ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON HIGHER ORDER FIBONACCI HYPER COMPLEX NUMBERS

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

TİEKORO KONE

AUGUST 2021

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ZONGULDAK
August 2021

APPROVAL OF THE THESIS:

The thesis entitled “On Higher Order Fibonacci Hyper Complex Numbers” and submitted by Tiekoro KONE has been examined and accepted by the jury as a Master of Science thesis in Department of Mathematics, Graduate School of Natural and Applied Sciences, Zonguldak Bülent Ecevit University. 26/08/2021

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Approved by the Graduate School of Natural and Applied Sciences. ..../..../2021

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by the academic rules and ethical principles all works that are not result of this study are cited properly."

Tiekoro KONE

ABSTRACT

Master of Science Thesis

ON HIGHER ORDER FIBONACCI HYPER COMPLEX NUMBERS

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Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assoc. Prof. Dr. Can KIZILATEŞ
August 2021, 41 pages
This thesis deals with the study of the higher order Fibonacci quaternions and the higher order Fibonacci hyper complex numbers.

The thesis consists of five chapters.

Chapter 1 is a concise overview of what this thesis is about and also is a literature summary for Fibonacci numbers, Lucas numbers, quaternions, octonions, sedenions, and 2^m-ions.

In Chapter 2, we shall define the fundamental concepts, the definition of Fibonacci, Lucas, and higher order Fibonacci numbers. Their basic identities and the relations between these identities will be given. We will also define quaternions, octonions, and sedenions whose components Fibonacci numbers, and give some properties about these algebras.

In Chapter 3, we will define the higher order Fibonacci quaternions and study some properties of these quaternions.

ABSTRACT (continued)

In the Chapter 4, we will define and work on the higher order Fibonacci hyper complex numbers. Then many algebraic properties of the higher order Fibonacci 2^m -ions will be given. We will derive some identities such as Vajda's identity, Catalan's identity, Cassini's identity, and d'Ocagne's identity with the aid of the Binet formula. Moreover, we develop some matrix identities involving higher order Fibonacci 2^m -ions which allow us to obtain some properties of these higher order hyper complex numbers.

Keywords: Fibonacci numbers, Fibonacci quaternions, Fibonacci octonions, Fibonacci sedenions, Fibonacci 2^m-ions.

Science Code: 403.01.01

ÖZET

Yüksek Lisans Tezi

YÜKSEK MERTEBELİ FİBONACCİ HİPER KOMPLEKS SAYILARI ÜZERİNE

Tiekoro KONE

Zonguldak Bülent Ecevit Üniversitesi
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Tez Danışmanı: Doç. Dr. Can KIZILATEŞ

Ağustos 2021, 41 sayfa

Bu tezde, yüksek mertebeli Fibonacci kuaterniyonları ve hiper kompleks Fibonacci sayılarıyla ilgileneceğiz.

Tez beş bölümden oluşmaktadır.

Bölüm 1'de, bu tezin ne hakkında olduğuna dair kısa bir genel bakış ve aynı zamanda Fibonacci sayıların, Lucas sayıların, kuaterniyonların, oktoniyonların, sedeniyonların ve 2^m-iyonların literatür özetidir.

Bölüm 2'de, Fibonacci, Lucas ve yüksek mertebeli Fibonacci sayılarını tanımlayacağız. Temel özellikleri ve bu özelliklerin arasındaki bazı ilişkileri vereceğiz. Ayrıca bileşenleri Fibonacci sayıları olan kuaterniyonlar, oktoniyonlar ve sedeniyonları tanımlayacağız ve bu cebirler hakkında bazı özellikler vereceğiz.

Bölüm 3'de, yüksek mertebeli Fibonacci kuaterniyonları tanımlayacağız ve bu kuaterniyonlarla ilgili bazı özellikleri vereceğiz.

ÖZET (devam ediyor)

Bölüm 4'de, yüksek mertebeli Fibonacci hiper kompleks sayıları tanımlayıp üzerinde çalışacağız. Daha sonra yüksek mertebeli Fibonacci 2^m -iyonlarının birçok cebirsel özellikleri vericeğiz. Vajda eşitliği, Catalan eşitliği, Cassini eşitliği ve d'Ocagne eşitliği gibi bazı özellikleri Binet formülü yardımıyla türeteceğiz. Ayrıca, bu yüksek mertebeli hiper kompleks sayıların bazı özelliklerini elde etmemize yardımcı olan yüksek mertebeden Fibonacci 2^m-iyonlarını içeren bazı matris özellikleri ele alacağız.

Anahtar Kelimeler: Fibonacci sayıları, Fibonacci kuaterniyonları, Fibonacci oktoniyonları, Fibonacci sedeniyonları, Fibonacci 2^m-iyonları.

Bilim Kodu: 403.01.01

ACKNOWLEDGEMENTS

I would like to express the deepest appreciation to my advisor Assoc. Prof. Dr. Can KIZILATEŞ, who has the attitude and substance of a genius: he continually and convincingly conveyed a spirit of adventure in regard to research and scholarship, and an excitement in regard to teaching. I would like to thank him for supporting me and all thinks he did for me. And I also want to thank Murat HAMARAT for all his supports and all my colleagues.

I also would like to thank my family specially my parents who have a big role in my success for giving birth to me at the first place and supporting me spiritually throughout my life and for their support and encouragement throughout. I owe them everything and wish I could show them just how much I love and appreciate them.

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS

F_n	: n-th Fibonacci number
L_n	: n-th Lucas number
$F_n^{(s)}$	: n-th higher order Fibonacci number
$\mathbb{R}$	: Real numbers
$\mathbb{C}$	: Complex numbers
$\mathbb{Q}$	: Quaternions
$\mathbb{O}$	: Octonions
$\mathbb{S}$	: Sedenions
$\mathbb{Q}F_n$	: Fibonacci quaternions
$\mathbb{O}F_n$	: Fibonacci octonions
$\mathbb{S}F_n$	: Fibonacci sedenions
$\mathbb{Q}F_n^{(s)}$	: Higher order Fibonacci quaternions
$\mathbb{HC}_n^{(s)}$	: Higher order Fibonacci hyper complex numbers
$\mathbb{C}F_n^{(s)}$	: Higher order Fibonacci complex numbers
$\mathbb{C}F_n$	: Fibonacci complex numbers

CHAPTER 1

INTRODUCTION

Leonardo Fibonacci, also called Leonardo Pisano or Leonard of Pisa, was the most outstanding mathematician of the European Middle Ages. He was born around 1170 into the Bonacci family of Pisa. At age of 9, after his mother passed away, he was broughted by his father (Guglielmo) to the Algerian city of Bugia (now called Bougie) in order to learn the art of computation from a muslim schoolmaster, who introduced him to the Indian numeration system and Indian computation techniques. Fibonacci is also known as the person who tooks mathematics from the Arabs and made Europe recognize it.

Around 1200, at the age of 30, Fibonacci returned to Pisa. He was convinced of the elegance and practical superiority of the Indian numeration system over the Roman system then in use in Italy. In 1202, Fibonacci published one of his most important works Liber Abaci (The book of the Abacus). With this book, he introduced the modern number system and arithmetic algorithms to Europe. And so the Liber Abaci book becomes the handbook of all Europan scientists whom always used to work with the Roman number system.

Fibonacci's Liber Abaci contains many elementary problems but the most famous one is the rabbits problem. According to this problem, rabbits in the farm cannot breed in the first two months of their birth. From the third month, each pair of rabbits produces a pair of babies every month. Assuming that the rabbits are immortel, how many pairs of rabbits are in the farm in any n-th given month? Accordingly, the number of couples in any given month is equal to the sum of the previous two months. So the number of pairs of rabbits in the farm will be $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ It is known that the limit of this sequence of numbers obtained from the ratio of consecutive terms is related to the $\frac{1+\sqrt{5}}{2}$ irrational number called "golden ratio".

Many mathematicians have studied on Fibonacci numbers, one of them was the Italian mathematician Giovanni Domenico Cassini (1625-1712) who gave his name to the identity he has found. Later, the more general form of this identity was found by Eugene Catalan (1814-1894). For more details, please see [1].

Binet formula that allow us to find many identities with Fibonacci numbers, was proved for the first time by De Moivre and later by Jacques-Philippe-Marie Binet (1786-1856) by using the recurrence relation of Fibonacci numbers. The Binet formula allows us to calculate directly the n-th Fibonnaci number [2].

French mathematician Edward Lucas changed the initial conditions to define a sequence of number similar to the Fibonacci sequence which is called Lucas numbers. Researchers have proven very interesting identities between these two sequences. These numbers are sometimes used in nature and scientific fields [1].

In 1843, William Rowan Hamilton introduced quaternions as an extension of complex numbers, which is a $4 = 2^2$-dimensional associative and non-commutative algebra over $\mathbb{R}$. Shortly after this discovery, John Graves discovered the octonions, which is a $8 = 2^3$-dimensional non-associative and a non-commutative algebra over $\mathbb{R}$. In 1845, these algebras were rediscovered by Arthur Cayley. They are also known as the Cayley numbers. Passing from $\mathbb{R}$ to $\mathbb{C}$, from $\mathbb{C}$ to $\mathbb{Q}$, from $\mathbb{Q}$ to $\mathbb{O}$ is called the Cayley-Dickson doubling process or the Cayley-Dickson process. The next doubling process applied to $\mathbb{O}$ then produces an algebra $\mathbb{S}$, $2^4 = 16$-dimensional called sedenions. This doubling process can be extended beyond the sedenions to form what are known as the 2^m-ions. High order dimension algebras (or-hyper complex numbers) such as quaternions, octonions, sedenions emerging in many fields, especially in mathematics, coding theory, physics, robotics, computer science, etc. [3, 4, 5].

1.1 AIM OF THE THESIS

In this thesis, our main aim is to define the higher order Fibonacci quaternions and the higher order Fibonacci 2^m-ions (or higher order Fibonacci hyper complex numbers) and give some properties of these algebras. This thesis consists of five chapters:

The chapter 1 is a concise overview of what this thesis is about and also is a literature summary of Fibonacci numbers, Lucas numbers, quaternions, octonions, sedenions, and 2^m-ions.

In Chapter 2, we define the fundamental concepts, the definition of Fibonacci, Lucas, and higher order Fibonacci numbers. Their basic identities and the relations between these identities will be given. We will also give the definition of quaternions, octonions, sedenions whose components Fibonacci numbers, and give some properties about these algebras.

In Chapter 3, we define the higher order Fibonacci quaternions. Many proprieties of the higher order Fibonacci quaternions such as the Binet formula, the generating function, the exponential generating function, the recurrence relation, and the Vajda identity will be studied.

In the Chapter 4, we work on the higher order Fibonacci 2^m-ions. Many algebraic properties of the higher order Fibonacci hyper complex numbers will be given. We will derive some identities such as Vajda's identity, Catalan's identity, Cassini's identity, and d'Ocagne's identity with the aid of the Binet formula. Moreover, we develop some matrix identities involving higher order Fibonacci hyper complex numbers which allow us to obtain some properties of these higher order hyper complex numbers. And the final chapter consists of the conclusion.

1.2 REVIEW OF THE LITERATURE

In this section, informations will be given about references used in the thesis.

In [1] Koshy gave some basic properties of Fibonacci and Lucas numbers in the most comprehensive way.

In [15] Vajda gave some properties about Fibonacci and Lucas numbers in his "Fibonacci and Lucas Numbers and the Golden Section: Theory and Applications" book.

In [8] Özvatan and her M.S advisor Pashaev developed the Golden-Fibonacci calculus and obtained several applications of this calculus and they also defined the higher order Fibonacci numbers and gave some properties of these numbers.

In [16] Harman took a new approach toward the significant extension of Fibonacci numbers into the complex plane.

In [3] Hamilton worked on quaternions and gave in the most comprehensively way properties about it.

In [10, 17] Horadam defined the Fibonacci and Lucas quaternions and obtained some properties of the Fibonacci and Lucas quternions.

In [9] Halıcı investigated the Fibonacci and Lucas quaternions. She obtained some properties for these quaternions. Moreover, she derived some sums formulas for them.

In [4] Baez described the octonions and their relation to clifford algebras and spinors, Bott periodicity, projective and Lorentzian geometry, Jordan algebras, and the exceptional Lie groups. He also touch upon their applications in quantum logic, special relativity and supersymmetry.

In [5] Carmody studied and developed some properties about circular and hyperbolic quaternions, octonions, and sedenions.

In [11] Kecilioglu and Akkus introduced the Fibonacci and Lucas octonions and gave some miscellaneous properties for these octonions.

In [6] Imaeda and Imaeda defined and gave some properties related to sedenions.

In [12] Bilgici et al. introduced the Fibonacci and Lucas sedenions. They obtained generating functions and Binet formulas for the Fibonacci and Lucas sedenions, and derived some identities for Fibonacci and Lucas sedenions.

