HYPERDETERMINANTS, ENTANGLED STATES, AND INVARIANT THEORY

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By
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July, 2013

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In [1] and [2], A. Klyachko connects quantum entanglement and invariant theory so that entangled state of a quantum system can be explained by invariants of the system. After representing states in multidimensional matrices, this relation turns into finding multidimensional matrix invariants so called hyperdeterminants.

Here we provide a necessary and sufficient condition for existence of a hyperdeterminant of a multidimensional matrix of an arbitrary format. The answer is given in terms of the so called castling transform that relates hyperdeterminants of different formats. Among castling equivalent formats there is a unique castling reduced one, that has minimal number of entries. We prove the following theorem: "Multidimensional matrices of a given format admit a non-constant hyperdeterminant if and only if logarithm of dimensions of the castling reduced format satisfy polygonal inequalities."

Keywords: hyperdeterminants, quantum entanglement, invariant theory.

ÖZET

HİPERDETERMİNANTLAR, DOLANIK DURUMLAR VE DEĞİŞMEZLER TEORİSİ

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[1] ve [2] de, A. Klyachko, kuantum karmaşıklıkları ve değişmezler teorisini şu şekilde birbirine bağlamaktadır: bir kuantum sisteminin karmaşık durumu, o sistemin değişmezleriyle açıklanabilir. Durumları çok boyutlu matrisler ile temsil edersek, bu ilişki çok boyutlu matrislerin değişmezlerini diğer adıyla hiperdeterminantlarını bulmaya dönüşür.

Herhangi bir formattaki çok boyutlu bir matrisin hiperdeterminanta sahip olması için gerekli ve yeterli koşulları elde ediyoruz. Cevap, farklı formatlardaki hiperdeterminantları birbirine bağlayan rok dönüşümüyle verilmektedir. Denk rok formatları içinde en az elemana sahip biricik rok indirgenmiş format vardır. Şu teoremi ispatlıyoruz: "Verilen bir formattaki çok boyutlu matrisin bir hiperdeterminanta sahip olması için gerek ve yeter koşul matrisin rok indirgenmiş formatının boyutlarının logaritmasının çokgen eşitsizliğini sağlamasıdır."

Anahtar sözcükler: hiperdeterminantlar, kuantum dolanıklık, değişmezler teorisi.

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Contents

Chapter 1

Introduction

The state space of the quantum system A is an inner product space $\mathcal{H}_A$ over the complex field. In this thesis we only work with finite dimensional systems, $\dim \mathcal{H}_A < \infty$. A *pure state* is a unit vector $\psi \in \mathcal{H}_A$ or projector operator $|\psi\rangle\langle\psi|$ onto direction ψ. By fixing a non-zero vector ψ in the pure state, we can derive all information on the state of a system.

A *mixed state* ρ is a convex combination of the projectors, $\rho = \sum_i \rho_i |\psi\rangle\langle\psi|$, where ρ_i's are probabilities with $\sum_i \rho_i = 1$. Equivalently we have $\rho \geq 0$ with $\operatorname{Tr} \rho = 1$.

Hermitian operators play an important role in quantum mechanics, since physical quantities are given by Hermitian operators. An observable X is a Hermitian operator on $\mathcal{H}_A$. Measurement of X returns eigenvalue λ of X and it puts the system in eigenstate ψ, $X\psi = \lambda\psi$. The expectation value of X in the pure state ψ is given by $\langle\psi|X|\psi\rangle$. For a mixed state ρ, the expectation is given by trace $\operatorname{Tr}(\rho X)$. Assume that $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite dimensional state spaces of two quantum systems. Then the state space of the unified system lies in $\mathcal{H}_A \otimes \mathcal{H}_B$ according to the principle of superposition. This system is also called composite or two component system. If there are more than two quantum systems, it is called multicomponent system.

For a mixed state ρ_{AB} of composite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, there exists unique state ρ_A of component A such that for every observable $X_A : \mathcal{H}_A \mapsto \mathcal{H}_A$,

$$\mathrm{Tr}_{AB}\left(\rho_{AB} X_A\right) = \mathrm{Tr}_A\left(\rho_A X_A\right).$$

In this case ρ_A is *visible* state of component A and it is called *reduced* or *marginal* state of ρ_{AB}.

It is possible to write density matrix ρ_{AB} of composite system $\mathcal{H}_A \otimes \mathcal{H}_B$ such as:

$$\rho_{AB} = \sum_{\alpha} a_\alpha L_A^\alpha \otimes L_B^\alpha$$

where L_A, L_B are linear operators in $\mathcal{H}_A$ and $\mathcal{H}_B$. Its reduced matrices or marginal states are defined by:

$$\rho_A = \sum_{\alpha} \mathrm{Tr}\left(L_B^\alpha\right) L_A^\alpha := \mathrm{Tr}_B\left(\rho_{AB}\right)$$
$$\rho_B = \sum_{\alpha} \mathrm{Tr}\left(L_A^\alpha\right) L_B^\alpha := \mathrm{Tr}_A\left(\rho_{AB}\right)$$

With the following property,

$$\langle X_A \rangle_{\rho_{AB}} = \mathrm{Tr}\left(\rho_{AB} X_A\right) = \mathrm{Tr}\left(\rho_A X_A\right) = \langle X_A \rangle_{\rho_A}, \quad X_A : \mathcal{H}_A \longmapsto \mathcal{H}_A$$

we obtain that observation of subsystem A gives the same results as if A would be in reduced state ρ_A.
If there are three states, we can define reduced matrices in a similar way:

$$\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C = \mathcal{H}_A \otimes \mathcal{H}_{BC}$$

with $\rho_A = \mathrm{Tr}_{BC}\left(\rho_{ABC}\right)$. Other projections are made similarly.
If the margins ρ_A, ρ_B, ρ_C are scalar, local observations provide no information. By this fact we describe completely entangled states as in the following:

Definition 1.0.1. State $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ is completely entangled if and only if the margins ρ_A, ρ_B, ρ_C are scalar [1].

Solving the below problem is one of our aims in this thesis:

Problem 1.0.2. For which formats completely entangled states exists?

For two component system $\mathcal{H}_{AB}$, state ψ is completely entangled if and only if rows and columns of matrix representation of ψ are orthogonal and have the same norm. A state ψ is called *separable* if there is no quantum entanglement. Otherwise it is called *entangled*. In an entangled state, there are nonclassical correlations between substates. In a separable state, the information of the whole can be obtained by its parts. This distinction can be seen a tautological, however "Everybody knows, and nobody understand what is entanglement" [1].
We want to briefly describe quantum marginal problem. For a reduced state ρ_j, let λ_j be its spectra in nonincreasing order, $\lambda_j = \left(\lambda_j^1, \ldots, \lambda_j^k\right)$. Pure quantum marginal problem asks the following:

Problem 1.0.3. [1] Find conditions on ρ_A, ρ_B, ρ_C to be reduced states of $\psi \in \mathcal{H}_{ABC}$.

Answer depends only on spectra λ_A, λ_B, λ_C of ρ_A, ρ_B, ρ_C. We expand on quantum marginal problem in chapter 4.

Now we briefly deal with representation theory. We consider decomposition of $\mathcal{H}_{ABC}$ in Nth symmetric power, ie.

$$S^N \mathcal{H}_{ABC} = \sum_{\lambda,\mu,\nu \vdash N} g\left(\lambda,\mu,\nu\right) \mathcal{H}_A^\lambda \otimes \mathcal{H}_B^\mu \otimes \mathcal{H}_C^\nu$$

where $\mathcal{H}_A^\lambda, \mathcal{H}_B^\mu, \mathcal{H}_C^\nu$ irreducible representations with respective young diagrams λ, μ, ν. Multiplicity $g\left(\lambda,\mu,\nu\right)$ in the summation is called *Kronecker coefficient*. Let $\overline{\lambda}$ be normalized λ, like $\overline{\lambda} = \frac{\lambda}{N}$. In the previous decomposition , if there exists a nonzero Kronecker coefficient $g\left(\lambda,\mu,\nu\right)$ for rectangular diagrams of heights λ, μ, ν then normalized diagrams of Young diagrams are reduced spectra of some pure state $\psi \in \mathcal{H}_{ABC}$. Again, we canalize reader to the chapter 4 for detailed discussion.
If we put together the result of quantum marginal problem and existence of nonzero Kronecker coefficients we attain the bridge between quantum entanglement and invariant theory:

Theorem 1.0.4. *Entangled state exists if and only if Kronecker coefficients are nonzero.*

The existence of nonzero Kronecker coefficient has the following meaning: three component system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, briefly $\mathcal{H}_{ABC}$, has an invariant under the action of

$SL(\mathcal{H}_A) \times SL(\mathcal{H}_B) \times SL(\mathcal{H}_C)$. The tuple with entries as dimensions of multicomponent system is called *format*. In coordinates $\mathcal{H}_{ABC}$ can be represented as multidimensional matrix of complex numbers of format $a \times b \times c$. Hyperdeterminant in $\mathcal{H}_{ABC}$ is a polynomial function:

$$\mathrm{DET} : \mathcal{H}_{ABC} \longmapsto \mathbb{C}$$

which is invariant under the action of $SL(\mathcal{H}_A) \times SL(\mathcal{H}_B) \times SL(\mathcal{H}_C)$ which we denote it by DET. Since we are dealing with multidimensional matrices, it is appropriate to use the term hyperdeterminant.

Example 1.0.5. For a two component system, the classical determinant is the only invariant.

We solve the following problem which has a long story:

Problem 1.0.6. Let $G = SL(\mathcal{H}_1) \times SL(\mathcal{H}_2) \times \cdots \times SL(\mathcal{H}_n)$ act on the space $V = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ with $\dim \mathcal{H}_i < \infty$ for $i \in \{1, 2, \ldots, n\}$. For which multicomponent systems V under the action of G, there exists a non-constant hyperdeterminant?

In 1845, A. Cayley calculated hyperdeterminant for the format $2 \times 2 \times 2$ in a brilliant method and the result is:

$$a^2h^2 + b^2g^2 + c^2f^2 + d^2e^2 - 2ahbg - 2ahcf - 2ahde - 2bgcf - 2bgde - 2cfde + 4adfg + bech$$

where the values $a, b, \ldots$ can be considered in the corners of cubic matrix.

