KARADENİZ TECHNICAL UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS

MASTER THESIS

ARFAH ARFAH

JULY 2021
TRABZON

KARADENİZ TECHNICAL UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS

RICCI SOLITONS ON LIGHTLIKE HYPERSURFACE

Arfah ARFAH

This thesis is accepted to give the degree of
"MASTER OF SCIENCE"
By
The Graduate School of Natural and Applied Sciences at
Karadeniz Technical University

The Date of Submission : 20 / 04 /2021
The Date of Examination : 05 / 07 /2021

Supervisor : Dr. Ö

Trabzon 2021

ACKNOWLEDGEMENT

This thesis entitled “Ricci Solitons on Lightlike Hypersurfaces” is written as a requirement for completing the Master of Science in Mathematics Department of Karadeniz Technical University.

Infinity thanks to my master thesis supervisor, Asst. Prof. Dr. Gül TUĞ from my deepest heart for all supports, guidance, suggestions, and all help given to me. I would also thank to all lectures and staff in the Department of Mathematics for their guidance and helps during my study.

I would thank my lovely mother and my lovely father for all supports and motivation, which always given to me to reach my dream. Next, thanks to all my friends in Indonesia Student Association in Trabzon for togetherness, which always makes me feel at home.

Arfah ARFAH

Trabzon, 2021

DECLARATIONS

I declare that, this Master Thesis entitled "Ricci Solitons on Lightlike Hypersurface" has been completed under the guidance of my Master supervisor Asst. Prof. Dr. Gül TUĞ. All referred information in this thesis has been indicated in the text and cited in the reference list. I have obeyed all research and ethical rules during my research and I accept all responsibility if proven otherwise. 05/07/2021

Arfah Arfah

CONTENTS

Yüksek Lisans Tezi

ÖZET

LIGHTLİKE HİPERYÜZEYLER ÜZERİNDE RİCCİ SOLİTONLAR

Arfah ARFAH

Karadeniz Teknik Üniversitesi
Fen Bilimleri Enstitüsü
Matematık Anabilim Dalı
Danışman: Dr. Öğr. Üyesi Gül TUĞ
2021, 83 Sayfa

Bu tezde lightlike hiperyüzeyin, Ricci solitonun denklemini karşıladığını gösteriyoruz. Ayrıca, teoriyi destekleyen bazı teoremler ve sonuçlar veriyoruz. Ricci soliton denklemini sağlayan concircular vektör alanlarını kabul eden hafif benzeri bir hiper-yüzeyin gradyan Ricci soliton olma koşulunu da sağladığını ispatlıyoruz. Son bölümde, screen homotetik ve tamamen jeodezik lightlike hiperyüzeylerin Ricci solitonları ve gradyan Ricci solitonları olması için koşullar veriyoruz.

Anahtar Kelimeler: Lightlike hiperyüzey, Ricci solitonu, gradyan Ricci solitonu

Master Thesis

SUMMARY

RICCI SOLITONS ON LIGHTLIKE HYPERSURFACES

Arfah ARFAH

Karadeniz Technical University
The Graduate School of Natural and Applied Sciences
Mathematics Graduate Program
Supervisor: Asst. Prof. Dr. Gül TUĞ
2021, 83 Pages

In the present work, we show that a lightlike hypersurface satisfies Ricci soliton equation. Also, we give some Theorems and corollaries supporting theory. We prove that a lightlike hypersurface admitting concircular vector fields which is Ricci soliton satisfies the condition of gradient Ricci soliton as well. In the last part, we give conditions of the lightlike hypersurfaces which are screen homothetic and totally geodesic to be Ricci solitons and gradient Ricci solitons.

Keywords: Lightlike hypersurface, Ricci soliton, gradient Ricci soliton

NOTATIONS

∇	: Levi-Civita connection
R_{XY}	: Riemann Curvature Tensor
K	: Sectional Curvature
TM	: Tangent bundle
$Rad\ T_pM$	: Radical space at point $p \in M$
$S(TM)$	: Screen Distribution
$S(TM)^{\perp}$	: Orthogonal Complementary Vector Bundle to $S(TM)$
$tr(TM)$	: Transversal Vector Bundle
$ltr(TM)$	: Lightlike Transversal Vector Space
h	: Second Fundamental Form
A	: Shape Operator
B	: Local Second Fundamental Form

1. GENERAL INFORMATION

1.1. Introduction

In the early 1980s, Richard Hamilton, then at the University of California, San Diego, suggested utilizing a notion known as Ricci flow to prove Thurston's geometrization hypothesis. Every manifold has different kind of Riemannian metrics. The issue is that the measure might not be homogenous. Hamilton's notion allows the manifold to "find its own way" to metric which is homogeneous through enabling the evolution of metric in a term similar to how heat flows. The temperature will ultimately equilibrate if heat flows unobstructed across a space with no heat sources or sinks. If we do the same to a manifold, the equilibration of the curvature will hold which imply the manifold to be homogeneous.

A partial differential equation describes heat flow by equating the temperature-change rate in terms of time to its variation in space. Hamilton applied this concept to the metrics-related equation in Riemannian space. Suppose the change of metric's rate with respect to the time is proportional to a parameter known as the Ricci curvature. Ricci curvature is strongly connected to the concept of theory of general relativity of Einstein of space-bending curvature. The equation of Ricci flow stated by Hamilton has form: If $g = g(t)$ is the metric of Riemannian manifold at time t, it will satisfy $\frac{\partial g}{\partial t} = -2Ric(g)$, where Ric is the Ricci curvature (Mackenzie and Cipra, 2006).

Many mathematicians have been interested in the geometry of Ricci solitons throughout the last two decades. It became especially essential after Grigory Pereleman used Ricci solitons for solving the long-standing conjecture proposed by Poincare's in 1904. According to Perelman (2002), on compact simply connected Riemannian manifolds, the Ricci solitons are gradient Ricci solitons which satisfy the equation of Ricci flow. The study of Ricci solitons is divided into two parts: the effect of Ricci solitons on the topological properties of Riemannian manifold and the effect of Ricci solitons on its geometry. (Chen and Deshmukh, 2015).

For many years, theory of semi-Riemannian manifolds has studied by both mathematicians and physicists due to its wide range of applications. One of the most interesting theories of semi-Riemannian geometry is the concept of lightlike hypersurfaces. Lightlike hypersurfaces itself has a significant place in the general relativity theory as the event horizon region of the black hole. As Einstein's theory of gravitation was published in 1915, a considerable amount of studies about the theory of the black hole has conducted by scientists. Theory of the black holes and applications of lightlike hypersurfaces has been studied by Hawking (1973), Chandrasekhar (1983), and Chrusciel (2005). To study more about lightlike hypersurfaces in semi-Riemannian manifold we refer to the papers written by Duggal and Bejancu (1996), Duggal and Jin (2007), Güneş (2003), Atindogbe (2014), Şahin (2007).

Concircular vector fields are useful tools in many areas of mathematics and can be applied in other fields such as physics. In mathematics, concircular vector fields can be seen in concircular mapping study which is conformal and preserves the geodesic circle (Fialkow, 1939). Moreover, the concircular vector field is also important to study the projective and conformal transformation theory (Chen, 2014; Yano, 1940). In physics, the time-like concircular vector fields trajectories in the deSitter space show the lines of the world of receding or colliding galaxies which are fulfilling the postulate of Weyl, i.e., the orthogonality of the world lines orthogonal to a family of spacial hyperslices must be everywhere (Takeno, 1967). Some studies related to concircular vector fields can be seen in the studies by Brickell (1974), Kılıç (2021), Al-Dayel (2020), Chen (2016), and Chen (2015).

In this thesis, we specify the conditions for lightlike hypersurfaces admitting concircular vector fields to satisfies the Ricci soliton equation. Also, some theorems and corollaries supporting theory are given. We prove that Ricci solitons on lightlike hypersurface equipped with concircular vector fields satisfying the condition of a gradient Ricci soliton as well. In the final section, the conditions for the screen homothetic and totally geodesic lightlike hypersurfaces to be Ricci solitons and gradient Ricci solitons are given.

1.2. Topological Manifold

Definition 1.2.1. Let X be a set and τ be the collection of all subset of X. τ is called topology if it satisfies the following conditions:

(T1). $X, \emptyset \in \tau$,

(T2). $\forall A_1, A_2 \in \tau \Longrightarrow A_1 \cap A_2 \in \tau$,

(T3) $A_i \in \tau,\ i \in I \Longrightarrow \bigcup_{i \in I} A_i \in \tau$

where I is index set (Hacısalihoğlu, 2000).

Definition 1.2.2. A pair (X, τ) consisting of set X and a topology τ is called topological space (Hacısalihoğlu, 2000).

Definition 1.2.3. Given a nonempty set A and $m: A \times A \longrightarrow \mathbb{R}$ be a function. m is metric on X if it satisfies:

(i) For any $x \neq y$, $m(x, y) > 0$,
(ii) $m(x, y) = 0$ iff $x = y$,
(iii) $m(x, y) = m(y, x)$,
(iv) $m(x, y) \leq m(x, z) + m(z, y)$

for all $x, y, z \in X$.

Definition 1.2.3. Let X and Y be topological spaces. If a function $f: X \longrightarrow Y$ is bijective, continuous and its inverse f^{-1} is continuous as well, then f is said to be a homeomorphism (Şahin, 2012).

Definition 1.2.4. Given a topological space (X, τ). X is called Hausdorff space if for every two distinct elements in X, there will be open sets A and B such that $x \in A, y \in B$ and $A \cap B = \emptyset$ (Yıldız, 2005).

Definition 1.2.5. Given a topological space M. If the following conditions are satisfied, M is called n-dimensional topological manifold:

(M1) M is a Hausdorff space,

(M2) there is a neighborhood homeomorphic to open subset of $\mathbb{E}^n$ at each point in M,

(M3) M is covered by finite open sets (Hacısalihoğlu, 2000).

Definition 1.2.6. Let $U \subset \mathbb{E}^n$ be an open set. A map $f: U \rightarrow \mathbb{R}$ is C^k if and only if all partial derivatives of the order less or equal to k of the component functions f^i exist and continuous on U.

Definition 1.2.7. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *smooth* or C^∞ if f is C^k for all $k \geq 0$ (Renteln, 2014).

Definition 1.2.8. Let $U, V \subset \mathbb{R}^n$ be open sets and the function $f : U \rightarrow V$ be a homeomorphism. If f and f^{-1} are both smooth, then f is said to be diffeomorphism (Renteln, 2004).

Definition 1.2.9. Given a topological manifold M. M is said to be a differentiable manifold of equivalent class C^k if a differentiable structure C^k on M is defined. (Hacısalihoğlu, 2000).

Definition 1.2.10. Given an n-dimensional topological manifold M. A coordinate chart on M is a pair (U, φ), in which $U \subset M$ is an open set and $\varphi: U \rightarrow \widetilde{U}$ is a homeomorphism from U to $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^n$ which is also an open set (Lee, 2013).

Definition 1.2.11. An n-dimensional atlas $\mathcal{A}$ on a space S is a set of coordinate systems of dimension n in S such that

(A1) for each point of S is contained in the domain of some coordinate system in V, and

(A2) any two coordinate systems in $\mathcal{A}$ overlap smoothly.

(O'Neill, 1983).

Definition 1.2.12. Given an n-dimensional topological manifold M. The composition mapping $\psi \circ \varphi^{-1}: \varphi(U \cap V) \rightarrow \psi(U \cap V)$ is said to be the transition map from φ to ψ if (U, φ) and (V, ψ) are two disjoint chart (Lee, 2013).

Definition 1.2.13. For any point p on a manifold M, the tangent vector at point p on M is a function $w: M \to \mathbb{R}^n$ which satisfy:

(T1) $(af + bg) = kw(f) + lw(g)$

(T2) $w(fg) = w(f)g(p) + f(p)w(g)$ for all $k, l \in \mathbb{R}$ and $f, g \in C^\infty$.

(O'Neill, 1983).

Definition 1.2.15. Given a manifold M be a manifold with T_pM is the space of tangent vectors of M at point p. At each point $p \in M$, a differentiable map X on T_pM is called a vector field. Thus, the vector field of M can be written by

$X : M \longrightarrow \bigcup_{p\in M} T_pM$

(Şahin, 2012).

Definition 1.2.16. Given a smooth map of manifolds by $f: M \to N$ and a smooth map $g: N \to \mathbb{R}$. We define a new map $g^*h: M \to \mathbb{R}$ by $g^*f = f \circ g$. The map g^*f is the pullback of g by f because the function g is "pull back" from N to M (Renteln, 2004).

Definition 1.2.17. Given a mapping, $F: M \longrightarrow N$ be differentiable mapping and $X \in T_PM$ be the tangent vector of a curve at point $\alpha(t_0) = p$ on M. Then, the transformation $F_*: T_pM \longrightarrow T_{F(p)}N$ is called differential transformation (Şahin, 2012).

Definition 1.2.18. For smooth manifolds M and N of dimensions m and n, respectively with map $f: M \longrightarrow N$ be a smooth map. f is immersion at p if $m \leq n$ and the Jacobian matrix $\left(\frac{\partial y^i}{\partial x^j}\right)$ has a maximal rank (namely m) at $p \in M$. Furthermore, if f is an immersion for all $p \in M$ then M is called the submanifold of N (Renteln, 2014).

Definition 1.2.20. An injective immersion f is called embedding (or imbedding) if f maps M homeomorphically onto its image $f(M)$ in the induced topology (Renteln, 2014).

Definition 1.2.21. If F maps, m-dimensional manifold M into n-dimensional manifold N with $m \leq n$ then $n - m$ is called codimension of F (Şahin, 2012).

Definition 1.2.22. Given a surjective mapping F from a manifold M of dimension m to an n-dimensional manifold N with $n \geq m$. If mapping $F_*: T_pM \longrightarrow T_{F(p)}N$ at point $p \in M$ has rank n. Then, F is called submersion. (Şahin, 2012).

Definition 1.2.23. If f is submersion at $p \in M$ then point p is called regular on f, otherwise, it is called critical point of f (Renteln, 2014).

Definition 1.2.24. Let the function $f: \mathbb{E}^n \longrightarrow \mathbb{R}$ be differentiable and $\overrightarrow{v_P} \in T_{\mathbb{E}^n}(P)$. Then, $\vec{v} = \overrightarrow{PQ}$ such that $\overrightarrow{v_P}[f] = \frac{d}{dt}\Big(f\big(P_1 + t(Q_1 - P_1), \dots, P_n + (Q_n - P_n)\big)\Big)\,|_{t=0}$ denote the directional derivative of f according to the vector $\overrightarrow{v_P}$ (Hacısalihoğlu, 2000).

Definition 1.2.25. Let $X \in \chi(\mathbb{E}^n)$ and $f \in C(\mathbb{E}^n, \mathbb{R})$. For every $p \in \mathbb{E}^n$, $\big(X(f)\big)(p) = X_p[f]$, $X[f] \in C(\mathbb{E}^n, \mathbb{R})$ is called the derivative of function f in direction of the vector field X (Hacısalihoğlu, 2000).

Definition 1.2.26. For vector fields V and W on open subset U of a differentiable manifold M and $f \in C^\infty$, the mapping

$$[\,,] : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

$$[V, W]f = V(Wf) - W(Vf)$$

is called Lie bracket operator of vector fields V and W. Here $\Gamma(TM)$ is a vector space over a field $\mathbb{R}$ and $V(f)$ stands for the derivative of function f in direction of the vector field V (Şahin, 2012).

Lemma 1.2.1. The bracket operation on $\chi(M)$ has the following properties:

(1) $\mathbb{R}$-bilinearity : $[aX + bY, W] = a[X, W] + b[Y, W]$

$$[W, aX + bY] = a[W, X] + b[W, Y]$$

(2) Skew-symmetry : $[V, W] = -[W, V]$

(3) Jacobi identity : $\big[V, [W, X]\big] + \big[W, [X, V]\big] + \big[X, [V, W]\big] = 0$ (O'Neill, 1983).

Definition 1.2.27. On manifold M, a curve is a smooth mapping $\alpha : I \subset \mathbb{R} \longrightarrow M$ in which I is an open interval (O'Neill, 1983).

Definition 1.2.28. Let $\alpha : I \longrightarrow M$ be a curve.

$$\alpha'(t) = d\alpha\left(\frac{d}{dt}\bigg|_t\right) \in T_{\alpha(t)}(M)$$

is called tangent vector at point $\alpha(t)$ at $t \in I$ corresponding to parameter $t \in I$ (O'Neill, 1983).

Definition 1.2.29. Let $\alpha : I \longrightarrow M$ be a curve. If $\left|\left|\alpha'(t)\right|\right| = 1$ then α is called unit speed vector (Lee, 2009).