In [13], a rationalized algorithm for calculating the product of sedenions was presented by Cariow and Cariowa, which reduces the number of underlying multiplications.

In [20] Göcen and Soykan defined the Horadam 2^k–ions, which are generalization of the some above works, and examined their properties.

In [21] Köme and Kirik defined modified generalized Fibonacci 2^k–ions. Moreover they presented some properties of generalized Fibonacci 2^k–ions.

In [19] Bilgici and Daşdemir introduced unrestricted Fibonacci and Lucas hyper complex

numbers. Moreover they presented some algebraic properties of these type of numbers.
In [22] Kızılateş and Kone introduced higher order Fibonacci quaternions that generalize the Fibonacci quaternions studied by Horadam and Halıcı with the help of higher order Fibonacci numbers. We gave recurrence relation, Binet formula, generating and exponential generating functions and some other algebraic properties of higher order Fibonacci quaternions.
In [23] Kızılateş and Kone deals with developing a new class of quaternions, octonions and sedenions called higher order Fibonacci 2^m-ions (or higher order Fibonacci hyper complex numbers) whose components are higher order Fibonacci numbers. We obtained recurrence relation, Binet formula, generating function and exponential generating function of higher order Fibonacci 2^m-ions. We also derived some identities such as Vajda's identity, Catalan's identity, Cassini's identity, and d'Ocagne's identity are derived with the aid of the Binet formula. Finally, we developed some matrix identities involving higher order Fibonacci 2^m-ions which allow us to obtain some properties of these higher order hyper complex numbers.

CHAPTER 2

SOME BASIC DEFINITIONS

In this part of the thesis, we will give some basic definitions and properties related to our thesis.

2.1 FIBONACCI NUMBERS

Let F_n be the n-th Fibonacci number and $F_0 = 0, F_1 = 1$ be the initial conditions. For $n \geq 0$, the Fibonacci sequence is defined by the following recurrence relation:

$$F_{n+2} = F_{n+1} + F_n. \tag{2.1}$$

The first ten terms of Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34$.
From (2.1), the characteristic equation of Fibonacci sequence is $x^2 - x - 1 = 0$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of this equation. So the Binet formula of the Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{2.2}$$

The generating function and the exponential generating function of Fibonacci sequence are respectively

$$\sum_{n=0}^{\infty} F_n\, x^n = \frac{x}{1 - x - x^2},$$

$$\sum_{n=0}^{\infty} F_n\, \frac{x^n}{n!} = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}.$$

The Cassini identity for Fibonacci sequence is as follows:

$$F_{n+1}F_{n-1} - (F_n)^2 = (-1)^n.$$

For more details related to Fibonacci numbers, we refer to [1].

2.2 LUCAS NUMBERS

Let L_n be the n-th Lucas number and $L_0 = 2$ and $L_1 = 1$ be the initial conditions, the Lucas sequence is defined by the following recurrence relation:

$$L_{n+2} = L_{n+1} + L_n. \tag{2.3}$$

The first ten terms of Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76$.
The Binet formula for Lucas sequence is

$$L_n = \alpha^n + \beta^n. \tag{2.4}$$

The generating function and the exponential generating function of Lucas sequence are respectively

$$\sum_{n=0}^{\infty} L_n \; x^n = \frac{2 - x}{1 - x - x^2},$$

$$\sum_{n=0}^{\infty} L_n \; \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x}.$$

The Cassini identity for Lucas sequence is as follows:

$$L_{n-1}L_{n+1} + (L_n)^2 = 5\,(-1)^{n-1}.$$

For more properties and applications of Fibonacci and Lucas numbers, see [1, 7].

2.3 HIGHER ORDER FIBONACCI NUMBERS

The higher order Fibonacci numbers defined by Ozvatan and her M.S advisor Pashaev in [8], for $s \geq 1$ integer, as follows:

$$F_n^{(s)} = \frac{F_{ns}}{F_s} = \frac{(\alpha^s)^n - (\beta^s)^n}{\alpha^s - \beta^s}. \tag{2.5}$$

Since F_{ns} is divisible by F_s, the ratio $\frac{F_{ns}}{F_s}$ is an integer. So, all higher order Fibonacci numbers are integers. Let us state here that for $s = 1$, higher order Fibonacci number $F_n^{(1)}$, turns into ordinary Fibonacci number.
According to special cases of s and n, some higher order Fibonacci numbers are given in the following table:

Table 2.1 Some Higher Order Fibonacci Numbers

$n\backslash s$	1	2	3	4
1	$F_1^{(1)}=1$	$F_1^{(2)}=1$	$F_1^{(3)}=1$	$F_1^{(4)}=1$
2	$F_2^{(1)}=1$	$F_2^{(2)}=3$	$F_2^{(3)}=4$	$F_2^{(4)}=7$
3	$F_3^{(1)}=2$	$F_3^{(2)}=8$	$F_3^{(3)}=17$	$F_3^{(4)}=48$
4	$F_4^{(1)}=3$	$F_4^{(2)}=21$	$F_4^{(3)}=72$	$F_4^{(4)}=329$

The Binet type formula for higher order Fibonacci numbers is given by

$$F_n^{(s)} = \frac{(\alpha^s)^n - (\beta^s)^n}{\alpha^s - \beta^s}. \tag{2.6}$$

The recurrence relation for higher order Fibonacci numbers is

$$F_{n+1}^{(s)} = L_s F_n^{(s)} + (-1)^{s-1} F_{n-1}^{(s)}. \tag{2.7}$$

This formula is particular case of more general relation given below

$$F_{s(n+1)+\alpha} = L_s F_{sn+\alpha} + (-1)^{s-1} F_{s(n-1)+\alpha},$$

where $\alpha = 0, 1, ..., s-1$.

By extending n and s to negative integer numbers for higher order Fibonacci numbers $F_n^{(s)}$, the formulas can be derived as

$$F_{-n}^{(s)} = (-1)^{sn+1} F_n^{(s)},$$

$$F_n^{(-s)} = (-1)^{s(n+1)} F_n^{(s)},$$

$$F_{-n}^{(-s)} = (-1)^{s+1} F_n^{(s)}.$$

2.4 FIBONACCI QUATERNIONS

A quaternion p is a hyper-complex number and is defined by the following form [3]:

$$p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}, \tag{2.8}$$

where p_0, p_1, p_2, p_3 are real numbers and $\mathbf{e}_0 = \mathbf{1}$, $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, $\mathbf{e}_3 = \mathbf{k}$ are standard orthonormal basis in $\mathbb{R}^4$ which satisfy the quaternion multiplication rules as:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \tag{2.9}$$

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj},\ \mathbf{ki} = \mathbf{j} = -\mathbf{ik}. \tag{2.10}$$

Note that we can write $p = p_0 + u$ where $u = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$. The conjugate of the quaternion p is denoted by $p^* = p_0 - u$. The norm of a quaternion p is defined as $\|p\|^2 = p_0^2 + p_1^2 + p_2^2 + p_3^2$.

Now let's summarize some simple arithmetic operations provided by real quaternions. Let

$$p_1 = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k},$$

$$p_2 = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k},$$

and λ be a reel number, then

$$p_1 + p_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\mathbf{k},$$

$$p_1 - p_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\mathbf{j} + (d_1 - d_2)\mathbf{k},$$

$$\lambda p_1 = \lambda a_1 + (\lambda b_1)\,\mathbf{i} + (\lambda c_1)\,\mathbf{j} + (\lambda d_1)\,\mathbf{k},$$

$$\begin{aligned} p_1p_2 &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i} \\ &\quad + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\mathbf{j} + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathbf{k}. \end{aligned}$$

Many quaternions whose components are integer sequences were defined by researchers. For example, Horadam defined the n-th Fibonacci quaternion, for $n \geq 0$, as

$$\mathbb{Q}F_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k},$$

[10]. Here F_n is the n-th Fibonacci number. Also Horadam established a few relations for the Fibonacci quaternions [9]:

$$\begin{aligned} \mathbb{Q}F_n\left(\mathbb{Q}F_n\right)^* &= \sum_{i=0}^{3}\left(F_{n+i}\right)^2 = 3F_{2n+3} \\ \left(\mathbb{Q}F_n\right)^2 &= 2F_n\mathbb{Q}F_n - \mathbb{Q}F_n\left(\mathbb{Q}F_n\right)^* \\ \mathbb{Q}F_n + \left(\mathbb{Q}F_n\right)^* &= 2F_n. \end{aligned}$$

For $n > 0$, the Binet formula for the Fibonacci quaternions is

$$\mathbb{Q}F_n = \frac{\bar{\alpha}(\alpha)^n - \bar{\beta}(\beta)^n}{\alpha - \beta},$$

where $\bar{\alpha} = 1 + \alpha\mathbf{i}+\alpha^2\mathbf{j}+\alpha^3\mathbf{k}$ and $\bar{\beta} = 1 + \beta\mathbf{i}+\beta^2\mathbf{j}+\beta^3\mathbf{k}$. Halici [9] also derived generating functions and many other important identities of these quaternions.

The generating function for the Fibonacci quaternion $\mathbb{Q}F_n$ is

$$G\mathbb{Q}F_n = \frac{x + e_1 + (x+1)e_2 + (x+2)e_3}{1 - x - x^2}.$$

The exponential generating function for the Fibonacci quaternion $\mathbb{Q}F_n$ as

$$\sum_{n=0}^{\infty} \mathbb{Q}F_n \frac{x^n}{n!} = \frac{\bar{\alpha}e^{\alpha x} - \bar{\beta}e^{\beta x}}{\alpha - \beta}.$$

For $m, n \in Z$, the generating function of the quaternion $\mathbb{Q}F_{m+n}$ is

$$\sum_{n=0}^{\infty} \mathbb{Q}F_{m+n}x^n = \frac{\mathbb{Q}F_m + \mathbb{Q}F_{m-1}x}{1 - x - x^2}.$$

Then Halıcı [9] gave the Cassini identity for Fibonacci quaternions as

$$\mathbb{Q}F_{n-1}\mathbb{Q}F_{n+1} - (\mathbb{Q}F_n)^2 = (-1)^n(2\mathbb{Q}F_1 - 3k).$$

2.5 FIBONACCI OCTONIONS

John Graves discovered the octonions, which is a $8 = 2^3$-dimensional non-associative and a non-commutative algebra over $\mathbb{R}$. Short time after this discovery, these algebras were rediscovered by Arthur Cayley. John C. Baez studied octonions and gave their some important properties. For more details on octonions, see [4].

A p octonion, with real components $p_0, p_1, ..., p_7$, and $\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_7$, as base components, is defined by the following relation:

$$p = \sum_{s=0}^{7} p_s\mathbf{e}_s$$

where $\mathbf{e}_0 = 1, \mathbf{e}_1 = \mathbf{i}, \mathbf{e}_2 = \mathbf{j}, \mathbf{e}_3 = \mathbf{k}, \mathbf{e}_4 = \mathbf{e}, \mathbf{e}_5 = \mathbf{ie}, \mathbf{e}_6 = \mathbf{je}, \mathbf{e}_7 = \mathbf{ke}$. Note that we can write an octonion p as $p =$Re(p)+Im(p) where Re$(p) = p_0$ is the reel part and Im$(p) = \sum_{s=1}^{7} p_s\mathbf{e}_s$ is the imaginary part of the octonion. The set of octonions is denoted by $\mathbb{O}$.

For any $p, q \in \mathbb{O}$, and let

$$p = \sum_{i=0}^{7} p_i\mathbf{e}_i$$

and

$$q=\sum_{i=0}^{7} q_i \mathbf{e}_i,$$

$$p+q=\sum_{i=0}^{7} (p_i+q_i)\mathbf{e}_i,$$

$$p-q=\sum_{i=0}^{7} (p_i-q_i)\mathbf{e}_i.$$

And now let the conjugate of p be p^*, and the $p^*=$Re$(p)-$Im(p), so

$$pp^*=p^*p=\sum_{i=0}^{7} p_i^2.$$

The multiplication of two octonions is as following

$$pq=\left(\sum_{i=0}^{7}\alpha_i\mathbf{e}_i\right)\left(\sum_{j=0}^{7}\beta_j\mathbf{e}_j\right)=\sum_{i,j=0}^{7}\alpha_i\beta_j\left(\mathbf{e}_i\mathbf{e}_j\right),$$

where satisfy the octonion multiplication rules as:

Table 2.2 The Multiplication For The Basis Of Octonions

$\times$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_3$	-1	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	-1	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	-1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	-1	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	-1	$-\mathbf{e}_1$
$\mathbf{e}_7$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

The Fibonacci octonions is defined as follows [11]:

$$\mathbb{O}F_n=\sum_{j=0}^{7} F_{n+j}\mathbf{e}_j,$$

where $\mathbf{e}_0 = 1, \mathbf{e}_1 = \mathbf{i}, \mathbf{e}_2 = \mathbf{j}, \mathbf{e}_3 = \mathbf{k}, \mathbf{e}_4 = \mathbf{e}, \mathbf{e}_5 = \mathbf{ie}, \mathbf{e}_6 = \mathbf{je}, \mathbf{e}_7 = \mathbf{ke}$ and F_n is the n-th Fibonacci number.