In the first half of 1990's, The Three Russian Musketeers I. Gelfand, M. Kapranov and A. Zelevinsky revived the topic in a series of articles and they collected them in the book "Discriminants, Resultants, and Multidimensional Determinants, 1994 Birkhäuser". Although they explained the geometric machinery behind it by considering hyperdeterminants as defining equation of Segre embedding of several projective spaces, as we see in chapter 3, they calculated hyperdeterminants in some special cases which they called boundary formats. One reason for this, exact calculations of hyperdeterminants is difficult. Even in the format $2 \times 3 \times 4$, hyperdeterminant is of degree 12 with ≤ 124416 terms.

In this thesis we give necessary and sufficient conditions for the existence of hyperdeterminants. Our results exceed the previous attempts. For explicit calculations we use a method so called castling transform which links hyperdeterminant of the format $p \times q \times r$ to the format $p \times q \times pq - r$.
This thesis is devoted to prove this theorem:

Theorem 1.0.7. *The followings are equivalent for three component system $\mathcal{H}_{ABC}$ with $\dim \mathcal{H}_A = p$, $\dim \mathcal{H}_B = q$, $\dim \mathcal{H}_C = r$:*

i-) There exist a completely entangled state $\psi \in \mathcal{H}_{ABC}$ with scalar margins

ii-) There exist a hyperdeterminant of format $p \times q \times r$

iii-) The castling reduced system satisfies log polygonal inequalities ie. $p \leq qr$, $q \leq pr$, $r \leq pq$.

In Chapter 2, we give foundational material of projective geometry.
In Chapter 3, we discuss the previous works of Cayley and GKZ.
Chapter 4 is devoted to quantum mechanics and representation theory. We state quantum marginal problem [2], Kronecker coefficients and some related topics.
In chapter 5, we prove our fundamental theorem.
Chapters 6 and 7 are devoted to invariant theory and castling transform. Hyperdeterminants will be reconsidered as invariants. For the existence of nonconstant invariants of multicomponent systems, some necessary conditions will be explained. We heavily use the method "castling transform" to classify invariants. Also we calculate hyperdeterminants of format $2 \times n \times n$.
Conclusion chapter contains extended version of fundamental theorem and some remarks on it.

Chapter 2

Projective Geometry

2.1 Basics

Let F be a field. Projective space of dimension n over a field F is the set of all one dimensional subspaces of the n-tuple space F^{n+1}. We denote it by $\mathbb{P}^n$.

Definition 2.1.1. Projective space of dimension n is the quotient of $F^{n+1} - \{(0, \ldots, 0)\}$ by the action of the multiplicative group F by scalar multiplication.

We say that α is equivalent to β, ie. $\alpha \sim \beta$ if there exists a nonzero λ such that $\alpha = \lambda\beta$. Hence we have;

$$\mathbb{P}^n := (F^{n+1} - \{(0, \ldots, 0)\}) \, / \sim$$

A point p in $\mathbb{P}^n$ is written as $p = [x_0 : x_1 : \ldots : x_n]$. $x_0, \ldots, x_n$ are called *homogeneous coordinates.*

Definition 2.1.2. For any finite dimensional vector space V, *projectivization* of V is the set of all one dimensional subspaces of V. It is denoted by $\mathbb{P}(V)$.

If the field is $\mathbb{C}$, $\mathbb{P}^n$ is called complex projective space. It is obvious that complex projective space of dimension n is projectivization of the vector space $\mathbb{C}^{n+1}$.

Remark 2.1.3. Let V be a vector space and W be its subspace, ie. $W \subset V$. Then $\mathbb{P}(W) \subset \mathbb{P}(V)$, and $\mathbb{P}(W)$ is called *projective subspace.*

Projective subspaces of dimension one and two are called *line* and *plane* respectively. Projective subspace of codimension one is called *hyperplane.*

Let f be a polynomial of degree k in $F[x_0, x_1, \ldots, x_n]$. f is called *homogeneous* if it satisfies that $f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_0, x_1, \ldots, x_n)$ where λ is a nonzero element of F.

Definition 2.1.4. *Projective variety* is a subset $V \subset \mathbb{P}^n$, such that there is a set of homogeneous polynomials $T \subset F[x_0, x_1, \ldots, x_n]$ with $V = \{p \in \mathbb{P}^n \mid f(p) = 0 \ \forall f \in T\}$.

Example 2.1.5. Recall Veronese map:

$$\begin{aligned} &\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3 \\ &\phi([x : y]) = \left[x^3 : x^2 y : x y^2 : y^3\right] \end{aligned}$$

The image of the ϕ is a projective variety. It is the zero locus of the polynomials listed below.

i-) $f_1(z_0, z_1, z_2, z_3) = z_0 z_3 - z_1 z_2$

ii-) $f_2(z_0, z_1, z_2, z_3) = z_1^2 - z_0 z_2$

iii-) $f_3(z_0, z_1, z_2, z_3) = z_2^2 - z_1 z_3$

where $[z_0 : z_1 : z_2 : z_3]$ are homogeneous coordinates of $\mathbb{P}^3$.

2.1.1 Duality

A map from a vector space over a field to that field is called *functional.* The space of all linear functionals is called *dual space.* We denote it by putting star on it $*$ likely $V^* := \{\phi \mid \phi : V \mapsto F\}$. The dual space is also a vector space by summation

$(\sigma + \gamma)(x) = \sigma(x) + \gamma(x)$ and scalar multiplication $(\lambda\sigma)(x) = \lambda\sigma(x)$.
Let V be a vector space. Recall that its projectivization $\mathbb{P}(V)$ is the set of all one dimensional subspaces of V. We are ready to define *dual projective space*;

Definition 2.1.6. Dual projective space is the set of all codimension one subspaces of V. It is denoted by $(\mathbb{P}(V))^*$.

Let H be a hyperplane in $\mathbb{P}^n$ which is given by an equation $a_0x_0 + a_1x_1 + \ldots + a_nx_n = 0$. If we regard $[a_0 : a_1 : \ldots : a_n]$ as a point of $\mathbb{P}^n$, then the map sending $H = \{\sum_{i=0}^n a_ix_i = 0\}$ to $[a_0 : a_1 : \ldots : a_n]$ is a bijection between the set $(\mathbb{P}^n)^*$ of all hyperplanes in $\mathbb{P}^n$ and $\mathbb{P}^n$.

Remark 2.1.7. There is a natural bijection between $\mathbb{P}(V^*)$ and $(\mathbb{P}(V))^*$ given by associating to the span of a nonzero functional $f : V \mapsto \mathbb{C}$, the codimension one subspace which is the kernel of f. Hence $\mathbb{P}(V^*) \cong (\mathbb{P}(V))^*$.

By the above remark, dual projective space is the projectivization of the dual space V^*. If we apply the above map twice, we get $(\mathbb{P}^n)^{**} = \mathbb{P}^n$. We can extend the duality between points and lines to other linear subspaces of V. Let W be a subspace of V and consider their projectivizations $\mathbb{P}(W)$ and $\mathbb{P}(V)$ respectively. Then $\mathbb{P}(W)^*$ is the projectivization of the orthogonal complement $W^\perp \subset V^*$ [3].

Let X be a closed irreducible algebraic variety in $\mathbb{P}^n$. Recall that, a hyperplane $H \in \mathbb{P}^n$ is called to be *tangent* to variety X if there exists a smooth point $p \in X$ such that $p \in H$ and the tangent space to H at p contains the tangent space to X at p.

Definition 2.1.8. For a given variety X, the closure of the set of all hyperplanes tangent to X is called *projective dual variety.* It is denoted by $X^\vee$. Clearly we have $X^\vee \in (\mathbb{P}^n)^*$.

Let X be a hypersurface in $\mathbb{P}^n$ defined by homogeneous polynomial $F(x_0, x_1, \ldots, x_n) = 0$. Let $p = [a_0 : a_1 : \ldots : a_n]$ be a point which satisries $F(p) = 0$. We calculate the dual as in the following way;

$$[a_0 : a_1 : \ldots : a_n] \mapsto \left[\frac{\partial F}{\partial x_0}(p) : \frac{\partial F}{\partial x_1}(p) : \ldots : \frac{\partial F}{\partial x_n}(p)\right]$$

The image is in the dual space $(\mathbb{P}^n)^*$.

Example 2.1.9. [3, p. 16] Let x, y, z be homogeneous coordinates for $\mathbb{P}^2$ and p, q, r be homogeneous coordinates for dual space $(\mathbb{P}^2)^*$. Consider a plane curve C which is given by $f(x, y, z) = 0$. We choose an affine chart $\mathbb{C}^2 \subset \mathbb{P}^2$ by setting $z = 1$. Assume that we have a parametrization for the curve C given by $x = x(t)$ and $y = y(t)$. Then we obtain parametrization for p, q such that $p = p(t)$ and $q = q(t)$. By the definition $\frac{\partial f}{\partial x} = p(t)$ and $\frac{\partial f}{\partial y} = q(t)$, hence we get the equation;

$$p(t)\, x'(t) + q(t)\, y'(t) = 0$$

Moreover, we have the tangent equation which is given by;

$$p(t)\, x(t) + q(t)\, y(t) = 1$$

if we set $r = 1$. After calculations we derive parametric representation of $X^\vee$ as;

$$p(t) = \frac{-y'(t)}{x'(t)\, y(t) - x(t)\, y'(t)}, \qquad q(t) = \frac{x'(t)}{x'(t)\, y(t) - x(t)\, y'(t)}$$

If we apply this method to the equations in 2.3, we get equations of the form

$$u(t) = \frac{-q'(t)}{p'(t)\, q(t) - p(t)\, q'(t)}, \qquad v(t) = \frac{p'(t)}{p'(t)\, q(t) - p(t)\, q'(t)}$$

where u, v are coordinates in $\mathbb{P}^2$. If we put equations of p and q into 2.4, it turns out that $x = u$ and $y = v$, hence $X = X^{\vee\vee}$.

2.1.2 Veronese and Segre Maps

Previously we introduced Veronese map from $\mathbb{P}^1$ to $\mathbb{P}^3$. Now we consider generalized case;

$$\begin{aligned} &\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^n \\ &\phi([x : y]) = \left[x^n : x^{n-1}y : \ldots : x^{n-i}y^i : \ldots : y^n\right] \end{aligned}$$

The image is the zero locus of the homogeneous polynomials $f_{i,j}(z_0, \ldots, z_n) = z_i z_j - z_{i+1}z_{j-1}$ where $[z_0 : z_1 : \ldots : z_n]$ are homogeneous coordinates on $\mathbb{P}^n$.

Definition 2.1.10. Segre map is defined by sending a pair $([x], [y]) \in (\mathbb{P}^n \times \mathbb{P}^m)$ to the point in $\mathbb{P}^{(n+1)(m+1)-1}$ whose coordinates are the pairwise products of the coordinates of $[x]$ and $[y]$, ie.

