square in modulo p. In other words, the condition $\left(\frac{3}{p}\right) = -1$ implies the non-integerness of $h_{3p^\alpha}^{(r)}$, where $\left(\frac{\cdot}{p}\right)$ is the well-known Legendre symbol. Now, the Quadratic Reciprocity Law, which we mentioned in Section 2.2, indicates that

$$\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}} \cdot \left(\frac{p}{3}\right). \tag{3.86}$$

Therefore, taking $p \equiv \pm 5 \pmod{12}$ yields that $\left(\frac{3}{p}\right) = -1$, by (3.86). This shows that 3 is not a square in $\mathbb{F}_p$; and hence we deduce that $h_{3p^\alpha}^{(r)}$ is not an integer.

Next, we consider the case $k = 5$. Similar to the previous case, it is enough to find the necessary form of p, where $F_5(x) = 5x^4 - 15x^2 + 4$ does not have any roots in modulo p. Notice that $F_5(x) = 5t^2 - 15t + 4$, for $t = x^2$. Therefore, there does not exist any root of $F_5(x)$ modulo p, unless the discriminant Δ of $5t^2 - 15t + 4$ is a square in $\mathbb{F}_p$. A straightforward computation shows that $\Delta = 145 = 5 \cdot 29$. Also by the multiplicativity of the Legendre symbol and the Quadratic Reciprocity Law, we get that

$$\left(\frac{145}{p}\right) = \left(\frac{29}{p}\right) \cdot \left(\frac{5}{p}\right) = \left(\frac{p}{5}\right) \cdot \left(\frac{p}{29}\right),$$

as $5, 29 \equiv 1 \pmod{4}$. So, if $\left(\frac{145}{p}\right) = -1$, then $h_{5p^\alpha}^{(r)}$ is not an integer, by Theorem 3.17 again. Note that this case is satisfied, unless p is a square or a non-square in $\mathbb{F}_5$ and $\mathbb{F}_{29}$, simultaneously. Thanks to SageMath [Sage Developers, 2018], we compute the list of squares and non-squares in $\mathbb{F}_5$ and $\mathbb{F}_{29}$. Applying Chinese Remainder Theorem, we obtain the equivalence classes given in (3.84), where each of them leads to the non-integerness of $h_{5p^\alpha}^{(r)}$, for any $r \geq 2$. □

Finally, we can combine all of our results to obtain the rest of Theorem 1.3.

Corollary 3.20. *Suppose that $n \geq 2$ and $r \geq 1$ are two natural numbers. Then $h_n^{(r)}$ is not an integer, if*

1. *$n \leq 32$, or*
2. *$r \leq 20n$, or*
3. *$r \leq 35\,001$.*

Proof. By Theorem 3.9, we see that

$$N = \{1 < n \leq 32 \,|\, n \text{ is even or a prime power}\} \subseteq \left\{1 < n \leq 32 \,\middle|\, h_n^{(r)} \notin \mathbb{Z}\right\}.$$

Observe that $N = \{2, \ldots, 32\} \setminus \{15, 21\}$. Also for any $n \in \{15, 21\}$, we have $n = 3p$ such that $p \equiv \pm 5 \pmod{12}$. By Corollary 3.19, we obtain that $h_{15}^{(r)}, h_{21}^{(r)} \notin \mathbb{Z}$, for any natural number $r \geq 1$. Thus, we conclude the first part.

For the rest of the corollary, we will apply Theorem 3.6. So, we first evaluate the lower bound x_d, where for all $x \geq x_d$ there is a prime in $\left(\left(1 - \frac{1}{2d+7}\right) x, x\right]$. For this purpose, we use the bound in [Dusart, 1998, Theorem 1.9]. Next, for every $d \leq 20$, we calculate $n_d = \left\lceil \frac{9}{7} \cdot x_d \right\rceil$ and $r_d = d \cdot n_d$, as we did in Theorem 3.6. Table 3.1 gives us the corresponding values of x_d, n_d and r_d, for each $d \leq 20$. In order to prove that $h_n^{(r)}$ is not an integer for any $r \leq 20n$, we need to check all the lattice points in the region

$$R = \left\{(n, r) \in \mathbb{Z}^2 \,\middle|\, 1 < n \leq 1750,\, 1 \leq r \leq 35\,000\right\},$$

since $n_{20} = 1\,750$ and $r_{20} = 35\,000$. This will yield that $h_n^{(r)} \notin \mathbb{Z}$, for any $r \leq 35\,000$, as for all $n \geq n_{20}$ and $r \leq 20n$, we have the non-integerness of $h_n^{(r)}$ by Theorem 3.6. Also by Theorem 3.9 and Corollary 3.19, we have $1\,260$ many values of $n \leq 1750$, where $h_n^{(r)} \notin \mathbb{Z}$. Moreover, we can eliminate the odd values of $r \in [1, 35\,000]$, by Theorem 3.9. By SageMath [Sage Developers, 2018], we deduced that $h_n^{(r)} \notin \mathbb{Z}$, for any $(n, r) \in R$. Observe that if $r = 35\,001$ then $h_n^{(r)} \notin \mathbb{Z}$, by Theorem 3.9 again. Thus, we conclude that $h_n^{(r)} \notin \mathbb{Z}$, if $r \leq 35\,001$ or $r \leq 20n$, for any $n \in \mathbb{Z}_{>1}$. □

Table 3.1: The list of x_d, n_d and $r_d = d \cdot n_d$ values, where $d \leq 20$.

d	x_d	n_d	r_d
1	53	69	69
2	127	164	328
3	127	164	492
4	149	192	768
5	149	192	960
6	223	287	1.722
7	223	287	2.009
8	307	395	3.160
9	331	426	3.834
10	331	426	4.260
11	331	426	4.686
12	541	696	8.352
13	541	696	9.048
14	541	696	9.744
15	541	696	10.440
16	541	696	11.136
17	1361	1750	29.750
18	1361	1750	31.500
19	1361	1750	33.250
20	1361	1750	35.000