The conjugate of Fibonacci octonion $(\mathbb{O}F_n)^*$ is

$$(\mathbb{O}F_n)^* = F_n - \sum_{j=1}^{7} F_{n+j}\mathbf{e}_j.$$

The Binet formula for the Fibonacci octonions is

$$\mathbb{O}F_n = \frac{\alpha^*(\alpha)^n - \beta^*(\beta)^n}{\alpha - \beta},$$

where $\alpha^* = \sum_{j=0}^{7} \alpha^j \mathbf{e}_j$ and $\beta^* = \sum_{j=0}^{7} \beta^j \mathbf{e}_j$.

The generating function for the Fibonacci octonions is

$$G\mathbb{O}F_n = \frac{1}{1-x-x^2} \sum_{s=0}^{7} (F_s + F_{s-1}x)\,\mathbf{e}_s.$$

Cassini's identity for the Fibonacci octonions is as follows

$$\mathbb{O}F_{n+1}\mathbb{O}F_{n-1} - (\mathbb{O}F_n)^2 = (-1)^n \left[K_0 - \mathbb{O}F_0 + 14\mathbf{e}_5 + 14\mathbf{e}_6 + 7\mathbf{e}_7\right],$$

where K_n is the n-th Lucas octonions. For more details about Fibonacci octonions see [11].

2.6 FIBONACCI SEDENIONS

The next doubling process applied to $\mathbb{O}$ then produces an algebra $\mathbb{S}$, $2^4 = 16$-dimensional called sedenions. Sedenions were detailly studied by Imaeda K. and Imaeda M. and Cariow A. and Cariowa G. [6, 13]. Now let us define a sedenion p as follow.

A p sedenion, with real components $p_0, p_1, ..., p_{15}$, and $\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{15}$, as base components, is defined by the following relation:

$$p = \sum_{i=0}^{15} p_i \mathbf{e}_i.$$

Note that we can write an sedenion p as $p =$Re(p)+Im(p) where Re$(p) = p_0$ is the reel part and Im$(p) = \sum_{s=1}^{15} p_s \mathbf{e}_s$ is the imaginary part of the sedenion. The set of sedenion is denoted by $\mathbb{S}$.

For any $p, q \in \mathbb{S}$, and let

$$p = \sum_{i=0}^{15} p_i \mathbf{e}_i,$$

and

$$q=\sum_{i=0}^{15} q_i \mathbf{e}_i,$$

then,

$$p+q=\sum_{i=0}^{15}(p_i+q_i)\mathbf{e}_i,$$

$$p-q=\sum_{i=0}^{15}(p_i-q_i)\mathbf{e}_i.$$

and now let the conjugate of p be p^*, and the p^* =Re(p)−Im(p), so

$$pp^*=p^*p=\sum_{i=0}^{15} p_i^2.$$

The multiplication of two sedenions is as following:

$$pq=\left(\sum_{i=0}^{15}\alpha_i\mathbf{e}_i\right)\left(\sum_{j=0}^{15}\beta_j\mathbf{e}_j\right)=\sum_{i,j=0}^{15}\alpha_i\beta_j\left(\mathbf{e}_i\mathbf{e}_j\right),$$

where $\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{15}$ are called unit sedenion such that $\mathbf{e}_0$ is the unit element and $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{15}$ are imaginaries satisfying, for $i, j, k = 1, 2, \ldots, 15$ the following multiplication rules:

$$\mathbf{e}_0\mathbf{e}_i=\mathbf{e}_i\mathbf{e}_0=\mathbf{e}_i, \quad (\mathbf{e}_i)^2=-\mathbf{e}_0,$$

$$\mathbf{e}_i\mathbf{e}_j=-\mathbf{e}_j\mathbf{e}_i, \quad i\neq j,$$

$$\mathbf{e}_i\left(\mathbf{e}_j\mathbf{e}_k\right)=-\left(\mathbf{e}_i\mathbf{e}_j\right)\mathbf{e}_k, \quad i\neq j, \quad \mathbf{e}_i\mathbf{e}_j\neq\pm\mathbf{e}_k.$$

For more linear properties of sedenions, please see [6, 13, 14].

In [12] Bilgici have defined Fibonacci sedenions as follows:

$$\mathbb{S}F_n=\sum_{i=0}^{15} F_{n+i}\mathbf{e}_i.$$

The Binet formula for the Fibonacci sedenions is

$$\mathbb{S}F_n=\frac{\tilde{\alpha}(\alpha)^n-\tilde{\beta}(\beta)^n}{\alpha-\beta},$$

where $\tilde{\alpha}=\sum_{i=0}^{15}(\alpha)^i\mathbf{e}_i$ and $\tilde{\beta}=\sum_{i=0}^{15}(\beta)^i\mathbf{e}_i$.

The generating function for the Fibonacci sedenions is

$$G\mathbb{S}F_n = \frac{\mathbb{S}F_0 + \mathbb{S}F_{-1}x}{1 - x - x^2}.$$

Cassini's identity for the Fibonacci sedenions is as follows:

$$\mathbb{S}F_{m+1}\mathbb{S}F_{m-1} - (\mathbb{S}F_m)^2 = (-1)^m (2\mathbb{S}F_{-1} + \omega)$$

where $\omega = 94\mathbf{e}_9 + 94\mathbf{e}_{10} + 188\mathbf{e}_{11} + 282\mathbf{e}_{12} - 188\mathbf{e}_{13} + 94\mathbf{e}_{14} + 893\mathbf{e}_{15}$.
This doubling process can be extended beyond the sedenions to form what are known as the 2^m-ions.

2.7 FIBONACCI 2^m-IONS

The hyper-complex numbers known as 2^m-ions can be regarded as linear combinations of elements from a canonical basis set:

$$W = \sum_{i=0}^{2^m-1} w_i\mathbf{e}_i = w_0\mathbf{e}_0 + w_1\mathbf{e}_1 + ... + w_{2^m-1}\mathbf{e}_{2^m-1} \tag{2.11}$$

where the $\mathbf{e}_i$'s are elements of the basis set and the w_i's are real numbers [4, 20]. The conjugate of W is denoted by

$$W^* = w_0\mathbf{e}_0 - \sum_{i=1}^{2^m-1} w_i\mathbf{e}_i = w_0\mathbf{e}_0 - w_1\mathbf{e}_1 - ... - w_{2^m-1}\mathbf{e}_{2^m-1}. \tag{2.12}$$

According to choise of m in (2.11), we obtain the following special cases.

1. For $m = 2$, and $w_i = F_i$, we obtain the Fibonacci quaternions $\mathbb{Q}F_n$ [9, 10].

2. For $m = 3$, and $w_i = F_i$, we obtain the Fibonacci octonions $\mathbb{O}F_n$ [11].

3. For $m = 4$, and $w_i = F_i$, we obtain the Fibonacci sedenions $\mathbb{S}F_n$ [12].

CHAPTER 3

HIGHER ORDER FIBONACCI QUATERNIONS

Initially, we express that the results of this chapter are cited from [22] which has been published by us. In this part of our thesis, we will deal with the higher order Fibonacci quaternions. Many properties of the higher order Fibonacci quaternions such as the Binet formula, the generating function, the exponential generating function, the recurrence relation, the Vajda's identity, the Cassini's identity and the Catalan's identity will be studied.

Definition 3.1 *The higher order Fibonacci quaternions, $\mathbb{Q}F_n^{(s)}$, is defined by*

$$\mathbb{Q}F_n^{(s)} = F_n^{(s)} + F_{n+1}^{(s)}\mathbf{i} + F_{n+2}^{(s)}\mathbf{j} + F_{n+3}^{(s)}\mathbf{k}. \tag{3.1}$$

If we take $s = 1$ in (3.1), we get the Fibonacci quaternions defined and studied by Horadam [10] and Halıcı [9]. By virtue of (3.1), the higher order Fibonacci quaternions can be written as

$$\mathbb{Q}F_n^{(s)} = F_n^{(s)} + u,$$

where $u = F_{n+1}^{(s)}\mathbf{i} + F_{n+2}^{(s)}\mathbf{j} + F_{n+3}^{(s)}\mathbf{k}$.
The conjugate of the higher order Fibonacci quaternion $\mathbb{Q}F_n^{(s)}$ is denoted by $\left(\mathbb{Q}F_n^{(s)}\right)^*$ as

$$\left(\mathbb{Q}F_n^{(s)}\right)^* = F_n^{(s)} - u. \tag{3.2}$$

Proposition 3.1 *For the higher order Fibonacci quaternions, we have*

$$\mathbb{Q}F_n^{(s)} + \left(\mathbb{Q}F_n^{(s)}\right)^* = 2F_n^{(s)}. \tag{3.3}$$

Proof. Using (3.1), we find that

$$\begin{aligned}
\mathbb{Q}F_n^{(s)} + \left(\mathbb{Q}F_n^{(s)}\right)^* &= \left(F_n^{(s)} + F_{n+1}^{(s)}\mathbf{i} + F_{n+2}^{(s)}\mathbf{j} + F_{n+3}^{(s)}\mathbf{k}\right) + \left(F_n^{(s)} - F_{n+1}^{(s)}\mathbf{i} - F_{n+2}^{(s)}\mathbf{j} - F_{n+3}^{(s)}\mathbf{k}\right) \\
&= \left(F_n^{(s)} + F_n^{(s)}\right) + \left(F_{n+1}^{(s)} - F_{n+1}^{(s)}\right)\mathbf{i} + \left(F_{n+2}^{(s)} - F_{n+2}^{(s)}\right)\mathbf{j} + \left(F_{n+3}^{(s)} - F_{n+3}^{(s)}\right)\mathbf{k} \\
&= F_n^{(s)} + F_n^{(s)} \\
&= 2F_n^{(s)}.
\end{aligned}$$

■

Proposition 3.2 *The higher order Fibonacci Fibonacci quaternions satisfy the following identity:*

$$\left(\mathbb{Q}F_n^{(s)}\right)^2 = 2F_n^{(s)}\mathbb{Q}F_n^{(s)} - \mathbb{Q}F_n^{(s)}\left(\mathbb{Q}F_n^{(s)}\right)^*. \tag{3.4}$$

Proof. Using (3.1) and (3.2), we find that

$$\begin{aligned}
\left(\mathbb{Q}F_n^{(s)}\right)^2 &= \mathbb{Q}F_n^{(s)}\mathbb{Q}F_n^{(s)} \\
&= \left(F_n^{(s)} + \mathbf{i}F_{n+1}^{(s)} + \mathbf{j}F_{n+2}^{(s)} + \mathbf{k}F_{n+3}^{(s)}\right)\left(F_n^{(s)} + \mathbf{i}F_{n+1}^{(s)} + \mathbf{j}F_{n+2}^{(s)} + \mathbf{k}F_{n+3}^{(s)}\right) \\
&= F_n^{(s)}F_n^{(s)} + F_n^{(s)}\mathbf{i}F_{n+1}^{(s)} + F_n^{(s)}\mathbf{j}F_{n+2}^{(s)} + F_n^{(s)}\mathbf{k}F_{n+3}^{(s)} + \mathbf{i}F_{n+1}^{(s)}F_n^{(s)} + \mathbf{i}F_{n+1}^{(s)}\mathbf{i}F_{n+1}^{(s)} \\
&\quad + F_{n+1}^{(s)}\mathbf{i}F_{n+2}^{(s)}\mathbf{j} + F_{n+1}^{(s)}\mathbf{i}F_{n+3}^{(s)}\mathbf{k} + F_{n+2}^{(s)}\mathbf{j}F_n^{(s)} + F_{n+2}^{(s)}\mathbf{j}F_{n+1}^{(s)}\mathbf{i} + F_{n+2}^{(s)}\mathbf{j}F_{n+2}^{(s)}\mathbf{j} + F_{n+2}^{(s)}\mathbf{j}F_{n+3}^{(s)}\mathbf{k} \\
&\quad + F_{n+3}^{(s)}\mathbf{k}F_n^{(s)} + F_{n+3}^{(s)}\mathbf{k}F_{n+1}^{(s)}\mathbf{i} + F_{n+3}^{(s)}\mathbf{k}F_{n+2}^{(s)}\mathbf{j} + F_{n+3}^{(s)}\mathbf{k}F_{n+3}^{(s)}\mathbf{k} \\
&= \left(F_n^{(s)}F_n^{(s)} - F_{n+1}^{(s)}F_{n+1}^{(s)} - F_{n+2}^{(s)}F_{n+2}^{(s)} - F_{n+3}^{(s)}F_{n+3}^{(s)}\right) \\
&\quad + \left(F_n^{(s)}F_{n+1}^{(s)} + F_{n+1}^{(s)}F_n^{(s)} - F_{n+2}^{(s)}F_{n+3}^{(s)} + F_{n+3}^{(s)}F_{n+2}^{(s)}\right)\mathbf{i} \\
&\quad + \left(F_n^{(s)}F_{n+2}^{(s)} - F_{n+1}^{(s)}F_{n+3}^{(s)} + F_{n+2}^{(s)}F_n^{(s)} + F_{n+3}^{(s)}F_{n+1}^{(s)}\right)\mathbf{j} \\
&\quad + \left(F_n^{(s)}F_{n+3}^{(s)} + F_{n+1}^{(s)}F_{n+2}^{(s)} - F_{n+2}^{(s)}F_{n+1}^{(s)} + F_{n+3}^{(s)}F_n^{(s)}\right)\mathbf{k} \\
&= \left(F_n^{(s)}\right)^2 - \left(F_{n+1}^{(s)}\right)^2 - \left(F_{n+2}^{(s)}\right)^2 - \left(F_{n+3}^{(s)}\right)^2 + \left(2F_n^{(s)}F_{n+1}^{(s)}\right)\mathbf{i} \\
&\quad + \left(2F_n^{(s)}F_{n+2}^{(s)}\right)\mathbf{j} + \left(2F_n^{(s)}F_{n+3}^{(s)}\right)\mathbf{k}.
\end{aligned}$$