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \mapsto \mathbb{P}^{(n+1)(m+1)-1}$$
$$\sigma([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) \mapsto [x_0y_0 : \ldots : x_iy_j : \cdots : x_ny_m]$$

2.1.3 Grassmannian- Exterior Algebra- Plücker Embedding

Let V be a vector space of dimension n.

Definition 2.1.11. The set of all r dimensional subspaces of V is called *Grassmanian* and we denote it by $G_r(V)$.

For $r = 1$, we obtain $G_1(V) = \mathbb{P}(V)$. Also, for $r = n-1$, $G_{n-1}(V) = \mathbb{P}(V)^*$. We can obtain a natural isomorphism between $G_r(V)$ and $G_{n-r}(V^*)$ in this manner.

2.1.3.1 Exterior Algebra

Let R be a commutative ring with unit and M be an R module. We define *tensor algebra* of M over R as $T(M) = \bigoplus_{k=0}^{\infty} T^k(M)$ where $T^k(M) = \underbrace{M \otimes_R \ldots \otimes_R M}_{\text{k times}}$ is tensor product. Let $\mathcal{A}^n(M)$ be a submodule of $T^n(M)$ such that $\mathcal{A}^n(M) = \{x_1 \otimes x_2 \otimes \ldots \otimes x_n \in T^n(M) \mid x_i = x_j, i \neq j\}$. The quotient of $T^n(M)$ by $\mathcal{A}^n(M)$ is called *exterior algebra*, and it is denoted by $\bigwedge^n M$. Briefly we have

$$\bigwedge^n M = {}^{T^n(M)}/_{\mathcal{A}^n(M)}$$
$$x_1 \wedge \ldots \wedge x_n = x_1 \otimes \ldots \otimes x_n + \mathcal{A}^n(M)$$

2.1.3.2 Plücker Embedding

Let W be r dimensional subspace of V with basis $w_1, w_2, \ldots, w_r$. We consider the map;

$$\phi : G_r(V) \mapsto \mathbb{P}\left(\bigwedge^r V\right)$$

which sends span of $w_1, w_2, \ldots, w_r$ to $[w_1 \wedge w_2 \wedge \ldots \wedge w_r]$. The map is well defined since if we choose another basis which spans W, we get $v_1 \wedge v_2 \wedge \ldots \wedge v_r = \det A\,(w_1 \wedge w_2 \wedge \ldots \wedge w_r)$ where A is transformation matrix. So $[v_1 \wedge v_2 \wedge \ldots \wedge v_r] = \det A\,[w_1 \wedge w_2 \wedge \ldots \wedge w_r]$. This map is also injective, hence it is an embedding if we compare dimensions. This map is called *Plücker embedding*.

Chapter 3

Previous Studies for Hyperdeterminants

As the name of the chapter suggests, we introduce hyperdeterminants. First we mention the first article in which hyperdeterminants defined, then we examine the geometric machinery behind them.

3.1 Cayley's Work

In the paper [4], A. Cayley introduce the term *hyperdeterminant*. He focused on linear transformations of multilinear forms, and tried to find under which conditions this transformation preserves that form. The originality of his work is; he makes it without the concept of tensors. He started with bilinear form $U = \sum_i \sum_j r_i s_j x_i y_j$, then makes linear substitution. In modern language we have $A = [a_{ij}]$ where $a_{ij} = r_i s_j$, $X = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$. Then $U(A) = X^t A Y$. With new variables $X = [\lambda_{ij}] \dot{X}$ and $Y = [\sigma_{ij}] \dot{Y}$, we obtain $\dot{U} = \dot{X}^t \left([\lambda_{ij}]^t A [\sigma_{ij}] \right) \dot{Y}$. Then determinant of $\dot{A}$ is $\det \lambda \det \sigma \det A$.

Then, for the form $U = ax_1y_1z_1 + bx_2y_1z_1 + cx_1y_2z_1 + dx_2y_2z_1 + ex_1y_1z_2 + fx_2y_1z_2 + gx_1y_2z_2 + hx_2y_2z_2$, he calculates hyperdeterminant [4, p.(13) 89] which is:

$$a^2h^2 + b^2g^2 + c^2f^2 + d^2e^2 - 2ahbg - 2ahcf - 2ahde - 2bgcf - 2bgde - 2cfde + 4adfg + bech$$

Indeed, above variables are elements of three dimensional matrix of format $(2, 2, 2)$.
We can view it as:

$$\left[\begin{array}{cc|cc} a & c & e & g \\ b & d & f & h \end{array}\right]$$

where $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $\begin{bmatrix} e & g \\ f & h \end{bmatrix}$ are parallel slices in the third coordinate.

3.2 GKZ-Hyperdeterminants

Despite being creative and original at his time, Cayley's work is primitive for today's mathematical techniques. Here, we present hyperdeterminants in more precise way. According to the [3, p. 444], there are three different approach to define GKZ-hyperdeterminants, we give here geometric and analytic construction.

Let $V_1, V_2, \ldots, V_r$ be vector spaces with dimensions $k_1, k_2, \ldots, k_r$ respectively. Let X be the product of projectivizations of $V_1, V_2, \ldots, V_r$, ie. $X = \mathbb{P}(V_1^*) \times \mathbb{P}(V_2^*) \times \ldots \times \mathbb{P}(V_r^*)$. As we discussed previous chapter, the product X can be embedded (Segre) into projective space of dimension $k_1k_2 \cdots k_r - 1$. The hyperdeterminant of format $k_1 \times k_2 \times \ldots \times k_r$ is homogeneous polynomial function on $V_1 \otimes V_2 \otimes \ldots \otimes V_r$ which is a defining equation of the projective dual variety $X^\vee$. The existence of hyperdeterminant depends on whether $X^\vee$ is hypersurface in $\mathbb{P}^{k_1k_2\cdots k_r-1-1}$ or not. If $X^\vee$ is not hypersurface, we call it trivial case.
In coordinate wise interpretation, if we choose a basis for each V_i, we can represent any element in $V_1 \otimes \ldots \otimes V_r$ by a multidimensional matrix.

Let H be a hyperplane which lies in $X^\vee$. Hence H and its partial derivatives should vanish at some point of X. Let $x^j = \left(x_0^j, x_1^j, \ldots, x_{k_j}^j\right)$ be a coordinate system on V_j^*, then H becomes;

$$H\left(x^1, x^2, \ldots, x^r\right) = \sum_{i_1,\ldots,i_r} a_{i_1,\ldots,i_r} x_{i_1}^1 \ldots x_{i_r}^r$$

If hyperdeterminant is zero, we have nonzero solutions of those equations;

$$H\left(x^1, x^2, \ldots, x^r\right) = 0, \qquad \frac{\partial H\left(x^1, x^2, \ldots, x^r\right)}{\partial x_i^j} = 0$$

for all i, j.

Example 3.2.1. In this example we show that hyperdeterminant, as the term suggests, is a generalization [1] of classical determinant. Consider Segre embedding of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ into $\mathbb{P}^{n^2+2n}$. After restriction, any linear form f can be written as $f = \sum_{i,j} a_{ij} x_i y_j$. Let $A = [a_{ij}]$. The condition 3.2 turns into those equations:

$$\frac{\partial f}{\partial x_i} = 0 \Longrightarrow AY = 0$$
$$\frac{\partial f}{\partial y_j} = 0 \Longrightarrow AX = 0$$

where $X = [x_1, \ldots, x_n]^t$ and $Y = [y_1, \ldots, y_n]^t$. Nontrivial solution exists when $\det A = 0$.

Example 3.2.2. [3, p. 449] After applying (3.4) for $U = ax_1y_1z_1 + bx_2y_1z_1 + cx_1y_2z_1 + dx_2y_2z_1 + ex_1y_1z_2 + fx_2y_1z_2 + gx_1y_2z_2 + hx_2y_2z_2$, hyperdeterminant vanishes if the equations below have nontrivial solution:

$$ax_1y_1 + bx_2y_1 + cx_1y_2 + dx_2y_2 = 0$$
$$ex_1y_1 + fx_2y_1 + gx_1y_2 + hx_2y_2 = 0$$
$$ax_1z_1 + bx_2z_1 + ex_1z_2 + fx_2z_2 = 0$$
$$cx_1z_1 + dx_2z_1 + gx_1z_2 + hx_2z_2 = 0$$
$$ay_1z_1 + cy_2z_1 + ey_1z_2 + gy_2z_2 = 0$$
$$by_1z_1 + dy_2z_1 + fy_1z_2 + hy_2z_2 = 0$$

3.2.1 Existence of GKZ-Hyperdeterminants

In [3], authors states the theorem below which contains a sharp condition for the existence of hyperdeterminant for multicomponent systems. We weaken this condition for the existence of hyperdeterminants.

Theorem 3.2.3. *The GKZ-hyperdeterminant of the format $k_1 \times k_2 \times \ldots \times k_r$ is nontrivial if and only if $k_j - 1 \leq \sum_{i \neq j} (k_i - 1)$ for each j.*

[1] algebraic generalization not geometric, since we do not have volume of any object.

3.2.2 Degree Formulas For GKZ-Hyperdeterminants

We consider the GKZ-hyperdeterminant of the format $(k_1, \ldots, k_r)$. The generating function for the degrees $D(k_1, \ldots k_r)$ is given by

$$\sum_{k_1,\ldots k_r \geq 1} D(k_1, \ldots, k_r)\, z_1^{k_1-1} \cdots z_r^{k_r-1} = \frac{1}{\left(1 - \sum_{i=2}^{r} (i-2)\, e_i(z_1, \ldots, z_r)\right)^2},$$

where e_i is the i-th symmetric polynomial [3, p. 454]. This formula is compact however calculations are not easy. For some small formats we have;

Format	Degree
$(2, n, n)$	$2n(n-1)$
$(3, n, n)$	$3n(n-1)^2$
$(2, 2, 2, 2)$	24
$(2, 2, 3)$	6

The format $k_1 \times k_2 \times \ldots \times k_r$ is called *boundary format* if it satisfies $k_j - 1 = \sum_{i \neq j} k_i - 1$ for some j [3]. Similarly we define *log-boundary format* if logarithm of dimensions satisfy $\log k_j = \sum_{i \neq j} \log k_i$ for some j.