Chapter 4

DISPROOF OF THE CONJECTURE

From the previous chapter, it is known that $h_n^{(r)} \notin \mathbb{Z}$, for every $r \leq 35\,001$. Observe that this result does not depend on the choice of $n > 1$. However, the same non-integerness bound for n is rather small compared to r, i.e. $2 \leq n \leq 32$. The methods that were given in the previous chapter are not sufficient to show that $h_{33}^{(r)}$ is not an integer, for every $r \geq 1$. In this chapter, we will explain why the given methods cannot work. Namely, we will show that there are some values of $r > 1$ where $h_{33}^{(r)}$ is an integer. To obtain integer hyperharmonic numbers, we first observe that it is enough to get a non-negative p-adic valuation of $h_{33}^{(r)}$, for every prime $p \in [2, 31]$. After that, we give a general form of r which leads to a non-negative 2-adic valuation for $h_{33}^{(r)}$. Using a remark in [Göral and Sertbaş, 2017], we consider each prime $p \in [7, 31]$ except 11 and give sufficient conditions on $r \pmod p$ to get $\nu_p\left(h_{33}^{(r)}\right) \geq 0$. Combining the given general form of r with each of these conditions that come from different primes $p \in \{7, 13, 17, 19, 23, 29, 31\}$, we obtain several common solutions for r which satisfy $\nu_p\left(h_{33}^{(r)}\right) \geq 0$, for all primes $p \leq 33$ except $p = 3, 5, 11$. Analyzing each of the remaining prime cases together with the previous ones, we deduce that there are 153 different α values modulo 748 440 given in Appendix A, where the corresponding $r = 64 \cdot (2^\alpha - 1) + 32$ gives us an integer $h_{33}^{(r)}$.

4.1 Hyperharmonic Integers

As a consequence of Lemma 2.2, it is enough to consider the p-adic valuation of $h_{33}^{(r)}$ for all primes $p \leq 33$. So we begin with the case $p = 2$:

Proposition 4.1. *For any $\alpha \geq 0$, we have $\nu_2\left(h_{33}^{(r)}\right) = 0$, if $r = 64 \cdot (2^\alpha - 1) + 32$.*

Proof. By equation (1.2), we get

$$\nu_2\left(h_n^{(r)}\right) = \nu_2\left(\binom{n+r-1}{r-1}\right) + \nu_2\left(h_{n+r-1} - h_{r-1}\right). \tag{4.1}$$

So, the equality

$$\nu_2\left(\binom{r+32}{r-1}\right) = -\nu_2\left(h_{r+32} - h_{r-1}\right)$$

implies that $\nu_2\left(h_{33}^{(r)}\right) = 0$. Let $\theta = \mu_2\left(I(33,r)\right)$ where $\mu_2(\cdot)$ is given in Definition 2.4. Then there exists an odd integer c such that $c \cdot 2^\theta \in I(33,r)$, otherwise it contradicts the fact that $\theta = \mu_2\left(I(33,r)\right)$. Also by Remark 2.5, we know that $\mathcal{M}_2\left(I(33,r)\right) = 1$. This leads to the fact that $h_{r+32} - h_{r-1} = \dfrac{1}{c2^\theta} + Q_2$, for some $Q_2 \in \mathbb{Q}$ where $\nu_2\left(Q_2\right) \geq -(\theta - 1)$. By the non-Archimedean property of the 2-adic valuation, we conclude that

$$\nu_2\left(h_{r+32} - h_{r-1}\right) = -\theta. \tag{4.2}$$

So in order to get $\nu_2\left(h_{33}^{(r)}\right) = 0$, we need to have

$$\nu_2\left(\binom{r+32}{r-1}\right) = \theta = \mu_2\left(I(33,r)\right).$$

As it is given in Proposition 2.9, the latter case holds if we write $r-1$ and 33 in binary representations and obtain carries in all possible places after the addition of them. Observe that the condition on r is equivalent to $r - 1 = 2^6(2^\alpha - 1) + 31$, where $31 = (1,1,1,1,1)_2$. Therefore, we have

$$r - 1 = (\overbrace{1,1,\ldots,1}^{\alpha \text{ many}},0,1,1,1,1,1)_2$$
$$33 = (0,0,\ldots,0,1,0,0,0,0,1)_2.$$

Notice that we obtain a carry at each step in the addition of $r-1$ and 33. Thus, we conclude that $\nu_2\left(h_{33}^{(r)}\right) = 0$, if $r = 2^6(2^\alpha - 1) + 32$ where $\alpha \geq 0$. □

Remark 4.2. Note that the given form of r is compatible with the one that we obtained in the proof of Corollary 3.16. Moreover, observe that for any $n, r \geq 2$, we can extend the result in (4.2) to

$$\nu_2\left(\left(h_{n+r-1} - h_{r-1}\right)\right) = -\mu_2\left(I(n,r)\right),$$

since $\mathcal{M}_2\left(I(n,r)\right) = 1$ holds by Remark 2.5. Combining this fact with equation (4.1), we obtain that

$$\nu_2\left(h_n^{(r)}\right) = \nu_2\left(\binom{n+r-1}{r-1}\right) - \mu_2\left(I(n,r)\right).$$

As we know by Proposition 2.9 that the inequality

$$\nu_p\left(\binom{n+r-1}{r-1}\right) \leq \mu_p\left(I(n,r)\right)$$

is satisfied for any prime p. Hence, we conclude that $\nu_2\left(h_n^{(r)}\right) \leq 0$, for all $n, r \in \mathbb{N}_{\geq 1}$, as $h_1^{(r)} = 1$. One can check that the case $r = 1$ was obtained in [Theisinger, 1915] where it was shown that $\nu_2\left(h_n\right) \leq -1$, for any $n \geq 2$. So the form of r that we obtained in Proposition 4.1 is one of the best possible one in the sense of the 2-adic valuation.