then,

$$\begin{aligned}
&= \left(F_n^{(s)}\right)^2 - \left(F_{n+1}^{(s)}\right)^2 - \left(F_{n+2}^{(s)}\right)^2 - \left(F_{n+3}^{(s)}\right)^2 + 2F_n^{(s)}\left(F_{n+1}^{(s)}\mathbf{i} + F_{n+2}^{(s)}\mathbf{j} + F_{n+3}^{(s)}\mathbf{k}\right) \\
&= \left(F_n^{(s)}\right)^2 - \left(F_{n+1}^{(s)}\right)^2 - \left(F_{n+2}^{(s)}\right)^2 - \left(F_{n+3}^{(s)}\right)^2 \\
&\quad + 2F_n^{(s)}\left(F_n^{(s)} - F_n^{(s)} + F_{n+1}^{(s)}\mathbf{i} + F_{n+2}^{(s)}\mathbf{j} + F_{n+3}^{(s)}\mathbf{k}\right) \\
&= -\left(F_{n+1}^{(s)}\right)^2 - \left(F_{n+2}^{(s)}\right)^2 - \left(F_{n+3}^{(s)}\right)^2 - 2\left(F_n^{(s)}\right)^2 + \left(F_n^{(s)}\right)^2 \\
&\quad + 2F_n^{(s)}\left(F_n^{(s)} + F_{n+1}^{(s)}\mathbf{i} + F_{n+2}^{(s)}\mathbf{j} + F_{n+3}^{(s)}\mathbf{k}\right) \\
&= -\left(\left(F_n^{(s)}\right)^2 + \left(F_{n+1}^{(s)}\right)^2 + \left(F_{n+2}^{(s)}\right)^2 + \left(F_{n+3}^{(s)}\right)^2\right) + 2F_n^{(s)}\mathbb{Q}F_n^{(s)} \\
&= -\mathbb{Q}F_n^{(s)}\left(\mathbb{Q}F_n^{(s)}\right)^* + 2F_n^{(s)}\mathbb{Q}F_n^{(s)}.
\end{aligned}$$

■

Theorem 3.3 *The Binet type formula for higher order Fibonacci quaternions is*

$$\mathbb{Q}F_n^{(s)} = \frac{\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n}{\alpha^s - \beta^s} \tag{3.5}$$

where $\hat{\alpha} = 1 + \alpha^s\mathbf{i} + \alpha^{2s}\mathbf{j} + \alpha^{3s}\mathbf{k}$ *and* $\hat{\beta} = 1 + \beta^s\mathbf{i} + \beta^{2s}\mathbf{j} + \beta^{3s}\mathbf{k}$.

Proof. By using (2.5) and (3.1), we have

$$\begin{aligned}
\mathbb{Q}F_n^{(s)} &= F_n^{(s)} + F_{n+1}^{(s)}\mathbf{i} + F_{n+2}^{(s)}\mathbf{j} + F_{n+3}^{(s)}\mathbf{k} \\
&= \frac{(\alpha^s)^n - (\beta^s)^n}{\alpha^s - \beta^s} + \frac{(\alpha^s)^{n+1} - (\beta^s)^{n+1}}{\alpha^s - \beta^s}\mathbf{i} + \frac{(\alpha^s)^{n+2} - (\beta^s)^{n+2}}{\alpha^s - \beta^s}\mathbf{j} + \frac{(\alpha^s)^{n+3} - (\beta^s)^{n+3}}{\alpha^s - \beta^s}\mathbf{k} \\
&= \frac{1}{\alpha^s - \beta^s}\left((\alpha^s)^n\left(1 + \alpha^s\mathbf{i} + \alpha^{2s}\mathbf{j} + \alpha^{3s}\mathbf{k}\right) - (\beta^s)^n\left(1 + \beta^s\mathbf{i} + \beta^{2s}\mathbf{j} + \beta^{3s}\mathbf{k}\right)\right) \\
&= \frac{\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n}{\alpha^s - \beta^s}.
\end{aligned}$$

So the proof is completed. ■

Theorem 3.4 *The recurrence relation for higher order Fibonacci quaternions is given by*

$$\mathbb{Q}F_{n+1}^{(s)} = L_s\mathbb{Q}F_n^{(s)} + (-1)^{s+1}\,\mathbb{Q}F_{n-1}^{(s)}. \tag{3.6}$$

Proof. By using (2.4) and (3.5), we have

$$\begin{aligned}
\mathbb{Q}F_{n+1}^{(s)} &= \frac{\hat{\alpha}(\alpha^s)^{n+1} - \hat{\beta}(\beta^s)^{n+1}}{\alpha^s - \beta^s} \\
&= \frac{1}{\alpha^s - \beta^s}\left[\hat{\alpha}(\alpha^s)^{n+1} - \hat{\beta}(\beta^s)^{n+1}\right] \\
&= \frac{1}{\alpha^s - \beta^s}\left[\hat{\alpha}(\alpha^s)^n\alpha^s - \hat{\beta}(\beta^s)^n\beta^s\right] \\
&= \frac{1}{\alpha^s - \beta^s}\left[\hat{\alpha}(\alpha^s)^n\alpha^s + (-\hat{\beta}(\beta^s)^n\alpha^s + \hat{\beta}(\beta^s)^n\alpha^s) - \hat{\beta}(\beta^s)^n\beta^s\right] \\
&= \frac{1}{\alpha^s - \beta^s}\left[(\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n)\alpha^s + \hat{\beta}(\beta^s)^n\alpha^s - \hat{\beta}(\beta^s)^n\beta^s\right] \\
&= \frac{1}{\alpha^s - \beta^s}\left[(\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n)\right]\alpha^s + \frac{1}{\alpha^s - \beta^s}\left[\hat{\beta}(\beta^s)^n\alpha^s - \hat{\beta}(\beta^s)^n\beta^s\right] \\
&= \mathbb{Q}F_n^{(s)}\alpha^s + \frac{1}{\alpha^s - \beta^s}\left[\hat{\beta}(\beta^s)^n\alpha^s - \hat{\beta}(\beta^s)^n\beta^s\right] \\
&= \mathbb{Q}F_n^{(s)}(\alpha^s + \beta^s - \beta^s) + \frac{1}{\alpha^s - \beta^s}\left[\hat{\beta}(\beta^s)^n\alpha^s - \hat{\beta}(\beta^s)^n\beta^s\right] \\
&= \mathbb{Q}F_n^{(s)}(\alpha^s + \beta^s) - Q_n^{(s)}\beta^s + \frac{1}{\alpha^s - \beta^s}\left[\hat{\beta}(\beta^s)^n\alpha^s - \hat{\beta}(\beta^s)^n\beta^s\right].
\end{aligned}$$

also

$$
\begin{aligned}
&= L_s\mathbb{Q}F_n^{(s)} + \frac{1}{\alpha^s - \beta^s}\left[\hat{\beta}(\beta^s)^n\beta^s - \hat{\alpha}(\alpha^s)^n\beta^s + \hat{\beta}(\beta^s)^n\alpha^s - \hat{\beta}(\beta^s)^n\beta^s\right] \\
&= L_s\mathbb{Q}F_n^{(s)} + \frac{1}{\alpha^s - \beta^s}\left[\hat{\beta}(\beta^s)^n\alpha^s - \hat{\alpha}(\alpha^s)^n\beta^s\right] \\
&= L_s\mathbb{Q}F_n^{(s)} + \frac{(\alpha\beta)^s}{\alpha^s - \beta^s}\left[\hat{\beta}(\beta^s)^{n-1} - \hat{\alpha}(\alpha^s)^{n-1}\right] \\
&= L_s\mathbb{Q}F_n^{(s)} - \frac{(\alpha\beta)^s}{\alpha^s - \beta^s}\left[\hat{\alpha}(\alpha^s)^{n-1} - \hat{\beta}(\beta^s)^{n-1}\right] \\
&= L_s\mathbb{Q}F_n^{(s)} - (-1)^s\left[\frac{\hat{\alpha}(\alpha^s)^{n-1} - \hat{\beta}(\beta^s)^{n-1}}{\alpha^s - \beta^s}\right] \\
&= L_s\mathbb{Q}F_n^{(s)} - (-1)^s\mathbb{Q}F_{n-1}^{(s)} \\
&= L_s\mathbb{Q}F_n^{(s)} + (-1)^{s+1}\mathbb{Q}F_{n-1}^{(s)} \\
&= L_s\mathbb{Q}F_n^{(s)} + (-1)^{s+1}\,\mathbb{Q}F_{n-1}^{(s)},
\end{aligned}
$$

and that gives us the desired recursion formula (3.6). ■

Theorem 3.5 *By extending n and s to negative integer numbers for higher order Fibonacci quaternions $\mathbb{Q}F_n^{(s)}$; the following identities can be derived as*

$$\mathbb{Q}F_{-n}^{(s)} = (-1)^{sn}\frac{\hat{\alpha}(\beta^s)^n - \hat{\beta}(\alpha^s)^n}{\alpha^s - \beta^s}, \tag{3.7}$$

$$\mathbb{Q}F_{-n}^{(-s)} = (-1)^{s-1}\mathbb{Q}F_n^{(s)}, \tag{3.8}$$

$$\mathbb{Q}F_n^{(-s)} = (-1)^{s-1}\mathbb{Q}F_{-n}^{(s)}. \tag{3.9}$$

Proof. By using (3.5), we have

$$
\begin{aligned}
\mathbb{Q}F_{-n}^{(s)} &= \frac{\hat{\alpha}(\alpha^s)^{-n} - \hat{\beta}(\beta^s)^{-n}}{\alpha^s - \beta^s} \\
&= \frac{\frac{\hat{\alpha}(\beta^s)^n - \hat{\beta}(\alpha^s)^n}{(\alpha^s)^n(\beta^s)^n}}{\alpha^s - \beta^s} \\
&= (-1)^{sn}\frac{\hat{\alpha}(\beta^s)^n - \hat{\beta}(\alpha^s)^n}{\alpha^s - \beta^s},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Q}F_{-n}^{(-s)} &= \frac{\hat{\alpha}(\alpha^{-s})^{-n} - \hat{\beta}(\beta^{-s})^{-n}}{\alpha^{-s} - \beta^{-s}} \\
&= (\alpha\beta)^s \, \frac{\hat{\alpha}(\alpha^{s})^{n} - \hat{\beta}(\beta^{s})^{n}}{\beta^{s} - \alpha^{s}} \\
&= (-1)^s \frac{\hat{\alpha}(\alpha^{s})^{n} - \hat{\beta}(\beta^{s})^{n}}{\beta^{s} - \alpha^{s}} \\
&= (-1)^{s-1} \mathbb{Q}F_{n}^{(s)},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{Q}F_{n}^{(-s)} &= \frac{\hat{\alpha}(\alpha^{-s})^{n} - \hat{\beta}(\beta^{-s})^{n}}{\alpha^{-s} - \beta^{-s}} \\
&= \frac{(-1)^s}{\beta^{s} - \alpha^{s}} \frac{\hat{\alpha}(\beta^{s})^{n} - \hat{\beta}(\alpha^{s})^{n}}{(-1)^{sn}} \\
&= (-1)^{s(n+1)} \frac{\hat{\alpha}(\beta^{s})^{n} - \hat{\beta}(\alpha^{s})^{n}}{\beta^{s} - \alpha^{s}} \\
&= (-1)^{s-1} \mathbb{Q}F_{-n}^{(s)}.
\end{aligned}
$$

So the proof is completed. ■

3.1 GENERATING FUNCTION OF THE HIGHER ORDER FIBONACCI QUATERNIONS

In this section, we give the generating function of higher order Fibonacci quaternions. To obtain the generating function, we need the following lemma:

Lemma 3.6 *The following identities hold:*

$$\hat{\alpha} - \hat{\beta} = (\alpha^s - \beta^s)(\mathbf{i} + L_s\mathbf{j} + (L_{2s} + (-1)^{s+1})\mathbf{k}) \tag{3.10}$$

and

$$\hat{\alpha}\beta^s - \hat{\beta}\alpha^s = (-1)^s(\alpha^s - \beta^s)\left((-1)^{s+1} + \mathbf{j} + L_s\mathbf{k}\right), \tag{3.11}$$

where $\hat{\alpha} = 1 + \alpha^s\mathbf{i} + \alpha^{2s}\mathbf{j} + \alpha^{3s}\mathbf{k}$ *and* $\hat{\beta} = 1 + \beta^s\mathbf{i} + \beta^{2s}\mathbf{j} + \beta^{3s}\mathbf{k}$.

