ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON HIGHER ORDER JACOBSTHAL NUMBERS

DEPARTMENT OF MATHEMATICS

DOCTOR OF PHILOSOPHY THESIS

EVREN EYİCAN POLATLI

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ADVISOR : Prof. Dr. Yüksel SOYKAN

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by academic rules and ethical principles all works that are not result of this study are cited properly."

Evren EYİCAN POLATLI

ABSTRACT

Doctor of Philosophy Thesis

ON HIGHER ORDER JACOBSTHAL NUMBERS

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Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Prof. Dr. Yüksel SOYKAN

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In this thesis, we obtain various combinatorial formulas and matrix representations of higher order Jacobsthal numbers and their certain special cases. Then in the last chapter, we define generalized third order Jacobsthal matrix sequence and investigate their properties. This thesis consists of seven chapters.

The organization of this thesis is as follows:

Chapter 1 provides a concise overview of what this thesis is about. Besides, there are properties of generalized second and higher order Fibonacci, Pell and Jacobsthal sequences. The Fibonacci sequence is an important integer sequence with many interesting properties in the literature. This sequence has a number of applications within diverse disciplines. For example, Statistics, Biology, Physics, Finance, Architecture, Computer Science, Medicine are some application areas of this sequence. We begin the first chapter by presenting some properties of the Fibonacci sequence. Then, we present the properties of Pell numbers and some properties of Jacobsthal sequences that appear in many fields such as Fibonacci sequences.

ABSTRACT (continued)

Jacobsthal numbers, which are the subject of this thesis, have applications in many branches of mathematics such as group theory, calculus, applied mathematics, linear algebra.

Chapter 2 includes some basic definitions and some significant theorems used throughout the thesis. We present some basic definitions, theorems and properties of Jacobsthal and Jacobsthal-Lucas numbers. Jacobsthal sequence J_n and Jacobsthal-Lucas sequence j_n respectively, are defined by the following recurrence relations: $J_{n+2} = J_{n+1} + 2J_n$ where $J_0 = 0,\ J_1 = 1$ and $j_{n+2} = j_{n+1} + 2j_n$ where $j_0 = 2,\ j_1 = 1$.

Chapter 3 presents several combinatorial formulas and matrix representations of the generalized third order Jacobsthal sequences. A generalized third order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third order recurrence relations $V_n = V_{\text{n-1}} + V_{\text{n-2}} + 2V_{\text{n-3}}$ with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero. In the continuation of Chapter 3, we explore in detail four special cases of generalized third order Jacobsthal sequences: third order Jacobsthal sequence J_n, third order Jacobsthal-Lucas sequence j_n, modified third order Jacobsthal sequence K_n and third order Jacobsthal Perrin sequence Q_n. Throughout the chapter, some basic properties of generalized third order Jacobsthal numbers such as Binet formulas, generating functions, and Simson formulas are presented.

Chapter 4 includes the six special cases and some properties of generalized fourth order Jacobsthal sequences. A generalized fourth order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n\geq 0}$ is defined by the fourth order recurrence relations $V_n = V_{\text{n-1}} + V_{\text{n-2}} + V_{\text{n-3}} + 2V_{n-4}$ with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3$ not all being zero. . In the continuation of Chapter 4, we investigate, in detail, six special cases: fourth order Jacobsthal sequence J_n, fourth order Jacobsthal-Lucas sequence j_n, modified fourth order Jacobsthal sequence K_n, fourth order Jacobsthal Perrin sequence Q_n, adjusted fourth order Jacobsthal sequence S_n and modified fourth order Jacobsthal-Lucas sequence R_n.

ABSTRACT (continued)

Chapter 5 consists of the basic information about generalized fifth order Jacobsthal numbers. Firstly, in this chapter, we present the definition and some properties of generalized Pentanacci sequence. The generalized Pentanacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4)\}_{n\geq 0}$ is defined by the fifth order recurrence relations $W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$ with the initial values W_0, W_1, W_2, W_3, W_4 are arbitrary complex (or real) numbers and r, s, t, u, v are real numbers. In this chapter, we consider the case $r = s = t = u = 1, v = 2$ and in this case we write $W_n = V_n$. After that, we give six special cases of the fifth order Jacobsthal numbers and their properties.

Chapter 6 includes the some properties of the generalized sixth order Jacobsthal sequences and its six special cases. First we present the definition and some properties of a generalized Hexanacci sequence. The generalized Hexanacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n\geq 0}$ is defined by the sixth order recurrence relation $W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$ where $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, W_4 = c_4, W_5 = c_5$ are arbitrary real or complex numbers and r, s, t, u, v, y are real numbers. In this chapter, we consider the case $r = s = t = u = v = 1, y = 2$ and in this case we write $W_n = V_n$. After that, we give six special cases of the sixth order Jacobsthal numbers and their Binet formulas, generating functions, Simson formulas,etc.

Chapter 7 includes matrix sequences of generalized third order Jacobsthal numbers and to examine the relationship between generalized third order Jacobsthal matrix sequences and their special cases. For any integer $n \geq 0$, the third order Jacobsthal matrix sequence $(\mathcal{J}_n)$ and third order Jacobsthal-Lucas matrix sequence $(\mathcal{M}_n)$ and modified third order Jacobsthal matrix sequence $(\mathcal{K}_n)$ are defined by $\mathcal{J}_n = \mathcal{J}_{n-1} + \mathcal{J}_{n-2} + 2\mathcal{J}_{n-3}$, $\mathcal{M}_n = \mathcal{M}_{n-1} + \mathcal{M}_{n-2} + 2\mathcal{M}_{n-3}$ and $\mathcal{K}_n = \mathcal{K}_{n-1} + \mathcal{K}_{n-2} + 2\mathcal{K}_{n-3}$ respectively, with initial conditions

$$\mathcal{J}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{J}_1 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{J}_2 = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix},$$

ABSTRACT (continued)

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 4 & 4 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}, \mathcal{M}_1 = \begin{pmatrix} 5 & 5 & 2 \\ 1 & 4 & 4 \\ 2 & -1 & 2 \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} 10 & 7 & 10 \\ 5 & 5 & 2 \\ 1 & 4 & 4 \end{pmatrix},$$

$$\mathcal{K}_0 = \begin{pmatrix} 1 & 2 & 6 \\ 3 & -2 & -1 \\ -\frac{1}{2} & \frac{7}{2} & -\frac{3}{2} \end{pmatrix}, \mathcal{K}_1 = \begin{pmatrix} 3 & 7 & 2 \\ 1 & 2 & 6 \\ 3 & -2 & -1 \end{pmatrix}, \mathcal{K}_2 = \begin{pmatrix} 10 & 5 & 6 \\ 3 & 7 & 2 \\ 1 & 2 & 6 \end{pmatrix}.$$

In addition, in this chapter, we give the n-th (general) term of the generalized third order Jacobsthal matrix sequence with the following formula:

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}.$$

Keywords: Fibonacci numbers, Jacobsthal numbers, higher order Jacobsthal numbers, Simson formulas, Binet formulas, generating function, summation formulas, matrix sequence

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ÖZET

Doktora Tezi

YÜKSEK MERTEBEDEN JACOBSTHAL SAYILARI ÜZERİNE

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Tez Danışmanı: Prof. Dr. Yüksel SOYKAN
Haziran 2022, 157 sayfa

Bu tezde, yüksek mertebeden Jacobsthal sayıları ve bunların belirli özel durumlarının çeşitli kombinatoryal formülleri ve matris gösterimlerini elde ettik. Daha sonra son bölümde genelleştirilmiş üçüncü mertebeden Jacobsthal matris dizisini tanımladık ve özelliklerini inceledik. Bu tez yedi bölümden oluşmaktadır.

Bu tezin organizasyonu aşağıdaki gibidir:

Bölüm 1, bu tezin ne hakkında olduğuna dair kısa bir genel bakış sağlar. Ayrıca genelleştirilmiş ikinci ve daha yüksek mertebeden Fibonacci, Pell ve Jacobsthal dizilerinin özellikleri vardır. Fibonacci dizisi, literatürde birçok ilginç özelliği olan önemli bir tamsayı dizidir. Bu dizi, çeşitli disiplinlerde bir dizi uygulamaya sahiptir. Örneğin İstatistik, Biyoloji, Fizik, Finans, Mimarlık, Bilgisayar Bilimi, Tıp bu dizinin bazı uygulama alanlarıdır. İlk bölüme Fibonacci dizisinin bazı özelliklerini sunarak başlıyoruz. Ardından Pell sayılarının özelliklerini ve Fibonacci dizileri gibi birçok alanda karşımıza çıkan Jacobsthal dizilerinin bazı özelliklerini sunuyoruz. Bu tezin konusu olan Jacobsthal sayıları grup teorisi, kalkülüs, uygulamalı matematik, lineer cebir gibi matematiğin birçok dalında uygulamaları bulunmaktadır.

ÖZET (devam ediyor)

Bölüm 2, tez boyunca kullanılan bazı temel tanımları ve bazı önemli teoremleri içerir. Jacobsthal ve Jacobsthal-Lucas sayılarının bazı temel tanımlarını, teoremlerini ve özelliklerini sunuyoruz. Sırasıyla Jacobsthal dizisi ve Jacobsthal-Lucas dizisi, aşağıdaki rekürans bağıntılarıyla tanımlanır: $J_0 = 0,\ J_1 = 1$ olmak üzere $J_{n+2} = J_{n+1} + 2J_n$ ve $j_0 = 2,\ j_1 = 1$ iken $j_{n+2} = j_{n+1} + 2j_n$ dir.

Bölüm 3, genelleştirilmiş üçüncü mertebeden Jacobsthal dizilerinin birkaç kombinatoryal formülünü ve matris temsillerini sunar. Genelleştirilmiş bir üçüncü mertebeden Jacobsthal dizisi, $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ olmak üzere $V_0 = c_0,\ V_1 = c_1,\ V_2 = c_2$ başlangıç değerlerin tümü sıfır olmayan $V_n = V_{n-1} + V_{n-2} + 2V_{n-3}$ üçüncü mertebeden rekürans bağıntısı ile tanımlanır. Üçüncü bölümün devamında, genelleştirilmiş üçüncü mertebeden Jacobsthal dizilerinin dört özel durumunu ayrıntılı olarak araştırıyoruz: üçüncü mertebeden Jacobsthal dizisi J_n, üçüncü mertebeden Jacobsthal-Lucas dizisi j_n, modifiye edilmiş üçüncü mertebeden Jacobsthal dizisi K_n ve üçüncü mertebeden Jacobsthal Perrin dizisi Q_n. Bölüm boyunca, Binet formülleri, üreteç fonksiyonları, Simson formülleri gibi genelleştirilmiş üçüncü mertebeden Jacobsthal sayılarının bazı temel özelliklerini sunulmuştur.

Bölüm 4, genelleştirilmiş dördüncü mertebeden Jacobsthal dizilerini ve onun altı özel durumunu içerir. Genelleştirilmiş bir dördüncü mertebeden Jacobsthal dizisi, $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n\geq 0}$ olmak üzere $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3$ başlangıç değerlerin tümü sıfır olmayan dördüncü mertebeden $V_n = V_{n-1} + V_{n-2} + V_{n-3} + 2V_{n-4}$ rekürans bağıntısı ile tanımlanır. 4. Bölümün devamında, altı özel durumu ayrıntılı olarak inceliyoruz: dördüncü mertebeden Jacobsthal dizisi J_n, dördüncü mertebeden Jacobsthal-Lucas dizisi j_n, modifiye edilmiş dördüncü mertebeden Jacobsthal dizisi K_n, dördüncü mertebeden Jacobsthal Perrin dizisi Q_n, ayarlanmış dördüncü mertebeden Jacobsthal dizisi R_n ve modifiye edilmiş dördüncü mertebeden Jacobsthal-Lucas dizisi S_n dir.

Bölüm 5, genelleştirilmiş beşinci mertebeden Jacobsthal sayıları hakkında temel bilgilerden oluşmaktadır.

ÖZET (devam ediyor)

İlk olarak, bu bölümde genelleştirilmiş Pentanacci dizisinin tanımını ve bazı özelliklerini sunuyoruz. Genelleştirilmiş Pentanacci dizisi, $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4)\}_{n\geq 0}$ olmak üzere W_0, W_1, W_2, W_3, W_4 başlangıç değerleri keyfi karmaşık (veya gerçel) sayılar ve r, s, t, u, v gerçel sayılar olan beşinci mertebeden $W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$ rekürans bağıntısyla tanımlanır. Bu bölümde, $r = s = t = u = 1,\ v = 2$ durumunu ele alıyoruz ve bu durumda $W_n = V_n$ yazıyoruz. Daha sonra beşinci dereceden Jacobsthal sayılarının altı özel durumunu ve özelliklerini veriyoruz.

Bölüm 6, genelleştirilmiş altıncı mertebeden Jacobsthal dizilerinin bazı özelliklerini ve onun altı özel durumunu içerir. İlk önce genelleştirilmiş bir Hexanacci dizisinin tanımını ve bazı özelliklerini sunuyoruz. Genelleştirilmiş Hexanacci dizisi, $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n\geq 0}$ olmak üzere, $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, W_4 = c_4, W_5 = c_5$ keyfi gerçel veya karmaşık sayılar ve r, s, t, u, v, y gerçel sayılar olan altıncı mertebeden $W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$ rekürans bağıntısı ile tanımlanır. Bu bölümde, $r = s = t = u = v = 1,\ y = 2$ durumunu ele alıyoruz ve bu durumda $W_n = V_n$ yazıyoruz. Daha sonra altıncı dereceden Jacobsthal sayılarının altı özel durumunu ve bunların Binet formüllerini, üreteç fonksiyonlarını, Simson vb. formüllerini veriyoruz.

Bölüm 7, genelleştirilmiş üçüncü mertebeden Jacobsthal sayılarının matris dizilerini ve genelleştirilmiş üçüncü mertebeden Jacobsthal matris dizileri ile bunların özel durumları arasındaki ilişkiyi incelemeyi içerir. Herhangi bir $n \geq 0$ için, üçüncü mertebeden Jacobsthal matris dizisi $(\mathcal{J}_n)$, üçüncü mertebeden Jacobsthal-Lucas matris dizisi $(\mathcal{M}_n)$ ve üçüncü mertebeden modifiye edilmiş Jacobsthal matris dizisi $(\mathcal{K}_n)$,

$$\mathcal{J}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{J}_1 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{J}_2 = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix},$$

ÖZET (devam ediyor)

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 4 & 4 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}, \mathcal{M}_1 = \begin{pmatrix} 5 & 5 & 2 \\ 1 & 4 & 4 \\ 2 & -1 & 2 \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} 10 & 7 & 10 \\ 5 & 5 & 2 \\ 1 & 4 & 4 \end{pmatrix},$$

$$\mathcal{K}_0 = \begin{pmatrix} 1 & 2 & 6 \\ 3 & -2 & -1 \\ -\frac{1}{2} & \frac{7}{2} & -\frac{3}{2} \end{pmatrix}, \mathcal{K}_1 = \begin{pmatrix} 3 & 7 & 2 \\ 1 & 2 & 6 \\ 3 & -2 & -1 \end{pmatrix}, \mathcal{K}_2 = \begin{pmatrix} 10 & 5 & 6 \\ 3 & 7 & 2 \\ 1 & 2 & 6 \end{pmatrix}.$$

başlangıç koşullarıyla birlikte, sırasıyla $\mathcal{J}_n = \mathcal{J}_{n-1} + \mathcal{J}_{n-2} + 2\mathcal{J}_{n-3}$, $\mathcal{M}_n = \mathcal{M}_{n-1} + \mathcal{M}_{n-2} + 2\mathcal{M}_{n-3}$ ve $\mathcal{K}_n = \mathcal{K}_{n-1} + \mathcal{K}_{n-2} + 2\mathcal{K}_{n-3}$ rekürans bağıntıları ile tanımlanır.

Ayrıca bu bölümde genelleştirilmiş üçüncü mertebeden Jacobsthal matris dizisinin *n*. (genel) terimini aşağıdaki formülle veriyoruz:

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}.$$

Anahtar Kelimeler: Fibonacci sayıları, Jacobsthal sayıları, yüksek mertebeden Jacobsthal sayıları, Simson formülleri, Binet formülleri, üreteç fonksiyonu, toplam formülleri, matris dizisi

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS

$\{F_n\}$	: Fibonacci sequence
F_n	: *n*-th Fibonacci number
$\{J_n^{(k)}\}$	: *k*-step Jacobsthal sequence
J_n	: *n*-th Jacobsthal number
J_{-n}	: *n*-th Jacobsthal number with negative subscripts
$\{W_n^{(k)}\}$	: Generalized *k*-step sequence
Δ	: Discriminant
$\{U_n\}$	: *k*-step Fibonacci sequence
$\{P_n\}$	: Pell sequence
P_n	: *n*-th Pell number
$\{q_n\}$	: Modified Pell sequence
q_n	: *n*-th Modified Pell number
$\{Q_n\}$	: Pell-Lucas sequence
Q_n	: *n*-th Pell-Lucas number
$\{V_n^{(k)}\}$	: Generalized *k*-step Jacobsthal numbers
V_n	: *n*-th generalized Jacobsthal number
$\{j_n^{(k)}\}$	: *k*-step Jacobsthal-Lucas numbers
j_n	: *n*-th Jacobsthal-Lucas number
j_{-n}	: *n*-th Jacobsthal-Lucas number with negative subscripts
$\{K_n^{(k)}\}$	: Modified *k*-step Jacobsthal numbers

LIST OF SYMBOLS AND ABBREVIATIONS (continued)

$\left\{P_n^{(k)}\right\}$	: *k*-step Jacobsthal Perrin numbers
P_n	: *n*-th Jacobsthal Perrin numbers
P_{-n}	: *n*-th Jacobsthal Perrin number with negative subscripts
$\left\{R_n^{(k)}\right\}$	: Modified *k*-step Jacobsthal-Lucas numbers
R_n	: *n*-th modified Jacobsthal-Lucas numbers
R_{-n}	: *n*-th modified Jacobsthal-Lucas number with negative subscripts
$\left\{S_n^{(k)}\right\}$	: Adjusted *k*-step Jacobsthal numbers
S_n	: *n*-th adjusted Jacobsthal number
S_{-n}	: *n*-th adjusted Jacobsthal number with negative subscripts
$\left\{Q_n^{(k)}\right\}$	: *k*-step Jacobsthal-Perrin numbers
Q_n	: *n*-th Jacobsthal-Perrin number
Q_{-n}	: *n*-th Jacobsthal-Perrin number with negative subscripts
$\left\{\mathcal{V}_n\right\}$	: Generalized third order Jacobsthal matrix sequence
$\left\{\mathcal{J}_n\right\}$	: Third order Jacobsthal matrix sequence
$\left\{\mathcal{M}_n\right\}$	: Third order Jacobsthal-Lucas matrix sequence
$\left\{\mathcal{K}_n\right\}$	: Modified third order Jacobsthal matrix sequence

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEWS

The Fibonacci sequence is an important integer sequence with many interesting properties in the literature. This sequence has a number of applications within diverse disciplines. For example, Statistics, Biology, Physics, Finance, Architecture, Computer Science, Medicine are some application areas of this sequence. Also, it is possible to see Fibonacci numbers in nature. The presence of Fibonacci numbers is seen in the leaves and fruits of some plants and in the bark of some animals. On the other hand, in Mathematics, they also occur in Pascal's triangle, in Pythagorean triples, computer algorithms, graph theory, some areas of algebra, etc.

The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is a sequence starting with 0 and 1, where $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The first few Fibonacci numbers are:

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...$

Like the Fibonacci sequences, Jacobsthal sequences appear in many fields. Jacobsthal numbers have some applications in many branches of mathematics such as group theory, calculus, applied mathematics, linear algebra, etc. At the same time, it is well known that computers use conditional directives to change the flow of execution of a program. And also branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction. This brings out being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 cases on 5 bits, 21 cases on 6 bits,$\ldots$, which are exactly the Jacobsthal numbers. In 1973, the first use of these numbers appears "A Handbook of Integer Sequences" in a paper by Sloane by the title applications of Jacobsthal sequences to curves in [62]. After that, in 1988, Horadam [41] introduced and examined the Jacobsthal and Jacobsthal–Lucas sequences. Also in [40], he gave several properties of the second order Jacobsthal and Jacobsthal-Lucas numbers. The Jacobsthal sequence (sequence A001045 in [62]) $\{J_n\}$ is defined by the following recurrence relation, for $n \geq 0$,

$$J_{n+2} = J_{n+1} + 2J_n$$

where $J_0 = 0$ and $J_1 = 1$. The first few terms of the Jacobsthal sequence (second order Jacobsthal sequence) is

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10923, ...$$

In the literature, many researchers have studies on this sequence and its generalizations. For more detailed information, see for example [1, 2, 6, 14, 15, 21, 22, 27, 40, 41, 46, 47, 51]. For higher order Jacobsthal sequences, see [7, 8, 9, 10, 11, 12, 19, 86].

The purpose of this thesis is to obtain various combinatorial formulas and matrix representations of third, fourth, fifth and sixth order Jacobsthal sequences and their certain special cases and also in the last chapter to describe the generalized third order Jacobsthal matrix sequence and to investigate its properties. This thesis consists of seven chapters. The first chapter is a concise overview of what this thesis is about, as well as a short literature review of Fibonacci sequences, Pell sequences and Jacobsthal sequences. The second chapter consists of some basic definitions and some significant theorems used throughout the thesis. The third chapter presents several combinatorial formulas and matrix representations of the generalized third order Jacobsthal sequences. The fourth chapter contains the four special cases and some properties of generalized fourth order Jacobsthal sequences. The fifth chapter consists a note on generalized fifth order Jacobsthal sequences. The sixth chapter introduces the some properties of generalized sixth order Jacobsthal sequences. The seventh chapter presents generalized third order Jacobsthal matrix sequence and special cases of its and investigate their properties.

The following sections are summaries of Fibonacci-type sequences and higher order Fibonacci-type sequences relevant to our studies in this thesis.

1.1 LITERATURE REVIEW FOR FIBONACCI TYPE SEQUENCES

In this section, we have given some integer sequences that have an important role in the literature and some basic properties about them. We have divided the studies which we examine in literature into three subsections as review on Fibonacci sequences, review on Pell sequences and review on Jacobsthal sequences.

1.1.1 Review On Generalized Fibonacci Sequences

Fibonacci numbers and their generalization have many interesting properties and applications to almost every field of science and art. In [49, 50], Koshy has devoted nearly 1400 pages to the properties of Fibonacci and Lucas number. More links can also be seen in [32, 33, 92]. This subsection is concerned with the review of generalized Fibonacci sequences. Then, we have classified these studies into two subsubsections as some properties of generalized second order Fibonacci sequences and higher order Fibonacci sequences.

Some Properties Of Generalized Second Order Fibonacci Sequences

In [36, 37, 38, 39], Horadam obtained several fundamental properties of $W_n(a,b;r,s)$, where a,b are nonnegative integers and r,s are arbitrary integers.

- The recurrence relation of the sequence $W_n(a,b;r,s)$ is given by

  $W_n = rW_{n-1} + sW_{n-2},\ (n \geq 2)$

  where $W_0 = a$ and $W_1 = b$. For instance, Fibonacci and Lucas sequences are special cases of this sequence, $F_n = W_n(0,1;1,1)$ and $L_n = W_n(2,1;1,1)$.

- The characteristic equation of $W_n(a,b;r,s)$ is

  $x^2 - rx - s = 0.$

- When $\Delta = r^2 + 4s$, the roots of characteristic equation are $\alpha = \dfrac{r+\sqrt{\Delta}}{2}$ and $\beta = \dfrac{r-\sqrt{\Delta}}{2}$. Also

$$\begin{aligned}
\alpha+\beta &= r, \\
\alpha\beta &= -s, \\
(\alpha-\beta)^2 &= (\alpha+\beta)^2 - 4\alpha\beta = r^2+4s.
\end{aligned}$$

- Binet formula of generalized Fibonacci numbers are given by

  $W_n = \dfrac{b_1\alpha^n}{\alpha-\beta} + \dfrac{b_2\beta^n}{\beta-\alpha}$

  where $b_1 = W_1 - \beta W_0$ and $b_2 = W_1 - \alpha W_0.$

- Suppose that $f_{w_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Fibonacci sequence $\{W_n\}_{n\geq 0}$. Then

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \tag{1.1}$$

- In [68], Soykan gave the following summation formulas for $n \geq 1$:

If $(s+1)(r+s-1)(r-s+1) \neq 0$ then

$$\begin{aligned}\sum_{i=1}^{n} W_i^2 &= \frac{(1-s)W_{n+2}^2 + (1-s-r^2-r^2 s)W_{n+1}^2 + 2rsW_{n+1}W_{n+2}}{(s+1)(r+s-1)(r-s+1)} + \\ &\quad \frac{(s-1)W_1^2 + s^2(s-1)W_0^2 - 2rsW_1W_0}{(s+1)(r+s-1)(r-s+1)}.\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} W_{i+1}W_i &= \frac{rW_{n+2}^2 + rs^2W_{n+1}^2 + (1-r^2-s^2)W_{n+1}W_{n+2} - rW_1^2}{(s+1)(r+s-1)(r-s+1)} - \\ &\quad \frac{rs^2W_0^2 + s(-r^2+s^2-1)W_1W_0}{(s+1)(r+s-1)(r-s+1)}.\end{aligned}$$

For a proof, see [68].

Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence F_n, namely,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This formula can be written in terms of the following determinant equation:

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

A search of the literature turns up that there are many identities including Simson (Cassini), Catalan, d'Ocagne, Melham, Tagiuri, Gelin-Cesaro, Gould identities.

For all integers n, the Simson formula given above can be generalized for $W_n(W_0, W_1; r, s)$ as follows:

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = (-1)^n s^n \begin{vmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{vmatrix}, \tag{1.2}$$

see [64].

- The Catalan's identity

  $F_{n-r}F_{n+r} - F_n^2 = (-1)^{n-r+1} F_r^2$

  has several generalizations.

- The Gelin-Ces‘aro identity

  $F_{n-2}F_{n-1}F_{n+1}F_{n+2} = F_n^4 - 1$

- In [60], Silvester showed that several properties of the Fibonacci sequence can be derived from a matrix representation. If the following matrix is taken

  $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

  then

  $$A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} \tag{1.3}$$

  where u_n represents the nth Fibonacci number.

Some Properties Of Higher Order Fibonacci Sequences

In [64], Soykan investigated generalized k-step Fibonacci numbers and gave an attractive formula.

- For $k \geq 2$, the generalized k-step Fibonacci numbers,

$\{W_n(W_{0,}W_1, W_2, ..., W_{k-1}; r_{1,}r_2, ..., r_k)\}_{n\geq k}$ (or shortly $\{W_n\}_{n\geq k}$), is defined by the following k order linear recurrence relation:

$$W_n = \sum_{i=1}^{k} r_i W_{n-i} = r_1 W_{n-1} + r_2 W_{n-2} + r_3 W_{n-3} + ... + r_{k-1} W_{n-k-1} + r_k W_{n-k}$$

with k initial terms $W_0 = c_0$, $W_1 = c_1$, $W_2 = c_2, ...,$ $W_{k-1} = c_{k-1}$, where r_i, $1 \leq i \leq k$, are all real numbers and c_i, $0 \leq i \leq k-1$, are all real or complex numbers. This type of sequence is also called the generalized Fibonacci k-sequence or generalized k-nacci sequence. $\{W_n\}_{n\geq k}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r_{k-1}}{r_k} W_{-(n-1)} - \frac{r_{k-2}}{r_k} W_{-(n-2)} - \frac{r_{k-3}}{r_k} W_{-(n-3)} \cdots - \frac{r_1}{r_k} W_{-(n-(k-1))} + \frac{1}{r_k} W_{-(n-k)}$$

for $n = k-2,\ k-1,\ k,\ k+1....$

- For $k \geq 2$, the k order linear recurrence relation of the k step Fibonacci numbers, $U_n (n \geq k)$ is

$$U_n = \sum_{i=1}^{k} U_{n-i} = U_{k-1} + U_{k-2} + U_{k-3} + ... + U_{n-k}$$

with k initial conditions

$$\begin{pmatrix} U_m = 0, & -k+2 \leq m \leq 0 \\ U_{-m+1} = 1, & m = k \end{pmatrix}$$

- Indeed, Fibonacci numbers F_n ($k = 2$, $U = F$), Tribonacci numbers T_n ($k = 3$, $U = T$), Tetranacci numbers M_n ($k = 4$, $U = M$) and Pentanacci numbers $P_n (k = 5$, $U = P)$ are some of the well known members of this k-step Fibonacci numbers.

Table 1.1 Some special cases of k-step Fibonacci numbers

m	Name	n	-3	-2	-1	0	1	2	3	4	5	6	7	8
2	Fibonacci	F_n	2	-1	1	0	1	1	2	3	5	8	13	21
3	Tribonacci	T_n	-1	1	0	0	1	1	2	4	7	13	24	44
4	Tetranacci	M_n	1	0	0	0	1	1	2	4	8	15	29	56
5	Pentanacci	P_n	0	0	0	0	1	1	2	4	8	16	31	61

- In [24], Dresden and Du presented a Binet-like formula that can be used to produce the k generalized Fibonacci numbers for the nth k generalized Fibonacci numbers $F_n^{\{k\}}$, then

$$F_n^{\{k\}} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}$$

where $\alpha_1, ..., \alpha_k$ are the roots of the characteristic equation

$$x^k - x^{k-1} - ... - 1 = 0.$$

- In [43], Kalman defined the matrix A by

$$A = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{k-2} & c_{k-1} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{1.4}$$

Then, by an inductive argument, he obtained the generalization of (1.4) as follows:

$$A^n \begin{pmatrix} a_{k-1} \\ \vdots \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_{n+k-1} \\ \vdots \\ a_{n+1} \\ a_n \end{pmatrix}.$$

1.1.2 Review On Pell Sequence

In this subsection, we review some papers related to Pell Sequence in the literature. Then, we have classified these studies into two subsubsections as some properties of Pell sequences and some properties of higher order Pell sequences.

Some Properties Of Second Order Pell Sequences

The Pell sequence $\{P_n\}$ are defined by the following recurrence relation, for $n \geq 2$

$$P_n = 2P_{n-1} + P_{n-2}$$

where $P_0 = 0$ and $P_1 = 1$. The Pell-Lucas sequence $\{Q_n\}$ and modified Pell sequence $\{q_n\}$ satisfy the same recurrence with initial conditions $Q_0 = Q_1 = 2$ and $q_0 = q_1 = 1$, respectively. Actually,

$$Q_n = 2q_n.$$

Table 1.2 The first few terms of the P_n, Q_n and q_n

n	0	1	2	3	4	5	6	7	8	9	10
P_n	0	1	2	5	12	29	70	169	408	985	2378
Q_n	2	2	6	14	34	82	198	478	1154	2786	6726
q_n	1	1	3	7	17	41	99	239	577	1393	3363

We present the following properties of Pell numbers given by Horadam in [34] and [35]:

- Characteristic equation of Pell sequences is

  $$x^2 - 2x - 1 = 0,$$

- If the roots of the characteristic equation are α and β, then

  $$\alpha = 1 + \sqrt{2} \text{ and } \beta = 1 - \sqrt{2}.$$

Also

$$\begin{aligned} \alpha+\beta &= 2, \\ \alpha-\beta &= 2\sqrt{2}, \\ \alpha\beta &= -1. \end{aligned}$$

- Explicit Binet formsulas of $\{P_n\}$, $\{Q_n\}$ and $\{q_n\}$ are

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.5}$$

$$Q_n = \alpha^n + \beta^n,$$

$$q_n = \frac{\alpha^n + \beta^n}{2},$$

where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$.

- The generating functions of Pell, Pell-Lucas and modified Pell numbers as follow:

$$\sum_{n=1}^{\infty} P_n x^n = \frac{x}{1-2x-x^2},$$

$$\sum_{n=1}^{\infty} Q_n x^n = \frac{2-2x}{1-2x-x^2},$$

$$\sum_{n=1}^{\infty} q_n x^n = \frac{1-x}{1-2x-x^2}.$$

In [35] ,Horadam gave some results as follows:

- The following identities are true: ([35])

    * $q_n = P_{n+1} - P_n,$
    * $q_{n+1} = P_{n+1} + P_n,$
    * $P_{n+1} = \frac{q_{n+1}+q_n}{2},$
    * $q_{-n} = (-1)^n q,$
    * $P_{-n} = (-1)^{n+1} P_n,$
    * $2P_n = \frac{q_{n+1}+q_{n-1}}{2}.$

- The Simson Formula for the Pell sequence $\{P_n\}$, Pell-Lucas sequence $\{Q_n\}$ and modified Pell sequence $\{q_n\}$ are as follows: ([35])

  * $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$,
  * $Q_{n+1}Q_{n-1} - Q_n^2 = (-1)^{n+1} - 2P_n$,
  * $q_{n+1}q_{n-1} - q_n^2 = 2(-1)^{n+1}$.

- The summation formulas for the Pell sequence $\{P_n\}$, Pell-Lucas sequence $\{Q_n\}$ and modified Pell sequence $\{q_n\}$ are as follows: ([35])

  * $\sum_{i=1}^{n} q_i = \frac{q_{n+1}+q_n}{2} - 1 = P_{n+1} - 1$,
  * $\sum_{i=1}^{n} q_{2i} = \frac{q_{2n+1}-1}{2}$,
  * $\sum_{i=1}^{n} q_{2i-1} = \frac{q_{2n}-1}{2}$,
  * $\sum_{i=0}^{n-1} (-1)^i q_{-i} = \frac{(-1)^n(q_{-n}-q_{-n+1})}{2}$,
  * $\sum_{i=0}^{n} q_{-2i} = \frac{-q_{-2n-1}+1}{2}$,
  * $\sum_{i=1}^{n} q_{-2i+1} = \frac{-q_{-2n}+1}{2}$,
  * $\sum_{i=1}^{n} Q_i = \frac{q_{n+2}-1}{2} - (n+1)$,
  * $\sum_{i=1}^{n} P_i = \frac{P_{n+1}+P_n-1}{2}$.

- In [57], by the aid of (1.5), Santana and D'iaz Barrero obtained the sum of the first n nonzero Pell numbers which is denoted by S_n

  $$S_n = \frac{\alpha^{n+1} + \beta^{n+1} - 2}{4}.$$

  In the same paper, for all $n \geq 0$, they gave the following summation formulas:

  $$P_{2n-1} + P_{2n+1} = \sum_{r=0}^{n} \frac{2n}{2n-r} \binom{2n-r}{r} 2^{2n-2r},$$

  $$P_{2n} + P_{2n+2} = \sum_{r=0}^{n} \frac{2n+1}{2n+1-r} \binom{2n+1-r}{r} 2^{2n+1-2r},$$

  and

  $$P_{n+1} = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r}{r} 2^{n-2r}$$

  where

  $$\binom{m}{r} = \frac{m!}{(m-r)!r!}.$$

- In [3], Bicknell gave a matrix which generates the Pell sequence

  $$M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad M^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}$$

  which can be proved by mathematical induction.

- In [23], Dasdemir gave the matrix relation of Pell and modified Pell numbers as follows:

Firstly, he introduced the following N matrix

$$N = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

Considering the N matrix, he obtained the following equations:

$$\begin{pmatrix} q_{n+2} \\ q_{n+1} \end{pmatrix} = N \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} \text{ and } \begin{pmatrix} P_{n+2} \\ P_{n+1} \end{pmatrix} = \frac{1}{2} N \begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix}.$$

In the same paper, the E matrix was defined as follows:

$$E = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then he obtained

$$\begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix} = E \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} \text{ or } \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} = \frac{1}{2} E \begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix}.$$

In the same paper, Dasdemir also gave the following determinantal equations for all positive integers n and m :

* $P_{n+1}P_{n-1} + P_n^2 = \frac{1}{2}\left(q_{2n} + (-1)^n\right),$

* $q_{n+1}q_{n-1} + q_n^2 = q_{2n} + (-1)^{n+1},$

* $q_{m+n} = P_m q_{n+1} + P_{m-1} q_n,$

* $P_{2n} = 2P_n q_n.$

Some Properties Of Higher Order Pell Sequences

In [87], Tasci and Kilic defined the k sequences of the generalized k order Pell numbers as follows:

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + ... + P_{n-k}^i \tag{1.6}$$

for $n > 0$ and $1 \leq i \leq k$, with initial conditions

$$P_n^i = \begin{pmatrix} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise}, \end{pmatrix}$$

for $1 - k \leq n \leq 0$, where P_n^i is the nth term of the ith sequence. When $k = 2$, the generalized k order Pell sequence, $\left\{P_n^k\right\}$ is reduced to the usual Pell sequence. The above sequences are a special case of a sequence which is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + ... + c_k a_n,$$

where $c_1, c_2, ..., c_k$ are real constants.

Table 1.3 A few values of generalized k order Pell numbers

k	Name	1	2	3	4	5	6	7	8	9	10
2	Pell	1	2	5	12	29	70	169	408	985	2378
3	3 Pell	1	2	5	13	33	84	214	545	1388	3535
4	4 Pell	1	2	5	13	34	88	228	591	1532	3971
5	5 Pell	1	2	5	13	34	89	232	605	1578	4116

By the aid of (1.6), the following matrix representation can be written:

$$\begin{pmatrix} P_{n+1}^i \\ P_n^i \\ P_{n-1}^i \\ \vdots \\ P_{n-k+2}^i \end{pmatrix} = \begin{pmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} P_n^i \\ P_{n-1}^i \\ P_{n-2}^i \\ \vdots \\ P_{n-k+1}^i \end{pmatrix}.$$

The following R matrix is called generalized k order Pell matrix:

$$R = [r_{ij}]_{k\times k} = \begin{pmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

In [87], Tasci and Kilic obtained the following formulas related to higher order Pell numbers:

- If P_n^i is the nth term of the k order Pell number, for $1 \le i \le k$, then for all positive integers n and m

$$\begin{aligned} P_{n+m}^i &= \sum_{j=1}^{k} P_m^j P_{n-j+1}^i, \\ P_{n+m} &= P_{m+1}P_n + P_m P_{n-1}. \end{aligned}$$

- If P_n^i is given the generalized k order Pell number for $1 \le i \le k$, then

$$P_n^i = \sum_{(r_1,r_2,\dots,r_k)} \frac{r_k}{r_1+r_2+\dots+r_k} \times \binom{r_1+r_2+\dots+r_k}{r_1,r_2,\dots,r_k} 2^{r_1}$$

  where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + \dots + kr_k = n - i + k$.

- In [44], for $1 \le j \le 2k+1$, Kilic gave the generalized k order Pell numbers P_j^k in terms of the Fibonacci numbers as follows:

$$P_{k+1+t}^k = F_{2(k+1+t)} - \sum_{i=1}^{t} F_{2i-1}F_{2(t+1-i)}.$$

1.1.3 Review On Jacobsthal Sequences

In our thesis, we studied on higher order Jacobsthal numbers. However, in this section, we give a review of the studies on jacobsthal numbers in addition to our studies. The Jacobsthal numbers related to our studied are given in detail in the second chapter.

- In [40], Horadam introduced the recurrence relations of $\{J_n\}$ and $\{j_n\}$ for the Jacobsthal and Jacobsthal–Lucas sequences. He defined these sequences by the following recurrence relations:

$$\begin{aligned} J_n &= J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \; J_1 = 1, \text{ for } n \geq 2, \\ j_n &= j_{n-1} + 2j_{n-2}, \quad j_0 = 2, \; j_1 = 1, \text{ for } n \geq 2. \end{aligned}$$

  He also gave Cassini-like formulas as follows:

$$\begin{aligned} J_{n+1}J_{n-1} - J_n^2 &= (-1)^n \, 2^{n-1}, \\ j_{n+1}j_{n-1} - j_n^2 &= 3^2 \, (-1)^{n-1} \, 2^{n-1}. \end{aligned}$$

- In [47], Köken and Bozkurt defined the Jacobsthal J-matrix

$$J = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

  and they showed that

$$J^n = \begin{pmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{pmatrix} = J^{n-1} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$

  for any natural number n. Then, they gave the following matrix equation for any $n, s > 0$ and $n + s > 2$:

$$\begin{pmatrix} J_{n+s} & 2J_{n+s-1} \\ J_{n+s-1} & 2J_{n+s-2} \end{pmatrix} = J^{n+s-2} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

- Jacobsthal-Lucas B-matrix was given by Köken in [48] and was examined by Daşdemir in [21]:

$$B = \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$$

  and for $n, s > 0$ and $n + s > 2$,

$$\begin{pmatrix} j_{n+s} & 2j_{n+s-1} \\ j_{n+s-1} & 2j_{n+s-2} \end{pmatrix} = B^{n+s-2} \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}.$$

- In [28], Djordjevic defined the generalized Jacobsthal polynomials $\{J_{n,m}(x)\}$ and the generalized Jacobsthal-Lucas polynomials $\{j_{n,m}(x)\}$ as follows:

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x), \ n \geq m,$$

where $J_{0,m}(x) = 0$, $J_{n,m}(x) = 1$, $n = 1, 2, ..., m-1$, and

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x), \ n \geq m,$$

where $j_{0,m}(x) = 2$, $j_{n,m}(x) = 1$, $n = 1, 2, ..., m-1$. Djordjevic also gave following generating functions of the polynomials $\{J_{n,m}(x)\}$ and the polynomials $\{j_{n,m}(x)\}$

$$F(x,t) = (1 - t - 2xt^m)^{-1} = \sum_{n=1}^{\infty} J_{n,m}(x)\, t^{n-1}, \tag{1.7}$$

and

$$G(x,t) = \left(1 + 4xt^{m-1}\right)(1 - t - 2xt^m)^{-1} = \sum_{n=1}^{\infty} j_{n,m}(x)\, t^{n-1}. \tag{1.8}$$

- In the same paper, Djordjevic obtained the following explicit representations from (1.7) and (1.8) as follows:

$$J_{n,m}(x) = \sum_{k=0}^{\left\lfloor \frac{n-1}{m} \right\rfloor} \binom{n-1-(m-1)k}{k} (2x)^k,$$

and

$$j_{n,m}(x) = \sum_{k=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k.$$

- In [20], Cook et al, gave the generalized Jacobsthal numbers recurrence relation as follows:

$$J_{n+k}^{(k)} = \sum_{j=1}^{k-1} J_{n+k-j}^{(k)} + 2J_n^{(k)} \tag{1.9}$$

where $n \geq 0$ and $J_1^{(k)} = 0$, for $i = 0, 1, ..., k-2$ and $J_{k-1}^{(k)} = 1$. In this paper, they gave the characteristic equation for (1.9) as follows:

$$(x-2)\left(x^{k-1} + x^{k-2} + ... + x^2 + x + 1\right) = 0,$$

with eigenvalues 2 and $w_t = e^{\frac{2\pi i m}{k}}$ for $t = 1, 2, ..., k-1$. They also gave the Binet formulas for higher order Jacobsthal and Jacobsthal-Lucas numbers as follows:

$$J_n^{(k)} = \frac{1}{\prod_{t=1}^{k-1}(2-w_j)}\left(2^n - \sum_{t=1}^{k-1}\prod_{m\neq t}^{k-1}\frac{2-w_m}{wt-w_m}w_t^n\right),$$

$$j_n^{(k)} = \frac{1}{\prod_{t=1}^{k-1}(2-w_j)}\left(2^n - \sum_{t=1}^{k-1}\prod_{m\neq t}^{k-1}\frac{2-w_m}{wt-w_m}w_t^n\right).$$

- In [19], Cook and Bacon studied ordinary generating function of higher order Jacobsthal and Jacobsthal-Lucas numbers as follow:

$$\sum_{i=0}^{\infty} J_i^{(k)} x^i = \frac{J_0^{(k)} + \left(J_1^{(k)} - J_0^{(k)}\right)x + ... + \left(J_{k-1}^{(k)} - J_{k-2}^{(k)} - ... - 2J_0^{(k)}\right)x^{k-1}}{1 - x - x^2 - ... - 2x^k},$$

$$\sum_{i=0}^{\infty} j_i^{(k)} x^i = \frac{j_0^{(k)} + \left(j_1^{(k)} - j_0^{(k)}\right)x + ... + \left(j_{k-1}^{(k)} - j_{k-2}^{(k)} - ... - 2j_0^{(k)}\right)x^{k-1}}{1 - x - x^2 - ... - 2x^k}.$$

CHAPTER 2

THE FUNDAMENTAL DEFINITIONS AND THEOREMS

In this chapter, we summarize some fundamental definitions, theorems, and properties related to Jacobsthal and Jacobsthal-Lucas numbers given in [40, 41, 47, 64].

2.1 SOME PROPERTIES OF GENERALIZED SECOND ORDER JACOBSTHAL NUMBERS

In this section, we give some definitions and theorems related to Jacobsthal numbers.

- A generalized second order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the following recurrence relation

  $$V_n = V_{n-1} + 2V_{n-2} \tag{2.1}$$

  with the initial values $V_0 = c_0, V_1 = c_1$ not all being zero. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

  $$V_{-n} = -\frac{1}{2}V_{-(n-1)} + \frac{1}{2}V_{-(n-2)} \tag{2.2}$$

  for $n = 1, 2, 3, ...$ Therefore, recurrence (2.2) holds for all integer n.

- As $\{V_n\}$ is a second order recurrence sequence (difference equation), it's characteristic equation is

  $$x^2 - x - 2 = 0. \tag{2.3}$$

  The roots of Equation (2.3) are $\alpha = 2$ and $\beta = -1$. Note that the following identities hold:

  $$\begin{aligned} \alpha + \beta &= 1, \\ \alpha\beta &= -2, \\ \alpha - \beta &= 3. \end{aligned}$$

- The first few terms of generalized second order Jacobsthal numbers with positive and negative subscript are given in Table 2.1.

Table 2.1 The first few terms of generalized second order Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$\frac{1}{2}V_1 - \frac{1}{2}V_0$
2	$V_1 + 2V_0$	$-\frac{1}{4}V_1 + \frac{3}{4}V_0$
3	$3V_1 + 2V_0$	$\frac{3}{8}V_1 - \frac{5}{8}V_0$
4	$5V_1 + 6V_0$	$-\frac{5}{16}V_1 + \frac{11}{16}V_0$
5	$11V_1 + 10V_0$	$\frac{11}{32}V_1 - \frac{21}{32}V_0$
6	$20V_1 + 22V_0$	$-\frac{21}{64}V_1 + \frac{43}{64}V_0$
7	$42V_1 + 42V_0$	$\frac{43}{128}V_1 - \frac{85}{128}V_0$
8	$82V_1 + 86V_0$	$-\frac{85}{256}V_1 + \frac{171}{256}V_0$

- Two special cases of the sequence $\{V_n\}$, Jacobsthal sequence $\{J_n\}_{n\geq 0}$ and Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$, are given by

  $J_{n+2} = J_{n+1} + 2J_n$

  where $J_0 = 0$, $J_1 = 1$ and

  $j_{n+2} = j_{n+1} + 2j_n$

  where $j_0 = 2$, $j_1 = 1$, respectively.

- In the rest of the chapter, for easy writing, we drop the superscripts and write J_n and j_n for $J_n^{(2)}$ and $j_n^{(2)}$ respectively.