The following observation which was also given in [Göral and Sertbaş, 2017, Section 5], will be useful for the construction of hyperharmonic integers.

Lemma 4.3. *Let $n \geq 2$ be a given natural number. Suppose that there exists a prime number p such that $k = |I_p(n,1)| < p$. Moreover, assume that $r \geq 2$ is given so that $|I_p(n,r)| = k+1$ is satisfied. Then $\nu_p\left(h_n^{(r)}\right) \geq 0$.*

Proof. Consider

$$\begin{aligned} h_n^{(r)} &= \frac{\mathrm{N}_1}{\mathrm{D}_1} \cdot \frac{cp \cdot (c+1)p \cdots (c+k)p}{p \cdot 2p \cdots kp} \cdot \left(\frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n+r-1}\right) \\ &= \frac{\mathrm{N}_1}{\mathrm{D}_1} \cdot \frac{p^{k+1} \cdot c \cdot (c+1) \cdots (c+k)}{p^k \cdot k!} \\ &\qquad \cdot \left(\frac{1}{p} \cdot \left(\frac{1}{c} + \frac{1}{c+1} + \cdots + \frac{1}{c+k}\right) + \frac{\mathrm{N}_2}{\mathrm{D}_2}\right), \end{aligned}$$

where $\nu_p\left(\mathrm{N}_1\right) = \nu_p\left(\mathrm{D}_1\right) = \nu_p\left(\mathrm{D}_2\right) = 0 \leq \nu_p\left(\mathrm{N}_2\right)$. This gives us

$$\begin{aligned} h_n^{(r)} &= \frac{\mathrm{N}_1 \cdot p}{\mathrm{D}_1 \cdot k!} \cdot c \cdot (c+1) \cdots (c+k) \cdot \left(\frac{1}{p} \cdot \frac{\mathrm{A}}{c \cdot (c+1) \cdots (c+k)} + \frac{\mathrm{N}_2}{\mathrm{D}_2}\right) \\ &= \frac{\mathrm{N}_1 \cdot \mathrm{A}}{\mathrm{D}_1 \cdot k!} + \frac{\mathrm{N}_1 \cdot \mathrm{N}_2}{\mathrm{D}_1 \cdot \mathrm{D}_2} \cdot \frac{p \cdot c \cdot (c+1) \cdots (c+k)}{k!}, \end{aligned}$$

where

$$\nu_p\left(\mathrm{A}\right) = \nu_p\left(\sum_{i=0}^{k} \frac{c \cdot (c+1) \cdots (c+k)}{c+i}\right) \geq 0.$$

The assumption $k < p$ implies that $\nu_p(k!) = 0$. Hence, we get that $\nu_p\left(\frac{\mathrm{N}_1 \cdot \mathrm{A}}{\mathrm{D}_1 \cdot k!}\right) = \nu_p(\mathrm{A}) \geq 0$. For the second summand, we also have

$$\nu_p\left(\frac{\mathrm{N}_1 \cdot \mathrm{N}_2}{\mathrm{D}_1 \cdot \mathrm{D}_2} \cdot \frac{p \cdot c \cdot (c+1) \cdots (c+k)}{k!}\right)$$
$$= \nu_p(\mathrm{N}_1) + 1 + \nu_p(c \cdot (c+1) \cdots (c+k)) \geq 1.$$

The non-Archimedean property of the p-adic valuation yields that

$$\nu_p\left(h_n^{(r)}\right) \geq \min\left\{\nu_p\left(\frac{\mathrm{N}_1 \cdot \mathrm{A}}{\mathrm{D}_1 \cdot k!}\right), \nu_p\left(\frac{\mathrm{N}_1 \cdot \mathrm{N}_2}{\mathrm{D}_1 \cdot \mathrm{D}_2} \cdot \frac{p \cdot c \cdot (c+1) \cdots (c+k)}{k!}\right)\right\} \geq 0,$$

which concludes the lemma. □

Now, we consider the p-adic valuation of $h_{33}^{(r)}$ where the prime p satisfies the inequality $k_p := |I_p(33,1)| = \left\lfloor \frac{33}{p} \right\rfloor < p$. This will be a key step towards obtaining an integer $h_{33}^{(r)}$.

Corollary 4.4. *Let $p \in \{7, 13, 17, 19, 23, 29, 31\}$. Assume that $r \equiv 1 - b \pmod{p}$ for some $b \in \{1, \ldots, n_p\}$, where $n_p \equiv 33 \pmod{p}$ with $n_p \in \{1, \ldots, p-1\}$. Then $\nu_p\left(h_{33}^{(r)}\right) \geq 0$.*

Proof. We will show that the condition given on r implies that $|I_p(33,r)| = k_p + 1$, for any $p \in \{7, 13, 17, 19, 23, 29, 31\}$. Firstly, observe that $33 = k_p p + n_p$, where $0 < n_p < p$ and k_p as defined above. Note that $k_p \geq 1$ and $0 \geq 1 - b \geq 1 - n_p > -(p-1)$. If $r = cp + (1-b)$, then we have $cp - (p-1) < r \leq cp$. Since $k_p > 0$, we see that $r + 32 = (c + k_p)p + (n_p - b) \geq (c + k_p)p > cp \geq r$. Hence, we have $|I_p(33,r)| = k_p + 1$, as $cp, \ldots, (c+k_p)p \in I_p(33,r)$. By Lemma 4.3, we conclude that $\nu_p\left(h_{33}^{(r)}\right) \geq 0$, since $k_p < p$ for all $p \in \{7, 13, 17, 19, 23, 29, 31\}$. □

Finally, we can prove Theorem 1.4 which states the existence of hyperharmonic integers.

Theorem 4.5. *There are infinitely many values of $r \in \mathbb{Z}_{>0}$ such that $h_{33}^{(r)}$ is an integer. More precisely, $h_{33}^{(r)}$ is an integer for*

$$r = 64 \cdot (2^{\alpha + k \cdot 748\,440} - 1) + 32,$$

where $k \geq 0$ is an integer, α takes 153 different values in $\mathbb{Z} \cap [0, 748\,439]$ and the minimum of these r values is equal to $64 \cdot (2^{2659} - 1) + 32$.