Definition 1.2.30. W is called a vector space over field $\mathbb{F}$ if the operations:

i) Addition;
ii) Multiplication with scalar on $\mathbb{F}$,

satisfy the following conditions:

1) V is a commutative group according to the addition operation with the identity element 0 (zero vector);
2) For $\alpha, \beta \in \mathbb{F}, x, y \in W$,
   a) $\alpha(x + y) = \alpha x + \alpha y$,
   b) $(\alpha + \beta)x = \alpha x + \beta x,$
   c) $(\alpha\beta)x = \alpha(\beta x),$
   d) $0 \cdot x = 0, 1 \cdot x = x.$

(Chern, et al, 2000).

Definition 1.2.31. Let V, Z be vector spaces over the field $\mathbb{F}$ of finite dimension. A map $f: V \to Z$ is called linear if for any $v_1, v_2 \in V, \alpha^1, \alpha^2 \in \mathbb{F}$,

$$f(\alpha^1 v_1 + \alpha^2 v_2) = \alpha^1 f(v_1) + \alpha^2 f(v_2).$$

(Chern, et al, 2000).

Definition 1.2.32. Let V, W, Z be vector spaces over the field $\mathbb{F}$ of finite dimension. A map $f: V \times W \to Z$ is called bilinear if for any $v, v_1, v_2 \in V, w, w_1, w_2 \in W$ and $\alpha^1, \alpha^2 \in \mathbb{F}$ satisfy the following conditions.

i) $f(\alpha^1 v_1 + \alpha^2 v_2, w) = \alpha^1 f(v_1, w) + \alpha^2 f(v_2, w),$

ii) $f(v, \alpha^1 w_1 + \alpha^2 w_2) = \alpha^1 f(v, w_1) + \alpha^2 f(v, w_2)$.

Analogously, we define the r-linear map by $f: V_1 \times V_2 \times \ldots \times V_r \to Z$ where $V_1, V_2, \ldots, V_r$ are vector spaces over $\mathbb{F}$ (Chern, et al, 2000).

Remark 1.2.1. When $Z = \mathbb{F}$, the objects defined in definitions (1.2.31) and (1.2.32) is respectively called $\mathbb{F}$-valued linear functions, $\mathbb{F}$-valued bilinear function and $\mathbb{F}$-valued r-linear function.

Definition 1.2.33. The set of all linear function V^*on vector space V is called dual space of V, i.e., $V^* = \{f \mid f : V \to \mathbb{F}\}$ (Renteln, 2004).

Definition 1.2.34. If $\{e_i\}$ is a basis of V, there is a canonical dual basis or cobasis $\{\varepsilon_i\}$ of V^*, defined by $\langle e_i, \varepsilon_j \rangle = \delta_i^j$, where δ_i^j is the Kronecker delta:

$$\delta_i^j := \begin{cases} 1 & if\ i = j \\ 0 & otherwise. \end{cases}$$

(Renteln, 2004).

Definition 1.2.35. Let $\{e_i\}$ and $\{\tilde{e}_i\}$ be two bases of V. Each new (tilde) basis vector can be expressed as an old basis vector linear combination and by convention, we write

$$\tilde{e}_j = \Sigma_i e_i A_{ij},$$

for some nonsingular matrix $A = (A_{ij})$, called the change of basis matrix (Renteln, 2004).

Definition. 1.2.36. A covariant tensor of degree s ($(0, s)$-tensor) at a point p on a differentiable manifold M is a multilinear mapping

$$A_p: (T_p M) \times \ldots \times (T_p M) \to \mathbb{R}.$$

A basis of the space of all $(0, s)$-tensor is given by the set

$$\left(dx^{j_1}|_p \otimes \ldots \otimes dx^{j_s}|_p\right)_{j_1, \ldots, j_s = 1, \ldots, n},$$

where

$$\left(dx^{j_1} \otimes \ldots \otimes dx^{j_s}\right)\left(\frac{\partial}{\partial x^{l_1}}, \ldots, \frac{\partial}{\partial x^{l_s}}\right) = \delta_{l_1}^{j_1} \ldots \delta_{l_s}^{j_s}$$

(Kuhnel, 2013).

Definition 1.2.37. An s-covariant and r-contravariant tensor ((r,s)-tensor) at a point p on a differentiable manifold M is a multilinear mapping

$$A_p := \underbrace{(T_pM)^* \times \ldots \times (T_pM)^*}_{r} \times \underbrace{(T_pM) \times \ldots \times (T_pM)}_{s} \to \mathbb{R}$$

where $(T_pM)^* := Hom(T_pM; \mathbb{R})$ denotes the space of dual of the tangent space.

A basis (s,r)-tensor space is given by

$$\left(\frac{\partial}{\partial x^{i_1}}|_p \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}}|_p \otimes dx^{j_1}|_p \otimes \ldots \otimes dx^{j_s}|_p\right)_{i_1,\ldots,i_r,j_1,\ldots,j_s=1,\ldots,n}$$

where

$$\left(\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes dx^{j_s}\right)\left(dx^{j_1} \otimes \ldots \otimes dx^{j_s}\right)\left(\frac{\partial}{\partial x^{l_1}}, \ldots, \frac{\partial}{\partial x^{l_s}}\right) = \delta_{i_1}^{k_1} \ldots \delta_{i_s}^{k_s} . \delta_{l_1}^{j_1} \ldots \delta_{l_s}^{j_s}$$

(Kuhnel, 2013).

Definition 1.2.38. Given finite-dimensional spaces X, Y and linear mapping $\mathfrak{L}(X, Y; \mathbb{F})$ by $X \times Y \to \mathbb{R}$. The tensor product of V^* and W^* is defined by the mapping

$\otimes: X^* \times Y^* \to \mathfrak{L}(X, Y; \mathbb{F})$

$(x^*, y) \mapsto x^* \otimes y^* = x^*(x) \cdot y^*(y),$

where $x \in X$ and $y \in Y$ (Kuhnel, 2013).

Definition 1.2.39. Given finite-dimensional spaces X, Y and linear mapping $\mathfrak{L}(X, Y; \mathbb{F})$ by $X \times Y \to \mathbb{R}$. The tensor product of X^* and Y^* is defined by the mapping

$\otimes: X^* \times Y^* \to \mathfrak{L}(X, Y; \mathbb{F})$

$(x^*, y^*) \mapsto x^* \otimes y^* = x^*(x) \cdot y^*(y),$

where $x \in X$ and $y \in Y$ (O' Neill 1966).

Theorem 1.2.1. Given a differentiable real-valued function f, then 1-form df on E^3 can be expressed as $df = \Sigma f_i dx_i$, where $f_i = \frac{\partial f}{\partial x^i}$ (O' Neill 1966).

1.3. Riemann Manifold

Definition 1.3.1. The symmetric bilinear form b is:

(1) Positive (or negative) definite if $v \neq 0$ and $b(v,v) > 0$ $(or\ < 0)$,
(2) Positive (negative) semi-definite if $b(v,v) \geq 0$ (≤ 0),
(3) Nondegenerate if for all $w \in V$ $b(v,w) = 0$ then $v = 0$ (O'Neill, 1983).

Definition 1.3.2. Given a differentiable manifold M with differentiable vector fields $\chi(M)$. If the bilinear form

$$g: \chi(M) \times \chi(M) \longrightarrow C^{\infty}(M)$$

is symmetric and positive definite, that is for any vector fields $V, W \in \chi(M)$ the conditions

(a) $g(V,W) = g(W,V)$
(b) $g(V,V) \geq 0$ and $\forall V$ $g(V,V) = 0 \Leftrightarrow V = 0$

are satisfied, the bilinear form g is called Riemannian metric or metric. The pair (M,g) is called Riemannian manifold (Şahin, 2012).

Definition 1.3.3. The metric tensor components of g on U are $g_{ij} = \langle \partial_i, \partial_j \rangle$ $(1 \leq i,j \leq n)$ if $x_1, x_2, \ldots, x_n$ is a coordinate system on $U \subset M$ (O'Neill, 1983).

Definition 1.3.4. Given Riemannian manifold M with metric tensor $<,>$. If T is the tangent vector of curve α given by atlas $\{([a,b],\alpha)$, then the curve length from $\alpha(a)$ to $\alpha(b)$ is given by

$$|\alpha|_a^b = \int_a^b \sqrt{< T(t), T(t) >}\, dt, \quad t \in [a,b] \tag{1}$$

(Şahin, 2012).

Definition 1.3.5. The largest integer i.e., the dimension of a subspace $V \subset W$ on which $b|W$ is negative definite is called the index of a symmetric bilinear form b on V (O'Neill, 1983).

Definition 1.3.6. Let $u^1, u^2, \dots, u^n$ be the natural coordinates on $\mathbb{R}^n$. If V and $W = \sum W^i \partial_i \in \chi(M)$, the vector field $\nabla_V W = \sum V(W^i)\partial_i$ is called the natural covariant derivative of W with respect to V (O'Neill, 1983).

Definition 1.3.7. Let ∇: $\chi(M) \times \chi(M) \longrightarrow \chi(M)$ satisfying

(i) $\nabla_X Y$ is $C^\infty(M, \mathbb{R})$-linear in V;

(ii) $\nabla_X Y,$ is $\mathbb{R}$-linear in W;

(iii) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y,\ \forall f \in C^\infty(M, \mathbb{R})$.

∇ is called affine connection or linear connection and $\nabla_X Y$ stands for the covariant derivative of W in direction of V (Calin & Chang, 2005).

Theorem 1.3.1. Given a manifold (M, g). The Christoffel symbol of the first and second kind of metric g are respectively given by

$$\Gamma_{kij} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

and

$$\Gamma_{ij}^k = \frac{1}{2} g^{km}\left(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right)$$

where $g_{jk}g^{ki} = \delta_j^i$ (Ahsan, 2015).

Definition 1.3.8. The parametrized curve $p(t)$ is said to be a geodesic if its velocity vector is constant (parallel), that is, if it satisfies the condition $\frac{d}{dt}\frac{dp}{dt} = 0$ (Ahsan, 2015).

Theorem 1.3.2. Let the coordinates of a point P on the curve η be $x^i(\lambda)$, the curve η is given by real parameter λ. In terms of this parameter λ the tangent vector t^i at the point P has the component $t^i = \frac{dx^i}{d\lambda}$. Now, if the curve η does not change its direction as we move along it, then the change rate of t^i along t^i is zero, i.e.,

$$t^i t^j_{;i} = 0.$$

The equation of the geodesic is expressed as

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{kj}\frac{dx^k}{d\lambda}\frac{dx^j}{d\lambda} = 0$$

(Ahsan, 2015).

Definition 1.3.9. Given a liner connection ∇. The operator

$$T : \chi(M) \times \chi(M) \longrightarrow \chi(M)$$

$$T(V,W) = \nabla_V W - \nabla_W V - [V,W]$$

is called torsion tensor (Calin & Chang, 2005).

Definition 1.3.10. Given a liner connection ∇. The curvature tensor of the linear connection ∇ is given by

$$R : \chi(M) \times \chi(M) \times \chi(M) \longrightarrow \chi(M)$$

$$R(V,W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z \tag{6}$$

(Calin & Chang, 2005).

Theorem 1.3.3. Riemann curvature connection components can be expressed by

$$R^k_{ijl} = \frac{\partial}{\partial x^j}\Gamma^k_{il} - \frac{\partial}{\partial x^l}\Gamma^k_{ij} + \Gamma^m_{il}\Gamma^k_{mj} - \Gamma^m_{ij}\Gamma^k_{ml} \tag{7a}$$

or

$$R^k_{ijl} = \Gamma^k_{il,j} - \Gamma^k_{ij,l} + \Gamma^m_{il}\Gamma^k_{mj} - \Gamma^m_{ij}\Gamma^k_{ml} \tag{7b}$$

(Ahsan, 2015).

Theorem 1.3.4. The covariant form of the Riemann curvature tensor can be expressed by

$$R_{hijl} = \frac{\partial}{\partial x^j}\Gamma_{hil} - \frac{\partial}{\partial x^l}\Gamma_{hij} + g_{km}(\Gamma^k_{ij}\Gamma^m_{hl} - \Gamma^k_{il}\Gamma^m_{hj}) \tag{8a}$$

or

$$R_{hijl} = \frac{1}{2}\left(\frac{\partial^2 g_{hl}}{\partial x^j \partial x^i} + \frac{\partial^2 g_{ij}}{\partial x^l \partial x^h} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^h} - \frac{\partial^2 g_{hj}}{\partial x^l \partial x^i}\right) + g_{km}\left(\Gamma^k_{ij}\Gamma^m_{hl} - \Gamma^k_{il}\Gamma^m_{hj}\right). \tag{8b}$$

(Ahsan, 2015).

Theorem 1.3.5. The Riemann curvature tensor has the following properties:

(i) In the last two indices and in the first two indices, R^h_{ijl} and R_{hijl} are skew-symmetric, i.e.,

$R^h_{ijl} = -R^h_{ilj}$ and $R_{hijl} = -R_{hilj}$

$R_{hijl} = -R_{ihjl}$

(ii) R_{hijl} is symmetric according to an interchange of the indices first pair (hi) and the second pair of indices (jl), without changing the order of the indices in each pair, i.e.,

$R_{hijl} = R_{jlhi}$

(iii) If we permutate the last three indices (ijl), then

$R^k_{ijl} + R^k_{jli} + R^k_{lij} = 0$

$R_{hijl} + R_{hjli} + R_{hlij} = 0$

(iv) The Riemann tensor also satisfies the identities

$R_{hijk;l} + R_{hikl;j} + R_{hilj;k} = 0$ (Bianchi identities)

$R^h_{ijk;l} + R^h_{ikl;j} + R^h_{ilj;k} = 0.$

(Ahsan, 2015).

Definition 1.3.11. Let $\sigma \subset T_pM$ be a two-dimensional subset, spanned by V, W. Then, the sectional curvature is given by

$$K_\sigma = \frac{\langle R(V,W)W,V\rangle}{\langle V,V\rangle\langle W,W\rangle - \langle V,W\rangle^2}$$

(Ahsan, 2015).

Remark 1.3.1. If V, W are orthonormal, then one has simply

$$K_\sigma = \langle R(V,W)W,V\rangle$$

Definition 1.3.12. The Ricci tensor R_{ij} components are defined by

$$R_{ij} \coloneqq R^k_{ijk}.$$

(Ahsan, 2015).

Theorem 1.3.6. Ricci curvature tensor components can be expressed by

$$R_{ij} = \frac{\partial\Gamma^k_{ik}}{\partial x^j} - \frac{\partial\Gamma^k_{ij}}{\partial x^k} + \Gamma^m_{ik}\Gamma^k_{mj} - \Gamma^m_{ij}\Gamma^k_{mk} \tag{10}$$

(Ahsan, 2015).

Theorem 1.3.7. The first contraction of the Riemann curvature tensor $R(V,W)Z$ given by

$$Ric(V,W) = \sum_i \langle R(E_i,V)W, E_i \rangle,$$

where E_i, $1 \leq i \leq n = \dim(M)$ is called the Ricci tensor (Ahsan, 2015).

Definition 1.3.13. The Ricci curvature scalar is defined by

$$R \coloneqq g^{ij} R_{ij}.$$

Also, by the skew-symmetry properties, the only way to construct R from R_{kilj} is

$$R \equiv g^{kl} g^{ij} R_{kilj} = -g^{kl} g^{ij} R_{iklj}$$

while

$$g^{ki} g^{lj} R_{kilj} = 0.$$

(Ahsan, 2015).

Notes.

1. The Riemann tensor has the component R_{1111}, which is always zero in one dimension due to symmetry considerations.
2. In dimension two, the Riemann tensor has only one component, which may be taken as R_{1212}. But due to the symmetries of the Riemann tensor, the other components related to R_{1212} are
   $$R_{1212} = -R_{2112} = -R_{1221} = R_{2121}$$
   and
   $$R_{1111} = R_{1122} = R_{2211} = R_{2222} = 0.$$
   In any coordinate system, the relation among the components of the Riemann tensor
   $$R_{ijkl} = \alpha\left(g_{ik} g_{jl} - g_{il} g_{jk}\right), \qquad i,j = 1,2.$$
   but
   $$R = g_{ij} R^{ij} = R_{ij}^{ij} = \alpha\left(g_i^i g_j^j - g_j^i g_j^i\right) = \alpha(4-2) = 2\alpha.$$

Thus, in any coordinate system, we have

$R_{ijkl} = \frac{1}{2}R\left(g_{ij}g_{jl} - g_{il}g_{jk}\right),\ i,j = 1,2$

3. In three dimensions there are six components of the Riemann tensor which is independent and thus the curvature tensors are expressed in R_{ij} alone.