Table 2.2 The first few terms of the J_n and j_n

n	0	1	2	3	4	5	6	7	8	9	10
J_n	0	1	1	3	5	11	21	43	85	171	341
J_{-n}		0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{7}{16}$	$-\frac{9}{32}$	$-\frac{9}{64}$	$\frac{55}{128}$	$-\frac{73}{256}$	$-\frac{73}{512}$
j_n	2	1	5	7	17	31	65	127	257	511	1025
j_{-n}		$-\frac{1}{2}$	$\frac{5}{4}$	$-\frac{7}{8}$	$\frac{17}{16}$	$-\frac{31}{32}$	$\frac{65}{64}$	$-\frac{127}{128}$	$\frac{257}{256}$	$-\frac{511}{512}$	$\frac{1025}{1024}$

- Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized second order Jacobsthal sequence $\{V_n\}_{n\geq 0}$. Then, $f_{V_n}(x)$ is given by

$$f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x}{1 - x - 2x^2}.$$

- Binet formula of generalized second order Jacobsthal numbers $\{V_n\}$ is given by

$$V_n = \frac{d_1 \alpha^n}{(\alpha - \beta)} + \frac{d_2 \beta^n}{(\beta - \alpha)} \tag{2.4}$$

where $d_1 = V_0 \alpha + (V_1 - V_0)$ and $d_2 = V_0 \beta + (V_1 - V_0)$.

- For all integers n, Binet formulas of second order Jacobsthal and Jacobsthal-Lucas sequences are

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$j_n = \alpha^n + \beta^n.$$

- Matrix representations which are given in [45] for Pell numbers can be adapted to second order Jacobsthal numbers. If we take $k = i = 2$ in Corollary 3.1 in [45], then the following matrices are obtained:

$$\Lambda = \begin{pmatrix} \alpha & 1 \\ \beta & 1 \end{pmatrix}, \ \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & 1 \\ \beta^{n-1} & 1 \end{pmatrix}, \ \Lambda_2 = \begin{pmatrix} \alpha & \alpha^{n-1} \\ \beta & \beta^{n-1} \end{pmatrix}.$$

Thus, for all integers n, the Binet formula for second order Jacobsthal numbers is obtained alternatively as follows:

$$\begin{aligned} J_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{2} J_{3-j} \det(\Lambda_j) = \frac{1}{\det \Lambda} (J_2 \det(\Lambda_1) + J_1 \det(\Lambda_2)) \\ &= \frac{1}{\det(\Lambda)} (\det(\Lambda_1) + \det(\Lambda_2)) \\ &= \left(\begin{vmatrix} \alpha^{n-1} & 1 \\ \beta^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha & \alpha^{n-1} \\ \beta & \beta^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \alpha & 1 \\ \beta & 1 \end{vmatrix} \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta}. \end{aligned}$$

Similarly, for all integers n, the Binet formula for second order Jacobsthal-Lucas is obtained as follows:

$$\begin{aligned} j_n &= \frac{1}{\det \Lambda}(j_2 \det(\Lambda_1) + j_1 \det(\Lambda_2)) \\ &= \left(5 \begin{vmatrix} \alpha^{n-1} & 1 \\ \beta^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha & \alpha^{n-1} \\ \beta & \beta^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \alpha & 1 \\ \beta & 1 \end{vmatrix} \\ &= \alpha^n + \beta^n. \end{aligned}$$

- For all integers n, the Simson formula of generalized second order Jacobsthal numbers is given by

$$\begin{vmatrix} V_{n+1} & V_n \\ V_n & V_{n-1} \end{vmatrix} = (-1)^n 2^n \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}, \tag{2.5}$$

  see [64].

- The following summation formulas hold for $\{V_n\}_{n\geq 0}$:

  * $\sum_{k=0}^{n} V_k = \frac{1}{2}(V_{n+2} - V_1)$.
  * $\sum_{k=0}^{n} V_{2k} = \frac{1}{3}((n+3)\,V_{2n+2} - 2\,(n+2)\,V_{2n+1} + V_1 - 3V_0)$.
  * $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{3}(-\,(n+1)\,V_{2n+2} + 2\,(n+3)\,V_{2n+1} - 2V_1 + 2V_0$.
  * $\sum_{k=0}^{n}(-1)^k V_k = \frac{1}{3}((n+2)\,(-1)^n\,V_{n+2} + (2n+3)\,(-1)^{n+1}\,V_{n+1} + V_1 - V_0)$.
  * $\sum_{k=0}^{n}(-1)^k V_{2k} = \frac{1}{10}(3\,(-1)^n\,V_{2n+2} + 2\,(-1)^{n+1}\,V_{2n+1} - V_1 + 4V_0)$.
  * $\sum_{k=0}^{n}(-1)^k V_{2k+1} = \frac{1}{10}((-1)^n\,V_{2n+2} + 6\,(-1)^n\,V_{2n+1} + 3V_1 - 2V_0)$.

2.2 SOME PROPERTIES OF J_n AND j_n

Now, we can list some properties of J_n and j_n from [40] as follows:

- The Binet's formula:

  * $J_n = \dfrac{2^n - (-1)^n}{3}$,
  * $j_n = 2^n + (-1)^n$,

- Generating Functions:

  * $\sum_{k=0}^{\infty} J_k x^k = \dfrac{x}{1-x-2x^2}$,
  * $\sum_{k=0}^{\infty} j_k x^k = \dfrac{2-x}{1-x-2x^2}$,

- Catalan's Identity:

  * $J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1}$
  * $j_{n+1}j_{n-1} - j_n^2 = 9(-1)^{n-1} 2^{n-1} = -9\left(J_{n+1}J_{n-1} - J_n^2\right)$,

- Summation formulas:

  * $\sum_{k=0}^{n} J_k = \dfrac{J_{n+2}-1}{2}$,
  * $\sum_{k=0}^{n} J_{2k} = \dfrac{((n+3)J_{2n+2} - 2(n+2)J_{2n+1} + 1)}{3}$,
  * $\sum_{k=0}^{n} J_{2k+1} = \dfrac{(-(n+1)J_{2n+2} + 2(n+3)J_{2n+1} - 2)}{3}$,
  * $\sum_{k=0}^{n} j_k = \dfrac{j_{n+2}-1}{2}$,
  * $\sum_{k=0}^{n} j_{2k} = \dfrac{(-(n+1)J_{2n+2} + 2(n+3)J_{2n+1} - 2)}{3}$,
  * $\sum_{k=0}^{n} j_{2k+1} = \dfrac{(-(n+1)j_{2n+2} + 2(n+3)j_{2n+1} + 2)}{3}$,
  * $\sum_{k=0}^{n}(-1)^k J_k = \dfrac{1}{3}((n+2)(-1)^n J_{n+2} + (2n+3)(-1)^{n+1} J_{n+1} + 1)$,
  * $\sum_{k=0}^{n}(-1)^k J_{2k} = \dfrac{1}{10}(3(-1)^n J_{2n+2} + 2(-1)^{n+1} J_{2n+1} - 1)$,
  * $\sum_{k=0}^{n}(-1)^k J_{2k+1} = \dfrac{1}{10}((-1)^n J_{2n+2} + 6(-1)^n J_{2n+1} + 3)$,
  * $\sum_{k=0}^{n}(-1)^k j_k = \dfrac{1}{3}((n+2)(-1)^n j_{n+2} + (2n+3)(-1)^{n+1} j_{n+1} - 1)$,
  * $\sum_{k=0}^{n}(-1)^k j_{2k} = \dfrac{1}{10}(3(-1)^n j_{2n+2} + 2(-1)^{n+1} j_{2n+1} + 7)$,
  * $\sum_{k=0}^{n}(-1)^k j_{2k+1} = \dfrac{1}{10}((-1)^n j_{2n+2} + 6(-1)^n j_{2n+1} - 1)$.

- Other Basic Identities:

  * $j_n J_n = J_{2n}$,
  * $j_n = J_{n+1} + 2J_{n-1}$,

* $j_{n+1}+j_n=3\left(J_{n+1}+J_n\right)=3.2^n$,
* $j_{n+1}-j_n=3\left(J_{n+1}-J_n\right)+4\left(-1\right)^{n+1}=2^n+2\left(-1\right)^{n+1}$,
* $j_{n+r}+j_{n-r}=3\left(J_{n+r}+J_{n-r}\right)+4\left(-1\right)^{n-r}=2^{n-r}\left(2^{2r}+1\right)+2\left(-1\right)^{n-r}$,
* $j_{n+r}-j_{n-r}=3\left(J_{n+r}-J_{n-r}\right)=2^{n-r}\left(2^{2r}-1\right)$,
* $J_n+j_n=2J_{n+1}$,
* $J_{n+2}+2J_n=j_{n+1}$,
* $3J_n+j_n=2^{n+1}$,
* $2j_{n+1}+j_{n-1}=3\left(2J_{n+1}+J_{n-1}\right)+6\left(-1\right)^{n+1}$,
* $j_{n+2}j_{n-2}-j_n^2=-9\left(J_{n+2}J_{n-2}-J_n^2\right)=9\left(-1\right)^n 2^{n-2}$,
* $j_mj_n+9J_mJ_n=2j_{m+n}$,
* $J_mj_n-J_nj_m=\left(-1\right)^n 2^{n+1}J_{m-n}$,
* $j_mj_n-9J_mJ_n=\left(-1\right)^n 2^{n+1}j_{m+n}$,
* $9J_n=j_{n+1}+2j_{n-1}$,

- Limit Properties:

  * $\lim_{n\to\infty}\left(\frac{J_{n+1}}{J_n}\right)=\lim_{n\to\infty}\left(\frac{j_{n+1}}{j_n}\right)=2$,
  * $\lim_{n\to\infty}\left(\frac{j_n}{J_n}\right)=3$.

CHAPTER 3

GENERALIZED THIRD ORDER JACOBSTHAL NUMBERS

In this chapter, we introduce the generalized third order Jacobsthal sequences and we investigate, in detail, four special cases: third order Jacobsthal, third order Jacobsthal-Lucas, modified third order Jacobsthal and Jacobsthal Perrin sequences. The results given below are quoted from [83].

3.1 INTRODUCTION

In this section, we give basic information about generalized third order Jacobsthal numbers.

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{3.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers. This sequence and it's some special cases have been studied by many authors, see for example [4, 5, 16, 25, 26, 53, 55, 58, 59, 63, 85, 96, 99].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (3.1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{3.2}$$

whose roots are

$$
\begin{aligned}
\alpha &= \alpha(r,s,t) = \frac{r}{3} + A + B, \\
\beta &= \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B, \\
\gamma &= \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3} \\
\Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \quad \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).
\end{aligned}
$$

Here α, β and γ satisfy the following equalities:

$$
\begin{aligned}
\alpha + \beta + \gamma &= r, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -s, \\
\alpha\beta\gamma &= t.
\end{aligned}
$$

If $\Delta(r,s,t) > 0$, then the Equ. (3.2) has one real (α) and two nonreal solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet formula

$$
W_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{3.3}
$$

where

$$
b_1 = W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0, \ b_2 = W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0, \ b_3 = W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0.
$$

Note that the Binet form of a sequence satisfying (3.2) for non-negative integers is valid for all integers n. For a proof of this result, see [42]. This result of Howard and Saidak [42] is even true in the case of higher order recurrence relations.

In this chapter, we consider the case $r = s = 1$, $t = 2$ and in this case we write $V_n = W_n$. A generalized third order Jacobsthal sequence $\{V_n\}_{n \geq 0} = \{V_n(V_0, V_1, V_2)\}_{n \geq 0}$ is defined by the following recurrence relation

$$
V_n = V_{n-1} + V_{n-2} + 2V_{n-3} \tag{3.4}
$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can also be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} + \frac{1}{2}V_{-(n-3)}$$

for $n \geq 1$. Therefore, recurrence (3.4) holds for all integer n.
(3.3) can be used to obtain Binet formula of generalized third order Jacobsthal numbers. Binet formula of generalized third order Jacobsthal numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0, \tag{3.5}$$

$$b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0, \tag{3.6}$$

$$b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{3.7}$$

Here α, β and γ are the roots of the cubic equation $x^3 - x^2 - x - 2 = 0$ and

$$\begin{aligned} \alpha &= 2, \\ \beta &= \frac{-1+i\sqrt{3}}{2}, \\ \gamma &= \frac{-1-i\sqrt{3}}{2}. \end{aligned}$$

Also α, β and γ satisfy the following equalities:

$$\begin{aligned} \alpha+\beta+\gamma &= 1, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= -1, \\ \alpha\beta\gamma &= 2. \end{aligned}$$

The first few generalized third order Jacobsthal numbers with positive subscript and negative subscript are given in Table 3.1.

Table 3.1. The first few terms of generalized third order Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$\frac{1}{2}V_2-\frac{1}{2}V_1-\frac{1}{2}V_0$
2	V_2	$-\frac{1}{4}V_2+\frac{3}{4}V_1-\frac{1}{4}V_0$
3	$V_2+V_1+2V_0$	$-\frac{1}{8}V_2-\frac{1}{8}V_1+\frac{7}{8}V_0$
4	$2V_2+3V_1+2V_0$	$\frac{7}{16}V_2-\frac{9}{16}V_1-\frac{9}{16}V_0$
5	$5V_2+4V_1+4V_0$	$-\frac{9}{32}V_2+\frac{23}{32}V_1-\frac{9}{32}V_0$
6	$9V_2+9V_1+10V_0$	$-\frac{9}{64}V_2-\frac{9}{64}V_1+\frac{55}{64}V_0$
7	$18V_2+19V_1+18V_0$	$\frac{55}{128}V_2-\frac{73}{128}V_1-\frac{73}{128}V_0$
8	$37V_2+36V_1+36V_0$	$-\frac{73}{256}V_2+\frac{183}{256}V_1-\frac{73}{256}V_0$

Now we present four special case of the sequence $\{V_n\}$. Third order Jacobsthal sequence $\{J_n\}_{n\geq 0}$, third order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$, modified third order Jacobsthal sequence $\{K_n\}_{n\geq 0}$ and third order Jacobsthal Perrin sequence $\{Q_n\}_{n\geq 0}$ are defined, respectively, by the following recurrence relations:

$$J_{n+3}=J_{n+2}+J_{n+1}+2J_n, \quad J_0=0,\ J_1=1,\ J_2=1, \tag{3.8}$$

$$j_{n+3}=j_{n+2}+j_{n+1}+2j_n, \quad j_0=2,\ j_1=1,\ j_2=5 \tag{3.9}$$

and

$$K_{n+3}=K_{n+2}+K_{n+1}+2K_n, \quad K_0=3,\ K_1=1,\ K_2=3. \tag{3.10}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ are defined in [19] and $\{K_n\}_{n\geq 0}$ is given in [7]. In this chapter we introduce another third order sequence namely: third-order Jacobsthal Perrin sequence which is given by the third order reccurence relations

$$Q_{n+3}=Q_{n+2}+Q_{n+1}+2Q_n, \quad Q_0=3,\ Q_1=0,\ Q_2=2. \tag{3.11}$$

The sequences $\{J_n\}_{n\geq 0}$, $\{j_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$J_{-n}=-\frac{1}{2}J_{-(n-1)}-\frac{1}{2}J_{-(n-2)}+\frac{1}{2}J_{-(n-3)} \tag{3.12}$$

$$j_{-n}=-\frac{1}{2}j_{-(n-1)}-\frac{1}{2}j_{-(n-2)}+\frac{1}{2}j_{-(n-3)} \tag{3.13}$$

$$K_{-n} = -\frac{1}{2}K_{-(n-1)} - \frac{1}{2}K_{-(n-2)} + \frac{1}{2}K_{-(n-3)} \tag{3.14}$$

$$Q_{-n} = -\frac{1}{2}Q_{-(n-1)} - \frac{1}{2}Q_{-(n-2)} + \frac{1}{2}Q_{-(n-3)} \tag{3.15}$$

for $n \geq 1$, respectively. Therefore, recurrences (3.12), (3.13) (3.14) and (3.15) hold for all integer n.

In the rest of the chapter, for easy writing, we drop the superscripts and write J_n, j_n, K_n and Q_n for $J_n^{(3)}, j_n^{(3)}, K_n^{(3)}$ and $Q_n^{(3)}$ respectively.

Note that J_n is the sequence A077947 in [62] associated with the expansion of $1/(1 - x - x^2 - 2x^3)$, j_n is the sequence A226308 in [62] and K_n is the sequence A186575 in [62] associated with the expansion of $(1 + 2x + 6x^2)/(1 - x - x^2 - 2x^3)$ in powers of x. Q_n is not indexed in [62] yet.

Next, we present the first few values of the third order Jacobsthal, third order Jacobsthal-Lucas, modified third order Jacobsthal and Jacobsthal Perrin numbers with positive and negative subscripts:

Table 3.2. The first few terms of the J_n, j_n, K_n and Q_n

n	0	1	2	3	4	5	6	7	8	9	10
J_n	0	1	1	2	5	9	18	37	73	146	293
J_{-n}		0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{7}{16}$	$-\frac{9}{32}$	$-\frac{9}{64}$	$\frac{55}{128}$	$-\frac{73}{256}$	$-\frac{73}{512}$
j_n	2	1	5	10	17	37	74	145	293	586	1169
j_{-n}		1	-1	1	$\frac{1}{2}$	$-\frac{5}{4}$	$\frac{7}{8}$	$\frac{7}{16}$	$-\frac{41}{32}$	$\frac{55}{64}$	$\frac{55}{128}$
K_n	3	1	3	10	15	31	66	127	255	514	1023
K_{-n}		$-\frac{1}{2}$	$-\frac{3}{4}$	$\frac{17}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$\frac{129}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$\frac{1025}{512}$	$-\frac{1023}{1024}$
Q_n	3	0	2	8	10	22	48	90	182	368	730
Q_{-n}		$-\frac{1}{2}$	$-\frac{5}{4}$	$\frac{19}{8}$	$-\frac{13}{16}$	$-\frac{45}{32}$	$\frac{147}{64}$	$-\frac{109}{128}$	$-\frac{365}{256}$	$\frac{1171}{512}$	$-\frac{877}{1024}$

For all integers n, third order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal and Jacobsthal Perrin numbers (using initial conditions in (3.5)) can be expressed using Binet formulas as

$$J_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

$$j_n = \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},$$

$$K_n = \alpha^n + \beta^n + \gamma^n,$$

$$Q_n = \frac{(3\alpha^2 - 3\alpha - 1)\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(3\beta^2 - 3\beta - 1)\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{(3\gamma^2 - 3\gamma - 1)\gamma^n}{(\gamma - \alpha)(\gamma - \beta)}.$$

3.1.1 Generating Functions

In this subsection, we give the ordinary generating function of the sequence V_n.

Lemma 3.1 *Suppose that* $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized third order Jacobsthal sequence* $\{V_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} V_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - 2x^3}. \tag{3.16}$$

Proof. Using the definition of generalized third order Jacobsthal numbers, and substracting $x\sum_{n=0}^{\infty} V_n x^n$, $x^2 \sum_{n=0}^{\infty} V_n x^n$ and $2x^3 \sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$\begin{aligned}
(1 - x - x^2 - 2x^3)\sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty} V_n x^n - x\sum_{n=0}^{\infty} V_n x^n - x^2\sum_{n=0}^{\infty} V_n x^n - 2x^3\sum_{n=0}^{\infty} V_n x^n \\
&= \sum_{n=0}^{\infty} V_n x^n - \sum_{n=0}^{\infty} V_n x^{n+1} - \sum_{n=0}^{\infty} V_n x^{n+2} - 2\sum_{n=0}^{\infty} V_n x^{n+3} \\
&= \sum_{n=0}^{\infty} V_n x^n - \sum_{n=1}^{\infty} V_{n-1} x^n - \sum_{n=2}^{\infty} V_{n-2} x^n - 2\sum_{n=3}^{\infty} V_{n-3} x^n \\
&= (V_0 + V_1 x + V_2 x^2) - (V_0 x + V_1 x^2) - V_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (V_n - V_{n-1} - V_{n-2} - 2V_{n-3}) x^n \\
&= V_0 + V_1 x + V_2 x^2 - V_0 x - V_1 x^2 - V_0 x^2 \\
&= V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2.
\end{aligned}$$

Rearranging above equation, we get

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - 2x^3}.$$

■

The previous Lemma gives the following results as particular examples.

Corollary 3.1 *Generated functions of third order Jacobsthal, Jacobsthal-Lucas, modified*

Jacobsthal and Jacobsthal Perrin numbers are

$$
\begin{aligned}
\sum_{n=0}^{\infty} J_n x^n &= \frac{x}{1-x-x^2-2x^3}, \\
\sum_{n=0}^{\infty} j_n x^n &= \frac{2-x+2x^2}{1-x-x^2-2x^3}, \\
\sum_{n=0}^{\infty} K_n x^n &= \frac{3-2x-x^2}{1-x-x^2-2x^3}, \\
\sum_{n=0}^{\infty} Q_n x^n &= \frac{3-3x-x^2}{1-x-x^2-2x^3},
\end{aligned}
$$

respectively.

3.1.2 Obtaining Binet Formula From Generating Function

In this subsecion, we next find Binet formula of generalized third order Jacobsthal numbers $\{V_n\}$ by the use of generating function for V_n. In the rest of the thesis, we find the Binet formulas with the help of the way given in [30].

Theorem 3.2 *(Binet formula of generalized third order Jacobsthal numbers) For all integers n, we have*

$$V_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{3.17}$$

where

$$
\begin{aligned}
d_1 &= V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1 - V_0), \\
d_2 &= V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1 - V_0), \\
d_3 &= V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1 - V_0).
\end{aligned}
$$

Proof. Let

$$h(x) = 1 - x - x^2 - 2x^3.$$

Then for some α, β and γ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)$$

i.e.,

$$1 - x - x^2 - 2x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x) \tag{3.18}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}$, ve $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives α, β, and γ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{1}{x} - \frac{1}{x^2} - \frac{2}{x^3} = 0.$$

This implies $x^3 - x^2 - x - 2 = 0$. Now, by (3.16) and (3.18), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we write

$$\frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)}. \tag{3.19}$$

So

$$V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2 = A_1(1-\beta x)(1-\gamma x)+A_2(1-\alpha x)(1-\gamma x)+A_3(1-\alpha x)(1-\beta x).$$

If we consider $x = \frac{1}{\alpha}$, we get $V_0 + (V_1 - V_0)\frac{1}{\alpha} + (V_2 - V_1 - V_0)\frac{1}{\alpha^2} = A_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})$. This gives

$$A_1 = \frac{\alpha^2(V_0 + (V_1 - V_0)\frac{1}{\alpha} + (V_2 - V_1 - V_0)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)} = \frac{V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1 - V_0)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$A_2 = \frac{V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1 - V_0)}{(\beta-\alpha)(\beta-\gamma)}, A_3 = \frac{V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1 - V_0)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (3.19) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1} + A_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} V_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n$$

where

$$\begin{aligned} A_1 &= \frac{V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1 - V_0)}{(\alpha-\beta)(\alpha-\gamma)}, \\ A_2 &= \frac{V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1 - V_0)}{(\beta-\alpha)(\beta-\gamma)}, \\ A_3 &= \frac{V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1 - V_0)}{(\gamma-\alpha)(\gamma-\beta)} \end{aligned}$$

and then we get (3.17). ■

Note that from (3.5) and (3.17) we have

$$\begin{aligned}
V_2-(\beta+\gamma)V_1+\beta\gamma V_0 &= V_0\alpha^2+(V_1-V_0)\alpha+(V_2-V_1-V_0),\\
V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0 &= V_0\beta^2+(V_1-V_0)\beta+(V_2-V_1-V_0),\\
V_2-(\alpha+\beta)V_1+\alpha\beta V_0 &= V_0\gamma^2+(V_1-V_0)\gamma+(V_2-V_1-V_0).
\end{aligned}$$

Next, using Theorem 3.2, we present the Binet formulas of third order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal and Jacobsthal Perrin sequences.

Corollary 3.2 *For all integers n, Binet formulas of third order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal and Jacobsthal Perrin sequences are*

$$\begin{aligned}
J_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},\\
j_n &= \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},\\
K_n &= \alpha^n+\beta^n+\gamma^n,\\
Q_n &= \frac{(3\alpha^2-3\alpha-1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(3\beta^2-3\beta-1)\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{(3\gamma^2-3\gamma-1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}
\end{aligned}$$

respectively.

Matrix method which is given in [45] for Pell numbers can be adjusted to third order Jacobsthall numbers. Take $k=i=3$ in Corollary 3.1 in [45]. Let

$$\begin{aligned}
\Lambda &= \begin{pmatrix} \alpha^2 & \alpha & 1\\ \beta^2 & \beta & 1\\ \gamma^2 & \gamma & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha & 1\\ \beta^{n-1} & \beta & 1\\ \gamma^{n-1} & \gamma & 1 \end{pmatrix},\\
\Lambda_2 &= \begin{pmatrix} \alpha^2 & \alpha^{n-1} & 1\\ \beta^2 & \beta^{n-1} & 1\\ \gamma^2 & \gamma^{n-1} & 1 \end{pmatrix}, \Lambda_3 = \begin{pmatrix} \alpha^2 & \alpha & \alpha^{n-1}\\ \beta^2 & \beta & \beta^{n-1}\\ \gamma^2 & \gamma & \gamma^{n-1} \end{pmatrix}.
\end{aligned}$$

Then, for all integers n, the Binet formula for third order Jacobsthal numbers is

$$\begin{aligned}
J_n &= \frac{1}{\det(\Lambda)}\sum_{j=1}^{3} J_{4-j}\det(\Lambda_j) = \frac{1}{\det(\Lambda)}(J_3\det(\Lambda_1)+J_2\det(\Lambda_2)+J_1\det(\Lambda_3)) \\
&= \frac{1}{\det(\Lambda)}(2\det(\Lambda_1)+\det(\Lambda_2)+\det(\Lambda_3)) \\
&= \left(2\begin{vmatrix}\alpha^{n-1} & \alpha & 1\\ \beta^{n-1} & \beta & 1\\ \gamma^{n-1} & \gamma & 1\end{vmatrix} + \begin{vmatrix}\alpha^{2} & \alpha^{n-1} & 1\\ \beta^{2} & \beta^{n-1} & 1\\ \gamma^{2} & \gamma^{n-1} & 1\end{vmatrix} + \begin{vmatrix}\alpha^{2} & \alpha & \alpha^{n-1}\\ \beta^{2} & \beta & \beta^{n-1}\\ \gamma^{2} & \gamma & \gamma^{n-1}\end{vmatrix}\right) \Big/ \begin{vmatrix}\alpha^{2} & \alpha & 1\\ \beta^{2} & \beta & 1\\ \gamma^{2} & \gamma & 1\end{vmatrix} \\
&= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}.
\end{aligned}$$

Similarly, for all integers n, we obtain the Binet formula for third order Jacobsthal-Lucas, modified third order Jacobsthal and Jacobsthal Perrin numbers as

$$\begin{aligned}
j_n &= \frac{1}{\det(\Lambda)}(j_3\det(\Lambda_1)+j_2\det(\Lambda_2)+j_1\det(\Lambda_3)) \\
&= \left(10\begin{vmatrix}\alpha^{n-1} & \alpha & 1\\ \beta^{n-1} & \beta & 1\\ \gamma^{n-1} & \gamma & 1\end{vmatrix} + 5\begin{vmatrix}\alpha^{2} & \alpha^{n-1} & 1\\ \beta^{2} & \beta^{n-1} & 1\\ \gamma^{2} & \gamma^{n-1} & 1\end{vmatrix} + \begin{vmatrix}\alpha^{2} & \alpha & \alpha^{n-1}\\ \beta^{2} & \beta & \beta^{n-1}\\ \gamma^{2} & \gamma & \gamma^{n-1}\end{vmatrix}\right) \Big/ \begin{vmatrix}\alpha^{2} & \alpha & 1\\ \beta^{2} & \beta & 1\\ \gamma^{2} & \gamma & 1\end{vmatrix} \\
&= \frac{(2\alpha^2-\alpha+2)\alpha^{n}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2-\beta+2)\beta^{n}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(2\gamma^2-\gamma+2)\gamma^{n}}{(\gamma-\alpha)(\gamma-\beta)}
\end{aligned}$$

and

$$\begin{aligned}
K_n &= \frac{1}{\Lambda}(K_3\det(\Lambda_1)+K_2\det(\Lambda_2)+K_1\det(\Lambda_3)) \\
&= \left(10\begin{vmatrix}\alpha^{n-1} & \alpha & 1\\ \beta^{n-1} & \beta & 1\\ \gamma^{n-1} & \gamma & 1\end{vmatrix} + 3\begin{vmatrix}\alpha^{2} & \alpha^{n-1} & 1\\ \beta^{2} & \beta^{n-1} & 1\\ \gamma^{2} & \gamma^{n-1} & 1\end{vmatrix} + \begin{vmatrix}\alpha^{2} & \alpha & \alpha^{n-1}\\ \beta^{2} & \beta & \beta^{n-1}\\ \gamma^{2} & \gamma & \gamma^{n-1}\end{vmatrix}\right) \Big/ \begin{vmatrix}\alpha^{2} & \alpha & 1\\ \beta^{2} & \beta & 1\\ \gamma^{2} & \gamma & 1\end{vmatrix} \\
&= \alpha^n+\beta^n+\gamma^n
\end{aligned}$$

and

$$\begin{aligned}
Q_n &= \frac{1}{\det(\Lambda)}(Q_3\det(\Lambda_1)+Q_2\det(\Lambda_2)+Q_1\det(\Lambda_3)) \\
&= \left(8\begin{vmatrix}\alpha^{n-1} & \alpha & 1\\ \beta^{n-1} & \beta & 1\\ \gamma^{n-1} & \gamma & 1\end{vmatrix} + 2\begin{vmatrix}\alpha^{2} & \alpha^{n-1} & 1\\ \beta^{2} & \beta^{n-1} & 1\\ \gamma^{2} & \gamma^{n-1} & 1\end{vmatrix}\right) \Big/ \begin{vmatrix}\alpha^{2} & \alpha & 1\\ \beta^{2} & \beta & 1\\ \gamma^{2} & \gamma & 1\end{vmatrix} \\
&= \frac{(3\alpha^2-3\alpha-1)\alpha^{n}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(3\beta^2-3\beta-1)\beta^{n}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(3\gamma^2-3\gamma-1)\gamma^{n}}{(\gamma-\alpha)(\gamma-\beta)}
\end{aligned}$$

respectively.

3.1.3 Simson Formulas

In this subsection, the following theorem gives the generalization of Simson formula of generalized third order Jacobsthal sequence $\{V_n\}_{n\geq 0}$.

Theorem 3.3 (Simson Formula of Generalized Third Order Jacobsthal Numbers) *For all integers n, we have*

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = 2^n \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{3.20}$$

For the proof of (3.20), see [64]. ■

The previous theorem gives the following results as particular examples.

Corollary 3.3 *For all integers n, Simson formula of third order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal, Jacobsthal Perrin numbers are given as*

$$\begin{vmatrix} J_{n+2} & J_{n+1} & J_n \\ J_{n+1} & J_n & J_{n-1} \\ J_n & J_{n-1} & J_{n-2} \end{vmatrix} = -2^{n-1},$$

and

$$\begin{vmatrix} j_{n+2} & j_{n+1} & j_n \\ j_{n+1} & j_n & j_{n-1} \\ j_n & j_{n-1} & j_{n-2} \end{vmatrix} = -9 \times 2^{n+1},$$

and

$$\begin{vmatrix} K_{n+2} & K_{n+1} & K_n \\ K_{n+1} & K_n & K_{n-1} \\ K_n & K_{n-1} & K_{n-2} \end{vmatrix} = -147 \times 2^{n-2},$$

and

$$\begin{vmatrix} Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+1} & Q_n & Q_{n-1} \\ Q_n & Q_{n-1} & Q_{n-2} \end{vmatrix} = -35 \times 2^n,$$

respectively.

3.1.4 Some Identities

In this subsection, we obtain some identities of third order Jacobsthal, third order Jacobsthal-Lucas, modified third order Jacobsthal and third order Jacobsthal Perrin numbers. Firstly, we give some fundamental relations between $\{J_n\}$ and $\{j_n\}$.

Lemma 3.4 *For all integers n, the following equalities are true:*

$$
\begin{aligned}
j_n &= J_{n+4} - 2J_{n+3} + J_{n+2}, && (3.21)\\
j_n &= -J_{n+3} + 2J_{n+2} + 2J_{n+1}, && (3.22)\\
j_n &= J_{n+2} + J_{n+1} - 2J_n, && (3.23)\\
j_n &= 2J_{n+1} - J_n + 2J_{n-1}, && (3.24)\\
j_n &= J_n + 4J_{n-1} + 4J_{n-2}, && (3.25)
\end{aligned}
$$

and

$$
\begin{aligned}
48J_n &= 5j_{n+4} - 11j_{n+3} + 5j_{n+2}, && (3.26)\\
24J_n &= -3j_{n+3} + 5j_{n+2} + 5j_{n+1}, && (3.27)\\
12J_n &= j_{n+2} + j_{n+1} - 3j_n, && (3.28)\\
6J_n &= j_{n+1} - j_n + j_{n-1}, && (3.29)\\
3J_n &= j_{n-1} + j_{n-2}, && (3.30)
\end{aligned}
$$

Proof. Note that all the identities hold for all integers n. We only prove (3.21). If we write

$$j_n = a \times J_{n+4} + b \times J_{n+3} + c \times J_{n+2}$$

and solve the system of equations

$$
\begin{aligned}
j_0 &= a \times J_4 + b \times J_3 + c \times J_2\\
j_1 &= a \times J_5 + b \times J_4 + c \times J_3\\
j_2 &= a \times J_6 + b \times J_5 + c \times J_4
\end{aligned}
$$

then we get $a = 1$, $b = -2$, $c = 1$. The other equalities can be proved similarly. ∎

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present some fundamental relations between $\{J_n\}$ and $\{K_n\}$.

Lemma 3.5 *For all integers n, the following equalities hold:*

$$
\begin{aligned}
8K_n &= 17J_{n+4} - 23J_{n+3} - 15J_{n+2}, \\
4K_n &= -3J_{n+3} + J_{n+2} + 17J_{n+1}, \\
2K_n &= -J_{n+2} + 7J_{n+1} - 3J_n, \\
K_n &= 3J_{n+1} - 2J_n - J_{n-1}, \\
K_n &= J_n + 2J_{n-1} + 6J_{n-2}
\end{aligned}
$$

and

$$
\begin{aligned}
294J_n &= -K_{n+4} - 15K_{n+3} + 55K_{n+2}, \\
147J_n &= -8K_{n+3} + 27K_{n+2} - K_{n+1} \\
147J_n &= 19K_{n+2} - 9K_{n+1} - 16K_n, \\
147J_n &= 10K_{n+1} + 3K_n + 38K_{n-1}. \\
147J_n &= 13K_n + 48K_{n-1} + 20K_{n-2}.
\end{aligned}
$$

Thirdly, we give some fundamental relations between $\{j_n\}$ and $\{K_n\}$.

Lemma 3.6 *For all integers n, the following equalities hold:*

$$
\begin{aligned}
49j_n &= 15K_{n+3} - 20K_{n+2} + 8K_{n+1}, \\
49j_n &= -5K_{n+2} + 23K_{n+1} + 30K_n, \\
49j_n &= 18K_{n+1} + 25K_n - 10K_{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
48K_n &= 19j_{n+3} + 3j_{n+2} - 61j_{n+1}, \\
24K_n &= 11j_{n+2} - 21j_{n+1} + 19j_n, \\
12K_n &= -5j_{n+1} + 15j_n + 11j_{n-1}.
\end{aligned}
$$

Fourthly, we give some fundamental relations between $\{j_n\}$ and $\{Q_n\}$.

Lemma 3.7 *For all integers n, the following equalities hold:*

$$
\begin{aligned}
70j_n &= 4Q_{n+4} + 19Q_{n+3} - 26Q_{n+2}, \\
70j_n &= 23Q_{n+3} - 22Q_{n+2} + 8Q_{n+1}, \\
70j_n &= Q_{n+2} + 31Q_{n+1} + 46Q_n, \\
70j_n &= 32Q_{n+1} + 47Q_n + 2Q_{n-1}, \\
70j_n &= 79Q_n + 34Q_{n-1} + 64Q_{n-2},
\end{aligned}
$$

and

$$\begin{aligned}
96Q_n &= -71j_{n+4} + 121j_{n+3} + 57j_{n+2}, \\
48Q_n &= 25j_{n+3} - 7j_{n+2} - 71j_{n+1}, \\
24Q_n &= 9j_{n+2} - 23j_{n+1} + 25j_n, \\
12Q_n &= -7j_{n+1} + 17j_n + 9j_{n-1}, \\
6Q_n &= 5j_n + j_{n-1} - 7j_{n-2}.
\end{aligned}$$

Fifthly, we give some fundamental relations between $\{K_n\}$ and $\{Q_n\}$

Lemma 3.8 *For all integers n, the following equalities hold:*

$$\begin{aligned}
140K_n &= -33Q_{n+4} + 97Q_{n+3} - 13Q_{n+2}, \\
70K_n &= 32Q_{n+3} - 23Q_{n+2} - 33Q_{n+1}, \\
70K_n &= 9Q_{n+2} - Q_{n+1} + 64Q_n, \\
70K_n &= 8Q_{n+1} + 73Q_n + 18Q_{n-1}, \\
70K_n &= 81Q_n + 26Q_{n-1} + 16Q_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
588Q_n &= -145K_{n+4} + 471K_{n+3} - 257K_{n+2}, \\
294Q_n &= 163K_{n+3} - 201K_{n+2} - 145K_{n+1}, \\
147Q_n &= -19K_{n+2} + 9K_{n+1} + 163K_n, \\
147Q_n &= -10K_{n+1} + 144K_n - 38K_{n-1}, \\
147Q_n &= 134K_n - 48K_{n-1} - 20K_{n-2}.
\end{aligned}$$

Sixthly, we give some fundamental relations between $\{J_n\}$ and $\{Q_n\}$

Lemma 3.9 *For all integers n, the following equalities hold:*

$$\begin{aligned}
70J_n &= Q_{n+4} - 4Q_{n+3} + 11Q_{n+2}, \\
70J_n &= -3Q_{n+3} + 12Q_{n+2} + 2Q_{n+1}, \\
70J_n &= 9Q_{n+2} - Q_{n+1} - 6Q_n, \\
70J_n &= 8Q_{n+1} + 3Q_n + 18Q_{n-1}, \\
70J_n &= 11Q_n + 26Q_{n-1} + 16Q_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
8Q_n &= 19J_{n+4} - 29J_{n+3} - 13J_{n+2}, \\
4Q_n &= -5J_{n+3} + 3J_{n+2} + 19J_{n+1}, \\
2Q_n &= -J_{n+2} + 7J_{n+1} - 5J_n, \\
Q_n &= 3J_{n+1} - 3J_n - J_{n-1}, \\
Q_n &= 2J_{n-1} + 6J_{n-2}.
\end{aligned}$$

3.1.5 Linear Sums

In this subsection, we present some formulas of generalized third order Jacobsthal numbers with positive subscripts.

Proposition 3.10 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k = \frac{1}{3}(V_{n+3} - V_{n+1} - V_2 + V_0)$.

(b) $\sum_{k=0}^{n} V_{2k} = \frac{1}{3}(V_{2n+1} + 2V_{2n} - V_1 + V_0)$.

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{3}(V_{2n+2} + 2V_{2n+1} - V_2 + V_1)$.

Proof. See [67].

As special cases of above proposition, we have the following four corollaries. First one presents some summation formulas of third order Jacobsthal numbers. (take $V_n = J_n$ with $J_0 = 0,\ J_1 = 1,\ J_2 = 1$).

Corollary 3.4 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} J_k = \frac{1}{3}(J_{n+3} - J_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} J_{2k} = \frac{1}{3}(J_{2n+1} + 2J_{2n} - 1)$.

(c) $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{3}(J_{2n+2} + 2J_{2n+1})$.

Second one presents some summation formulas of third order Jacobsthal-Lucas numbers. (take $V_n = j_n$ with $j_0 = 2,\ j_1 = 1,\ j_2 = 5$).

Corollary 3.5 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} j_k = \frac{1}{3}(j_{n+3} - j_{n+1} - 3)$.

(b) $\sum_{k=0}^{n} j_{2k} = \frac{1}{3}(j_{2n+1} + 2j_{2n} + 1)$.

(c) $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{3}(j_{2n+2} + 2j_{2n+1} - 4)$.

Third one presents some summation formulas of modified third order Jacobsthal numbers. (take $V_n = K_n$ with $K_0 = 3,\ K_1 = 1,\ K_2 = 3$).

Corollary 3.6 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} K_k = \frac{1}{3}(K_{n+3} - K_{n+1})$.

(b) $\sum_{k=0}^{n} K_{2k} = \frac{1}{3}(K_{2n+1} + 2K_{2n} + 2)$.

(c) $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{3}(K_{2n+2} + 2K_{2n+1} - 2)$.

Fourth one presents some summation formulas of third order Jacobsthal Perrin numbers. (take $V_n = Q_n$ with $Q_0 = 3,\ Q_1 = 0,\ Q_2 = 2$).

Corollary 3.7 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{3}(Q_{n+3} - Q_{n+1} + 1)$.

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{3}(Q_{2n+1} + 2Q_{2n} + 3)$.

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{3}(Q_{2n+2} + 2Q_{2n+1} - 2)$.

The following proposition presents some formulas of generalized third order Jacobsthal numbers with negative subscripts.

Proposition 3.11 *For $n \geq 1$, we have the following formulas:*

(a) $\sum_{k=1}^{n} V_{-k} = \frac{1}{3}(-4V_{-n-1} - 3V_{-n-2} - 2V_{-n-3} + V_2 - V_0)$.

(b) $\sum_{k=1}^{n} V_{-2k} = \frac{1}{3}\left(-V_{-2n+1} + V_{-2n} + V_1 - V_0\right)$.

(c) $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{3}\left(-V_{-2n} - 2V_{-2n-1} + V_2 - V_1\right)$.

Proof. See [67].

Taking $V_n = J_n$ with $J_0 = 0$, $J_1 = 1$, $J_2 = 1$ in the last proposition, we have the following corollary which gives linear summation formulas of third order Jacobsthal numbers.

Corollary 3.8 *For $n \geq 1$, we have the following properties:*

(a) $\sum_{k=1}^{n} J_{-k} = \frac{1}{3}(-4J_{-n-1} - 3J_{-n-2} - 2J_{-n-3} + 1)$.

(b) $\sum_{k=1}^{n} J_{-2k} = \frac{1}{3}(-J_{-2n+1} + J_{-2n} + 1)$.

(c) $\sum_{k=1}^{n} J_{-2k+1} = \frac{1}{3}(-J_{-2n} - 2J_{-2n-1})$.

From the last proposition, we have the following corollary which gives linear summation formulas of third order Jacobsthal-Lucas numbers (take $V_n = j_n$ with $j_0 = 2$, $j_1 = 1$, $j_2 = 5$).

Corollary 3.9 *For $n \geq 1$, we have the following properties:*

(a) $\sum_{k=1}^{n} j_{-k} = \frac{1}{3}(-4j_{-n-1} - 3j_{-n-2} - 2j_{-n-3} + 3)$.

(b) $\sum_{k=1}^{n} j_{-2k} = \frac{1}{3}(-j_{-2n+1} + j_{-2n} - 1)$.

(c) $\sum_{k=1}^{n} j_{-2k+1} = \frac{1}{3}(-j_{-2n} - 2j_{-2n-1} + 4)$.

Third one presents some summation formulas of modified third order Jacobsthal numbers. (take $V_n = K_n$ with $K_0 = 3$, $K_1 = 1$, $K_2 = 3$).

Corollary 3.10 *For $n \geq 1$, we have the following properties:*

(a) $\sum_{k=1}^{n} K_{-k} = \frac{1}{3}(-4K_{-n-1} - 3K_{-n-2} - 2K_{-n-3})$.

(b) $\sum_{k=1}^{n} K_{-2k} = \frac{1}{3}(-K_{-2n+1} + K_{-2n} - 2)$.

(c) $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{3}(-K_{-2n} - 2K_{-2n-1} + 2)$.

Fourth one presents some summation formulas of third order Jacobsthal Perrin numbers. (take $V_n = Q_n$ with $Q_0 = 3$, $Q_1 = 0$, $Q_2 = 2$).

Corollary 3.11 *For $n \geq 1$, we have the following properties:*

(a) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{3}(-4Q_{-n-1} - 3Q_{-n-2} - 2Q_{-n-3} - 1)$.

(b) $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{3}(-Q_{-2n+1} + Q_{-2n} - 3)$.

(c) $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{3}(-Q_{-2n} - 2Q_{-2n-1} + 2)$.

3.1.6 Matrices related with Generalized Third Order Jacobsthal numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{3.31}$$

For matrix formulation of (3.31), see [43]. In fact, Kalman gave the formula in the following form:

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We consider the following square matrix A of order 3

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. From (3.4) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \tag{3.32}$$

By using (3.32) and from induction, we get

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = J$ in (3.32), then we have

$$\begin{pmatrix} J_{n+2} \\ J_{n+1} \\ J_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} J_{n+1} \\ J_n \\ J_{n-1} \end{pmatrix}. \tag{3.33}$$

Now we consider the following matrices:

$$B_n = \begin{pmatrix} J_{n+1} & J_n + 2J_{n-1} & 2J_n \\ J_n & J_{n-1} + 2J_{n-2} & 2J_{n-1} \\ J_{n-1} & J_{n-2} + 2J_{n-3} & 2J_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}.$$

Theorem 3.12 *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (3.33) to 3-columns and multiplying the obtained on the right-hand side by A, we get

$B_n = AB_{n-1}$.

By induction argument, from the last equation, we obtain

$B_n = A^{n-1}B_1$.

Since $B_1 = A$, then it follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3} + 2V_{n-4} & 2V_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Thus

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

■

Note that the matrix A^n satisfies the following equalities:

$$A^n = A^{n-1} + A^{n-2} + 2A^{n-3}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

for all integer m and n.

Theorem 3.13 *For $m, n \geq 0$ we have*

$$\begin{aligned} V_{n+m} &= V_nJ_{m+1} + 2J_{m-1}V_{n-1} + J_m(V_{n-1} + 2V_{n-2}) \quad (3.34) \\ &= V_nJ_{m+1} + (V_{n-1} + 2V_{n-2})J_m + 2J_{m-1}V_{n-1} \quad (3.35) \end{aligned}$$

Proof. From the equation $C_{n+m} = C_nB_m = B_mC_n$, we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof. ■

Remark 3.1 *By induction, it can be proved that for all integers $m, n \leq 0$, (3.34) holds. So for all integers m, n, (3.34) is true.*

Corollary 3.12 *For all integers m, n, we have*

$$\begin{aligned} J_{n+m} &= J_nJ_{m+1} + (J_{n-1} + 2J_{n-2})J_m + 2J_{m-1}J_{n-1} \quad (3.36) \\ j_{n+m} &= j_nJ_{m+1} + (j_{n-1} + 2j_{n-2})J_m + 2J_{m-1}j_{n-1} \quad (3.37) \\ K_{n+m} &= K_nJ_{m+1} + (K_{n-1} + 2K_{n-2})J_m + 2J_{m-1}K_{n-1} \quad (3.38) \\ Q_{n+m} &= Q_nJ_{m+1} + (Q_{n-1} + 2Q_{n-2})J_m + 2J_{m-1}Q_{n-1} \quad (3.39) \end{aligned}$$

CHAPTER 4

GENERALIZED FOURTH ORDER JACOBSTHAL NUMBERS

The aim of this chapter is to study the generalized fourth order Jacobsthal sequences and we investigate, in detail, six special cases: fourth order Jacobsthal, fourth order Jacobsthal-Lucas, modified fourth order Jacobsthal, fourth order Jacobsthal Perrin, adjusted fourth order Jacobsthal and modified fourth order Jacobsthal-Lucas sequences. We state that the results of this chapter are cited from [82] which has been published by Soykan and Eyican Polatli.

4.1 INTRODUCTION

In this section, we give basic information about generalized fourth order Jacobsthal numbers.