Proof. We combine Proposition 4.1 and Corollary 4.4 to obtain a common solution r for which $\nu_p\left(h_{33}^{(r)}\right) \geq 0$, where $p \in \{2, 7, 13, 17, 19, 23, 29, 31\}$. Let

$$\mathcal{P} = \{7, 13, 17, 19, 23, 29, 31\}.$$

For all prime numbers $p \in \mathcal{P}$, we need to find a common solution for the congruence

$$64 \cdot (2^{\alpha_p} - 1) + 32 \equiv 1 - b_p \pmod{p}, \tag{4.3}$$

where $b_p \in \{1, \ldots, n_p\}$ and $n_p \equiv 33 \pmod{p}$ with $n_p \in \{1, \ldots, p-1\}$. Note that

$$2^{\alpha_p} \equiv 1 - 2^{-6}(b_p + 31) \pmod{p}. \tag{4.4}$$

For different values of b_p, we find all solutions of congruence (4.3) where

$$\alpha_p \equiv \operatorname{dlog}_2\left(1 - 2^{-6}(b_p + 31), p\right) \pmod{(\operatorname{ord}_p(2))}. \tag{4.5}$$

As we mentioned in Section 1.1, $\operatorname{ord}_p(2)$ denotes the order of 2 in $\left(\mathbb{Z}/p\mathbb{Z}\right)^\times$ and $\operatorname{dlog}_2(1 - 2^{-6}(b_p + 31), p)$ represents the solution of congruence (4.4) modulo $\operatorname{ord}_2(p)$, if it exists. In order to find all common values of α_0 satisfying congruence (4.5) for all $p \in \mathcal{P}$, we will use the Chinese Remainder Theorem for non-coprime moduli. To do so, we first pick two primes $p_1, p_2 \in \mathcal{P}$ and take solutions α_p satisfying (4.5) together with the corresponding modulus $\operatorname{ord}_p(2)$, where $p \in \{p_1, p_2\}$. Applying the general version of the Chinese Remainder Theorem for them, we obtain a set of α_ℓ values satisfying congruence (4.5) for $p = p_1, p_2$, and a common modulus $\ell = \operatorname{lcm}(\operatorname{ord}_{p_1}(2), \operatorname{ord}_{p_2}(2))$. Then, we take $p_3 \in \mathcal{P} \setminus \{p_1, p_2\}$ and combine the solutions α_{p_3} with the values of α_ℓ via Chinese Remainder Theorem again. This leads to a set of common solutions for the modulus $\operatorname{lcm}(\ell, \operatorname{ord}_{p_3}(2))$. Continuing this process until there is no prime left in the set $\mathcal{P}$, we get all common solutions $\alpha_0 \pmod{L}$, where $L = \operatorname{lcm}\{\operatorname{ord}_p(2) \mid p \in \mathcal{P}\} = 27\,720$ is the common modulus and each of α_0 is congruent to some $\alpha_p \pmod{(\operatorname{ord}_p(2))}$, for any $p \in \mathcal{P}$. Thanks to SageMath [Sage Developers, 2018], we see that the set

$$\mathcal{A}_0 := \{0 \leq \alpha_0 < 27\,720 \mid \forall p \in \mathcal{P}\ \alpha_0 \text{ satisfies (4.5) for some } b_p \in \{1, \ldots, n_p\}\}$$

contains 196 elements. Thus, for any prime $p \in \{2, 7, 13, 17, 19, 23, 29, 31\}$, any integer $k \geq 0$ and $\alpha_0 \in \mathcal{A}_0$, we have $\nu_p\left(h_{33}^{(r)}\right) \geq 0$, where $r = 64 \cdot (2^{\alpha_0 + k \cdot 27\,720} - 1) + 32$.

Now, we consider these solutions for the prime $p = 11$. As $p \mid 33$, we know by Lemma 2.3 that $|I_{11}(33, r)| = 3$. Let $c_{11} = \left\lceil \frac{r}{11} \right\rceil$. Then by equation (1.2), we have

$$\begin{aligned} h_{33}^{(r)} &= \frac{A_{11}}{B_{11}} \cdot \frac{11c_{11} \cdot 11(c_{11}+1) \cdot 11(c_{11}+2)}{11 \cdot 22 \cdot 33} \left(\frac{1}{11} \sum_{i=0}^{2} \frac{1}{c_{11}+i} + q_{11} \right) \\ &= \frac{A_{11}}{B_{11}} \cdot \frac{3c_{11}^2 + 6c_{11} + 2}{66} + \frac{A_{11}}{B_{11}} \cdot \frac{c_{11}(c_{11}+1)(c_{11}+2)}{6} \cdot q_{11}, \end{aligned}$$

where $\nu_{11}(A_{11}) = \nu_{11}(B_{11}) = 0 \leq \nu_{11}(q_{11})$ and $\nu_{11}(c_{11}(c_{11}+1)(c_{11}+2)) \geq 0$. Therefore, $3c_{11}^2 + 6c_{11} + 2 \equiv 0 \pmod{11}$ implies that $\nu_{11}\left(h_{33}^{(r)}\right) \geq 0$. Note that $x \equiv 1, 8 \pmod{11}$ are the only two roots of the polynomial $3x^2 + 6x + 2$ modulo 11. So, if

$$\begin{aligned} r \equiv\ & 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 78, 79, \\ & 80, 81, 82, 83, 84, 85, 86, 87, 88 \pmod{121}, \end{aligned} \tag{4.6}$$

then $c_{11} = \left\lceil \frac{r}{11} \right\rceil \equiv 1, 8 \pmod{11}$. According to the computations performed by SageMath [Sage Developers, 2018], we deduce that there are only 52 different elements in $\mathcal{A}_0$ which satisfy congruence (4.6). Define the set