   To preserve the symmetry properties, we assume the following relation between Riemann and Ricci tensors:

   $R_{hijk} = g_{hj}R_{ik} + g_{ik}T_{hj} - g_{ij}R_{hk} - g_{hk}R_{ij}.$

   Contracting the indices h and j, we get,

   $R_{ik} = R_{ik} + g_{ik}T$

   where $T = g^{hk}R_{hk}$. Multiplying this equation by g^{ik} and using $g^{ik}g_{ik} = 3$, we get $R = 4T$. Now put the value of R, we get after simplification

   $R_{ik} = R_{ik} - \dfrac{1}{4}Rg_{ik}.$

   When this value of T_{ik} is substituted into equation (26), we get Riemann tensor in terms of the Ricci tensor and scalar curvature as

   $R_{hijk} = \left(g_{hj}R_{ik} + g_{ik}R_{hj} - g_{ij}R_{hk} - g_{hk}R_{ij}\right) - \frac{R}{2}\left(g_{hj}g_{ik} - g_{hk}g_{ij}\right).$

(Ahsan, 2015).

Example 1.3.1. Calculate the components of the Riemann tensor for the metric

$ds^2 = dx^2 + G(x,y)dy^2.$

Solution. Here

$$g_{11} = 1, \qquad g_{22} = G(x,y) = G, \qquad g^{11} = 1, \qquad g^{22} = \frac{1}{G},\ \ g_{ij} = 0, for\ i \neq j$$

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = 0$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2G}\frac{\partial G}{\partial x}, \qquad \Gamma^1_{22} = -\frac{1}{2}\frac{\partial G}{\partial x}, \qquad \Gamma^2_{22} = \frac{1}{2G}\frac{\partial G}{\partial y}.$$

In two dimensions, the only component of the Riemann tensor is R_{1212}, which can be calculated from equation (20b), and we have

$$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} + \frac{\partial^2 g_{21}}{\partial x^2 \partial x^1} - \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}\right) + g_{km}\left(\Gamma^k_{21}\Gamma^m_{12} - \Gamma^k_{22}\Gamma^m_{11}\right)$$

$= -\frac{1}{2}\left(\frac{\partial^2 G}{\partial x^2}\right) + g_{11}(\Gamma_{21}^1\Gamma_{12}^1 - \Gamma_{22}^1\Gamma_{11}^1) + g_{22}(\Gamma_{21}^2\Gamma_{12}^2 - \Gamma_{22}^2\Gamma_{11}^2)$

$= -\frac{1}{2}\left(\frac{\partial^2 G}{\partial x^2}\right) + \frac{1}{4G}\left(\frac{\partial G}{\partial x}\right)^2.$

Example 1.3.2. For the metric $ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\phi^2$, we have

$$g_{11} = a^2, \qquad g_{22} = a^2 \sin^2\theta\,, \qquad g^{11} = \frac{1}{a^2}, \qquad g^{22} = \frac{1}{a^2}\sin^2\theta\,, \qquad g_{ij} = 0$$

for $i \neq j$. The non-vanishing components of the Christoffel symbols are

$$\Gamma_{22}^1 = -\sin\theta\cos\theta\,, \qquad \Gamma_{12}^2 = \Gamma_{21}^2 = \cot\theta.$$

The only non-zero component of the Riemann tensor is R_{1212}, which can be calculated using equation (20b), and we have

$$R_{1212} = -a^2\sin^2\theta$$

and the components of Ricci tensor, from equation (22), are

$$R_{11} = -1, \qquad R_{22} = -\sin^2\theta$$

while the scalar curvature,

$$R = -\frac{1}{a^2}.$$

(Ahsan, 2015).

1.4. Lie Derivative and Differential Operators

There are two sorts of transformations in Euclidean space: (a) discrete, such as reflection, and (b) continuous, such as translation and rotation. The continuous transformations are significant, and their applications may be found in the general theory of relativity.

Consider the coordinate transformation

$$\tilde{x}^i = \tilde{x}^i(\epsilon; x^j)$$

where

$$x^i = \tilde{x}^i(0; x^j)$$

and ϵ is a parameter. The equation above describes a one parameter set of transformations $x^i \to \tilde{x}^i$, which can be explained as follow.

Assume that a given point A is labeled by a set of four coordinates x^i. Assign another point B in the same coordinate system to this point A. As a result, each point in space may be assigned to another point in space using the same coordinate system, and the equation above explains the space's mapping into itself.

Consider the special case, of the infinitesimal transformation

$$x^i \to \tilde{x}^i = x^i + \epsilon V^i(x)$$

where ϵ is a parameter (small an arbitrary) and $V^i(x)$ is a continuous vector field, which may be defined as

$$V^i(x) = \left(\frac{\partial \tilde{x}^i}{\partial \epsilon}\right)_{\epsilon=0}.$$

The transformation is called infinitesimal mapping, which means that “to each A (with coordinates x^i) of the space there corresponds another point B when the same coordinate system used”.

Now let there be a tensor field $T(x)$, which can be evaluated at point B in two different manners, i.e., first, the value $T(\tilde{x})$ of $T(x)$ at point B and secondly, the value of $\tilde{T}(\tilde{x})$ of the transform tensor $\tilde{T}$ (using the usual transformation laws of tensor) at point B. The difference between these two values of the tensor field T evaluated at the point B with the coordinate $\tilde{x}$ refer to the concept of Lie derivative of T.

Definition 1.5.1. Lie derivative of a general tensor $T(x)$ according to the vector field ξ^i is defined as

$$\mathcal{L}_\xi T(x) = \lim_{\epsilon \to 0} \frac{T(x) - \tilde{T}(x)}{\epsilon} \tag{12}$$

(Ahsan, 2015).

Theorem 1.5.1. The Lie derivative of the scalar function f according to the vector ξ^i is the scalar product of ξ^i with the gradient of f. Therefore, it can be written as

$$\mathcal{L}_\xi f = \xi^k \frac{\partial f}{\partial x^k} \tag{13}$$

(Ahsan, 2015).

Theorem 1.5.2. Suppose v^i is a contravariant vector. Its Lie derivative according to the vector field ξ is given by

$$\mathcal{L}_\xi v^i = \xi^k \frac{\partial v^i}{\partial x^k} - v^k \frac{\partial \xi^i}{\partial x^k} \tag{14}$$

(Ahsan, 2015).

Theorem 1.5.3. Given a covariant vector v_i. Its Lie derivative according to the vector field ξ is given by

$$\mathcal{L}_\xi v_j = \xi^i \frac{\partial v_j}{\partial x^i} + v_i \frac{\partial \xi^i}{\partial x^j} \tag{15}$$

(Ahsan, 2015).

Theorem 1.5.4. The Lie derivative of a vector field X on M according to the vector fied V on M is given by

$$\mathcal{L}_V X = [V, X] \tag{16}$$

(Duggal & Sahin, 2010).

Theorem 1.5.5. The Lie derivative of tensor T of type (r, s) according to the vector field V is given by

$$\begin{aligned}(\mathcal{L}_V T)(\omega^1, \dots, \omega^r, X_1, \dots . X_s) &= V\big(T(\omega^1, \dots, \omega^r, X_1, \dots . X_s)\big) \\ &\quad - \textstyle\sum_{a=1}^{r} T(\omega^1, \dots, \mathcal{L}_V \omega^a, \dots, \omega^r, X_1, \dots . X_s)) \\ &\quad - \textstyle\sum_{A=1}^{r} T(\omega^1, \dots, \omega^r, [V, X_A], \dots . X_s)\ ,\end{aligned} \tag{17}$$

where $\omega^1, \dots, \omega^r$ is r 1-forms and and $X_1, \dots . X_s$ are vector fields (Duggal & Sahin, 2010).

Theorem 1.5.6 The Lie derivative of the 1-form $\omega = \omega_i dx^i$ according to the vector field V is given by

$$(\mathcal{L}_V\omega)(X) = V\big(\omega(X)\big) - \omega[V,X] \tag{18}$$

(Duggal and Sahin, 2010).

Theorem 1.5.7. Given a tensor T_1^1 of type (1,1), then its Lie derivative according to the vector field V is given by

$$(\mathcal{L}_V T)(W) = V\big(T(W)\big) - T([V,W]) \tag{19}$$

(Duggal & Sahin, 2010).

Theorem 1.5.8. Given a p-form $\omega = a_{i_1 \dots i_p} dx^{i_i} \wedge \dots \wedge dx^{i_p}$. Its Lie derivative according to the vector field V is given by

$$\mathcal{L}_V\omega = \left(\mathcal{L}_V a_{i_1,\dots,i_p}\right) dx^{i_i} \wedge \dots \wedge dx^{i_p} \tag{20}$$

(Duggal & Sahin, 2010).

Theorem 1.5.9. Given a metric tensor g. Its Lie derivative according to the vector field V is given by

$$(\mathcal{L}_X g)(V,W) = X\big(g(V,W)\big) - g([X,V],W) - g(V,[X,W])\text{`} \tag{21}$$

(Duggal & Sahin, 2010).

Theorem 1.5.10. The Lie derivative of Levi-Civita connection ∇ for arbitrary vector field V, W on M, according to the vector field X is given by

$$(\mathcal{L}_X\nabla)(V,W) = \nabla_V\nabla_W X - \nabla_{\nabla_V W}X + R(X,V)W \tag{22}$$

(Duggal & Sahin, 2010).

Theorem 1.5.11. Given vector fields X, Y, Z on M. Then,

i) $(\mathcal{L}_V\nabla)(X,Y) = \mathcal{L}_V(\nabla_X Y) - \nabla_X(\mathcal{L}_V Y) - \nabla_{[V,X]}Y,$

ii) $-\omega\big((\mathcal{L}_V\nabla)(X,Y)\big) = \big(\mathcal{L}_V(\nabla_X\omega) - \nabla_X(\mathcal{L}_V\omega) - \nabla_{[V,X]}\omega\big)Y,$

iii) $(\mathcal{L}_V\nabla)(X,TY) - T((\mathcal{L}_V\nabla)(X,Y)) = \big(\mathcal{L}_V(\nabla_X T) - \nabla_X(\mathcal{L}_V T) - \nabla_{[V,X]}T\big)Y$,

iv) $(\mathcal{L}_V R)(X,Y,Z) = \nabla_X(\mathcal{L}_V\nabla)(Y,Z) - \nabla_Y(\mathcal{L}_V\nabla)(X,Z)$ö

where ω is a 1-form and T is a $(1,1)$ tensor field. In local coordinates, these formulas can be written as follow:

i) $\left(\mathcal{L}_V\Gamma_{ij}^k\right)Y^j = \mathcal{L}_V(\nabla_i Y^k) - \nabla_i(\mathcal{L}_V Y^k)$;

ii) $-\left(\mathcal{L}_V\Gamma_{ij}^k\right)\omega_k = \mathcal{L}_V\left(\nabla_i \omega_j\right) - \nabla_i\left(\mathcal{L}_V \omega_j\right)$;

iii) $\left(\mathcal{L}_V\Gamma_{im}^j\right)T_k^m - T_m^j\mathcal{L}_V(\Gamma_{ik}^m) = \mathcal{L}_V\left(\nabla_i T_k^j\right) - \nabla_i\left(\mathcal{L}_V T_k^j\right)$;

iv) $\mathcal{L}_V R_{mik}^j = \nabla_i\mathcal{L}_V\Gamma_{km}^j - \nabla_k\mathcal{L}_V\Gamma_{im}^j$.

(Duggal & Sahin, 2010).

Theorem 1.5.12. Lie derivative satisfies the following properties:

i) $\mathcal{L}_V(aX + bY) = a\mathcal{L}_V X + b\mathcal{L}_V Y$, for all $a, b \in \mathbb{R}$ and $X, Y \in \Gamma(TM)$: (linearity)

ii) $\mathcal{L}_V(T \otimes S) = (\mathcal{L}_V T) \otimes S + T \otimes \mathcal{L}_V S$, where $\otimes$ denotes the product of tensor T and S

iii) $\mathcal{L}_V$ is commutative according to the contraction operator

(Duggal & Sahin, 2010).

Theorem 1.5.13. Suppose f is a smooth function. Then its gradient denoted by $grad\ f$, is given by

$$grad\ f = \sum_{i,j=1}^n g^{ij}\, \partial_i f \partial_j \tag{23}$$

(Duggal & Sahin, 2010).

Theorem 1.5.14. The divergence X denoted by $div\ X$ is given by

$$div\ X = X^m; m = \frac{\partial X^m}{\partial x^m} + \Gamma_{km}^m X^k, \tag{24}$$

(Duggal & Sahin, 2010).

Theorem 1.5.15. The Laplacian of f, denoted by Δf, is given by

$$\Delta f = div(grad\ f)$$

$$= \sum_{i,j=0}^n g^{ij}\left\{\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\right\} \tag{25}$$

(Duggal & Sahin, 2010).

Theorem 1.5.16. The Hessian of a function f, denoted by $Hess\ (f) = \nabla(\nabla f)$, is given by

$$Hess_f(V, W) = VWf - (\nabla_V W)f = g(\nabla_V(grad\ f), W), \tag{26}$$

for all $V, W \in \Gamma(TM)$ (Duggal & Sahin, 2010).

1.5. Riemann Submanifold

Given a manifold $\bar{M}$ of dimension n immersed in manifold M of dimension m where $m > n$. Suppose that $\bar{g}$ is the Riemannian metric of $\bar{M}$, the submanifold M is a Riemannian manifold as well equipped with Riemannian metric g called the induced metric satisfying

$$g(X, Y) = \bar{g}(X, Y)$$

for any vector field X, Y in M.

Suppose $TM^{\perp}$ is the normal vector bundle of M in $\bar{M}$. The tangent bundle of $\bar{M}$, restricted to M, can be expressed as the direct sum between the tangent bundle TM and the normal bundle $TM^{\perp}$ of M in $\bar{M}$, that is

$$T_p\bar{M} = T_pM \oplus TM^{\perp} \tag{27a}$$

or

$$T\bar{M} = TM \oplus TM^{\perp}. \tag{27b}$$

Vector $V \in T_pM^{\perp}$ is called a normal vector. The collection of normal vector fields is labeled by $\chi(M)^{\perp}$ or $\Gamma(TM^{\perp})$. Now, for the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$, we define

$$(\bar{\nabla}_X Y)^T = \nabla_X Y, \;\; (\bar{\nabla}_X V)^{\perp} = \nabla_X^{\perp} V \tag{28}$$

for all $X, Y \in \chi(M), V \in \chi(M)^{\perp}$. T and $\perp$ respectively indicate the projection of the tangent bundle and the normal bundle of the submanifold. On the other side define

$$(\bar{\nabla}_X Y)^{\perp} = h(X, Y) \tag{29a}$$

and

$$(\bar{\nabla}_X V)^T = -A_V X. \tag{29b}$$

It is easy to show that h is symmetric and bilinear such that

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{30a}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla_X^{\perp} V. \quad (30b)$$

Lemma 1.6.1. For all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$ on submanifold M, the equation

$$g(A_V X, Y) = g(h(X,Y), V) \quad (31)$$

is satisfied (Şahin, 2012).

Definition 1.6.1. Given a submanifold $(M, g)$of Riemannian manifold $(\overline{M}, \bar{g})$ with $\{e_1, .., e_n\}$ as the orthonormal basis of the submanifold. The mean curvature vector field of M is expressed by

$$H = \frac{1}{n} trace(h) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_j). \quad (32)$$

The mean curvature vector field is independent of the orthonormal basis collection (Şahin, 2012).

Lemma 1.6.2. Given a submanifold (M, g) of Riemannian manifold $(\overline{M}, \bar{g})$ with $\{e_1, .., e_n\}$ as the orthonormal basis of the submanifold. Then,

$$H = \frac{1}{n} \sum_{j=1}^{m} \left(trace\ A_{n_j} \right) n_j \quad (33)$$

(Şahin, 2012)

Definition 1.6.2. M is said to be totally geodesic if h, the second fundamental form of the submanifold (M, g) vanishes identically or $h = 0$ (Chen, 1973).

Definition 1.6.3. A normal section N on M is said to be umbilical, if A_N is proportional to the identity transformation I everywhere, that is

$$A = \rho I$$

for some functions ρ. In addition, M is said to be totally umbilical if the submanifold M is umbilical according to all local normal sections in M (Chen, 1973).

Theorem 1.6.1. The necessary and sufficient conditions of totally umbilical properties for submanifold M is

$$h(X, Y) = g(X, Y) H \quad (34)$$

for all $X, Y \in \chi(M)$, where H stands for the mean curvature of the submanifold M (Şahin, 2012).