A generalized Tetranacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3; r_1, r_2, r_3, r_4)\}_{n\geq 0}$ is defined by the fourth order recurrence relations

$$W_n = r_1 W_{n-1} + r_2 W_{n-2} + r_3 W_{n-3} + r_4 W_{n-4}, \qquad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3 \quad (4.1)$$

where the initial values W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r_1, r_2, r_3, r_4 are real numbers. Tetranacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [31, 52, 54, 61, 65, 93, 94].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r_3}{r_4} W_{-(n-1)} - \frac{r_2}{r_4} W_{-(n-2)} - \frac{r_1}{r_4} W_{-(n-3)} + \frac{1}{r_4} W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (4.1) holds for all integer n.

In this chapter we consider the case $r_1 = r_2 = r_3 = 1$, $r_4 = 2$ and in this case we write $V_n = W_n$. A generalized fourth order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n\geq 0}$ is defined by the fourth order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + 2V_{n-4} \tag{4.2}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$, $V_3 = c_3$ not all being zero. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} - \frac{1}{2}V_{-(n-3)} + \frac{1}{2}V_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (4.2) holds for all integer n.

As $\{V_n\}$ is a fourth order recurrence sequence (difference equation), it's characteristic equation is

$$x^4 - x^3 - x^2 - x - 2 = 0. \tag{4.3}$$

The roots α, β, γ and δ of equation (4.3) are given by

$$\alpha = -1,$$

$$\beta = 2,$$

$$\gamma = i,$$

$$\delta = -i.$$

Here α, β, γ and δ satisfy the following equalities:

$$\alpha + \beta + \gamma + \delta = 1,$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -1,$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 1,$$

$$\alpha\beta\gamma\delta = -2.$$

The first few generalized fourth order Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 4.1.

Table 4.1. The first few terms of generalized fourth order Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$\frac{1}{2}V_3 - \frac{1}{2}V_1 - \frac{1}{2}V_2 - \frac{1}{2}V_0$
2	V_2	$\frac{3}{4}V_2 - \frac{1}{4}V_1 - \frac{1}{4}V_0 - \frac{1}{4}V_3$
3	V_3	$\frac{7}{8}V_1 - \frac{1}{8}V_0 - \frac{1}{8}V_2 - \frac{1}{8}V_3$
4	$2V_0 + V_1 + V_2 + V_3$	$\frac{15}{16}V_0 - \frac{1}{16}V_1 - \frac{1}{16}V_2 - \frac{1}{16}V_3$
5	$2V_0 + 3V_1 + 2V_2 + 2V_3$	$\frac{15}{32}V_3 - \frac{17}{32}V_1 - \frac{17}{32}V_2 - \frac{17}{32}V_0$
6	$4V_0 + 4V_1 + 5V_2 + 4V_3$	$\frac{47}{64}V_2 - \frac{17}{64}V_1 - \frac{17}{64}V_0 - \frac{17}{64}V_3$
7	$8V_0 + 8V_1 + 8V_2 + 9V_3$	$\frac{111}{128}V_1 - \frac{17}{128}V_0 - \frac{17}{128}V_2 - \frac{17}{128}V_3$
8	$18V_0 + 17V_1 + 17V_2 + 17V_3$	$\frac{239}{256}V_0 - \frac{17}{256}V_1 - \frac{17}{256}V_2 - \frac{17}{256}V_3$
9	$34V_0 + 35V_1 + 34V_2 + 34V_3$	$\frac{239}{512}V_3 - \frac{273}{512}V_1 - \frac{273}{512}V_2 - \frac{273}{512}V_0$
10	$68V_0 + 68V_1 + 73V_2 + 68V_3$	$\frac{751}{1024}V_2 - \frac{273}{1024}V_1 - \frac{273}{1024}V_0 - \frac{273}{1024}V_3$

Now we define six special cases of the sequence $\{V_n\}$. Fourth order Jacobsthal sequence $\{J_n^{(4)}\}_{n\geq 0}$, fourth order Jacobsthal-Lucas sequence $\{j_n^{(4)}\}_{n\geq 0}$, modified fourth order Jacobsthal sequence $\{K_n^{(4)}\}_{n\geq 0}$, fourth order Jacobsthal Perrin sequence $\{Q_n^{(4)}\}_{n\geq 0}$, adjusted fourth order Jacobsthal sequence $\{S_n^{(4)}\}_{n\geq 0}$ and modified fourth order Jacobsthal-Lucas sequence $\{R_n^{(4)}\}_{n\geq 0}$ are defined, respectively, by the following recurrence relations:

$$J_{n+4} = J_{n+3} + J_{n+2} + J_{n+1} + 2J_n, \quad J_0 = 0,\ J_1 = 1,\ J_2 = 1,\ J_3 = 2, \tag{4.4}$$

$$j_{n+4} = j_{n+3} + j_{n+2} + j_{n+1} + 2j_n, \quad j_0 = 2,\ j_1 = 1,\ j_2 = 5,\ j_3 = 10, \tag{4.5}$$

$$K_{n+4} = K_{n+3} + K_{n+2} + K_{n+1} + 2K_n, \quad K_0 = 3,\ K_1 = 1,\ K_2 = 3,\ K_3 = 10, \tag{4.6}$$

$$Q_{n+4} = Q_{n+3} + Q_{n+2} + Q_{n+1} + 2Q_n, \quad Q_0 = 3,\ Q_1 = 0,\ Q_2 = 2,\ Q_3 = 8, \tag{4.7}$$

$$S_{n+4} = S_{n+3} + S_{n+2} + S_{n+1} + 2S_n, \quad S_0 = 0,\ S_1 = 1,\ S_2 = 1,\ S_3 = 2, \tag{4.8}$$

$$R_{n+4} = R_{n+3} + R_{n+2} + R_{n+1} + 2R_n, \quad R_0 = 4,\ R_1 = 1,\ R_2 = 3,\ R_3 = 7. \tag{4.9}$$

The sequences $\{J_n^{(4)}\}_{n\geq 0}$, $\{j_n^{(4)}\}_{n\geq 0}$,$\{K_n^{(4)}\}_{n\geq 0}, Q_n^{(4)}\}_{n\geq 0}$, $\{S_n^{(4)}\}_{n\geq 0}$ and $\{R_n^{(4)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$
\begin{aligned}
J_{-n}^{(4)} &= -\frac{1}{2}J_{-(n-1)}^{(4)} - \frac{1}{2}J_{-(n-2)}^{(4)} - \frac{1}{2}J_{-(n-3)}^{(4)} + \frac{1}{2}J_{-(n-4)}^{(4)}, && (4.10)\\
j_{-n}^{(4)} &= -\frac{1}{2}j_{-(n-1)}^{(4)} - \frac{1}{2}j_{-(n-2)}^{(4)} - \frac{1}{2}j_{-(n-3)}^{(4)} + \frac{1}{2}j_{-(n-4)}^{(4)}, && (4.11)\\
K_{-n}^{(4)} &= -\frac{1}{2}K_{-(n-1)}^{(4)} - \frac{1}{2}K_{-(n-2)}^{(4)} - \frac{1}{2}K_{-(n-3)}^{(4)} + \frac{1}{2}K_{-(n-4)}^{(4)}, && (4.12)\\
Q_{-n}^{(4)} &= -\frac{1}{2}Q_{-(n-1)}^{(4)} - \frac{1}{2}Q_{-(n-2)}^{(4)} - \frac{1}{2}Q_{-(n-3)}^{(4)} + \frac{1}{2}Q_{-(n-4)}^{(4)}, && (4.13)\\
S_{-n}^{(4)} &= -\frac{1}{2}S_{-(n-1)}^{(4)} - \frac{1}{2}S_{-(n-2)}^{(4)} - \frac{1}{2}S_{-(n-3)}^{(4)} + \frac{1}{2}S_{-(n-4)}^{(4)}, && (4.14)\\
R_{-n}^{(4)} &= -\frac{1}{2}R_{-(n-1)}^{(4)} - \frac{1}{2}R_{-(n-2)}^{(4)} - \frac{1}{2}R_{-(n-3)}^{(4)} + \frac{1}{2}R_{-(n-4)}^{(4)} && (4.15)
\end{aligned}
$$

for $n \geq 1$, respectively. Therefore, recurrences (4.4)-(4.9) hold for all integer n.

Next, we present the first few values of the fourth order Jacobsthal, fourth order Jacobsthal-Lucas, modified fourth order Jacobsthal , fourth order Jacobsthal Perrin, adjusted fourth order Jacobsthal and modified fourth order Jacobsthal-Lucas numbers with positive and negative subscripts:

Table 4.2. The first few terms of the J_n, j_n, K_n, Q_n, S_n and R_n

n	0	1	2	3	4	5	6	7	8	9	10
$J_n^{(4)}$	0	1	1	1	3	7	13	25	51	103	205
$J_{-n}^{(4)}$		$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{8}$	$-\frac{3}{16}$	$-\frac{19}{32}$	$\frac{13}{64}$	$\frac{77}{128}$	$-\frac{51}{256}$	$-\frac{307}{512}$	$\frac{205}{1024}$
$j_n^{(4)}$	2	1	5	10	20	37	77	154	308	613	1229
$j_{-n}^{(4)}$		1	$\frac{1}{2}$	$-\frac{5}{4}$	$\frac{7}{8}$	$\frac{7}{16}$	$\frac{7}{32}$	$-\frac{89}{64}$	$\frac{103}{128}$	$\frac{103}{256}$	$\frac{103}{512}$
$K_n^{(4)}$	3	1	3	10	20	35	71	146	292	579	1159
$K_{-n}^{(4)}$		$\frac{3}{2}$	$-\frac{5}{4}$	$-\frac{9}{8}$	$\frac{31}{16}$	$\frac{31}{32}$	$-\frac{97}{64}$	$-\frac{161}{128}$	$\frac{479}{256}$	$\frac{479}{512}$	$-\frac{1569}{1024}$
$Q_n^{(4)}$	3	0	2	8	16	26	54	112	224	442	886
$Q_{-n}^{(4)}$		$\frac{3}{2}$	$-\frac{5}{4}$	$-\frac{13}{8}$	$\frac{35}{16}$	$\frac{35}{32}$	$-\frac{93}{64}$	$-\frac{221}{128}$	$\frac{547}{256}$	$\frac{547}{512}$	$-\frac{1501}{1024}$
$S_n^{(4)}$	0	1	1	2	4	9	17	34	68	137	273
$S_{-n}^{(4)}$		0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$\frac{15}{32}$	$-\frac{17}{64}$	$-\frac{17}{128}$	$-\frac{17}{256}$
$R_n^{(4)}$	4	1	3	7	19	31	63	127	259	511	1023
$R_{-n}^{(4)}$		$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$\frac{49}{16}$	$-\frac{31}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$\frac{769}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$

In the rest of the chapter, for easy writing, we drop the superscripts and write J_n, j_n, K_n, Q_n, S_n and R_n for $J_n^{(4)}, j_n^{(4)}, K_n^{(4)}, Q_n^{(4)}, S_n^{(4)}$ and $R_n^{(4)}$, respectively. Note that J_n, j_n and S_n are the sequences A007909, A226309 and A115451 in [62], respectively. K_n, Q_n and R_n sequence aren't in the database of http://oeis.org [62], yet.

4.1.1 Generating Functions

In this subsection, we give the ordinary generating function of the sequence V_n.

Lemma 4.1 *Suppose that* $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized fourth order Jacobsthal sequence* $\{V_n\}_{n\geq 0}$*. Then,* $\sum_{n=0}^{\infty} V_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - x^3 - 2x^4} + \frac{(V_3 - V_2 - V_1 - V_0)x^3}{1 - x - x^2 - x^3 - 2x^4}. \tag{4.16}$$

*Proof***.** Using the definition of generalized fourth order Jacobsthal numbers, and substracting $2xf(x)$, $x^2 f(x)$, $x^3 f(x)$ and $x^4 f(x)$ from $f(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
& (1 - x - x^2 - x^3 - 2x^4) f_{V_n}(x) \\
= {} & \sum_{n=0}^{\infty} V_n x^n - x \sum_{n=0}^{\infty} V_n x^n - x^2 \sum_{n=0}^{\infty} V_n x^n - x^3 \sum_{n=0}^{\infty} V_n x^n - 2x^4 \sum_{n=0}^{\infty} V_n x^n \\
= {} & \sum_{n=0}^{\infty} V_n x^n - \sum_{n=0}^{\infty} V_n x^{n+1} - \sum_{n=0}^{\infty} V_n x^{n+2} - \sum_{n=0}^{\infty} V_n x^{n+3} - 2\sum_{n=0}^{\infty} V_n x^{n+4} \\
= {} & \sum_{n=0}^{\infty} V_n x^n - \sum_{n=1}^{\infty} V_{n-1} x^n - \sum_{n=2}^{\infty} V_{n-2} x^n - \sum_{n=3}^{\infty} V_{n-3} x^n - 2\sum_{n=4}^{\infty} V_{n-4} x^n \\
= {} & (V_0 + V_1 x + V_2 x^2 + V_3 x^3) - (V_0 x + V_1 x^2 + V_2 x^3) - (V_0 x^2 + V_1 x^3) - V_0 x^3 \\
& + \sum_{n=4}^{\infty} (V_n - V_{n-1} - V_{n-2} - V_{n-3} - 2V_{n-4}) x^n \\
= {} & (V_0 + V_1 x + V_2 x^2 + V_3 x^3) - (V_0 x + V_1 x^2 + V_2 x^3) - (V_0 x^2 + V_1 x^3) - V_0 x^3 \\
= {} & V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3
\end{aligned}$$

Rearranging above equation, we get

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3}{1 - x - x^2 - x^3 - 2x^4}.$$

■

The previous Lemma gives the following results as particular examples.

Corollary 4.1 *Generated functions of fourth order Jacobsthal, fourth order Jacobsthal-Lucas, modified fourth order Jacobsthal, fourth order Jacobsthal Perrin, adjusted fourth* *order Jacobsthal and modified fourth order Jacobsthal-Lucas sequences are*

$$\begin{aligned}
\sum_{n=0}^{\infty} J_n x^n &= \frac{x - x^3}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} j_n x^n &= \frac{2 - x + 2x^2 + 2x^3}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} K_n x^n &= \frac{3 - 2x - x^2 + 3x^3}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} Q_n x^n &= \frac{3 - 3x - x^2 + 3x^3}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} S_n x^n &= \frac{x}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} R_n x^n &= \frac{4 - 3x - 2x^2 - x^3}{1 - x - x^2 - x^3 - 2x^4},
\end{aligned}$$

respectively.

4.1.2 Obtaining Binet Formula From Generating Function

In this subsection, we next find Binet formula of generalized fourth order Jacobsthal numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 4.2 *(Binet formula of generalized fourth order Jacobsthal numbers)*

$$\begin{aligned}
V_n &= \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \\
&\quad \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{d_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}
\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}
d_1 &= V_0\alpha^3 + (V_1 - V_0)\alpha^2 + (V_2 - V_1 - V_0)\alpha + (V_3 - V_2 - V_1 - V_0), \\
d_2 &= V_0\beta^3 + (V_1 - V_0)\beta^2 + (V_2 - V_1 - V_0)\beta + (V_3 - V_2 - V_1 - V_0), \\
d_3 &= V_0\gamma^3 + (V_1 - V_0)\gamma^2 + (V_2 - V_1 - V_0)\gamma + (V_3 - V_2 - V_1 - V_0), \\
d_4 &= V_0\delta^3 + (V_1 - V_0)\delta^2 + (V_2 - V_1 - V_0)\delta + (V_3 - V_2 - V_1 - V_0).
\end{aligned}$$

Proof. Let

$h(x) = 1 - x - x^2 - x^3 - 2x^4.$

Then for some α, β, γ and δ we write

$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)$

i.e.,

$$1-x-x^2-x^3-2x^4=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \tag{4.18}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ ve $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x})=1-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\frac{2}{x^4}=0.$$

This implies $x^4-x^3-x^2-x-2=0$. Now, by (4.16) and (4.18), it follows that

$$\sum_{n=0}^{\infty} V_n x^n=\frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} \tag{4.19}$$

$$= \frac{A_1}{(1-\alpha x)}+\frac{A_2}{(1-\beta x)}+\frac{A_3}{(1-\gamma x)}+\frac{A_4}{(1-\delta x)} \tag{4.20}$$

So

$$\begin{aligned}
& V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3 \\
= \; & A_1(1-\beta x)(1-\gamma x)(1-\delta x)+A_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\
& +A_3(1-\alpha x)(1-\beta x)(1-\delta x)+A_4(1-\alpha x)(1-\beta x)(1-\gamma x).
\end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get $V_0+(V_1-V_0)\frac{1}{\alpha}+(V_2-V_1-V_0)\frac{1}{\alpha^2}+(V_3-V_2-V_1-V_0)\frac{1}{\alpha^3}=A_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})$. This gives

$$\begin{aligned}
A_1 &= \frac{\alpha^3(V_0+(V_1-V_0)\frac{1}{\alpha}+(V_2-V_1-V_0)\frac{1}{\alpha^2}+(V_3-V_2-V_1-V_0)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\
&= \frac{V_0\alpha^3+(V_1-V_0)\alpha^2+(V_2-V_1-V_0)\alpha+(V_3-V_2-V_1-V_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
A_2 &= \frac{V_0\beta^3+(V_1-V_0)\beta^2+(V_2-V_1-V_0)\beta+(V_3-V_2-V_1-V_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\
A_3 &= \frac{V_0\gamma^3+(V_1-V_0)\gamma^2+(V_2-V_1-V_0)\gamma+(V_3-V_2-V_1-V_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\
A_4 &= \frac{V_0\delta^3+(V_1-V_0)\delta^2+(V_2-V_1-V_0)\delta+(V_3-V_2-V_1-V_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.
\end{aligned}$$

Thus (4.18) can be written as

$$\sum_{n=0}^{\infty} V_n x^n=A_1(1-\alpha x)^{-1}+A_2(1-\beta x)^{-1}+A_3(1-\gamma x)^{-1}+A_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned}\sum_{n=0}^{\infty} V_n x^n &= A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n + A_4 \sum_{n=0}^{\infty} \delta^n x^n \\ &= \sum_{n=0}^{\infty} (A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n + A_4 \delta^n) x^n.\end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n + A_4 \delta^n$$

where

$$\begin{aligned}A_1 &= \frac{V_0 \alpha^3 + (V_1 - V_0)\alpha^2 + (V_2 - V_1 - V_0)\alpha + (V_3 - V_2 - V_1 - V_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}, \\ A_2 &= \frac{V_0 \beta^3 + (V_1 - V_0)\beta^2 + (V_2 - V_1 - V_0)\beta + (V_3 - V_2 - V_1 - V_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ A_3 &= \frac{V_0 \gamma^3 + (V_1 - V_0)\gamma^2 + (V_2 - V_1 - V_0)\gamma + (V_3 - V_2 - V_1 - V_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ A_4 &= \frac{V_0 \delta^3 + (V_1 - V_0)\delta^2 + (V_2 - V_1 - V_0)\delta + (V_3 - V_2 - V_1 - V_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},\end{aligned}$$

and then we get (4.17). ■

Next, using Theorem 4.2, we present the Binet formulas of fourth order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal, Jacobsthal Perrin, adjusted Jacobsthal and modified Jacobsthal-Lucas sequences.

Corollary 4.2 *Binet formulas of fourth order Jacobsthal, fourth order Jacobsthal-Lucas, modified fourth order Jacobsthal, fourth order Jacobsthal Perrin, adjusted fourth order Jacobsthal and modified fourth order Jacobsthal-Lucas sequences are*

$$\begin{aligned}J_n &= \frac{(\alpha^2-1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(\beta^2-1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &\quad + \frac{(\gamma^2-1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(\delta^2-1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n,\end{aligned}$$

$$\begin{aligned}j_n &= \frac{(2\alpha^3-\alpha^2+2\alpha+2)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(2\beta^3-\beta^2+2\beta+2)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &\quad + \frac{(2\gamma^3-\gamma^2+2\gamma+2)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(2\delta^3-\delta^2+2\delta+2)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n,\end{aligned}$$

$$\begin{aligned}K_n &= \frac{(3\alpha^3-2\alpha^2-\alpha+3)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(3\beta^3-2\beta^2-\beta+3)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &\quad + \frac{(3\gamma^3-2\gamma^2-\gamma+3)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(3\delta^3-2\delta^2-\delta+3)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n,\end{aligned}$$

$$
\begin{aligned}
Q_n &= \frac{(3\alpha^3-3\alpha^2-\alpha+3)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n+\frac{(3\beta^3-3\beta^2-\beta+3)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\
&\quad +\frac{(3\gamma^3-3\gamma^2-\gamma+3)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n+\frac{(3\delta^3-3\delta^2-\delta+3)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n,
\end{aligned}
$$

$$
\begin{aligned}
S_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^{n+2}+\frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^{n+2} \\
&\quad +\frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^{n+2}+\frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^{n+2},
\end{aligned}
$$

$$
\begin{aligned}
R_n &= \frac{(4\alpha^3-3\alpha^2-2\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n+\frac{(4\beta^3-3\beta^2-2\beta-1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\
&\quad +\frac{(4\gamma^3-3\gamma^2-2\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n+\frac{(4\delta^3-3\delta^2-2\delta-1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n
\end{aligned}
$$

respectively.

The above formulas can be written as follows

$J_n = \frac{1}{5}\beta^n + (-\frac{1}{10} - \frac{3}{10}i)\gamma^n + (-\frac{1}{10} + \frac{3}{10}i)\delta^n,$

$j_n = \frac{1}{2}\alpha^n + \frac{6}{5}\beta^n + (\frac{3}{20} + \frac{9}{20}i)\gamma^n + (\frac{3}{20} - \frac{9}{20}i)\delta^n,$

$K_n = \frac{1}{6}\alpha^n + \frac{17}{15}\beta^n + (\frac{17}{20} + \frac{11}{20}i)\gamma^n + (\frac{17}{20} - \frac{11}{20}i)\delta^n,$

$Q_n = \frac{1}{3}\alpha^n + \frac{13}{15}\beta^n + (\frac{9}{10} + \frac{7}{10}i)\gamma^n + (\frac{9}{10} - \frac{7}{10}i)\delta^n,$

$S_n = -\frac{1}{6}\alpha^n + \frac{4}{15}\beta^n + (-\frac{1}{20} - \frac{3}{20}i)\gamma^n + (-\frac{1}{20} + \frac{3}{20}i)\delta^n,$

$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n.$

We can also find Binet formulas by using matrix method which is given in [45]. Take $k = i = 4$ in Corollary 3.1 in [45]. Let

$$
\Lambda = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^2 & \delta & 1 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^3 & \delta^{n-1} & \delta & 1 \end{pmatrix},
$$

$$
\Lambda_3 = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^3 & \delta^2 & \delta^{n-1} & 1 \end{pmatrix}, \Lambda_4 = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^3 & \beta^2 & \beta & \beta^{n-1} \\ \gamma^3 & \gamma^2 & \gamma & \gamma^{n-1} \\ \delta^3 & \delta^2 & \delta & \delta^{n-1} \end{pmatrix}.
$$

Then the Binet formula for fourth order Jacobsthal numbers is

$$
\begin{aligned}
J_n &= \frac{1}{\det(\Lambda)}\sum_{j=1}^{4} P_{5-j}\det(\Lambda_j) \\
&= \frac{1}{\Lambda}(J_4\det(\Lambda_1)+J_3\det(\Lambda_2)+J_2\det(\Lambda_3)+J_1\det(\Lambda_4)) \\
&= \frac{1}{\det(\Lambda)}(3\det(\Lambda_1)+\det(\Lambda_2)+\det(\Lambda_3)+\det(\Lambda_4)) \\
&= \left(3\begin{vmatrix} \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^2 & \delta & 1 \end{vmatrix} + \begin{vmatrix} \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^3 & \delta^{n-1} & \delta & 1 \end{vmatrix} + \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^3 & \delta^2 & \delta^{n-1} & 1 \end{vmatrix}\right. \\
&\quad \left. + \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^3 & \beta^2 & \beta & \beta^{n-1} \\ \gamma^3 & \gamma^2 & \gamma & \gamma^{n-1} \\ \delta^3 & \delta^2 & \delta & \delta^{n-1} \end{vmatrix}\right) \Big/ \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{vmatrix}.
\end{aligned}
$$

Similarly, we obtain the Binet formula for fourth order Jacobsthal, fourth order Jacobsthal-Lucas, modified fourth order Jacobsthal, fourth order Jacobsthal Perrin, adjusted fourth order Jacobsthal and modified fourth order Jacobsthal-Lucas sequences as

$$
\begin{aligned}
j_n &= \frac{1}{\det\Lambda}(20\det(\Lambda_1)+10\det(\Lambda_2)+5\det(\Lambda_3)+\det(\Lambda_4)), \\
K_n &= \frac{1}{\det\Lambda}(20\det(\Lambda_1)+10\det(\Lambda_2)+3\det(\Lambda_3)+\det(\Lambda_4)), \\
Q_n &= \frac{1}{\det\Lambda}(16\det(\Lambda_1)+8\det(\Lambda_2)+2\det(\Lambda_3)), \\
S_n &= \frac{1}{\det\Lambda}(4\det(\Lambda_1)+2\det(\Lambda_2)+\det(\Lambda_3)+\det(\Lambda_4)), \\
R_n &= \frac{1}{\det\Lambda}(19\det(\Lambda_1)+7\det(\Lambda_2)+3\det(\Lambda_3)+\det(\Lambda_4)),
\end{aligned}
$$

respectively.

4.1.3 Simson Formulas

Now, the following theorem gives the generalization of Simson formula of generalized Tetranacci sequence $\{W_n\}$.

Theorem 4.3 (Simson Formula of Generalized Tetranacci Numbers) *For all in-*

tegers n we have

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n r_4^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{4.21}$$

For the proof of (4.21) see [64]. ■

A special case of the above theorem is the following theorem which gives Simson formula of the generalized fourth order Jacobsthal sequence $\{V_n\}$.

Theorem 4.4 (Simson Formula of Generalized Fourth Order Jacobsthal Numbers) *For all integers n we have*

$$\begin{vmatrix} V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} \end{vmatrix} = (-1)^n 2^n \begin{vmatrix} V_3 & V_2 & V_1 & V_0 \\ V_2 & V_1 & V_0 & V_{-1} \\ V_1 & V_0 & V_{-1} & V_{-2} \\ V_0 & V_{-1} & V_{-2} & V_{-3} \end{vmatrix}.$$

The previous theorem gives the following results as particular examples.

Corollary 4.3 *Simson formula of fourth order Jacobsthal sequence, fourth order Jacobsthal-Lucas sequence, modified fourth order Jacobsthal sequence, fourth order Jacobsthal Perrin sequence, adjusted fourth order Jacobsthal sequence and modified fourth order Jacobsthal-Lucas sequence are given as*

$$\begin{vmatrix} J_{n+3} & J_{n+2} & J_{n+1} & J_n \\ J_{n+2} & J_{n+1} & J_n & J_{n-1} \\ J_{n+1} & J_n & J_{n-1} & J_{n-2} \\ J_n & J_{n-1} & J_{n-2} & J_{n-3} \end{vmatrix} = 0,$$

$$\begin{vmatrix} j_{n+3} & j_{n+2} & j_{n+1} & j_n \\ j_{n+2} & j_{n+1} & j_n & j_{n-1} \\ j_{n+1} & j_n & j_{n-1} & j_{n-2} \\ j_n & j_{n-1} & j_{n-2} & j_{n-3} \end{vmatrix} = \frac{243}{4}(-1)^n 2^n,$$

$$\begin{vmatrix} K_{n+3} & K_{n+2} & K_{n+1} & K_n \\ K_{n+2} & K_{n+1} & K_n & K_{n-1} \\ K_{n+1} & K_n & K_{n-1} & K_{n-2} \\ K_n & K_{n-1} & K_{n-2} & K_{n-3} \end{vmatrix} = \frac{697}{8}(-1)^n 2^n,$$

$$\begin{vmatrix} Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \end{vmatrix} = 169(-1)^n 2^n,$$

$$\begin{vmatrix} S_{n+3} & S_{n+2} & S_{n+1} & S_n \\ S_{n+2} & S_{n+1} & S_n & S_{n-1} \\ S_{n+1} & S_n & S_{n-1} & S_{n-2} \\ S_n & S_{n-1} & S_{n-2} & S_{n-3} \end{vmatrix} = (-1)^{n+1} 2^{n-1},$$

$$\begin{vmatrix} R_{n+3} & R_{n+2} & R_{n+1} & R_n \\ R_{n+2} & R_{n+1} & R_n & R_{n-1} \\ R_{n+1} & R_n & R_{n-1} & R_{n-2} \\ R_n & R_{n-1} & R_{n-2} & R_{n-3} \end{vmatrix} = 450(-1)^n 2^n,$$

respectively.

4.1.4 Some Identities

In this subsection, we obtain some identities of fourth order Jacobsthal, fourth order Jacobsthal-Lucas, modified fourth order Jacobsthal, fourth order Jacobsthal Perrin, adjusted fourth order Jacobsthal and modified fourth order Jacobsthal-Lucas numbers. Next, we give some fundamental relations between $\{J_n\}$ and $\{j_n\}$.

Lemma 4.5 *The following equalities are true:*

$$\begin{aligned} 36J_n &= -7j_{n+4} + 5j_{n+3} + 17j_{n+2} + 5j_{n+1}, \qquad (4.22) \\ 18J_n &= -j_{n+3} + 5j_{n+2} - j_{n+1} - 7j_n, \\ 9J_n &= 2j_{n+2} - j_{n+1} - 4j_n - j_{n-1}, \\ 9J_n &= j_{n+1} - 2j_n + j_{n-1} + 4j_{n-2}, \\ 9J_n &= -j_n + 2j_{n-1} + 5j_{n-2} + 2j_{n-3} \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We only prove (4.22). If we write

$J_n = a \times j_{n+4} + b \times j_{n+3} + c \times j_{n+2} + d \times j_{n+1}$

and solve the system of equations

$$\begin{aligned}
J_0 &= a \times j_4 + b \times j_3 + c \times j_2 + d \times j_1 \\
J_1 &= a \times j_5 + b \times j_4 + c \times j_3 + d \times j_2 \\
J_2 &= a \times j_6 + b \times j_5 + c \times j_4 + d \times j_3 \\
J_3 &= a \times j_7 + b \times j_6 + c \times j_5 + d \times j_4
\end{aligned}$$

then we get $a = -\frac{7}{36}$, $b = \frac{5}{36}$, $c = \frac{17}{36}$, $d = \frac{5}{36}$. The other equalities can be proved similarly.

■

Note that all the identities in the above Lemma can be proved by induction as well.

We present some fundamental relations between $\{J_n\}$ and $\{K_n\}$.

Lemma 4.6 *The following equalities are true:*

$$\begin{aligned}
1394J_n &= -121K_{n+4} + 185K_{n+3} + 219K_{n+2} - 87K_{n+1} \\
697J_n &= 32K_{n+3} + 49K_{n+2} - 104K_{n+1} - 121K_n, \\
697J_n &= 81K_{n+2} - 72K_{n+1} - 89K_n + 64K_{n-1}, \\
697J_n &= 9K_{n+1} - 8K_n + 145K_{n-1} + 162K_{n-2}, \\
697J_n &= K_n + 154K_{n-1} + 171K_{n-2} + 18K_{n-3}.
\end{aligned}$$

We give some fundamental relations between $\{J_n\}$ and $\{Q_n\}$

Lemma 4.7 *The following equalities are true:*

$$\begin{aligned}
26J_n &= -2Q_{n+4} + 3Q_{n+3} + 4Q_{n+2} - Q_{n+1}, \\
26J_n &= Q_{n+3} + 2Q_{n+2} - 3Q_{n+1} - 4Q_n, \\
26J_n &= 3Q_{n+2} - 2Q_{n+1} - 3Q_n + 2Q_{n-1}, \\
26J_n &= Q_{n+1} + 5Q_{n-1} + 6Q_{n-2}, \\
26J_n &= Q_n + 6Q_{n-1} + 7Q_{n-2} + 2Q_{n-3}.
\end{aligned}$$

We present some fundamental relations between $\{J_n\}$ and $\{S_n\}$.

Lemma 4.8 *The following equalities are true:*

$$\begin{aligned}
8J_n &= 5S_{n+4} - 3S_{n+3} - 11S_{n+2} - 3S_{n+1}, \\
4J_n &= S_{n+3} - 3S_{n+2} + S_{n+1} + 5S_n, \\
2J_n &= -S_{n+2} + S_{n+1} + 3S_n + S_{n-1}, \\
J_n &= S_n - S_{n-2}.
\end{aligned}$$

We give some fundamental relations between $\{J_n\}$ and $\{R_n\}$

Lemma 4.9 *The following equalities are true:*

$$\begin{aligned}
150J_n &= -8R_{n+4} + 22R_{n+3} + 7R_{n+2} - 23R_{n+1}, \\
150J_n &= 14R_{n+3} - R_{n+2} - 31R_{n+1} - 16R_n, \\
150J_n &= 13R_{n+2} - 17R_{n+1} - 2R_n + 28R_{n-1}, \\
150J_n &= -4R_{n+1} + 11R_n + 41R_{n-1} + 26R_{n-2}, \\
150J_n &= 7R_n + 37R_{n-1} + 22R_{n-2} - 8R_{n-3}.
\end{aligned}$$

We present some fundamental relations between $\{K_n\}$ and $\{j_n\}$.

Lemma 4.10 *The following equalities are true:*

$$\begin{aligned}
697j_n &= 470K_{n+4} - 805K_{n+3} + 215K_{n+2} - 601K_{n+1}, \\
697j_n &= -335K_{n+3} + 685K_{n+2} - 131K_{n+1} + 940K_n - 10K_{n-1}, \\
697j_n &= 350K_{n+2} - 466K_{n+1} + 605K_n - 670K_{n-1}, \\
697j_n &= -116K_{n+1} + 955K_n - 320K_{n-1} + 700K_{n-2}, \\
697j_n &= 839K_n - 436K_{n-1} + 584K_{n-2} - 232K_{n-3}
\end{aligned}$$

and

$$\begin{aligned}
24K_n &= -3j_{n+4} - 11j_{n+3} + 571j_{n+2} - 431j_{n+1}, \\
54K_n &= 31j_{n+3} - 41j_{n+2} - 41j_{n+1} + 49j_n, \\
27K_n &= -5j_{n+2} - 5j_{n+1} + 40j_n + 31j_{n-1}, \\
27K_n &= -10j_{n+1} + 35j_n + 26j_{n-1} - 10j_{n-2}, \\
27K_n &= 25j_n + 16j_{n-1} - 20j_{n-2} - 20j_{n-3}.
\end{aligned}$$

We give some fundamental relations between $\{j_n\}$ and $\{Q_n\}$.

Lemma 4.11 *The following equalities are true:*

$$\begin{aligned}
26j_n &= 11Q_{n+4} - 16Q_{n+3} + 2 \times Q_{n+2} - 10Q_{n+1}, \\
26j_n &= 11Q_{n+4} - 16Q_{n+3} + 2Q_{n+2} - 10Q_{n+1}, \\
26j_n &= -5Q_{n+3} + 13Q_{n+2} + Q_{n+1} + 22Q_n, \\
26j_n &= 8Q_{n+2} - 4Q_{n+1} + 17Q_n - 10Q_{n-1}, \\
26j_n &= 4Q_{n+1} + 25Q_n - 2Q_{n-1} + 16Q_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
108Q_n &= 65j_{n+4} - 7j_{n+3} - 151j_{n+2} - 151j_{n+1}, \\
54Q_n &= 29j_{n+3} - 43j_{n+2} - 43j_{n+1} + 65j_n, \\
54Q_n &= 29j_{n+3} - 43j_{n+2} - 43j_{n+1} + 65j_n, \\
27Q_n &= -7j_{n+2} - 7j_{n+1} + 47j_n + 29j_{n-1}, \\
27Q_n &= -14j_{n+1} + 40j_n + 22j_{n-1} - 14j_{n-2}.
\end{aligned}$$

We present some fundamental relations between $\{j_n\}$ and $\{S_n\}$.

Lemma 4.12 *The following equalities are true:*

$$\begin{aligned}
4j_n &= -5S_{n+4} + 7S_{n+3} + 7S_{n+2} + 7S_{n+1} \\
2j_n &= S_{n+3} + S_{n+2} + S_{n+1} - 5S_n \\
j_n &= S_{n+2} + S_{n+1} - 2S_n + S_{n-1} \\
j_n &= 2S_{n+1} - S_n + 2S_{n-1} + 2S_{n-2} \\
j_n &= S_n + 4S_{n-1} + 4S_{n-2} + 4S_{n-3}
\end{aligned}$$

and

$$\begin{aligned}
27S_n &= -4j_{n+4} + 5j_{n+3} + 5j_{n+2} + 5j_{n+1} \\
27S_n &= j_{n+3} + j_{n+2} + j_{n+1} - 8j_n \\
27S_n &= 2j_{n+2} + 2j_{n+1} - 7j_n + 2j_{n-1} \\
27S_n &= 4j_{n+1} - 5j_n + 4j_{n-1} + 4j_{n-2} \\
27S_n &= -j_n + 8j_{n-1} + 8j_{n-2} + 8j_{n-3}
\end{aligned}$$

We give some fundamental relations between $\{j_n\}$ and $\{R_n\}$.

Lemma 4.13 *The following equalities are true:*

$$\begin{aligned}
300j_n &= 64R_{n+4} - 101R_{n+3} + 19R_{n+2} + 34R_{n+1}, \\
300j_n &= -37R_{n+3} + 83R_{n+2} + 98R_{n+1} + 128R_n, \\
300j_n &= 46R_{n+2} + 61R_{n+1} + 91R_n - 74R_{n-1}, \\
300j_n &= 107R_{n+1} + 137R_n - 28R_{n-1} + 92R_{n-2}, \\
300j_n &= 244R_n + 79R_{n-1} + 199R_{n-2} + 214R_{n-3},
\end{aligned}$$

and

$$\begin{aligned}
36R_n &= 13j_{n+4} + j_{n+3} - 11j_{n+2} - 71j_{n+1}, \\
18R_n &= 7j_{n+3} + j_{n+2} - 29j_{n+1} + 13j_n, \\
9R_n &= 4j_{n+2} - 11j_{n+1} + 10j_n + 7j_{n-1}, \\
9R_n &= -7j_{n+1} + 14j_n + 11j_{n-1} + 8j_{n-2}, \\
9R_n &= 7 \times j_n + 4j_{n-1} + j_{n-2} - 14j_{n-3}.
\end{aligned}$$

We give some fundamental relations between $\{K_n\}$ and $\{Q_n\}$.

Lemma 4.14 *The following equalities are true:*

$$\begin{aligned}
26K_n &= 10Q_{n+4} - 7Q_{n+3} - 13Q_{n+2} - 9Q_{n+1}, \\
26K_n &= 3Q_{n+3} - 3Q_{n+2} + Q_{n+1} + 2Q_n, \\
26K_n &= 4Q_{n+1} + 23Q_n + 6Q_{n-1}, \\
26K_n &= 27Q_n + 10Q_{n-1} + 4Q_{n-2} + 8Q_{n-3},
\end{aligned}$$

and

$$\begin{aligned}
1394Q_n &= 983K_{n+4} - 1261K_{n+3} - 581K_{n+2} - 1125K_{n+1}, \\
697Q_n &= -139K_{n+3} + 201K_{n+2} - 71K_{n+1} + 983K_n, \\
697Q_n &= 62K_{n+2} - 210K_{n+1} + 844K_n - 278K_{n-1}, \\
697Q_n &= -148K_{n+1} + 906K_n - 216K_{n-1} + 124K_{n-2}, \\
697Q_n &= 758K_n - 364K_{n-1} - 24K_{n-2} - 296K_{n-3}.
\end{aligned}$$

We present some fundamental relations between $\{S_n\}$ and $\{R_n\}$.

Lemma 4.15 *The following equalities are true:*

$$\begin{aligned}
900S_n &= -44R_{n+4} + 121R_{n+3} + R_{n+2} - 14R_{n+1}, \\
900S_n &= 77R_{n+3} - 43R_{n+2} - 58R_{n+1} - 88R_n, \\
900S_n &= 34R_{n+2} + 19R_{n+1} - 11R_n + 154R_{n-1}, \\
900S_n &= 53R_{n+1} + 23R_n + 188R_{n-1} + 68R_{n-2}, \\
900S_n &= 76R_n + 241R_{n-1} + 121R_{n-2} + 106R_{n-3},
\end{aligned}$$

and

$$\begin{aligned}
8R_n &= -7S_{n+4} + S_{n+3} + 9S_{n+2} + 49S_{n+1}, \\
4R_n &= -3S_{n+3} + S_{n+2} + 21S_{n+1} - 7S_n, \\
2R_n &= -S_{n+2} + 9S_{n+1} - 5S_n - 3S_{n-1}, \\
R_n &= 4S_{n+1} - 3S_n - 2S_{n-1} - S_{n-2}, \\
R_n &= S_n + 2S_{n-1} + 3S_{n-2} + 8S_{n-3}.
\end{aligned}$$

We now present a few special identities for the fourth order Jacobsthal sequence J_n.

Theorem 4.16 *(Catalan's identity) For all natural numbers n and m, the following identity holds*

$$J_{n+m}J_{n-m} - J_n^2 = (S_{n+m} - S_{n+m-2})(S_{n-m} - S_{n-m-2}) - (S_n - S_{n-2})^2.$$

Proof. We use the identity

$$J_n = S_n - S_{n-2}.$$

■

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified fourth order Jacobsthal sequence.

Corollary 4.4 *(Cassini's identity) For all natural numbers n and m, the following identity holds*

$$J_{n+1}J_{n-1} - J_n^2 = (S_{n+1} - S_{n-1})(S_{n-1} - S_{n-3}) - (S_n - S_{n-2})^2.$$

The d'Ocagne, Gelin-Cesàro and Melham identities can also be obtained by using $J_n = S_n - S_{n-2}$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified fourth order Jacobsthal sequence $\{J_n\}$.

Theorem 4.17 *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne identity)*

$$J_{m+1}J_n - J_mJ_{n+1} = (S_{m+1} - S_{m-1})(S_n - S_{n-2}) - (S_m - S_{m-2})(S_{n+1} - S_{n-1}).$$

(b) *(Gelin-Cesàro identity)*

$$J_{n+2}J_{n+1}J_{n-1}J_{n-2} - J_n^4 = (S_{n+2} - S_n)(S_{n+1} - S_{n-1})(S_{n-1} - S_{n-3})(S_{n-2} - S_{n-4}) - (S_n - S_{n-2})^4.$$

(c) *(Melham identity)*

$$J_{n+1}J_{n+2}J_{n+6} - J_{n+3}^3 = (S_{n+1} - S_{n-1})(S_{n+2} - S_n)(S_{n+6} - S_{n+4}) - (S_{n+3} - S_{n+1})^3.$$

Proof. Use the identity $J_n = S_n - S_{n-2}$. ∎

4.1.5 Linear Sums

In this subsection, presents summing formulas of generalized fourth order Jacobsthal numbers.

Proposition 4.18 *For $n \geq 0$, we have the following formula:*

$$\sum_{k=0}^{n} W_k = \frac{1}{4}(W_{n+4} - W_{n+2} - 2W_{n+1} - W_3 + W_1 + 2W_0).$$

Proof. See [66].

Corollary 4.5 *We have the following properties:*

(a) $\sum_{k=0}^{n} J_k = \frac{1}{4}(J_{n+4} - J_{n+2} - 2J_{n+1})$.

(b) $\sum_{k=0}^{n} j_k = \frac{1}{4}(j_{n+4} - j_{n+2} - 2j_{n+1} - 5)$.

(c) $\sum_{k=0}^{n} K_k = \frac{1}{4}(K_{n+4} - K_{n+2} - 2K_{n+1} - 3)$.

(d) $\sum_{k=0}^{n} Q_k = \frac{1}{4}(Q_{n+4} - Q_{n+2} - 2Q_{n+1} - 2)$.

(e) $\sum_{k=0}^{n} S_k = \frac{1}{4}(S_{n+4} - S_{n+2} - 2S_{n+1} - 1)$.

(f) $\sum_{k=0}^{n} R_k = \frac{1}{4}(R_{n+4} - R_{n+2} - 2R_{n+1} + 2)$.

Proof.

(a) Take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$ in the last proposition.

(b) Take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$ in the last proposition.

(c) Take $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10$ in the last proposition.

(d) Take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8$ in the last proposition.

(e) Take $S_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2$ in the last proposition.

(f) Take $S_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the last proposition.

The following proposition presents summation formulas of generalized fourth order Jacobsthal numbers with negative subscripts.

Proposition 4.19 *For $n \geq 1$, we have the following formula:*

$$\sum_{k=1}^{n} W_{-k} = \frac{1}{4}(-W_{-n+3} + W_{-n+1} + 2W_{-n} + W_3 - W_1 - 2W_0).$$

Proof. See [66].

Corollary 4.6 *We have the following properties:*

(a) $\sum_{k=1}^{n} J_{-k} = \frac{1}{4}(-J_{-n+3} + J_{-n+1} + 2J_{-n})$.

(b) $\sum_{k=1}^{n} j_{-k} = \frac{1}{4}(-j_{-n+3} + j_{-n+1} + 2j_{-n} + 5)$.

(c) $\sum_{k=1}^{n} K_{-k} = \frac{1}{4}(-K_{-n+3} + K_{-n+1} + 2K_{-n} + 3)$.

(d) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{4}(-Q_{-n+3} + Q_{-n+1} + 2Q_{-n} + 2)$.

(e) $\sum_{k=1}^{n} S_{-k} = \frac{1}{4}(-S_{-n+3} + S_{-n+1} + 2S_{-n} + 1)$.

(f) $\sum_{k=1}^{n} R_{-k} = \frac{1}{4}(-R_{-n+3} + R_{-n+1} + 2R_{-n} - 2)$.

4.1.6 Matrices Related with Generalized Fourth Order Jacobsthal numbers

In this subsection, we consider the following the square matrix A of order 4 as:

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -2$. From (4.2) we have

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{4.23}$$

using (4.23) and from induction we get

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = S$ in (4.23), then we have

$$\begin{pmatrix} S_{n+3} \\ S_{n+2} \\ S_{n+1} \\ S_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} S_{n+2} \\ S_{n+1} \\ S_n \\ S_{n-1} \end{pmatrix}. \tag{4.24}$$

We consider the following matrices:

$$B_n = \begin{pmatrix} S_{n+1} & S_n + S_{n-1} + 2S_{n-2} & S_n + 2S_{n-1} & 2S_n \\ S_n & S_{n-1} + S_{n-2} + 2S_{n-3} & S_{n-1} + 2S_{n-2} & 2S_{n-1} \\ S_{n-1} & S_{n-2} + S_{n-3} + 2S_{n-4} & S_{n-2} + 2S_{n-3} & 2S_{n-2} \\ S_{n-2} & S_{n-3} + S_{n-4} + 2S_{n-5} & S_{n-3} + 2S_{n-4} & 2S_{n-3} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} + 2V_{n-2} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + V_{n-2} + 2V_{n-3} & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + 2V_{n-4} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + 2V_{n-5} & V_{n-3} + 2V_{n-4} & 2V_{n-3} \end{pmatrix}.$$

Theorem 4.20 *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (4.24) to 4-colums and multiplying the obtained on the right-hand side by A, we get

$B_n = AB_{n-1}$.

By induction argument, from the last equation, we obtain

$B_n = A^{n-1} B_1$.