$$\mathcal{A}_0' := \{\alpha_0 \in \mathcal{A}_0 \mid r = 64 \cdot (2^{\alpha_0} - 1) + 32 \text{ satisfies congruence (4.6)}\}$$

and take any $\alpha_0 \in \mathcal{A}_0'$. Since the order of 2 in $\left(\mathbb{Z}/121\mathbb{Z}\right)^\times$ is $\mathrm{ord}_{121}(2) = 110$, we get that

$$r = 2^6 \cdot (2^{\alpha_0 + k \cdot 27\,720} - 1) + 32 \equiv 64 \cdot (2^{\alpha_0} - 1) + 32 \pmod{121},$$

for any integer $k \geq 1$. This indicates that $\nu_{11}\left(h_{33}^{(r)}\right) \geq 0$ for any $r = 2^6 \cdot (2^{\alpha_0 + k \cdot 27\,720} - 1) + 32$, where $\alpha_0 \in \mathcal{A}_0'$ and $k \geq 0$.

Next, we deal with the case $p = 5$. By SageMath [Sage Developers, 2018], we obtain that $r = 64 \cdot (2^{\alpha_0} - 1) + 32 \equiv 0, 19 \pmod{25}$, for any $\alpha_0 \in \mathcal{A}_0'$. For these

values of r, we also observed that

$$\begin{aligned} h_{33}^{(r)} &= \binom{r+32}{r-1}(h_{r+32}-h_{r-1}) \\ &= \frac{A_5}{B_5}\cdot\frac{5c_5\cdot 5(c_5+1)\cdots 5(c_5+6)}{5\cdot 10\cdots 30}\left(\frac{1}{5}\sum_{i=0}^{6}\frac{1}{c_5+i}+q_5\right) \\ &= \frac{A_5}{B_5}\cdot\frac{5c_5\cdot 5(c_5+1)\cdots 5(c_5+6)}{5^7\cdot 144} \\ &\qquad\cdot\frac{7{c_5}^6+126{c_5}^5+875{c_5}^4+2940{c_5}^3+4872{c_5}^2+3528c_5+720}{5c_5(c_5+1)\cdots(c_5+6)}+Q_5 \\ &= \frac{A_5}{B_5}\cdot\frac{7{c_5}^6+126{c_5}^5+875{c_5}^4+2940{c_5}^3+4872{c_5}^2+3528c_5+720}{5\cdot 144}+Q_5 \end{aligned}$$

is satisfied, where $c_5=\left\lceil\frac{r}{5}\right\rceil$ and $\nu_5(A_5)=\nu_5(B_5)=0\le\nu_5(q_5)$. Here, Q_5 denotes the rational number $\frac{A_5}{B_5}\cdot\frac{c_5(c_5+1)\cdots(c_5+6)}{2^4\cdot 3^2}\cdot q_5$, and this yields that $\nu_5(Q_5)\ge 0$. Observe that if the polynomial

$$g_5(x):=7x^6+126x^5+875x^4+2940x^3+4872x^2+3528x+720$$

is divisible by 5 for $x=c_5$, then we get that $\nu_5\left(h_{33}^{(r)}\right)\ge 0$. If $25\mid r$, then we see that $g_5(c_5)\equiv 0 \pmod 5$, as $c_5\equiv 0 \pmod 5$ is a root of $g_5(x)$ in $\mathbb{Z}/5\mathbb{Z}$. Similarly, for $r\equiv 19 \pmod{25}$, we have $c_5=\left\lceil\frac{r}{5}\right\rceil\equiv 4 \pmod 5$ which is also a root of the polynomial $g_5(x)$ modulo 5. Hence, for all previously found $\alpha_0\in\mathcal{A}_0'$, we have $\nu_5\left(h_{33}^{(r)}\right)\ge 0$, where $r=64\cdot(2^{\alpha_0}-1)+32$. Also, for any integer $k\ge 1$ and $\alpha_0\in\mathcal{A}_0'$, notice that

$$r=64\cdot(2^{\alpha_0+k\cdot 27\,720}-1)+32\equiv 64\cdot(2^{\alpha_0}-1)+32\equiv 0,19 \pmod{25},$$

as $\operatorname{ord}_{25}(2)=20$. Thus, for any $r=64\cdot(2^{\alpha_0+k\cdot 27\,720}-1)+32$, we conclude that $\nu_5\left(h_{33}^{(r)}\right)\ge 0$, where $k\in\mathbb{Z}_{\ge 0}$ and $\alpha_0\in\mathcal{A}_0'$.

Finally, let $p=3$. Define $f_3(x)=x(x+1)\cdots(x+10)$. Consider

$$\begin{aligned} h_{33}^{(r)} &= \frac{A_3}{B_3}\cdot\frac{3c_3\cdot 3(c_3+1)\cdots 3(c_3+10)}{33!}\left(\frac{1}{3}\sum_{i=0}^{10}\frac{1}{c_3+i}+q_3\right) \\ &= \frac{A_3}{B_3}\cdot\frac{3^{11}\cdot f_3(c_3)}{3^{15}\cdot D_3}\cdot\frac{1}{3}\cdot\frac{g_3(c_3)}{f_3(c_3)}+\frac{A_3}{B_3}\cdot\frac{3^{11}\cdot f_3(c_3)}{3^{15}\cdot D_3}\cdot q_3 \qquad (4.7) \\ &= \frac{A_3}{B_3}\cdot\frac{g_3(c_3)}{3^5\cdot D_3}+\frac{A_3}{B_3}\cdot\frac{f_3(c_3)}{3^4\cdot D_3}\cdot q_3, \qquad (4.8) \end{aligned}$$

where $c_3 = \left\lceil \frac{r}{3} \right\rceil$ and $\nu_3(A_3) = \nu_3(B_3) = \nu_3(D_3) = 0 \leq \nu_3(q_3)$. Here, $g_3(x)$ denotes the derivative of the polynomial $f_3(x)$. To see that equation (4.7) holds, observe that

$$\log f_3(x) = \sum_{i=0}^{10} \log(x+i), \tag{4.9}$$

where $\log(\cdot)$ denotes the natural logarithm. Taking the derivative of both sides with respect to x in equation (4.9) yields that $\frac{g_3(x)}{f_3(x)} = \sum_{i=0}^{10} \frac{1}{x+i}$. Also, note that $\nu_3(f_3(c_3)) \geq 4$, since

$$\binom{c_3+10}{10} = \frac{c_3(c_3+1)\cdots(c_3+10)}{10!} = \frac{f_3(c_3)}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7} \in \mathbb{Z}.$$