Chapter 4

Quantum Mechanics and Representation Theory

We roughly mentioned very fundamental part of quantum mechanics in the introduction. In this chapter, we extend it. The list below contains some basics of quantum mechanics:

- A pure state $\psi \in \mathcal{H}_A$ of a quantum system A is a unit vector in its Hilbert space $\mathcal{H}_A$, or the projector $|\psi\rangle\langle\psi|$ onto direction ψ, if the phase factor is unimportant.
- A mixed state ρ is a convex combination of pure states $\rho = \sum_i \rho_i |\psi_i\rangle\langle\psi_i|$ with probabilities ρ_i. Clearly $\rho \geq 0$ is nonnegative Hermitian opertor in $\mathcal{H}_A$ of unit trace $\operatorname{Tr}\rho = 1$.
- An observable X_A is a Hermitian operator in $\mathcal{H}_A$. Its measurement on the system in state ψ produces a random quantity with the expectation value $\langle\psi|X_A|\psi\rangle$. For mixed state ρ the expectation is given by the trace $\operatorname{Tr}(\rho X_A)$.
- Superposition principle $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.
- For mixed state ρ_{AB} of a composite system AB there exists unique state ρ_A of the system A s.t.$\operatorname{Tr}(\rho_{AB}X_A) = \operatorname{Tr}(\rho_A X_A)$, $\forall$ observables X_A, i.e. ρ_A is a visible state of the subsystem A, called reduced or marginal state of A. The reduction $\rho_{AB} \mapsto \rho_A$ is just another name for the contraction of a tensor.

In the simplest form, tensor contraction defined as a map $\mathcal{H}^* \otimes \mathcal{H} \mapsto \mathbb{C}$, with $(f, v) = f(v)$, $f(v)$ is a bilinear form. After fixing basis, we can represent an element from $\mathcal{H}^* \otimes \mathcal{H}$ by a matrix $\left[\alpha_i^j\right]$, where i and j corresponds to row and columns respectively. Then the above map turns into trace: $\sum_i a_i^i = \operatorname{Tr}\left(\left[\alpha_i^j\right]\right)$. If we have tensor of (m, n) type ie. $(\mathcal{H}^*)^{\otimes m} \otimes (\mathcal{H}^{\otimes n})$, we get multidimensional matrix $A = \left[a_{i_1 \ldots i_n}^{j_1 \ldots j_m}\right]$, and contraction on each component is defined similar to the above construction.

4.1 Quantum Marginal Problem

In the introduction, we considered at most three component systems. Generalization of this is the following:

$$\begin{aligned} &\mathcal{H}_I = \bigotimes_{i \in I} \mathcal{H}_i = \mathcal{H}_J \otimes \mathcal{H}_{\overline{J}}, \quad J \subset I, \quad \overline{J} = I \setminus J \\ &\rho_J = \operatorname{Tr}_{\overline{J}}(\rho_I) \end{aligned} .$$

However, we want to focus on three component systems for now. For a reduced state ρ_j, let λ_j be its spectra in nonincreasing order, $\lambda_j = \left(\lambda_j^1, \ldots, \lambda_j^k\right)$. Pure quantum marginal problem asks the following:

Problem 4.1.1. Find conditions on ρ_A, ρ_B, ρ_C to be reduced states of $\psi \in \mathcal{H}_{ABC}$.

Answer depends only on spectra λ_A, λ_B, λ_C of ρ_A, ρ_B, ρ_C.

Investigating two component systems $\mathcal{H} \otimes \mathcal{H}_B$ is useful. Let e_i and f_j be orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$. Then $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written in matrix $[\psi_{ij}]$ where $\psi = \sum_{i,j} \psi_{ij} e_i \otimes f_j$. Margins of ψ in bases are:

$$\rho_A = \psi^* \psi, \qquad \rho_B = \psi \psi^*$$

After discarding zero eigenvalues, margins of pure state ψ have the common eigenvalues λ_i, $\operatorname{spec} \rho_A = \operatorname{spec} \rho_B$. Then the state $\psi = \sum_i \lambda_i e_i \otimes f_i$ has margins ρ_A, ρ_B. This system is completely entangled if and only if $\dim \mathcal{H}_A = \dim \mathcal{H}_B$, and $[\psi_{ij}]$ is unitary matrix. We make this characterization due to the entanglement equation [1] which can be paraphrased as: State $\psi \in \mathcal{H}$ is completely entangled if all observables X have zero expectation in state ψ

$$\langle\psi|X|\psi\rangle = 0.$$

For our two component system this amounts to the identification:

Theorem 4.1.2. *State ψ is completely entangled if and only if rows and columns of $[\psi_{ij}]$ are orthogonal and have the same norm.*

For ρ_A and ρ_B, normalization factors are $\frac{1}{\dim \mathcal{H}_A}$ and $\frac{1}{\dim \mathcal{H}_B}$. Thus they have to be same if the system is entangled.

For three component system we have similar identification, however this time we are dealing with three dimensional matrices:

Theorem 4.1.3. *State $\psi \in \mathcal{H}_{ABC}$ is completely entangled if and only if in each direction parallel slices are orthogonal.*

Hence, $\rho^A_{ij} = \mathrm{slice}_i\, \overline{(\mathrm{slice}_j)}$. In this view, it is possible to consider state ψ as a multidimensional analogue of unitary matrix.

4.1.1 Polygonal Inequality

Log-polygonal inequality comes from this fact: we can consider three component system $\mathcal{H}_{ABC} = \mathcal{H}_{AB} \otimes \mathcal{H}_C$. So ρ_{AB} and ρ_C have the same spectra, it follows that rank of ρ_C equals rank of ρ_{AB} and the last one is smaller or equal to product of ranks ie. $\mathrm{Rank}\, \rho_C = \mathrm{Rank}\, \rho_{AB} \leq \mathrm{Rank}\, \rho_A \times \mathrm{Rank}\, \rho_B$. This is same for other identifications hence we get $d_A \leq d_B \times d_C$, $d_C \leq d_B \times d_A$, $d_B \leq d_A \times d_C$ where $d_X = \mathrm{Rank}\, \rho_X$ for $X \in \{A, B, C\}$.

4.1.2 Criterion For Entanglement

It is appropriate to mention briefly the relationship between quantum entanglement and invariant theory.

According to the article "Dynamical Symmetry Approach to Entanglement[1]", *quantum dynamical system* $G : \mathcal{H}$ is defined in the following way: state space $\mathcal{H}$ with the action of dynamical symmetry group G which corresponds to local measurements.

Example 4.1.4. If we are restricted to local measurements of a system consisting of two remote components A, B with full access to the local degrees of freedom then the dynamical group is $SU(\mathcal{H}_A) \times SU(\mathcal{H}_B)$ acting on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

In the same paper [1], A. Klyachko makes a characterization for entanglement as in the following:

Theorem 4.1.5. *State $\phi \in \mathcal{H}$ is entangled if and only if it can be separated from zero by G-invariant polynomial*

$$f(\phi) \neq f(0), f(g.x) = f(x), \forall g \in G, x \in \mathcal{H}.$$

Example 4.1.6. [1] State $\psi \in \mathcal{H}_{AB}$ is entangled if and only if $\det[\psi_{ij}]$ is nonzero.

As we seen in the example, this criteria enables us to reduce problems of quantum entanglement into invariant theoretical settings. To illuminate quantum entanglement, one need to describe invariants.However, working with special unitary group may cause some difficulties. To exceed it, there are two important tools [1]:

Theorem 4.1.7 (Kempf-Ness)**.** *State $\phi \in \mathcal{H}$ is completely entangled if and only if it has minimal length in its complex orbit*

$$|\psi| \leq |g.\psi|, \forall g \in G^c$$

Complex orbit $G^c\psi$ contains a completely entangled state if and only if it is closed. In this case the completely entangled state is unique up to action of G.

Theorem 4.1.8. *Complex stabilizer $(G^c)_\psi$ of stable state ψ coincides with complexification of its compact stabilizer $(G_\psi)^c$.*

Complexification of SU_n is SL_n, and this corresponds to stochastic local operations and classical communication (SLOCC) which can be explained by this: two states belong to the same class under SLOCC if and only if they are converted by an invertible local operation. We can work on group SL instead of SU.

4.2 Representation Theory

Recall that a partition λ of n is non-increasing sequence of numbers whose sum is n, ie. $\sum_i \lambda_i = n$. It is denoted by $\lambda \vdash n$. We also represent them by using Young diagram of height i, of lengths λ_i.

Example 4.2.1. Let $\lambda = (4, 2, 1)$ be partition of 7, diagram is:

$$\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array}$$

Young diagrams are very essential tool for representations of symmetric group and general linear group. Consider symmetric group of four letters. There are five irreducible representations We have:

- Trivial representation corresponds to $\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline\end{array}$
- Sign or alternating representation: $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$
- Standard representation: $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$
- Tensor product of standard representation and sign representation: $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1}\end{array}$
- The last one: $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}$

Dimensions of representations of symmetric group can be calculated by hook-length formula. Hook length of a square in diagram is summation of number of boxes in the same column, row plus one. For example in standard representation hook lengths are:

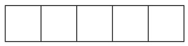
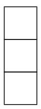
$$\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}.$$

Theorem 4.2.2. $\dim S^\lambda = \dfrac{n!}{\prod HookLengths}$

In the previous example, dimension is 3.

Let $\mathcal{H}^{\lambda}$ be representation of $GL(\mathcal{H})$. It can be parametrized by Young diagrams. Row diagram with lenght n corresponds to nth syymetric power. Column diagram with length n corresponds to nth alternating power.

Example 4.2.3.

-  corresponds to $S^5\mathcal{H}$
-  corresponds to $\Lambda^3\mathcal{H}$

Let λ be a column diagram. It produces one dimensional representation given by determinant if $\dim \mathcal{H} = \dim \lambda$. Rectangular block $n \times k$ gives $\det^k$.

Definition 4.2.4. [5, 149] Let V be a G-module. The space of G-invariants in V, V^G is isotypic component of the trivial representation in V.

We consider $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ as a $GL(\mathcal{H}_A) \times GL(\mathcal{H}_B) \times GL(\mathcal{H}_C)$ module. It is possible to decompose $S^N(\mathcal{H}_{ABC})$ as:

$$S^N(\mathcal{H}_{ABC}) = \bigoplus_{\lambda,\mu,\nu \vdash N} \left(\mathcal{H}_A^{\lambda} \otimes \mathcal{H}_B^{\mu} \otimes \mathcal{H}_C^{\nu}\right)^{\oplus \dim(\lambda\otimes\mu\otimes\nu)^{S_N}}$$

where $(\lambda \otimes \mu \otimes \nu)^{S_N}$ denotes the space of S_N invariants. The multiplicity is nothing but so called Kronecker coefficient, $g(\lambda, \mu, \nu) = \dim(\lambda \otimes \mu \otimes \nu)^{S_N}$. Numerical calculations are deeply related to the character table of S_N, since we have:

$$\dim(\lambda \otimes \mu \otimes \nu)^{S_N} = \frac{1}{N!} \sum_{\sigma \in S_N} \chi_{\lambda}(\sigma)\, \chi_{\mu}(\sigma)\, \chi_{\nu}(\sigma)$$

where χ_{λ} is character of representation λ.