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$\begin{aligned}
AC_{n-1} &= \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & V_{n-1}+V_{n-2}+2V_{n-3} & V_{n-1}+2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2}+V_{n-3}+2V_{n-4} & V_{n-2}+2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3}+V_{n-4}+2V_{n-5} & V_{n-3}+2V_{n-4} & 2V_{n-3} \\ V_{n-3} & V_{n-4}+V_{n-5}+2V_{n-6} & V_{n-4}+2V_{n-5} & 2V_{n-4} \end{pmatrix} \\
&= \begin{pmatrix} V_{n+1} & V_n+V_{n-1}+2V_{n-2} & V_n+2V_{n-1} & 2V_n \\ V_n & V_{n-1}+V_{n-2}+2V_{n-3} & V_{n-1}+2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2}+V_{n-3}+2V_{n-4} & V_{n-2}+2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3}+V_{n-4}+2V_{n-5} & V_{n-3}+2V_{n-4} & 2V_{n-3} \end{pmatrix} = C_n.
\end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1} C_1$. Thus

$C_{n+m} = A^{n+m-1} C_1 = A^{n-1} A^m C_1 = A^{n-1} C_1 A^m = C_n B_m$

and similarly

$$C_{n+m} = B_m C_n.$$

■

Note that the matrix A^n satisfies the following equalities:

$$A^n = A^{n-1} + A^{n-2} + A^{n-3} + 2A^{n-4}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

for all integer m and n.

Theorem 4.21 *For $m, n \geq 0$ we have*

$$V_{n+m} = V_n S_{m+1} + (V_{n-1} + V_{n-2} + 2V_{n-3}) S_m + (V_{n-1} + 2V_{n-2}) S_{m-1} + 2V_{n-1} S_{m-2}. \quad (4.25)$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m.We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. ■

Remark 4.1 *By induction, it can be proved that for all integers $m, n \leq 0$, (4.25) holds. So for all integers m, n (4.25) is true.*

Corollary 4.7 *For all integers m, n, we have*

$$\begin{aligned}
J_{n+m} &= J_n S_{m+1} + (J_{n-1} + J_{n-2} + 2J_{n-3}) S_m + (J_{n-1} + 2J_{n-2}) S_{m-1} + 2J_{n-1} S_{m-2}, \\
j_{n+m} &= j_n S_{m+1} + (j_{n-1} + j_{n-2} + 2j_{n-3}) S_m + (j_{n-1} + 2j_{n-2}) S_{m-1} + 2j_{n-1} S_{m-2}, \\
K_{n+m} &= K_n S_{m+1} + (K_{n-1} + K_{n-2} + 2K_{n-3}) S_m + (K_{n-1} + 2K_{n-2}) S_{m-1} + 2K_{n-1} S_{m-2}, \\
Q_{n+m} &= Q_n S_{m+1} + (Q_{n-1} + Q_{n-2} + 2Q_{n-3}) S_m + (Q_{n-1} + 2Q_{n-2}) S_{m-1} + 2Q_{n-1} S_{m-2}, \\
S_{n+m} &= S_n S_{m+1} + (S_{n-1} + S_{n-2} + 2S_{n-3}) S_m + (S_{n-1} + 2S_{n-2}) S_{m-1} + 2S_{n-1} S_{m-2}, \\
R_{n+m} &= R_n S_{m+1} + (R_{n-1} + R_{n-2} + 2R_{n-3}) S_m + (R_{n-1} + 2R_{n-2}) S_{m-1} + 2R_{n-1} S_{m-2}.
\end{aligned}$$

CHAPTER 5

GENERALIZED FIFTH ORDER JACOBSTHAL NUMBERS

In this chapter, we introduce the generalized fifth order Jacobsthal sequences and we investigate, in detail, six special cases which we call them fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas sequences. We state that the results of this chapter are cited from [79] which was published by Soykan and Eyican Polatli.

5.1 INTRODUCTION

In this section, we give basic information about generalized fifth order Jacobsthal numbers. First, we recall the definition and some properties of generalized Pentanacci sequence.

The generalized Pentanacci sequence (or the generalized (r, s, t, u, v) sequence or 5-step Fibonacci sequence)

$$\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$$

is defined by the fifth order recurrence relations

$$\begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \qquad (5.1)\\ W_0 &= a, W_1 = b, W_2 = c, W_3 = d, W_4 = e \end{aligned}$$

where the initial values W_0, W_1, W_2, W_3, W_4 are arbitrary complex (or real) numbers and r, s, t, u, v are real numbers. Generalized Pentanacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [52, 54, 56, 76]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-(n-1)} - \frac{t}{v}W_{-(n-2)} - \frac{s}{v}W_{-(n-3)} - \frac{r}{v}W_{-(n-4)} + \frac{1}{v}W_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (5.1) holds for all integer n.

As $\{W_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5 - rx^4 - sx^3 - tx^2 - ux - v = 0 \tag{5.2}$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda$. Note that we have the following identities:

$$\alpha + \beta + \gamma + \delta + \lambda = r,$$

$$\alpha\beta + \alpha\lambda + \alpha\gamma + \beta\lambda + \alpha\delta + \beta\gamma + \lambda\gamma + \beta\delta + \lambda\delta + \gamma\delta = -s,$$

$$\alpha\beta\lambda + \alpha\beta\gamma + \alpha\lambda\gamma + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\gamma + \alpha\gamma\delta + \beta\lambda\delta + \beta\gamma\delta + \lambda\gamma\delta = t,$$

$$\alpha\beta\lambda\gamma + \alpha\beta\lambda\delta + \alpha\beta\gamma\delta + \alpha\lambda\gamma\delta + \beta\lambda\gamma\delta = -u$$

$$\alpha\beta\gamma\delta\lambda = v.$$

Generalized Pentanacci numbers can be expressed, for all integers n, using Binet formula.

Theorem 5.1 *[76] (Binet formula of generalized (r, s, t, u, v) numbers (generalized Pentanacci numbers))*

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad + \frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} p_1 &= W_4 - (\beta+\gamma+\delta+\lambda)W_3 + (\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_2 \\ &\quad -(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_1 + (\beta\lambda\gamma\delta)W_0, \\ p_2 &= W_4 - (\alpha+\gamma+\delta+\lambda)W_3 + (\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)W_2 \\ &\quad -(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)W_1 + (\alpha\lambda\gamma\delta)W_0, \\ p_3 &= W_4 - (\alpha+\beta+\delta+\lambda)W_3 + (\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)W_2 \\ &\quad -(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)W_1 + (\alpha\beta\lambda\delta)W_0, \\ p_4 &= W_4 - (\alpha+\beta+\gamma+\lambda)W_3 + (\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)W_2 \\ &\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)W_1 + (\alpha\beta\lambda\gamma)W_0, \\ p_5 &= W_4 - (\alpha+\beta+\gamma+\delta)W_3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)W_2 \\ &\quad -(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)W_1 + (\alpha\beta\gamma\delta)W_0. \end{aligned}$$

Usually, it is customary to choose r, s, t, u, v so that the Equ. (5.2) has at least one real (say α) solutions.

Note that the Binet formula of a sequence satisfying (5.2) for non-negative integers is valid for all integers n, (see [42], this result of Howard and Saidak [42] is even true in the case of higher order recurrence relations).

Next, we give the ordinary generating function of the sequence W_n.

Lemma 5.2 *[76] Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r, s, t, u, v) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{\Omega}{1 - rx - sx^2 - tx^3 - ux^4 - vx^5} \tag{5.4}$$

where

$$\begin{aligned}\Omega &= W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 \\ &\quad + (W_3 - rW_2 - sW_1 - tW_0)x^3 + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)x^4.\end{aligned}$$

We next give Binet formula of generalized (r, s, t, u, v) numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 5.3 *[76] (Binet formula of generalized (r, s, t, u, v) numbers)*

$$\begin{aligned}W_n &= \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad + \frac{q_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\end{aligned} \tag{5.5}$$

where

$$
\begin{aligned}
q_1 &= W_0\alpha^4+(W_1-rW_0)\alpha^3+(W_2-rW_1-sW_0)\alpha^2 \\
&\quad +(W_3-rW_2-sW_1-tW_0)\alpha+(W_4-rW_3-sW_2-tW_1-vW_0), \\
q_2 &= W_0\beta^4+(W_1-rW_0)\beta^3+(W_2-rW_1-sW_0)\beta^2 \\
&\quad +(W_3-rW_2-sW_1-tW_0)\beta+(W_4-rW_3-sW_2-tW_1-vW_0), \\
q_3 &= W_0\gamma^4+(W_1-rW_0)\gamma^3+(W_2-rW_1-sW_0)\gamma^2 \\
&\quad +(W_3-rW_2-sW_1-tW_0)\gamma+(W_4-rW_3-sW_2-tW_1-vW_0), \\
q_4 &= W_0\delta^4+(W_1-rW_0)\delta^3+(W_2-rW_1-sW_0)\delta^2 \\
&\quad +(W_3-rW_2-sW_1-tW_0)\delta+(W_4-rW_3-s3W_2-tW_1-vW_0), \\
q_5 &= W_0\lambda^4+(W_1-rW_0)\lambda^3+(W_2-rW_1-sW_0)\lambda^2 \\
&\quad +(W_3-rW_2-sW_1-tW_0)\lambda+(W_4-rW_3-sW_2-tW_1-vW_0).
\end{aligned}
$$

In this paper we consider the case $r=s=t=u=1, v=2$ and in this case we write $V_n = W_n$. A generalized fifth order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0,V_1,V_2,V_3,V_4)\}_{n\geq 0}$ is defined by the fifth order recurrence relations

$$V_n = V_{n-1}+V_{n-2}+V_{n-3}+V_{n-4}+2V_{n-5} \tag{5.6}$$

with the initial values $V_0=c_0, V_1=c_1, V_2=c_2$, $V_3=c_3$, $V_4=c_4$ not all being zero.
The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)}-\frac{1}{2}V_{-(n-2)}-\frac{1}{2}V_{-(n-3)}-\frac{1}{2}V_{-(n-4)}+\frac{1}{2}V_{-(n-5)}$$

for $n=1,2,3,....$ Therefore, recurrence (5.6) holds for all integer n.
As $\{V_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5-x^4-x^3-x^2-x-2=(x-2)(x^4+x^3+x^2+x+1)=0. \tag{5.7}$$

The roots α, β, γ, δ and λ of Equation (5.7) are given by:

$\alpha = 2,$

$\beta = \frac{1}{4}(\sqrt{5}-1)+\frac{\sqrt{2\sqrt{5}+10}}{4}i,$

$\gamma = \frac{1}{4}(\sqrt{5}-1)-\frac{\sqrt{2\sqrt{5}+10}}{4}i,$

$$\delta = -\frac{1}{4}(\sqrt{5}+1) + \frac{\sqrt{-2\sqrt{5}+10}}{4}i,$$

$$\lambda = -\frac{1}{4}(\sqrt{5}+1) - \frac{\sqrt{-2\sqrt{5}+10}}{4}i.$$

Here α, β, γ, δ and λ satisfy the following equalities:

$$\alpha+\beta+\gamma+\delta+\lambda = 1,$$

$$\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta = -1,$$

$$\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta = 1,$$

$$\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta = -1,$$

$$\alpha\beta\gamma\delta\lambda = 2.$$

The first few generalized fifth order Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 5.1.

Table 5.1. The first few terms of generalized fifth order Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$\frac{1}{2}V_4-\frac{1}{2}V_1-\frac{1}{2}V_2-\frac{1}{2}V_3-\frac{1}{2}V_0$
2	V_2	$\frac{3}{4}V_3-\frac{1}{4}V_1-\frac{1}{4}V_2-\frac{1}{4}V_0-\frac{1}{4}V_4$
3	V_3	$\frac{7}{8}V_2-\frac{1}{8}V_1-\frac{1}{8}V_0-\frac{1}{8}V_3-\frac{1}{8}V_4$
4	V_4	$\frac{15}{16}V_1-\frac{1}{16}V_0-\frac{1}{16}V_2-\frac{1}{16}V_3-\frac{1}{16}V_4$
5	$2V_0+V_1+V_2+V_3+V_4$	$\frac{31}{32}V_0-\frac{1}{32}V_1-\frac{1}{32}V_2-\frac{1}{32}V_3-\frac{1}{32}V_4$
6	$2V_0+3V_1+2V_2+2V_3+2V_4$	$\frac{31}{64}V_4-\frac{33}{64}V_1-\frac{33}{64}V_2-\frac{33}{64}V_3-\frac{33}{64}V_0$
7	$4V_0+4V_1+5V_2+4V_3+4V_4$	$\frac{95}{128}V_3-\frac{33}{128}V_1-\frac{33}{128}V_2-\frac{33}{128}V_0-\frac{33}{128}V_4$
8	$8V_0+8V_1+8V_2+9V_3+8V_4$	$\frac{223}{256}V_2-\frac{33}{256}V_1-\frac{33}{256}V_0-\frac{33}{256}V_3-\frac{33}{256}V_4$
9	$16V_0+16V_1+16V_2+16V_3+17V_4$	$\frac{479}{512}V_1-\frac{33}{512}V_0-\frac{33}{512}V_2-\frac{33}{512}V_3-\frac{33}{512}V_4$
10	$34V_0+33V_1+33V_2+33V_3+33V_4$	$\frac{991}{1024}V_0-\frac{33}{1024}V_1-\frac{33}{1024}V_2-\frac{33}{1024}V_3-\frac{33}{1024}V_4$
11	$66V_0+67V_1+66V_2+66V_3+66V_4$	$\frac{991}{2048}V_4-\frac{1057}{2048}V_1-\frac{1057}{2048}V_2-\frac{1057}{2048}V_3-\frac{1057}{2048}V_0$
12	$132V_0+132V_1+133V_2+132V_3+132V_4$	$\frac{3039}{4096}V_3-\frac{1057}{4096}V_1-\frac{1057}{4096}V_2-\frac{1057}{4096}V_0-\frac{1057}{4096}V_4$
13	$264V_0+264V_1+264V_2+265V_3+264V_4$	$\frac{7135}{8192}V_2-\frac{1057}{8192}V_1-\frac{1057}{8192}V_0-\frac{1057}{8192}V_3-\frac{1057}{8192}V_4$

Now we define six special cases of the sequence $\{V_n\}$. Fifth order Jacobsthal sequence $\{J_n^{(5)}\}_{n\geq0}$, fifth order Jacobsthal-Lucas sequence $\{j_n^{(5)}\}_{n\geq0}$, modified fifth order Jacobsthal sequence $\{K_n^{(5)}\}_{n\geq 0}$, fifth order Jacobsthal Perrin sequence $\{Q_n^{(5)}\}_{n\geq 0}$, adjusted fifth order Jacobsthal sequence $\{S_n^{(5)}\}_{n\geq 0}$ and modified fifth order Jacobsthal-Lucas sequence $\{R_n^{(5)}\}_{n\geq 0}$ are defined, respectively, by the fifth order recurrence relations

$$\begin{aligned} J_{n+5}^{(5)} &= J_{n+4}^{(5)} + J_{n+3}^{(5)} + J_{n+2}^{(5)} + J_{n+1}^{(5)} + 2J_n^{(5)}, \\ J_0^{(5)} &= 0, J_1^{(5)} = 1, J_2^{(5)} = 1, J_3^{(5)} = 1, J_4^{(5)} = 1, \end{aligned} \tag{5.8}$$

$$\begin{aligned} j_{n+5}^{(5)} &= j_{n+4}^{(5)} + j_{n+3}^{(5)} + j_{n+2}^{(5)} + j_{n+1}^{(5)} + 2j_n^{(5)}, \\ j_0^{(5)} &= 2, j_1^{(5)} = 1, j_2^{(5)} = 5, j_3^{(5)} = 10, j_4^{(5)} = 20, \end{aligned} \tag{5.9}$$

$$\begin{aligned} K_{n+5}^{(5)} &= K_{n+4}^{(5)} + K_{n+3}^{(5)} + K_{n+2}^{(5)} + K_{n+1}^{(5)} + 2K_n^{(5)}, \\ K_0^{(5)} &= 3, K_1^{(5)} = 1, K_2^{(5)} = 3, K_3^{(5)} = 10, K_4^{(5)} = 20, \end{aligned} \tag{5.10}$$

$$\begin{aligned} Q_{n+5}^{(5)} &= Q_{n+4}^{(5)} + Q_{n+3}^{(5)} + Q_{n+2}^{(5)} + Q_{n+1}^{(5)} + 2Q_n^{(5)}, \\ Q_0^{(5)} &= 3, Q_1^{(5)} = 0, Q_2^{(5)} = 2, Q_3^{(5)} = 8, Q_4^{(5)} = 16, \end{aligned} \tag{5.11}$$

$$\begin{aligned} S_{n+5}^{(5)} &= S_{n+4}^{(5)} + S_{n+3}^{(5)} + S_{n+2}^{(5)} + S_{n+1}^{(5)} + 2S_n^{(5)}, \\ S_0^{(5)} &= 0, S_1^{(5)} = 1, S_2^{(5)} = 1, S_3^{(5)} = 2, S_4^{(5)} = 4, \end{aligned} \tag{5.12}$$

$$\begin{aligned} R_{n+5}^{(5)} &= R_{n+4}^{(5)} + R_{n+3}^{(5)} + R_{n+2}^{(5)} + R_{n+1}^{(5)} + 2R_n^{(5)}, \\ R_0^{(5)} &= 5, R_1^{(5)} = 1, R_2^{(5)} = 3, R_3^{(5)} = 7, R_4^{(5)} = 19. \end{aligned} \tag{5.13}$$

The sequences $\{J_n^{(5)}\}_{n\geq 0}$, $\{j_n^{(5)}\}_{n\geq 0}$,$\{K_n^{(5)}\}_{n\geq 0}, Q_n^{(5)}\}_{n\geq 0}$, $\{S_n^{(5)}\}_{n\geq 0}$ and $\{R_n^{(5)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$J_{-n}^{(5)} = -\frac{1}{2}J_{-(n-1)}^{(5)} - \frac{1}{2}J_{-(n-2)}^{(5)} - \frac{1}{2}J_{-(n-3)}^{(5)} - J_{-(n-4)}^{(5)} + \frac{1}{2}J_{-(n-5)}^{(5)}, \tag{5.14}$$

$$j_{-n}^{(5)} = -\frac{1}{2}j_{-(n-1)}^{(5)} - \frac{1}{2}j_{-(n-2)}^{(5)} - \frac{1}{2}j_{-(n-3)}^{(5)} - \frac{1}{2}j_{-(n-4)}^{(5)} + \frac{1}{2}j_{-(n-5)}^{(5)}, \tag{5.15}$$

$$K_{-n}^{(5)} = -\frac{1}{2}K_{-(n-1)}^{(5)} - \frac{1}{2}K_{-(n-2)}^{(5)} - \frac{1}{2}K_{-(n-3)}^{(5)} - \frac{1}{2}K_{-(n-4)}^{(5)} + \frac{1}{2}K_{-(n-5)}^{(5)}, \tag{5.16}$$

$$Q_{-n}^{(5)} = -\frac{1}{2}Q_{-(n-1)}^{(5)} - \frac{1}{2}Q_{-(n-2)}^{(5)} - \frac{1}{2}Q_{-(n-3)}^{(5)} - \frac{1}{2}Q_{-(n-4)}^{(5)} + \frac{1}{2}Q_{-(n-5)}^{(5)}, \tag{5.17}$$

$$S_{-n}^{(5)} = -\frac{1}{2}S_{-(n-1)}^{(5)} - \frac{1}{2}S_{-(n-2)}^{(5)} - \frac{1}{2}S_{-(n-3)}^{(5)} - \frac{1}{2}S_{-(n-4)}^{(5)} + \frac{1}{2}S_{-(n-5)}^{(5)}, \tag{5.18}$$

$$R_{-n}^{(5)} = -\frac{1}{2}R_{-(n-1)}^{(5)} - \frac{1}{2}R_{-(n-2)}^{(5)} - \frac{1}{2}R_{-(n-3)}^{(5)} - \frac{1}{2}R_{-(n-4)}^{(5)} + \frac{1}{2}R_{-(n-5)}^{(5)}, \tag{5.19}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (5.8)-(5.13) hold for all integer n.

In the rest of the chapter, for easy writing, we drop the superscripts and write J_n, j_n, K_n, Q_n, S_n and R_n for $J_n^{(5)}, j_n^{(5)}$, $K_n^{(5)}, Q_n^{(5)}, S_n^{(5)}$ and $R_n^{(5)}$, respectively. Note that J_n and j_n are the sequences A226310, A226311 in [62], respectively.

Next, we present the first few values of the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers with positive and negative subscripts:

Table 5.2. The first few terms of the J_n, j_n, K_n, Q_n, S_n and R_n

n	0	1	2	3	4	5	6	7	8	9	10
$J_n^{(5)}$	0	1	1	1	1	4	9	17	33	65	132
$J_{-n}^{(5)}$		-1	0	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{8}$	$-\frac{17}{16}$	$-\frac{1}{32}$	$\frac{31}{64}$	$\frac{95}{128}$	$-\frac{33}{256}$
$j_n^{(5)}$	2	1	5	10	20	40	77	157	314	628	1256
$j_{-n}^{(5)}$		1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{11}{8}$	$\frac{13}{16}$	$\frac{13}{32}$	$\frac{13}{64}$	$\frac{13}{128}$	$-\frac{371}{256}$	$\frac{397}{512}$
$K_n^{(5)}$	3	1	3	10	20	40	75	151	306	612	1224
$K_{-n}^{(5)}$		$\frac{3}{2}$	$\frac{3}{4}$	$-\frac{13}{8}$	$-\frac{21}{16}$	$\frac{59}{32}$	$\frac{59}{64}$	$\frac{59}{128}$	$-\frac{453}{256}$	$-\frac{709}{512}$	$\frac{1851}{1024}$
$Q_n^{(5)}$	3	0	2	8	16	32	58	118	240	480	960
$Q_{-n}^{(5)}$		$\frac{3}{2}$	$\frac{3}{4}$	$-\frac{13}{8}$	$-\frac{29}{16}$	$\frac{67}{32}$	$\frac{67}{64}$	$\frac{67}{128}$	$-\frac{445}{256}$	$-\frac{957}{512}$	$\frac{2115}{1024}$
$S_n^{(5)}$	0	1	1	2	4	8	17	33	66	132	264
$S_{-n}^{(5)}$		0	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{1}{32}$	$\frac{31}{64}$	$-\frac{33}{128}$
$R_n^{(5)}$	5	1	3	7	15	36	63	127	255	511	1028
$R_{-n}^{(5)}$		$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$\frac{129}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$\frac{4097}{1024}$

Next, we give the ordinary generating function of the sequence V_n.

Lemma 5.4 *Suppose that* $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized fifth order Jacobsthal sequence* $\{V_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} V_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3+(V_4-V_3-V_2-V_1-V_0)x^4}{1-x-x^2-x^3-x^4-2x^5}$$

Proof. Take $r = s = t = u = 1, v = 2$ in Lemma 5.2. $\square$

The previous Lemma gives the following results as particular examples.

Corollary 5.1 *Generating functions of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order*

Jacobsthal and modified fifth order Jacobsthal-Lucas numbers are

$$\sum_{n=0}^{\infty} J_n x^n = \frac{x - x^3 - 2x^4}{1 - x - x^2 - x^3 - x^4 - 2x^5},$$

$$\sum_{n=0}^{\infty} j_n x^n = \frac{2 - x + 2x^2 + 2x^3 + 2x^4}{1 - x - x^2 - x^3 - x^4 - 2x^5},$$

$$\sum_{n=0}^{\infty} K_n x^n = \frac{3 - 2x - x^2 + 3x^3 + 3x^4}{1 - x - x^2 - x^3 - x^4 - 2x^5},$$

$$\sum_{n=0}^{\infty} Q_n x^n = \frac{3 - 3x - x^2 + 3x^3 + 3x^4}{1 - x - x^2 - x^3 - x^4 - 2x^5},$$

$$\sum_{n=0}^{\infty} S_n x^n = \frac{x}{1 - x - x^2 - x^3 - x^4 - 2x^5},$$

$$\sum_{n=0}^{\infty} R_n x^n = \frac{5 - 4x - 3x^2 - 2x^3 - x^4}{1 - x - x^2 - x^3 - x^4 - 2x^5},$$

respectively.

We next give Binet formula of generalized fifth order Jacobsthal numbers $\{V_n\}$ by the use of Theorems 5.1 and 5.3.

Theorem 5.5 *(Binet formula of generalized fifth order Jacobsthal numbers)*

$$\begin{aligned} V_n &= \frac{c_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{c_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{c_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad + \frac{c_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{c_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} c_1 &= V_4 - (\beta+\gamma+\delta+\lambda)V_3 + (\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)V_2 \\ &\quad -(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)V_1 + (\beta\lambda\gamma\delta)V_0 \\ &= V_0\alpha^4 + (V_1 - V_0)\alpha^3 + (V_2 - V_1 - V_0)\alpha^2 \\ &\quad +(V_3 - V_2 - V_1 - V_0)\alpha + (V_4 - V_3 - V_2 - V_1 - V_0), \end{aligned}$$

$$\begin{aligned} c_2 &= V_4 - (\alpha+\gamma+\delta+\lambda)V_3 + (\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)V_2 \\ &\quad -(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)V_1 + (\alpha\lambda\gamma\delta)V_0 \\ &= V_0\beta^4 + (V_1 - V_0)\beta^3 + (V_2 - V_1 - V_0)\beta^2 \\ &\quad +(V_3 - V_2 - V_1 - V_0)\beta + (V_4 - V_3 - V_2 - V_1 - V_0), \end{aligned}$$

$$\begin{aligned}
c_3 &= V_4-(\alpha+\beta+\delta+\lambda)V_3+(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)V_2\\
&\quad -(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)V_1+(\alpha\beta\lambda\delta)V_0\\
&= V_0\gamma^4+(V_1-V_0)\gamma^3+(V_2-V_1-V_0)\gamma^2\\
&\quad +(V_3-V_2-V_1-V_0)\gamma+(V_4-V_3-V_2-V_1-V_0),
\end{aligned}$$

$$\begin{aligned}
c_4 &= V_4-(\alpha+\beta+\gamma+\lambda)V_3+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)V_2\\
&\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)V_1+(\alpha\beta\lambda\gamma)V_0\\
&= V_0\delta^4+(V_1-V_0)\delta^3+(V_2-V_1-V_0)\delta^2\\
&\quad +(V_3-V_2-V_1-V_0)\delta+(V_4-V_3-V_2-V_1-V_0),
\end{aligned}$$

$$\begin{aligned}
c_5 &= V_4-(\alpha+\beta+\gamma+\delta)V_3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)V_2\\
&\quad -(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)V_1+(\alpha\beta\gamma\delta)V_0\\
&= V_0\lambda^4+(V_1-V_0)\lambda^3+(V_2-V_1-V_0)\lambda^2\\
&\quad +(V_3-V_2-V_1-V_0)\lambda+(V_4-V_3-V_2-V_1-V_0).
\end{aligned}$$

Next, using Theorem 5.5, we present the Binet formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas sequences.

Corollary 5.2 *Binet formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal , fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers are*

$$\begin{aligned}
J_n &= \frac{(\alpha^3-\alpha-2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{(\beta^3-\beta-2)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad +\frac{(\gamma^3-\gamma-2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{(\delta^3-\delta-2)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}\\
&\quad +\frac{(\lambda^3-\lambda-2)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned}$$

$$\begin{aligned}
j_n &= \frac{(\alpha^4+4\alpha^3+4\alpha^2+4\alpha+4)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{(\beta^4+4\beta^3+4\beta^2+4\beta+4)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad +\frac{(\gamma^4+4\gamma^3+4\gamma^2+4\gamma+4)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{(\delta^4+4\delta^3+4\delta^2+4\delta+4)\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}\\
&\quad +\frac{(\lambda^4+4\lambda^3+4\lambda^2+4\lambda+4)\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned}$$

$$
\begin{aligned}
K_n &= \frac{(\alpha^4+2\alpha^3+6\alpha^2+6\alpha+6)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{(\beta^4+2\beta^3+6\beta^2+6\beta+6)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\
&\quad + \frac{(\gamma^4+2\gamma^3+6\gamma^2+6\gamma+6)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{(\delta^4+2\delta^3+6\delta^2+6\delta+6)\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\
&\quad + \frac{(\lambda^4+2\lambda^3+6\lambda^2+6\lambda+6)\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned}
$$

$$
\begin{aligned}
Q_n &= \frac{2(\alpha^3+3\alpha^2+3\alpha+3)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{2(\beta^3+3\beta^2+3\beta+3)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\
&\quad + \frac{2(\gamma^3+3\gamma^2+3\gamma+3)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{2(\delta^3+3\delta^2+3\delta+3)\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\
&\quad + \frac{2(\lambda^3+3\lambda^2+3\lambda+3)\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned}
$$

$$
\begin{aligned}
S_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\
&\quad + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\
&\quad + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned}
$$

$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n$

respectively.

Binet formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers can be given in the following forms:

$$
\begin{aligned}
J_n &= \frac{4}{31}\alpha^n - \frac{1}{155}((6\sqrt{5}+5) + 2\sqrt{2}\sqrt{\sqrt{5}+5}(6+\sqrt{5})i)\beta^n \\
&\quad + \frac{1}{155}(-(6\sqrt{5}+5) + 2\sqrt{2}\sqrt{\sqrt{5}+5}(6+\sqrt{5})i)\gamma^n \\
&\quad + \frac{1}{155}((6\sqrt{5}-5) + 2\sqrt{2}\sqrt{5-\sqrt{5}}(\sqrt{5}-6)i)\delta^n \\
&\quad + \frac{1}{155}((6\sqrt{5}-5) + 2\sqrt{2}\sqrt{5-\sqrt{5}}(-\sqrt{5}+6)i)\lambda^n,
\end{aligned}
$$

$$
\begin{aligned}
j_n &= \frac{38}{31}\alpha^n + \frac{1}{1240}(12(20-7\sqrt{5}) + \sqrt{\sqrt{5}+5}(111\sqrt{2}+3\sqrt{10})i)\beta^n \\
&\quad + \frac{1}{1240}(12(20-7\sqrt{5}) - \sqrt{\sqrt{5}+5}(111\sqrt{2}+3\sqrt{10})i)\gamma^n \\
&\quad + \frac{1}{1240}(12(20+7\sqrt{5}) + \sqrt{5-\sqrt{5}}(111\sqrt{2}-3\sqrt{10})i)\delta^n \\
&\quad + \frac{1}{1240}(12(20+7\sqrt{5}) + \sqrt{5-\sqrt{5}}(-111\sqrt{2}+3\sqrt{10})i)\lambda^n,
\end{aligned}
$$

$$
\begin{aligned}
K_n &= \frac{37}{31}\alpha^n + \frac{1}{1240}(4(13\sqrt{5}+140) + \sqrt{2}\sqrt{\sqrt{5}+5}(73i+69i\sqrt{5}))\beta^n \\
&+\frac{1}{1240}(4(13\sqrt{5}+140) - \sqrt{2}\sqrt{\sqrt{5}+5}(73i+69i\sqrt{5}))\gamma^n \\
&+\frac{1}{1240}(4(-13\sqrt{5}+140) + \sqrt{2}\sqrt{5-\sqrt{5}}(73i-69i\sqrt{5}))\delta^n \\
&+\frac{1}{1240}(4(-13\sqrt{5}+140) + \sqrt{2}\sqrt{5-\sqrt{5}}(-73i+69i\sqrt{5}))\lambda^n,
\end{aligned}
$$

$$
\begin{aligned}
Q_n &= \frac{29}{31}\alpha^n + \frac{1}{1240}(8(80+3\sqrt{5}) + 10\sqrt{2}\sqrt{\sqrt{5}+5}(11+7\sqrt{5})i)\beta^n \\
&+\frac{1}{1240}(8(80+3\sqrt{5}) - 10\sqrt{2}\sqrt{\sqrt{5}+5}(11+7\sqrt{5})i)\gamma^n \\
&+\frac{1}{1240}(8(80-3\sqrt{5}) + 10\sqrt{2}\sqrt{5-\sqrt{5}}(11-7\sqrt{5})i)\delta^n \\
&+\frac{1}{1240}(8(80-3\sqrt{5}) + 10\sqrt{2}\sqrt{5-\sqrt{5}}(-11+7\sqrt{5})i)\lambda^n,
\end{aligned}
$$

$$
\begin{aligned}
S_n &= \frac{8}{31}\alpha^n + \frac{1}{1240}(4(-20+7\sqrt{5}) - \sqrt{2}\sqrt{\sqrt{5}+5}(37i+i\sqrt{5}))\beta^n \\
&+\frac{1}{1240}(4(-20+7\sqrt{5}) + \sqrt{2}\sqrt{\sqrt{5}+5}(37i+i\sqrt{5}))\gamma^n \\
&+\frac{1}{1240}(-4(20+7\sqrt{5}) + \sqrt{2}\sqrt{\sqrt{5}+5}(21-19\sqrt{5})i)\delta^n \\
&+\frac{1}{1240}(-4(20+7\sqrt{5}) + \sqrt{2}\sqrt{\sqrt{5}+5}(-21+19\sqrt{5})i)\lambda^n,
\end{aligned}
$$

$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n.$

5.1.1 Simson Formulas

The following theorem gives the generalization of Simson formula of generalized Pentanacci sequence $\{W_n\}$.

Theorem 5.6 (Simson Formula of Generalized Pentanacci Numbers) *For all integers n, we have*

$$
\begin{vmatrix}
W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n \\
W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} \\
W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} \\
W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} \\
W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4}
\end{vmatrix}
= v^n
\begin{vmatrix}
W_4 & W_3 & W_2 & W_1 & W_0 \\
W_3 & W_2 & W_1 & W_0 & W_{-1} \\
W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\
W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\
W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4}
\end{vmatrix}. \tag{5.21}
$$

For the proof of (6.15) see [64, Theorem 5]. $\square$

A special case of the above theorem is the following theorem which gives Simson formula of the generalized fifth order Jacobsthal sequence $\{V_n\}$.

Theorem 5.7 (Simson Formula of Generalized Fifth Order Jacobsthal Numbers) *For all integers n, we have*

$$\begin{vmatrix} V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \end{vmatrix} = 2^n \begin{vmatrix} V_4 & V_3 & V_2 & V_1 & V_0 \\ V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \end{vmatrix}. \tag{5.22}$$

The previous Theorem gives the following results as particular examples.

Corollary 5.3 *Simson formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas are given as*

$$\begin{vmatrix} J_{n+4} & J_{n+3} & J_{n+2} & J_{n+1} & J_n \\ J_{n+3} & J_{n+2} & J_{n+1} & J_n & J_{n-1} \\ J_{n+2} & J_{n+1} & J_n & J_{n-1} & J_{n-2} \\ J_{n+1} & J_n & J_{n-1} & J_{n-2} & J_{n-3} \\ J_n & J_{n-1} & J_{n-2} & J_{n-3} & J_{n-4} \end{vmatrix} = 11 \times 2^{n-2}$$

$$\begin{vmatrix} j_{n+4} & j_{n+3} & j_{n+2} & j_{n+1} & j_n \\ j_{n+3} & j_{n+2} & j_{n+1} & j_n & j_{n-1} \\ j_{n+2} & j_{n+1} & j_n & j_{n-1} & j_{n-2} \\ j_{n+1} & j_n & j_{n-1} & j_{n-2} & j_{n-3} \\ j_n & j_{n-1} & j_{n-2} & j_{n-3} & j_{n-4} \end{vmatrix} = 1539 \times 2^{n-3}$$

$$\begin{vmatrix} K_{n+4} & K_{n+3} & K_{n+2} & K_{n+1} & K_n \\ K_{n+3} & K_{n+2} & K_{n+1} & K_n & K_{n-1} \\ K_{n+2} & K_{n+1} & K_n & K_{n-1} & K_{n-2} \\ K_{n+1} & K_n & K_{n-1} & K_{n-2} & K_{n-3} \\ K_n & K_{n-1} & K_{n-2} & K_{n-3} & K_{n-4} \end{vmatrix} = 17057 \times 2^{n-4}$$

$$\begin{vmatrix} Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} \end{vmatrix} = 1595 \times 2^n$$

$$\begin{vmatrix} S_{n+4} & S_{n+3} & S_{n+2} & S_{n+1} & S_n \\ S_{n+3} & S_{n+2} & S_{n+1} & S_n & S_{n-1} \\ S_{n+2} & S_{n+1} & S_n & S_{n-1} & S_{n-2} \\ S_{n+1} & S_n & S_{n-1} & S_{n-2} & S_{n-3} \\ S_n & S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} \end{vmatrix} = 2^{n-1}$$

$$\begin{vmatrix} R_{n+4} & R_{n+3} & R_{n+2} & R_{n+1} & R_n \\ R_{n+3} & R_{n+2} & R_{n+1} & R_n & R_{n-1} \\ R_{n+2} & R_{n+1} & R_n & R_{n-1} & R_{n-2} \\ R_{n+1} & R_n & R_{n-1} & R_{n-2} & R_{n-3} \\ R_n & R_{n-1} & R_{n-2} & R_{n-3} & R_{n-4} \end{vmatrix} = 120125 \times 2^{n-4}$$

respectively.

5.1.2 Some Identities

In this subsection, we obtain some identities of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers. Firstly, we give some fundamental relations between $\{J_n\}$ and $\{j_n\}$.

Lemma 5.8 *The following equalities are true:*

$$\begin{aligned} 57J_n &= -9j_{n+4} + 10j_{n+3} + 29j_{n+2} - 9j_{n+1} - 28j_n, && (5.23) \\ 57J_n &= j_{n+3} + 20j_{n+2} - 18j_{n+1} - 37j_n - 18j_{n-1}, \\ 57J_n &= 21j_{n+2} - 17j_{n+1} - 36j_n - 17j_{n-1} + 2j_{n-2}, \\ 57J_n &= 4j_{n+1} - 15j_n + 4j_{n-1} + 23j_{n-2} + 42j_{n-3}, \\ 57J_n &= -11j_n + 8j_{n-1} + 27j_{n-2} + 46j_{n-3} + 8j_{n-4}, \end{aligned}$$

and

$$\begin{aligned}
22j_n &= -7J_{n+4}+41J_{n+3}-13J_{n+2}+23J_{n+1}-J_n,\\
11j_n &= 17J_{n+3}-10J_{n+2}+8J_{n+1}-4J_n-7J_{n-1},\\
11j_n &= 7J_{n+2}+25J_{n+1}+13J_n+10J_{n-1}+34J_{n-2},\\
11j_n &= 32J_{n+1}+20J_n+17J_{n-1}+41J_{n-2}+14J_{n-3},\\
11j_n &= 52J_n+49J_{n-1}+73J_{n-2}+46J_{n-3}+11J_{n-4}.
\end{aligned}$$

Proof. Note that all the identities hold for all integers n. We only prove (5.23). If we write

$$J_n = a\times j_{n+4}+b\times j_{n+3}+c\times j_{n+2}+d\times j_{n+1}+e\times j_n$$

and solve the system of equations

$$\begin{aligned}
J_0 &= a\times j_4+b\times j_3+c\times j_2+d\times j_1+e\times j_0\\
J_1 &= a\times j_5+b\times j_4+c\times j_3+d\times j_2+e\times j_1\\
J_2 &= a\times j_6+b\times j_5+c\times j_4+d\times j_3+e\times j_2\\
J_3 &= a\times j_7+b\times j_6+c\times j_5+d\times j_4+e\times j_3\\
J_4 &= a\times j_8+b\times j_7+c\times j_6+d\times j_5+e\times j_4
\end{aligned}$$

then we get $a=-\frac{3}{19}$, $b=\frac{10}{57}$, $c=\frac{29}{57}$, $d=-\frac{3}{19}$, $e=-\frac{28}{57}$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{J_n\}$ and $\{S_n\}$.

Lemma 5.9 *The following equalities are true:*

$$\begin{aligned}
2J_n &= S_{n+4}-S_{n+3}-3S_{n+2}+S_{n+1}+3S_n,\\
J_n &= -S_{n+2}+S_{n+1}+2S_n+S_{n-1},\\
J_n &= S_n-S_{n-2}-2S_{n-3},
\end{aligned}$$

and

$$
\begin{aligned}
11S_n &= 3J_{n+4} - 5J_{n+3} + 4J_{n+2} - 2J_{n+1} + 2J_n, \\
11S_n &= -2J_{n+3} + 7J_{n+2} + J_{n+1} + 5J_n + 6J_{n-1}, \\
11S_n &= 5J_{n+2} - J_{n+1} + 3J_n + 4J_{n-1} - 4J_{n-2}, \\
11S_n &= 4J_{n+1} + 8J_n + 9J_{n-1} + J_{n-2} + 10J_{n-3}, \\
11S_n &= 12J_n + 13J_{n-1} + 5J_{n-2} + 14J_{n-3} + 8J_{n-4}.
\end{aligned}
$$

Thirdly, we give a few basic relations between $\{J_n\}$ and $\{R_n\}$.

Lemma 5.10 *The following equalities are true:*

$$
\begin{aligned}
4805J_n &= 319R_{n+4} + 71R_{n+3} - 425R_{n+2} - 1417R_{n+1} - 518R_n, \\
4805J_n &= 390R_{n+3} - 106R_{n+2} - 1098R_{n+1} - 199R_n + 638R_{n-1}, \\
4805J_n &= 284R_{n+2} - 708R_{n+1} + 191R_n + 1028R_{n-1} + 780R_{n-2}, \\
4805J_n &= -424R_{n+1} + 475R_n + 1312R_{n-1} + 1064R_{n-2} + 568R_{n-3}, \\
4805J_n &= 51R_n + 888R_{n-1} + 640R_{n-2} + 144R_{n-3} - 848R_{n-4},
\end{aligned}
$$

and

$$
\begin{aligned}
44R_n &= -97J_{n+4} + 191J_{n+3} - J_{n+2} + 127J_{n+1} + 115J_n, \\
22R_n &= 47J_{n+3} - 49J_{n+2} + 15J_{n+1} + 9J_n - 97J_{n-1}, \\
11R_n &= -J_{n+2} + 31J_{n+1} + 28J_n - 25J_{n-1} + 47J_{n-2}, \\
11R_n &= 30J_{n+1} + 27J_n - 26J_{n-1} + 46J_{n-2} - 2J_{n-3}, \\
11R_n &= 57J_n + 4J_{n-1} + 76J_{n-2} + 28J_{n-3} + 60J_{n-4}.
\end{aligned}
$$

Next, we present a few basic relations between $\{j_n\}$ and $\{S_n\}$.

Lemma 5.11 *The following equalities are true:*

$$
\begin{aligned}
4j_n &= S_{n+4} + S_{n+3} + S_{n+2} + S_{n+1} - 11S_n, \\
2j_n &= S_{n+3} + S_{n+2} + S_{n+1} - 5S_n + S_{n-1}, \\
j_n &= S_{n+2} + S_{n+1} - 2S_n + S_{n-1} + S_{n-2}, \\
j_n &= 2S_{n+1} - S_n + 2S_{n-1} + 2S_{n-2} + 2S_{n-3}, \\
j_n &= S_n + 4S_{n-1} + 4S_{n-2} + 4S_{n-3} + 4S_{n-4},
\end{aligned}
$$

and

$$\begin{aligned}
57S_n &= j_{n+4}+j_{n+3}+j_{n+2}+j_{n+1}-18j_n,\\
57S_n &= 2j_{n+3}+2j_{n+2}+2j_{n+1}-17j_n+2j_{n-1},\\
57S_n &= 4j_{n+2}+4j_{n+1}-15j_n+4j_{n-1}+4j_{n-2},\\
57S_n &= 8j_{n+1}-11j_n+8j_{n-1}+8j_{n-2}+8j_{n-3},\\
57S_n &= -3j_n+16j_{n-1}+16j_{n-2}+16j_{n-3}+16j_{n-4}.
\end{aligned}$$

Now, we give a few basic relations between $\{j_n\}$ and $\{R_n\}$.

Lemma 5.12 *The following equalities are true:*

$$\begin{aligned}
4805j_n &= -581R_{n+4}+907R_{n+3}+1000R_{n+2}+1186R_{n+1}+1558R_n,\\
4805j_n &= 326R_{n+3}+419R_{n+2}+605R_{n+1}+977R_n-1162R_{n-1},\\
4805j_n &= 745R_{n+2}+931R_{n+1}+1303R_n-836R_{n-1}+652R_{n-2},\\
4805j_n &= 1676R_{n+1}+2048R_n-91R_{n-1}+1397R_{n-2}+1490R_{n-3},\\
4805j_n &= 3724R_n+1585R_{n-1}+3073R_{n-2}+3166R_{n-3}+3352R_{n-4},
\end{aligned}$$

and

$$\begin{aligned}
114R_n &= 41j_{n+4}+3j_{n+3}-35j_{n+2}-263j_{n+1}+79j_n,\\
57R_n &= 22j_{n+3}+3j_{n+2}-111j_{n+1}+60j_n+41j_{n-1},\\
57R_n &= 25j_{n+2}-89j_{n+1}+82j_n+63j_{n-1}+44j_{n-2},\\
57R_n &= -64j_{n+1}+107j_n+88j_{n-1}+69j_{n-2}+50j_{n-3},\\
57R_n &= 43j_n+24j_{n-1}+5j_{n-2}-14j_{n-3}-128j_{n-4}.
\end{aligned}$$

Next, we present a few basic relations between $\{S_n\}$ and $\{R_n\}$.

Lemma 5.13 *The following equalities are true:*

$$\begin{aligned}
4805S_n &= 297R_{n+4}-199R_{n+3}-230R_{n+2}-292R_{n+1}-416R_n,\\
4805S_n &= 98R_{n+3}+67R_{n+2}+5R_{n+1}-119R_n+594R_{n-1},\\
4805S_n &= 165R_{n+2}+103R_{n+1}-21R_n+692R_{n-1}+196R_{n-2},\\
4805S_n &= 268R_{n+1}+144R_n+857R_{n-1}+361\times R_{n-2}+330R_{n-3},\\
4805S_n &= 412R_n+1125R_{n-1}+629R_{n-2}+598\times R_{n-3}+536\times R_{n-4},
\end{aligned}$$

and

$$\begin{aligned}
8R_n &= -7S_{n+4}+S_{n+3}+9S_{n+2}+57S_{n+1}-15S_n,\\
4R_n &= -3S_{n+3}+S_{n+2}+25S_{n+1}-11S_n-7S_{n-1},\\
2R_n &= -S_{n+2}+11S_{n+1}-7S_n-5S_{n-1}-3S_{n-2},\\
R_n &= 5S_{n+1}-4S_n-3S_{n-1}-2S_{n-2}-S_{n-3},\\
R_n &= S_n+2S_{n-1}+3S_{n-2}+4S_{n-3}+10S_{n-4}.
\end{aligned}$$

Note that we also can give some other identities which related to $\{K_n\}$ and $\{Q_n\}$. For example, we have

$$\begin{aligned}
8K_n &= -13S_{n+4}+19S_{n+3}+19S_{n+2}+19S_{n+1}-21S_n,\\
8Q_n &= -13S_{n+4}+19S_{n+3}+19S_{n+2}+19S_{n+1}-29S_n.
\end{aligned}$$

5.1.3 Linear Sums

Sums of Terms with Positive Subscripts:

The following proposition presents some formulas of generalized fifth order Jacobsthal numbers with positive subscripts.

Proposition 5.14 *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k = \frac{1}{5}(V_{n+5}-V_{n+3}-2V_{n+2}-3V_{n+1}-V_4+V_2+2V_1+3V_0)$.

(b) $\sum_{k=0}^{n} V_{2k} = \frac{1}{15}(-V_{2n+2}+5V_{2n+1}+11V_{2n}+2V_{2n-1}+8V_{2n-2}+V_4-5V_3+4V_2-2V_1+7V_0)$.

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{15}(4V_{2n+2}+10V_{2n+1}+V_{2n}+7V_{2n-1}-2V_{2n-2}-4V_4+5V_3-V_2+8V_1+2V_0)$.

Proof. Take $r=1, s=1, t=1, u=1, v=2$, in Theorem 2.1 in [77].

From the last proposition, we have the following corollary which gives sum formulas of fifth order Jacobsthal numbers (take $V_n = J_n$ with $J_0=0, J_1=1, J_2=1, J_3=1, J_4=1$).