Proof. *From the definitions of* $\hat{\alpha}$ *and* $\hat{\beta}$, *we have*

$$
\begin{aligned}
\hat{\alpha} - \hat{\beta} &= (1 + \alpha^s\mathbf{i} + \alpha^{2s}\mathbf{j} + \alpha^{3s}\mathbf{k}) - (1 + \beta^s\mathbf{i} + \beta^{2s}\mathbf{j} + \beta^{3s}\mathbf{k}) \\
&= (1-1) + (\alpha^s - \beta^s)\mathbf{i} + (\alpha^{2s} - \alpha^{2s})\mathbf{j} + (\alpha^{3s} - \beta^{3s})\mathbf{k} \\
&= (\alpha^s - \beta^s)(\mathbf{i} + L_s\mathbf{j} + (L_{2s} + (-1)^{s+1})\mathbf{k}),
\end{aligned}
$$

and

$$\begin{aligned}\left(\hat{\alpha}\beta^s-\hat{\beta}\alpha^s\right) &= (\beta^s+\alpha^s\beta^s\mathbf{i}+\alpha^{2s}\beta^s\mathbf{j}+\alpha^{3s}\beta^s\mathbf{k})-(\alpha^s+\beta^s\alpha^s\mathbf{i}+\beta^{2s}\alpha^s\mathbf{j}+\beta^{3s}\alpha^s\mathbf{k})\\ &= (\beta^s-\alpha^s)+(-1)^s\left(\alpha^s-\beta^s\right)\mathbf{j}+(-\mathbf{1})^s\left(\alpha^{2s}-\beta^{2s}\right)\mathbf{k}\\ &= (-1)^s(\alpha^s-\beta^s)\left((-1)^{s+1}+\mathbf{j}+L_s\mathbf{k}\right).\end{aligned}$$

Theorem 3.7 *The generating function of the higher order Fibonacci quaternions $\mathbb{Q}F_n^{(s)}$ is given by*

$$G^{(s)}(x)=\frac{(\mathbf{i}+L_s\mathbf{j}+(L_{2s}+(-1)^s)\mathbf{k})+(-1)^{s+1}\left((-1)^{s+1}+\mathbf{j}+L_s\mathbf{k}\right)x}{1-L_sx+(-1)^sx^2}. \tag{3.12}$$

■

Proof. Let

$$G^{(s)}(x)=\sum_{n=0}^{\infty}\mathbb{Q}F_n^{(s)}x^n$$

be the generating function of $\mathbb{Q}F_n^{(s)}$, then using (2.5) and (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{Q}F_n^{(s)}x^n &= \sum_{n=0}^{\infty}(F_n^{(s)}+F_{n+1}^{(s)}\mathbf{i}+F_{n+2}^{(s)}\mathbf{j}+F_{n+3}^{(s)}\mathbf{k})\ x^n\\ &= \frac{1}{\alpha^s-\beta^s}\sum_{n=0}^{\infty}\left(\alpha^{ns}(1+\alpha^s\mathbf{i}+\alpha^{2s}\mathbf{j}+\alpha^{3s}\mathbf{k})-\beta^{ns}(1+\beta^s\mathbf{i}+\beta^{2s}\mathbf{j}+\beta^{3s}\mathbf{k})\right)x^n\\ &= \frac{1}{\alpha^s-\beta^s}\sum_{n=0}^{\infty}((\alpha^s)^n\hat{\alpha})x^n-\frac{1}{\alpha^s-\beta^s}\sum_{n=0}^{\infty}((\beta^s)^n\hat{\beta})x^n\\ &= \frac{\hat{\alpha}}{\alpha^s-\beta^s}\sum_{n=0}^{\infty}(\alpha^s)^nx^n-\frac{\hat{\beta}}{\alpha^s-\beta^s}\sum_{n=0}^{\infty}(\beta^s)^nx^n\\ &= \frac{\hat{\alpha}}{\alpha^s-\beta^s}\frac{1}{1-\alpha^sx}-\frac{\hat{\beta}}{\alpha^s-\beta^s}\frac{1}{1-\beta^sx}\\ &= \frac{\hat{\alpha}-\hat{\alpha}\beta^sx-\hat{\beta}+\hat{\beta}\alpha^sx}{(\alpha^s-\beta^s)(1-\alpha^sx)(1-\beta^sx)}\\ &= \frac{(\hat{\alpha}-\hat{\beta})-(\hat{\alpha}\beta^s-\hat{\beta}\alpha^s)x}{(\alpha^s-\beta^s)(1-L_sx+(-1)^sx^2)}.\end{aligned}$$

Then using (3.10) and (3.11), we have

$$\begin{aligned}&= \frac{(\alpha^s-\beta^s)(\mathbf{i}+L_s\mathbf{j}+(L_{2s}+(-1)^s)\mathbf{k})-(-1)^s(\alpha^s-\beta^s)\left((-1)^{s+1}+\mathbf{j}+L_s\mathbf{k}\right)x}{(\alpha^s-\beta^s)(1-L_sx+(-1)^sx^2)}\\ &= \frac{(\mathbf{i}+L_s\mathbf{j}+(L_{2s}+(-1)^s)\mathbf{k})-(-1)^s\left((-1)^{s+1}+\mathbf{j}+L_s\mathbf{k}\right)x}{1-L_sx+(-1)^sx^2}\\ &= \frac{(\mathbf{i}+L_s\mathbf{j}+(L_{2s}+(-1)^s)\mathbf{k})+(-1)^{s+1}\left((-1)^{s+1}+\mathbf{j}+L_s\mathbf{k}\right)x}{1-L_sx+(-1)^sx^2}.\end{aligned}$$

Now let us give the general formula of generatig function of the higher order Fibonacci quaternions. ■

Theorem 3.8 *For $m, n \in Z$, the generating function of the higher order Fibonacci quaternions $\mathbb{Q}F_{m+n}^{(s)}$ is*

$$\sum_{n=0}^{\infty} \mathbb{Q}F_{m+n}^{(s)} x^n = \frac{\mathbb{Q}F_m^{(s)} + (-1)^{s+1}\mathbb{Q}F_{m-1}^{(s)} x}{1 - L_s x + (-1)^s x^2}. \tag{3.13}$$

Proof. Using (3.5) we have,

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{Q}F_{m+n}^{(s)} x^n &= \sum_{n=0}^{\infty} \left(\frac{\hat{\alpha}(\alpha^s)^{m+n} - \hat{\beta}(\beta^s)^{m+n}}{\alpha^s - \beta^s} \right) x^n \\
&= \frac{1}{\alpha^s - \beta^s} \left[\sum_{n=0}^{\infty} \hat{\alpha}(\alpha^s)^{m+n} x^n - \sum_{n=0}^{\infty} \hat{\beta}(\beta^s)^{m+n} x^n \right] \\
&= \frac{1}{\alpha^s - \beta^s} \left[\hat{\alpha}(\alpha^s)^m \sum_{n=0}^{\infty} (\alpha^s)^n x^n - \hat{\beta}(\beta^s)^m \sum_{n=0}^{\infty} (\beta^s)^n x^n \right] \\
&= \frac{1}{\alpha^s - \beta^s} \left[\hat{\alpha}(\alpha^s)^m \frac{1}{1 - \alpha^s x} - \hat{\beta}(\beta^s)^m \frac{1}{1 - \beta^s x} \right] \\
&= \frac{1}{\alpha^s - \beta^s} \left[\frac{\hat{\alpha}(\alpha^s)^m (1 - \beta^s x) - \hat{\beta}(\beta^s)^m (1 - \alpha^s x)}{(1 - \alpha^s x)(1 - \beta^s x)} \right] \\
&= \frac{1}{\alpha^s - \beta^s} \left[\frac{\hat{\alpha}(\alpha^s)^m - \hat{\alpha}(\alpha^s)^m \beta^s x - \hat{\beta}(\beta^s)^m + \hat{\beta}(\beta^s)^m \alpha^s x}{1 - L_s x + (-1)^s x^2} \right] \\
&= \frac{\hat{\alpha}(\alpha^s)^m - \hat{\beta}(\beta^s)^m}{(\alpha^s - \beta^s)(1 - L_s x + (-1)^s x^2)} - \frac{\alpha^s \beta^s \left(\hat{\alpha}(\alpha^s)^{m-1} - \hat{\beta}(\beta^s)^{m-1} \right) x}{(\alpha^s - \beta^s)(1 - L_s x + (-1)^s x^2)} \\
&= \mathbb{Q}F_m^{(s)} \frac{1}{1 - L_s x + (-1)^s x^2} - \frac{(\hat{\alpha}(\alpha^s)^{m-1} - \hat{\beta}(\beta^s)^{m-1})}{(\alpha^s - \beta^s)} \frac{(-1)^s x}{1 - L_s x + (-1)^s x^2} \\
&= \frac{\mathbb{Q}F_m^{(s)}}{1 - L_s x + (-1)^s x^2} - \frac{\mathbb{Q}F_{m-1}^{(s)} (-1)^s x}{1 - L_s x + (-1)^s x^2} \\
&= \frac{\mathbb{Q}F_m^{(s)} + (-1)^{s+1} \mathbb{Q}F_{m-1}^{(s)} x}{1 - L_s x + (-1)^s x^2}.
\end{aligned}$$

■

Now, we shall give the exponential generating function of the higher order Fibonacci quaternions.

Theorem 3.9 *Exponential generating function of the higher order Fibonacci quaternions $\mathbb{Q}F_n^{(s)}$ is*

$$\sum_{n=0}^{\infty} \mathbb{Q}F_n^{(s)} \frac{x^n}{n!} = \frac{\hat{\alpha} e^{\alpha^s x} - \hat{\beta} e^{\beta^s x}}{\alpha^s - \beta^s}. \tag{3.14}$$

Proof. Let

$$U^{(s)}(x) = \sum_{n=0}^{\infty} \mathbb{Q}F_n^{(s)} \frac{x^n}{n!}$$

be the exponential generating function of $\mathbb{Q}F_n^{(s)}$, then using (3.5), we have

$$\begin{aligned} U^{(s)}(x) &= \sum_{n=0}^{\infty} \mathbb{Q}F_n^{(s)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n}{\alpha^s - \beta^s} \right) \frac{x^n}{n!} \\ &= \frac{1}{\alpha^s - \beta^s} \left[\hat{\alpha} \sum_{n=0}^{\infty} \frac{(\alpha^s)^n}{n!} x^n - \hat{\beta} \sum_{n=0}^{\infty} \frac{(\beta^s)^n}{n!} x^n \right] \\ &= \frac{1}{\alpha^s - \beta^s} \left[\hat{\alpha} e^{\alpha^s x} - \hat{\beta} e^{\beta^s x} \right] \\ &= \frac{\hat{\alpha} e^{\alpha^s x} - \hat{\beta} e^{\beta^s x}}{\alpha^s - \beta^s}. \end{aligned}$$

■

3.2 SOME ADDITIONAL IDENTITIES FOR HIGHER ORDER FIBONACCI QUATERNIONS

In this part of our thesis, we derive some identities involving higher order Fibonacci quaternions. In order to compute our next result, we will give the following lemma.