Remark 4.2.5. We treat young diagram $\lambda = (k_1, k_2, \ldots, k_r)$ with $\lambda \vdash n$ as spectra and normalized them to unit trace $\overline{\lambda} = \left(\frac{k_1}{n}, \frac{k_2}{n}, \ldots, \frac{k_r}{n}\right)$.

Remark 4.2.6. Let $\psi \in \mathcal{H}_{ABC}$ with reduced matrices ρ_A, ρ_B, ρ_C. Their spectra $\left(\lambda^A, \lambda^B, \lambda^C\right)$ forms a convex polytope which is called *moment polytope*.

The following theorem [6] combines representations and quantum marginal problem:

Theorem 4.2.7.

- *$g(\lambda, \mu, \nu) \neq 0 \Rightarrow$ normalized diagrams of young diagrams are reduced spectra of some pure state $\psi \in \mathcal{H}_{ABC}$ [2].*
- *The whole moment polytope is a convex hull of such normalized spectra which are also everywhere dense in it.*

The interpretation of the problem is this: an entangled states ψ exists in $\mathcal{H}_{ABC}$ if and only if $g(\lambda, \mu, \nu) \neq 0$ for some scalar spectra $\left(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}\right)$. Now the invariants arrive on the scene, the term $\mathcal{H}_A^\lambda \otimes \mathcal{H}_B^\mu \otimes \mathcal{H}_C^\nu$ is $SL(\mathcal{H}_A) \times SL(\mathcal{H}_A) \times SL(\mathcal{H}_C)$-invariant.

Chapter 5

Main Result

Let G be a connected semisimple group with representation V, $\dim V = n$. Consider the representation $\left(G \times SL_k, V \otimes \mathbb{C}^k\right)$ with $1 \leq k < n$. The representation $\left(G \times SL_{n-k}, V^* \otimes \mathbb{C}^{n-k}\right)$ is said to be obtained by the *castling transform* [7]. Those systems are called *castling equivalent*
We discuss most of the properties of castling transform in the chapter "Castling Transform". However, to state our main theorem we give an important property of castling transform [8]:

Theorem 5.0.8. *The algebras of invariants of castling equivalent systems are isomorphic.*

We prove it later. The theorem above and partial list of Littelmann [8] enable us to prove our main theorem:

Theorem 5.0.9. *Consider three component system of format* (p, q, r) *such that* $G : \mathcal{H}_{ABC}$ *where* $G = SL(\mathcal{H}_A) \times SL(\mathcal{H}_B) \times SL(\mathcal{H}_C)$ *and* $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ *with* $\dim \mathcal{H}_A = p$, $\dim \mathcal{H}_B = q$, $\dim \mathcal{H}_C = r$. *The followings are equivalent*

i-) *There exists completely entangled state* $\psi \in \mathcal{H}_{ABC} \iff$

ii-) *There exists nontrivial* G*-invariant* $\iff$

iii-) *Logarithm of the dimensions of castling reduced system satisfy polygonal inequality.*

Proof. As we discussed in the section quantum marginal problem, completely entangled state exists if and only if margins are scalar. Another characterization of entanglement, by theorem 4.2.7, is: an entangled state exists if and only if respective Kronecker coefficients are nonzero. This implies existence of a G-invariant and $(i) \iff (ii)$. Castling equivalent systems have the same algebra of invariants with different grading. In this chain, take the one which has minimal dimensions. For the existence of invariants necessary condition is this: logarithm of dimensions of castling reduced system satisfies polygonal inequality. This is demonstrated in subsection "Polygonal inequality". Connected and semisimple groups with irreducible representations, which have polynomial algebra of invariants, were classified by Littelmann in [8]. In our case, the required ones can be collected in the table:

G	V	Con.	$\dim V$	$\dim V//G$	Degree
$SL_{n+1} \times F$	$V_{n+1} \otimes V_d$	$d < n+1$	$(n+1)\,d$	0	0
$SL_{n+1} \times F$	$V_{n+1} \otimes V_d$	$d = n+1$	$(n+1)\,d$	1	$n+1$
$SL_4 \times SL_4 \times SL_2$	$V_4 \otimes V_4 \otimes V_2$	-	32	2	8,12
$SL_3 \times SL_3 \times SL_2$	$V_3 \otimes V_3 \otimes V_2$	-	18	1	12
$SL_3 \times SL_3 \times SL_3$	$V_3 \otimes V_3 \otimes V_3$	-	27	3	6,9,12
$SL_2 \times SL_2 \times SL_2$	$V_2 \otimes V_2 \otimes V_2$	-	8	1	4
$SL_2 \times SL_2 \times SL_2 \times SL_2$	$V_2 \otimes V_2 \otimes V_2 \otimes V_2$	-	16	4	2,4,4,6

where F is any connected semisimple group with representation of total dimension d. This table consists of castling reduced systems. Besides the first two, are sporadic cases. By the first entry in the table, if dimensions do not satisfy log-polygonal inequality, they can not have invariant. Also, existence of invariants implies log-polygonal inequalities. To sum up we have the equivalent statements:

- State ψ is completely entangled $\iff$
- Margins are scalar $\iff$
- Kronecker coefficients are nonzero $\iff$
- There exist invariants $\iff$

- Log-polygonal inequalities are satisfied

This proves our theorem. $\square$

In the chapter "Conclusion", we generalize this theorem for n-component systems with $n \geq 4$.

Chapter 6

Invariant Theory

In this part, we introduce foundations of invariant theory and explain some methods which are necessary to understand hyperdeterminants. Unless stated differently, our ground field $k = \mathbb{C}$.

6.1 Basics

Let V be a finite dimensional vector space over field k. Then, $GL(V)$ is the group of automorphisms of V. If dimension of V is n, we have $GL(V) = GL(n, k)$, where $GL(n, k)$ is group of invertible $n \times n$ matrices. For a given group G and vector space V, we consider group homomorphism $\rho : G \to GL(V)$. ρ is called *representation* of G. Most of the time we call V representation, and we use $g.v$ instead of $\rho(g) v$ for $g \in G$, $v \in V$.

A subrepresentation of a V is G-invariant vector subspace W of V. A representation V is called irreducible if there is no G-invariant subspace W of V.

Example 6.1.1. Let $\rho : G \to GL(V)$ be a representation. Then, the dual representation $\rho^* : G \to GL(V^*)$ is given by:

$$\rho^*(g) = \rho\left(g^{-1}\right)^t$$

for all $g \in G$.

A function $f : V \mapsto k$ is called polynomial on V if f is a polynomial in the coordinates with respect to basis of V. Polynomials on V forms an algebra over k which is denoted by $k[V]$. It is also called coordinate ring [9]. Assume that V^* has basis $x_1, x_2, \ldots, x_n$ which are called coordinate functions, then we get $k[V] = k[x_1, x_2, \ldots, x_n]$.

Remark 6.1.2. $k[V]$ is graded algebra with respect to degree, ie. $k[V] = \bigoplus_d k[V]_d$ where $k[V]_d$ is the subspace of homogeneous polynomials of degree d. Hence degree one elements of $k[V]$ is V^*. Also we have $k[V]_d = S^d(V^*)$, where $S(V)$ is the symmetric algebra of V.

A polynomial $f \in k[V]$ is called invariant if $f(gv) = f(v)$ for all $g \in G$ and $v \in V$. We define the action of G on $k[V]$ by $(gf)(v) = f(g^{-1}v)$. We use g^{-1} inside f, to make this left action on the space of functions on V.

Let G be a group acting on V, ie. $G \subset GL(V)$. The ring

$$k[V]^G = \{f \in k[V] \mid \sigma f = f, \forall \sigma \in G\}$$

is called *algebra of invariants* or *invariant ring*. It is subalgebra of $k[V]$.
I give here some classical examples of invariant theory [10, p. 134,140].

Example 6.1.3. i-) We consider the action of $G = S_n$ on $V = \mathbb{C}^n$ by permutation of coordinates as $\sigma(x_1, \ldots, x_n) = \left(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\right)$, $\sigma \in S_n$ Then $\mathbb{C}[V]^G$ is the algebra of symmetric polynomials. Moreover, by the fundamental theorem of symmetric functions, every symmetric polynomial can be expressed in the elementary symmetric polynomials: $e_d = \sum_{i_1<\ldots<i_d} x_{i_1} \cdots x_{i_d}$, for all $1 \leq d \leq n$. Hence $k[V]^{S_n} = k[e_1, e_2, \ldots e_n]$.

ii-) One of the oldest and richest topic in invariant theory is invariants of binary forms. Let $V = V_d$ be the vector space of binary forms $f(x, y) = \sum_{i=0}^{d} \alpha_i x^{d-i} y^i$ of degree d with coordinates $x, y \in \mathbb{C}^2$. The action of $G = SL_2(\mathbb{C})$ is given by:

$$(gf)(x, y) = f(dx - by, -cx + ay), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

The inverse matrix g is used in the action to make it left action in the space of functions. *Invariants of a binary form of degree* d are the invariants of the action of G on V_d. For every binary form, its discriminant is invariant. For small degrees, we give invariants in the part 4.5 .

6.2 Tensor Invariants

Let $V^{\otimes k}$ denote $\underbrace{V \otimes \ldots \otimes V}_{\text{k times}}$ for a vector space V. We consider the action of $GL(V)$ on rank one elements of $V^{\otimes k}$ which is given by:

$$g(v_1 \otimes \ldots \otimes v_k) = (gv_1) \otimes \ldots \otimes (gv_k)$$

for $g \in GL(V)$, $v_1 \otimes \ldots \otimes v_k \in V^{\otimes k}$. This action can be extended linearly, hence we obtain $GL(V)$ action on $V^{\otimes k}$.