Corollary 5.4 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} J_k = \frac{1}{5}(J_{n+5}-J_{n+3}-2J_{n+2}-3J_{n+1}+2)$.

(b) $\sum_{k=0}^{n} J_{2k} = \frac{1}{15}(-J_{2n+2}+5J_{2n+1}+11J_{2n}+2J_{2n-1}+8J_{2n-2}-2)$.

(c) $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{15}(4J_{2n+2} + 10J_{2n+1} + J_{2n} + 7J_{2n-1} - 2J_{2n-2} + 8)$.

Taking $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$ in the last proposition, we have the following corollary which presents sum formulas of fifth order Jacobsthal-Lucas numbers.

Corollary 5.5 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} j_k = \frac{1}{5}(j_{n+5} - j_{n+3} - 2j_{n+2} - 3j_{n+1} - 7)$.

(b) $\sum_{k=0}^{n} j_{2k} = \frac{1}{15}(-j_{2n+2} + 5j_{2n+1} + 11j_{2n} + 2j_{2n-1} + 8j_{2n-2} + 2)$.

(c) $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{15}(4j_{2n+2} + 10j_{2n+1} + j_{2n} + 7j_{2n-1} - 2j_{2n-2} - 23)$.

From the last proposition, we have the following corollary which gives sum formulas of modified fifth order Jacobsthal numbers (take $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20$).

Corollary 5.6 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} K_k = \frac{1}{5}(K_{n+5} - K_{n+3} - 2K_{n+2} - 3K_{n+1} - 6)$.

(b) $\sum_{k=0}^{n} K_{2k} = \frac{1}{15}(-K_{2n+2} + 5K_{2n+1} + 11K_{2n} + 2K_{2n-1} + 8K_{2n-2} + 1)$.

(c) $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{15}(4K_{2n+2} + 10K_{2n+1} + K_{2n} + 7K_{2n-1} - 2K_{2n-2} - 19)$.

Taking $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16$ in the last proposition, we have the following corollary which presents sum formulas of fifth order Jacobsthal Perrin numbers.

Corollary 5.7 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{5}(Q_{n+5} - Q_{n+3} - 2Q_{n+2} - 3Q_{n+1} - 5)$.

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{15}(-Q_{2n+2} + 5Q_{2n+1} + 11Q_{2n} + 2Q_{2n-1} + 8Q_{2n-2} + 5)$.

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{15}(4Q_{2n+2} + 10Q_{2n+1} + Q_{2n} + 7Q_{2n-1} - 2Q_{2n-2} - 20)$.

From the last proposition, we have the following corollary which gives sum formulas of adjusted fifth order Jacobsthal numbers (take $V_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$).

Corollary 5.8 *For $n \geq 0$, adjusted fifth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} S_k = \frac{1}{5}(S_{n+5} - S_{n+3} - 2S_{n+2} - 3S_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} S_{2k} = \frac{1}{15}(-S_{2n+2} + 5S_{2n+1} + 11S_{2n} + 2S_{2n-1} + 8S_{2n-2} - 4)$.

(c) $\sum_{k=0}^{n} S_{2k+1} = \frac{1}{15}(4S_{2n+2} + 10S_{2n+1} + S_{2n} + 7S_{2n-1} - 2S_{2n-2} + 1)$.

Taking $V_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$ in the last proposition, we have the following corollary which presents sum formulas of modified fifth order Jacobsthal-Lucas numbers.

Corollary 5.9 *For $n \geq 0$, modified fifth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} R_k = \frac{1}{5}(R_{n+5} - R_{n+3} - 2R_{n+2} - 3R_{n+1} + 5)$.

(b) $\sum_{k=0}^{n} R_{2k} = \frac{1}{15}(-R_{2n+2} + 5R_{2n+1} + 11R_{2n} + 2R_{2n-1} + 8R_{2n-2} + 25)$.

(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{15}(4R_{2n+2} + 10R_{2n+1} + R_{2n} + 7R_{2n-1} - 2R_{2n-2} - 10)$.

Sums of Terms with Negative Subscripts

The following proposition presents some formulas of generalized fifth order Jacobsthal numbers with negative subscripts.

Proposition 5.15 *For $n \geq 1$, we have the following formulas:*

(a) $\sum_{k=1}^{n} V_{-k} = \frac{1}{5}(-V_{-n+4} + V_{-n+2} + 2V_{-n+1} + 3V_{-n} + V_4 - V_2 - 2V_1 - 3V_0)$.

(b) $\sum_{k=1}^{n} V_{-2k} = \frac{1}{15}(-4V_{-2n+3} + 5V_{-2n+2} - V_{-2n+1} + 8V_{-2n} + 2V_{-2n-1} - V_4 + 5V_3 - 4V_2 + 2V_1 - 7V_0)$.

(c) $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{15}(V_{-2n+3} - 5V_{-2n+2} + 4V_{-2n+1} - 2V_{-2n} - 8V_{-2n-1} + 4V_4 - 5V_3 + V_2 - 8V_1 - 2V_0)$.

Proof. Take $r = 1, s = 1, t = 1, u = 1, v = 2$ in Theorem 3.1 in [77].

From the last proposition, we have the following corollary which gives sum formulas of fifth order Jacobsthal numbers (take $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1$).

Corollary 5.10 *For $n \geq 1$, fifth order Jacobsthal numbers have the following properties.*

(a) $\sum_{k=1}^{n} J_{-k} = \frac{1}{5}(-J_{-n+4} + J_{-n+2} + 2J_{-n+1} + 3J_{-n} - 2)$.

(b) $\sum_{k=1}^{n} J_{-2k} = \frac{1}{15}(-4J_{-2n+3} + 5J_{-2n+2} - J_{-2n+1} + 8J_{-2n} + 2J_{-2n-1} + 2)$.

(c) $\sum_{k=1}^{n} J_{-2k+1} = \frac{1}{15}(J_{-2n+3} - 5J_{-2n+2} + 4J_{-2n+1} - 2J_{-2n} - 8J_{-2n-1} - 8)$.

Taking $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$ in the last proposition, we have the following corollary which presents sum formulas of fifth order Jacobsthal-Lucas numbers.

Corollary 5.11 *For $n \geq 1$, fifth order Jacobsthal-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} j_{-k} = \frac{1}{5}(-j_{-n+4} + j_{-n+2} + 2j_{-n+1} + 3j_{-n} + 7)$.

(b) $\sum_{k=1}^{n} j_{-2k} = \frac{1}{15}(-4j_{-2n+3} + 5j_{-2n+2} - j_{-2n+1} + 8j_{-2n} + 2j_{-2n-1} - 2)$.

(c) $\sum_{k=1}^{n} j_{-2k+1} = \frac{1}{15}(j_{-2n+3} - 5j_{-2n+2} + 4j_{-2n+1} - 2j_{-2n} - 8j_{-2n-1} + 23)$.

From the last proposition, we have the following corollary which gives sum formulas of modified fifth order Jacobsthal numbers (take $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20$).

Corollary 5.12 *For $n \geq 1$, modified fifth order Jacobsthal numbers have the following properties.*

(a) $\sum_{k=1}^{n} K_{-k} = \frac{1}{5}(-K_{-n+4} + K_{-n+2} + 2K_{-n+1} + 3K_{-n} + 6)$.

(b) $\sum_{k=1}^{n} K_{-2k} = \frac{1}{15}(-4K_{-2n+3} + 5K_{-2n+2} - K_{-2n+1} + 8K_{-2n} + 2K_{-2n-1} - 1)$.

(c) $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{15}(K_{-2n+3} - 5K_{-2n+2} + 4K_{-2n+1} - 2K_{-2n} - 8K_{-2n-1} + 19)$.

Taking $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16$ in the last proposition, we have the following corollary which presents sum formulas of fifth order Jacobsthal Perrin numbers.

Corollary 5.13 *For $n \geq 1$, fifth order Jacobsthal Perrin numbers have the following properties.*

(a) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{5}(-Q_{-n+4} + Q_{-n+2} + 2Q_{-n+1} + 3Q_{-n} + 5)$.

(b) $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{15}(-4Q_{-2n+3} + 5Q_{-2n+2} - Q_{-2n+1} + 8Q_{-2n} + 2Q_{-2n-1} - 5)$.

(c) $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{15}(Q_{-2n+3} - 5Q_{-2n+2} + 4Q_{-2n+1} - 2Q_{-2n} - 8Q_{-2n-1} + 20)$.

From the last proposition, we have the following corollary which gives sum formulas of adjusted fifth order Jacobsthal numbers (take $V_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$).

Corollary 5.14 *For $n \geq 1$, adjusted fifth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n} S_{-k} = \frac{1}{5}(-S_{-n+4} + S_{-n+2} + 2S_{-n+1} + 3S_{-n} + 1)$.

(b) $\sum_{k=1}^{n} S_{-2k} = \frac{1}{15}(-4S_{-2n+3} + 5S_{-2n+2} - S_{-2n+1} + 8S_{-2n} + 2S_{-2n-1} + 4)$.

(c) $\sum_{k=1}^{n} S_{-2k+1} = \frac{1}{15}(S_{-2n+3} - 5S_{-2n+2} + 4S_{-2n+1} - 2S_{-2n} - 8S_{-2n-1} - 1)$.

Taking $V_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$ in the last proposition, we have the following corollary which presents sum formulas of modified fifth order Jacobsthal-Lucas numbers.

Corollary 5.15 *For $n \geq 1$, modified fifth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} R_{-k} = \frac{1}{5}(-R_{-n+4} + R_{-n+2} + 2R_{-n+1} + 3R_{-n} - 5)$.

(b) $\sum_{k=1}^{n} R_{-2k} = \frac{1}{15}(-4R_{-2n+3} + 5R_{-2n+2} - R_{-2n+1} + 8R_{-2n} + 2R_{-2n-1} - 25)$.

(c) $\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{15}(R_{-2n+3} - 5R_{-2n+2} + 4R_{-2n+1} - 2R_{-2n} - 8R_{-2n-1} + 10)$.

5.1.4 A Sum Formula

The formula in the following proposition can be used to calculate the sum of the generalized fifth order Jacobsthal sequence.

Proposition 5.16 *For all integers m and j with $S_m^3 - 3S_m^2 - 3S_{2m}S_m + 2S_{3m} + 3S_{2m} + 6S_m - 6 \times 2^m \times (S_{-m} - 1) - 6 \neq 0$, we have*

$$\sum_{k=0}^{n} V_{mk+j} = \frac{\Lambda + \Psi_1}{\Omega} \tag{5.24}$$

where

$\Lambda = 6V_{mn+3m+j} + 6(1 - R_m)V_{mn+2m+j} + 3(R_m^2 - R_{2m} - 2R_m + 2)V_{mn+m+j} + 6V_{mn-m+j} \times 2^m - 6 \times 2^m \times (R_{-m} - 1)\, V_{mn+j}$,

$\Psi_1 = -6V_{3m+j} - 6(1 - R_m)V_{2m+j} - 3(R_m^2 - R_{2m} - 2R_m + 2)V_{m+j} - 6V_{-m+j} \times 2^m + (R_m^3 - 3R_m^2 - 3R_{2m}R_m + 2R_{3m} + 3R_{2m} + 6R_m - 6)V_j$,

$\Omega = R_m^3 - 3R_m^2 - 3R_{2m}R_m + 2R_{3m} + 3R_{2m} + 6R_m - 6 \times 2^m \times (R_{-m} - 1) - 6.$

Proof. Take $r = 1, s = 1, t = 1, u = 1, v = 2$ and $R_n = H_n$ in Soykan [76, Theorem 13.]. □

Note that (5.24) can be written in the following form:

$$\sum_{k=1}^{n} V_{mk+j} = \frac{\Lambda + \Psi_2}{\Omega}$$

where

$\Psi_2 = -6V_{3m+j} - 6(1 - R_m)V_{2m+j} - 3(R_m^2 - R_{2m} - 2R_m + 2)V_{m+j} - 6V_{-m+j} \times 2^m + 6 \times 2^m \times (R_{-m} - 1)\, V_j$.

As special cases of the above proposition, we have the following identities.

Corollary 5.16 *The following identities hold:*

1. $m = 1,\ j = 0$.

   (a) $\sum_{k=0}^{n} J_k = \frac{1}{5}(J_{n+3} - J_{n+1} + 3J_n + 2J_{n-1} + 2)$.

   (b) $\sum_{k=0}^{n} j_k = \frac{1}{5}(j_{n+3} - j_{n+1} + 3j_n + 2j_{n-1} - 7)$.

   (c) $\sum_{k=0}^{n} K_k = \frac{1}{5}(K_{n+3} - K_{n+1} + 3K_n + 2K_{n-1} - 6)$.

   (d) $\sum_{k=0}^{n} Q_k = \frac{1}{5}(Q_{n+3} - Q_{n+1} + 3Q_n + 2Q_{n-1} - 5)$.

(e) $\sum_{k=0}^{n} S_k = \frac{1}{5}(S_{n+3} - S_{n+1} + 3S_n + 2S_{n-1} - 1)$.

(f) $\sum_{k=0}^{n} R_k = \frac{1}{5}(R_{n+3} - R_{n+1} + 3R_n + 2R_{n-1} + 5)$.

2. $m = -1,\ j = 0$.

(a) $\sum_{k=0}^{n} J_{-k} = \frac{1}{5}\left(-J_{-n+1} - 4J_{-n-1} - 3J_{-n-2} - 2J_{-n-3} - 2\right)$.

(b) $\sum_{k=0}^{n} j_{-k} = \frac{1}{5}\left(-j_{-n+1} - 4j_{-n-1} - 3j_{-n-2} - 2j_{-n-3} + 17\right)$.

(c) $\sum_{k=0}^{n} K_{-k} = \frac{1}{5}\left(-K_{-n+1} - 4K_{-n-1} - 3K_{-n-2} - 2K_{-n-3} + 21\right)$.

(d) $\sum_{k=0}^{n} Q_{-k} = \frac{1}{5}\left(-Q_{-n+1} - 4Q_{-n-1} - 3Q_{-n-2} - 2Q_{-n-3} + 20\right)$.

(e) $\sum_{k=0}^{n} S_{-k} = \frac{1}{5}\left(-S_{-n+1} - 4S_{-n-1} - 3S_{-n-2} - 2S_{-n-3} + 1\right)$.

(f) $\sum_{k=0}^{n} R_{-k} = \frac{1}{5}\left(-R_{-n+1} - 4R_{-n-1} - 3R_{-n-2} - 2R_{-n-3} + 20\right)$.

3. $m = 3,\ j = 1$.

(a) $\sum_{k=0}^{n} J_{3k+1} = \frac{1}{35}(J_{3n+10} - 6J_{3n+7} - 13J_{3n+4} + 15J_{3n+1} + 8J_{3n-2} + 3)$.

(b) $\sum_{k=0}^{n} j_{3k+1} = \frac{1}{35}(j_{3n+10} - 6j_{3n+7} - 13j_{3n+4} + 15j_{3n+1} + 8j_{3n-2} - 38)$.

(c) $\sum_{k=0}^{n} K_{3k+1} = \frac{1}{35}(K_{3n+10} - 6K_{3n+7} - 13K_{3n+4} + 15K_{3n+1} + 8K_{3n-2} - 44)$.

(d) $\sum_{k=0}^{n} Q_{3k+1} = \frac{1}{35}(Q_{3n+10} - 6Q_{3n+7} - 13Q_{3n+4} + 15Q_{3n+1} + 8Q_{3n-2} - 50)$.

(e) $\sum_{k=0}^{n} S_{3k+1} = \frac{1}{35}(S_{3n+10} - 6S_{3n+7} - 13S_{3n+4} + 15S_{3n+1} + 8S_{3n-2} + 6)$.

(f) $\sum_{k=0}^{n} R_{3k+1} = \frac{1}{35}(R_{3n+10} - 6R_{3n+7} - 13R_{3n+4} + 15R_{3n+1} + 8R_{3n-2} - 45)$.

4. $m = -3,\ j = -1$.

(a) $\sum_{k=0}^{n} J_{-3k-1} = \frac{1}{35}(-J_{-3n+2} + 6J_{-3n-1} - 22J_{-3n-4} - 15J_{-3n-7} - 8J_{-3n-10} - 13)$.

(b) $\sum_{k=0}^{n} j_{-3k-1} = \frac{1}{35}(-j_{-3n+2} + 6j_{-3n-1} - 22j_{-3n-4} - 15j_{-3n-7} - 8j_{-3n-10} + 13)$.

(c) $\sum_{k=0}^{n} K_{-3k-1} = \frac{1}{35}(-K_{-3n+2} + 6K_{-3n-1} - 22K_{-3n-4} - 15K_{-3n-7} - 8K_{-3n-10} + 39)$.

(d) $\sum_{k=0}^{n} Q_{-3k-1} = \frac{1}{35}(-Q_{-3n+2} + 6Q_{-3n-1} - 22Q_{-3n-4} - 15Q_{-3n-7} - 8Q_{-3n-10} + 30)$.

(e) $\sum_{k=0}^{n} S_{-3k-1} = \frac{1}{35}(-S_{-3n+2} + 6S_{-3n-1} - 22S_{-3n-4} - 15S_{-3n-7} - 8S_{-3n-10} + 9)$.

(f) $\sum_{k=0}^{n} R_{-3k-1} = \frac{1}{35}(-R_{-3n+2} + 6R_{-3n-1} - 22R_{-3n-4} - 15R_{-3n-7} - 8R_{-3n-10} - 15)$.

5. $m = 4,\ j = -2$.

(a) $\sum_{k=0}^{n} J_{4k-2} = \frac{1}{300}(4J_{4n+10} - 56J_{4n+6} - 116J_{4n+2} + 124J_{4n-2} + 64J_{4n-6} + 160).$

(b) $\sum_{k=0}^{n} j_{4k-2} = \frac{1}{300}(4j_{4n+10} - 56j_{4n+6} - 116j_{4n+2} + 124j_{4n-2} + 64j_{4n-6} - 70).$

(c) $\sum_{k=0}^{n} K_{4k-2} = \frac{1}{300}(4K_{4n+10} - 56K_{4n+6} - 116K_{4n+2} + 124K_{4n-2} + 64K_{4n-6} - 275).$

(d) $\sum_{k=0}^{n} Q_{4k-2} = \frac{1}{300}(4Q_{4n+10} - 56Q_{4n+6} - 116Q_{4n+2} + 124Q_{4n-2} + 64Q_{4n-6} - 295).$

(e) $\sum_{k=0}^{n} S_{4k-2} = \frac{1}{300}(4S_{4n+10} - 56S_{4n+6} - 116S_{4n+2} + 124S_{4n-2} + 64S_{4n-6} + 20).$

(f) $\sum_{k=0}^{n} R_{4k-2} = \frac{1}{300}(4R_{4n+10} - 56R_{4n+6} - 116R_{4n+2} + 124R_{4n-2} + 64R_{4n-6} - 305).$

6. $m = -4,\ j = -2.$

(a) $\sum_{k=0}^{n} J_{-4k-2} = \frac{1}{75}(14J_{4n-2} - 16J_{4n-14} - J_{-4n+2} - 46J_{-4n-6} - 31J_{-4n-10} - 40).$

(b) $\sum_{k=0}^{n} j_{-4k-2} = \frac{1}{75}(14j_{4n-2} - 16j_{4n-14} - j_{-4n+2} - 46j_{-4n-6} - 31j_{-4n-10} + 55).$

(c) $\sum_{k=0}^{n} K_{-4k-2} = \frac{1}{75}(14K_{4n-2} - 16K_{4n-14} - K_{-4n+2} - 46K_{-4n-6} - 31K_{-4n-10} + 125).$

(d) $\sum_{k=0}^{n} Q_{-4k-2} = \frac{1}{75}(14Q_{4n-2} - 16Q_{4n-14} - Q_{-4n+2} - 46Q_{-4n-6} - 31Q_{-4n-10} + 130).$

(e) $\sum_{k=0}^{n} S_{-4k-2} = \frac{1}{75}(14S_{4n-2} - 16S_{4n-14} - S_{-4n+2} - 46S_{-4n-6} - 31S_{-4n-10} - 5).$

(f) $\sum_{k=0}^{n} R_{-4k-2} = \frac{1}{75}(14R_{4n-2} - 16R_{4n-14} - R_{-4n+2} - 46R_{-4n-6} - 31R_{-4n-10} + 20).$

5.1.5 Matrices Related with Generalized Fifth Order Jacobsthal Numbers

In this subsection, we consider the following square matrix A of order 5 as:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. We consider the following matrices:

$$B_n = \begin{pmatrix} S_{n+1} & S_n + S_{n-1} + S_{n-2} + 2S_{n-3} & S_n + S_{n-1} + 2S_{n-2} & S_n + 2S_{n-1} & 2S_n \\ S_n & S_{n-1} + S_{n-2} + S_{n-3} + 2S_{n-4} & S_{n-1} + S_{n-2} + 2S_{n-3} & S_{n-1} + 2S_{n-2} & 2S_{n-1} \\ S_{n-1} & S_{n-2} + S_{n-3} + S_{n-4} + 2S_{n-5} & S_{n-2} + S_{n-3} + 2S_{n-4} & S_{n-2} + 2S_{n-3} & 2S_{n-2} \\ S_{n-2} & S_{n-3} + S_{n-4} + S_{n-5} + 2S_{n-6} & S_{n-3} + S_{n-4} + 2S_{n-5} & S_{n-3} + 2S_{n-4} & 2S_{n-3} \\ S_{n-3} & S_{n-4} + S_{n-5} + S_{n-6} + 2S_{n-7} & S_{n-4} + S_{n-5} + 2S_{n-6} & S_{n-4} + 2S_{n-5} & 2S_{n-4} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} + V_{n-2} + 2V_{n-3} & V_n + V_{n-1} + 2V_{n-2} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + V_{n-2} + V_{n-3} + 2V_{n-4} & V_{n-1} + V_{n-2} + 2V_{n-3} & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + V_{n-4} + 2V_{n-5} & V_{n-2} + V_{n-3} + 2V_{n-4} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + V_{n-5} + 2V_{n-6} & V_{n-3} + V_{n-4} + 2V_{n-5} & V_{n-3} + 2V_{n-4} & 2V_{n-3} \\ V_{n-3} & V_{n-4} + V_{n-5} + V_{n-6} + 2V_{n-7} & V_{n-4} + V_{n-5} + 2V_{n-6} & V_{n-4} + 2V_{n-5} & 2V_{n-4} \end{pmatrix}.$$

Theorem 5.17 *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 2$ and $S_n = G_n$ in Soykan [76, Theorem 16.]. □

Theorem 5.18 *For all integers m, n, we have*

$$\begin{aligned} V_{n+m} &= V_n S_{m+1} + V_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + V_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + V_{n-3}(S_m + 2S_{m-1}) + 2V_{n-4}S_m. \end{aligned}$$

Proof. Take $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 2$ and $S_n = G_n$ in Soykan [76, Theorem 17.]. □

Corollary 5.17 *For all integers m, n, we have*

$$\begin{aligned} J_{n+m} &= J_n S_{m+1} + J_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + J_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + J_{n-3}(S_m + 2S_{m-1}) + 2J_{n-4}S_m, \end{aligned}$$

$$\begin{aligned} j_{n+m} &= j_n S_{m+1} + j_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + j_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + j_{n-3}(S_m + 2S_{m-1}) + 2j_{n-4}S_m, \end{aligned}$$

$$\begin{aligned} K_{n+m} &= K_n S_{m+1} + K_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + K_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + K_{n-3}(S_m + 2S_{m-1}) + 2K_{n-4}S_m, \end{aligned}$$

$$
\begin{aligned}
Q_{n+m} &= Q_n S_{m+1} + Q_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + Q_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + Q_{n-3}(S_m + 2S_{m-1}) + 2Q_{n-4}S_m,
\end{aligned}
$$

$$
\begin{aligned}
S_{n+m} &= S_n S_{m+1} + S_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + S_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + S_{n-3}(S_m + 2S_{m-1}) + 2S_{n-4}S_m,
\end{aligned}
$$

$$
\begin{aligned}
R_{n+m} &= R_n S_{m+1} + R_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + R_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + R_{n-3}(S_m + 2S_{m-1}) + 2R_{n-4}S_m.
\end{aligned}
$$

CHAPTER 6

GENERALIZED SIXTH ORDER JACOBSTHAL NUMBERS

In this chapter, we introduce the generalized sixth order Jacobsthal sequences and we investigate, in detail, six special cases which we call them sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas sequences. Initially, we express that the results of this chapter are cited from [80] which was published by Soykan and Eyican Polatli.

6.1 INTRODUCTION

In this section, we give basic information about generalized sixth order Jacobsthal numbers. First we recall the definition and some properties of a generalized Hexanacci sequence. The generalized Hexanacci sequence (or the generalized (r, s, t, u, v, y) sequence or 6-step Fibonacci sequence)

$$\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined by the sixth order recurrence relation

$$\begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}, \qquad (6.1)\\ W_0 &= c_0, W_1 = c_1, W_2 = c_2,\ W_3 = c_3, W_4 = c_4, W_5 = c_5,\ n \geq 6 \end{aligned}$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and r, s, t, u, v, y are real numbers. Generalized Hexanacci sequence has been studied by many authors, see for example [54, 56, 73, 74, 75]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5} + \frac{1}{y}W_{-n+6}$$

for $n = 1, 2, 3, ...$ when $y \neq 0$. Therefore, recurrence (6.1) holds for all integers n.

As $\{W_n\}$ is a sixth order recurrence sequence (difference equation), it's characteristic equation is

$$x^6 - rx^5 - sx^4 - tx^3 - ux^2 - vx - y = 0 \tag{6.2}$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda, \mu$.

Note that we have the following identities:

$\alpha + \beta + \gamma + \delta + \lambda + \mu = r,$

$\alpha\beta + \alpha\lambda + \alpha\gamma + \alpha\mu + \beta\lambda + \alpha\delta + \beta\gamma + \beta\mu + \lambda\gamma + \lambda\mu + \beta\delta + \lambda\delta + \gamma\mu + \gamma\delta + \mu\delta = -s,$

$\alpha\beta\lambda + \alpha\beta\gamma + \alpha\beta\mu + \alpha\lambda\gamma + \alpha\lambda\mu + \alpha\beta\delta + \alpha\lambda\delta + \alpha\gamma\mu + \beta\lambda\gamma + \beta\lambda\mu + \alpha\gamma\delta + \alpha\mu\delta + \beta\lambda\delta + \beta\gamma\mu + \lambda\gamma\mu + \beta\gamma\delta + \beta\mu\delta + \lambda\gamma\delta + \lambda\mu\delta + \gamma\mu\delta = t,$

$\alpha\beta\lambda\gamma + \alpha\beta\lambda\mu + \alpha\beta\lambda\delta + \alpha\beta\gamma\mu + \alpha\lambda\gamma\mu + \alpha\beta\gamma\delta + \alpha\beta\mu\delta + \alpha\lambda\gamma\delta + \alpha\lambda\mu\delta + \beta\lambda\gamma\mu + \alpha\gamma\mu\delta + \beta\lambda\gamma\delta + \beta\lambda\mu\delta + \beta\gamma\mu\delta + \lambda\gamma\mu\delta = -u$

$\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta = v,$

$\alpha\beta\lambda\gamma\mu\delta = -y.$

Throughout the paper, we use the following notations interchangeable.

$$r = r_1, s = r_2, t = r_3, u = r_4, v = r_5, y = r_6$$

and

$$\alpha = \alpha_1, \beta = \alpha_2, \gamma = \alpha_3, \delta = \alpha_4, \lambda = \alpha_5, \mu = \alpha_6.$$

So (6.1) and (6.2) can be written as follows:

$$W_n = r_1 W_{n-1} + r_2 W_{n-2} + r_3 W_{n-3} + r_4 W_{n-4} + r_5 W_{n-5} + r_6 W_{n-6}$$

and

$$x^6 - r_1 x^5 - r_2 x^4 - r_3 x^3 - r_4 x^2 - r_5 x - r_6 = 0$$

respectively.

Generalized hexanacci numbers can be expressed, for all integers n, using Binet formula.

Theorem 6.1 *[73](Binet formula of generalized (r, s, t, u, v, y) numbers (generalized Hexanacci numbers))*

$$\begin{aligned} W_n \;=\; & \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ & + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ & + \frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)} + \frac{p_6\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)} \end{aligned} \tag{6.3}$$

where

$$\begin{aligned}
p_1 &= W_5-(\beta+\gamma+\delta+\lambda+\mu)W_4\\
&+(\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\beta\delta+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3\\
&-(\beta\lambda\gamma+\beta\lambda\mu+\beta\lambda\delta+\beta\gamma\mu+\lambda\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2\\
&+(\beta\lambda\gamma\mu+\beta\lambda\gamma\delta+\beta\lambda\mu\delta+\beta\gamma\mu\delta+\lambda\gamma\mu\delta)W_1-\beta\lambda\gamma\mu\delta W_0,
\end{aligned}$$

$$\begin{aligned}
p_2 &= W_5-(\alpha+\gamma+\delta+\lambda+\mu)W_4\\
&+(\alpha\lambda+\alpha\gamma+\alpha\mu+\alpha\delta+\lambda\gamma+\lambda\mu+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3\\
&-(\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\lambda\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\lambda\gamma\mu+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2\\
&+(\alpha\lambda\gamma\mu+\alpha\lambda\gamma\delta+\alpha\lambda\mu\delta+\alpha\gamma\mu\delta+\lambda\gamma\mu\delta)W_1-\alpha\lambda\gamma\mu\delta W_0,
\end{aligned}$$

$$\begin{aligned}
p_3 &= W_5-(\alpha+\beta+\delta+\lambda+\mu)W_4+(\alpha\beta+\alpha\lambda+\alpha\mu+\beta\lambda+\alpha\delta\\
&+\beta\mu+\lambda\mu+\beta\delta+\lambda\delta+\mu\delta)W_3\\
&-(\alpha\beta\lambda+\alpha\beta\mu+\alpha\lambda\mu+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\mu+\alpha\mu\delta+\beta\lambda\delta+\beta\mu\delta+\lambda\mu\delta)W_2\\
&+(\alpha\beta\lambda\mu+\alpha\beta\lambda\delta+\alpha\beta\mu\delta+\alpha\lambda\mu\delta+\beta\lambda\mu\delta)W_1-\alpha\beta\lambda\mu\delta W_0,
\end{aligned}$$

$$\begin{aligned}
p_4 &= W_5-(\alpha+\beta+\gamma+\lambda+\mu)W_4\\
&+(\alpha\beta+\alpha\lambda+\alpha\gamma+\alpha\mu+\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\gamma\mu)W_3\\
&-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\beta\mu+\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\gamma\mu+\beta\lambda\gamma+\beta\lambda\mu+\beta\gamma\mu+\lambda\gamma\mu)W_2\\
&+(\alpha\beta\lambda\gamma+\alpha\beta\lambda\mu+\alpha\beta\gamma\mu+\alpha\lambda\gamma\mu+\beta\lambda\gamma\mu)W_1-\alpha\beta\lambda\gamma\mu W_0,
\end{aligned}$$

$$\begin{aligned}
p_5 &= W_5-(\alpha+\beta+\gamma+\delta+\mu)W_4\\
&+(\alpha\beta+\alpha\gamma+\alpha\mu+\alpha\delta+\beta\gamma+\beta\mu+\beta\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3\\
&-(\alpha\beta\gamma+\alpha\beta\mu+\alpha\beta\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\beta\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\gamma\mu\delta)W_2\\
&+(\alpha\beta\gamma\mu+\alpha\beta\gamma\delta+\alpha\beta\mu\delta+\alpha\gamma\mu\delta+\beta\gamma\mu\delta)W_1-\alpha\beta\gamma\mu\delta W_0,
\end{aligned}$$

$$\begin{aligned}
p_6 &= W_5-(\alpha+\beta+\gamma+\delta+\lambda)W_4\\
&+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_3\\
&-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_2\\
&+(\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta)W_1-\alpha\beta\lambda\gamma\delta W_0.
\end{aligned}$$

Usually, it is customary to choose $r_1, r_2, r_3, r_4, r_5, r_6$ so that the Equ. (6.2) has at least one real (say α_1) solutions.

Note that the Binet form of a sequence satisfying (6.2) for non-negative integers is valid for all integers n, (see [42], this result of Howard and Saidak [42] is even true in the case of higher-order recurrence relations).

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 6.2 *[73]Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r,s,t,u,v,y) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{\Lambda}{1 - rx - sx^2 - tx^3 - ux^4 - vx^5 - yx^6} \tag{6.4}$$
$$= \frac{\Lambda}{1 - r_1x - r_2x^2 - r_3x^3 - r_4x^4 - r_5x^5 - r_6x^6}$$

where

$$\begin{aligned}\Lambda &= W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3 \\ &\quad +(W_4 - rW_3 - sW_2 - tW_1 - uW_0)x^4 \\ &\quad +(W_5 - rW_4 - sW_3 - tW_2 - uW_1 - vW_0)x^5\end{aligned}$$

or (in short formula)

$$\begin{aligned}\Lambda &= W_0 + (W_1 - r_1W_0)x + (W_2 - r_1W_1 - r_2W_0)x^2 \\ &\quad +(W_3 - r_1W_2 - r_2W_1 - r_3W_0)x^3 \\ &\quad +(W_4 - r_1W_3 - r_2W_2 - r_3W_1 - r_4W_0)x^4 \\ &\quad +(W_5 - r_1W_4 - r_2W_3 - r_3W_2 - r_4W_1 - r_5W_0)x^5 \\ &= W_0 + \sum_{i=1}^{6-1} x^i \left(W_i - \sum_{j=1}^{i} r_j W_{i-j} \right).\end{aligned}$$

We next give another form of Binet formula of generalized (r,s,t,u,v,y) numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 6.3 *[73](Binet formula of generalized (r,s,t,u,v,y) numbers)*

$$W_n = \sum_{k=1}^{6} \frac{q_k \alpha_k^n}{\prod_{\substack{j=1 \\ k\neq j}}^{6} (\alpha_k - \alpha_j)} \tag{6.5}$$

where

$$q_l = W_0\alpha_l^{6-1} + \sum_{i=1}^{6-1}\alpha_l^{6-1-i}\left[W_i - \sum_{j=1}^{i} r_j W_{i-j}\right], \quad 1 \le l \le m = 6.$$

In this chapter we consider the case $r_1 = r_2 = r_3 = r_4 = r_5 = 1$, $r_6 = 2$ and in this case we write $V_n = W_n$. A generalized sixth order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4, V_5)\}_{n\geq 0}$ is defined by the sixth order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + 2V_{n-6} \tag{6.6}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$, $V_3 = c_3$, $V_4 = c_4$, $V_5 = c_5$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} - \frac{1}{2}V_{-(n-3)} - \frac{1}{2}V_{-(n-4)} - \frac{1}{2}V_{-(n-5)} + \frac{1}{2}V_{-(n-6)}$$

for $n \geq 1$. Therefore, recurrence (6.6) holds for all integer n.

As $\{V_n\}$ is a sixth order recurrence sequence (difference equation), it's characteristic equation is

$$x^6 - x^5 - x^4 - x^3 - x^2 - x - 2 = (x-2)(x+1)(x^4 + x^2 + 1) = 0. \tag{6.7}$$

The roots $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 of Equation (6.7) are given by

$$\theta_1 = \alpha = 2,$$

$$\theta_2 = \beta = -1,$$

$$\theta_3 = \gamma = \frac{1 + i\sqrt{3}}{2},$$

$$\theta_4 = \delta = \frac{1 - i\sqrt{3}}{2},$$

$$\theta_5 = \lambda = \frac{-1 + i\sqrt{3}}{2},$$

$$\theta_6 = \mu = \frac{-1 - i\sqrt{3}}{2}.$$

Note that we have the following identities:

$\alpha + \beta + \gamma + \delta + \lambda + \mu = 1,$

$\alpha\beta + \alpha\lambda + \alpha\gamma + \alpha\mu + \beta\lambda + \alpha\delta + \beta\gamma + \beta\mu + \lambda\gamma + \lambda\mu + \beta\delta + \lambda\delta + \gamma\mu + \gamma\delta + \mu\delta = -1,$

$\alpha\beta\lambda + \alpha\beta\gamma + \alpha\beta\mu + \alpha\lambda\gamma + \alpha\lambda\mu + \alpha\beta\delta + \alpha\lambda\delta + \alpha\gamma\mu + \beta\lambda\gamma + \beta\lambda\mu + \alpha\gamma\delta + \alpha\mu\delta + \beta\lambda\delta + \beta\gamma\mu + \lambda\gamma\mu + \beta\gamma\delta + \beta\mu\delta + \lambda\gamma\delta + \lambda\mu\delta + \gamma\mu\delta = 1,$

$\alpha\beta\lambda\gamma + \alpha\beta\lambda\mu + \alpha\beta\lambda\delta + \alpha\beta\gamma\mu + \alpha\lambda\gamma\mu + \alpha\beta\gamma\delta + \alpha\beta\mu\delta + \alpha\lambda\gamma\delta + \alpha\lambda\mu\delta + \beta\lambda\gamma\mu + \alpha\gamma\mu\delta + \beta\lambda\gamma\delta + \beta\lambda\mu\delta + \beta\gamma\mu\delta + \lambda\gamma\mu\delta = -1$

$\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta = 1,$

$\alpha\beta\lambda\gamma\mu\delta = -2.$

Now we define six special cases of the sequence $\{V_n\}$. Sixth order Jacobsthal sequence $\{J_n^{(6)}\}_{n\geq 0}$, sixth order Jacobsthal-Lucas sequence $\{j_n^{(6)}\}_{n\geq 0}$, modified sixth order Jacobsthal sequence $\{K_n^{(6)}\}_{n\geq 0}$, sixth order Jacobsthal Perrin sequence $\{Q_n^{(6)}\}_{n\geq 0}$, adjusted sixth order Jacobsthal sequence $\{S_n^{(6)}\}_{n\geq 0}$ and modified sixth order Jacobsthal-Lucas sequence $\{R_n^{(6)}\}_{n\geq 0}$ are defined, respectively, by the sixth order recurrence relations

$$\begin{aligned} J_{n+6}^{(6)} &= J_{n+5}^{(6)} + J_{n+4}^{(6)} + J_{n+3}^{(6)} + J_{n+2}^{(6)} + J_{n+1}^{(6)} + 2J_n^{(6)}, \\ J_0^{(6)} &= 0, J_1^{(6)} = 1, J_2^{(6)} = 1, J_3^{(6)} = 1, J_4^{(6)} = 1, J_5^{(6)} = 1, \end{aligned} \tag{6.8}$$

$$\begin{aligned} j_{n+6}^{(6)} &= j_{n+5}^{(6)} + j_{n+4}^{(6)} + j_{n+3}^{(6)} + j_{n+2}^{(6)} + j_{n+1}^{(6)} + 2j_n^{(6)}, \\ j_0^{(6)} &= 2, j_1^{(6)} = 1, j_2^{(6)} = 5, j_3^{(6)} = 10, j_4^{(6)} = 20, j_5^{(6)} = 40, \end{aligned} \tag{6.9}$$

$$\begin{aligned} K_{n+6}^{(6)} &= K_{n+5}^{(6)} + K_{n+4}^{(6)} + K_{n+3}^{(6)} + K_{n+2}^{(6)} + K_{n+1}^{(6)} + 2K_n^{(6)}, \\ K_0^{(6)} &= 3, K_1^{(6)} = 1, K_2^{(6)} = 3, K_3^{(6)} = 10, K_4^{(6)} = 20, K_5^{(6)} = 40, \end{aligned} \tag{6.10}$$

$$\begin{aligned} Q_{n+6}^{(6)} &= Q_{n+5}^{(6)} + Q_{n+4}^{(6)} + Q_{n+3}^{(6)} + Q_{n+2}^{(6)} + Q_{n+1}^{(6)} + 2Q_n^{(6)}, \\ Q_0^{(6)} &= 3, Q_1^{(6)} = 0, Q_2^{(6)} = 2, Q_3^{(6)} = 8, Q_4^{(6)} = 16, Q_5^{(6)} = 32, \end{aligned} \tag{6.11}$$

$$\begin{aligned} S_{n+6}^{(6)} &= S_{n+5}^{(6)} + S_{n+4}^{(6)} + S_{n+3}^{(6)} + S_{n+2}^{(6)} + S_{n+1}^{(6)} + 2S_n^{(6)}, \\ S_0^{(6)} &= 0, S_1^{(6)} = 1, S_2^{(6)} = 1, S_3^{(6)} = 2, S_4^{(6)} = 4, S_5^{(6)} = 8, \end{aligned} \tag{6.12}$$

$$\begin{aligned} R_{n+6}^{(6)} &= R_{n+5}^{(6)} + R_{n+4}^{(6)} + R_{n+3}^{(6)} + R_{n+2}^{(6)} + R_{n+1}^{(6)} + 2R_n^{(6)}, \\ R_0^{(6)} &= 6, R_1^{(6)} = 1, R_2^{(6)} = 3, R_3^{(6)} = 7, R_4^{(6)} = 15, R_5^{(6)} = 31. \end{aligned} \tag{6.13}$$

The sequences $\{J_n^{(6)}\}_{n\geq 0}$, $\{j_n^{(6)}\}_{n\geq 0}$,$\{K_n^{(6)}\}_{n\geq 0}, Q_n^{(6)}\}_{n\geq 0}$, $\{S_n^{(6)}\}_{n\geq 0}$ and $\{R_n^{(6)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
J_{-n}^{(6)} &= -\frac{1}{2}J_{-(n-1)}^{(6)} - \frac{1}{2}J_{-(n-2)}^{(6)} - \frac{1}{2}J_{-(n-3)}^{(6)} - \frac{1}{2}J_{-(n-4)}^{(6)} - \frac{1}{2}J_{-(n-5)}^{(6)} + \frac{1}{2}J_{-(n-6)}^{(6)} \\
j_{-n}^{(6)} &= -\frac{1}{2}j_{-(n-1)}^{(6)} - \frac{1}{2}j_{-(n-2)}^{(6)} - \frac{1}{2}j_{-(n-3)}^{(6)} - \frac{1}{2}j_{-(n-4)}^{(6)} - \frac{1}{2}j_{-(n-5)}^{(6)} + \frac{1}{2}j_{-(n-6)}^{(6)} \\
K_{-n}^{(6)} &= -\frac{1}{2}K_{-(n-1)}^{(6)} - \frac{1}{2}K_{-(n-2)}^{(6)} - \frac{1}{2}K_{-(n-3)}^{(6)} - \frac{1}{2}K_{-(n-4)}^{(6)} - \frac{1}{2}K_{-(n-5)}^{(6)} + \frac{1}{2}K_{-(n-6)}^{(6)} \\
Q_{-n}^{(6)} &= -\frac{1}{2}Q_{-(n-1)}^{(6)} - \frac{1}{2}Q_{-(n-2)}^{(6)} - \frac{1}{2}Q_{-(n-3)}^{(6)} - \frac{1}{2}Q_{-(n-4)}^{(6)} - \frac{1}{2}Q_{-(n-5)}^{(6)} + \frac{1}{2}Q_{-(n-6)}^{(6)} \\
S_{-n}^{(6)} &= -\frac{1}{2}S_{-(n-1)}^{(6)} - \frac{1}{2}S_{-(n-2)}^{(6)} - \frac{1}{2}S_{-(n-3)}^{(6)} - \frac{1}{2}S_{-(n-4)}^{(6)} - \frac{1}{2}S_{-(n-5)}^{(6)} + \frac{1}{2}S_{-(n-6)}^{(6)} \\
R_{-n}^{(6)} &= -\frac{1}{2}R_{-(n-1)}^{(6)} - \frac{1}{2}R_{-(n-2)}^{(6)} - \frac{1}{2}R_{-(n-3)}^{(6)} - \frac{1}{2}R_{-(n-4)}^{(6)} - \frac{1}{2}R_{-(n-5)}^{(6)} + \frac{1}{2}R_{-(n-6)}^{(6)}
\end{aligned}$$

for $n \geq 1$, respectively. Therefore, recurrences (6.8)- (6.13) hold for all integer n.

Next, we present the first few values of the sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas numbers with positive and negative subscripts:

Table 6.1. The first few terms of the J_n, j_n, K_n, Q_n, S_n and R_n

n	0	1	2	3	4	5	6	7	8	9	10
$J_n^{(6)}$	0	1	1	1	1	1	5	11	21	41	81
$J_{-n}^{(6)}$		$-\frac{3}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	$\frac{11}{16}$	$\frac{27}{32}$	$-\frac{5}{64}$	$-\frac{197}{128}$	$-\frac{69}{256}$	$\frac{187}{512}$	$\frac{699}{1024}$
$j_n^{(6)}$	2	1	5	10	20	40	80	157	317	634	1268
$j_{-n}^{(6)}$		1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{23}{16}$	$\frac{25}{32}$	$\frac{25}{64}$	$\frac{25}{128}$	$\frac{25}{256}$	$\frac{25}{512}$
$K_n^{(6)}$	3	1	3	10	20	40	80	155	311	626	1252
$K_{-n}^{(6)}$		$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$-\frac{29}{16}$	$-\frac{45}{32}$	$\frac{115}{64}$	$\frac{115}{128}$	$\frac{115}{256}$	$\frac{115}{512}$	$-\frac{1933}{1024}$
$Q_n^{(6)}$	3	0	2	8	16	32	64	122	246	496	992
$Q_{-n}^{(6)}$		$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$-\frac{29}{16}$	$-\frac{61}{32}$	$\frac{131}{64}$	$\frac{131}{128}$	$\frac{131}{256}$	$\frac{131}{512}$	$-\frac{1917}{1024}$
$S_n^{(6)}$	0	1	1	2	4	8	16	33	65	130	260
$S_{-n}^{(6)}$		0	0	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{1}{32}$	$-\frac{1}{64}$
$R_n^{(6)}$	6	1	3	7	15	31	69	127	255	511	1023
$R_{-n}^{(6)}$		$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$\frac{321}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$

In the rest of the chapter, for easy writing, we drop the superscripts and write J_n, j_n, K_n, Q_n, S_n and R_n for $J_n^{(6)}$, $j_n^{(6)}$, $K_n^{(6)}$, $Q_n^{(6)}$, $S_n^{(6)}$ and $R_n^{(6)}$, respectively.

Next, we give the ordinary generating function of the sequence V_n.

Lemma 6.4 *Suppose that* $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized sixth order Jacobsthal sequence* $\{V_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} V_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{\Lambda}{1 - x - x^2 - x^3 - x^4 - x^5 - 2x^6} \tag{6.14}$$

where

$$\begin{aligned}\Lambda &= V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3 \\ &\quad +(V_4 - V_3 - V_2 - V_1 - V_0)x^4 + (V_5 - V_4 - V_3 - V_2 - V_1 - V_0)x^5 \\ &= V_0 + \sum_{i=1}^{6-1} x^i \left(V_i - \sum_{j=1}^{i} V_{i-j} \right).\end{aligned}$$

Proof. Take $r = s = t = u = v = 1$, $y = 2$ in Lemma 6.2. $\square$

The previous Lemma gives the following results as particular examples.