Lemma 3.10 *The following identities hold for*

$$\hat{\alpha}\hat{\beta} = A - \Delta B \tag{3.15}$$

and

$$\hat{\beta}\hat{\alpha} = A + \Delta B, \tag{3.16}$$

where $\Delta = \alpha^s - \beta^s$, $A = 2(-1)^{s+1} + L_s\mathbf{i} + L_{2s}\mathbf{j} + L_{3s}\mathbf{k}$ *and* $B = F_1^{(s)}\mathbf{i} + (-1)^{s+1} F_2^{(s)}\mathbf{j} + (-1)^s F_1^{(s)}\mathbf{k}$.

Proof. By virtue of (2.9) and (2.10), we get

$$\begin{aligned} \hat{\alpha}\hat{\beta} &= \left(1 + \alpha^s\mathbf{i} + \alpha^{2s}\mathbf{j} + \alpha^{3s}\mathbf{k}\right)\left(1 + \beta^s\mathbf{i} + \beta^{2s}\mathbf{j} + \beta^{3s}\mathbf{k}\right) \\ &= 2(-1)^{s+1} + (L_s + \beta^s - \alpha^s)\mathbf{i} + \left(L_{2s} + (-1)^s\left(\alpha^{2s} - \beta^{2s}\right)\right)\mathbf{j} \\ &\quad + (L_{3s} + (-1)^s(\beta^s - \alpha^s))\mathbf{k} \\ &= 2(-1)^{s+1} + L_s\mathbf{i} + L_{2s}\mathbf{j} + L_{3s}\mathbf{k} - \Delta F_1^{(s)}\mathbf{i} + (-1)^s \Delta F_2^{(s)}\mathbf{j} + (-1)^{s+1} \Delta F_1^{(s)}\mathbf{k} \\ &= 2(-1)^{s+1} + \left(L_s - \Delta F_1^{(s)}\right)\mathbf{i} + \left(L_{2s} + (-1)^s \Delta F_2^{(s)}\right)\mathbf{j} + \left(L_{3s} - (-1)^s \Delta F_1^{(s)}\right)\mathbf{k} \\ &= A - \Delta B. \end{aligned}$$

and

$$\begin{aligned}
\hat{\beta}\hat{\alpha} &= \left(1+\beta^s\mathbf{i}+\beta^{2s}\mathbf{j}+\beta^{3s}\mathbf{k}\right)\left(1+\alpha^s\mathbf{i}+\alpha^{2s}\mathbf{j}+\alpha^{3s}\mathbf{k}\right) \\
&= 2\left(-1\right)^{s+1}+\left(L_s-\beta^s+\alpha^s\right)\mathbf{i}+\left(L_{2s}+\left(-1\right)^s\left(-\alpha^{2s}+\beta^{2s}\right)\right)\mathbf{j} \\
&\quad +\left(L_{3s}+\left(-1\right)^s\left(-\beta^s+\alpha^s\right)\right)\mathbf{k} \\
&= 2\left(-1\right)^{s+1}+\left(L_s+\Delta F_1^{(s)}\right)\mathbf{i}+\left(L_{2s}-\left(-1\right)^s\Delta F_2^{(s)}\right)\mathbf{j}+\left(L_{3s}+\left(-1\right)^s\Delta F_1^{(s)}\right)\mathbf{k} \\
&= A+\Delta B.
\end{aligned}$$

■

We now give the following result that we will use to obtain a number of special identities.

Theorem 3.11 *(Vajda's identity) For any integers n, m, and r, we have*

$$\mathbb{Q}F_{m+n}^{(s)}\mathbb{Q}F_{n+r}^{(s)}-\mathbb{Q}F_{n}^{(s)}\mathbb{Q}F_{m+n+r}^{(s)}=\left(-1\right)^{ns}F_m^{(s)}\left[AF_r^{(s)}+BL_{sr}\right].$$

Proof. *By using the Binet formula for the higher order Fibonacci quaternions, we find that*

$$\begin{aligned}
&\mathbb{Q}F_{m+n}^{(s)}\mathbb{Q}F_{n+r}^{(s)}-\mathbb{Q}F_{n}^{(s)}\mathbb{Q}F_{m+n+r}^{(s)} \\
&= \frac{1}{\Delta^2}\left[\begin{array}{c}\left(\hat{\alpha}(\alpha^s)^{n+m}-\hat{\beta}(\beta^s)^{n+m}\right)\left(\hat{\alpha}(\alpha^s)^{n+r}-\hat{\beta}(\beta^s)^{n+r}\right) \\ -\left(\hat{\alpha}(\alpha^s)^{n}-\hat{\beta}(\beta^s)^{n}\right)\left(\hat{\alpha}(\alpha^s)^{n+m+r}-\hat{\beta}(\beta^s)^{n+m+r}\right)\end{array}\right] \\
&= \frac{1}{\Delta^2}\left[-\hat{\alpha}\hat{\beta}(\alpha^s)^{n+m}(\beta^s)^{n+r}-\hat{\beta}\hat{\alpha}(\alpha^s)^{n+r}(\beta^s)^{n+m}+\hat{\alpha}\hat{\beta}(\alpha^s)^{n}(\beta^s)^{n+m+r}+\hat{\beta}\hat{\alpha}(\alpha^s)^{n+m+r}(\beta^s)^{n}\right] \\
&= \frac{1}{\Delta^2}\left[(\alpha^s\beta^s)^n\left(-\hat{\alpha}\hat{\beta}(\alpha^s)^{m}(\beta^s)^{r}-\hat{\beta}\hat{\alpha}(\alpha^s)^{r}(\beta^s)^{m}+\hat{\alpha}\hat{\beta}(\beta^s)^{m+r}+\hat{\beta}\hat{\alpha}(\alpha^s)^{m+r}\right)\right] \\
&= \frac{\left(-1\right)^{ns}}{\Delta^2}\left[-\hat{\alpha}\hat{\beta}(\beta^s)^{r}\left((\alpha^s)^m-(\beta^s)^m\right)+\hat{\beta}\hat{\alpha}(\alpha^s)^{r}\left((\alpha^s)^m-(\beta^s)^m\right)\right] \\
&= \frac{\left(-1\right)^{ns}}{\Delta^2}\left[\left((\alpha^s)^m-(\beta^s)^m\right)\left(\hat{\beta}\hat{\alpha}(\alpha^s)^{r}-\hat{\alpha}\hat{\beta}(\beta^s)^{r}\right)\right] \\
&= \frac{\left(-1\right)^{ns}F_m^{(s)}}{\Delta}\left[\left(A+\Delta B\right)\left(\alpha^s\right)^r-\left(A-\Delta B\right)\left(\beta^s\right)^r\right] \\
&= \frac{\left(-1\right)^{ns}F_m^{(s)}}{\Delta}\left[A(\alpha^s)^r+\Delta B(\alpha^s)^r-A(\beta^s)^r+\Delta B(\beta^s)^r\right] \\
&= \frac{\left(-1\right)^{ns}F_m^{(s)}}{\Delta}\left[A\left((\alpha^s)^r-(\beta^s)^r\right)+\Delta B\left((\alpha^s)^r+(\beta^s)^r\right)\right] \\
&= \left(-1\right)^{ns}F_m^{(s)}\left[AF_r^{(s)}+BL_{sr}\right].
\end{aligned}$$

So the proof is completed. ■

Now, we have the following special cases from the Vajda's identity:

Corollary 3.1 *For $m=-r$, we have Catalan's identity*

$$\mathbb{Q}F_{n-r}^{(s)}\mathbb{Q}F_{n+r}^{(s)}-\left(\mathbb{Q}F_{n}^{(s)}\right)^{2}=(-1)^{ns}F_{-r}^{(s)}\left[AF_{r}^{(s)}+BL_{sr}\right].$$

Corollary 3.2 *For $r=-m=1$, we give Cassini's identity*

$$\mathbb{Q}F_{n-1}^{(s)}\mathbb{Q}F_{n+1}^{(s)}-\left(\mathbb{Q}F_{n}^{(s)}\right)^{2}=(-1)^{ns+s+1}\left[A+BL_{s}\right].$$

Corollary 3.3 *For $r=j-n$, and $m=1$, we reduce to d'Ocagne's identity*

$$\mathbb{Q}F_{n+1}^{(s)}\mathbb{Q}F_{j}^{(s)}-\mathbb{Q}F_{n}^{(s)}\mathbb{Q}F_{j+1}^{(s)}=(-1)^{ns}\left[AF_{j-n}^{(s)}+BL_{s(j-n)}\right].$$

CHAPTER 4

HIGHER ORDER FIBONACCI HYPER COMPLEX NUMBERS

Initially, we express that the results of this chapter are cited from [23] which has been published by us.

In this chapter, we will work on the higher order Fibonacci 2^m-ions. Many identities about the higher order Fibonacci 2^m-ions will be given and also the relations between the higher order Fibonacci quaternions and the higher order Fibonacci 2^m-ions will be found. We now define the higher order Fibonacci 2^m-ions and derive some new identities concerning higher Fibonacci 2^m-ions.

Definition 4.1 *The higher order Fibonacci 2^m-ions $\mathbb{HC}_n^{(s)}$ (or higher order Fibonacci hyper complex numbers), are defined by*

$$\mathbb{HC}_n^{(s)} = \sum_{h=0}^{2^m-1} F_{n+h}^{(s)}\mathbf{e}_h = F_n^{(s)}\mathbf{e}_0 + F_{n+1}^{(s)}\mathbf{e}_1 + \cdots + F_{n+2^m-1}^{(s)}\mathbf{e}_{2^m-1}, \tag{4.1}$$

where the $\mathbf{e}_h$'s are elements of the basis set and $F_n^{(s)}$ is the n-th higher order Fibonacci number.

According to choice of s and m in (4.1), we obtain the following special cases.

1. For $m = 0$, we obtain the original higher order Fibonacci numbers $F_n^{(s)}$ [8].

2. For $m = 1$, we obtain the higher order Fibonacci complex numbers $\mathbb{C}F_n^{(s)}$.

3. For $m = 2$, we obtain the higher order Fibonacci quaternions $\mathbb{Q}F_n^{(s)}$ [22].

4. For $m = 3$, we obtain the higher order Fibonacci octonions $\mathbb{O}F_n^{(s)}$.

5. For $m = 4$, we obtain the higher order Fibonacci sedenions $\mathbb{S}F_n^{(s)}$.

6. For $s = 1$ and $m = 0$, we obtain the original Fibonacci numbers F_n [1, 15].

7. For $s=1$ and $m=1$, we obtain the Fibonacci complex numbers $\mathbb{C}F_n$ [16].

8. For $s=1$ and $m=2$, we obtain the Fibonacci quaternions $\mathbb{Q}F_n$ [9, 10].

9. For $s=1$ and $m=3$, we obtain the Fibonacci octonions $\mathbb{O}F_n$ [11].

10. For $s=1$ and $m=4$, we obtain the Fibonacci sedenions $\mathbb{S}F_n$ [12].

The conjugate of the higher order Fibonacci 2^m-ions, $\mathbb{HC}_n^{(s)}$ is denoted by $\left(\mathbb{HC}_n^{(s)}\right)^*$ as

$$\left(\mathbb{HC}_n^{(s)}\right)^* = F_n^{(s)}\mathbf{e}_0 - \sum_{h=1}^{2^m-1} F_{n+h}^{(s)}\mathbf{e}_h = F_n^{(s)}\mathbf{e}_0 - F_{n+1}^{(s)}\mathbf{e}_1 - \cdots - F_{n+2^m-1}^{(s)}\mathbf{e}_{2^m-1}. \tag{4.2}$$

Proposition 4.1 *For the higher order Fibonacci 2^m-ions, we have*

$$\mathbb{HC}_n^{(s)} + \left(\mathbb{HC}_n^{(s)}\right)^* = 2F_n^{(s)}\mathbf{e}_0. \tag{4.3}$$

Proof. Using (4.1) and (4.2), we find that

$$\begin{aligned}\mathbb{HC}_n^{(s)} + \left(\mathbb{HC}_n^{(s)}\right)^* &= \left(F_n^{(s)}\mathbf{e}_0 + \cdots + F_{n+2^m-1}^{(s)}\mathbf{e}_{2^m-1}\right) + \left(F_n^{(s)}\mathbf{e}_0 - \cdots - F_{n+2^m-1}^{(s)}\mathbf{e}_{2^m-1}\right)\\ &= F_n^{(s)}\mathbf{e}_0 + F_n^{(s)}\mathbf{e}_0\\ &= 2F_n^{(s)}\mathbf{e}_0.\end{aligned}$$

■

Theorem 4.2 *Binet type formula for the higher order Fibonacci 2^m-ions is*

$$\mathbb{HC}_n^{(s)} = \frac{\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n}{\alpha^s - \beta^s}, \tag{4.4}$$

where

$$\hat{\alpha} = \sum_{h=0}^{2^m-1} (\alpha^s)^h \mathbf{e}_h = \left(\mathbf{e}_0 + (\alpha^s)\mathbf{e}_1 + (\alpha^s)^2\mathbf{e}_2 + \cdots + (\alpha^s)^{2^m-1}\mathbf{e}_{2^m-1}\right),$$

and

$$\hat{\beta} = \sum_{h=0}^{2^m-1} (\beta^s)^h \mathbf{e}_h = \left(\mathbf{e}_0 + (\beta^s)\mathbf{e}_1 + (\beta^s)^2\mathbf{e}_2 + \cdots + (\beta^s)^{2^m-1}\mathbf{e}_{2^m-1}\right).$$