1.6. Semi-Riemann Manifold

Definition 1.7.1. The largest integer which is the dimension of a subspace $W \subset V$ on which $b|W$ in negative definite is the index of a symmetric bilinear form b on V(O'Neill, 1983).

Definition 1.7.2. Metric tensor $\bar{g}$ on differentiable manifold $\bar{M}$ is non-degenerate $(0,2)$ the tensor field which is symmetric on M of a constant index. In other words, $\bar{g}$ smoothly assigned to each point p of M a scalar product $\bar{g}_p$ on the tangent space $T_p(M)$, and the index of $\bar{g}_p$ is the same for every p (O'Neill, 1983).

Definition 1.7.3. The common value of v of index $\bar{g}_p$ is called the index of $\bar{M}$ where $0 \leq v \leq n = \dim M$. In case $v = 0$, $\bar{M}$ is called a semi-Riemannian manifold and if $v = 1$ with $n \geq 2$, $\bar{M}$ is called Lorentz manifold (O'Neill, 1983).

For an integer v with $0 \leq v \leq n$, changing the first v having a plus sign to minus yields a metric tensor

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{v} v^i w^i + \sum_{j=v+1}^{n} v^j w^j.$$

The Euclidean space is derived when $\mathbb{R}_v^n$ reduce to $\mathbb{R}^n$ if $v = 0$. For $n \geq 2$, $\mathbb{R}_1^n$ is called Minkowski n-space. For case $n = 4$, we sometimes refer to a simple example of a spacetime (O'Neill, 1983).

Definition 1.7.4. On semi-Riemannian $\bar{M}$, the tangent vector v is said to be

1) spacelike when $\langle v, v \rangle > 0$ or $v = 0$,
2) timelike when $\langle v, v \rangle < 0$,
3) null when $\langle v, v \rangle = 0$ and $v \neq 0$.

(O'Neill, 1983).

Definition 1.7.5.

i. If x and y is timelike vectors in $\mathbb{R}_1^3$ then $\exists!\,\theta \geq 0$, the hyperbolic angle between vectors x and y such that, $\langle x, y\rangle = -\|x\|\|y\| \cosh\theta$.
ii. If x and y is spacelike vectors spanning timelike subspace in R_1^3 then $\exists!\,\theta \geq 0$, origin angel between x and y such that $\langle x, y\rangle = \|x\|\|y\| \cosh\theta$.
iii. If x and y is spacelike vectors spanning spacelike subspace in R_1^3 then $\exists!\,\theta \geq 0$, the spacelike angle between x and y such that $\langle x, y\rangle = \|x\|\|y\| \cos\theta$.
iv. If x and y are timelike vectors in R_1^3, $\exists!\,\theta \geq 0$, Lorentzian timelike angle between x and y such that $\langle x, y\rangle = \|x\|\|y\| \sinh\theta$.

(Birman and Nomizo, 1984).

Definition 1.7.6. A curve $\alpha(s)$ on a semi-Riemannian manifold $\bar{M}$ is spacelike, timelike or lightlike (null) if all tangent vector along the curve is spacelike, timelike or lightlike (null), respectively (O'Neill, 1983).

Theorem 1.7.1. $\bar{\nabla}$ is called Levi-Civita connection on $\bar{M}$ if for any vector fields $X, V, W \in \chi(\bar{M})$,

1) $[V, W] = \nabla_V W - \nabla_W V$,
2) $X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$

are satisfied. (O'Neill, 1983).

Definition 1.7.7. A connection ∇ on a smooth manifold $\bar{M}$ is defined by,

$$\nabla: \chi(\bar{M}) \times \chi(\bar{M}) \to \chi(\bar{M})$$

satisfying

1) $\nabla_V W$, is $C^\infty(M, \mathbb{R})$-lineer in V,
2) $\nabla_V W$, is $\mathbb{R}$-linear in W,
3) $\nabla_V(fW) = (Vf)W + f\nabla_V W$ for $f \in C^\infty(M, \mathbb{R})$

(O'Neill, 1983).

1.7. Lightlike Hypersurfaces of Semi-Riemannian Manifolds

For $\forall p \in M$, T_pM denotes a hyperplane of the semi-Euclidean space $\left(T_p, \bar{M}, \bar{g}_p\right)$. We define the normal bundle

$$T_pM^{\perp} = \left\{v \in T_p\bar{M} \middle| g_p(v, w) = 0, \forall w \in T_pM\right\}. \tag{37}$$

Let the metric g of M be degenerate. In this case, vector field $\xi \neq 0$ on M exists so that for all $X \in \chi(TM)$ it satisfies $g(\xi, X) = 0$ (Duggal and Bejancu, 1996).

Definition 1.8.1. The radical (null) space T_uM is defined by

$$Rad\ T_pM = \left\{\xi \in T_pM \middle| g_p(\xi, X) = 0, \forall X \in T_pM\right\} \tag{38}$$

The dimension of $Rad\,T_pM$ is called nullity degree of g, denoted by $null\,T_pM$. (Duggal and Bejancu, 1996).

For all point $p \in M$, the normal space $T_pM^{\perp}$ is also degenerate, which can be written by

$$Rad\ T_pM = T_pM \cap T_pM^{\perp}.$$

For hyperspace M, since the dimension $T_pM^{\perp}$ is 1 then $dim\left(Rad\,T_pM\right) = 1$. Therefore, $Rad\ T_pM = T_pM^{\perp}$. $Rad\ TM$ is called radical (null) distribution of M.

The tangent bundle of M is denoted by TM and the normal bundle of M is denoted $TM^{\perp}$. Therefore, we will let a complementary vector bundle $S(TM)$of $TM^{\perp}$ in TM, i.e.,

$$TM = Rad\,TM \oplus S(TM) \tag{39}$$

(Duggal and Bejancu, 1996).

Definition 1.8.2. $S(TM)$ is a non-degenerate subspace called screen distribution on M (Duggal and Bejancu, 1996).

The complementary vector bundle which is orthogonal to $S(TM)$ in $T\bar{M}|_M$ is labeled by $S(TM)^{\perp}$. $S(TM)^{\perp}$ is a non-degenerate vector bundle with $rank = 2$. Therefore, we have the following decomposition

$$T\bar{M}|_M = S(TM) \perp S(TM)^{\perp} \text{ where } S(TM) \cap S(TM)^{\perp} \neq \{0\}. \tag{40}$$

Moreover, $Rad\,TM = TM^{\perp}$ vector bundle is a subbundle of $S(TM)^{\perp}$.

Lightlike hypersurface of semi-Riemannian manifold $(\overline{M},\bar{g})$ of dimension $(m+2)$is denoted by $\big(M, gS(TM)\big)$ (Duggal and Bejancu, 1996).

Theorem 1.8.1. There exists a unique vector bundle $tr(TM)$ of rank 1 over lightlike hypersurface $\big(M, g, S(TM)\big)$, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $U \subset M$, the equation

$$\bar{g}(N,\xi) = 1,\ \bar{g}(N,N) = \bar{g}(N,W) = 0,\ \forall X \in \chi\big(S(TM)\big)|_U, \tag{41}$$

are satisfied for unique section M of $tr(TM)$ on U (Duggal and Bejancu, 1996).

Consider a complementary vector bundle F of $TM^{\perp}$ in $S(TM)^{\perp}$ and take $V \in \chi(F|_U), V \neq 0$. Then, $\bar{g}(V,\xi) \neq 0$ on U, otherwise $S(TM)^{\perp}$ would be degenerate at a point of U. As in case of null curves, we have

$$N = \frac{1}{\bar{g}(\xi,N)}\left\{V - \frac{\bar{g}(V,V)}{2\bar{g}(\xi,V)}\xi\right\}. \tag{42}$$

For any $u \in M$, we call $tr(TM)$ the lightllike transversal vector bundle where $tr(TM)_p \cap T_pM = \{0\}$. Therefore, we have decomposition

$$T\overline{M}|_M = TM \oplus tr(TM). \tag{43}$$

(Duggal and Bejancu, 1996).

The Gauss formula and the Weingarten formulas of M are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{44a}$$

$$\overline{\nabla}_X V = -A_V X + \nabla_X^t V \tag{44b}$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$, $X, Y \in \chi(TM)$ and $V \in \chi\big(tr(TM)\big)$. $\nabla_X Y$ and $A_V X$ are elements of $\Gamma(TM)$ while $h(X,Y)$ and $\nabla_X^t V$ elements of $\Gamma\big(tr(TM)\big)$. It is clear that ∇ is a linear connection on M with zero torsion. h is called the second fundamental form and A_V is called the shape operator of the lightlike immersion of M in $\overline{M}$. ∇ and ∇_X^t are respectively called the induced connection on M and $tr(TM)$. (Duggal and Bejancu, 1996).

In local coordinate, if $\{\xi, N\}$ is a pair of sections on $U \subset M$. Then, a simetric bilinear form B and a 1-form τ on U is defined by

$$B(X,Y) = \bar{g}(h(X,Y),\xi) \tag{45a}$$

$$\tau(X) = \bar{g}(\nabla_X^t N,\xi). \tag{45b}$$

Therefore,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N \tag{46a}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N. \tag{46b}$$

Since B is the only component of h on U with respect to N, we call B the local second fundamental form of M. Furthermore, since $\overline{\nabla}$ is a metric connection, then for any $X \in \chi(TM|_U)$

$$B(X,\xi) = 0 \tag{47}$$

is satisfied (Duggal and Bejancu, 1996).

Proposition 1.8.1. Given two screen distributions, namely $S(TM)$ and $S(TM)'$ and second fundamental forms with respect to $tr(TM)$ and $tr(TM)'$, namely h and h' on M, respectively. Then, $B = B'$ on U, which means that the local second fundamental form of M on U is independent of the choice of screen distribution.

Corollary 1.8.1. The lightlike hypersurface second fundamental form is degenerate. (Duggal and Sahin, 2010).

Assume P is the projection morphism of TM on $S(TM)$ according to the decomposition (39). Then, we obtain

$$\nabla_X PY = \nabla_X^* PY + h^*(X,PY) \tag{48a}$$

$$\nabla_X W = -A_W^* X + \nabla_X^{*t} W. \tag{48b}$$

where $\nabla_X^* Y$ and $A_W^* X$ are element of $\Gamma\big(S(TM)\big)$ whereas $h^*(X,Y)$ and $\nabla_X^{*t} W$ are elements of $\Gamma(TM^{\perp})$. ∇^* and ∇^{*t} are linear connection on vector bundles $S(TM)$ and $TM^{\perp}$, respectively. h^* is called the second fundamental form and A_W^* is called the shape operator of screen distribution $S(TM)$. With the same way, for all $X, Y \in \chi(TM)$, we define on U

$$C(X,PY) = \bar{g}(h^*(X,PY),N) \tag{49a}$$

$$\varepsilon(X) = \bar{g}(\nabla_X^{*t}\xi,N). \tag{49b}$$

Hence, $\varepsilon(X) = -\tau(X)$ and we have,

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)\xi \tag{50a}$$

$$\nabla_X \xi = -A_\xi^* X - \tau(X)\xi. \tag{50b}$$

$C(X, PY)$ is called local screen second fundamental form of screen distribution $S(TM)$. The second fundamental form and the shape operator of M and $S(TM)$ satisfy

$$B(X, Y) = g\left(A_\xi^* X, Y\right) \tag{51a}$$

$$C(X, PY) = g(A_N X, PY) \tag{51b}$$

such that $\bar{g}\left(A_\xi^* X, N\right) = 0$ and $\bar{g}(A_N Y, N) = 0$ (Duggal and Sahin, 2010).

Proposition 1.8.2.

1. The linear connection ∇^* is a metric connection on $S(TM)$.
2. The induced connection ∇ on M satisfies,
   $$(\nabla_X g)(Y, Z) = B(X, Y)\bar{g}(Z, N) + B(X, Z)\bar{g}(Y, N) \tag{52}$$
   for any $X, Y, Z \in \Gamma(TM)$ (Duggal and Sahin,2010).

Proposition 1.8.3. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, the following assertions are equivalent:

1. The shape operator A_N of M has a zero eigen value.
2. The second form of screen distribution $S(TM)$ is degenerate.
3. The shape operator A_ξ^* of screen distribution $S(TM)$ is symmetric according to the second fundamental form of M.
4. The connection ∇^* on $S(TM)$ is a metric connection.
5. The sectional integral curve $\xi \in \chi(Rad\,TM)$ is a null geodesic based on the connection ∇ and $\bar{\nabla}$ on M and $\bar{M}$, respectively (Duggal and Sahin, 2010).

Theorem 1.8.2. The Lie derivative of metric tensor g is given by

$$(\mathcal{L}_Z g)(X, Y) = B(Z, Y)\eta(X) + B(X, Z)\eta(Y) + g(\nabla_X Z, Y) + g(X, \nabla_Y Z), \tag{53}$$

where $\eta(X) = \bar{g}(X, N)$ (Duggal and Bejancu, 1996).

Theorem 1.8.3. Let R and $\bar{R}$ be the curvature tensor on lightlike hypersurface $\big(M, g, S(TM)\big)$ and semi-Riemannian manifold $(\bar{M}, \bar{g})$, respectively. Then, the relation

$$\bar{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \qquad (54)$$

is obtained (Duggal and Bejancu, 1996).

Theorem 1.8.4. Suppose $\{e_1, \dots, e_n, \xi, N\}$ is the orthonormal basis of TM and $\{e_1, \dots, e_n\}$ is the orthonormal basis on $\Gamma\big(S(TM)\big)$. The induces Ricci tensor of M is given by

$$Ric(X,Y) = \sum_{j=1}^{n} g\big(R(e_j, X)Y, e_j\big) + g(R(\xi, X)Y, N) \qquad (55)$$

for all $X, Y \in TM$ (Duggal and Bejancu, 1996).

Theorem 1.8.5. Let $\big(M, g, S(TM)\big)$ be a lightlike hypersurface of $(n+2)$ - dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then,

$$Ric(X,Y) = ncg(PX, PY) - B(X,Y)\alpha + \sum_{i=1}^{m} \epsilon_i \, B(w_i, Y)C(X, w_i) \qquad (56)$$

where $\alpha = \sum_{i=1} \epsilon_i g(A_N w_i, w_i)$, $\{w_i\}, i \in \{1, 2, \dots, n\}$ and $X, Y \in \Gamma(TM)$ (Duggal and Bejancu, 1996).

Definition 1.8.3. The mean curvature of a lightlike hypersurface $\big(M, g, S(TM)\big)$ is given by

$$H = \sum_{a=1}^{n} C\big(e_i, e_j\big). \qquad (57)$$

where $\{e_1, \dots, e_n\}$ is the orthonormal basis on $\Gamma\big(S(TM)\big)$ (Güneş, Şahin, and Kılıç, 2003).

Definition 1.8.4. A lightlike hypersurface $\big(M, g, S(TM)\big)$ is totally umbilical if there exist a smooth function F such that

$$B(X,Y) = Fg(X,Y), \; X, Y \in \Gamma(TM). \qquad (58)$$

(Duggal and Bejancu, 1996).

2. RESULT AND DISCUSSIONS

2.1. Ricci Flow

Definition 2.1.1. Suppose M is a manifold and $g(t), t \in (0, T)$ is a one-parameter family of Riemann metrics on M. $g(t)$ is said to be a solution of Ricci flow if

$$\frac{\partial}{\partial t} g(t) = -2Ric\big(g(t)\big) \tag{59}$$

(Brendle, 2010).

Because it does not conserve volume in general, the preceding formulation of Ricci flow is known as a Ricci flow equation which is unnormalized. As a result, it is frequently beneficial to investigate the normalized Ricci flow by imposing the condition $Vol(M, g) = 1$. The Ricci flow equation possesses the form of a limited system with a Lagrange multiplier which enters a vector field that is time-dependent to a parameter and has the goal of volume conservation.

For any $g \in M$ there exist a unique volume element $d\mu_g$ on M. Locally, we have $(x^1, x^2, \dots, x^n)$, $d\mu_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \dots \wedge dx^n$. Let $vol(M, g) = \int_M d\mu_g$ be the Riemannian manifold (M, g) volume, $\mathbb{R}^+ = (0, \infty)$, and

$$vol: M \to \mathbb{R}^+, \qquad g \to \int_M d\mu_g = vol(M, g)$$

express the Riemannian metrics space of volume one.

Now, we write the Ricci flow equation as a constrained dynamical system

$$\frac{\partial g}{\partial t} = -2Ric(g) + cg$$

$$vol(M, g) = 1$$

for curves

$g: [0, T) \to M,\ t \longmapsto g(t),$ with $g(0) = g_0 \in M$, and

$c: [0, T) \to \mathbb{R}, \qquad t \mapsto c(t),$

continues on the semi-closed interval $[0, T)$ and smooth on $(0, T), 0 < T \leq \infty$.