When we set $k = 2$, we can identify $V \otimes V$ with $n \times n$ matrix where n is dimension of V. In this case, we have;

$$\begin{aligned} &Mat_n(k) \to Mat_n(k) \\ &X \to gXg^t \end{aligned}$$

By the inspiration of previous group action, we can similarly consider the action of the group $GL(V_1) \times GL(V_2) \times \ldots \times GL(V_k)$ on $V_1 \otimes V_2 \otimes \ldots \otimes V_k$ which is given by;

$$(g_1, g_2, \ldots, g_k)(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = (g_1v_1) \otimes (g_2v_2) \otimes \ldots \otimes (g_kv_k)$$

for $(g_1, g_2, \ldots, g_k) \in GL(V_1) \times GL(V_2) \times \ldots \times GL(V_k)$, $v_1 \otimes \ldots \otimes v_k \in V_1 \otimes V_2 \otimes \ldots \otimes V_k$

After setting $k = 2$ and $\dim V_1 = \dim V_2 = n$, we obtain:

$$\begin{aligned} &Mat_n(k) \to Mat_n(k) \\ &X \to AXB^{-1} \end{aligned}$$

It is obvious that determinant of X is semi-invariant under above action since we have multiplication by nonzero scalars. To get rid of this, we choose elements A and B from $SL(V)$. In conclusion determinant is invariant. Indeed, determinant is the only invariant for the previous construction.

Proposition 6.2.1. $k[Mat_n(k)]^{SL_n \times SL_n} = k[\det]$.

Proof. Let X be a generic matrix in $Mat_n(k)$. The left multiplication by $A \in SL_n$ corresponds to row operations, similarly the action of $B \in SL_n$ corresponds to column operations. Hence X can be written in diagonal form. More precisely we get:

$$X' = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

where λ is determinant of X'. Let $f(X) = \det X$. It is clear that f is invariant with the restriction $f|_{X'} = \lambda$. The algebra of invariants of those kind of matrices is $k[t]$ with trivial action. Since diagonal matrices are dense in $Mat_n(k)$, determinant is the only invariant function. Or it can be seen directly that the only parameter in X' is $\det X'$. □

Proposition 6.2.2. *Let X be $m \times n$ matrix. If $m \neq n$, there exists no invariant for the action of $SL_m \times SL_n$.*

Proof. Under the action of $SL_m \times SL_n$, X can be transformed into:

$$X = \begin{bmatrix} \lambda_1 & 0 & \cdots & & & 0 \\ 0 & \lambda_2 & \cdots & & & 0 \\ \vdots & & \ddots & & & 0 \\ 0 & \cdots & & \lambda_k & \cdots & 0 \end{bmatrix}$$

or its transpose, with $k \leq \min(m, n)$.
Closure of each orbit contains zero. Any invariant function f assumes same values on orbits, so it is zero. □

Corollary 6.2.3. *It is obvious that orbits are closed if and only if $m = n$.*

Indeed, for the action of general linear group, orbits are ranks. Two matrices are in the same orbit if and only if they have the same rank.
The hyperdeterminant, defined over $V_1 \otimes V_2 \otimes \ldots \otimes V_k$, is invariant under the action of the group $SL(V_1) \times SL(V_2) \times \ldots \times SL(V_k)$. Most of the time we focus three component systems. Cayley's hyperdeterminant is simply $SL_2 \times SL_2 \times SL_2$ invariant.

6.3 Russian Conjecture

First we introduce some general properties of invariant rings. According to Popov and Vinberg [10], page 239, there are various good (from the point of view of invariant theory) properties that reductive linear groups can possess. Those are:

- (FA) the algebra of invariants $k[V]^G$ is free, ie it has algebraically independent homogeneous generators
- (ED) All fibers of the quotient morphism $\phi : V \to V//G$ have the same dimension.
- (FO) Each fiber of the morphism ϕ contains only a finite number of orbits.
- (ST) The stabilizer of a point of V in general position is nontrivial.
- (FM) The algebra $k[V]$ is a free $k[V]^G$-module

Linear groups satisfying (FA), (ED), (FO) are called coregular, equidimensional, visible respectively.

Theorem 6.3.1. *[10, p. 247] For connected simple irreducible linear groups the properties (FA), (ED), (FO), (ST) are equivalent.*

The previous theorem become invalid for connected semisimple linear groups. There are counterexamples for the implications (FA)⇒(ED), (ED)⇒(FO), (ST)⇒(FA), (ED)⇒(ST) [10].

We give here the statement of Russian conjecture:

Conjecture 6.3.2. If G is connected and semisimple and V is a representation, then (ED) implies (FA).

We have some classifications, and completed classifications for some groups. Assume G is connected and semisimple and V is irreducible. We have the following results:

- All (G, V) satisfying (FA) and (ED) were classified by Littelmann [8],[9]
- (FO) were classified by Kac [9]

6.4 Categorical Quotient

The ring $k[X]^G$ is finitely generated when G is reductive group and X is a G variety [9, p. 51]. We define $X//G$ as the affine variety which corresponds to $k[X]^G$. Since there is the inclusion $k[X]^G \subseteq k[X]$, it turns into a morphism $\phi : X \twoheadrightarrow X//G$. $X//G$ is called *categorical quotient*. In other words, ϕ sends orbits of G to the points in $X//G$.

Remark 6.4.1. We give here some properties of categorical quotient:

i-) Categorical quotient exists if and only if algebra of invariants are finitely generated.

ii-) Let X be a G-variety. Then the map $\phi : X \to X//G$ is surjective.

iii-) Let Y_1, Y_2 be subvariety of X which are stable under G action. Then $\overline{\phi(Y_1)} \cap \overline{\phi(Y_2)} = \overline{\phi(Y_1 \cap Y_2)}$.

iv-) For every $x \in X//G$, the fiber $\phi^{-1}(x)$ contains exactly one closed orbit.

Example 6.4.2. Assume that G is multiplicative group k^* acting on $V = k^n$ by

$$\lambda.(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n) \qquad \lambda \in G, (x_1, \ldots x_n) \in k^n.$$

It is obvious that $k[V]^G = k$, hence $V//G$ is just a point. Nonzero orbits are lines though origin with the origin removed. Closure of orbits contains zero.

Example 6.4.3. Assume that G is multiplicative group k^* acting on $V = k^2$ by

$$\lambda.(x, y) = \left(\lambda x, \lambda^{-1} y\right) \qquad \lambda \in G, (x, y) \in k^2$$

Then $k[V]^G = k[xy]$, hence $V//G = k$. The orbits of $(x,) \neq (0, 0)$ are one dimensional and closed (parts of hyperbola). The orbits of points on x and y axes have dimension one but their closure contain $\{(0, 0)\}$. Moreover we have:

$$\begin{aligned} &\phi : V \to V//G \\ &(x, y) \to xy \end{aligned}$$

Chapter 7

Castling Transform

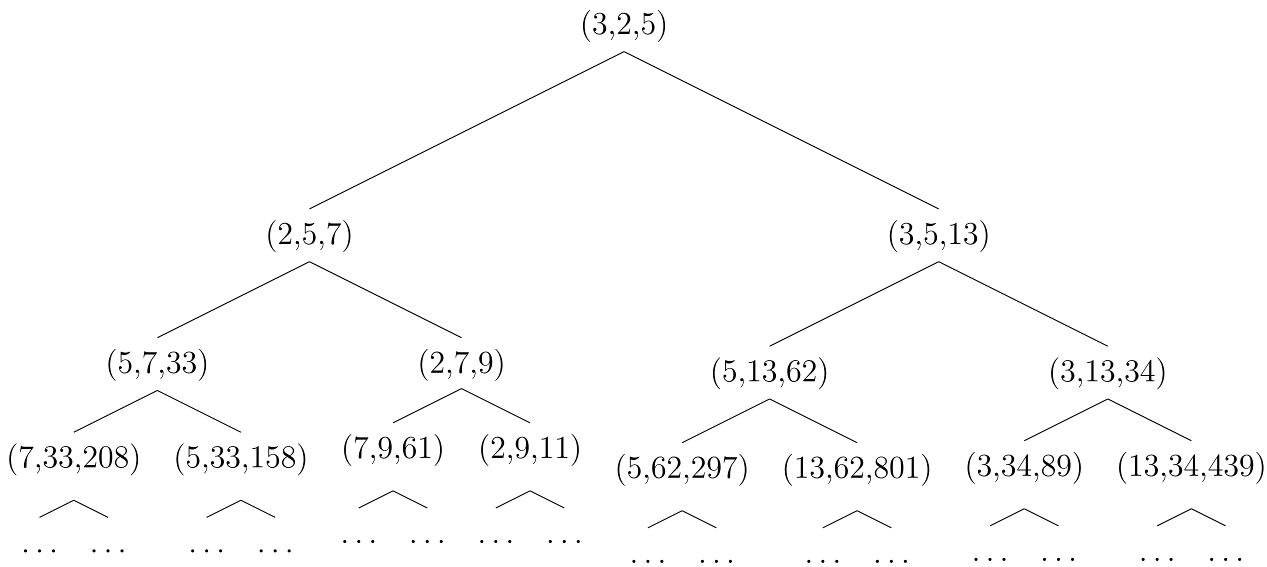
To study properties of invariants, we introduce an essential tool which is called castling transform.

Let G be a connected semisimple group with representation V, and $\dim V = n$. By using this, it is possible to construct a representation $V \otimes k^d$ for the group $G \times SL_d\left(k^d\right)$ where $1 \leq d < n$. For the sake of the notation, (G, V) denotes for any group G with representation V.

Definition 7.0.4. For a given tuple $\left(G \times SL_d, V \otimes k^d\right)$, a new one can generated as $\left(G \times SL_{n-d}, V^* \otimes k^{n-d}\right)$, it is said to be obtained from $\left(G \times SL_d, V \otimes k^d\right)$ by the *castling transform.*

$\left(G \times SL_d, V \otimes k^d\right)$ and $\left(G \times SL_{n-d}, V^* \otimes k^{n-d}\right)$ are called castling equivalent to each other. This method enables us to exploit arithmetical calculations. By this, we can make chains of castling castling transforms by increasing the total degree of representation. Degrees of representations in the tuples $\left(G \times SL_d, V \otimes k^d\right)$,$\left(G \times SL_{n-d}, V^* \otimes k^{n-d}\right)$ are nd and $n^2 - nd$. After choosing $d \leq \frac{n}{2}$, degree is increased.