Corollary 6.1 *Generating functions of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas numbers are*

$$\sum_{n=0}^{\infty} J_n x^n = \frac{x - x^3 - 2x^4 - 3x^5}{1 - x - x^2 - x^3 - x^4 - x^5 - 2x^6},$$

$$\sum_{n=0}^{\infty} j_n x^n = \frac{2 - x + 2x^2 + 2x^3 + 2x^4 + 2x^5}{1 - x - x^2 - x^3 - x^4 - x^5 - 2x^6},$$

$$\sum_{n=0}^{\infty} K_n x^n = \frac{3 - 2x - x^2 + 3x^3 + 3x^4 + 3x^5}{1 - x - x^2 - x^3 - x^4 - x^5 - 2x^6},$$

$$\sum_{n=0}^{\infty} Q_n x^n = \frac{3 - 3x - x^2 + 3x^3 + 3x^4 + 3x^5}{1 - x - x^2 - x^3 - x^4 - x^5 - 2x^6},$$

$$\sum_{n=0}^{\infty} S_n x^n = \frac{x}{1 - x - x^2 - x^3 - x^4 - x^5 - 2x^6},$$

$$\sum_{n=0}^{\infty} R_n x^n = \frac{6 - 5x - 4x^2 - 3x^3 - 2x^4 - x^5}{1 - x - x^2 - x^3 - x^4 - x^5 - 2x^6},$$

respectively.

We next give Binet formula of generalized sixth order Jacobsthal numbers $\{V_n\}$ by the use of Theorems 6.1 and 6.3.

Theorem 6.5 *(Binet formula of generalized sixth order Jacobsthal numbers)*

$$\begin{aligned} V_n &= \frac{c_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)} + \frac{c_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ &+ \frac{c_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)} + \frac{c_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ &+ \frac{c_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)} + \frac{c_6\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)} \end{aligned}$$

where

$$\begin{aligned} c_1 &= V_5 - (\beta+\gamma+\delta+\lambda+\mu)V_4 \\ &+(\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\beta\delta+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)V_3 \\ &-(\beta\lambda\gamma+\beta\lambda\mu+\beta\lambda\delta+\beta\gamma\mu+\lambda\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)V_2 \\ &+(\beta\lambda\gamma\mu+\beta\lambda\gamma\delta+\beta\lambda\mu\delta+\beta\gamma\mu\delta+\lambda\gamma\mu\delta)V_1 - \beta\lambda\gamma\mu\delta V_0 \\ &= V_0\alpha^5 + (V_1-V_0)\alpha^4 + (V_2-V_1-V_0)\alpha^3 + (V_3-V_2-V_1-V_0)\alpha^2 \\ &+(V_4-V_3-V_2-V_1-V_0)\alpha + (V_5-V_4-V_3-V_2-V_1-V_0), \end{aligned}$$

$$\begin{aligned} c_2 &= V_5 - (\alpha+\gamma+\delta+\lambda+\mu)V_4 \\ &+(\alpha\lambda+\alpha\gamma+\alpha\mu+\alpha\delta+\lambda\gamma+\lambda\mu+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)V_3 \\ &-(\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\lambda\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\lambda\gamma\mu+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)V_2 \\ &+(\alpha\lambda\gamma\mu+\alpha\lambda\gamma\delta+\alpha\lambda\mu\delta+\alpha\gamma\mu\delta+\lambda\gamma\mu\delta)V_1 - \alpha\lambda\gamma\mu\delta V_0 \\ &= V_0\beta^5 + (V_1-V_0)\beta^4 + (V_2-V_1-V_0)\beta^3 + (V_3-V_2-V_1-V_0)\beta^2 \\ &+(V_4-V_3-V_2-V_1-V_0)\beta + (V_5-V_4-V_3-V_2-V_1-V_0), \end{aligned}$$

$$\begin{aligned} c_3 &= V_5 - (\alpha+\beta+\delta+\lambda+\mu)V_4 \\ &+(\alpha\beta+\alpha\lambda+\alpha\mu+\beta\lambda+\alpha\delta+\beta\mu+\lambda\mu+\beta\delta+\lambda\delta+\mu\delta)V_3 \\ &-(\alpha\beta\lambda+\alpha\beta\mu+\alpha\lambda\mu+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\mu+\alpha\mu\delta+\beta\lambda\delta+\beta\mu\delta+\lambda\mu\delta)V_2 \\ &+(\alpha\beta\lambda\mu+\alpha\beta\lambda\delta+\alpha\beta\mu\delta+\alpha\lambda\mu\delta+\beta\lambda\mu\delta)V_1 - \alpha\beta\lambda\mu\delta V_0 \\ &= V_0\gamma^5 + (V_1-V_0)\gamma^4 + (V_2-V_1-V_0)\gamma^3 + (V_3-V_2-V_1-V_0)\gamma^2 \\ &+(V_4-V_3-V_2-V_1-V_0)\gamma + (V_5-V_4-V_3-V_2-V_1-V_0), \end{aligned}$$

$$
\begin{aligned}
c_4 &= V_5 - (\alpha+\beta+\gamma+\lambda+\mu)V_4 \\
&\quad +(\alpha\beta+\alpha\lambda+\alpha\gamma+\alpha\mu+\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\gamma\mu)V_3 \\
&\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\beta\mu+\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\gamma\mu+\beta\lambda\gamma+\beta\lambda\mu+\beta\gamma\mu+\lambda\gamma\mu)V_2 \\
&\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\mu+\alpha\beta\gamma\mu+\alpha\lambda\gamma\mu+\beta\lambda\gamma\mu)V_1 - \alpha\beta\lambda\gamma\mu V_0 \\
&= V_0\delta^5 + (V_1-V_0)\delta^4 + (V_2-V_1-V_0)\delta^3 + (V_3-V_2-V_1-V_0)\delta^2 \\
&\quad +(V_4-V_3-V_2-V_1-V_0)\delta + (V_5-V_4-V_3-V_2-V_1-V_0),
\end{aligned}
$$

$$
\begin{aligned}
c_5 &= V_5 - (\alpha+\beta+\gamma+\delta+\mu)V_4 \\
&\quad +(\alpha\beta+\alpha\gamma+\alpha\mu+\alpha\delta+\beta\gamma+\beta\mu+\beta\delta+\gamma\mu+\gamma\delta+\mu\delta)V_3 \\
&\quad -(\alpha\beta\gamma+\alpha\beta\mu+\alpha\beta\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\beta\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\gamma\mu\delta)V_2 \\
&\quad +(\alpha\beta\gamma\mu+\alpha\beta\gamma\delta+\alpha\beta\mu\delta+\alpha\gamma\mu\delta+\beta\gamma\mu\delta)V_1 - \alpha\beta\gamma\mu\delta V_0 \\
&= V_0\lambda^5 + (V_1-V_0)\lambda^4 + (V_2-V_1-V_0)\lambda^3 + (V_3-V_2-V_1-V_0)\lambda^2 \\
&\quad +(V_4-V_3-V_2-V_1-V_0)\lambda + (V_5-V_4-V_3-V_2-V_1-V_0),
\end{aligned}
$$

$$
\begin{aligned}
c_6 &= V_5 - (\alpha+\beta+\gamma+\delta+\lambda)V_4 \\
&\quad +(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)V_3 \\
&\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)V_2 \\
&\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta)V_1 - \alpha\beta\lambda\gamma\delta V_0 \\
&= V_0\mu^5 + (V_1-V_0)\mu^4 + (V_2-V_1-V_0)\mu^3 + (V_3-V_2-V_1-V_0)\mu^2 \\
&\quad +(V_4-V_3-V_2-V_1-V_0)\mu + (V_5-V_4-V_3-V_2-V_1-V_0).
\end{aligned}
$$

Next, using Theorem 6.5, we present the Binet formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas sequences.

Corollary 6.2 *Binet formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas sequences are*

$$
\begin{aligned}
J_n &= \frac{(\alpha^4-\alpha^2-2\alpha-3)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)} + \frac{(\beta^4-\beta^2-2\beta-3)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\
&\quad +\frac{(\gamma^4-\gamma^2-2\gamma-3)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)} + \frac{(\delta^4-\delta^2-2\delta-3)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\
&\quad +\frac{(\lambda^4-\lambda^2-2\lambda-3)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)} + \frac{(\mu^4-\mu^2-2\mu-3)\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},
\end{aligned}
$$

$$
\begin{aligned}
j_n &= \frac{(2\alpha^5-\alpha^4+2\alpha^3+2\alpha^2+2\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{(2\beta^5-\beta^4+2\beta^3+2\beta^2+2\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\
&\quad+\frac{(2\gamma^5-\gamma^4+2\gamma^3+2\gamma^2+2\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{(2\delta^5-\delta^4+2\delta^3+2\delta^2+2\delta+2)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\
&\quad+\frac{(2\lambda^5-\lambda^4+2\lambda^3+2\lambda^2+2\lambda+2)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{(2\mu^5-\mu^4+2\mu^3+2\mu^2+2\mu+2)\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},
\end{aligned}
$$

$$
\begin{aligned}
K_n &= \frac{(3\alpha^5-2\alpha^4-\alpha^3+3\alpha^2+3\alpha+3)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{(3\beta^5-2\beta^4-\beta^3+3\beta^2+3\beta+3)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\
&\quad+\frac{(3\gamma^5-2\gamma^4-\gamma^3+3\gamma^2+3\gamma+3)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{(3\delta^5-2\delta^4-\delta^3+3\delta^2+3\delta+3)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\
&\quad+\frac{(3\lambda^5-2\lambda^4-\lambda^3+3\lambda^2+3\lambda+3)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{(3\mu^5-2\mu^4-\mu^3+3\mu^2+3\mu+3)\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},
\end{aligned}
$$

$$
\begin{aligned}
Q_n &= \frac{(3\alpha^5-3\alpha^4-\alpha^3+3\alpha^2+3\alpha+3)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{(3\beta^5-3\beta^4-\beta^3+3\beta^2+3\beta+3)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\
&\quad+\frac{(3\gamma^5-3\gamma^4-\gamma^3+3\gamma^2+3\gamma+3)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{(3\delta^5-3\delta^4-\delta^3+3\delta^2+3\delta+3)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\
&\quad+\frac{(3\lambda^5-3\lambda^4-\lambda^3+3\lambda^2+3\lambda+3)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{(3\mu^5-3\mu^4-\mu^3+3\mu^2+3\mu+3)\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},
\end{aligned}
$$

$$
\begin{aligned}
S_n &= \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\
&\quad+\frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{\delta^{n+4}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\
&\quad+\frac{\lambda^{n+4}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{\mu^{n+4}}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},
\end{aligned}
$$

$R_n=\alpha^n+\beta^n+\gamma^n+\delta^n+\lambda^n+\mu^n,$

respectively.

Binet formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas sequences can be given in the following forms:

$$
\begin{aligned}
J_n &= \frac{5}{63}\alpha^n+\frac{1}{9}\beta^n+\left(-\frac{5}{18}i\sqrt{3}-\frac{1}{6}\right)\gamma^n \\
&\quad+\left(\frac{5}{18}i\sqrt{3}-\frac{1}{6}\right)\delta^n+\left(\frac{1}{14}-\frac{5}{42}i\sqrt{3}\right)\lambda^n+\left(\frac{5}{42}i\sqrt{3}+\frac{1}{14}\right)\mu^n,
\end{aligned}
$$

$$
\begin{aligned}
j_n &= \frac{26}{21}\alpha^n+\frac{1}{3}\beta^n+\frac{1}{6}i\sqrt{3}\gamma^n-\frac{1}{6}i\sqrt{3}\delta^n+\left(\frac{1}{7}i\sqrt{3}+\frac{3}{14}\right)\lambda^n \\
&\quad+\left(\frac{3}{14}-\frac{1}{7}i\sqrt{3}\right)\mu^n,
\end{aligned}
$$

$$K_n = \frac{11}{9}\alpha^n + \frac{1}{9}\beta^n + \left(\frac{7}{18}i\sqrt{3} + \frac{1}{3}\right)\gamma^n + \left(\frac{1}{3} - \frac{7}{18}i\sqrt{3}\right)\delta^n + \frac{1}{2}\lambda^n + \frac{1}{2}\mu^n,$$

$$\begin{aligned} Q_n &= \frac{61}{63}\alpha^n + \frac{2}{9}\beta^n + \left(\frac{4}{9}i\sqrt{3} + \frac{1}{3}\right)\gamma^n + \left(\frac{1}{3} - \frac{4}{9}i\sqrt{3}\right)\delta^n \\ &\quad + \left(\frac{1}{21}i\sqrt{3} + \frac{4}{7}\right)\lambda^n + \left(\frac{4}{7} - \frac{1}{21}i\sqrt{3}\right)\mu^n, \end{aligned}$$

$$\begin{aligned} S_n &= \frac{1}{63}\alpha^{n+4} - \frac{1}{9}\beta^{n+4} + \left(\frac{1}{36}i\sqrt{3} + \frac{1}{12}\right)\gamma^{n+4} + \left(\frac{1}{12} - \frac{1}{36}i\sqrt{3}\right)\delta^{n+4} \\ &\quad + \left(\frac{5}{84}i\sqrt{3} - \frac{1}{28}\right)\lambda^{n+4} + \left(-\frac{5}{84}i\sqrt{3} - \frac{1}{28}\right)\mu^{n+4}, \\ &= \frac{16}{63}\alpha^n - \frac{1}{9}\beta^n - \frac{1}{18}i\sqrt{3}\gamma^n + \frac{1}{18}i\sqrt{3}\delta^n + (-\frac{1}{21}i\sqrt{3} - \frac{1}{14})\lambda^n \\ &\quad +(\frac{1}{21}i\sqrt{3} - \frac{1}{14})\mu^n, \end{aligned}$$

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n + \mu^n.$$

6.1.1 Simson Formulas

The following theorem gives the generalization of Simson formula of generalized Hexanacci sequence $\{W_n\}$.

Theorem 6.6 (Simson Formula of Generalized Hexanacci Numbers) *For all integers n we have*

$$\begin{vmatrix} W_{n+5} & W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} & W_{n-5} \end{vmatrix} \tag{6.15}$$

$$= (-1)^n y^6 \begin{vmatrix} W_5 & W_4 & W_3 & W_2 & W_1 & W_0 \\ W_4 & W_3 & W_2 & W_1 & W_0 & W_{-1} \\ W_3 & W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\ W_2 & W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\ W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} \\ W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} & W_{-5} \end{vmatrix}.$$

Proof. (6.15) is given in Soykan [64]. □

A special case of the above theorem is the following theorem which gives Simson formula of the generalized sixth order Jacobsthal sequence $\{V_n\}$.

Theorem 6.7 (Simson Formula of Generalized Sixth Order Jacobsthal Numbers) *For all integers n we have*

$$\begin{vmatrix} V_{n+5} & V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} & V_{n-5} \end{vmatrix} = (-1)^n \times 2^n \begin{vmatrix} V_5 & V_4 & V_3 & V_2 & V_1 & V_0 \\ V_4 & V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_3 & V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} & V_{-5} \end{vmatrix}.$$

The previous Theorem gives the following results as particular examples.

Corollary 6.3 *Simson formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas numbers are given as*

$$\begin{vmatrix} J_{n+5} & J_{n+4} & J_{n+3} & J_{n+2} & J_{n+1} & J_n \\ J_{n+4} & J_{n+3} & J_{n+2} & J_{n+1} & J_n & J_{n-1} \\ J_{n+3} & J_{n+2} & J_{n+1} & J_n & J_{n-1} & J_{n-2} \\ J_{n+2} & J_{n+1} & J_n & J_{n-1} & J_{n-2} & J_{n-3} \\ J_{n+1} & J_n & J_{n-1} & J_{n-2} & J_{n-3} & J_{n-4} \\ J_n & J_{n-1} & J_{n-2} & J_{n-3} & J_{n-4} & J_{n-5} \end{vmatrix} = 35(-1)^{n+1} \times 2^{n-1},$$

$$\begin{vmatrix} j_{n+5} & j_{n+4} & j_{n+3} & j_{n+2} & j_{n+1} & j_n \\ j_{n+4} & j_{n+3} & j_{n+2} & j_{n+1} & j_n & j_{n-1} \\ j_{n+3} & j_{n+2} & j_{n+1} & j_n & j_{n-1} & j_{n-2} \\ j_{n+2} & j_{n+1} & j_n & j_{n-1} & j_{n-2} & j_{n-3} \\ j_{n+1} & j_n & j_{n-1} & j_{n-2} & j_{n-3} & j_{n-4} \\ j_n & j_{n-1} & j_{n-2} & j_{n-3} & j_{n-4} & j_{n-5} \end{vmatrix} = 9477(-1)^{n+1} \times 2^{n-4},$$

$$\begin{vmatrix} K_{n+5} & K_{n+4} & K_{n+3} & K_{n+2} & K_{n+1} & K_n \\ K_{n+4} & K_{n+3} & K_{n+2} & K_{n+1} & K_n & K_{n-1} \\ K_{n+3} & K_{n+2} & K_{n+1} & K_n & K_{n-1} & K_{n-2} \\ K_{n+2} & K_{n+1} & K_n & K_{n-1} & K_{n-2} & K_{n-3} \\ K_{n+1} & K_n & K_{n-1} & K_{n-2} & K_{n-3} & K_{n-4} \\ K_n & K_{n-1} & K_{n-2} & K_{n-3} & K_{n-4} & K_{n-5} \end{vmatrix} = 98637(-1)^{n+1} \times 2^{n-5},$$

$$\begin{vmatrix} Q_{n+5} & Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} & Q_{n-5} \end{vmatrix} = 8113(-1)^{n+1} \times 2^{n},$$

$$\begin{vmatrix} S_{n+5} & S_{n+4} & S_{n+3} & S_{n+2} & S_{n+1} & S_n \\ S_{n+4} & S_{n+3} & S_{n+2} & S_{n+1} & S_n & S_{n-1} \\ S_{n+3} & S_{n+2} & S_{n+1} & S_n & S_{n-1} & S_{n-2} \\ S_{n+2} & S_{n+1} & S_n & S_{n-1} & S_{n-2} & S_{n-3} \\ S_{n+1} & S_n & S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} \\ S_n & S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} & S_{n-5} \end{vmatrix} = (-1)^{n} \times 2^{n-1},$$

$$\begin{vmatrix} R_{n+5} & R_{n+4} & R_{n+3} & R_{n+2} & R_{n+1} & R_n \\ R_{n+4} & R_{n+3} & R_{n+2} & R_{n+1} & R_n & R_{n-1} \\ R_{n+3} & R_{n+2} & R_{n+1} & R_n & R_{n-1} & R_{n-2} \\ R_{n+2} & R_{n+1} & R_n & R_{n-1} & R_{n-2} & R_{n-3} \\ R_{n+1} & R_n & R_{n-1} & R_{n-2} & R_{n-3} & R_{n-4} \\ R_n & R_{n-1} & R_{n-2} & R_{n-3} & R_{n-4} & R_{n-5} \end{vmatrix} = 321489(-1)^{n+1} \times 2^{n-1}.$$

6.1.2 Some Identities

In this subsection, we obtain some identities of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas numbers. Firstly, we give some fundamental relations between $\{J_n\}$ and $\{j_n\}$.

Lemma 6.8 *The following equalities are true:*

$$
\begin{aligned}
234J_n &= -53j_{n+5} + 25j_{n+4} + 103j_{n+3} + 181j_{n+2} - 53j_{n+1} - 131j_n, \qquad (6.16)\\
117J_n &= -14j_{n+4} + 25j_{n+3} + 64j_{n+2} - 53j_{n+1} - 92j_n - 53j_{n-1},\\
117J_n &= 11j_{n+3} + 50j_{n+2} - 67j_{n+1} - 106j_n - 67j_{n-1} - 28j_{n-2},\\
117J_n &= 61j_{n+2} - 56j_{n+1} - 95j_n - 56j_{n-1} - 17j_{n-2} + 22j_{n-3},\\
117J_n &= 5j_{n+1} - 34j_n + 5j_{n-1} + 44j_{n-2} + 83j_{n-3} + 122j_{n-4},
\end{aligned}
$$

and

$$
\begin{aligned}
70j_n &= -26J_{n+5} + 109J_{n+4} - 11J_{n+3} + 49J_{n+2} + 19J_{n+1} + 34J_n,\\
70j_n &= 83J_{n+4} - 37J_{n+3} + 23J_{n+2} - 7J_{n+1} + 8J_n - 52J_{n-1},\\
70j_n &= 46J_{n+3} + 106J_{n+2} + 76J_{n+1} + 91J_n + 31J_{n-1} + 166J_{n-2},\\
70j_n &= 152J_{n+2} + 122J_{n+1} + 137J_n + 77J_{n-1} + 212J_{n-2} + 92J_{n-3},\\
70j_n &= 274J_{n+1} + 289J_n + 229J_{n-1} + 364J_{n-2} + 244J_{n-3} + 304J_{n-4}.
\end{aligned}
$$

Proof. Note that all the identities hold for all integers n. We only prove (6.16). If we write

$$J_n = a \times j_{n+5} + b \times j_{n+4} + c \times j_{n+3} + d \times j_{n+2} + e \times j_{n+1} + f \times j_n$$

and solve the system of equations

$$
\begin{aligned}
J_0 &= a \times j_5 + b \times j_4 + c \times j_3 + d \times j_2 + e \times j_1 + f \times j_0\\
J_1 &= a \times j_6 + b \times j_5 + c \times j_4 + d \times j_3 + e \times j_2 + f \times j_1\\
J_2 &= a \times j_7 + b \times j_6 + c \times j_5 + d \times j_4 + e \times j_3 + f \times j_2\\
J_3 &= a \times j_8 + b \times j_7 + c \times j_6 + d \times j_5 + e \times j_4 + f \times j_3\\
J_4 &= a \times j_9 + b \times j_8 + c \times j_7 + d \times j_6 + e \times j_5 + f \times j_4\\
J_5 &= a \times j_{10} + b \times j_9 + c \times j_8 + d \times j_7 + e \times j_6 + f \times j_5
\end{aligned}
$$

then we get $a = -\frac{53}{234}$, $b = \frac{25}{234}$, $c = \frac{103}{234}$, $d = \frac{181}{234}$, $e = -\frac{53}{234}$, $f = -\frac{131}{234}$. The other equalities can be proved similarly. $\square$

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present some fundamental relations between $\{J_n\}$ and $\{S_n\}$.

Lemma 6.9 *The following equalities are true:*

$$\begin{aligned}
16J_n &= 11S_{n+5} - 5S_{n+4} - 21S_{n+3} - 37S_{n+2} + 11S_{n+1} + 27S_n, \\
8J_n &= 3S_{n+4} - 5S_{n+3} - 13S_{n+2} + 11S_{n+1} + 19S_n + 11S_{n-1}, \\
4J_n &= -S_{n+3} - 5S_{n+2} + 7S_{n+1} + 11S_n + 7S_{n-1} + 3S_{n-2}, \\
2J_n &= -3S_{n+2} + 3S_{n+1} + 5S_n + 3S_{n-1} + S_{n-2} - S_{n-3}, \\
J_n &= S_n - S_{n-2} - 2S_{n-3} - 3S_{n-4},
\end{aligned}$$

and

$$\begin{aligned}
70S_n &= 18J_{n+5} - 27J_{n+4} + 13J_{n+3} - 7J_{n+2} + 3J_{n+1} - 2J_n, \\
70S_n &= -9J_{n+4} + 31J_{n+3} + 11J_{n+2} + 21J_{n+1} + 16J_n + 36J_{n-1}, \\
70S_n &= 22J_{n+3} + 2J_{n+2} + 12J_{n+1} + 7J_n + 27J_{n-1} - 18J_{n-2}, \\
70S_n &= 24J_{n+2} + 34J_{n+1} + 29J_n + 49J_{n-1} + 4J_{n-2} + 44J_{n-3}, \\
70S_n &= 58J_{n+1} + 53J_n + 73J_{n-1} + 28J_{n-2} + 68J_{n-3} + 48J_{n-4}.
\end{aligned}$$

Thirdly, we give some fundamental relations between $\{J_n\}$ and $\{R_n\}$.

Lemma 6.10 *The following equalities are true:*

$$\begin{aligned}
3969J_n &= 178R_{n+5} + 73R_{n+4} - 137R_{n+3} - 557R_{n+2} - 1397R_{n+1} - 431R_n, \\
3969J_n &= 251R_{n+4} + 41R_{n+3} - 379R_{n+2} - 1219R_{n+1} - 253R_n + 356R_{n-1}, \\
3969J_n &= 292R_{n+3} - 128R_{n+2} - 968R_{n+1} - 2R_n + 607R_{n-1} + 502R_{n-2}, \\
3969J_n &= 164R_{n+2} - 676R_{n+1} + 290R_n + 899R_{n-1} + 794R_{n-2} + 584R_{n-3}, \\
3969J_n &= -512R_{n+1} + 454R_n + 1063R_{n-1} + 958R_{n-2} + 748R_{n-3} + 328R_{n-4},
\end{aligned}$$

and

$$\begin{aligned}
35R_n &= -73J_{n+5} + 127J_{n+4} + 27J_{n+3} + 77J_{n+2} + 52J_{n+1} + 117J_n, \\
35R_n &= 54J_{n+4} - 46J_{n+3} + 4J_{n+2} - 21J_{n+1} + 44J_n - 146J_{n-1}, \\
35R_n &= 8J_{n+3} + 58J_{n+2} + 33J_{n+1} + 98J_n - 92J_{n-1} + 108J_{n-2}, \\
35R_n &= 66J_{n+2} + 41J_{n+1} + 106J_n - 84J_{n-1} + 116J_{n-2} + 16J_{n-3}, \\
35R_n &= 107J_{n+1} + 172J_n - 18J_{n-1} + 182J_{n-2} + 82J_{n-3} + 132J_{n-4}.
\end{aligned}$$

Next, we present some fundamental relations between $\{j_n\}$ and $\{S_n\}$.

Lemma 6.11 *The following equalities are true:*

$$
\begin{aligned}
8j_n &= S_{n+5} + S_{n+4} + S_{n+3} + S_{n+2} + S_{n+1} - 23S_n, \\
4j_n &= S_{n+4} + S_{n+3} + S_{n+2} + S_{n+1} - 11S_n + S_{n-1}, \\
2j_n &= S_{n+3} + S_{n+2} + S_{n+1} - 5S_n + S_{n-1} + S_{n-2}, \\
j_n &= S_{n+2} + S_{n+1} - 2S_n + S_{n-1} + S_{n-2} + S_{n-3}, \\
j_n &= 2S_{n+1} - S_n + 2S_{n-1} + 2S_{n-2} + 2S_{n-3} + 2S_{n-4},
\end{aligned}
$$

and

$$
\begin{aligned}
117S_n &= j_{n+5} + j_{n+4} + j_{n+3} + j_{n+2} + j_{n+1} - 38j_n, \\
117S_n &= 2j_{n+4} + 2j_{n+3} + 2j_{n+2} + 2j_{n+1} - 37j_n + 2j_{n-1}, \\
117S_n &= 4j_{n+3} + 4j_{n+2} + 4j_{n+1} - 35j_n + 4j_{n-1} + 4j_{n-2}, \\
117S_n &= 8j_{n+2} + 8j_{n+1} - 31j_n + 8j_{n-1} + 8j_{n-2} + 8j_{n-3}, \\
117S_n &= 16j_{n+1} - 23j_n + 16j_{n-1} + 16j_{n-2} + 16j_{n-3} + 16j_{n-4}.
\end{aligned}
$$

Now, we give some fundamental relations between $\{j_n\}$ and $\{R_n\}$.

Lemma 6.12 *The following equalities are true:*

$$
\begin{aligned}
2646j_n &= -295R_{n+5} + 377R_{n+4} + 398R_{n+3} + 440R_{n+2} + 524R_{n+1} + 692R_n \\
2646j_n &= 82R_{n+4} + 103R_{n+3} + 145R_{n+2} + 229R_{n+1} + 397R_n - 590R_{n-1} \\
2646j_n &= 185R_{n+3} + 227R_{n+2} + 311R_{n+1} + 479R_n - 508R_{n-1} + 164R_{n-2} \\
2646j_n &= 412R_{n+2} + 496R_{n+1} + 664R_n - 323R_{n-1} + 349R_{n-2} + 370R_{n-3} \\
2646j_n &= 908R_{n+1} + 1076R_n + 89R_{n-1} + 761R_{n-2} + 782R_{n-3} + 824R_{n-4}
\end{aligned}
$$

and

$$
\begin{aligned}
78R_n &= 27j_{n+5} + j_{n+4} - 25j_{n+3} - 51j_{n+2} - 233j_{n+1} + 53j_n, \\
39R_n &= 14j_{n+4} + j_{n+3} - 12j_{n+2} - 103j_{n+1} + 40j_n + 27j_{n-1}, \\
39R_n &= 15j_{n+3} + 2j_{n+2} - 89j_{n+1} + 54j_n + 41j_{n-1} + 28j_{n-2}, \\
39R_n &= 17j_{n+2} - 74j_{n+1} + 69j_n + 56j_{n-1} + 43j_{n-2} + 30 \times j_{n-3}, \\
39R_n &= -57j_{n+1} + 86j_n + 73j_{n-1} + 60j_{n-2} + 47j_{n-3} + 34j_{n-4}.
\end{aligned}
$$

Next, we present some fundamental relations between $\{S_n\}$ and $\{R_n\}$.

Lemma 6.13 *The following equalities are true:*

$$
\begin{aligned}
7938S_n &= 379R_{n+5} - 293R_{n+4} - 314R_{n+3} - 356R_{n+2} - 440R_{n+1} - 608R_n, \\
7938S_n &= 86R_{n+4} + 65R_{n+3} + 23R_{n+2} - 61R_{n+1} - 229R_n + 758R_{n-1}, \\
7938S_n &= 151R_{n+3} + 109R_{n+2} + 25R_{n+1} - 143R_n + 844R_{n-1} + 172R_{n-2}, \\
7938S_n &= 260R_{n+2} + 176R_{n+1} + 8R_n + 995R_{n-1} + 323R_{n-2} + 302R_{n-3}, \\
7938S_n &= 436R_{n+1} + 268R_n + 1255R_{n-1} + 583R_{n-2} + 562R_{n-3} + 520R_{n-4},
\end{aligned}
$$

and

$$
\begin{aligned}
16R_n &= -15S_{n+5} + S_{n+4} + 17S_{n+3} + 33S_{n+2} + 145S_{n+1} - 31S_n, \\
8R_n &= -7S_{n+4} + S_{n+3} + 9S_{n+2} + 65S_{n+1} - 23S_n - 15S_{n-1}, \\
4R_n &= -3S_{n+3} + S_{n+2} + 29S_{n+1} - 15S_n - 11S_{n-1} - 7S_{n-2}, \\
2R_n &= -S_{n+2} + 13S_{n+1} - 9S_n - 7S_{n-1} - 5S_{n-2} - 3S_{n-3}, \\
R_n &= 6S_{n+1} - 5S_n - 4S_{n-1} - 3S_{n-2} - 2S_{n-3} - S_{n-4}.
\end{aligned}
$$

Note that we also can give some other identities which related to $\{K_n\}$ and $\{Q_n\}$. For example, we have

$$
\begin{aligned}
16K_n &= -29S_{n+5} + 35S_{n+4} + 35S_{n+3} + 35S_{n+2} + 35S_{n+1} - 45S_n, \\
16Q_n &= -29S_{n+5} + 35S_{n+4} + 35S_{n+3} + 35S_{n+2} + 35S_{n+1} - 61S_n.
\end{aligned}
$$

6.1.3 Linear Sums

Sums of Terms with Positive Subscripts:

The following proposition presents some formulas of generalized sixth order Jacobsthal numbers with positive subscripts.

Proposition 6.14 *For $n \geq 0$ we have the following formula:*

$$\sum_{k=0}^{n} V_k = \frac{1}{6}(V_{n+6} - V_{n+4} - 2V_{n+3} - 3V_{n+2} - 4V_{n+1} - V_5 + V_3 + 2V_2 + 3V_1 + 4V_0)$$

Proof. Take $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 1$, $y = 2$, in Theorem 2.1 in [78]. □

From the last proposition, we have the following corollary which gives sum formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas numbers, respectively.

Corollary 6.4 *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} J_k = \frac{1}{6}(J_{n+6} - J_{n+4} - 2J_{n+3} - 3J_{n+2} - 4J_{n+1} + 5)$.

(b) $\sum_{k=0}^{n} j_k = \frac{1}{6}(j_{n+6} - j_{n+4} - 2j_{n+3} - 3j_{n+2} - 4j_{n+1} - 9)$.

(c) $\sum_{k=0}^{n} K_k = \frac{1}{6}(K_{n+6} - K_{n+4} - 2K_{n+3} - 3K_{n+2} - 4K_{n+1} - 9)$.

(d) $\sum_{k=0}^{n} Q_k = \frac{1}{6}(Q_{n+6} - Q_{n+4} - 2Q_{n+3} - 3Q_{n+2} - 4Q_{n+1} - 8)$.

(e) $\sum_{k=0}^{n} S_k = \frac{1}{6}(S_{n+6} - S_{n+4} - 2S_{n+3} - 3S_{n+2} - 4S_{n+1} - 1)$.

(f) $\sum_{k=0}^{n} R_k = \frac{1}{6}(R_{n+6} - R_{n+4} - 2R_{n+3} - 3R_{n+2} - 4R_{n+1} + 9)$.

Sums of Terms with Negative Subscripts

The following proposition presents some formulas of generalized sixth order Jacobsthal numbers with negative subscripts.

Proposition 6.15 *For $n \geq 1$ we have the following formula:*

$$\sum_{k=1}^{n} V_{-k} = \frac{1}{6}(-V_{-n+5} + V_{-n+3} + 2V_{-n+2} + 3V_{-n+1} + 4V_{-n} + V_5 - V_3 - 2V_2 - 3V_1 - 4V_0).$$

Proof. Take $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 1$, $y = 2$, in Theorem 3.1 in [78]. $\square$

From the last proposition, we have the following corollary which gives sum formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas numbers, respectively.

Corollary 6.5 *For $n \geq 1$, we have the following formulas:*

(a) $\sum_{k=1}^{n} J_{-k} = \frac{1}{6}(-J_{-n+5} + J_{-n+3} + 2J_{-n+2} + 3J_{-n+1} + 4J_{-n} - 5)$.

(b) $\sum_{k=1}^{n} j_{-k} = \frac{1}{6}(-j_{-n+5} + j_{-n+3} + 2j_{-n+2} + 3j_{-n+1} + 4j_{-n} + 9)$.

(c) $\sum_{k=1}^{n} K_{-k} = \frac{1}{6}(-K_{-n+5} + K_{-n+3} + 2K_{-n+2} + 3K_{-n+1} + 4K_{-n} + 9)$.

(d) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{6}(-Q_{-n+5} + Q_{-n+3} + 2Q_{-n+2} + 3Q_{-n+1} + 4Q_{-n} + 8)$.

(e) $\sum_{k=1}^{n} S_{-k} = \frac{1}{6}(-S_{-n+5} + S_{-n+3} + 2S_{-n+2} + 3S_{-n+1} + 4S_{-n} + 1)$.

(f) $\sum_{k=1}^{n} R_{-k} = \frac{1}{6}(-R_{-n+5} + R_{-n+3} + 2R_{-n+2} + 3R_{-n+1} + 4R_{-n} - 9)$.

6.1.4 A Sum Formula

The formula in the following proposition can be used to calculate the sum of the generalized sixth order Jacobsthal numbers.

Proposition 6.16 *For all integers m and j with $R_m^4-4R_m^3+3R_{2m}^2+12R_m^2-6R_m^2R_{2m}+8R_{3m}R_m+12R_{2m}R_m-6R_{4m}-8R_{3m}-12R_{2m}-24R_m-24\left(-2\right)^m\left(R_{-m}-1\right)+24\neq 0$, we have*

$$\sum_{k=0}^{n}V_{mk+j}=\frac{\Lambda+\Psi_1}{\Omega} \tag{6.17}$$

where

$\Lambda=-24V_{mn+5m+j}+24(R_m-1)V_{mn+4m+j}-12(R_m^2-2R_m-R_{2m}+2)V_{mn+3m+j}+4(R_m^3-3R_m^2-3R_{2m}R_m+3R_{2m}+2R_{3m}+6R_m-6)V_{mn+2m+j}+(-R_m^4+4R_m^3+6R_m^2R_{2m}-12R_m^2-3R_{2m}^2-12R_{2m}R_m-8R_{3m}R_m+6R_{4m}+8R_{3m}+12R_{2m}+24R_m-24)V_{mn+m+j}+24\left(-2\right)^mV_{mn+j}$

$\Psi_1=24V_{5m+j}-24(R_m-1)V_{4m+j}+12(R_m^2-2R_m-R_{2m}+2)V_{3m+j}-4(R_m^3-3R_m^2-3R_{2m}R_m+3R_{2m}+2R_{3m}+6R_m-6)V_{2m+j}-(-R_m^4+4R_m^3+6R_m^2R_{2m}-12R_m^2-3R_{2m}^2-12R_{2m}R_m-8R_{3m}R_m+6R_{4m}+8R_{3m}+12R_{2m}+24R_m-24)V_{m+j}-(-R_m^4+4R_m^3-12R_m^2+6R_m^2R_{2m}-3R_{2m}^2-8R_{3m}R_m-12R_{2m}R_m+6R_{4m}+8R_{3m}+12R_{2m}+24R_m+24\left(-2\right)^mR_{-m}-24)V_j$

$\Omega=R_m^4-4R_m^3+3R_{2m}^2+12R_m^2-6R_m^2R_{2m}+8R_{3m}R_m+12R_{2m}R_m-6R_{4m}-8R_{3m}-12R_{2m}-24R_m-24\left(-2\right)^m\left(R_{-m}-1\right)+24$

Proof. Take $r=1,s=1,t=1,u=1,v=1,y=2$ and $R_n=H_n$ in Soykan [73, Theorem 13]. □

Note that (6.17) can be written in the following form:

$$\sum_{k=1}^{n}V_{mk+j}=\frac{\Lambda+\Psi_2}{\Omega}$$

where

$\Psi_2=24V_{5m+j}-24(R_m-1)V_{4m+j}+12(R_m^2-2R_m-R_{2m}+2)V_{3m+j}-4(R_m^3-3R_m^2-3R_{2m}R_m+3R_{2m}+2R_{3m}+6R_m-6)V_{2m+j}-(-R_m^4+4R_m^3+6R_m^2R_{2m}-12R_m^2-3R_{2m}^2-12R_{2m}R_m-8R_{3m}R_m+6R_{4m}+8R_{3m}+12R_{2m}+24R_m-24)V_{m+j}-24\left(-y\right)^mV_j.$

As special cases of the above proposition, we have the following identities.

Corollary 6.6 *The following identities hold:*

1. $m=1,\ j=3.$

(a) $\sum_{k=0}^{n} J_{k+3} = \frac{1}{6}(J_{n+8} - J_{n+6} - 2J_{n+5} - 3J_{n+4} + 2J_{n+3} - 7).$

(b) $\sum_{k=0}^{n} j_{k+3} = \frac{1}{6}(j_{n+8} - j_{n+6} - 2j_{n+5} - 3j_{n+4} + 2j_{n+3} - 57).$

(c) $\sum_{k=0}^{n} K_{k+3} = \frac{1}{6}(K_{n+8} - K_{n+6} - 2K_{n+5} - 3K_{n+4} + 2K_{n+3} - 51).$

(d) $\sum_{k=0}^{n} Q_{k+3} = \frac{1}{6}(Q_{n+8} - Q_{n+6} - 2Q_{n+5} - 3Q_{n+4} + 2Q_{n+3} - 38).$

(e) $\sum_{k=0}^{n} S_{k+3} = \frac{1}{6}(S_{n+8} - S_{n+6} - 2S_{n+5} - 3S_{n+4} + 2S_{n+3} - 13).$

(f) $\sum_{k=0}^{n} R_{k+3} = \frac{1}{6}(R_{n+8} - R_{n+6} - 2R_{n+5} - 3R_{n+4} + 2R_{n+3} - 51).$

2. $m = -1,\ j = 0.$

(a) $\sum_{k=0}^{n} J_{-k} = \frac{1}{6}\left(-J_{-n} - 6J_{-n-1} - 5J_{-n-2} - 4J_{-n-3} - 3J_{-n-4} - 2J_{-n-5} - 5\right).$

(b) $\sum_{k=0}^{n} j_{-k} = \frac{1}{6}\left(-j_{-n} - 6j_{-n-1} - 5j_{-n-2} - 4j_{-n-3} - 3j_{-n-4} - 2j_{-n-5} + 21\right).$

(c) $\sum_{k=0}^{n} K_{-k} = \frac{1}{6}\left(-K_{-n} - 6K_{-n-1} - 5K_{-n-2} - 4K_{-n-3} - 3K_{-n-4} - 2K_{-n-5} + 27\right).$

(d) $\sum_{k=0}^{n} Q_{-k} = \frac{1}{6}\left(-Q_{-n} - 6Q_{-n-1} - 5Q_{-n-2} - 4Q_{-n-3} - 3Q_{-n-4} - 2Q_{-n-5} + 26\right).$

(e) $\sum_{k=0}^{n} S_{-k} = \frac{1}{6}\left(-S_{-n} - 6S_{-n-1} - 5S_{-n-2} - 4S_{-n-3} - 3S_{-n-4} - 2S_{-n-5} + 1\right).$

(f) $\sum_{k=0}^{n} R_{-k} = \frac{1}{6}\left(-R_{-n} - 6R_{-n-1} - 5R_{-n-2} - 4R_{-n-3} - 3R_{-n-4} - 2R_{-n-5} + 27\right).$

6.1.5 Matrices Related with Generalized Sixth Order Jacobsthal numbers

In this subsection, we consider the following square matrix A of order 6 as:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -2$. We consider the following matrices:

$$B_n = \begin{pmatrix} S_{n+1} & \sum\limits_{k=3}^{7} r_{k-1}S_{n+3-k} & \sum\limits_{k=3}^{6} r_{k}S_{n+3-k} & \sum\limits_{k=3}^{5} r_{k+1}S_{n+3-k} & \sum\limits_{k=3}^{4} r_{k+2}S_{n+3-k} & r_6S_n \\ S_{n} & \sum\limits_{k=4}^{8} r_{k-2}S_{n+3-k} & \sum\limits_{k=4}^{7} r_{k-1}S_{n+3-k} & \sum\limits_{k=4}^{6} r_{k}S_{n+3-k} & \sum\limits_{k=4}^{5} r_{k+1}S_{n+3-k} & r_6S_{n-1} \\ S_{n-1} & \sum\limits_{k=5}^{9} r_{k-3}S_{n+3-k} & \sum\limits_{k=5}^{8} r_{k-2}S_{n+3-k} & \sum\limits_{k=5}^{7} r_{k-1}S_{n+3-k} & \sum\limits_{k=5}^{6} r_{k}S_{n+3-k} & r_6S_{n-2} \\ S_{n-2} & \sum\limits_{k=6}^{10} r_{k-4}S_{n+3-k} & \sum\limits_{k=6}^{9} r_{k-3}S_{n+3-k} & \sum\limits_{k=6}^{8} r_{k-2}S_{n+3-k} & \sum\limits_{k=6}^{7} r_{k-1}S_{n+3-k} & r_6S_{n-3} \\ S_{n-3} & \sum\limits_{k=7}^{11} r_{k-5}S_{n+3-k} & \sum\limits_{k=7}^{10} r_{k-4}S_{n+3-k} & \sum\limits_{k=7}^{9} r_{k-3}S_{n+3-k} & \sum\limits_{k=7}^{8} r_{k-2}S_{n+3-k} & r_6S_{n-4} \\ S_{n-4} & \sum\limits_{k=8}^{12} r_{k-6}S_{n+3-k} & \sum\limits_{k=8}^{11} r_{k-5}S_{n+3-k} & \sum\limits_{k=8}^{10} r_{k-4}S_{n+3-k} & \sum\limits_{k=8}^{9} r_{k-3}S_{n+3-k} & r_6S_{n-5} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & \sum\limits_{k=3}^{7} r_{k-1}V_{n+3-k} & \sum\limits_{k=3}^{6} r_{k}V_{n+3-k} & \sum\limits_{k=3}^{5} r_{k+1}V_{n+3-k} & \sum\limits_{k=3}^{4} r_{k+2}V_{n+3-k} & r_6V_n \\ V_{n} & \sum\limits_{k=4}^{8} r_{k-2}V_{n+3-k} & \sum\limits_{k=4}^{7} r_{k-1}V_{n+3-k} & \sum\limits_{k=4}^{6} r_{k}V_{n+3-k} & \sum\limits_{k=4}^{5} r_{k+1}V_{n+3-k} & r_6V_{n-1} \\ V_{n-1} & \sum\limits_{k=5}^{9} r_{k-3}V_{n+3-k} & \sum\limits_{k=5}^{8} r_{k-2}V_{n+3-k} & \sum\limits_{k=5}^{7} r_{k-1}V_{n+3-k} & \sum\limits_{k=5}^{6} r_{k}V_{n+3-k} & r_6V_{n-2} \\ V_{n-2} & \sum\limits_{k=6}^{10} r_{k-4}V_{n+3-k} & \sum\limits_{k=6}^{9} r_{k-3}V_{n+3-k} & \sum\limits_{k=6}^{8} r_{k-2}V_{n+3-k} & \sum\limits_{k=6}^{7} r_{k-1}V_{n+3-k} & r_6V_{n-3} \\ V_{n-3} & \sum\limits_{k=7}^{11} r_{k-5}V_{n+3-k} & \sum\limits_{k=7}^{10} r_{k-4}V_{n+3-k} & \sum\limits_{k=7}^{9} r_{k-3}V_{n+3-k} & \sum\limits_{k=7}^{8} r_{k-2}V_{n+3-k} & r_6V_{n-4} \\ V_{n-4} & \sum\limits_{k=8}^{12} r_{k-6}V_{n+3-k} & \sum\limits_{k=8}^{11} r_{k-5}V_{n+3-k} & \sum\limits_{k=8}^{10} r_{k-4}V_{n+3-k} & \sum\limits_{k=8}^{9} r_{k-3}V_{n+3-k} & r_6V_{n-5} \end{pmatrix}$$

where

$r_1 = 1, r_2 = 1, r_3 = 1, r_4 = 1, r_5 = 1, r_6 = 2.$

Theorem 6.17 *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$

(b) $C_1A^n = A^nC_1$

(c) $C_{n+m} = C_nB_m = B_mC_n.$

Proof. Take $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 1$, $y = 2$ and $S_n = G_n$ in Soykan [73, Theorem 16]. $\square$

Theorem 6.18 *For all integers m, n, we have*

$$\begin{aligned} V_{n+m} &= V_n S_{m+1} + V_{n-1}(S_m + S_{m-1} + S_{m-2} + S_{m-3} + 2S_{m-4}) \\ &\quad + V_{n-2}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + V_{n-3}(S_m + S_{m-1} + 2S_{m-2}) + V_{n-4}(S_m + 2S_{m-1}) + 2S_m V_{n-5}. \end{aligned}$$

Proof. Take $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 1$, $y = 2$ and $S_n = G_n$ in Soykan [73, Theorem 17]. □

Corollary 6.7 *For all integers m, n, we have*

$$\begin{aligned} J_{n+m} &= J_n S_{m+1} + J_{n-1}(S_m + S_{m-1} + S_{m-2} + S_{m-3} + 2S_{m-4}) \\ &\quad + J_{n-2}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + J_{n-3}(S_m + S_{m-1} + 2S_{m-2}) + J_{n-4}(S_m + 2S_{m-1}) \\ &\quad + 2S_m J_{n-5}, \end{aligned}$$

$$\begin{aligned} j_{n+m} &= j_n S_{m+1} + j_{n-1}(S_m + S_{m-1} + S_{m-2} + S_{m-3} + 2S_{m-4}) \\ &\quad + j_{n-2}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + j_{n-3}(S_m + S_{m-1} + 2S_{m-2}) + j_{n-4}(S_m + 2S_{m-1}) \\ &\quad + 2S_m j_{n-5}, \end{aligned}$$

$$\begin{aligned} K_{n+m} &= K_n S_{m+1} + K_{n-1}(S_m + S_{m-1} + S_{m-2} + S_{m-3} + 2S_{m-4}) \\ &\quad + K_{n-2}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + K_{n-3}(S_m + S_{m-1} + 2S_{m-2}) + K_{n-4}(S_m + 2S_{m-1}) \\ &\quad + 2S_m K_{n-5}, \end{aligned}$$

$$\begin{aligned} Q_{n+m} &= Q_n S_{m+1} + Q_{n-1}(S_m + S_{m-1} + S_{m-2} + S_{m-3} + 2S_{m-4}) \\ &\quad + Q_{n-2}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + Q_{n-3}(S_m + S_{m-1} + 2S_{m-2}) + Q_{n-4}(S_m + 2S_{m-1}) \\ &\quad + 2S_m Q_{n-5}, \end{aligned}$$

$$\begin{aligned} S_{n+m} &= S_n S_{m+1} + S_{n-1}(S_m + S_{m-1} + S_{m-2} + S_{m-3} + 2S_{m-4}) \\ &\quad + S_{n-2}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + S_{n-3}(S_m + S_{m-1} + 2S_{m-2}) + S_{n-4}(S_m + 2S_{m-1}) \\ &\quad + 2S_m S_{n-5}, \end{aligned}$$

$$
\begin{aligned}
R_{n+m} &= R_n S_{m+1} + R_{n-1}(S_m + S_{m-1} + S_{m-2} + S_{m-3} + 2S_{m-4}) \\
&\quad + R_{n-2}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + R_{n-3}(S_m + S_{m-1} + 2S_{m-2}) + R_{n-4}(S_m + 2S_{m-1}) \\
&\quad + 2S_m R_{n-5}.
\end{aligned}
$$

CHAPTER 7

MATRIX SEQUENCES OF GENERALIZED THIRD ORDER JACOBSTHAL NUMBERS

Our purpose in this chapter is to construct matrix sequences of generalized third order Jacobsthal numbers and to examine the relationship between generalized third order Jacobsthal matrix sequences and their special cases. The matrix sequences have taken so much interest for different type of numbers. Here are some researches on matrix sequences of numbers: [17, 18, 29, 88, 89, 90, 91, 95, 97] for generalized Fibonacci numbers, [13, 70, 71, 84, 98, 100] for generalized Tribonacci numbers, [69] for generalized Tetranacci numbers. We state that the results of this chapter are cited from [81] which has been published by Soykan and Eyican Polatli.