To preserve the constrain equation $vol(M, g) = 1$, the value of $c: [0, T) \to \mathbb{R}$ which is Lagrange multiplier depends on time. As a result, it can be eliminated. Differentiating the time yields

$$0 = \tfrac{d}{dt} vol(M, g) = \tfrac{d}{dt} \int_M d\mu_g = \int_M \tfrac{\partial}{\partial t} d\mu_g = \int_M \tfrac{1}{2} tr_g \left(\tfrac{\partial g}{\partial t}\right) d\mu_g$$

$$= \int_M \tfrac{1}{2} tr_g(-2Ric(g) + cg) d\mu_g = -\int_M R(g) d\mu_g + \tfrac{n}{2} c \int_M d\mu_g$$

$$= -R_{total}(g) + \tfrac{n}{2} c\, vol(M, g), \text{ ....*}$$

where $R_{total}(g) = \int_M R(g)\, d\mu_g$ denotes the total scalar curvature. Therefore,

$$c = \frac{2}{n} \frac{R_{total}(g)}{vol(M, g)} = \frac{2}{n} \bar{R}(g)$$

where $\bar{R}(g) = \frac{R_{total}(g)}{vol(M,g)}$ express the total scalar curvature volume average of g. As c applying integrals over M, it is a global function of g and depends on g at every point of M.

Substituting c into the dynamical system of Ricci flow yields

$$\frac{\partial g}{\partial t} = -2Ric(g) + \frac{2}{n} \bar{R}(g) g, \qquad g_0 \in M$$

where $vol(M, g_0) = 1$ is a constrain and only applied to the initial condition g_0.

The constrain on the whole flow g_t is derived from the last equation since

$$\frac{d}{dt} vol(M, g) = \frac{1}{2} \int_M tr_g \left(\frac{\partial g}{\partial t}\right) d\mu_g = \frac{1}{2} \int_M tr_g \left(-2Ric(g) + \frac{2}{n} \bar{R}(g) g\right) d\mu_g$$

$$-\int_M R(g)\, d\mu_g + \frac{n}{2} \frac{2}{n} \bar{R}(g) \int_M d\mu_g = -R_{total}(g) + R_{total}(g) = 0$$

so that from the initial constrain $g_0 \in M$, $vol(M, g_t) = 1$ (Fischer, 2004)

Theorem 2.1.1. Invariant of the curvature tensor R holds that is local isometry $\varphi: (M, g) \to (\widetilde{M}, \tilde{g})$ satisfies $\varphi^* \tilde{R} = R$ (Andrews and Hopper, 2011).

Therefore, in case the diffeomorphism $\varphi: M \to \widetilde{M}$ is a time-independent such that $g(t) = \varphi^* \tilde{g}(t)$ and $\tilde{g}$ satisfies the Ricci flow, it can be seen that

$$\frac{\partial}{\partial t} g = \varphi^* \left(\frac{\partial}{\partial t} \tilde{g} \right)$$

$$= \varphi^* \left(-2Ric(\tilde{g}) \right)$$

$$= -2Ric(\varphi^* \tilde{g})$$

$$= -2Ric(g),$$

In conclusion, g is a solution to Ricci flow as well. In other words, the invariant property of Ricci flow under all diffeomorphism groups holds.

Besides, the Ricci flow possesses scaling features that are needed for bow-up singularity analysis. Changes in spatial coordinates are permitted due to the time-independent diffeomorphism invariance. It is able to move in the time coordinates that is when $g(x, t)$ fulfills the Ricci flow, $g(x, t - t_0)$ is also satisfies the Ricci soliton for any $t_0 \in \mathbb{R}$. In addition, the Ricci flow possess a scale invariance property that is when g satisfies the Ricci flow with $\lambda > 0$, then so does g_λ, where

$$g_\lambda(x, t) = \lambda^2 g\left(x, \frac{1}{\lambda^2} \right).$$

This property is important in singularity analysis developing under Ricci flow. In the condition of the curvature limit to infinity, we apply a rescaling to find bounded-curvature metrics and find a smooth limit of those metrics (Andrews and Hoper, 2011).

2.1. Ricci Solitons

In this part, we will look at the Ricci flow special solution. The most common type of Ricci flow solution is self-similar solutions, also known as Ricci solitons, which can be thought of as flow with generalized fixed points. The Ricci flow solution is classified into three forms based on its time interval parameter: eternal solution, ancient solution, and immortal solution.

2.1.1. Generalized Fixed Points

Definition 2.2.1.1. Suppose that $(M, g(t))$ is an unnormalized Ricci flow solution (α, ω) containing 0, and set $g_0 = g(0)$. One says $g(t)$ is a Ricci flow self-similar solution if there exist scalar $\sigma(t)$ and diffeomorphisms ψ_t of M such that

$$g(t) = \sigma\, \psi_t^*(g_0) \tag{60}$$

for all $t \in (\alpha, \omega)$. Thus, a self-similar solution is a solution of Ricci flow equation that has not been normalized and satisfies the equation (60) (Chow and Knopf, 2004).

Let $\mathcal{m}$ be the Riemannian metrics space on a differentiable manifold M, and assume $\mathcal{d}$ to be the diffeomorphisms group of M. Take the quotient map $\pi: \mathcal{m} \to \mathcal{m}/\mathcal{d} \times \mathbb{R}_+$, with $\mathbb{R}_+$ is scalar. We can show that a Ricci soliton $g(t)$ is the Ricci flow solution where $\pi(g(t))$ is time-independent.

We begin from the initial conditions producing Ricci solitons. By taking the derivative of equation (59) yields

$$-2\, Ric(g(t)) = \dot{\sigma}(t)\psi_t^*(g_0) + \sigma(t)\psi_t^*\left(\frac{\partial g}{\partial t}\right) + \sigma(t)\psi_t^*(\mathcal{L}_X g).$$

This follows the definition of Lie derivative. (It may help us to write $\psi_t^*(g(t)) = \psi_t^*(g(t) - g(s)) + \psi_t^*(g(s))$ and differentiate at $t = s$) As consequence

$$-2\, Ric(g(t)) = \dot{\sigma}(t)\psi_t^*(g_0) + \sigma(t)\psi_t^*(\mathcal{L}_X g), \tag{61}$$

where $g_0 = g(0)$, $\mathcal{L}$ stand for the Lie derivative, X is a vector field depending on time such that $X(\psi(t)(p)) = \frac{d}{dt}(\psi(t)(p))$ for any $p \in M$, and $\dot{\sigma} \coloneqq \frac{d\sigma}{dt}$ (Chow and Knopf, 2004).

Definition 2.1.1.2. If $\dot{\sigma}(t_0)$ is $> 0, = 0$ or < 0, the Ricci flow $g(t)$ is said to be expanding, steady, or shrinking at time t_0, respectively (Chow et. al., 2007).

Since $Ric(g(t)) = \psi^* Ric(g_0)$, we can drop the pullbacks in (61) and get

$$-2Ric(g_0) = \dot{\sigma}(t)g_0 + \mathcal{L}_{\tilde{X}(t)}g_0, \tag{62}$$

where $\tilde{X}(t) = \sigma(t)X(t)$. Although g_0 is time-independent, $\dot{\sigma}(t)$ and $\tilde{X}(t)$ still might be time-dependent. For instance, the Euclidean space $\left(M^n, g(t)\right) \equiv (\mathbb{R}, g_{can})$ is the solution which is stationary to the Ricci flow and categorized as a steady Ricci soliton. However, this solution can be expanding or shrinking as well based on the diffeomorphism. Given any function $\sigma(t) > 0$ and $\sigma(0) = 1$, take the diffeomorphisms $\psi(t): \mathbb{R}^n \to \mathbb{R}^n$ defined by $\psi(t)(x) = \sigma(t)^{-1/2}x$ for $x \in \mathbb{R}^n$. Then, $\psi(t)^* g_{can} = \sigma(t)^{-1} g_{can}$. Since $g(t) \equiv g_{can}$, we may rewrite this as

$$g(t) = g_{can} = \sigma(t)\psi(t)^* g(0). \tag{63}$$

By choosing $\sigma(t)$ so that the sign of $\dot{\sigma}(t)$ changes, this metric can expand or shrink at distinct times.

On the other hand, in general, a Ricci soliton solution is evolving purely by scaling (modulo diffeomorphisms). As the derivative of the metric according to the time is equal to the negative of twice the Ricci tensor, it is thus natural to ask whether one can put the general soliton equation in a canonical form where $\sigma(t)$ is equal to a linear function (Chow and Knopf, 2004).

Proposition 2.1.1.1. Suppose that $(\mathrm{M}^n, \mathrm{g(t)})$ is a Ricci soliton with unique initial metric $g_0 = g(0)$ among the solutions of the soliton. Then, we can find a diffeomorphism $\varphi(t): \mathrm{M} \to \mathrm{M}$ and a constant $\varepsilon \in \mathbb{R}$ satisfying the equation

$$g(t) = (1 + \varepsilon t)\varphi(t)^* g_0 \tag{64}$$

(Chow et.all, 2007).

Proof. Differentiating (62) with respect to time gives

$$\ddot{\sigma}(t)g_0 + \mathcal{L}_{\dot{\tilde{X}}(t)}g_0 = 0. \tag{65}$$

Case1. If $\ddot{\sigma}(t) \equiv 0$, then $\sigma(t) = 1 + \varepsilon t$ for some constant ε. Hence, by (60), we may simply take $\varphi(t) = \psi(t)$ to obtain (64).

Case2. If $\ddot{\sigma}(t)$ is not identically zero, then let $Y_0 = -\dot{\tilde{X}}(t_0)/\ddot{\sigma}(t_0)$ at some t_0 where $\ddot{\sigma}(t_0) \neq 0$. We then have

$$\mathcal{L}_{Y_0}g_0 = g_0 \tag{66}$$

Substituting (66) into (62), we have

$$-2Ric(g_0) = \mathcal{L}_{\dot{\sigma}(t)Y_0+\tilde{X}(t)}g_0$$

for every t. Take the vector field

$$X_0 \coloneqq \dot{\sigma}(0)Y_0 + \tilde{X}(0).$$

Consequently,

$$-2Ric(g_0) = \mathcal{L}_{X_0}g_0$$

Suppose generates X_0 the 1-parameter diffeomorphism group $\varphi(t)$. Then, it is clear that $\tilde{g}(t) = \varphi_t^* g_0$ is also the steady solution of the Ricci flow with g_0 as the same initial conditions. Therefore, by our assumption about the uniqueness of Ricci soliton solutions with initial metric g_0, we may change $\psi(t)$ with $\varphi(t)$ so that $\sigma(t) \equiv 1$ in (60) is obtained.

As the geometric properties of every Ricci soliton $g(t)$ is the same as that of its initial condition, we can use g_0 and construct another Ricci soliton in canonical form. Take arbitrary time t_0 and take $\varepsilon \coloneqq \dot{\sigma}(t_0)$ and $X_0 \coloneqq \tilde{X}(t_0)$. As a result, the equation (62) becomes

$$-2Ric(g_0) = \varepsilon g_0 + \mathcal{L}_{X_0}g_0.$$

at $t = t_0$. Using indices, the equation above can be written as

$$-2R_{ij} = \nabla_i X_j + \nabla_j X_i + \varepsilon g_{ij}, \tag{67}$$

where $X_i = g_{ij}X^j$, the one-form derived from X by the way of lowering indices using g. The equation (67) is called Ricci soliton structure denoted by (M, g, X, ε).

If we set $\sigma(t) = 1 - 2\lambda t$ and a vector field X on M, so that $\varepsilon = \dot{\sigma}(t_0) = -2\lambda$. So that, the equation of Ricci soliton structure or equation becomes

$$-2Ric(g_0) = \mathcal{L}_X g_0 - 2\lambda g_0 \tag{68}$$

or

$$-2R_{ij} = \nabla_i X_j + \nabla_j X_i - 2\lambda g_{ij}.$$

Note that the flow is steady, expanding or shrinking Ricci soliton depending on whether $\lambda = 0, \lambda < 0$ or $\lambda > 0$, respectively (Chow and Knopf, 2004).

Lemma 2.1.1.1. Assume that (M, g) is a Ricci flow solution with a special form given by equation (60). Then, we can find a vector field X on M such that (M, g_0, X) is the solution of Ricci soliton (68). Conversely, given any solution (M, g_0, X) of the Ricci soliton (68), then we can find a family of 1-parameter of scalar $\sigma(t)$ and diffeomorphisms ψ_t of M so that $(M, g(t))$ is a Ricci flow solution in case of $g(t)$ given by equation (60) (Chow et.all, 2007).

Proof. Let $(M, g(t))$ be a Ricci flow solution with form given in equation (60). Without loss of generality, take $\sigma(0) = 1$ and $\psi_0 = id$. Therefore, we have

$$-2Ric(g_0) = \frac{\partial}{\partial t} g(t)|_{t=0} = \dot{\sigma}(0) g_0 + \mathcal{L}_{Y(0)} g_0,$$

where $Y(t)$ is the vector fields family generating the diffeomorphisms ψ_t. This implies that g_0 satisfies (68) with $\lambda = -\frac{1}{2}\ \dot{\sigma}(0)$ and $X = Y_0$.

Conversely, suppose that g_0 satisfies (68). Define

$$\sigma(t) \coloneqq 1 - 2\lambda t,$$

and define a family of a 1-parameter of vector fields Y_t, where $\psi_0 = id_M$, and define a family of smooth 1-parameter of metrics on M by

$$g(t) \coloneqq \sigma(t) \psi_t^*(g_0).$$

Then, $g(t)$ has the special form (60). The computation

$$\frac{\partial}{\partial t} g = \frac{d\sigma}{dt} \cdot \psi_t^*(g_0) + \sigma(t) \cdot \psi_t^*\left(\mathcal{L}_{Y_t} g_0\right) = \psi_t^*(-2\lambda g_0 + \mathcal{L}_X g_0)$$

implies by (68) that

$$\frac{\partial}{\partial t} g = \psi_t^*\left(-2Ric(g_0)\right) = -2Ric(g).$$

Hence, that $g(t)$ satisfies the Ricci flow.

Definition 2.1.1.3. A Ricci soliton (M, g, X) is called gradient soliton if we can find a smooth function $f: M \to \mathbb{R}$ such that $X^b = df$. This function is called the potential function (Chow et al., 2007).

In this case (68), becomes

$$R_{ij} + \nabla_i \nabla_j f + \frac{\varepsilon}{2} g_{ij} = 0. \tag{69}$$

or

$$R_{ij} + Hes_{g_{ij}}(f) = \lambda g_0,$$

where $Hes_{g_0}(f) = \nabla_i \nabla_j f = \frac{1}{2}\mathcal{L}_X g_0$ and $\varepsilon = -2\lambda$.

Proposition 2.1.1.2. Given a complete gradient Ricci soliton structure $(M, g_0, \nabla f_0, \varepsilon)$ is on M. Then, we can find a solution of Ricci flow that is $g(t)$ with $g(0) = g_0$, a diffeomorphism $\varphi(t)$ with $\varphi(0) = id_M$, and functions $f(t)$ with $f(0) = f_0$ defined for all t with

$$\tau(t) \coloneqq 1 + \varepsilon t > 0,$$

satisfying

(1) $\varphi(t): M \to M$ is the diffeomorphisms family of 1-parameter derived from $X(t) \coloneqq \frac{1}{\tau(t)} grad_{g_0} f_0$; that is,
$$\frac{\partial}{\partial t}\varphi(t)(x) = \frac{1}{\tau(t)}\left(grad_{g_0} f_0\right)\left(\varphi(t)(x)\right),$$
(2) $g(t)$ is the pull-back by $\varphi(t)$ of g_0 up to the $\tau(t)$ as the scale factor,
$$g(t) = \tau(t)\varphi(t)^* g_0,$$
(3) $f(t)$ is the pull-back by $\varphi(t)$ of f_0:
$$f(t) = f_0 \circ \varphi(t) = \varphi(t)^* f_0.$$

Moreover,

$$Ric\big(g(t)\big) + \nabla^{g(t)}\nabla^{g(t)} f(t) + \frac{\varepsilon}{2\tau} g(t) = 0,$$

where $\nabla^{g(t)}$ stand for the covariant derivative according to $g(t)$, and

$$\frac{\partial f}{\partial t}(t) = \left|grad_{g(t)} f(t)\right|^2_{g(t)}.$$

(Chow et.all, 2007).

Proposition 2.1.1.3. Suppose $g(t) = \sigma(t)\psi_t^*(g_0)$ is a steady gradient soliton with potential f. Then, the equations

i) $Ric + Hess(f) = 0,$

ii) $R + \Delta f = 0,$

iii) $|\nabla f|^2 + R = constant$,

iv) $\partial_t f = |\nabla f|^2$

are satisfied (Muller, 2006).