Proof. By using (2.5) and (4.1), we have

$$\begin{aligned}
\mathbb{HC}_n^{(s)} &= \sum_{h=0}^{2^m-1} F_{n+h}^{(s)}\mathbf{e}_h = F_n^{(s)}\mathbf{e}_0 + F_{n+1}^{(s)}\mathbf{e}_1 + F_{n+2}^{(s)}\mathbf{e}_2 + \cdots + F_{n+2^m-1}^{(s)}\mathbf{e}_{2^m-1} \\
&= \left(\frac{(\alpha^s)^n - (\beta^s)^n}{\alpha^s - \beta^s}\right)\mathbf{e}_0 + \left(\frac{(\alpha^s)^{n+1} - (\beta^s)^{n+1}}{\alpha^s - \beta^s}\right)\mathbf{e}_1 + \left(\frac{(\alpha^s)^{n+2} - (\beta^s)^{n+2}}{\alpha^s - \beta^s}\right)\mathbf{e}_2 + \cdots \\
&\quad + \left(\frac{(\alpha^s)^{n+2^m-1} - (\beta^s)^{n+2^m-1}}{\alpha^s - \beta^s}\right)\mathbf{e}_{2^m-1} \\
&= \frac{1}{\alpha^s - \beta^s}\left(\begin{array}{c} (\alpha^s)^n\left(\mathbf{e}_0 + (\alpha^s)\mathbf{e}_1 + (\alpha^s)^2\mathbf{e}_2 + \cdots + (\alpha^s)^{2^m-1}\mathbf{e}_{2^m-1}\right) - \\ (\beta^s)^n\left(\mathbf{e}_0 + (\beta^s)\mathbf{e}_1 + (\beta^s)^2\mathbf{e}_2 + \cdots + (\beta^s)^{2^m-1}\mathbf{e}_{2^m-1}\right) \end{array}\right) \\
&= \frac{\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n}{\alpha^s - \beta^s}.
\end{aligned}$$

So the proof is completed. ∎

Now we will give some special cases of the Binet-type formula for the higher order Fibonacci 2^m-ions.

Corollary 4.1 *For special choices of s and m, the following cases can be obtained.*

1. *If $s = 1$ and $m = 2$ are selected, we have the Binet-like formula for the Fibonacci quaternions as*

$$\mathbb{Q}F_n = \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}, \tag{4.5}$$

where $\bar{\alpha} = \sum_{h=0}^{3} \alpha^h \mathbf{e}_h$ and $\bar{\beta} = \sum_{h=0}^{3} \beta^h \mathbf{e}_h$.

2. *If $s = 1$ and $m = 3$ are selected, we have the Binet-like formula for the Fibonacci octonions as*

$$\mathbb{O}F_n = \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha - \beta} \tag{4.6}$$

where $\alpha^* = \sum_{h=0}^{7} \alpha^h \mathbf{e}_h$ and $\beta^* = \sum_{h=0}^{7} \beta^h \mathbf{e}_h$.

3. *If $s = 1$ and $m = 4$ are selected, we have the Binet-like formula for the Fibonacci sedenions as*

$$\mathbb{S}F_n = \frac{\tilde{\alpha}\alpha^n - \tilde{\beta}\beta^n}{\alpha - \beta}, \tag{4.7}$$

where $\tilde{\alpha} = \sum_{h=0}^{15} \alpha^h \mathbf{e}_h$ and $\tilde{\beta} = \sum_{h=0}^{15} \beta^h \mathbf{e}_h$.

Theorem 4.3 *For $n \geq 1$, the recurrence relation of higher order Fibonacci 2^m-ions is given by*

$$\mathbb{HC}_{n+1}^{(s)} = L_s \mathbb{HC}_n^{(s)} + (-1)^{s+1} \mathbb{HC}_{n-1}^{(s)}. \tag{4.8}$$

Proof. By using (2.4) and (4.4), we have

$$\begin{aligned}
\mathbb{HC}_{n+1}^{(s)} &= \frac{\hat{\alpha}(\alpha^s)^{n+1} - \hat{\beta}(\beta^s)^{n+1}}{\alpha^s - \beta^s} \\
&= \frac{1}{\alpha^s - \beta^s}\left(\hat{\alpha}(\alpha^s)^n \alpha^s - \hat{\beta}(\beta^s)^n \beta^s\right) \\
&= \frac{1}{\alpha^s - \beta^s}\left[\left(\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n\right)\alpha^s + \hat{\beta}(\beta^s)^n \alpha^s - \hat{\beta}(\beta^s)^n \beta^s\right] \\
&= \frac{1}{\alpha^s - \beta^s}\left(\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n\right)\alpha^s + \frac{1}{\alpha^s - \beta^s}\left(\hat{\beta}(\beta^s)^n \alpha^s - \hat{\beta}(\beta^s)^n \beta^s\right) \\
&= \mathbb{HC}_n^{(s)} \alpha^s + \frac{1}{\alpha^s - \beta^s}\left(\hat{\beta}(\beta^s)^n \alpha^s - \hat{\beta}(\beta^s)^n \beta^s\right) \\
&= \mathbb{HC}_n^{(s)} (\alpha^s + \beta^s) - \mathbb{HC}_n^{(s)} \beta^s + \frac{1}{\alpha^s - \beta^s}\left(\hat{\beta}(\beta^s)^n \alpha^s - \hat{\beta}(\beta^s)^n \beta^s\right) \\
&= L_s \mathbb{HC}_n^{(s)} + \frac{1}{\alpha^s - \beta^s}\left(\hat{\beta}(\beta^s)^n \beta^s - \hat{\alpha}(\alpha^s)^n \beta^s + \hat{\beta}(\beta^s)^n \alpha^s - \hat{\beta}(\beta^s)^n \beta^s\right) \\
&= L_s \mathbb{HC}_n^{(s)} + \frac{1}{\alpha^s - \beta^s}\left(\hat{\beta}(\beta^s)^n \alpha^s - \hat{\alpha}(\alpha^s)^n \beta^s\right) \\
&= L_s \mathbb{HC}_n^{(s)} + \frac{(\alpha\beta)^s}{\alpha^s - \beta^s}\left(\hat{\beta}(\beta^s)^{n-1} - \hat{\alpha}(\alpha^s)^{n-1}\right) \\
&= L_s \mathbb{HC}_n^{(s)} - (-1)^s \left(\frac{\hat{\alpha}(\alpha^s)^{n-1} - \hat{\beta}(\beta^s)^{n-1}}{\alpha^s - \beta^s}\right) \\
&= L_s \mathbb{HC}_n^{(s)} + (-1)^{s+1} \mathbb{HC}_{n-1}^{(s)},
\end{aligned}$$

and that gives us the desired recursion formula (4.8). ■

Now, we will give the generating functions of higher order Fibonacci 2^m-ions in the following theorem.

Theorem 4.4 *The generating function of the higher order Fibonacci 2^m-ions $\mathbb{HC}_n^{(s)}$ is given by*

$$G^{(s)}(x) = \frac{(\hat{\alpha} - \hat{\beta}) - (\hat{\alpha}\beta^s - \hat{\beta}\alpha^s)x}{(\alpha^s - \beta^s)(1 - L_s x + (-1)^s x^2)}. \tag{4.9}$$

Proof. Let

$$G^{(s)}(x) = \sum_{n=0}^{\infty} \mathbb{HC}_n^{(s)} x^n \tag{4.10}$$

be the generating function of $\mathbb{HC}_n^{(s)}$, then using (2.4), (4.1) and (4.4) respectively, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{HC}_n^{(s)}x^n &= \sum_{n=0}^{\infty}(F_n^{(s)}\mathbf{e}_0+F_{n+1}^{(s)}\mathbf{e}_1+\cdots+F_{n+2^m-1}^{(s)}\mathbf{e}_{2^m-1})\ x^n\\
&= \frac{1}{\alpha^s-\beta^s}\sum_{n=0}^{\infty}((\alpha^s)^n\hat{\alpha})x^n-\frac{1}{\alpha^s-\beta^s}\sum_{n=0}^{\infty}((\beta^s)^n\hat{\beta})x^n\\
&= \frac{\hat{\alpha}}{\alpha^s-\beta^s}\sum_{n=0}^{\infty}(\alpha^s)^n x^n-\frac{\hat{\beta}}{\alpha^s-\beta^s}\sum_{n=0}^{\infty}(\beta^s)^n x^n\\
&= \frac{\hat{\alpha}}{\alpha^s-\beta^s}\frac{1}{1-\alpha^s x}-\frac{\hat{\beta}}{\alpha^s-\beta^s}\frac{1}{1-\beta^s x}\\
&= \frac{\hat{\alpha}-\hat{\alpha}\beta^s x-\hat{\beta}+\hat{\beta}\alpha^s x}{(\alpha^s-\beta^s)(1-\alpha^s x)(1-\beta^s x)}\\
&= \frac{(\hat{\alpha}-\hat{\beta})-(\hat{\alpha}\beta^s-\hat{\beta}\alpha^s)x}{(\alpha^s-\beta^s)(1-L_s x+(-1)^s x^2)}.
\end{aligned}$$

■

Corollary 4.2 *For special choices of s and m, the following cases can be obtained:*

1. *If $s=1$ and $m=2$ are selected, we have the generating function for the Fibonacci quaternions as*

   $$G\mathbb{Q}F_n=\frac{x+\mathbf{e}_1+(x+1)\mathbf{e}_2+(x+2)\mathbf{e}_3}{1-x-x^2}.$$

2. *If $s=1$ and $m=3$ are selected, we have the generating function for the Fibonacci octonions as*

   $$G\mathbb{O}F_n=\frac{1}{1-x-x^2}\sum_{s=0}^{7}(F_s+F_{s-1}x)\,\mathbf{e}_s.$$

3. *If $s=1$ and $m=4$ are selected, we have the generating function for the Fibonacci sedenions as*

   $$G\mathbb{S}F_n=\frac{\mathbb{S}F_0+\mathbb{S}F_{-1}x}{1-x-x^2}.$$

Theorem 4.5 *For n, $k\in\mathbb{Z}$, the generating function of the higher order Fibonacci 2^m-ions $\mathbb{HC}_{k+n}^{(s)}$ is*

$$\sum_{n=0}^{\infty}\mathbb{HC}_{k+n}^{(s)}\ x^n=\frac{\mathbb{HC}_k^{(s)}+(-1)^{s+1}\mathbb{HC}_{k-1}^{(s)}x}{1-L_s x+(-1)^s x^2}. \tag{4.11}$$

Proof. Using (4.4) we have,

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{HC}_{k+n}^{(s)}\, x^n &= \sum_{n=0}^{\infty} \left(\frac{\hat{\alpha}(\alpha^s)^{k+n} - \hat{\beta}(\beta^s)^{k+n}}{\alpha^s - \beta^s} \right) x^n \\
&= \frac{1}{\alpha^s - \beta^s} \left[\sum_{n=0}^{\infty} \hat{\alpha}(\alpha^s)^{k+n} x^n - \sum_{n=0}^{\infty} \hat{\beta}(\beta^s)^{k+n} x^n \right] \\
&= \frac{1}{\alpha^s - \beta^s} \left[\hat{\alpha}(\alpha^s)^{k} \sum_{n=0}^{\infty} (\alpha^s)^n x^n - \hat{\beta}(\beta^s)^{k} \sum_{n=0}^{\infty} (\beta^s)^n x^n \right] \\
&= \frac{1}{\alpha^s - \beta^s} \left[\hat{\alpha}(\alpha^s)^{k} \frac{1}{1-\alpha^s x} - \hat{\beta}(\beta^s)^{k} \frac{1}{1-\beta^s x} \right] \\
&= \frac{1}{\alpha^s - \beta^s} \left[\frac{\hat{\alpha}(\alpha^s)^{k}(1-\beta^s x) - \hat{\beta}(\beta^s)^{k}(1-\alpha^s x)}{(1-\alpha^s x)(1-\beta^s x)} \right] \\
&= \frac{1}{\alpha^s - \beta^s} \left[\frac{\hat{\alpha}(\alpha^s)^{k} - \hat{\alpha}(\alpha^s)^{k}\beta^s x - \hat{\beta}(\beta^s)^{k} + \hat{\beta}(\beta^s)^{k}\alpha^s x}{1 - L_s x + (-1)^s x^2} \right] \\
&= \frac{\hat{\alpha}(\alpha^s)^{k} - \hat{\beta}(\beta^s)^{k}}{(\alpha^s - \beta^s)\left(1 - L_s x + (-1)^s x^2\right)} - \frac{\alpha^s \beta^s \left(\hat{\alpha}(\alpha^s)^{k-1} - \hat{\beta}(\beta^s)^{k-1} \right) x}{(\alpha^s - \beta^s)\left(1 - L_s x + (-1)^s x^2\right)} \\
&= \frac{\mathbb{HC}_{k}^{(s)}}{1 - L_s x + (-1)^s x^2} - \frac{(\hat{\alpha}(\alpha^s)^{k-1} - \hat{\beta}(\beta^s)^{k-1})}{(\alpha^s - \beta^s)} \frac{(-1)^s x}{1 - L_s x + (-1)^s x^2} \\
&= \frac{\mathbb{HC}_{k}^{(s)}}{1 - L_s x + (-1)^s x^2} - \frac{\mathbb{HC}_{k-1}^{(s)}(-1)^s x}{1 - L_s x + (-1)^s x^2} \\
&= \frac{\mathbb{HC}_{k}^{(s)} + (-1)^{s+1}\mathbb{HC}_{k-1}^{(s)} x}{1 - L_s x + (-1)^s x^2}.
\end{aligned}$$