7.1 THE MATRIX SEQUENCES OF THIRD ORDER JACOBSTHAL AND THIRD ORDER JACOBSTHAL-LUCAS NUMBERS

In this section, we define generalized third order Jacobsthal matrix sequence and investigate their properties.

Definition 7.1 *For any integer $n \geq 0$, the third order Jacobsthal matrix sequence $(\mathcal{V}_n)$ and third order Jacobsthal-Lucas matrix sequence $(\mathcal{M}_n)$ are defined by*

$$\mathcal{V}_n = \mathcal{V}_{n-1} + \mathcal{V}_{n-2} + 2\mathcal{V}_{n-3} \tag{7.1}$$

with initial conditions

$$\mathcal{V}_0 = \begin{pmatrix} V_1 & V_2 - V_1 & 2V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 - V_0 \\ \frac{1}{2}(V_2 - V_1 - V_0) & \frac{1}{2}(3V_0 + V_1 - V_2) & \frac{1}{2}(3V_1 - V_0 - V_2) \end{pmatrix},$$

$$\mathcal{V}_1 = \begin{pmatrix} V_2 & 2V_0 + V_1 & 2V_1 \\ V_1 & V_2 - V_1 & 2V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 - V_0 \end{pmatrix},$$

$$\mathcal{V}_2 = \begin{pmatrix} 2V_0+V_1+V_2 & 2V_1+V_2 & 2V_2 \\ V_2 & 2V_0+V_1 & 2V_1 \\ V_1 & V_2-V_1 & 2V_0 \end{pmatrix}.$$

The sequence $\{\mathcal{V}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{V}_{-n} = -\frac{1}{2}\mathcal{V}_{-(n-1)} - \frac{1}{2}\mathcal{V}_{-(n-2)} + \frac{1}{2}\mathcal{V}_{-(n-3)}$$

for $n \geq 1$. Therefore, recurrence (7.1) holds for all integers n.

Three special cases of generalized third order Jacobsthal matrix sequence (take $V_n = J_n$, $V_n = j_n$, $V_n = K_n$, respectively) can be defined as follows.

Definition 7.2 *For any integer $n \geq 0$, the third order Jacobsthal matrix sequence $(\mathcal{J}_n)$ and third order Jacobsthal-Lucas matrix sequence $(\mathcal{M}_n)$ and modified third order Jacobsthal matrix sequence $(\mathcal{K}_n)$ are defined by*

$$\begin{aligned} \mathcal{J}_n &= \mathcal{J}_{n-1} + \mathcal{J}_{n-2} + 2\mathcal{J}_{n-3}, \\ \mathcal{M}_n &= \mathcal{M}_{n-1} + \mathcal{M}_{n-2} + 2\mathcal{M}_{n-3}, \\ \mathcal{K}_n &= \mathcal{K}_{n-1} + \mathcal{K}_{n-2} + 2\mathcal{K}_{n-3}, \end{aligned}$$

respectively, with initial conditions

$$\mathcal{J}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{J}_1 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{J}_2 = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 4 & 4 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}, \mathcal{M}_1 = \begin{pmatrix} 5 & 5 & 2 \\ 1 & 4 & 4 \\ 2 & -1 & 2 \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} 10 & 7 & 10 \\ 5 & 5 & 2 \\ 1 & 4 & 4 \end{pmatrix},$$

$$\mathcal{K}_0 = \begin{pmatrix} 1 & 2 & 6 \\ 3 & -2 & -1 \\ -\frac{1}{2} & \frac{7}{2} & -\frac{3}{2} \end{pmatrix}, \mathcal{K}_1 = \begin{pmatrix} 3 & 7 & 2 \\ 1 & 2 & 6 \\ 3 & -2 & -1 \end{pmatrix}, \mathcal{K}_2 = \begin{pmatrix} 10 & 5 & 6 \\ 3 & 7 & 2 \\ 1 & 2 & 6 \end{pmatrix}.$$

The sequences $\{\mathcal{J}_n\}_{n\geq 0}$, $\{\mathcal{M}_n\}_{n\geq 0}$ and $\{\mathcal{K}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{J}_{-n} = -\frac{1}{2}\mathcal{J}_{-(n-1)} - \frac{1}{2}\mathcal{J}_{-(n-2)} + \frac{1}{2}\mathcal{J}_{-(n-3)},$$

$$\mathcal{M}_{-n} = -\frac{1}{2}\mathcal{M}_{-(n-1)} - \frac{1}{2}\mathcal{M}_{-(n-2)} + \frac{1}{2}\mathcal{M}_{-(n-3)},$$

$$\mathcal{K}_{-n} = -\frac{1}{2}\mathcal{K}_{-(n-1)} - \frac{1}{2}\mathcal{K}_{-(n-2)} + \frac{1}{2}\mathcal{K}_{-(n-3)},$$

for $n = 1, 2, 3, ...$ respectively.

The following theorem gives the nth general terms of the generalized third order Jacobsthal matrix sequence.

Theorem 7.1 *For any integer n, the nth general term of the generalized third order Jacobsthal matrix sequence is*

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}. \tag{7.2}$$

Proof. Suppose that $n \geq 0$. We prove (7.2) by strong mathematical induction on n. If $n = 0$ then, since $V_{-1} = \frac{1}{2}V_2 - \frac{1}{2}V_1 - \frac{1}{2}V_0$, $V_{-2} = -\frac{1}{4}V_2 + \frac{3}{4}V_1 - \frac{1}{4}V_0$, $V_{-3} = -\frac{1}{8}V_2 - \frac{1}{8}V_1 + \frac{7}{8}V_0$, we have

$$\begin{aligned} \mathcal{V}_0 &= \begin{pmatrix} V_1 & V_0 + 2V_{-1} & 2V_0 \\ V_0 & V_{-1} + 2V_{-2} & 2V_{-1} \\ V_{-1} & V_{-2} + 2V_{-3} & 2V_{-2} \end{pmatrix} \\ &= \begin{pmatrix} V_1 & V_2 - V_1 & 2V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 - V_0 \\ \frac{1}{2}(V_2 - V_1 - V_0) & \frac{1}{2}(3V_0 + V_1 - V_2) & \frac{1}{2}(3V_1 - V_0 - V_2) \end{pmatrix} \end{aligned}$$

which is true. Assume that the equality holds for $n \le k$. For $n = k+1$, we have

$$\begin{aligned}
\mathcal{V}_{k+1} &= \mathcal{V}_{k+1-1} + \mathcal{V}_{k+1-2} + 2\mathcal{V}_{k+1-3} \\
&= \mathcal{V}_{k} + \mathcal{V}_{k-1} + 2\mathcal{V}_{k-2} \\
&= \begin{pmatrix} V_{k+1} & V_k + 2V_{k-1} & 2V_k \\ V_k & V_{k-1} + 2V_{k-2} & 2V_{k-1} \\ V_{k-1} & V_{k-2} + 2V_{k-3} & 2V_{k-2} \end{pmatrix} + \begin{pmatrix} V_{k-1+1} & V_{k-1} + 2V_{k-1-1} & 2V_{k-1} \\ V_{k-1} & V_{k-1-1} + 2V_{k-1-2} & 2V_{k-1-1} \\ V_{k-1-1} & V_{k-1-2} + 2V_{k-1-3} & 2V_{k-1-2} \end{pmatrix} \\
&\quad +2 \begin{pmatrix} V_{k-2+1} & V_{k-2} + 2V_{k-2-1} & 2V_{k-2} \\ V_{k-2} & V_{k-2-1} + 2V_{k-2-2} & 2V_{k-2-1} \\ V_{k-2-1} & V_{k-2-2} + 2V_{k-2-3} & 2V_{k-2-2} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+1} + V_k + 2V_{k-1} & V_k + 3V_{k-1} + 4V_{k-2} + 4V_{k-3} & 2V_k + 2V_{k-1} + 4V_{k-2} \\ V_k + V_{k-1} + 2V_{k-2} & V_{k-1} + 3V_{k-2} + 4V_{k-3} + 4V_{k-4} & 2V_{k-1} + 2V_{k-2} + 4V_{k-3} \\ V_{k-1} + V_{k-2} + 2V_{k-3} & V_{k-2} + 3V_{k-3} + 4V_{k-4} + 4V_{k-5} & 2V_{k-2} + 2V_{k-3} + 4V_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+2} & V_{k+1} + 2V_k & 2V_{k+1} \\ V_{k+1} & V_k + 2V_{k-1} & 2V_k \\ V_k & V_{k-1} + 2V_{k-2} & 2V_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+1+1} & V_{k+1} + 2V_{k+1-1} & 2V_{k+1} \\ V_{k+1} & V_{k+1-1} + 2V_{k+1-2} & 2V_{k+1-1} \\ V_{k+1-1} & V_{k+1-2} + 2V_{k+1-3} & 2V_{k+1-2} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on $k+1$, this proves (7.2).

For the case $n \le 0$, similarly, (7.2) can be proved by strong mathematical induction on n. □

The following theorem gives the nth general terms of the third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal matrix sequences.

Corollary 7.1 *For any integer n, we have the following formulas of the matrix sequences:*

$$\mathcal{J}_n = \begin{pmatrix} J_{n+1} & J_n + 2J_{n-1} & 2J_n \\ J_n & J_{n-1} + 2J_{n-2} & 2J_{n-1} \\ J_{n-1} & J_{n-2} + 2J_{n-3} & 2J_{n-2} \end{pmatrix},$$

$$\mathcal{M}_n = \begin{pmatrix} j_{n+1} & j_n + 2j_{n-1} & 2j_n \\ j_n & j_{n-1} + 2j_{n-2} & 2j_{n-1} \\ j_{n-1} & j_{n-2} + 2j_{n-3} & 2j_{n-2} \end{pmatrix},$$

$$\mathcal{K}_n = \begin{pmatrix} K_{n+1} & K_n + 2K_{n-1} & 2K_n \\ K_n & K_{n-1} + 2K_{n-2} & 2K_{n-1} \\ K_{n-1} & K_{n-2} + 2K_{n-3} & 2K_{n-2} \end{pmatrix}.$$

We now give the Binet formula for the generalized third order Jacobsthal matrix sequence.

Theorem 7.2 *For every integer n, the Binet formula of the generalized third order Jacobsthal matrix sequence are given by*

$$\mathcal{V}_n = A\alpha^n + B\beta^n + C\gamma^n$$

where

$$A = \frac{\alpha\mathcal{V}_2 + \alpha(\alpha-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)},$$

$$B = \frac{\beta\mathcal{V}_2 + \beta(\beta-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\beta(\beta-\gamma)(\beta-\alpha)},$$

$$C = \frac{\gamma\mathcal{V}_2 + \gamma(\gamma-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.$$

Proof. We need to prove the theorem only for $n \geq 0$. By the assumption, the characteristic equation of (7.1) is $x^3 - x^2 - x - 2 = 0$ and the roots of it are α, β and γ. So it's general solution is given by

$$\mathcal{V}_n = A\alpha^n + B\beta^n + C\gamma^n.$$

Using initial condition which is given in Definition 7.1, and also applying lineer algebra operations, we obtain the matrices A, B, C as desired. This gives the formula for $\mathcal{V}_n$. □

The following theorem gives the Binet formulas of the third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal matrix sequences.

Corollary 7.2 *For every integer n, the Binet formulas of the third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal matrix sequences are given by*

$$\mathcal{J}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n,$$

$$\mathcal{M}_n = A_2\alpha^n + B_2\beta^n + C_2\gamma^n,$$

$$\mathcal{K}_n = A_3\alpha^n + B_3\beta^n + C_3\gamma^n,$$

where

$$\begin{aligned}
A_1 &= \frac{\alpha\mathcal{J}_2+\alpha(\alpha-1)\mathcal{J}_1+2\mathcal{J}_0}{\alpha\left(\alpha-\gamma\right)\left(\alpha-\beta\right)}, B_1=\frac{\beta\mathcal{J}_2+\beta(\beta-1)\mathcal{J}_1+2\mathcal{J}_0}{\beta\left(\beta-\gamma\right)\left(\beta-\alpha\right)},\\
C_1 &= \frac{\gamma\mathcal{J}_2+\gamma(\gamma-1)\mathcal{J}_1+2\mathcal{J}_0}{\gamma\left(\gamma-\beta\right)\left(\gamma-\alpha\right)},
\end{aligned}$$

$$\begin{aligned}
A_2 &= \frac{\alpha\mathcal{M}_2+\alpha(\alpha-1)\mathcal{M}_1+2\mathcal{M}_0}{\alpha\left(\alpha-\gamma\right)\left(\alpha-\beta\right)}, B_2=\frac{\beta\mathcal{M}_2+\beta(\beta-1)\mathcal{M}_1+2\mathcal{M}_0}{\beta\left(\beta-\gamma\right)\left(\beta-\alpha\right)},\\
C_2 &= \frac{\gamma\mathcal{M}_2+\gamma(\gamma-1)\mathcal{M}_1+2\mathcal{M}_0}{\gamma\left(\gamma-\beta\right)\left(\gamma-\alpha\right)},
\end{aligned}$$

$$\begin{aligned}
A_3 &= \frac{\alpha\mathcal{K}_2+\alpha(\alpha-1)\mathcal{K}_1+2\mathcal{K}_0}{\alpha\left(\alpha-\gamma\right)\left(\alpha-\beta\right)}, B_3=\frac{\beta\mathcal{K}_2+\beta(\beta-1)\mathcal{K}_1+2\mathcal{K}_0}{\beta\left(\beta-\gamma\right)\left(\beta-\alpha\right)},\\
C_3 &= \frac{\gamma\mathcal{K}_2+\gamma(\gamma-1)\mathcal{K}_1+2\mathcal{K}_0}{\gamma\left(\gamma-\beta\right)\left(\gamma-\alpha\right)}.
\end{aligned}$$

Now we will obtain these functions in terms of generalized third order Jacobsthal matrix sequence as a consequence of Theorems 7.1 and 7.2. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

Corollary 7.3 *For every integers n, the Binet formula for the generalized third order Jacobsthal numbers is given as*

$$V_n=\frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$\begin{aligned}
b_1 &= V_2-(\beta+\gamma)V_1+\beta\gamma V_0,\ b_2=V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0,\\
b_3 &= V_2-(\alpha+\beta)V_1+\alpha\beta V_0.
\end{aligned}$$

Proof. From Theorem 7.2, we have

$$\begin{aligned}
\mathcal{V}_n &= A\alpha^n + B\beta^n + C\gamma^n \\
&= \frac{\alpha\mathcal{V}_2 + \alpha(\alpha-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}\alpha^n + \frac{\beta\mathcal{V}_2 + \beta(\beta-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n \\
&\quad + \frac{\gamma\mathcal{V}_2 + \gamma(\gamma-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\
&= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}\begin{pmatrix} . & . & . \\ \alpha V_2 + \alpha(\alpha-1)V_1 + 2V_0 & . & . \\ . & . & . \end{pmatrix} \\
&\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}\begin{pmatrix} . & . & . \\ \beta V_2 + \beta(\beta-1)V_1 + 2V_0 & . & . \\ . & . & . \end{pmatrix} \\
&\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}\begin{pmatrix} . & . & . \\ \gamma V_2 + \gamma(\gamma-1)V_1 + 2V_0 & . & . \\ . & . & . \end{pmatrix}.
\end{aligned}$$

(we only write the 2nd row and 1st column entries of the matrices). By Theorem 7.1, we know that

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$\begin{aligned}
V_n &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}(\alpha V_2 + \alpha(\alpha-1)V_1 + 2V_0) \\
&\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}(\beta V_2 + \beta(\beta-1)V_1 + 2V_0) \\
&\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}(\gamma V_2 + \gamma(\gamma-1)V_1 + 2V_0) \\
&= \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0, \\
b_3 &= V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0.
\end{aligned}$$

Note that

$$\begin{aligned}
\alpha V_2+\alpha\left(\alpha-1\right)V_1+2V_0 &= \alpha(V_2+\left(\alpha-1\right)V_1+\frac{2}{\alpha}V_0) \\
&= \alpha(V_2-(\beta+\gamma)V_1+\beta\gamma V_0)=\alpha b_1, \\
\beta V_2+\beta\left(\beta-1\right)V_1+2V_0 &= \beta(V_2+\left(\beta-1\right)V_1+\frac{2}{\beta}V_0) \\
&= \beta(V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0)=\beta b_2, \\
\gamma V_2+\gamma\left(\gamma-1\right)V_1+2V_0 &= \gamma(V_2+\left(\gamma-1\right)V_1+\frac{2}{\gamma}V_0) \\
&= \gamma(V_2-(\alpha+\beta)V_1+\alpha\beta V_0)=\gamma b_3.
\end{aligned}$$

□

Now, we present summation formulas for the generalized third order Jacobsthal matrix sequence.

Theorem 7.3 *For all integers m,j we have*

$$\sum_{k=0}^{n-1}\mathcal{V}_{mk+j}=\frac{\mathcal{V}_{mn+m+j}+2^m\mathcal{V}_{mn-m+j}+(1-K_m)\mathcal{V}_{mn+j}-\mathcal{V}_{m+j}-2^m\mathcal{V}_{j-m}+(K_m-1)\mathcal{V}_j}{K_m+2^m(1-K_{-m})-1} \tag{7.3}$$

Proof. Note that

$$\begin{aligned}
\sum_{i=0}^{n-1}\mathcal{V}_{mi+j} &= \sum_{i=0}^{n-1}(A\alpha^{mi+j}+B\beta^{mi+j}+C\gamma^{mi+j}) \\
&= A\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right)+B\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right)+C\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right)
\end{aligned}$$

Simplifying and rearranging the last equalities in the last two expression imply (7.3) as required. □

As in Corollary 7.3, in the proof of next corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices to obtain summation formula for the generalized third order Jacobsthal sequence.

Corollary 7.4 *For all integers m,j we have*

$$\sum_{k=0}^{n-1}V_{mk+j}=\frac{V_{mn+m+j}+2^mV_{mn-m+j}+(1-K_m)V_{mn+j}-V_{m+j}-2^mV_{j-m}+(K_m-1)V_j}{K_m+2^m(1-K_{-m})-1}.$$

We now give generating functions of $\mathcal{V}_n$.

Theorem 7.4 *The generating function for the generalized third order Jacobsthal matrix sequence is given as*

$$\sum_{n=0}^{\infty} \mathcal{V}_n x^n = \frac{\mathcal{V}_0 + (\mathcal{V}_1 - \mathcal{V}_0)x + (\mathcal{V}_2 - \mathcal{V}_1 - \mathcal{V}_0)x^2}{1 - x - x^2 - 2x^3}$$

$$= \frac{1}{1 - x - x^2 - 2x^3} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$\begin{aligned}
a_{11} &= V_1 + (-V_1 + V_2)x + 2V_0 x^2 \\
a_{21} &= V_0 + (-V_0 + V_1)x + (-V_0 - V_1 + V_2)x^2 \\
a_{31} &= \frac{1}{2}(V_2 - V_1 - V_0) + \frac{1}{2}(3V_0 + V_1 - V_2)\, x + \frac{1}{2}(-V_2 + 3V_1 - V_0)x^2 \\
a_{12} &= V_2 - V_1 + (2V_0 + 2V_1 - V_2)x + (-2V_0 + 2V_1)x^2 \\
a_{22} &= V_1 - V_0 + (V_0 - 2V_1 + V_2)x + (3V_0 + V_1 - V_2)x^2 \\
a_{32} &= \frac{1}{2}(3V_2 - 5V_1 - V_0)x^2 + \frac{1}{2}(V_2 + V_1 - 5V_0)\, x + \frac{1}{2}(3V_0 + V_1 - V_2) \\
a_{13} &= (-2V_0 - 2V_1 + 2V_2)x^2 + (-2V_0 + 2V_1)x + 2V_0 \\
a_{23} &= V_2 - V_1 - V_0 + (3V_0 + V_1 - V_2)x + (-V_0 + 3V_1 - V_2)x^2 \\
a_{33} &= \frac{1}{2}(3V_2 - 5V_1 - V_0)x + \frac{1}{2}(7V_0 - V_1 - V_2)x^2 + \frac{1}{2}(-V_2 + 3V_1 - V_0)
\end{aligned}$$

Proof. Suppose that $g(x) = \sum_{n=0}^{\infty} \mathcal{V}_n x^n$ is the generating function for the sequence $\{\mathcal{V}_n\}_{n\geq 0}$. Using the definition of the matrix sequence of generalized third order Jacobsthal numbers (7.1), and substracting $x\sum_{n=0}^{\infty} \mathcal{V}_n x^n$, $x^2\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ and $2x^3\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ from $\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ we obtain

$$\begin{aligned}
(1 - x - x^2 - 2x^3)\sum_{n=0}^{\infty} \mathcal{V}_n x^n &= \sum_{n=0}^{\infty} \mathcal{V}_n x^n - x\sum_{n=0}^{\infty} \mathcal{V}_n x^n - x^2\sum_{n=0}^{\infty} \mathcal{V}_n x^n - 2x^3\sum_{n=0}^{\infty} \mathcal{V}_n x^n \\
&= \sum_{n=0}^{\infty} \mathcal{V}_n x^n - \sum_{n=0}^{\infty} \mathcal{V}_n x^{n+1} - \sum_{n=0}^{\infty} \mathcal{V}_n x^{n+2} - 2\sum_{n=0}^{\infty} \mathcal{V}_n x^{n+3} \\
&= \sum_{n=0}^{\infty} \mathcal{V}_n x^n - \sum_{n=1}^{\infty} \mathcal{V}_{n-1} x^n - \sum_{n=2}^{\infty} \mathcal{V}_{n-2} x^n - 2\sum_{n=3}^{\infty} \mathcal{V}_{n-3} x^n \\
&= (\mathcal{V}_0 + \mathcal{V}_1 x + \mathcal{V}_2 x^2) - (\mathcal{V}_0 x + \mathcal{V}_1 x^2) - \mathcal{V}_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (\mathcal{V}_n - \mathcal{V}_{n-1} - \mathcal{V}_{n-2} - 2\mathcal{V}_{n-3}) x^n \\
&= \mathcal{V}_0 + \mathcal{V}_1 x + \mathcal{V}_2 x^2 - \mathcal{V}_0 x - \mathcal{V}_1 x^2 - \mathcal{V}_0 x^2 \\
&= \mathcal{V}_0 + (\mathcal{V}_1 - \mathcal{V}_0)x + (\mathcal{V}_2 - \mathcal{V}_1 - \mathcal{V}_0)x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} \mathcal{V}_n x^n = \frac{\mathcal{V}_0 + (\mathcal{V}_1 - \mathcal{V}_0)x + (\mathcal{V}_2 - \mathcal{V}_1 - \mathcal{V}_0)x^2}{1 - x - x^2 - 2x^3}$$

which equals the $\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ in the Theorem. This completes the proof. $\square$

The following corollary gives the generating functions of the third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal matrix sequences.

Corollary 7.5 *The generating functions for the third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal matrix sequences are given as*

$$\sum_{n=0}^{\infty} \mathcal{J}_n x^n = \frac{1}{1 - x - x^2 - 2x^3} \begin{pmatrix} 1 & 2x^2 + x & 2x \\ x & 1 - x & 2x^2 \\ x^2 & x - x^2 & -x^2 - x + 1 \end{pmatrix},$$

$$\sum_{n=0}^{\infty} \mathcal{M}_n x^n = \frac{1}{1 - x - x^2 - 2x^3} \begin{pmatrix} 4x^2 + 4x + 1 & -2x^2 + x + 4 & 4x^2 - 2x + 4 \\ 2x^2 - x + 2 & 2x^2 + 5x - 1 & -4x^2 + 2x + 2 \\ -2x^2 + x + 1 & 4x^2 - 2x + 1 & 4x^2 + 4x - 2 \end{pmatrix},$$

$$\sum_{n=0}^{\infty} \mathcal{K}_n x^n = \frac{1}{1 - x - x^2 - 2x^3} \begin{pmatrix} 6x^2 + 2x + 1 & -4x^2 + 5x + 2 & -2x^2 - 4x + 6 \\ -x^2 - 2x + 3 & 7x^2 + 4x - 2 & -3x^2 + 7x - 1 \\ -\frac{3}{2}x^2 + \frac{7}{2}x - \frac{1}{2} & \frac{1}{2}x^2 - \frac{11}{2}x + \frac{7}{2} & \frac{17}{2}x^2 + \frac{1}{2}x - \frac{3}{2} \end{pmatrix}.$$

The well known generating function for generalized third order Jacobsthal numbers is as in (3.16). However, we will obtain these functions in terms of generalized third order Jacobsthal matrix sequences as a consequence of Theorem 7.4. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 7.4. Thus we have the following corollary.

Corollary 7.6 *The generating function for the generalized third order Jacobsthal sequence $\{V_n\}$ is given as*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - 2x^3}.$$

Using Theorem 7.1 and Corollary 7.1, we see that

$$\mathcal{V}_{-1} = \begin{pmatrix} V_0 & V_1 - V_0 & V_2 - V_1 - V_0 \\ \frac{1}{2}(V_2 - V_1 - V_0) & \frac{1}{2}(3V_0 + V_1 - V_2) & \frac{1}{2}(3V_1 - V_0 - V_2) \\ \frac{1}{4}(3V_1 - V_0 - V_2) & \frac{1}{4}(3V_2 - 5V_1 - V_0) & \frac{1}{4}(7V_0 - V_1 - V_2) \end{pmatrix},$$

$$\mathcal{V}_{-2} = \begin{pmatrix} \frac{1}{2}(V_2 - V_1 - V_0) & \frac{1}{2}(3V_0 + V_1 - V_2) & \frac{1}{2}(3V_1 - V_0 - V_2) \\ \frac{1}{4}(3V_1 - V_0 - V_2) & \frac{1}{4}(3V_2 - 5V_1 - V_0) & \frac{1}{4}(7V_0 - V_1 - V_2) \\ \frac{1}{8}(7V_0 - V_1 - V_2) & \frac{1}{8}(7V_1 - 9V_0 - V_2) & \frac{1}{8}(7V_2 - 9V_1 - 9V_0) \end{pmatrix},$$

and

$$\begin{aligned}
\mathcal{J}_{-1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \ \mathcal{J}_{-2} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \end{pmatrix}, \\
\mathcal{M}_{-1} &= \begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & -2 \\ -1 & 2 & 2 \end{pmatrix}, \ \mathcal{M}_{-2} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 2 \\ 1 & -2 & 1 \end{pmatrix}, \\
\mathcal{K}_{-1} &= \begin{pmatrix} 3 & -2 & -1 \\ -\frac{1}{2} & \frac{7}{2} & -\frac{3}{2} \\ -\frac{3}{4} & \frac{1}{4} & \frac{17}{4} \end{pmatrix}, \ \mathcal{K}_{-2} = \begin{pmatrix} -\frac{1}{2} & \frac{7}{2} & -\frac{3}{2} \\ -\frac{3}{4} & \frac{1}{4} & \frac{17}{4} \\ \frac{17}{8} & -\frac{23}{8} & -\frac{15}{8} \end{pmatrix}.
\end{aligned}$$

We now give generating functions of the generalized third order Jacobsthal matrix sequence $\mathcal{V}_n$ for negative indices.

Theorem 7.5 *For negative indices, the generating function for the generalized third order Jacobsthal matrix sequence is given as*

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{V}_{-n} x^n &= \frac{2\mathcal{V}_0 + (\mathcal{V}_2 - \mathcal{V}_1)x + \mathcal{V}_1 x^2}{2 + x + x^2 - x^3} \\
&= \frac{1}{2 + x + x^2 - x^3} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
b_{11} &= 2V_1 + (2V_0 + V_1)\, x + V_2 x^2 \\
b_{21} &= 2V_0 + (-V_1 + V_2)\, x + V_1 x^2 \\
b_{31} &= V_2 - V_1 - V_0 + (-V_0 + V_1)\, x + V_0 x^2
\end{aligned}$$

and

$$\begin{aligned}
b_{12} &= 2V_2 - 2V_1 + x\,(V_1 - 2V_0 + V_2) + x^2\,(2V_0 + V_1) \\
b_{22} &= 2V_1 - 2V_0 + (2V_0 + 2V_1 - V_2)\, x + (-V_1 + V_2)\, x^2 \\
b_{32} &= -V_2 + V_1 + 3V_0 + (V_0 - 2V_1 + V_2)\, x + (-V_0 + V_1)\, x^2
\end{aligned}$$

and

$$\begin{aligned}
b_{13} &= 4V_0 + (-2V_1 + 2V_2)\,x + 2V_1x^2 \\
b_{23} &= 2V_2 - 2V_1 - 2V_0 + (-2V_0 + 2V_1)\,x + 2V_0x^2 \\
b_{33} &= -V_2 + 3V_1 - V_0 + (3V_0 + V_1 - V_2)\,x + (-V_0 - V_1 + V_2)\,x^2
\end{aligned}$$

Proof. Then, using Definition 7.1, and substracting $-\frac{1}{2}x\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$, $-\frac{1}{2}x^2\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$ and $\frac{1}{2}x^3\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$ from $\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$, we obtain

$$\begin{aligned}
(1+\frac{1}{2}x+\frac{1}{2}x^2-\frac{1}{2}x^3)\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n &= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n+\frac{1}{2}x\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n+\frac{1}{2}x^2\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n \\
&\quad -\frac{1}{2}x^3\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n \\
&= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n+\frac{1}{2}\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+1}+\frac{1}{2}\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+2} \\
&\quad -\frac{1}{2}\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+3} \\
&= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n+\frac{1}{2}\sum_{n=1}^{\infty}\mathcal{V}_{-n+1}x^n+\frac{1}{2}\sum_{n=2}^{\infty}\mathcal{V}_{-n+2}x^n \\
&\quad -\frac{1}{2}\sum_{n=3}^{\infty}\mathcal{V}_{-n+3}x^n \\
&= (\mathcal{V}_0+\mathcal{V}_{-1}x+\mathcal{V}_{-2}x^2)+\frac{1}{2}(\mathcal{V}_0x+\mathcal{V}_{-1}x^2) \\
&\quad +\frac{1}{2}\mathcal{V}_0x^2 \\
&\quad +\sum_{n=3}^{\infty}(\mathcal{V}_{-n}+\frac{1}{2}\mathcal{V}_{-n+1}+\frac{1}{2}\mathcal{V}_{-n+2}-\frac{1}{2}\mathcal{V}_{-n+3})x^n \\
&= (\mathcal{V}_0+\mathcal{V}_{-1}x+\mathcal{V}_{-2}x^2)+\frac{1}{2}(\mathcal{V}_0x+\mathcal{V}_{-1}x^2) \\
&\quad +\frac{1}{2}\mathcal{V}_0x^2 \\
&= \mathcal{V}_0+\frac{1}{2}(\mathcal{V}_2-\mathcal{V}_1)x+\frac{1}{2}\mathcal{V}_1x^2
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{V}_{-1} &= \frac{1}{t}(\mathcal{V}_2-r\mathcal{V}_1-s\mathcal{V}_0), \\
\mathcal{V}_{-2} &= \frac{1}{t^2}(-s\mathcal{V}_2+t\mathcal{V}_1+s^2\mathcal{V}_0+rs\mathcal{V}_1-rt\mathcal{V}_0).
\end{aligned}$$

Rearranging above equation, we get

$$\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n=\frac{2\mathcal{V}_0+(\mathcal{V}_2-\mathcal{V}_1)x+\mathcal{V}_1x^2}{2+x+x^2-x^3}$$

which equals the $\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n}$ in the Theorem. $\square$

The following corollary gives the generating functions of the third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal matrix sequences with negative indices .

Corollary 7.7 *The generating functions for the third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal matrix sequences with negative indices are given as*

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{J}_{-n}x^{n} &= \frac{1}{2+x+x^{2}-x^{3}}\begin{pmatrix} x^{2}+x+2 & x^{2}+2x & 2x^{2} \\ x^{2} & x+2 & 2x \\ x & x^{2}-x & 2 \end{pmatrix}, \\
\sum_{n=0}^{\infty}\mathcal{M}_{-n}x^{n} &= \frac{1}{2+x+x^{2}-x^{3}}\begin{pmatrix} 5x^{2}+5x+2 & 5x^{2}+2x+8 & 2x^{2}+8x+8 \\ x^{2}+4x+4 & 4x^{2}+x-2 & 4x^{2}-2x+4 \\ 2x^{2}-x+2 & -x^{2}+5x+2 & 2x^{2}+2x-4 \end{pmatrix}, \\
\sum_{n=0}^{\infty}\mathcal{K}_{-n}x^{n} &= \frac{1}{2+x+x^{2}-x^{3}}\begin{pmatrix} 3x^{2}+7x+2 & 7x^{2}-2x+4 & 2x^{2}+4x+12 \\ x^{2}+2x+6 & 2x^{2}+5x-4 & 6x^{2}-4x-2 \\ 3x^{2}-2x-1 & -2x^{2}+4x+7 & -x^{2}+7x-3 \end{pmatrix},
\end{aligned}
$$

respectively.

Now, we will obtain generating functions for generalized third order Jacobsthal numbers in terms of generalized third order Jacobsthal matrix sequences with negative indices as a consequence of Theorem 7.5. To do this, we will again compare the 2nd row and 1st column entries with the matrices in Theorem 7.5. Thus we have the following corollary.

Corollary 7.8 *The generating function for the generalized third order Jacobsthal sequence $\{V_{-n}\}_{n\geq 0}$ is given as*

$$
\sum_{n=0}^{\infty}V_{-n}x^{n}=\frac{2V_{0}+(-V_{1}+V_{2})\,x+V_{1}x^{2}}{2+x+x^{2}-x^{3}}.
$$

The previous corollary gives the following results as particular examples.

Corollary 7.9 *Generating functions of third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal numbers with negative indices are*

$$\begin{aligned}
\sum_{n=0}^{\infty} J_{-n}x^n &= \frac{2J_0 + (-J_1 + J_2)\,x + J_1x^2}{2 + x + x^2 - x^3} = \frac{x^2}{2 + x + x^2 - x^3}, \\
\sum_{n=0}^{\infty} j_{-n}x^n &= \frac{2j_0 + (-j_1 + j_2)\,x + j_1x^2}{2 + x + x^2 - x^3} = \frac{4 + 4x + x^2}{2 + x + x^2 - x^3}, \\
\sum_{n=0}^{\infty} K_{-n}x^n &= \frac{2K_0 + (-K_1 + K_2)\,x + K_1x^2}{2 + x + x^2 - x^3} = \frac{6 + 2x + x^2}{2 + x + x^2 - x^3},
\end{aligned}$$

respectively.

7.1.1 Some Identities

In this section, we assume that m and n are arbitrary integers, unless otherwise mentioned. We obtain some identities of generalized third order Jacobsthal and third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal numbers. First, we can give some fundamental relations between $\{V_n\}$ and $\{J_n\}$.

Lemma 7.6 *The following equalities are true:*

(a) $8V_n = (7V_0 - V_1 - V_2)J_{n+4} + (-9V_0 + 7V_1 - V_2)J_{n+3} + (-9V_0 - 9V_1 + 7V_2)J_{n+2}.$

(b) $4V_n = (-V_0 + 3V_1 - V_2)J_{n+3} + (-V_0 - 5V_1 + 3V_2)J_{n+2} + (7V_0 - V_1 - V_2)J_{n+1}.$

(c) $2V_n = (-V_0 - V_1 + V_2)J_{n+2} + (3V_0 + V_1 - V_2)J_{n+1} + (-V_0 + 3V_1 - V_2)J_n.$

(d) $V_n = V_0J_{n+1} + (V_1 - V_0)J_n + (V_2 - V_1 - V_0)J_{n-1}.$

(e) $V_n = V_1J_n + (V_2 - V_1)J_{n-1} + 2V_0J_{n-2}.$

Proof. Note that all the identities hold for all integers n. We prove (a). If we write

$$V_n = a \times J_{n+4} + b \times J_{n+3} + c \times J_{n+2}$$

and solve the system of equations

$$\begin{aligned}
V_0 &= a \times J_4 + b \times J_3 + c \times J_2 \\
V_1 &= a \times J_5 + b \times J_4 + c \times J_3 \\
V_2 &= a \times J_6 + b \times J_5 + c \times J_4
\end{aligned}$$

then we get $a = \frac{1}{8}(7V_0 - V_1 - V_2)$, $b = \frac{1}{8}(-9V_0 + 7V_1 - V_2)$, $c = \frac{1}{8}(7V_2 - 9V_1 - 9V_0)$. The other equalities can be proved similarly. $\square$

Note that all the identities in the above lemma can be proved by induction as well.

Next, we present some fundamental relations between $\{J_n\}$ and $\{V_n\}$.

Lemma 7.7 *The following equalities are true:*

(a) $2(4V_0^3+4V_0^2V_1+2V_0^2V_2+3V_0V_1^2-5V_0V_1V_2-V_0V_2^2+3V_1^3-2V_1V_2^2+V_2^3)J_n = (-V_1^2-V_1V_2-2V_0V_1+V_2^2)V_{n+4}+(3V_1^2+V_1V_2+2V_0V_1-V_2^2-2V_0V_2)V_{n+3}+(4V_0^2+4V_0V_1+2V_0V_2+V_1^2-V_1V_2-V_2^2)V_{n+2}$.

(b) $(4V_0^3+4V_0^2V_1+2V_0^2V_2+3V_0V_1^2-5V_0V_1V_2-V_0V_2^2+3V_1^3-2V_1V_2^2+V_2^3)J_n = (V_1^2-V_0V_2)V_{n+3}+(2V_0^2+V_0V_1+V_0V_2-V_1V_2)V_{n+2}+(-V_1^2-V_1V_2-2V_0V_1+V_2^2)V_{n+1}$.

(c) $(4V_0^3+4V_0^2V_1+2V_0^2V_2+3V_0V_1^2-5V_0V_1V_2-V_0V_2^2+3V_1^3-2V_1V_2^2+V_2^3)J_n = (2V_0^2+V_0V_1+V_1^2-V_2V_1)V_{n+2}+(V_2^2-2V_0V_1-V_0V_2-V_1V_2)V_{n+1}+2(V_1^2-V_0V_2)V_n$.

(d) $(4V_0^3+4V_0^2V_1+2V_0^2V_2+3V_0V_1^2-5V_0V_1V_2-V_0V_2^2+3V_1^3-2V_1V_2^2+V_2^3)J_n = (2V_0^2-V_0V_1-V_0V_2+V_1^2-2V_1V_2+V_2^2)V_{n+1}+(2V_0^2+V_0V_1-2V_2V_0+3V_1^2-V_2V_1)V_n+2(2V_0^2+V_0V_1+V_1^2-V_2V_1)V_{n-1}$.

(e) $(4V_0^3+4V_0^2V_1+2V_0^2V_2+3V_0V_1^2-5V_0V_1V_2-V_0V_2^2+3V_1^3-2V_1V_2^2+V_2^3)J_n = (4V_0^2-3V_0V_2+4V_1^2-3V_1V_2+V_2^2)V_n+(6V_0^2+V_0V_1-V_0V_2+3V_1^2-4V_1V_2+V_2^2)V_{n-1}+2(2V_0^2-V_0V_1-V_0V_2+V_1^2-2V_1V_2+V_2^2)V_{n-2}$.

Now, we give some fundamental relations between $\{V_n\}$ and $\{j_n\}$.

Lemma 7.8 *The following equalities are true:*

(a) $96V_n = (-27V_0+5V_1+5V_2)j_{n+4}+(37V_0-27V_1+5V_2)j_{n+3}+(37V_0+37V_1-27V_2)j_{n+2}$.

(b) $48V_n = (5V_0-11V_1+5V_2)j_{n+3}+(5V_0+21V_1-11V_2)j_{n+2}+(-27V_0+5V_1+5V_2)j_{n+1}$.

(c) $24V_n = (5V_0+5V_1-3V_2)j_{n+2}+(-11V_0-3V_1+5V_2)j_{n+1}+(5V_0-11V_1+5V_2)j_n$.

(d) $12V_n = (-3V_0+V_1+V_2)j_{n+1}+(5V_0-3V_1+V_2)j_n+(5V_0+5V_1-3V_2)j_{n-1}$.

(e) $6V_n = (V_0 - V_1 + V_2)j_n + (V_0 + 3V_1 - V_2)j_{n-1} + (-3V_0 + V_1 + V_2)j_{n-2}$.

Next, we present some fundamental relations between $\{j_n\}$ and $\{V_n\}$.

Lemma 7.9 *The following equalities are true:*

(a) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)j_n = (2V_0^2 + 3V_1^2 - V_2^2 + 3V_0V_1 - V_0V_2)V_{n+4} + (2V_0^2 - 3V_1^2 + 2V_2^2 - 3V_0V_1 + 2V_0V_2 - 3V_1V_2)V_{n+3} + 2(-2V_0^2 - 2V_0V_2 - 3V_1V_0 + V_2^2)V_{n+2}$.

(b) $(V_0+V_1+V_2)(4V_0^2+3V_1^2+V_2^2-2V_0V_2-3V_1V_2)j_n = (4V_0^2+V_2^2+V_0V_2-3V_1V_2)V_{n+3} + (-2V_0^2 - 3V_0V_1 - 5V_0V_2 + 3V_1^2 + V_2^2)V_{n+2} + 2(2V_0^2 + 3V_1^2 - V_2^2 + 3V_0V_1 - V_0V_2)V_{n+1}$.

(c) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)j_n = (2V_0^2 + 3V_1^2 + 2V_2^2 - 3V_0V_1 - 4V_0V_2 - 3V_1V_2)V_{n+2} + (8V_0^2 + 6V_1^2 - V_2^2 + 6V_0V_1 - V_0V_2 - 3V_1V_2)V_{n+1} + 2(4V_0^2 + V_2^2 + V_0V_2 - 3V_1V_2)V_n$.

(d) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)j_n = (10V_0^2 + 9V_1^2 + V_2^2 + 3V_0V_1 - 5V_0V_2 - 6V_1V_2)V_{n+1} + (10V_0^2 + 3V_1^2 + 4V_2^2 - 3V_0V_1 - 2V_0V_2 - 9V_1V_2)V_n + 2(2V_0^2 + 3V_1^2 + 2V_2^2 - 3V_0V_1 - 4V_0V_2 - 3V_1V_2)V_{n-1}$.

(e) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)j_n = (20V_0^2 + 12V_1^2 + 5V_2^2 - 7V_0V_2 - 15V_1V_2)V_n + (14V_0^2 + 15V_1^2 + 5V_2^2 - 3V_0V_1 - 13V_0V_2 - 12V_1V_2)V_{n-1} + 2(10V_0^2 + 9V_1^2 + V_2^2 + 3V_0V_1 - 5V_0V_2 - 6V_1V_2)V_{n-2}$.

Now, we give some fundamental relations between $\{V_n\}$ and $\{K_n\}$.

Lemma 7.10 *The following equalities are true:*

(a) $588V_n = (-85V_0 - 57V_1 + 55V_2)K_{n+4} + (195V_0 + 27V_1 - 57V_2)K_{n+3} + (-29V_0 + 195V_1 - 85V_2)K_{n+2}$.

(b) $294V_n = (55V_0 - 15V_1 - V_2)K_{n+3} + (-57V_0 + 69V_1 - 15V_2)K_{n+2} + (-85V_0 - 57V_1 + 55V_2)K_{n+1}$.

(c) $147V_n = (-V_0 + 27V_1 - 8V_2)K_{n+2} + (-15V_0 - 36V_1 + 27V_2)K_{n+1} + (55V_0 - 15V_1 - V_2)K_n$.

(d) $147V_n = (-16V_0-9V_1+19V_2)K_{n+1}+(54V_0+12V_1-9V_2)K_n+2(-V_0+27V_1-8V_2)K_{n-1}.$

(e) $147V_n = (38V_0 + 3V_1 + 10V_2)K_n + (-18V_0 + 45V_1 + 3V_2)K_{n-1} + 2(-16V_0 - 9V_1 + 19V_2)K_{n-2}.$

Next, we present some fundamental relations between $\{K_n\}$ and $\{V_n\}$.

Lemma 7.11 *The following equalities are true:*

(a) $4(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)K_n = (-4V_0^2 + 4V_0V_1 - 14V_0V_2 + 15V_1^2 + 5V_1V_2 - 3V_2^2)V_{n+4} + (28V_0^2 + 12V_0V_1 + 22V_0V_2 - 9V_1^2 - 15V_1V_2 + V_2^2)V_{n+3} + (-12V_0^2 - 40V_0V_1 - 2V_0V_2 - 21V_1^2 - 11V_1V_2 + 17V_2^2)V_{n+2}.$

(b) $2(4V_0^3+4V_0^2V_1+2V_0^2V_2+3V_0V_1^2-5V_0V_1V_2-V_0V_2^2+3V_1^3-2V_1V_2^2+V_2^3)K_n = (12V_0^2 + 8V_0V_1 + 4V_0V_2 + 3V_1^2 - 5V_1V_2 - V_2^2)V_{n+3} + (-8V_0^2 - 18V_0V_1 - 8V_0V_2 - 3V_1^2 - 3V_1V_2 + 7V_2^2)V_{n+2} + (-4V_0^2 + 4V_0V_1 - 14V_0V_2 + 15V_1^2 + 5V_1V_2 - 3V_2^2)V_{n+1}.$

(c) $(4V_0^3+4V_0^2V_1+2V_0^2V_2+3V_0V_1^2-5V_0V_1V_2-V_0V_2^2+3V_1^3-2V_1V_2^2+V_2^3)K_n = (2V_0^2 - 2V_0V_2 - 5V_1V_0 + 3V_2^2 - 4V_1V_2)V_{n+2} + (4V_0^2 + 6V_0V_1 - 5V_0V_2 + 9V_1^2 - 2V_2^2)V_{n+1} + (12V_0^2 + 8V_0V_1 + 4V_0V_2 + 3V_1^2 - 5V_1V_2 - V_2^2)V_n.$

(d) $(4V_0^3+4V_0^2V_1+2V_0^2V_2+3V_0V_1^2-5V_0V_1V_2-V_0V_2^2+3V_1^3-2V_1V_2^2+V_2^3)K_n = (6V_0^2 + V_0V_1 - 7V_0V_2 + 9V_1^2 - 4V_1V_2 + V_2^2)V_{n+1} + (14V_0^2 + 3V_0V_1 + 2V_0V_2 + 3V_1^2 - 9V_1V_2 + 2V_2^2)V_n + 2(2V_0^2 - 2V_0V_2 - 5V_1V_0 + 3V_2^2 - 4V_1V_2)V_{n-1}.$

(e) $(4V_0^3+4V_0^2V_1+2V_0^2V_2+3V_0V_1^2-5V_0V_1V_2-V_0V_2^2+3V_1^3-2V_1V_2^2+V_2^3)K_n = (20V_0^2 + 4V_0V_1 - 5V_0V_2 + 12V_1^2 - 13V_1V_2 + 3V_2^2)V_n + (10V_0^2 - 9V_0V_1 - 11V_0V_2 + 9V_1^2 - 12V_1V_2 + 7V_2^2)V_{n-1} + 2(6V_0^2 + V_0V_1 - 7V_0V_2 + 9V_1^2 - 4V_1V_2 + V_2^2)V_{n-2}.$

7.1.2 Relation Between Generalized third order Jacobsthal Matrix Sequences and It's Special Cases

In this subsection, we assume that m and n are arbitrary integers, unless otherwise mentioned.