Proof. If we differentiate $g(t)\psi_t^*(g_0)$ with respect to t, using Ricci flow equation and the definition of Lie derivative $\mathcal{L}_X g$ yields

$$-2R_{ij} = \partial_t g_{ij} = (\mathcal{L}_X g)_{ij} = -2div^* X = \nabla_i X_j + \nabla_j X_i.$$

Hence, $-2R_{ij} = 2\nabla_i \nabla_j f$ and the first equation holds. The calculation of $\mathcal{L}_X g$, where g is a tensor of type (0,2). To find the second equation we can trace the first equation. Next, to find equation iii, taking the covariant derivative of part i, permutating the indices, and substracting the two following equations:

$$\nabla_k R_{ij} + \nabla_k \nabla_i \nabla_j f = 0$$

$$\nabla_i R_{kj} + \nabla_i \nabla_k \nabla_j f = 0$$

$$\Longrightarrow \nabla_k R_{ij} - \nabla_i R_{kj} + R_{kijp} \nabla_p f = 0.$$

Taking the trace the equation above with g^{kj} we find

$$\nabla_j R_{ij} - \nabla_i R + R_{ip} \nabla_p f = 0.$$

By using the second Bianchi identity and the equation i) we have $\nabla_j R_{ij} = \frac{1}{2}\nabla_i R$ and $R_{ip} = -\nabla_i \nabla_p f$, so that

$$0 = -\frac{1}{2}\nabla_i R - \nabla_i \nabla_p f \nabla_p f = -\frac{1}{2}\nabla_i (R + |\nabla f|^2).$$

which is nothing but equation iii). Next, with using the definition

$$f(t,p) = f\big(0, \psi_t(p)\big), \text{ i.e., } f(t,\cdot) = \psi_t^*\big(f(0,\ .)\big)$$

and differentiating it according to t we find iv), namely

$$\partial_t f = \mathcal{L}_X f = X.\nabla f = |\nabla f|^2.$$

Analogously to the previous case, a Ricci soliton is shrinking if we can find potential f generated by vector field $X(p) \coloneqq -\partial_t \psi_t(p)$. By setting $\sigma(t) = (T-t)$ we have $g(t) = (T-t)\psi_t^*(g_0)$. Without loss of generality we write $(T-t)$ by σ.

Proposition 2.1.1.4. Suppose $g(t) = \sigma(t)\psi_t^*(g_0)$ is a shrinking gradient soliton with potential function $f, g(t) = (T - t)\psi_t^*(g_0)$. Then, f satisfies the equations

i) $Ric + Hess(f) - \frac{g}{2\sigma} = 0,$

ii) $R + \Delta f - \frac{n}{2\sigma} = 0,$

iii) $|\nabla f|^2 + R - \frac{f}{\sigma} = constant,$

iv) $\partial_t f = |\nabla f|^2 = -\partial_\sigma f$

(Muller, 2006).

Proof. Differentiating $g(t) = (T - t)\psi_t^*(g_0)$ and using the equation of Ricci flow implies

$$-2R_{ij} = -\frac{g_{ij}}{\sigma} + (\mathcal{L}_X g)_{ij} = -\frac{g_{ij}}{\sigma} + \nabla_i X_j + \nabla_j X_i,$$

hence $-2R_{ij} = 2\nabla_i\nabla_j f - \frac{g_{ij}}{\sigma}$, which implies i). To find the second equation, we can trace the first equation. Since $\nabla g = 0$, taking the covariant derivative of the first equation we get

$$\nabla_j R_{ij} - \nabla_i R + \frac{R_{ip}}{\nabla}_p f = 0,$$

where $R_{ip} = -\nabla_i\nabla_p f + \frac{g_{ip}}{2\sigma}$. As a result, we have

$$0 = -\frac{1}{2}\nabla_i R - \nabla_i\nabla_p f\nabla_p f - \frac{g_{ip}}{2\sigma}\nabla_p f = -\frac{1}{2}\nabla_i\left(R + |\nabla f|^2 + \frac{f}{\sigma}\right).$$

Hence the third equation is proven. Finally, the last one follows the steady case. Q.E.D.

The expanded Ricci soliton is analogous to the shrinking Ricci soliton, with a different sign. If $t > T$, and $g(t)$ is a expanding gradient Ricci soliton then $g(t) = (t - T)\psi_t^*(g_0)$ and we can find a potential function f generated by $X(p) \coloneqq \partial_t\psi_t(p)$. We usually write τ instead of $(t - T)$ in this case.

Replacing σ with $-\tau$ and ∂_σ by $-\partial_\tau$ in the previous proposition, we get

Proposition 2.1.1.4. Suppose $g(t)\psi_t^*(g_0)$ is a gradient expander soliton with potential $f, g(t) = (t - T)\psi_t^*(g_0)$. Then, f satisfies the equations

i) $Ric + Hess(f) + \frac{g}{2\tau} = 0,$

ii) $R + \Delta f + \frac{n}{2\tau} = 0,$

iii) $|\nabla f|^2 + R + \frac{f}{\tau} = constant,$

iv) $\partial_t f = |\nabla f|^2 = \partial_\tau f$

(Muller, 2006).

2.1.2. Eternal Solution

The eternal solution of a Ricci flow is a solution that exists time $-\infty < t < \infty$. On an eternal solution, one can look arbitrarily far backward or forwards in time without encountering any concentration phenomena. This implies that such a solution must be extremely stable, with no concentration of curvature at any finite point in the past or future.

One of the eternal solutions of Ricci flow is Hamilton's Cigar soliton. This soliton is complete Riemannian surface $(\mathbb{R}^2, g_\Sigma)$, which is given by the metric

$$g_\Sigma(t) = \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}.$$

To see that $g_\Sigma(t)$ is a Ricci flow solution, we compute its Ricci tensor with respect to the coordinate system $z_1 = x, z_2 = y$. The Christoffel symbols are computed by the formula

$$\Gamma_{ij}^k = \frac{1}{2}\sum_k g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

and it is easy to find

$$\Gamma_{11}^1 = -\frac{x}{e^{4t} + x^2 + y^2}, \qquad \Gamma_{12}^1 = -\frac{y}{e^{4t} + x^2 + y^2}, \qquad \Gamma_{22}^1 = \frac{x}{e^{4t} + x^2 + y^2}$$

$$\Gamma_{11}^2 = \frac{y}{e^{4t} + x^2 + y^2}, \qquad \Gamma_{12}^2 = -\frac{x}{e^{4t} + x^2 + y^2}, \qquad \Gamma_{22}^2 = -\frac{y}{e^{4t} + x^2 + y^2}.$$

In terms of Christoffel symbols, the Ricci tensor has components

$$R_{ij} = \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{kp}^k \Gamma_{ij}^p - \Gamma_{jp}^k \Gamma_{ik}^p.$$

Using Christoffel symbols for $g_\Sigma(t)$ we obtain

$$Ric(t) = \frac{2e^{4t}}{e^{4t} + x^2 + y^2}(dx^2 + dy^2).$$

On the other hand, we have

$$g'_\Sigma(t) = -\frac{4e^{4t}}{e^{4t} + x^2 + y^2}(dx^2 + dy^2).$$

Thus, $g_\Sigma(t)$ satisfies the Ricci flow equation.

By making the change of parameter group of diffeomorphisms $\psi: \mathbb{R}^2 \to \mathbb{R}^2$ by $\psi_t(e^{-2t}x, e^{-2t}y)$, and $\sigma(t) = \frac{1}{4}$, so that

$g_\Sigma(t) = \sigma(t)\psi_t^*(g_0)$ and $\dot{\sigma} = 0$.

Therefore, $g_\Sigma(t)$ is a steady eternal solution of Ricci flow and soliton.

Remark 2.2.2.1. The cigar soliton is actually a Kahler metric

$$g_\Sigma = \frac{dzd\bar{z}}{1 + |z|^2}$$

On $\mathbb{C} \approx \mathbb{R}^2$, and hence a Kahler-Ricci soliton. In fact, the cigar is the simplest representative of an entire family of Kahler-Ricci soliton that exists on $\mathbb{C}^{2m}$ for all $m \in \mathbb{N}$ and on certain other complex manifolds.

The cigar metric on the topological cylinder

$$C = \{(x, y) : x \in \mathbb{R}\ and\ y \in \mathbb{R}/\mathbb{Z}\}$$

is

$$\frac{dx^2 + dy^2}{1 + e^{-2x}}.$$

2.1.3. Ancient Solution

The ancient solution of the Ricci flow is a solution that exists on time interval $-\infty < t < \omega < \infty$. Some metrics, which are categorized into the ancient solution of Ricci flow, are Einstein manifold, the round sphere, and the Rosenau solution.

If (M, g) is Einstein, i.e., $Ric_g = \lambda g$, then $g(t) = (1 - 2\lambda t)g_0$ is a Ricci flow because

$$\frac{\partial}{\partial t} g(t) = -2\lambda g_0 = -2Ric(g_0) = -2Ric\big(g(t)\big).$$

When $\lambda > 0$ Ricci flow can only be defined up to time $t = \frac{1}{2\lambda}$. After which it becomes extinct.

Let g_{can} denote the standard round metric on S^n of radius 1, and taking the family of 1-parameter conformally equivalent to the metrics

$$g(t) \coloneqq r(t)^2 g_{can},$$

where $r(t)$ is to be determined. Observe that $g(t)$ is a Ricci flow solution iff

$$2r\frac{dr}{dt} g_{can} = \frac{\partial}{\partial t} g = -2Ric(g) = -2Ric[g_{can}] = -2(n-1)g_{can},$$

Note that an n-dimensional sphere of radius 1, the sectional curvatures are all 1. Thus, for any unit vector v, $Ric(v, v) = n - 1$ and

$$Ric(g) = (n-1)g.$$

Hence, if only if $r(t)$ is a solution of the ODE

$$\frac{dr}{dt} = -\frac{(n-1)}{r}$$

that is

$$r(t) = \sqrt{r_0 - 2(n-1)t},$$

where r_0 is the initial radius of the sphere. The manifold shrinks to a point as $t \to \frac{r_0^2}{2(n-1)}$ (Chow and Knopf, 2004).

Now, we will consider the Rosenau solution as the Ricci flow ancient solution. For each $t \in (-\infty, 0)$, we define a metric on $\mathbb{R}^2$ by

$$g(t) = \frac{8\sinh(-t)}{1 + 2\cosh(-t)\,|x|^2 + |x|^4} h,$$

where h is the flat metric on the manifold $\mathcal{M}^2 = \mathbb{R} \times S_1^1$ and for $x \in \mathbb{R}^2$. Note that $g(t)$ extends to a smooth metric on S^2. The scalar curvature of $g(t)$ is given by

$$Scal = \frac{\cosh(-t)}{\sinh(-t)} - \frac{2\sinh(-t)}{1 + 2\cosh(-t)\,|x|^2 + |x|^4}.$$

This implies

$$\frac{\partial}{\partial t} g(t) = -scal_{g(t)} g(t) = -2Ric_{g(t)}.$$

Consequently, the metric $g(t), t \in (-\infty, 0)$ form a Ricci flow solution (Brendle, 2010).

2.1.4. Immortal solution

The immortal solution of the Ricci flow is the solution that exists on time interval $-\infty < \alpha < t < -\infty$. One of the examples categorized into immortal Ricci flow solution which is expanding is hyperbolic n-space.

Let g_{can} denote the standard round metric on H^n of radius 1, and taking the family of 1-parameter which is conformally equivalent to the metrics

$$g(t) \coloneqq r(t)^2 g_{can},$$

where $r(t)$ is to be determined. Observe that $g(t)$ is a Ricci flow solution iff

$$2r\frac{dr}{dt} g_{can} = \frac{\partial}{\partial t} g = -2Ric(g) = -2Ric[g_{can}] = 2(n-1)g_{can},$$

Note that an n-dimensional hyperbolic of radius 1, the sectional curvatures are all 1. Thus, for any unit vector v, $Ric(v, v) = -(n-1)$ and

$$Ric(g) = -(n-1)g.$$

Hence, $r(t)$ is a solution of the ODE if only if

$$\frac{dr}{dt} = \frac{(n-1)}{r}$$

that is

$$r(t) = \sqrt{r_0 + 2(n-1)t},$$

where r_0 is the initial radius of the hyperbolic. The manifold expands to infinity (Brendle, 2010).

2.2. Euclidean Hypersurface as Ricci Soliton

Given m-dimensional Riemann manifold $(\bar{M}, \bar{g})$ and n-dimensional Riemann manifold (M, g). Suppose $\phi: M \to \bar{M}$ is immersion from M into $\bar{M}$ and ∇ and $\bar{\nabla}$ are the Levi-Civita connections on (M, g) and $(\bar{M}, \bar{g})$, respectively.

For any vector fields $X, Y \in \chi(M)$ and vector field N normal to M. Then, the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{70}$$

$$\bar{\nabla}_X N = -A_N X + D_X N, \tag{71}$$

respectively where $\nabla_X Y$ is the tangent component and $h(X, Y)$ is the normal components of $\bar{\nabla}_X Y$ while $-A_N X$ is the tangential component and $D_X N$ is the normal components of $\bar{\nabla}_X N$. h is the second fundamental form, A is the shape operator, and D is the normal connection of M in the ambient manifold $\bar{M}$. Note here that $(\bar{\nabla}_X Y)^T = \nabla_X Y, (\bar{\nabla}_X Y)^\perp = h(X, Y)$, $(\bar{\nabla}_X N)^T = -A_N X$ and $(\bar{\nabla}_X N)^\perp = D_X N$, where T and $\perp$ denote the tangent bundle and normal bundle, respectively (Al-Sodais, Deshmukh and Alodan, 2014).

For a normal vector $N \in T_p M^\perp, p \in M$, A_N is a self-adjoint endomorphism that has relation to the second fundamental form by

$$\bar{g}(h(X, Y), N) = g(A_N X, Y) \tag{72}$$

Then, for hypersurface in M, we have the Gauss formula and Weingarten formula as follow:

$$\bar{\nabla}_X Y = \nabla_X Y + g(A_N X, Y) N. \tag{73}$$

$$\overline{\nabla}_X N = -A_N X. \tag{74}$$

The mean curvature of hypersurface M in $\overline{\mathrm{M}}$ is defined by

$$H = \frac{1}{n} trace(h). \tag{75}$$

The equations of Gauss and Codazzi are respectively given by

$$g(R(X,Y),Z,W) = \bar{g}(\bar{R}(X,Y),Z,W) + \bar{g}\big(h(X,W),h(Y,Z)\big)$$
$$-\bar{g}\big(h(X,Z),h(Y,w)\big) \tag{76}$$

$$(\bar{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_X h)(X,Z) \tag{77}$$

for vectors X, Y, Z, W tangent to M, $(\bar{R}(X,Y)Z)^{\perp}$ is the normal component of $(\bar{R}(X,Y)Z)$ and $\overline{\nabla} h$ is defined by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{78}$$

The hypersurface Ricci tensor Ric is given by

$$Ric(X,Y) = n\alpha(A_N X, Y) - g(A_N X, A_N Y) \tag{79}$$

$$(\nabla A_N)(X,Y) = (\nabla A_N)(Y,X) \tag{80}$$

where $X, Y \in \chi(M)$ and $n\alpha = trace(A_N)$. The equation of Codazzi of the hypersurface is given by

$$(\nabla A_N)(X,Y) = (\nabla A_N)(Y,X), \tag{81}$$

where the covariant derivative $(\nabla A_N)(X,Y) = \nabla_X(A_N Y) - A_N(\nabla_X Y)$.

Now, assume an orientable Riemannian manifold (M,g) of dimension n is immersed into a Euclid $\mathbb{R}^{n+1}$ isometrically with immersion $\psi: M \to \mathbb{R}^{n+1}$. Then, ψ express the position of vector field of M in $\mathbb{R}^{n+1}$ so that it can be written as $\psi: t + \rho N$. Here $t \in \chi(M)$ is the tangential component on M and $\rho = g(\psi, N)$ is called support function. As a result, we have

$$\nabla_X t = X + \rho A_N X \text{ and } \nabla\rho = -A_N X \tag{82}$$

where $\nabla\rho$ denotes the support function gradient (Al-Sodais, Deshmukh, and Alodan, 2014).