Example 7.0.5. We consider the group SL_3 acting $\mathbb{C}^3$. It is obvious that $(SL_3, \mathbb{C}^3)$ is the same with $(SL_3 \times SL_1, \mathbb{C}^3 \otimes \mathbb{C}1)$. This tuple is castling equivalent to $\left(SL_3 \times SL_2, (\mathbb{C}^3)^* \otimes \mathbb{C}^2\right)$. Again expanding previous expression by $\times SL_1$ and $\otimes \mathbb{C}$, we obtain triple $(3, 2, 1)$ which demonstrates degrees. By this process we make the rest of chain which contains only triplets as in the following:



Even without expanding triplet to four tuple by $\times SL_1$ and $\otimes\mathbb{C}$, degrees grow fast. Any triplet (p, q, r) is interpreted that $SL_p \times SL_q \times SL_r$ acts on $V_p \otimes V_q \otimes V_r$. Castling equivalent systems share very important property; their algebra of invariants are the same [7, p. 206], [8]. Hence, in order to study invariants of $\bigotimes_{i\in\mathbb{N}} V_i$ under the action of $\prod_{i\in\mathbb{N}} SL(V_i)$, it is enough to investigate the system with minimal degree for easier computations.

Remark 7.0.6. Historically, the method castling transform were introduced by M. Sato and T. Kimura [11] in the article: "A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. Journal 65 (1977)". Let G be a connected linear algebraic group with representation V. The tuple (G, V) is called prehomogeneous vector space if there is a dense G-orbit in V, ie. $\overline{\rho(G)\,v} = V$. They gave a theorem: $\left(G \times SL_d, V \otimes k^d\right)$ is a prehomogeneous vector space if and only if $\left(G \times SL_{n-d}, V^* \otimes k^{n-d}\right)$ is [11, p. 225].

7.1 Algebra of Invariants of Castling Equivalent Systems

We consider the group $G = H \times SL_k(\mathbb{C})$ with representation $W \otimes \mathbb{C}^k$ where $\dim W = n > k$. As we discussed, its castling transform is the tuple $\left(H \times SL_{n-k}(\mathbb{C}), W \otimes \mathbb{C}^{n-k}\right)$. Under this settings we have a theorem:

Theorem 7.1.1. *The algebra of invariants of castling equivalent systems are isomorphic.*

Proof. Let X be k-dimensional Grassmannian of W,$X = G_k(W)$. We consider the morphism:

$$\phi_1 : W \otimes \mathbb{C}^k \longmapsto X$$
$$\phi_1\left(\sum_{i=1}^{k} w_i \otimes e_i\right) = w_1 \wedge \cdots \wedge w_k$$

where $\{e_1, \ldots, e_k\}$ is basis of $\mathbb{C}^k$. The map ϕ corresponds to taking quotient by SL_k. Hence $\left(W \otimes \mathbb{C}^k\right) //G \cong X//H$. By using the fact $G_k(W) \cong G_{n-k}(W^*)$ under H, we have $\left(W \otimes \mathbb{C}^k\right) //G \cong \left(W^* \otimes \mathbb{C}^{n-k}\right) //G'$, $G' = H \times SL_{n-k}$. $\square$

Remark 7.1.2. Although the algebra of invariants of castling equivalent systems are isomorphic, invariants have different degree. Let f and f' be invariants of castling equivalent systems. If $\deg f = dn$, $\deg f'$ is $d(m-n)$ [11, p.227].

Example 7.1.3. We consider multidimensional matrix of format $2 \times 2 \times 3$

$$A = \left[\begin{array}{cc|cc|cc} a & b & e & f & i & j \\ c & d & g & h & k & l \end{array}\right]$$

The first slice is the classical 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So identify it with $[x_{ij}]$ notation.

Lemma 7.1.4. *Hyperdeterminant of this format is given by the resultant [3, p. 483]*

$$\begin{bmatrix} a & e & i & 0 & 0 & 0 \\ c & g & k & 0 & 0 & 0 \\ b & f & j & a & e & i \\ d & h & l & c & g & k \\ 0 & 0 & 0 & b & f & j \\ 0 & 0 & 0 & d & h & l \end{bmatrix}$$

Then computations show that determinant is given by six degree equation

$$-a^2fgl^2 + a^2fhkl + a^2ghjl - a^2h^2jk + abegl^2 - abehkl + abfgkl - abfhk^2 - abg^2jl - abghil + abghjk + abh^2ik + acefl^2 - acehjl - acf^2kl + acfgjl - acfhil + acfhjk - acghj^2 + ach^2ij - adefkl - adegjl + 2\,adehjk + adf^2k^2 + 2\,adfgil - 2\,adfgjk - adfhik + adg^2j^2 - adghij - b^2egkl + b^2ehk^2 + b^2g^2il - b^2ghik - bce^2l^2 + bcefkl + bcegjl + 2\,bcehil - 2\,bcehjk - 2\,bcfgil + bcfhik + bcghij - bch^2i^2 + bde^2kl - bdefk^2 - bdegil + bdegjk - bdehik + bdfgik - bdg^2ij + bdghi^2 - c^2efjl + c^2ehj^2 + c^2f^2il - c^2fhij + cde^2jl - cdefil + cdefjk - cdegj^2 - cdehij - cdf^2ik + cdfgij + cdfhi^2 - d^2e^2jk + d^2efik + d^2egij - d^2fgi^2$$

In [3], authors consider this format as boundary format, and they focus on boundary formats in detailed way. Since this format is castling equivalent to 2×2 matrix $[x_{ij}]$, we can use the previous theorem to obtain the same result with multiplicated by -1.

Let M be 4×4 matrix:

$$M = \begin{bmatrix} x_{11} & x_{12} & x_{21} & x_{22} \\ a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

It is clear that invariant of $[x_{ij}]$ is $x_{11}x_{22} - x_{12}x_{21}$. We replace the variables x_{ij} with cofactors of M in the following way:

$$x_{11} = \det \begin{bmatrix} b & c & d \\ f & g & h \\ j & k & l \end{bmatrix}, \qquad x_{12} = -\det \begin{bmatrix} a & c & d \\ e & g & h \\ i & k & l \end{bmatrix}$$

$$x_{21} = -\det \begin{bmatrix} a & b & d \\ e & f & h \\ i & j & l \end{bmatrix}, \qquad x_{22} = \det \begin{bmatrix} a & b & c \\ e & f & g \\ i & j & k \end{bmatrix}$$

Remark 7.1.5. Bremner's article is devoted to calculate this result [12].

This example can be extended to the system of format $2 \times 3 \times 4$, since $(2,3,4) \overset{ct}{\to} (2,3,2) \overset{ct}{\to} (2,2,1)$. We should get degree 12 invariant.

We have the format $2 \times 3 \times 4$:

$$B = \left[\begin{array}{cccc|cccc} a_{111} & a_{121} & a_{131} & a_{141} & a_{112} & a_{122} & a_{132} & a_{142} \\ a_{211} & a_{221} & a_{231} & a_{241} & a_{212} & a_{222} & a_{232} & a_{241} \\ a_{311} & a_{321} & a_{331} & a_{341} & a_{312} & a_{322} & a_{332} & a_{342} \end{array}\right]$$

The castling transform of format $(2,3,2)$ to $(2,3,4)$ fixes $(2,3)$ part. And we have to relate coordinates such that 2×2 submatrices corresponds to 4×4 submatrices of 6×6 matrix. After rewriting B in the form $4 \times 3 \times 2$ and combining with A, we get:

$$X = [x_{ij}] = \begin{bmatrix} a & b & e & f & i & j \\ c & d & g & h & k & l \\ a_{111} & a_{121} & a_{131} & a_{211} & a_{221} & a_{231} \\ a_{112} & a_{122} & a_{132} & a_{212} & a_{222} & a_{232} \\ a_{113} & a_{123} & a_{133} & a_{213} & a_{223} & a_{233} \\ a_{114} & a_{124} & a_{134} & a_{214} & a_{224} & a_{234} \end{bmatrix}$$

Let X^1_{ij} and X^2_{ij} denote;

$$X^1_{ij} = \begin{bmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{bmatrix} \qquad X^2_{ij} = \begin{bmatrix} x_{3m} & x_{3n} & x_{3y} & x_{3t} \\ x_{4m} & x_{4n} & x_{4y} & x_{4t} \\ x_{5m} & x_{5n} & x_{5y} & x_{5t} \\ x_{6m} & x_{6n} & x_{6y} & x_{6t} \end{bmatrix}$$

where m, n, y, t are indexes from 1 to 6 and non-equal to i or j. X^1_{ij} and X^2_{ij} satisfies $\det X^1_{ij} = \det X^2_{ij}$. There are 15 equations in total. There is no straightforward way to substitute new variables. However, after using Laplace formula for W_{ijk} with choosing minors appropriately, we calculate hyperdeterminant:

$$\begin{aligned} &D^2_{12}D^2_{35}D^2_{46} - D^2_{23}D^2_{35}D^2_{26} - D^2_{25}D^2_{35}D^2_{24} - D^2_{14}D^2_{15}D^2_{46} + D^2_{34}D^2_{15}D^2_{26} \\ &+ D^2_{45}D^2_{15}D^2_{24} + D^2_{16}D^2_{13}D^2_{46} - D^2_{36}D^2_{13}D^2_{26} + D^2_{56}D^2_{13}D^2_{24} \end{aligned}$$

where $D^2_{ij} = \det X^2_{ij}$.

Remark 7.1.6. Hyperdeterminant of format $(2,3,4)$ has $\leq 9\,(4!)^3 = 124416$ terms of degree 12.

Example 7.1.7. Consider $m \times m$ matrix $M = [x_{ij}]$. This is indeed format of $m \times m \times 1$, hence its castling transform is format $m \times m \times (m^2 - 1)$. To get explicitly invariants of this system, one need to construct $m^2 \times m^2$ matrix where the first row is entries of M. After calculating cofactors of $(1, j)$ and putting them into $\det M$, one obtain invariants. In practise, complexity of this operation is high since there are $m!$ terms, which are subject to replacement with $(m^2 - 1)!$ terms, in the determinant of M.

7.2 Invariants of Three Component Systems

There are three possible type of reduced forms:

i-) (p, q, r), where $2p < qr, 2q < pr, 2r < pq$

ii-) (p, q, r), where $p > qr$, up to permutation

iii-) (p, q, r), where $p = qr$

We mostly focus on the first one. Since, by [8] there is no invariant for the second system. And the last one is called log-boundary format, which has free algebra of invariants with one generator.