Therefore, we have $\nu_3\left(\frac{A_3}{B_3} \cdot \frac{f_3(c_3)}{3^4 \cdot D_3} \cdot q_3\right) \geq 0$. In order to obtain $\nu_3\left(h_{33}^{(r)}\right) \geq 0$, it is enough to have $g_3(c_3) \equiv 0 \pmod{3^5}$, by equation (4.8). Using SageMath [Sage Developers, 2018], we find that there are 18 roots of the polynomial $g_3(x)$ modulo 3^5. Hence, if s is one of these roots and $r \equiv 3s-2,\, 3s-1,\, 3s \pmod{3^6}$, then we have $(s-1) + \ell \cdot 3^5 < \frac{r}{3} \leq s + \ell \cdot 3^5$, for some $\ell \in \mathbb{Z}_{\geq 1}$. This indicates that $c_3 = \left\lceil \frac{r}{3} \right\rceil \equiv s \pmod{243}$. As we did in congruence (4.3), we will solve

$$64 \cdot (2^{\alpha_3} - 1) + 32 \equiv s_3 \pmod{729}$$

for

$$\alpha_3 \equiv \operatorname{dlog}_2\left(1 + 2^{-6}(s_3 - 32), 729\right) \pmod{486}, \tag{4.10}$$

to obtain common solutions for primes 2 and 3, where $s_3 \in \{3s-2, 3s-1, 3s\}$ and $\operatorname{ord}_{729}(2) = 486$. By SageMath [Sage Developers, 2018] again, we obtain 36 different solutions modulo 486 for congruence (4.10). Note that for any α_3 satisfying congruence (4.10), we have $\nu_3\left(h_{33}^{(r)}\right) \geq 0$, as $r = 64 \cdot (2^{\alpha_3} - 1) + 32$. Also recall that for any $\alpha_0 \in \mathcal{A}_0'$, any integer $k \geq 0$ and any prime $p \in [2, 31] \setminus \{3\}$, we have $\nu_p\left(h_{33}^{(r)}\right) \geq 0$, where $r = 64 \cdot (2^{\alpha_0 + k \cdot 27\,720} - 1) + 32$. So, we can see each different $\alpha_0 \in \mathcal{A}_0'$ as an element from the corresponding equivalence class of α_0 in $\mathbb{Z}/(27\,720)\mathbb{Z}$. To obtain common solutions, we use generalized Chinese Remainder Theorem where

we combine 52 different α_0 values modulo 27 720 and 36 different α_3 values modulo 486, as $|\mathcal{A}'_0| = 52$. Computations give us 153 different common solutions α modulo $\operatorname{lcm}(27\,720, 486) = 748\,440$ for which the values $r = 64(2^\alpha - 1) + 32$ yield integer hyperharmonic numbers. As given in Appendix A, the smallest of these solutions is $\alpha = 2659$. Thus, we conclude that the smallest $r = 64 \cdot (2^{2659} - 1) + 32$, which is equal to

$$
\begin{aligned}
&175767446922889262206950794396365498056824255243025745088406\ldots\\
&495235754683352511692771314082693096092434704498758186181734\ldots\\
&800673704588668158380132949004211981935601929561769349593909\ldots\\
&451404512483238973702896610605956536893825250774917281516851\ldots\\
&151471560164451949451536810027813184714057210592464334291704\ldots\\
&621019661914345082077986712528439592368554441893396739145289\ldots\\
&365353722249760028982589210332188602724061055343355391610108\ldots\\
&138084504454268860733051604126249367862365349860077740756864\ldots\\
&467903116884656100242467955478128495828844502790059006425387\ldots\\
&127839619512712808222493423312318696244292911687949266584000\ldots\\
&396761753699582164597999382572096007734848046894136975363735\ldots\\
&104366648097401402827026050035496610552324757871922645454498\ldots\\
&799354723890282942105015978414530434976065482299854768579339\ldots\\
&1558048254536798175200.
\end{aligned}
$$

□

Chapter 5

CONCLUSION

In this thesis, we considered an open problem of Mező which presumes the non-existence of hyperharmonic integers, except 1. We attacked this problem from three different point of views: analytic, combinatorial and algebraic. As a result of analytic approach, we obtained that for any $n > 1$, there are infinitely many values of r such that $h_n^{(r)}$ is not an integer. Also, for any $r \geq 1$ and all sufficiently large n depending on r, we have the non-integerness property for $h_n^{(r)}$ again. Using the primes in short intervals, we proved that the probability of choosing an $(n, r) \in [1, x] \times [1, x]$ such that the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer, tends to 1, as x tends to infinity. In particular, we provide an error term on the number of exceptional tuples, which is $o(x^2)$. We also showed that this error term can also be improved under some plausible assumptions, like Riemann hypothesis and Cramér's conjecture. Note that the error term that we obtained by Cramér's conjecture may be the best possible one, as there are $x^2 + \mathcal{O}(x)$ many lattice points in $[1, x] \times [1, x]$ square. Apart from that, combinatorial methods yield that $h_n^{(r)}$ cannot be an integer, if n is even or a prime power, or r is odd. Moreover, there is a set of r values with density at least 99% such that $h_{33}^{(r)}$ is not an integer. From the algebraic perspective, we found several conditions for n, where each of them leads to a non-integer $h_n^{(r)}$. More precisely, if

- $n = 3p^\alpha$ where $\alpha \geq 1$, p is prime and $p \equiv \pm 5 \pmod{12}$, or
- $n = 5p^\alpha$ where $\alpha \geq 1$, p is prime and congruent to 56 different equivalence classes modulo 145,

then $h_n^{(r)} \notin \mathbb{Z}$. Finally, supporting our techniques with computational power, we get the following conditions on n and r which make $h_n^{(r)}$ a non-integer rational number:

- $r \leq 20n$.
- $1 < n \leq 32$.
- $r \leq 35.001$.