2.2.1. Conditions for Ricci Solitons

Theorem 2.3.1.1. Given Mas an orientable hypersurface of the Euclidean space $\mathbb{R}^{n+1}$, immersion $\psi: M \to \mathbb{R}^{n+1}, \psi = t + \rho N$ and A_N as the shape operator. (M, g, t, λ) is a Ricci soliton if and only if we can find a constant λ satisfying

$$(A_N)^2 - (\rho + n\alpha)A_N + (\lambda - 1)I = 0 \tag{83}$$

where α is the mean curvature of the hypersurface. Moreover, M is also satisfying gradient Ricci soliton (Al-Sodais, Deshmukh, and Alodan, 2014).

Proof. Given an immersion $\psi: M \to \mathbb{R}^{n+1}$ of the hypersurface M. By using the equation (81) the Lie derivative $(\mathcal{L}_t g)(X, Y)$ is given by

$$\begin{aligned}
(\mathcal{L}_t g)(X, Y) &= t\big(g(X, Y)\big) - g(\mathcal{L}_t X, Y) - g(X, \mathcal{L}_t Y) \\
&= -g(\nabla_X t, Y) - g(X, \nabla_Y t) - g(\nabla_t X - \nabla_X t, Y) - g(X, \nabla_t Y - \nabla_Y t) \\
&= g(\nabla_X t, Y) + g(X, \nabla_Y t) \\
&= g(X + \rho A_N X, Y) + g(X, Y + \rho A_N Y) \\
&= 2g(X, Y) + \rho g(A_N X, Y) + \rho g(X, A_N Y) \\
&= 2g(X, Y) + 2\rho g(A_N X, Y).
\end{aligned}$$

Therefore, together with the equation (79), we may have,

$$\begin{aligned}
Ric(X, Y) + \tfrac{1}{2}(\mathcal{L}_t g)(X, Y) &= n\alpha(A_N X, Y) - g(A_N X, A_N Y) + g(X, Y) + \rho g(A_N X, Y) \\
&= g(X, Y) + (\rho + n\alpha)g(A_N X, Y) - g((A_N)^2 X, Y) \\
&= g(X, Y) + g\Big(\big((\rho + n\alpha)A_N - (A_N)^2 + 1\big)(X), Y\Big).
\end{aligned}$$

Assume that we have the equation (83), then we can find a constant λ so that A_N fulfils the equation (83), then, the last equation above gives

$$g\big(((A_N)^2 - (\rho + n\alpha)A_N + (\lambda - 1)I)(X), Y\big) = 0$$

$$g\big(((A_N)^2 - (\rho + n\alpha)A_N - 1)(X), Y\big) + \lambda g(X, Y) = 0$$

$$g\big((-(A_N)^2 + (\rho + n\alpha)A_N + 1)(X), Y\big) = \lambda g(X, Y)$$

$$Ric(X, Y) + \frac{1}{2}(\mathcal{L}_t g)(X, Y) = \lambda g(X, Y).$$

Hence, M is a Ricci soliton with the potential vector field t.

Conversely, let (M, g, t, λ) be the Ricci soliton. By working from the end part, we have the equation (83).

In addition, $M$is also satisfying the condition of gradient Ricci soliton. Taking $f = \frac{1}{2}\|\psi\|^2$, we have $X(f) = g(X, t), X \in \chi(M)$, which implies $t = \nabla f$. This shows that M has a potential function f which implies that M is gradient Ricci solitons.

Theorem 2.3.1.2. Given Mas an orientable hypersurface of the Euclidean space $\mathbb{R}^{n+1}$, support function ρ and shape operator A_N. (M, g, ρ, λ) is a gradient Ricci soliton iff we can find a constant λ satisfying

$$(\nabla A_N)(X, t) + (1 - n\alpha)A_N X + (1 + \rho)(A_N)^2 X = -\lambda X \tag{84}$$

(Al-Sodais, Deshmukh and Alodan, 2014).

Proof. Given an immersion $\psi: M \to \mathbb{R}^{n+1}$ of the hypersurface M. By applying the equation (82), the Hessian $Hess_\rho$ can be calculated as follow

$$\begin{aligned} Hess_\rho(X, Y) &= g\big(Hess_\rho X, Y\big) = g(\nabla_X \nabla\rho, Y) \\ &= g(\nabla_X(-A_N t), Y) \\ &= -g(\nabla_X A_N t - A_N X - \rho(A_N)^2 X + A_N X + \rho(A_N)^2 X, Y) \\ &= -g(\nabla_X A_N t - A_N(X + \rho A_N X) + A_N X + \rho(A_N)^2 X, Y) \\ &= -g(\nabla_X A_N t - A_N(\nabla_X t) + A_N X + \rho(A_N)^2 X, Y) \\ &= -g\big((A_N)(X, t) + A_N X + \rho(A_N)^2 X, Y\big). \end{aligned}$$

Therefore,

$$\begin{aligned} &Ric(X, Y) + Hess_\rho(X, Y) \\ &\qquad = n\alpha g(A_N X, Y) - g((A_N)^2 X, Y) - g((\nabla A_N)(X, t) + A_N X \\ &\qquad + \rho(A_N)^2 X, Y) \end{aligned}$$

$$= -g\big((\nabla A_N)(X,t),Y\big) - g\big((1-n\alpha)(A_N X) - (1+\rho)(A_N)^2 X, Y\big)$$

$$= -g\big((\nabla A_N)(X,t),Y\big) + g((1-n\alpha)(A_N X) + (1+\rho)(A_N)^2 X, Y) \qquad (85)$$

In case we find a constant λ satisfying the condition given in equation (84), we obtain

$Ric\,(X,Y) + Hess_\rho(X,Y) = \lambda g(X,Y)$.

Conversely, assume that (M,g,ρ,λ) is a gradient Ricci soliton. By applying the equation (85) above and working backward, we can easily find the equation (84).

Next, let M be a hypersurface that is orientable in $S^{n+1}(1)$ with immersion $\psi\colon M \to S^{n+1}$ and Z be vector field which is constant on the Euclidean space $\mathbb{R}^{n+2}$. If $\overline{N}$ is a unit normal vector field on $S^{n+1}(1)$ in $\mathbb{R}^{n+2}$, we will have smooth functions f and σ on the hypersurface M defined by $f = g(Z,M\,)|_M$ and $\sigma = g(Z,\overline{N})|_M$. As a result, we can write vector field Z as $Z|_M = u + fN + \sigma\overline{N}$, for $u \in \chi(M)$ (Al-Sodais, Deshmukh and Alodan, 2014).

Theorem 2.3.1.3. Given an orientable hypersurface$M$ of S^{n+1}. Let $\psi\colon M \to S^{n+1}$ be immersion and Z be a constant vector field on $\mathbb{R}^{n+2}$. Then, (M,g,ρ,λ) is a Ricci soliton iff it satisfies $(A_N)^2 - (n\alpha + f)A_N + (\rho + 1 + \lambda - n)I = 0$. Furthermore, with the potential function σ, M satisfies the condition of gradient Ricci soliton (Al-Sodais, Deshmukh, and Alodan, 2014).

Proof. Suppose that $\nabla, \overline{\nabla}$ and D are the Riemannian connection on M, S^{n+1} and $\mathbb{R}^{n+2}$, respectively and Z be a constant constant vector field on $\mathbb{R}^{n+2}$. Then it can be written as

$$Z|_{S^{n+1}} = \xi + \sigma\overline{N} \qquad (86)$$

where $\xi|_M = u + fN$ is the tangential component of Z. If we take the covariant derivative of the equation (86) , we obtain

$$D_X Z = 0 = D_X(\xi + \sigma\overline{N}), \qquad X \in \chi(S^{n+1}).$$

which gives $\overline{\nabla}_X\xi - g(X,\xi)\overline{N} + X(\sigma)\overline{N} + \sigma X = 0$. Thus, equating tangential and normal components on $S^{n+1}(1)$ as hypersurface of $\mathbb{R}^{n+2}$, $\overline{\nabla}_X\xi = -\sigma X$, $X(\sigma) = g(X,\xi), X \in \chi(S^{n+1})$. Therefore, $\overline{\nabla}\sigma = \xi$, with $\overline{\nabla}\sigma$ on S^{n+1}. Next, since $\overline{\nabla}_X\xi = -\sigma X$ and substituting in $\xi = u + fN$ we obtain $\overline{\nabla}_X\xi = \overline{\nabla}_X u + X(f)N - fA_N X$ which implies $-\sigma X = \nabla_X u + g(A_N u, X)N + X(f) - fA_N X$. By this, equating tangential and normal components of M, we obtain $\nabla_X u = -\sigma X + f A_N X, \nabla f = -A_N u$, where A_N is the shape operator of M. Next, the Lie derivative can be computed as

$$(\mathcal{L}_u g)(X,Y) = -2\sigma g(X,Y) + 2fg(A_N X,Y), \qquad X,Y \in \chi(M).$$

Applying the equation of Gauss for M in S^{n+1} yields

$$Ric(X,Y) = (n-1)g(X,Y) + n\alpha g(A_N X,Y) - g(A_N X, A_N Y).$$

Therefore, by these two equations, we get

$$Ric(X,Y) + \frac{1}{2}(\mathcal{L}_u g)(X,Y)$$
$$= (n-1-\sigma)g(X,Y) + (n\alpha + f)g(A_N X,Y) - g((A_N)^2 X,Y)$$

Assume that we have a constant λ satisfying the given condition, then we easily find $Ric(X,Y) + \frac{1}{2}(\mathcal{L}_u g)(X,Y) = \lambda g(X,Y)$ which implies that (M,g,u,λ) is a Ricci soliton. By working backward, the converse if the part above is obvious.

Furthermore, consider that $\overline{\nabla}\sigma = \xi$. As a consequet, for $X \in \chi(M), X(\sigma) = g(X,\xi) = g(X, u + fN) = g(X,u)$, that is $u = \nabla\rho$. Therefore, M satisfies gradient Ricci soliton.

2.2.2. Compact Ricci Solitons

Theorem 2.3.2.1. Given a compact steady Ricci soliton(M,g,t,λ). Then, we cannot find any isometric immersion $\psi: M \to \mathbb{R}^{n+1}$ given by $\psi = t + \rho N$ (Al-Sodais, Deshmukh and Alodan, 2014).

Proof. Given an isometric immersion $\psi: M \to \mathbb{R}^{n+1}$, then since $f = \frac{1}{2}\|\psi\|^2$ implies $t = \nabla f$ and (M,g,t,λ) is the gradient Ricci soliton. For steady Ricci soliton, we have

$$\Delta f = -Scal, \tag{87}$$

where $Scal$ and Δf denote the scalar curvature of M and the Laplacian of the smooth function f. By applying the equation (80) together with the Schwarz inequality $\|B\|^2 \geq n\alpha^2$, we find $S \leq n\alpha^2(n-1)$, which together with the equation (87) gives $\Delta f \geq n\alpha^2(n-1)$.

Taking the integral of this equation over the compact manifold M, we get $0 \geq \int n\alpha^2(n-1)dv$, which gives us $\alpha = 0$. As a result, the hypersurface is minimal which implies a contradiction as there is no compact minimal hypersurface in $\mathbb{R}^{n+1}$ and this completed the proof.

Theorem 2.3.2.2. Let (M, g, t, λ) be a compact Ricci soliton which is immersed into $\mathbb{R}^{n+1}$ isometrically by $\psi: M \to \mathbb{R}^{n+1}, \psi = t + \rho N$ and α be the constant mean curvature of hypersurface of M. Then $\lambda \leq (n-1)\alpha^2$. (Al-Sodais, Deshmukh and Alodan, 2014).

Proof. Taking the trace of the gradient Ricci soliton equation in Riemannian space M yields $Scal + \Delta f = \lambda n$ or $\lambda n - \Delta f = n^2\alpha^2 - \|B\|^2$. Applying the Schwarz inequality to this equation yields

$$\lambda n \leq n\alpha^2(n-1) + \Delta f \tag{88}$$

Taking the integral of the above equation and assuming α constant, the proof is completed.

Theorem 2.3.2.3. Given a compact Ricci soliton (M, g, t, λ) which is isometrically immersed into $\mathbb{R}^{n+1}$by $\psi: M \to \mathbb{R}^{n+1}$, $\psi = t + \rho N$. The mean curvature α is a constant and M is isometric to $S^n(c)$ where $c = \alpha^2$ if α satisfies $\lambda \geq (n-1)\alpha^2$ (Al-Sodais, Deshmukh and Alodan, 2014).

Proof. Since M is immersed into $\mathbb{R}^{n+1}$ isometrically, from equation (88) we have $0 \leq \int \big((n-1)\alpha^2 - \lambda\big)dv$. If we assume the condition holds, the inequality above gives us $(n-1)\alpha^2 = \lambda$, which implies that α is a constant. Since $t = \nabla f, f = \frac{1}{2}\|\psi\|^2$ is the potential vector field, applying the above value of λ of the gradient Ricci soliton in the previous theorem with potential function f, we have

$$\Delta f = \|B\|^2 - n\alpha^2 \tag{89}$$

It can be seen that the Schwarz inequality gives us $\|B\|^2 \geq n\alpha^2$ and the equality holds iff $B = \alpha I$. Taking the integral of the equation (89) and applying the Schwarz inequality we have $B = \alpha I$. Therefore, the hypersurface M is totally umbilical and isometric to the round sphere $S^n(\alpha^2)$.

2.3. Ricci Soliton on Lorentzian manifolds

Definition 2.4.1. A semi-Riemannian manifold (M, g) is a Ricci soliton if it admits a smooth vector field X satisfying

$$\mathcal{L}_X g + Ric = \lambda g, \tag{90}$$

where $\mathcal{L}_X$ is the Lie derivative in the direction of X, Ric denotes the Ricci tensor and λ is a real number (Vazquez, et.al., 2012).

Definition 2.4.2. A Ricci soliton (M, g) is called gradient Ricci soliton if it admits a vector field X satisfying $X = grad\ f$, for some smooth function f. In this case, equation (90) can be expressed by

$$2\ Hes_f + Ric = \lambda g, \tag{91}$$

where Hes_f denotes the Hessian of f (Vazquez, et.al., 2012).

A gradient Ricci soliton on a semi-Riemannian manifold with potential function f satisfies the equation

$$Sc + 2\|grad\ f\|^2 - 2\lambda f = C, \tag{92}$$

where C is a constant. As a result, when the scalar curvature is constant, by performing the rescaling of the potential function, can find $\|grad\ f\|^2 = \lambda h$, which implies that the soliton is steady if and only if $grad\ f$ to be a null vector field on (M, g).

Riemannian manifolds equipped with an isometries group of dimensions at minimum $\frac{1}{2}n(n-1) + 1$ are of constant curvature or products of constant curvature space of dimension (n-1) with a line or a circle. In this thesis, we will provide examples of Ricci soliton on the Lorentzian manifold.

Definition 2.4.3. Egorov space is a space in Lorentzian manifold (M, g_f) equipped with Egorov metric defined as

$$g_f = (u, v, x_1, \dots, x_n) = dudv + f(u) \textstyle\sum_{i=1}^{n} (dx_i)^2, \tag{93}$$

where f is a positive real-valued function with a real variable (Vazquez, et.al., 2012).