■

Theorem 4.6 *Exponential generating function of the higher order Fibonacci 2^m-ions $\mathbb{HC}_n^{(s)}$ is*

$$\sum_{n=0}^{\infty} \mathbb{HC}_{n}^{(s)} \frac{x^n}{n!} = \frac{\hat{\alpha} e^{\alpha^s x} - \hat{\beta} e^{\beta^s x}}{\alpha^s - \beta^s}. \tag{4.12}$$

Proof. Let

$$\hat{U}^{(s)}(x) = \sum_{n=0}^{\infty} \mathbb{HC}_{n}^{(s)} \frac{x^n}{n!}$$

be the exponential generating function of $\mathbb{HC}_n^{(s)}$, then using (4.4), we have

$$\begin{aligned}
\hat{U}^{(s)}(x) &= \sum_{n=0}^{\infty} \mathbb{HC}_n^{(s)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n}{\alpha^s - \beta^s} \right) \frac{x^n}{n!} \\
&= \frac{1}{\alpha^s - \beta^s} \left[\hat{\alpha} \sum_{n=0}^{\infty} \frac{(\alpha^s)^n}{n!} x^n - \hat{\beta} \sum_{n=0}^{\infty} \frac{(\beta^s)^n}{n!} x^n \right] \\
&= \frac{1}{\alpha^s - \beta^s} \left[\hat{\alpha} e^{\alpha^s x} - \hat{\beta} e^{\beta^s x} \right] \\
&= \frac{\hat{\alpha} e^{\alpha^s x} - \hat{\beta} e^{\beta^s x}}{\alpha^s - \beta^s}.
\end{aligned}$$

■

4.1 SOME ADDITIONAL IDENTITIES FOR HIGHER ORDER FIBONACCI HYPER COMPLEX NUMBERS

In this part of our thesis, we derive some identities involving higher order Fibonacci 2^m-ions.

Theorem 4.7 *(Vajda's identity) For any integers n, k, and r, we have*

$$\mathbb{HC}_{n+k}^{(s)} \mathbb{HC}_{n+r}^{(s)} - \mathbb{HC}_n^{(s)} \mathbb{HC}_{n+k+r}^{(s)} = \frac{(-1)^{ns} F_k^{(s)}}{\alpha^s - \beta^s} \left(\hat{\beta}\hat{\alpha}(\alpha^s)^r - \hat{\alpha}\hat{\beta}(\beta^s)^r \right). \tag{4.13}$$

Proof. By using the Binet formula for the higher order Fibonacci 2^m-ions, we find that

$$\begin{aligned}
&\mathbb{HC}_{n+k}^{(s)} \mathbb{HC}_{n+r}^{(s)} - \mathbb{HC}_n^{(s)} \mathbb{HC}_{n+k+r}^{(s)} \\
&= \frac{1}{(\alpha^s - \beta^s)^2} \left(\begin{array}{c} \left(\hat{\alpha}(\alpha^s)^{n+k} - \hat{\beta}(\beta^s)^{n+k} \right) \left(\hat{\alpha}(\alpha^s)^{n+r} - \hat{\beta}(\beta^s)^{n+r} \right) \\ - \left(\hat{\alpha}(\alpha^s)^n - \hat{\beta}(\beta^s)^n \right) \left(\hat{\alpha}(\alpha^s)^{n+k+r} - \hat{\beta}(\beta^s)^{n+k+r} \right) \end{array} \right) \\
&= \frac{1}{(\alpha^s - \beta^s)^2} \left(-\hat{\alpha}\hat{\beta}(\alpha^s)^{n+k}(\beta^s)^{n+r} - \hat{\beta}\hat{\alpha}(\alpha^s)^{n+r}(\beta^s)^{n+k} + \hat{\alpha}\hat{\beta}(\alpha^s)^n(\beta^s)^{n+k+r} + \hat{\beta}\hat{\alpha}(\alpha^s)^{n+k+r}(\beta^s)^n \right) \\
&= \frac{1}{(\alpha^s - \beta^s)^2} \left((\alpha^s\beta^s)^n \left(-\hat{\alpha}\hat{\beta}(\alpha^s)^k(\beta^s)^r - \hat{\beta}\hat{\alpha}(\alpha^s)^r(\beta^s)^k + \hat{\alpha}\hat{\beta}(\beta^s)^{k+r} + \hat{\beta}\hat{\alpha}(\alpha^s)^{k+r} \right) \right) \\
&= \frac{(-1)^{ns}}{(\alpha^s - \beta^s)^2} \left(-\hat{\alpha}\hat{\beta}(\beta^s)^r \left((\alpha^s)^k - (\beta^s)^k \right) + \hat{\beta}\hat{\alpha}(\alpha^s)^r \left((\alpha^s)^k - (\beta^s)^k \right) \right) \\
&= \frac{(-1)^{ns}}{(\alpha^s - \beta^s)^2} \left(\left((\alpha^s)^k - (\beta^s)^k \right) \left(\hat{\beta}\hat{\alpha}(\alpha^s)^r - \hat{\alpha}\hat{\beta}(\beta^s)^r \right) \right) \\
&= \frac{(-1)^{ns} F_k^{(s)}}{\alpha^s - \beta^s} \left(\hat{\beta}\hat{\alpha}(\alpha^s)^r - \hat{\alpha}\hat{\beta}(\beta^s)^r \right).
\end{aligned}$$

So the proof is completed. ■

Now, we have the following special cases from the Vajda's identity:

Corollary 4.3 *For $k = -r$, we have Catalan's identity*

$$\mathbb{HC}_{n-r}^{(s)}\mathbb{HC}_{n+r}^{(s)} - \left(\mathbb{HC}_{n}^{(s)}\right)^2 = \frac{(-1)^{ns}F_{-r}^{(s)}}{\alpha^s - \beta^s}\left(\hat{\beta}\hat{\alpha}(\alpha^s)^r - \hat{\alpha}\hat{\beta}(\beta^s)^r\right). \tag{4.14}$$

Corollary 4.4 *For $r = -k = 1$, we give Cassini's identity*

$$\mathbb{HC}_{n-1}^{(s)}\mathbb{HC}_{n+1}^{(s)} - \left(\mathbb{HC}_{n}^{(s)}\right)^2 = \frac{(-1)^{ns+s+1}}{\alpha^s - \beta^s}\left(\hat{\beta}\hat{\alpha}\alpha^s - \hat{\alpha}\hat{\beta}\beta^s\right). \tag{4.15}$$

Corollary 4.5 *For $r = j - n$, and $k = 1$, we reduce to d'Ocagne's identity*

$$\mathbb{HC}_{n+1}^{(s)}\mathbb{HC}_{j}^{(s)} - \mathbb{HC}_{n}^{(s)}\mathbb{HC}_{j+1}^{(s)} = \frac{(-1)^{ns}}{\alpha^s - \beta^s}\left(\hat{\beta}\hat{\alpha}(\alpha^s)^{j-n} - \hat{\alpha}\hat{\beta}(\beta^s)^{j-n}\right). \tag{4.16}$$

4.2 A MATRIX REPRESENTATION FOR HIGHER ORDER FIBONACCI 2^m-IONS

In this part of the the thesis, we derive the matrix representation of the higher order Fibonacci 2^m-ions. We first introduce two matrix $\mathbf{Q}^{(s)}$ and $\mathbf{P}^{(s)}$ as follows:

$$\mathbf{Q}^{(s)} = \begin{pmatrix} L_s & (-1)^{s+1} \\ 1 & 0 \end{pmatrix}, \tag{4.17}$$

and

$$\mathbf{P}^{(s)} = \begin{pmatrix} \mathbb{HC}_{2}^{(s)} & \mathbb{HC}_{1}^{(s)} \\ \mathbb{HC}_{1}^{(s)} & \mathbb{HC}_{0}^{(s)} \end{pmatrix}, \tag{4.18}$$

where L_s is the Lucas numbers. Now we give the following theorem regarding our result.

Theorem 4.8 *For $n \geq 0$, the following equality holds:*

$$\left(\mathbf{Q}^{(s)}\right)^n \mathbf{P}^{(s)} = \begin{pmatrix} \mathbb{HC}_{n+2}^{(s)} & \mathbb{HC}_{n+1}^{(s)} \\ \mathbb{HC}_{n+1}^{(s)} & \mathbb{HC}_{n}^{(s)} \end{pmatrix}. \tag{4.19}$$

Proof. For the proof, we use induction method on n. The equality holds for $n = 0$. Assume that our assertion holds for $n = k$. Namely,

$$\left(\mathbf{Q}^{(s)}\right)^k \mathbf{P}^{(s)} = \begin{pmatrix} \mathbb{HC}_{k+2}^{(s)} & \mathbb{HC}_{k+1}^{(s)} \\ \mathbb{HC}_{k+1}^{(s)} & \mathbb{HC}_{k}^{(s)} \end{pmatrix}.$$

Then for $n = k+1$, we have

$$\begin{aligned}
\left(\mathbf{Q}^{(s)}\right)^{k+1}\mathbf{P}^{(s)} &= \begin{pmatrix} L_s & (-1)^{s+1} \\ 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \mathbb{HC}_2^{(s)} & \mathbb{HC}_1^{(s)} \\ \mathbb{HC}_1^{(s)} & \mathbb{HC}_0^{(s)} \end{pmatrix} \\
&= \begin{pmatrix} L_s & (-1)^{s+1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{HC}_{k+2}^{(s)} & \mathbb{HC}_{k+1}^{(s)} \\ \mathbb{HC}_{k+1}^{(s)} & \mathbb{HC}_k^{(s)} \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{HC}_{k+3}^{(s)} & \mathbb{HC}_{k+2}^{(s)} \\ \mathbb{HC}_{k+2}^{(s)} & \mathbb{HC}_{k+1}^{(s)} \end{pmatrix}.
\end{aligned}$$

So the proof is completed. ■

With the help of the matrices above, we can also give the Cassini identity for higher order Fibonacci 2^m-ions in the following theorem.

Theorem 4.9 *For $n \geq 1$, we have*

$$\mathbb{HC}_{n+1}^{(s)}\mathbb{HC}_{n-1}^{(s)} - \mathbb{HC}_n^{(s)} = (-1)^{sn-s}\left(\mathbb{HC}_2^{(s)}\mathbb{HC}_0^{(s)} - \left(\mathbb{HC}_1^{(s)}\right)^2\right).$$

Proof. If we take the determinant both sides in Equation (4.19), the proof is completed.

■

CHAPTER 5

CONCLUSION

In this thesis, our main aim is to define the higher order Fibonacci quaternions and the higher order Fibonacci 2^m-ions and give some properties of these algebras.
Firstly, we have introduced and studied higher order Fibonacci quaternions which are defined by means of the higher order Fibonacci numbers. Then we have derived several fundamental properties of these higher order Fibonacci quaternion.
We have also introduced and studied higher order Fibonacci 2^m-ions which are defined by means of the higher order Fibonacci numbers. Then we have derived several fundamental properties of these higher order Fibonacci 2^m-ions such as Binet formula, generating function and exponential generating function. With the help of the Binet formula, we derive Vajda's identity, Catalan's identity, Cassini identity, and d'Ocagne's identity. Also by means of the matrix representation of the higher order Fibonacci 2^m-ions, we prove the another type Cassini identity. The algebra of hyper complex numbers is noncommutative, whereas the algebra of hyper-dual numbers is commutative. For the interest of the readers of our paper, the results presented here have the potential to motivate further researchers of the subject of the higher order Fibonacci hyper-dual numbers.

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