7.2.1 Invariants of Format $2 \times n \times n$

Notice that this system is castling reduced one for any $n \geq 2$. Assume that $SL_n \times SL_n \times SL_2$ acts on $V_1 \otimes V_2 \otimes V_3$ as in given (4.6) where $\dim V_1 = \dim V_2 = n$ and $\dim V_3 = 2$. In matrix notation this action can be given by

$$(g_1, g_2, g_3)(A, B) = \left(f_{11} g_1 A g_2^{-1} + f_{12} g_1 B g_2^{-1}, f_{21} g_1 A g_2^{-1} + f_{22} g_1 B g_2^{-1}\right),$$

where $g_3^{-1} = [f_{ij}]$, [13]. Let V_n be space of binary forms of degree n. We consider the map:

$$\pi : (A, B) \mapsto V_n$$
$$\pi(A, B) = \det(xA + yB)$$

Theorem 7.2.1. *With the above settings, algebra of invariants of the format $2 \times n \times$ is isomorphic to algebra of invariants of binary forms.*

Proof. A, B are generic matrices, hence $\det(xA + yB) = \det(xI + yD)$, where D is diagonalization of BA^{-1}. Hence $\det(xI + yD) = \prod_{i=1}^{n}(x + yd_i) = \sum_k^n e_k(d_1, \ldots, d_n)$ with $D = diag(d_1, \ldots, d_n)$. Then it is obtained that π is surjective and SL_2-equivariant morphism.
We consider the induced homomorphism $\pi^* : \mathbb{C}[V_n]^{SL_2} \mapsto \mathbb{C}[(A, B)]^{SL_n \times SL_n \times SL_2}$, $\pi^*(f) = f \circ \pi$. Since π is dominant, images of π^* are algebraically independent. $\square$

We want to combine the result of the above theorem and Cayley's hyperdeterminant:

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, B = \begin{bmatrix} e & g \\ f & h \end{bmatrix}.$$

$\det(xA + yB) = adx^2 - cbx^2 + ahxy + edxy - cfxy - gbxy + ehy^2 - gfy^2$. The only invariant for binary forms of degree two is discriminant Δ, and the result of $\Delta(\det(xA + yB))$ is:

$a^2h^2 - 2\,abgh - 2\,acfh - 2\,adeh + 4\,adfg + b^2g^2 + 4\,bceh - 2\,bcfg - 2\,bdeg + c^2f^2 - 2\,cdef + d^2e^2$.

7.2.1.1 Other Examples

In degree 3, the only invariant is again discriminant. If we have the form $u = \sum_{i=0}^{4} \alpha_i x^{d-i} y^i$, then its discriminant is given by:

$$\Delta(u) = \alpha_1^2\alpha_2^2 - 4\alpha_0\alpha_2^3 - 4\alpha_1^3\alpha_3 - 27\alpha_0^2\alpha_3^2 + 18\alpha_0\alpha_1\alpha_2\alpha_3.$$

Let A and B be 3×3 matrices, $A = [a_{ij}]$, $B = [b_{ij}]$. Then determinant of $xA + yB$ is; $x^3a_{11}a_{22}a_{33} - x^3a_{11}a_{32}a_{23} - x^3a_{21}a_{12}a_{33} + x^3a_{21}a_{32}a_{13} + x^3a_{31}a_{12}a_{23} - x^3a_{31}a_{22}a_{13} + x^2ya_{11}a_{22}b_{33} - x^2ya_{11}a_{32}b_{32} - x^2ya_{11}a_{23}b_{32} + x^2ya_{11}a_{33}b_{22} - x^2ya_{21}a_{12}b_{33} + x^2ya_{21}a_{32}b_{31} + x^2ya_{21}a_{13}b_{32} - x^2ya_{21}a_{33}b_{12} + x^2ya_{31}a_{12}b_{32} - x^2ya_{31}a_{22}b_{31} - x^2ya_{31}a_{13}b_{22} + x^2ya_{31}a_{23}b_{12} + x^2ya_{12}a_{23}b_{31} - x^2ya_{12}a_{33}b_{21} - x^2ya_{22}a_{13}b_{31} + x^2ya_{22}a_{33}b_{11} + x^2ya_{32}a_{13}b_{21} - x^2ya_{32}a_{23}b_{11} + xy^2a_{11}b_{22}b_{33} - xy^2a_{11}{b_{32}}^2 + xy^2a_{21}b_{31}b_{32} - xy^2a_{21}b_{12}b_{33} - xy^2a_{31}b_{31}b_{22} + xy^2a_{31}b_{12}b_{32} - xy^2a_{12}b_{21}b_{33} + xy^2a_{12}b_{31}b_{32} + xy^2a_{22}b_{11}b_{33} - xy^2a_{22}{b_{31}}^2 - xy^2a_{32}b_{11}b_{32} + xy^2a_{32}b_{21}b_{31} +$

$xy^2a_{13}b_{21}b_{32} - xy^2a_{13}b_{31}b_{22} - xy^2a_{23}b_{11}b_{32} + xy^2a_{23}b_{31}b_{12} + xy^2a_{33}b_{11}b_{22} - xy^2a_{33}b_{21}b_{12} +$
$y^3b_{11}b_{22}b_{33} - y^3b_{11}{b_{32}}^2 + y^3b_{21}b_{31}b_{32} - y^3b_{21}b_{12}b_{33} - y^3{b_{31}}^2b_{22} + y^3b_{31}b_{12}b_{32}$

After evaluating Δ with these coefficients we get hyperdeterminant.

For degree 4, there are two basic invariants [10, p.142]

$$P(u) = \alpha_0\alpha_4 - \frac{1}{4}\alpha_1\alpha_3 + \frac{1}{12}\alpha_2^2 \qquad H(u) = \begin{bmatrix} \alpha_0 & \alpha_1/4 & \alpha_2/6 \\ \alpha_1/4 & \alpha_2/6 & \alpha_3/4 \\ \alpha_2/6 & \alpha_3/4 & \alpha_4 \end{bmatrix}$$

Therefore degrees of invariants of format $4 \times 4 \times 2$ are 8 and 12.

Remark 7.2.2. Discriminant of a binary form of degree 4 is $\Delta(u) = 2^8\left(P^3(u) - 27H^2(u)\right)$.

For $n \geq 5$, calculations are similar, one need to know explicit expressions of invariants of binary forms and powerful computer. Even in the case of format $2 \times 3 \times 3$, finding the exact form is complicated.

Remark 7.2.3. Formats $2 \times n \times n$ with $n \geq 5$ are not in the Littelmann's list since some of their generators are not algebraically independent. In [10, 142] authors gave a list, where f_d denotes fundamental invariant of degree d:

- $n = 5$, $k[V_5]^G = k[f_4, f_8, f_{12}, f_{18}]$, where $f_{18}^2 \in k[f_4, f_8, f_{12}]$ and f_4, f_8, f_{12} are algebraically independent.
- $n = 6$, $k[V_6]^G = k[f_2, f_4, f_6, f_{10}, f_{15}]$, where $f_{15}^2 \in k[f_2, f_4, f_6, f_{10}]$ and f_2, f_4, f_6, f_{10} are algebraically independent.
- $n = 8$, $k[V_8]^G = k[f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10},]$ and $f_2, \ldots, f_{10}$ are connected by five more relations.

7.2.2 Invariants of Format $3 \times 3 \times 3$

System of format $3 \times 3 \times 3$ is castling reduced. Let R denote algebra of invariants for that system. Its Hilbert series is given by:

$$h(t) = \frac{1}{(1-t^6)(1-t^9)(1-t^{12})}.$$

With $h(t) = \sum_{d \geq 0} \dim R_d t^d$, there are three basic invariants of degree 6,9 and 12 [14].

Remark 7.2.4. GKZ-Hyperdeterminant of previous system has degree 36, hence it is not a basic invariant.

7.3 Invariants of Log-Boundary Format

Consider three component system $V_p \otimes V_q \otimes V_{pq}$ under the action of $SL_p \times SL_q \times SL_{pq}$, where subscripts denotes dimension. This system is casting reduced. There is only one invariant for this system and it has degree pq. We can obtain this by identifying $V_p \otimes V_q$ with V_{pq}, hence $V_p \otimes V_q \otimes V_{pq} \cong V_{pq} \otimes V_{pq}$. Under the action of $SL_{pq} \times SL_{pq}$, determinant is the only invariant.

Theorem 7.3.1. *Let $SL_{p_1} \times \ldots \times SL_{p_n} \times SL_{p_1 \cdots p_n}$ act on $V_{p_1} \otimes \ldots \otimes V_{p_n} \otimes V_{p_1 \cdots p_n}$ with $p_i \geq 2$ for all i. This system is castling reduced and it has one invariant.*

Proof. It is obvious that system is castling reduced. If we make identification $V_{p_1} \otimes \ldots \otimes V_{p_n}$ with $V_{p_1 \cdots p_n}$, we get system $V_{p_1 \cdots p_n} \otimes V_{p_1 \cdots p_n}$, hence determinant is invariant. $\square$

Chapter 8

Conclusion

Here is the extended version of our fundamental theorem:

Theorem 8.0.2. *Consider a multicomponent system $G : \mathcal{H}$ where $G = \prod_i SL(\mathcal{H}_i)$ and $\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i$ with $\dim \mathcal{H}_i = D_i$. The followings are equivalent:*

- $\bigotimes_{i=1}^N \mathcal{H}_i$ *has a completely entangled state* $\Longleftrightarrow$
- $\exists$ *nontrivial G-invariant* $\Longleftrightarrow$
- $d_i = \log D_i$ *satisfy polygonal inequality ie.* $d_i \leq \sum_{j \neq i} d_j$

Proof. The proof is similar to the proof of fundamental theorem. In this case we have n-dimensional matrix. Assume that we have an entangled state $\psi \in \mathcal{H}$. This amounts to parallel slices in $[\psi]$ are orthogonal and the same trace. In decomposition of $\mathcal{H}$, we get other multiplicities instead of Kronecker coefficients. Again, nonzero coefficients exists if and only if there exists G-invariant polynomial. The rest is completed by consulting Littelman's paper [8]. $\square$

Remark 8.0.3. For the existence of invariants, the castling reduced form have to satisfy log-polygonal inequality. For instance, $(2, 3, 5)$ satisfies log-polygonal inequalities, however $(2, 3, 5) \xrightarrow{\text{c.t.}} (2, 3, 1) \xrightarrow{\text{c.t.}} (2, 3)$, $(2, 3)$ has no invariant.

Remark 8.0.4. Systems in the table satisfies Russian conjecture [9].

Corollary 8.0.5. *The format $(p_1, \ldots, p_k, p_1 \cdots p_k)$ has a hyperdeterminant which is not a GKZ-hyperdeterminant.*

Determinant of a square matrix is the only invariant for the action of $SL_n \times SL_n$. In the same manner, we can use the term hyperdeterminant for multidimensional matrices under the action of $\prod_i SL_{k_i}$ although it is not unique.

This theorem enables us to extend boundaries of theorem of GKZ. There is no GKZ-hyperdeterminant in format $(3, 4, 10)$, however its castling reduced form is $(2, 2)$ which has determinant as hyperdeterminant.

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