However, none of the prescribed methods guarantees the non-existence of hyperharmonic integers. In fact, the reason of this may be the existence of such integers. As we indicated in Chapter 4, there are infinitely many of them, even when $n = 33$. Observe that this does not contradict the error terms that we found in Theorem 3.8: since $h_{33}^{(r)} \in \mathbb{Z}$ when $r = 64 \cdot (2^{\alpha + k \cdot 748\,440} - 1) + 32$, for some $\alpha \in \{0, \ldots, 748\,439\}$ and $k \in \mathbb{Z}$, we see that there are $\mathcal{O}(\log x)$ many possible values for $r \leq x$ where $h_{33}^{(r)}$ is an integer. This yields that there are only $\mathcal{O}(\log x)$ exceptions in $[1, x] \times [1, x]$ square. So, even if this bound for exceptional r values holds for each $n \leq x$, then the number of such (n, r) tuples is at most $\mathcal{O}(x \log x)$, whereas the best error term that we obtained (under Cramér's conjecture) for the number of exceptional (n, r) tuples is $\mathcal{O}\left(x \log^3 x\right)$.

BIBLIOGRAPHY

[Alkan and Göral, 2018] Alkan, E. and Göral, H. (2018). Structural properties of Dirichlet series with harmonic coefficients. *The Ramanujan Journal*, 45(2):569–600.

[Alkan et al., 2018] Alkan, E., Göral, H., and Sertbaş, D. C. (2018). Hyperharmonic numbers can be rarely integers. *INTEGERS*, 18:Paper No. A43.

[Amrane and Belbachir, 2010] Amrane, R. A. and Belbachir, H. (2010). Non-integerness of class of hyperharmonic numbers. *Annales Mathematicae et Informaticae*, 37:7 – 10.

[Amrane and Belbachir, 2011] Amrane, R. A. and Belbachir, H. (2011). Are the hyperharmonics integral? a partial answer via the small intervals containing primes. *Comptes Rendus Mathematique*, 349(3):115 – 117.

[Apostol, 1976] Apostol, T. M. (1976). *Introduction to Analytic Number Theory*. Springer-Verlag, New York, NY, USA, 1 edition.

[Baker et al., 2001] Baker, R. C., Harman, G., and Pintz, J. (2001). The difference between consecutive primes, II. *Proceedings of the London Mathematical Society*, 83(3):532–562.

[Benjamin et al., 2003] Benjamin, A. T., Gaebler, D., and Gaebler, R. (2003). A combinatorial approach to hyperharmonic numbers. *INTEGERS: Electronic Journal of Combinatorial Number Theory*, 3(#A15):1–9.

[Berndt, 1985] Berndt, B. C. (1985). *Ramanujan's Notebooks: Part I*. Springer-Verlag, New York, NY, USA, 1 edition.

[Berndt, 1998] Berndt, B. C. (1998). *Ramanujan's Notebooks: Part V*. Springer-Verlag, New York, NY, USA, 1 edition.

[Bertrand, 1845] Bertrand, J. (1845). Mémoire sur le nombre de valeurs que peut prendre une fonction quand on y permute les lettres qu'elle renferme. *Journal de l'Ecole Royale Polytechnique*, 18(Cahier 30):123–140.

[Boyadzhiev and Dil, 2015] Boyadzhiev, K. N. and Dil, A. (2015). Euler sums of hyperharmonic numbers. *Journal of Number Theory*, 147:490 – 498.

[Chebyshev, 1852] Chebyshev, P. L. (1852). Mémoire sur les nombres premiers. *Journal de mathématiques pures et appliquées*, Sér. 1:366–390.

[Conway and Guy, 1996] Conway, J. H. and Guy, R. K. (1996). *The Book of Numbers*. Springer-Verlag, New York, NY, USA, 1 edition.

[Cramér, 1936] Cramér, H. (1936). On the order of magnitude of the difference between consecutive prime numbers. *Acta Arithmetica*, 2(1):23–46.

[Davenport, 1982] Davenport, H. (1982). *Multiplicative Number Theory*. Springer-Verlag, New York, NY, USA, 2 edition.

[de La Vallée Poussin, 1896] de La Vallée Poussin, C. (1896). Recherches analytiques sur la théorie des nombres; Première partie: La fonction $\zeta(s)$ de Riemann et les nombres premiers en général. *Annales de la Soc. scientifique de Bruxelles*, 20:183–256.

[de La Vallée Poussin, 1899] de La Vallée Poussin, C. (1899). Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférierus à une limite donné. *C. Mém. Couronnés Acad. Roy. Belgique*, 59:1–74.

[Dil and Mező, 2010] Dil, A. and Mező, I. (2010). Hyperharmonic series involving Hurwitz zeta function. *Journal of Number Theory*, 130(2):360 – 369.

[Dusart, 1998] Dusart, P. (1998). *Autour de la fonction qui compte le nombre de nombres premiers*. PhD thesis, Universite de Limoges.

[Erdős, 1932] Erdős, P. (1932). Beweis eines satzes von Tschebyschef (On a proof of a theorem of Chebyshev, in German). *Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae Sectio Scientiarum Mathematicum (Szeged)*, 5:194–198.

[Erdős, 1935] Erdős, P. (1935). On The Difference of Consecutive Primes. *The Quarterly Journal of Mathematics*, os-6(1):124–128.