The following theorem shows that there always exist interrelation between generalized third order Jacobsthal and third order Jacobsthal matrix sequences.

Theorem 7.12 *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{J}_n\}$, we have the following identities:*

(a) $8\mathcal{V}_n = (7V_0 - V_1 - V_2)\mathcal{J}_{n+4} + (-9V_0 + 7V_1 - V_2)\mathcal{J}_{n+3} + (-9V_0 - 9V_1 + 7V_2)\mathcal{J}_{n+2}$.

(b) $4\mathcal{V}_n = (-V_0 + 3V_1 - V_2)\mathcal{J}_{n+3} + (-V_0 - 5V_1 + 3V_2)\mathcal{J}_{n+2} + (7V_0 - V_1 - V_2)\mathcal{J}_{n+1}$.

(c) $2\mathcal{V}_n = (-V_0 - V_1 + V_2)\mathcal{J}_{n+2} + (3V_0 + V_1 - V_2)\mathcal{J}_{n+1} + (-V_0 + 3V_1 - V_2)\mathcal{J}_n$.

(d) $\mathcal{V}_n = V_0\mathcal{J}_{n+1} + (V_1 - V_0)\mathcal{J}_n + (V_2 - V_1 - V_0)\mathcal{J}_{n-1}$.

(e) $\mathcal{V}_n = V_1\mathcal{J}_n + (V_2 - V_1)\mathcal{J}_{n-1} + +2V_0\mathcal{J}_{n-2}$.

(f) $2(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (-V_1^2 - V_1V_2 - 2V_0V_1 + V_2^2)\mathcal{V}_{n+4} + (3V_1^2 + V_1V_2 + 2V_0V_1 - V_2^2 - 2V_0V_2)\mathcal{V}_{n+3} + (4V_0^2 + 4V_0V_1 + 2V_0V_2 + V_1^2 - V_1V_2 - V_2^2)\mathcal{V}_{n+2}$.

(g) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (V_1^2 - V_0V_2)\mathcal{V}_{n+3} + (2V_0^2 + V_0V_1 + V_0V_2 - V_1V_2)\mathcal{V}_{n+2} + (-V_1^2 - V_1V_2 - 2V_0V_1 + V_2^2)\mathcal{V}_{n+1}$.

(h) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (2V_0^2 + V_0V_1 + V_1^2 - V_2V_1)\mathcal{V}_{n+2} + (V_2^2 - 2V_0V_1 - V_0V_2 - V_1V_2)\mathcal{V}_{n+1} + 2(V_1^2 - V_0V_2)\mathcal{V}_n$.

(i) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (2V_0^2 - V_0V_1 - V_0V_2 + V_1^2 - 2V_1V_2 + V_2^2)\mathcal{V}_{n+1} + (2V_0^2 + V_0V_1 - 2V_2V_0 + 3V_1^2 - V_2V_1)\mathcal{V}_n + 2(2V_0^2 + V_0V_1 + V_1^2 - V_2V_1)\mathcal{V}_{n-1}$.

(j) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (4V_0^2 - 3V_0V_2 + 4V_1^2 - 3V_1V_2 + V_2^2)\mathcal{V}_n + (6V_0^2 + V_0V_1 - V_0V_2 + 3V_1^2 - 4V_1V_2 + V_2^2)\mathcal{V}_{n-1} + 2(2V_0^2 - V_0V_1 - V_0V_2 + V_1^2 - 2V_1V_2 + V_2^2)\mathcal{V}_{n-2}$.

Proof. From Lemmas 7.6 and 7.7, (a)-(j) follow. □

The following theorem shows that there always exist interrelation between generalized third order Jacobsthal and third order Jacobsthal-Lucas matrix sequences.

Theorem 7.13 *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{M}_n\}$, we have the following identities:*

(a) $96\mathcal{V}_n = (-27V_0 + 5V_1 + 5V_2)\mathcal{M}_{n+4} + (37V_0 - 27V_1 + 5V_2)\mathcal{M}_{n+3} + (37V_0 + 37V_1 - 27V_2)\mathcal{M}_{n+2}$.

(b) $48\mathcal{V}_n = (5V_0 - 11V_1 + 5V_2)\mathcal{M}_{n+3} + (5V_0 + 21V_1 - 11V_2)\mathcal{M}_{n+2} + (-27V_0 + 5V_1 + 5V_2)\mathcal{M}_{n+1}$.

(c) $24\mathcal{V}_n = (5V_0 + 5V_1 - 3V_2)\mathcal{M}_{n+2} + (-11V_0 - 3V_1 + 5V_2)\mathcal{M}_{n+1} + (5V_0 - 11V_1 + 5V_2)\mathcal{M}_n$.

(d) $12\mathcal{V}_n = (-3V_0 + V_1 + V_2)\mathcal{M}_{n+1} + (5V_0 - 3V_1 + V_2)\mathcal{M}_n + (5V_0 + 5V_1 - 3V_2)\mathcal{M}_{n-1}$.

(e) $6\mathcal{V}_n = (V_0 - V_1 + V_2)\mathcal{M}_n + (V_0 + 3V_1 - V_2)\mathcal{M}_{n-1} + (-3V_0 + V_1 + V_2)\mathcal{M}_{n-2}$.

(f) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (2V_0^2 + 3V_1^2 - V_2^2 + 3V_0V_1 - V_0V_2)\mathcal{V}_{n+4} + (2V_0^2 - 3V_1^2 + 2V_2^2 - 3V_0V_1 + 2V_0V_2 - 3V_1V_2)\mathcal{V}_{n+3} + 2(-2V_0^2 - 2V_0V_2 - 3V_1V_0 + V_2^2)\mathcal{V}_{n+2}$.

(g) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (4V_0^2 + V_2^2 + V_0V_2 - 3V_1V_2)\mathcal{V}_{n+3} + (-2V_0^2 - 3V_0V_1 - 5V_0V_2 + 3V_1^2 + V_2^2)\mathcal{V}_{n+2} + 2(2V_0^2 + 3V_1^2 - V_2^2 + 3V_0V_1 - V_0V_2)\mathcal{V}_{n+1}$.

(h) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (2V_0^2 + 3V_1^2 + 2V_2^2 - 3V_0V_1 - 4V_0V_2 - 3V_1V_2)\mathcal{V}_{n+2} + (8V_0^2 + 6V_1^2 - V_2^2 + 6V_0V_1 - V_0V_2 - 3V_1V_2)\mathcal{V}_{n+1} + 2(4V_0^2 + V_2^2 + V_0V_2 - 3V_1V_2)\mathcal{V}_n$.

(i) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (10V_0^2 + 9V_1^2 + V_2^2 + 3V_0V_1 - 5V_0V_2 - 6V_1V_2)\mathcal{V}_{n+1} + (10V_0^2 + 3V_1^2 + 4V_2^2 - 3V_0V_1 - 2V_0V_2 - 9V_1V_2)\mathcal{V}_n + 2(2V_0^2 + 3V_1^2 + 2V_2^2 - 3V_0V_1 - 4V_0V_2 - 3V_1V_2)\mathcal{V}_{n-1}$.

(j) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (20V_0^2 + 12V_1^2 + 5V_2^2 - 7V_0V_2 - 15V_1V_2)\mathcal{V}_n + (14V_0^2 + 15V_1^2 + 5V_2^2 - 3V_0V_1 - 13V_0V_2 - 12V_1V_2)\mathcal{V}_{n-1} + 2(10V_0^2 + 9V_1^2 + V_2^2 + 3V_0V_1 - 5V_0V_2 - 6V_1V_2)\mathcal{V}_{n-2}$.

Proof. From Lemmas 7.8 and 7.9, (a)-(j) follow. □

The following theorem shows that there always exist interrelation between generalized third order Jacobsthal and modified third order Jacobsthal matrix sequences.

Theorem 7.14 *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{K}_n\}$ we have the following identities:*

(a) $588\mathcal{V}_n = (-85V_0 - 57V_1 + 55V_2)\mathcal{K}_{n+4} + (195V_0 + 27V_1 - 57V_2)\mathcal{K}_{n+3} + (-29V_0 + 195V_1 - 85V_2)\mathcal{K}_{n+2}$.

(b) $294\mathcal{V}_n = (55V_0 - 15V_1 - V_2)\mathcal{K}_{n+3} + (-57V_0 + 69V_1 - 15V_2)\mathcal{K}_{n+2} + (-85V_0 - 57V_1 + 55V_2)\mathcal{K}_{n+1}$.

(c) $147\mathcal{V}_n = (-V_0 + 27V_1 - 8V_2)\mathcal{K}_{n+2} + (-15V_0 - 36V_1 + 27V_2)\mathcal{K}_{n+1} + (55V_0 - 15V_1 - V_2)\mathcal{K}_n$.

(d) $147V_n = (-16V_0 - 9V_1 + 19V_2)\mathcal{K}_{n+1} + (54V_0 + 12V_1 - 9V_2)\mathcal{K}_n + 2(-V_0 + 27V_1 - 8V_2)\mathcal{K}_{n-1}$.

(e) $147\mathcal{V}_n = (38V_0 + 3V_1 + 10V_2)\mathcal{K}_n + (-18V_0 + 45V_1 + 3V_2)\mathcal{K}_{n-1} + 2(-16V_0 - 9V_1 + 19V_2)\mathcal{K}_{n-2}$.

(f) $4(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (-4V_0^2 + 4V_0V_1 - 14V_0V_2 + 15V_1^2 + 5V_1V_2 - 3V_2^2)\mathcal{V}_{n+4} + (28V_0^2 + 12V_0V_1 + 22V_0V_2 - 9V_1^2 - 15V_1V_2 + V_2^2)\mathcal{V}_{n+3} + (-12V_0^2 - 40V_0V_1 - 2V_0V_2 - 21V_1^2 - 11V_1V_2 + 17V_2^2)\mathcal{V}_{n+2}$.

(g) $2(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (12V_0^2 + 8V_0V_1 + 4V_0V_2 + 3V_1^2 - 5V_1V_2 - V_2^2)\mathcal{V}_{n+3} + (-8V_0^2 - 18V_0V_1 - 8V_0V_2 - 3V_1^2 - 3V_1V_2 + 7V_2^2)\mathcal{V}_{n+2} + (-4V_0^2 + 4V_0V_1 - 14V_0V_2 + 15V_1^2 + 5V_1V_2 - 3V_2^2)\mathcal{V}_{n+1}$.

(h) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (2V_0^2 - 2V_0V_2 - 5V_1V_0 + 3V_2^2 - 4V_1V_2)\mathcal{V}_{n+2} + (4V_0^2 + 6V_0V_1 - 5V_0V_2 + 9V_1^2 - 2V_2^2)\mathcal{V}_{n+1} + (12V_0^2 + 8V_0V_1 + 4V_0V_2 + 3V_1^2 - 5V_1V_2 - V_2^2)\mathcal{V}_n$.

(i) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (6V_0^2 + V_0V_1 - 7V_0V_2 + 9V_1^2 - 4V_1V_2 + V_2^2)\mathcal{V}_{n+1} + (14V_0^2 + 3V_0V_1 + 2V_0V_2 + 3V_1^2 - 9V_1V_2 + 2V_2^2)\mathcal{V}_n + 2(2V_0^2 - 2V_0V_2 - 5V_1V_0 + 3V_2^2 - 4V_1V_2)\mathcal{V}_{n-1}$.

(j) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (20V_0^2 + 4V_0V_1 - 5V_0V_2 + 12V_1^2 - 13V_1V_2 + 3V_2^2)\mathcal{V}_n + (10V_0^2 - 9V_0V_1 - 11V_0V_2 + 9V_1^2 - 12V_1V_2 + 7V_2^2)\mathcal{V}_{n-1} + 2(6V_0^2 + V_0V_1 - 7V_0V_2 + 9V_1^2 - 4V_1V_2 + V_2^2)\mathcal{V}_{n-2}$.

Proof. From Lemmas 7.10 and 7.11, (a)-(j) follow. $\square$

To prove the following Lemma 7.16 (c) we need the next lemma.

Lemma 7.15 *Let A, B, C as in Theorem 7.2 and $A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3$ as in Corollary 7.2. Then the following relations hold:*

$$
\begin{aligned}
A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\
AB &= BA = AC = CA = CB = BC = (0)\,, \\
A_1B_1 &= B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0)\,, \\
A_2B_2 &= B_2A_2 = A_2C_2 = C_2A_2 = C_2B_2 = B_2C_2 = (0)\,, \\
A_3B_3 &= B_3A_3 = A_3C_3 = C_3A_3 = C_3B_3 = B_3C_3 = (0)\,.
\end{aligned}
$$

Proof. Using $\alpha+\beta+\gamma = 1$, $\alpha\beta+\alpha\gamma+\beta\gamma = -1$ and $\alpha\beta\gamma = 2$, required equalities can be established by matrix calculations. □

Lemma 7.16 *For all integers m and n, we have the following identities:*

(a) $\mathcal{J}_0\mathcal{V}_n = \mathcal{V}_n\mathcal{J}_0 = \mathcal{V}_n$.

(b) $\mathcal{V}_0\mathcal{J}_n = \mathcal{J}_n\mathcal{V}_0 = \mathcal{V}_n$.

(c) $\mathcal{J}_m\mathcal{J}_n = \mathcal{J}_n\mathcal{J}_m = \mathcal{J}_{m+n}$

(d) $\mathcal{J}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{J}_m = \mathcal{V}_{m+n}$.

(e) $\mathcal{J}_m\mathcal{M}_n = \mathcal{M}_n\mathcal{J}_m = \mathcal{M}_{m+n}$.

(f) $\mathcal{J}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{J}_m = \mathcal{K}_{m+n}$.

(g) $\mathcal{V}_0\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_0$.

(h) $\mathcal{V}_n\mathcal{V}_m = \mathcal{V}_m\mathcal{V}_n = \mathcal{V}_0\mathcal{V}_{m+n}$.

(i) $\mathcal{J}_{-n} = (\mathcal{J}_n)^{-1}$.

(j) $\mathcal{V}_{-n} = (\mathcal{V}_0)^{1-n}(\mathcal{V}_{-1})^n$

Proof. Identities can be established easily.

(a) Since $\mathcal{J}_0$ is the identity matrix, (a) follows.

(b) It can be seen by using Lemma 7.6.

(c) (c) is given in [84]. We supply the proof for completeness. Using Lemma 7.15 we obtain

$$
\begin{aligned}
\mathcal{J}_m\mathcal{J}_n &= (A_1\alpha^m + B_1\beta^m + C_1\gamma^m)(A_1\alpha^n + B_1\beta^n + C_1\gamma^n) \\
&= A_1^2\alpha^{m+n} + B_1^2\beta^{m+n} + C_1^2\gamma^{m+n} + A_1B_1\alpha^m\beta^n + B_1A_1\alpha^n\beta^m \\
&\quad + A_1C_1\alpha^m\gamma^n + C_1A_1\alpha^n\gamma^m + B_1C_1\beta^m\gamma^n + C_1B_1\beta^n\gamma^m \\
&= A_1\alpha^{m+n} + B_1\beta^{m+n} + C_1\gamma^{m+n} \\
&= \mathcal{J}_{m+n}.
\end{aligned}
$$

(d) From (b), we have

$$\mathcal{J}_m\mathcal{V}_n = \mathcal{J}_m\mathcal{J}_n\mathcal{V}_0.$$

Now from (c) and again from (b), we obtain $\mathcal{J}_m\mathcal{V}_n = \mathcal{J}_{m+n}\mathcal{V}_0 = \mathcal{V}_{m+n}$.

It can be shown similarly that $\mathcal{V}_n\mathcal{J}_m = \mathcal{V}_{m+n}$.

(e) Take $\mathcal{V}_n = \mathcal{M}_n$ in (d).

(f) Take $\mathcal{V}_n = \mathcal{K}_n$ in (d).

(g) After matrix multiplication, just compare the row and column entries of the matrices.

(h) Using (d) and (g) and (b) we get

$$\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_0\mathcal{V}_n\mathcal{J}_m = \mathcal{V}_n\mathcal{V}_0\mathcal{J}_m = \mathcal{V}_n\mathcal{V}_m.$$

Again, using (d) and (g) and (b), we obtain

$$\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_0\mathcal{V}_m\mathcal{J}_n = \mathcal{V}_m\mathcal{V}_0\mathcal{J}_n = \mathcal{V}_m\mathcal{V}_n.$$

This completes the proof of (h).

(i) Suppose first that $n \geq 0$. We prove by mathematical induction. If $n = 0$ then we have

$$\mathcal{J}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = (\mathcal{J}_0)^{-1}$$

which is true and

$$\mathcal{J}_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = (\mathcal{J}_1)^{-1}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k + 1$, by using (c), we obtain

$$(\mathcal{J}_{k+1})^{-1}$$

$$= (\mathcal{J}_k \mathcal{J}_1)^{-1} = (\mathcal{J}_1)^{-1} (\mathcal{J}_k)^{-1} = \mathcal{J}_{-1} \mathcal{J}_{-k}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} J_{-k+1} & J_{-k} + 2J_{-k-1} & 2J_{-k} \\ J_{-k} & J_{-k-1} + 2J_{-k-2} & 2J_{-k-1} \\ J_{-k-1} & J_{-k-2} + 2J_{-k-3} & 2J_{-k-2} \end{pmatrix}$$

$$= \begin{pmatrix} J_{-k} & J_{-k-1} + 2J_{-k-2} & 2J_{-k-1} \\ J_{-k-1} & J_{-k-2} + 2J_{-k-3} & 2J_{-k-2} \\ \frac{1}{2}J_{-k+1} - \frac{1}{2}J_{-k-1} - \frac{1}{2}J_{-k} & \frac{1}{2}J_{-k} + \frac{1}{2}J_{-k-1} - \frac{3}{2}J_{-k-2} - J_{-k-3} & J_{-k} - J_{-k-1} - J_{-k-2} \end{pmatrix}$$

$$= \begin{pmatrix} J_{-k} & J_{-k-1} + 2J_{-k-2} & 2J_{-k-1} \\ J_{-k-1} & J_{-k-2} + 2J_{-k-3} & 2J_{-k-2} \\ J_{-k-2} & J_{-k-3} + 2J_{-k-4} & 2J_{-k-3} \end{pmatrix}$$

$$= \begin{pmatrix} J_{-(k+1)+1} & J_{-(k+1)} + 2J_{-(k+1)-1} & 2J_{-(k+1)} \\ J_{-(k+1)} & J_{-(k+1)-1} + 2J_{-(k+1)-2} & 2J_{-(k+1)-1} \\ J_{-(k+1)-1} & J_{-(k+1)-2} + 2J_{-(k+1)-3} & 2J_{-(k+1)-2} \end{pmatrix}$$

$$= \mathcal{J}_{-(k+1)}$$

Thus, by induction on n, this proves (g) for $n \geq 0$. Suppose now that $n \leq 0$. Say $m = -n$. Then (g) can be written as

$$\mathcal{J}_m = (\mathcal{J}_{-m})^{-1}$$

and we prove this. Since $m \geq 0$, from the first part of the proof, we have

$$\mathcal{J}_{-m} = (\mathcal{J}_m)^{-1}$$

and so

$$(\mathcal{J}_{-m})^{-1} = ((\mathcal{J}_m)^{-1})^{-1} = \mathcal{J}_m$$

which completes the proof.

(j) Taking $-n+1$ for m and 1 for n in $\mathcal{V}_0\ \mathcal{V}_{m+n} = \mathcal{V}_m\mathcal{V}_n$ which is given in (h), we obtain that

$$\mathcal{V}_0\mathcal{V}_{-n} = \mathcal{V}_{-n+1}\mathcal{V}_{-1}. \tag{7.4}$$

If we multiply both side of the equation (7.4) with $\mathcal{V}_0$ we have the relation

$$\begin{aligned} \mathcal{V}_0\mathcal{V}_0\mathcal{V}_{-n} &= \mathcal{V}_0\mathcal{V}_{-n+1}\mathcal{V}_{-1} \\ &= \mathcal{V}_{-n+2}\mathcal{V}_{-1}\mathcal{V}_{-1}. \end{aligned}$$

Repeating this process we obtain

$$\mathcal{V}_0^{n-1}\mathcal{V}_{-n} = \mathcal{V}_{-1}^n.$$

Thus, it follows that

$$\mathcal{V}_{-n} = \mathcal{V}_0^{1-n}\mathcal{V}_{-1}^n.$$

This completes the proof. $\Box$

Note that using Lemma 7.16 (j) and (d), we obtain

$$\mathcal{V}_{-n} = (\mathcal{V}_0)^{1-n}(\mathcal{V}_{-1})^n = (\mathcal{V}_n\mathcal{J}_{-n})^{1-n}\mathcal{V}_{-1}^n = \mathcal{J}_{-n}^{1-n}\mathcal{V}_n^{1-n}\mathcal{V}_{-1}^n$$

and then by Lemma (i), we get

$$\mathcal{V}_{-n} = \mathcal{J}_n^{n-1}\mathcal{V}_n^{1-n}\mathcal{V}_{-1}^n.$$

Using Lemma 7.16 and comparing matrix entries, we have next result.

Corollary 7.10 *For generalized third order Jacobsthal, third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal numbers, we have the following identities:*

(a) $V_{m+n} = J_m V_{n+1} + (J_{m-1} + 2J_{m-2}) V_n + 2J_{m-1}V_{n-1} = J_{m+1}V_n + J_m (V_{n-1} + 2V_{n-2}) + 2J_{m-1}V_{n-1}$.

(b) $J_{m+n} = J_m J_{n+1} + (J_{m-1} + 2J_{m-2}) J_n + 2J_{m-1}J_{n-1} = J_{m+1}J_n + J_m (J_{n-1} + 2J_{n-2}) + 2J_{m-1}J_{n-1}$.

(c) $j_{m+n} = J_m j_{n+1} + (J_{m-1} + 2J_{m-2}) j_n + 2J_{m-1}j_{n-1} = J_{m+1}j_n + J_m (j_{n-1} + 2j_{n-2}) + 2J_{m-1}j_{n-1}$.

(d) $K_{m+n} = J_m K_{n+1} + (J_{m-1} + 2J_{m-2}) K_n + 2J_{m-1}K_{n-1} = J_{m+1}K_n + J_m (K_{n-1} + 2K_{n-2}) + 2J_{m-1}K_{n-1}$.

(e) $V_0V_{m+n+1} + (V_1 - V_0)V_{m+n} + (V_2 - V_1 - V_0)V_{m+n-1} = V_{m+1}V_n + (V_m + 2V_{m-1})V_{n-1} + 2V_mV_{n-2} = V_mV_{n+1} + (V_{m-1} + 2V_{m-2})V_n + 2V_{m-1}V_{n-1}$.

(f) $J_{m+n} = J_{m+1}J_n + (J_m + 2J_{m-1})J_{n-1} + 2J_mJ_{n-2} = J_mJ_{n+1} + (J_{m-1} + 2J_{m-2})J_n + 2J_{m-1}J_{n-1}$.

(g) $2j_{m+n+1} - j_{m+n} + 2j_{m+n-1} = j_{m+1}j_n + (j_m + 2j_{m-1})j_{n-1} + 2j_mj_{n-2} = j_mj_{n+1} + (j_{m-1} + 2j_{m-2})j_n + 2j_{m-1}j_{n-1}$.

(h) $3K_{m+n+1} - 2K_{m+n} - K_{m+n-1} = K_{m+1}K_n + (K_m + 2K_{m-1})K_{n-1} + 2K_mK_{n-2} = K_mK_{n+1} + (K_{m-1} + 2K_{m-2})K_n + 2K_{m-1}K_{n-1}$.

Proof. We prove (a) and (e) by using Lemma 7.16 (d) and (h). The others are special cases of (a) and (e). Lemma 7.16 (d), i.e., $\mathcal{J}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{J}_m = \mathcal{V}_{m+n}$, can be writtten as

$$\begin{pmatrix} J_{m+1} & J_m + 2J_{m-1} & 2J_m \\ J_m & J_{m-1} + 2J_{m-2} & 2J_{m-1} \\ J_{m-1} & J_{m-2} + 2J_{m-3} & 2J_{m-2} \end{pmatrix} \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix} \begin{pmatrix} J_{m+1} & J_m + 2J_{m-1} & 2J_m \\ J_m & J_{m-1} + 2J_{m-2} & 2J_{m-1} \\ J_{m-1} & J_{m-2} + 2J_{m-3} & 2J_{m-2} \end{pmatrix}$$

$$= \begin{pmatrix} V_{m+n+1} & V_{m+n} + 2V_{m+n-1} & 2V_{m+n} \\ V_{m+n} & V_{m+n-1} + 2V_{m+n-2} & 2V_{m+n-1} \\ V_{m+n-1} & V_{m+n-2} + 2V_{m+n-3} & 2V_{m+n-2} \end{pmatrix}$$

Now, by multiplying the matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identities in (a).

Lemma 7.16 (h), i.e., $\mathcal{V}_n\mathcal{V}_m = \mathcal{V}_m\mathcal{V}_n = \mathcal{V}_0\mathcal{V}_{m+n}$, can be writtten as

$$\begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix} \begin{pmatrix} V_{m+1} & V_m + 2V_{m-1} & 2V_m \\ V_m & V_{m-1} + 2V_{m-2} & 2V_{m-1} \\ V_{m-1} & V_{m-2} + 2V_{m-3} & 2V_{m-2} \end{pmatrix}$$

$$= \begin{pmatrix} V_{m+1} & V_m + 2V_{m-1} & 2V_m \\ V_m & V_{m-1} + 2V_{m-2} & 2V_{m-1} \\ V_{m-1} & V_{m-2} + 2V_{m-3} & 2V_{m-2} \end{pmatrix} \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} V_1 & V_2 - V_1 & 2V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 - V_0 \\ \frac{1}{2}(V_2 - V_1 - V_0) & \frac{1}{2}(3V_0 + V_1 - V_2) & \frac{1}{2}(3V_1 - V_0 - V_2) \end{pmatrix}$$

$$\times \begin{pmatrix} V_{m+n+1} & V_{m+n} + 2V_{m+n-1} & 2V_{m+n} \\ V_{m+n} & V_{m+n-1} + 2V_{m+n-2} & 2V_{m+n-1} \\ V_{m+n-1} & V_{m+n-2} + 2V_{m+n-3} & 2V_{m+n-2} \end{pmatrix}$$

Now, by multiplying the matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identities in (e). □

As an application of Lemma 7.16 (i) and Corollary 7.10 (b), we present the following example.

Example 7.1 *For all integers n, we have the following identities.*

$J_{-n} = \dfrac{1}{2^{n-1}}(J_{n-1}^2 - J_nJ_{n-2})$

and

$$(J_{n+2}+J_{n+1}+J_n)(J_{n+2}^2+3J_{n+1}^2+4J_n^2-3J_{n+2}J_{n+1}-2J_{n+2}J_n)=2^{n+1}.$$

Solution. Note that for all integers n, we have

$$\begin{aligned} J_{n-1} &= \frac{1}{2}(J_{n+2}-J_{n+1}-J_n), && (7.5)\\ J_{n-2} &= \frac{1}{4}\left(-J_{n+2}+3J_{n+1}-J_n\right), && (7.6)\\ J_{n-3} &= \frac{1}{8}\left(-J_{n+2}-J_{n+1}+7J_n\right). && (7.7)\end{aligned}$$

By using (taking $m=n$) Corollary 7.10 (b) and (7.5)-(7.6) we get

$$J_{2n}=\frac{1}{2}J_{n+2}^2+\frac{1}{2}J_{n+1}^2-\frac{1}{2}J_n^2+3J_{n+1}J_n-J_{n+2}J_n-J_{n+2}J_{n+1}. \tag{7.8}$$

In [[72], Corollary 12 (a)], the following formula is presented for J_{-n} :

$$J_{-n}=\frac{1}{2^{n+1}}(3J_n^2+2J_{2n}+J_{n+2}J_n-7J_{n+1}J_n).$$

Using (7.8), we obtain

$$J_{-n}=\frac{1}{2^{n+1}}(J_{n+2}^2+J_{n+1}^2+2J_n^2-2J_{n+2}J_{n+1}-J_{n+2}J_n-J_{n+1}J_n). \tag{7.9}$$

By comparing the 2nd rows and 1st columns entries of both sides of the relation $\mathcal{J}_{-n}=(\mathcal{J}_n)^{-1}$ which is given in Lemma 7.16 (i), we get

$$J_{-n}=\frac{1}{2}\frac{J_{n-1}^2-J_nJ_{n-2}}{J_{n-1}^3+J_n^2J_{n-3}+J_{n-2}^2J_{n+1}-J_{n+1}J_{n-1}J_{n-3}-2J_nJ_{n-1}J_{n-2}}. \tag{7.10}$$

Note that by using (7.5)-(7.7) we get

$$J_{n-1}^2-J_nJ_{n-2}=\frac{1}{4}(J_{n+2}^2+J_{n+1}^2+2J_n^2-2J_{n+2}J_{n+1}-J_{n+2}J_n-J_{n+1}J_n)$$

and

$$\begin{aligned} & J_{n-1}^3+J_n^2J_{n-3}+J_{n-2}^2J_{n+1}-J_{n+1}J_{n-1}J_{n-3}-2J_nJ_{n-1}J_{n-2}\\ = & \frac{1}{8}(4J_n^3+4J_n^2J_{n+1}+2J_n^2J_{n+2}+3J_nJ_{n+1}^2\\ & -5J_nJ_{n+1}J_{n+2}-J_nJ_{n+2}^2+3J_{n+1}^3-2J_{n+1}J_{n+2}^2+J_{n+2}^3)\\ = & \frac{1}{8}(J_{n+2}+J_{n+1}+J_n)(J_{n+2}^2+3J_{n+1}^2+4J_n^2-3J_{n+2}J_{n+1}-2J_{n+2}J_n)\end{aligned}$$

and so (7.10) can be written as

$$J_{-n}=\frac{J_{n+2}^2+J_{n+1}^2+2J_n^2-2J_{n+2}J_{n+1}-J_{n+2}J_n-J_{n+1}J_n}{(J_{n+2}+J_{n+1}+J_n)(J_{n+2}^2+3J_{n+1}^2+4J_n^2-3J_{n+2}J_{n+1}-2J_{n+2}J_n)}. \tag{7.11}$$

So the rights sides of the equations (7.9) and (7.11) must be equal. This completes the solution. □

Theorem 7.17 *For all integers m and n, we have the following identities:*

(a) $4\mathcal{V}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_m = (V_0+V_1-V_2)^2\mathcal{J}_{m+n+4} - 2(3V_0+V_1-V_2)(V_0+V_1-V_2)\mathcal{J}_{m+n+3} + (11V_0^2-5V_1^2-V_2^2+2V_0V_1-6V_0V_2+6V_1V_2)\mathcal{J}_{m+n+2} - 2(V_0-3V_1+V_2)(3V_0+V_1-V_2)\mathcal{J}_{m+n+1} + (V_0-3V_1+V_2)^2\mathcal{J}_{m+n}$.

(b) $2\mathcal{V}_m\mathcal{V}_n = 2\mathcal{V}_n\mathcal{V}_m = (-V_0-V_1+V_2)\mathcal{V}_{m+n+2} + (3V_0+V_1-V_2)\mathcal{V}_{m+n+1} + (-V_0+3V_1-V_2)\mathcal{V}_{m+n}$.

(c) $2\mathcal{J}_m\mathcal{V}_n = 2\mathcal{V}_n\mathcal{J}_m = (-V_0-V_1+V_2)\mathcal{J}_{m+n+2} + (3V_0+V_1-V_2)\mathcal{J}_{m+n+1} + (-V_0+3V_1-V_2)\mathcal{J}_{m+n}$.

(d) $24\mathcal{J}_m\mathcal{V}_n = 24\mathcal{V}_n\mathcal{J}_m = (5V_0+5V_1-3V_2)\mathcal{M}_{m+n+2} + (-11V_0-3V_1+5V_2)\mathcal{M}_{m+n+1} + (5V_0-11V_1+5V_2)\mathcal{M}_{m+n}$.

(e) $147\mathcal{J}_m\mathcal{V}_n = 147\mathcal{V}_n\mathcal{J}_m = (-V_0+27V_1-8V_2)\mathcal{K}_{m+n+2} + 3(-5V_0-12V_1+9V_2)\mathcal{K}_{m+n+1} + (55V_0-15V_1-V_2)\mathcal{K}_{m+n}$.

Proof.

(a) It follows from Theorem 7.12 (c) and Lemma 7.16 (c).

(b) It follows from Theorem 7.12 (c) and Lemma 7.16 (d).

(c) It follows from Theorem 7.12 (c) and Lemma 7.16 (c).

(d) It follows from Theorem 7.13 (c) and Lemma 7.16 (e).

(e) It follows from Theorem 7.14 (c) and Lemma 7.16 (f). □

Note that in Theorem 7.17 we use (c)'s of Theorems 7.12, 7.13 and 7.14. Using (a), (b), (d), (e), (f), (g), (h), (i), (j)'s of Theorems 7.12, 7.13 and 7.14 we can establish other recurrence relations.

Using Theorem 7.17 and comparing matrix entries, we have next result.

Theorem 7.18 *For generalized third order Jacobsthal, third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal numbers, we have the following identities:*

(a) $4(V_mV_{n+1} + (V_{m-1} + 2V_{m-2})V_n + 2V_{m-1}V_{n-1}) = 4(V_{m+1}V_n + V_m(V_{n-1} + 2V_{n-2}) + 2V_{m-1}V_{n-1}) = (V_0 + V_1 - V_2)^2 J_{m+n+4} - (6V_0 + 2V_1 - 2V_2)(V_0 + V_1 - V_2)J_{m+n+3} - (-11V_0^2 - 2V_0V_1 + 6V_0V_2 + 5V_1^2 - 6V_1V_2 + V_2^2)J_{m+n+2} - (2V_0 - 6V_1 + 2V_2)(3V_0 + V_1 - V_2)J_{m+n+1} + (V_0 - 3V_1 + V_2)^2 J_{m+n}$.

(b) $2(V_mV_{n+1} + (V_{m-1} + 2V_{m-2})V_n + 2V_{m-1}V_{n-1}) = 2(V_{m+1}V_n + V_m(V_{n-1} + 2V_{n-2}) + 2V_{m-1}V_{n-1}) = (-V_0 - V_1 + V_2)V_{m+n+2} + (3V_0 + V_1 - V_2)V_{m+n+1} + (-V_0 + 3V_1 - V_2)V_{m+n}$.

(c) $2(J_mV_{n+1} + ((J_{m-1} + 2J_{m-2})V_n + 2J_{m-1}V_{n-1}) = 2(J_{m+1}V_n + J_m(V_{n-1} + 2V_{n-2}) + 2J_{m-1}V_{n-1}) = (-V_0 - V_1 + V_2)J_{m+n+2} + (3V_0 + V_1 - V_2)J_{m+n+1} + (-V_0 + 3V_1 - V_2)J_{m+n}$.

(d) $24(J_mV_{n+1} + (J_{m-1} + 2J_{m-2})V_n + 2J_{m-1}V_{n-1}) = 24(J_{m+1}V_n + J_m(V_{n-1} + 2V_{n-2}) + 2J_{m-1}V_{n-1}) = (5V_0 + 5V_1 - 3V_2)j_{m+n+2} + (-11V_0 - 3V_1 + 5V_2)j_{m+n+1} + (5V_0 - 11V_1 + 5V_2)j_{m+n}$.

(e) $147(J_mV_{n+1} + (J_{m-1} + 2J_{m-2})V_n + 2J_{m-1}V_{n-1}) = 147(J_{m+1}V_n + J_m(V_{n-1} + 2V_{n-2}) + 2J_{m-1}V_{n-1}) = (-V_0 + 27V_1 - 8V_2)K_{m+n+2} + (-15V_0 - 36V_1 + 27V_2)K_{m+n+1} + (-15V_1 + 55V_0 - V_2)K_{m+n}$.

Proof. By multiplying matrices and then by comparing the 2nd rows and 1st columns entries in Theorem 7.17 (a), we get the required identities in (a). The remaining of identities can be proved by considering again Theorem 7.17. □

Taking $V_n = J_n$ in Theorem 7.18, we obtain the following corollary.

Corollary 7.11 *For third order Jacobsthal numbers, we have the following identities:*

(a) $J_mJ_{n+1} + (J_{m-1} + 2J_{m-2})J_n + 2J_{m-1}J_{n-1} = J_{m+1}J_n + J_m(J_{n-1} + 2J_{n-2}) + 2J_{m-1}J_{n-1} = J_{m+n}$.

(b) $12(J_mJ_{n+1} + (J_{m-1} + 2J_{m-2})J_n + 2J_{m-1}J_{n-1}) = 12(J_{m+1}J_n + J_m(J_{n-1} + 2J_{n-2}) + 2J_{m-1}J_{n-1}) = j_{m+n+2} + j_{m+n+1} - 3j_{m+n}$.

(c) $147(J_mJ_{n+1} + (J_{m-1} + 2J_{m-2})J_n + 2J_{m-1}J_{n-1}) = 147(J_{m+1}J_n + J_m(J_{n-1} + 2J_{n-2}) + 2J_{m-1}J_{n-1}) = 19K_{m+n+2} - 9K_{m+n+1} - 16K_{m+n}$.

Taking $V_n = j_n$ in Theorem 7.18, we get the following corollary.

Corollary 7.12 *For third order Jacobsthal-Lucas numbers, we have the following identities:*

(a) $j_m j_{n+1} + (j_{m-1} + 2j_{m-2})j_n + 2j_{m-1}j_{n-1} = j_{m+1}j_n + j_m(j_{n-1} + 2j_{n-2}) + 2j_{m-1}j_{n-1} = J_{m+n+4} + 2J_{m+n+3} - 3J_{m+n+2} - 4J_{m+n+1} + 4J_{m+n}$.

(b) $j_m j_{n+1} + (j_{m-1} + 2j_{m-2})j_n + 2j_{m-1}j_{n-1} = j_{m+1}j_n + j_m(j_{n-1} + 2j_{n-2}) + 2j_{m-1}j_{n-1} = j_{m+n+2} + j_{m+n+1} - 2j_{m+n}$.

(c) $J_m j_{n+1} + ((J_{m-1} + 2J_{m-2})\, j_n + 2J_{m-1}j_{n-1} = J_{m+1}j_n + J_m(j_{n-1} + 2j_{n-2}) + 2J_{m-1}j_{n-1} = J_{m+n+2} + J_{m+n+1} - 2J_{m+n}$.

(d) $J_m j_{n+1} + (J_{m-1} + 2J_{m-2})j_n + 2J_{m-1}j_{n-1} = J_{m+1}j_n + J_m(j_{n-1} + 2j_{n-2}) + 2J_{m-1}j_{n-1} = j_{m+n}$.

(e) $49(J_m j_{n+1} + (J_{m-1} + 2J_{m-2})j_n + 2J_{m-1}j_{n-1}) = 49(J_{m+1}j_n + J_m(j_{n-1} + 2j_{n-2}) + 2J_{m-1}j_{n-1}) = 30K_{m+n} + 23K_{m+n+1} - 5K_{m+n+2}$.

Taking $V_n = K_n$ in Theorem 7.18, we obtain the following corollary.

Corollary 7.13 *For modified third order Jacobsthal numbers, we have the following identities:*

(a) $4(K_m K_{n+1} + (K_{m-1} + 2K_{m-2})K_n + 2K_{m-1}K_{n-1}) = 4(K_{m+1}K_n + K_m(K_{n-1} + 2K_{n-2}) + 2K_{m-1}K_{n-1}) = J_{m+n+4} - 14J_{m+n+3} + 55J_{m+n+2} - 42J_{m+n+1} + 9J_{m+n}$.

(b) $2(K_m K_{n+1} + (K_{m-1} + 2K_{m-2})K_n + 2K_{m-1}K_{n-1}) = 2(K_{m+1}K_n + K_m(K_{n-1} + 2K_{n-2}) + 2K_{m-1}K_{n-1}) = -K_{m+n+2} + 7K_{m+n+1} - 3K_{m+n}$.

(c) $2(J_m K_{n+1} + ((J_{m-1} + 2J_{m-2})\, K_n + 2J_{m-1}K_{n-1}) = 2(J_{m+1}K_n + J_m(K_{n-1} + 2K_{n-2}) + 2J_{m-1}K_{n-1}) = -J_{m+n+2} + 7J_{m+n+1} - 3J_{m+n}$.

(d) $24(J_m K_{n+1} + (J_{m-1} + 2J_{m-2})K_n + 2J_{m-1}K_{n-1}) = 24(J_{m+1}K_n + J_m(K_{n-1} + 2K_{n-2}) + 2J_{m-1}K_{n-1}) = 11j_{m+n+2} - 21j_{m+n+1} + 19j_{m+n}$.

(e) $J_m K_{n+1} + (J_{m-1} + 2J_{m-2})K_n + 2J_{m-1}K_{n-1} = J_{m+1}K_n + J_m(K_{n-1} + 2K_{n-2}) + 2J_{m-1}K_{n-1} = K_{m+n}$.

The next two theorems provide us the convenience to obtain the powers of generalized third order Jacobsthal, third order Jacobsthal, third order Jacobsthal-Lucas and modified third order Jacobsthal matrix sequences.

Theorem 7.19 *For all integers m, n and r, the following identities hold:*

(a) $\mathcal{J}_n^m = \mathcal{J}_{mn}$,

(b) $\mathcal{J}_{n+1}^m = \mathcal{J}_1^m \mathcal{J}_{mn}$,

(c) $\mathcal{J}_{n-r}\mathcal{J}_{n+r} = \mathcal{J}_n^2 = \mathcal{J}_2^n$.

Proof. We prove for $m, n, r \geq 0$. The other cases can be proved similarly.

(a) We can write $\mathcal{J}_n^m$ as

$\mathcal{J}_n^m = \mathcal{J}_n\mathcal{J}_n...\mathcal{J}_n$ (m times).

Using Lemma 7.16 (c) iteratively, we obtain the required result:

$$\begin{aligned}
\mathcal{J}_n^m &= \underbrace{\mathcal{J}_n\mathcal{J}_n...\mathcal{J}_n}_{m \text{ times}} \\
&= \mathcal{J}_{2n}\underbrace{\mathcal{J}_n\mathcal{J}_n...\mathcal{J}_n}_{m-1 \text{ times}} \\
&= \mathcal{J}_{3n}\underbrace{\mathcal{J}_n\mathcal{J}_n...\mathcal{J}_n}_{m-2 \text{ times}} \\
&\vdots \\
&= \mathcal{J}_{(m-1)n}\mathcal{J}_n \\
&= \mathcal{J}_{mn}.
\end{aligned}$$

(b) As a similar approach in (a) we have

$$\mathcal{J}_{n+1}^m = \mathcal{J}_{n+1}.\mathcal{J}_{n+1}...\mathcal{J}_{n+1} = \mathcal{J}_{m(n+1)} = \mathcal{J}_m\mathcal{J}_{mn} = \mathcal{J}_1\mathcal{J}_{m-1}\mathcal{J}_{mn}.$$

Using Theorem 7.16 (c), we can write iteratively $\mathcal{J}_m = \mathcal{J}_1\mathcal{J}_{m-1}, \mathcal{J}_{m-1} = \mathcal{J}_1\mathcal{J}_{m-2}, ...,$ $\mathcal{J}_2 = \mathcal{J}_1\mathcal{J}_1$. Now it follows that

$$\mathcal{J}_{n+1}^m = \underbrace{\mathcal{J}_1\mathcal{J}_1...\mathcal{J}_1}_{m \text{ times}}\mathcal{J}_{mn} = \mathcal{J}_1^m\mathcal{J}_{mn}.$$

(c) Lemma 7.16 (c) gives

$$\mathcal{J}_{n-r}\mathcal{J}_{n+r} = \mathcal{J}_{2n} = \mathcal{J}_n\mathcal{J}_n = \mathcal{J}_n^2$$

and also

$$\mathcal{J}_{n-r}\mathcal{J}_{n+r} = \mathcal{J}_{2n} = \underbrace{\mathcal{J}_2\mathcal{J}_2...\mathcal{J}_2}_{n \text{ times}} = \mathcal{J}_2^n.$$

We have analogues results for the matrix sequence $\mathcal{V}_n$.

Theorem 7.20 *For all integers m, n and r, the following identities hold:*

(a) $\mathcal{V}_{n-r}\mathcal{V}_{n+r} = \mathcal{V}_n^2,$

(b) $\mathcal{V}_n^m = \mathcal{V}_0^m\mathcal{J}_{m+n}.$

Proof.

(a) We use Binet formula of generalized third order Jacobsthal sequence which is given in Theorem 7.2. So

$$\begin{aligned}
&\mathcal{V}_{n-r}\mathcal{V}_{n+r} - \mathcal{V}_n^2 \\
= \;& (A\alpha^{n-r} + B\beta^{n-r} + C\gamma^{n-r})(A\alpha^{n+r} + B\beta^{n+r} + C\gamma^{n+r}) - (A\alpha^n + B\beta^n + C\gamma^n)^2 \\
= \;& AB\alpha^{n-r}\beta^{n-r}(\alpha^r - \beta^r)^2 + AC\alpha^{n-r}\gamma^{n-r}(\alpha^r - \gamma^r)^2 + BC\beta^{n-r}\gamma^{n-r}(\beta^r - \gamma^r)^2 \\
= \;& 0
\end{aligned}$$

since $AB = AC = BC = 0$ (see Lemma 7.15). Now we get the result as required.

(b) By Theorem 7.19, we have

$$\mathcal{V}_0^m\mathcal{J}_{mn} = \underbrace{\mathcal{V}_0\mathcal{V}_0...\mathcal{V}_0}_{m \text{ times}}\underbrace{\mathcal{J}_n\mathcal{J}_n...\mathcal{J}_n}_{m \text{ times}}.$$

When we apply Lemma 7.16 (b) iteratively, it follows that

$$\begin{aligned}
\mathcal{V}_0^m\mathcal{J}_{mn} &= (\mathcal{V}_0\mathcal{J}_n)(\mathcal{V}_0\mathcal{J}_n)...(\mathcal{V}_0\mathcal{J}_n) \\
&= \mathcal{V}_n\mathcal{V}_n...\mathcal{V}_n = \mathcal{V}_n^m.
\end{aligned}$$

This completes the proof. □

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