[Erdős and Niven, 1946] Erdős, P. and Niven, I. (1946). Some properties of partial sums of the harmonic series. *Bulletin of the American Mathematical Society*, 52(4):248–251.

[Euler, 1776] Euler, L. (1776). Meditationes circa singulare serierum genus. *Novi Commentarii Academiae Scientiarum Petropolitanae*, 20:140–186.

[Ford et al., 2018] Ford, K., Green, B., Konyagin, S., Maynard, J., and Tao, T. (2018). Long gaps between primes. *J. Amer. Math. Soc.*, 31(1):65–105.

[Göral and Sertbaş, 2017] Göral, H. and Sertbaş, D. C. (2017). Almost all hyperharmonic numbers are not integers. *Journal of Number Theory*, 171:495 – 526.

[Hadamard, 1896] Hadamard, J. (1896). Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *Bulletin de la Société Mathématique de France*, 24:199–220.

[Harman, 2007] Harman, G. (2007). *Prime-Detecting Sieves. (LMS-33)*. Princeton University Press.

[Jia, 1996] Jia, C. (1996). Almost all short intervals containing prime numbers. *Acta Arithmetica*, 76(1):21–84.

[Kamano, 2011] Kamano, K. (2011). Dirichlet series associated with hyperharmonic numbers. *Memoirs of the Osaka Institute of Technology, Series A*, 56(2):11–15.

[Knuth and Wilf, 1989] Knuth, D. E. and Wilf, H. S. (1989). The power of a prime that divides a generalized binomial coefficient. *Journal für die Reine und Angewandte Mathematik*, 396:212–219.

[Koblitz, 1994] Koblitz, N. (1994). *A course in number theory and cryptography*. Springer-Verlag, New York, NY, USA, 2 edition.

[Kummer, 1852] Kummer, E. E. (1852). Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen. *Journal für die Reine und Angewandte Mathematik*, 44:93–146.

[Kürschák, 1918] Kürschák, J. (1918). On the harmonic series. *Matematikai és Fizikai Lapok*, 27:299–300.

[Mező, 2007] Mező, I. (2007). About the non-integer property of hyperharmonic numbers. *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös Nominatae Sectio Mathematica*, 50:13–20.

[Nagura, 1952] Nagura, J. (1952). On the interval containing at least one prime number. *Proceedings of the Japan Academy*, 28(4):177–181.

[Pambuccian, 2008] Pambuccian, V. (2008). The sum of irreducible fractions with consecutive denominators is never an integer in PA^-. *Notre Dame Journal of Formal Logic*, 49(4):425–429.

[Pintz, 1997] Pintz, J. (1997). Very large gaps between consecutive primes. *Journal of Number Theory*, 63(2):286 – 301.

[Ramanujan, 1919] Ramanujan, S. (1919). A proof of Bertrand's postulate. *Journal of the Indian Mathematical Society*, 11(134):181–182.

[Rankin, 1938] Rankin, R. A. (1938). The difference between consecutive prime numbers. *Journal of the London Mathematical Society*, s1-13(4):242–247.

[Sage Developers, 2018] Sage Developers, T. (2018). *SageMath, the Sage Mathematics Software System (Version 8.3)*. thttps://www.sagemath.org.

[Selberg, 1943] Selberg, A. (1943). On the normal density of primes in small intervals, and the difference between consecutive primes. *Archiv for Mathematik og Naturvidenskab*, 47:87–105.

[Theisinger, 1915] Theisinger, L. (1915). Bemerkung über die harmonische Reihe. *Monatshefte für Mathematik und Physik*, 26(1):132–134.

[Westzynthius, 1931] Westzynthius, E. (1931). Über die Verteilung der Zahlen, die zu den n ersten Primzahlen teilerfremd sind. *Commentat. Phys.-Math.*, 5(25):1–37.

Appendix A

THE ORDER OF HYPERHARMONIC INTEGERS

As we proved in Chapter 4, there are 153 different $\alpha \in \{0, \ldots, 748\,439\}$ such that $h_{33}^{(r)} \in \mathbb{Z}$, for $r = 64 \cdot (2^{\alpha + k \cdot 748\,440} - 1) + 32$ and $k \in \mathbb{Z}_{\geq 0}$. The set of these α values can be given as follows:

$$\{2659, 23039, 28979, 30599, 30739, 36539, 36679, 38299, 44239, 64619, 70559,$$
$$72179, 72319, 78114, 78119, 78259, 79879, 85819, 106199, 112139, 113759,$$
$$113899, 119699, 119839, 121459, 127399, 147779, 153719, 155339, 155479,$$
$$161274, 161279, 161419, 195299, 196919, 197059, 202859, 202999, 204619,$$
$$210559, 230939, 236879, 238499, 238639, 163039, 168979, 189359, 244434,$$
$$244439, 244579, 246199, 252139, 272519, 278459, 280079, 280219, 286019,$$
$$286159, 287779, 293719, 314099, 320039, 321659, 321799, 327594, 327599,$$
$$327739, 329359, 335299, 355679, 361619, 363239, 363379, 369179, 369319,$$
$$370939, 376879, 397259, 403199, 404819, 404959, 410754, 410759, 410899,$$
$$412519, 418459, 438839, 444779, 446399, 446539, 452339, 452479, 454099,$$
$$460039, 480419, 486359, 487979, 488119, 493914, 493919, 494059, 495679,$$
$$501619, 521999, 527939, 529559, 529699, 535499, 535639, 537259, 543199,$$
$$563579, 569519, 571139, 571279, 577074, 577079, 577219, 578839, 584779,$$
$$605159, 611099, 612719, 612859, 618659, 618799, 620419, 626359, 646739,$$
$$652679, 654299, 654439, 660234, 660239, 660379, 661999, 667939, 688319,$$
$$694259, 695879, 696019, 701819, 701959, 703579, 709519, 729899, 735839,$$
$$737459, 737599, 743394, 743399, 743539, 745159\}.$$