Let $(\mathbb{R}^{n+2}, g_f), n \geq 1$ be an Egorov space. Denote ∇ the Levi Civita connection of $(\mathbb{R}^{n+2}, g_f)$ and R is the curvature tensor defined by

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \tag{94}$$

According to the basis of coordinate vector fields $\left\{\partial_i = \frac{\partial}{\partial x_i}\right\}$, so that the metric components are

$$\begin{cases} g_{f_{ij}} = \delta_{ij} f \quad for\ i,j \leq n \\ \quad g_{f_{uu}} = g_{f_{vv}} = 0 \\ \quad g_{f_{uv}} = g_{f_{vu}} = 1. \end{cases} \tag{95}$$

By equation (95), a direct calculation, the only non-vanishing Christoffel symbols are

$$\Gamma_{ij}^{v} = -\frac{f'}{2},\ \Gamma_{iv}^{i} = \frac{f'}{2f},\ i = 1, \dots, n-2 \tag{96}$$

Therefore, the non-vanishing covariant derivatives of the coordinate vector fields are

$$\nabla_{\partial_i}\partial_i = -\frac{f'}{2}\partial_v,\ \nabla_{\partial_i}\partial_u = \frac{f'}{2f}\partial_i,\ i = 1, \dots, n \tag{97}$$

Using the equations (94) and (97) the only non-vanishing curvature components are

$$R_{iui}^{v} = \frac{1}{4f}[(f')^2 - 2ff''],\ R_{iuu}^{i} = -\frac{1}{4f^2}[(f')^2 - 2ff''],\ i = 1, \dots, n-2. \tag{98}$$

As a result, the components of the $(4-0)$ curvature tensor $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, with respect to $\{\partial_i\}$ are given by

$$\begin{cases} R_{iuiu} = \frac{1}{4f}[(f')^2 - 2ff''],\ \ i = 1, \dots, n \\ \qquad R_{ijkh} = 0,\ \ \text{otherwise}. \end{cases} \tag{99}$$

We can now calculate the components of the covariant derivative ∇R with respect to $\{\partial_i\}$. By equations (97) and (98), we easily get

$$(\nabla_{\partial_u} R)(\partial_i, \partial_u, \partial_u) = \frac{1}{2f^3}[(f')^3 + f^2 f''' - 2ff'f'']\,\partial_i\ , i = 1, \dots, n$$

$$(\nabla_{\partial_u} R)(\partial_i, \partial_u, \partial_i) = -\frac{1}{2f^2}[(f')^3 + f^2 f''' - 2ff'f'']\partial_{n-1}\ ,\ i = 1 \dots, n$$

$$(\nabla_{\partial_i} R)(\partial_j, \partial_k, \partial_u) = 0, \quad \text{otherwise}.$$

Note that $\frac{(f')^3+f^2f'''-2ff'f'''}{2f^3}$ is the first derivative of $-\frac{(f')^2-2ff''}{4f^2}$. Next, the Ricci tensor Ric is derived by contracting the curvature tensor such that

$$Ric(X,Y) = tr(R(X,.,Y,.)$$

and in local coordinate, it can be written as

$$Ric_{ij} = Ric\left(\partial_i, \partial_j\right) = \Sigma R^m_{imj}.$$

Therefore, by equation (98) we then obtain

$$Ric_{uu} = \frac{n}{4f^2}[(f')^2 - 2ff''],\ Ric_{ij} = 0\ aksi\ takdirde. \tag{100}$$

Moreover, from equations (97) and (100) we get

$$\nabla_u Ric_{uu} = -\frac{n}{2f^3}[(f')^3 + f^2f''' - 2ff'f''], \qquad \nabla_i Ric_{jk} = 0\ otherwise.$$

Next, we will show that Egorov metric is Ricci soliton and gradient Ricci solitons.

Theorem 2.4.4. Let $\left(\mathbb{R}^{n+2}, g_f\right)$ be Egorov space, then Egorov space is Ricci soliton (Vazquez, et.al., 2012).

Proof. Suppose that g_f is Ricci soliton and $X = \sum_{l=u,v,1,\ldots,n} X_l\partial_l$ be a vector field on $\left(\mathbb{R}^{n+2}, g_f\right)$. Recall the Lie derivative of (0,2) tensor g_{ij} is given by

$$\mathcal{L}_X g_{ij} = X_m\partial_m g_{ij} + g_{il}\partial_j X_l + g_{kj}\partial_i X_k$$

Consequently, for $1 \leq i \neq j \leq n$

$$\begin{aligned}\mathcal{L}_X g_{ij} &= X_m\partial_m g_{ij} + g_{il}\partial_j X_l + g_{kj}\partial_i X_k \\ &= g_{ii}\partial_j X_i + g_{jj}\partial_i X_j \\ &= f\partial_j X_i + f\partial_i X_j.\end{aligned}$$

Since $R_{ij} = 0$ and $g_{ij} = 0$, then the equation (91) becomes

$$\begin{aligned}\mathcal{L}_X g_{ij} + R_{ij} = \lambda g_{ij} \Rightarrow \mathcal{L}_X g_{ij} &= 0 \\ &\Rightarrow f\partial_j X_i + f\partial_i X_j = 0 \\ &\Rightarrow \partial_j X_i + \partial_i X_j. \end{aligned} \tag{101}$$

For $1 \leq i \leq n$

$$\mathcal{L}_X g_{iu} = X_m \partial_m g_{iu} + g_{il} \partial_u X_l + g_{ku} \partial_i X_k$$

$$= g_{ii} \partial_u X_i + g_{vu} \partial_i X_v$$

$$= f \partial_u X_i + \partial_i X_v.$$

Since $R_{iu} = 0$ and $g_{iu} = 0$, then the equation (91) becomes

$$\mathcal{L}_X g_{iu} + R_{iu} = \lambda g_{iu} \Rightarrow \mathcal{L}_X g_{iu} = 0$$

$$\Rightarrow f \partial_u X_i + \partial_i X_v = 0 \tag{102}$$

Next,

$$\mathcal{L}_X g_{iv} = X_m \partial_m g_{iv} + g_{il} \partial_v X_l + g_{kv} \partial_i X_k$$

$$= g_{ii} \partial_v X_i + g \partial_i X_u$$

$$= f \partial_v X_i + \partial_i X_u.$$

Since $R_{iv} = 0$ and $g_{iv} = 0$, then the equation (91) becomes

$$\mathcal{L}_X g_{iv} + R_{iv} = \lambda g_{iv} \Rightarrow \mathcal{L}_X g_{iv} = 0$$

$$\Rightarrow f \partial_v X_i + \partial_i X_u = 0. \tag{103}$$

For $1 \leq i \leq n$

$$\mathcal{L}_X g_{ii} = X_m \partial_m g_{ii} + g_{il} \partial_i X_l + g_{ki} \partial_i X_k$$

$$= X_u \partial_u g_{ii} + g_{ii} \partial_v X_i + g_{uv} \partial_i X_u$$

$$= X_u f' + 2f \partial_i X_i.$$

Since $R_{ii} = 0$ and $g_{ii} = f$, then the equation (91) becomes

$$\mathcal{L}_X g_{ii} + R_{ii} = \lambda g_{iv} \Rightarrow \mathcal{L}_X g_{ii} = \lambda f$$

$$\Rightarrow f X_u f' + 2f \partial_i X_i = \lambda f. \tag{104}$$

Next,

$$\mathcal{L}_X g_{uu} = X_m \partial_m g_{uu} + g_{ul} \partial_u X_l + g_{ku} \partial_u X_k$$

$$= g_{uv} \partial_u X_v + g_{vu} \partial_u X_v$$

$$= 2\partial_u X_v.$$

Since $R_{uu} \neq 0$ and $g_{uu} = 0$, then the equation (91) becomes

$$\mathcal{L}_X g_{uu} + R_{uu} = \lambda g_{uu} \Rightarrow Ric_{uu} + 2\partial_u X_v = 0. \tag{105}$$

Now,

$$\mathcal{L}_X g_{uv} = X_m \partial_m g_{uv} + g_{ul}\partial_v X_l + g_{kv}\partial_u X_k$$

$$= g_{uv}\partial_v X_v + g_{uv}\partial_u X_u$$

$$= \partial_v X_v + \partial_u X_u.$$

Since $R_{uv} \neq 0$ and $g_{uv} = 1$, then the equation (91) becomes

$$\mathcal{L}_X g_{uv} + R_{uv} = \lambda g_{uv} \Rightarrow \partial_v X_v + \partial_u X_u = \lambda. \tag{106}$$

With the same way

$$Ric_{vv} + \mathcal{L}_X g_{vv} = \lambda g_{vv}$$

$$2\partial_v X_u = 0$$

$$\partial_v X_u = 0. \tag{107}$$

To show that Egorov space is Ricci soliton, take

$$X = \left(-\frac{1}{2}\int Ric_{uu}\, du + \lambda v\right)\partial_v + \sum_{i=1}^{n} \frac{\lambda}{2} x_i \partial_i.$$

It is clear that X satisfies the equations (l11) to (l17). Consequently, Egorov space is Ricci soliton.

Theorem 2.4.5. Egorov space $\left(\mathbb{R}^{n+2}, g_f\right)$ is a gradient Ricci soliton iff its metric satisfies the following equations.

$$\begin{cases} f'\partial_v h + 2\partial_{ii}^2 h = \lambda f\,, \;\; 1 \leq i \leq n \\ 2\partial_{iu}^2 h - \frac{f'}{f}\partial_i h = 0, \;\; 1 \leq i \leq n \\ 2\partial_{uu}^2 + Ric_{uu} = 0 \\ \partial_{uv}^2 h = \frac{\lambda}{2} \\ \partial_{ij}^2 h = \partial_{iv}^2 h = \partial_{uv}^2\, h = 0\,, \;\; 1 \leq i \neq j \leq n \end{cases} \tag{108}$$

(Vazquez, et.al., 2012).

Proof. ($\Rightarrow$) Given Egorov space $\left(\mathbb{R}^{n+2}, g_f\right)$ as a gradient Ricci soliton and $X = grad\ h$ as a vector field on $\left(\mathbb{R}^{n+2}, g_f\right)$ with potential function h. In the coordinates of Egorov space we have $grad\ h = \left(\partial_v h, \partial_u h, \frac{1}{f}\partial_i h, \dots \frac{1}{f}\partial_n h\right)$.

From equation (104), we have

$$X_u f' + 2f\partial_i X_i = \lambda f \Rightarrow f'\partial_v h + 2f\partial_i\left(\frac{1}{f}\partial_i h\right) = \lambda f \Rightarrow f'\partial_v h + 2\partial_{ii}^2 = \lambda f$$

From equation (105), we have

$$Ric_{uu} + 2\partial_u X_v = 0 \Rightarrow Ric_{uu} + 2\partial_{uu}^2 h = 0.$$

From equation (101), we have

$$\partial_i X_j + \partial_j X_i = 0 \Rightarrow \partial_{ij}^2 h + \partial_{ji}^2 h = 0 \Rightarrow 2\partial_{ij}^2 h = 0 \Rightarrow \partial_{ij}^2 = 0.$$

From equation (103), we have

$$\partial_i X_u + f\partial_v X_i = 0 \Rightarrow \partial_{iv}^2 h + f\partial_v\left(\frac{1}{f}\partial_i h\right) = 0 \Rightarrow \partial_{iv}^2 h + \partial_{iv}^2 h = 0 \Rightarrow \partial_{iv}^2 = 0.$$

From equation (107), we have

$$\partial_v X_u = 0 \Rightarrow \partial_{vv}^2 h = 0.$$

($\Leftarrow$) Let the equation (108) is held. Integrating the equation (108), we find that the potential function can be separated as $h(u, v, x_1, \dots x_n) = h_0(u, x_1, \dots, x_n) + h_1(u)v$ for some functions h_0, h_1 . Moreover, equations $\partial_{uv}^2 h = \frac{\lambda}{2}$ and $\partial_{ij}^2 h = 0$ so that $h(u, v, x_1, \dots, x_n) = h_0(u) + \Sigma_i h_i(u, x_i) + \left(\frac{\lambda}{2}u + \kappa\right)v$ for some constant κ and functions $h_i, i = 1, \dots, n$. Therefore, equation (108) can be reduced to,

$$\begin{cases} \left(\kappa + \frac{\lambda}{2}u\right)f' + 2\partial_{ii}^2 h_i = \lambda f, & 1 \le i \le n, \\ 2f\partial_{iu}^2 h_i = f'\partial_i h_i, & 1 \le i \le n, \\ Ric_{uu} + 2h_0'' + 2\Sigma_i \partial_{uu}^2 h_i = 0 & \end{cases} \tag{109}$$

Taking the integral of the first equations in (109), ones have

$$h_i(u, x_i) = \frac{1}{8}x_i^2(2\lambda f - (2\kappa + u\lambda)f') + x_i h_i(u) + k_i(u),$$

for some functions $h_i(u)$ and $k_i(u)$. Substituting the above into equation (109) and deriving the second equation $2f\partial_{iiu}^3 h_i = f'\partial_{ii}^2 h_i$, one gets

$$\left(\frac{\lambda}{2}u + \kappa\right)((f')^2 - 2ff'') = 0.$$

This equation allows us to take $\lambda = \kappa = 0$ only if the manifold is flat. Therefore, the potential function can be written as $h(u, v, x_1, \dots, x_n) = h_0(u) + \Sigma_i x_i h_i(u)$. Next, by the second equation in (109), we have

$$2fh_i' = f'h_i,$$

and hence $h_i(u) = c_i\sqrt{f(u)}$ for some constants c_i. After this, taking the derivative of the last equation in (109) according to x_i, we have $0 = c_i((f')^2 - 2ff'')$ so that all c_i must be zero. Therefore, the potential function can be expressed as a function of the single variable $u, h(u, v, x_1, \dots, x_n) = h_0(u)$. Finally, from the last constrain in (109) we find $Ric_{uu} + 2h_0'' = 0$. As a result, we can state the following corollary.

Corollary 2.4.1. With potential function $h(u)$ given by $h'' = -\frac{1}{2}Ric_{uu}$, non-flat Egorov space is gradient steady Ricci soliton (Vazquez, et.al., 2012).

Cahen-Wallach Symmetric Spaces

Definition 2.4.4. Cahen-Wallach space $(\mathbb{R}^{n+2}, g_{cw})$ is Lorentzian space, which is defined by metric tensor

$$g_{cw}(u, v, x_1, \dots, x_n) = (\textstyle\sum_{i=1}^n \kappa_i x_i^2)(du)^2 + dudv + \sum_{i=1}^n (dx_i)^2 \tag{110}$$

where $i = 1, \dots, n$ are non-zero real numbers (Vazquez, et.al., 2012).

The non zero Levi-Civita connection of the metric are given by

$$\nabla_{\partial_u}\partial_u = -\Sigma_i \kappa_i x_i \partial_i, \qquad \nabla_{\partial_u}\partial_i = \nabla_{\partial_i}\partial_u = \kappa_i x_i \partial_v.$$

Therefore, we have

$$R(\partial_u, \partial_i, \partial_u, \partial_i) = -\kappa_i, \qquad i = 1, \dots, n$$

and otherwise is zero. The Ricci tensor satisfies $Ric_{uu} = -\sum_{i=1}^{n} \kappa_i$, the other terms being zero. Furthermore, the scalar curvature is also 0. ε-space is a flat Cahen-Wallach symmetric space where $\kappa_1 = \cdots = \kappa_n$.

Theorem 2.4.6. Cahen-Wallach symmetric spaces are Ricci solitons (Vazquez, et.al., 2012).

Proof. Given a vector field $X = X_u\partial_u + X_v\partial_v + \Sigma_i X_i\partial_i$ on $\mathbb{R}^{n+2}$. By direct calculation the Lie derivatives of the components of the metric g_{cw} are given by

$$\mathcal{L}_X g_{cw} = \begin{bmatrix} 2(\Sigma_i\kappa_i x_i X_i + (\Sigma_i\kappa_i x_i^2)\,\partial_u X_u + \partial_u X_v) & A_{uv} & A_{uj} \\ A_{uv} & 2\partial_v X_u & \partial_j X_u + \partial_v X_j \\ A_{uj} & \partial_j X_u + \partial_v X_j & A_{jk} \end{bmatrix} \tag{111}$$

where

$A_{uv} = \partial_u X_u + \partial_v X_v + (\Sigma_i\kappa_i x_i^2)\,\partial_v X_v;$

$A_{uj} = (\Sigma_i\kappa_i x_i^2)\partial_j X_u + \partial_j X_v + \partial_u X_j;$

$A_{jk} = \partial_j X_k + \partial_k X_j.$

Note that $\lambda = \left(\frac{2}{n+2}\right) div\,(X)$ and $div\,(X) = \partial_u X_u + \partial_v X_v + \Sigma_i\partial_i X_i$. Thus, the Ricci soliton of equation (110) is equivalent to

$$\begin{cases} \Sigma_i\kappa_i - 2\Sigma_i\kappa_i x_i X_i - 2\partial_u X_v - (\Sigma_i\kappa_i x_i^2)(2\partial_u X_u - \lambda) = 0, \\ \partial_v X_u = 0, \\ (\Sigma_i\kappa_i x_i^2)\partial_v X_u + (\partial_u X_u + \partial_v X_v) - \lambda = 0, \\ (\Sigma_i\kappa_i x_i^2)\partial_j X_u + \partial_j X_v + \partial_u X_j = 0, \\ \partial_j X_u + \partial_v X_j = 0, \\ 2\partial_j X_j - \lambda = 0, \\ \partial_j X_k + \partial_k X_j = 0. \end{cases}$$

where $j, k = 1, \dots, n$ and $j \neq k$.

It can be seen that

$$X_{(u,v,x_1,\dots,x_n)} = \left(0, \frac{1}{2}(\Sigma_i\kappa_i)u + \lambda v, \frac{\lambda}{2}x_1, \dots, \frac{\lambda}{2}x_n\right)$$

is a solution of the previous system where λ is a constant. The choice of this particular spacelike vector fields family implies that $(\mathbb{R}^{n+2}, g_{cw})$ is an expanding, steady, or shrinking Ricci soliton.

Theorem 2.4.7. All the Cahen-Wallach symmetric spaces are steady gradient Ricci solitons (Vazquez, et.al., 2012).

Proof. Given a function h on $\mathbb{R}^{n+2}$. Then, the gradient of the function h according to the metric given in equation (110) is

$$grad(h) = (\partial_v h, (-\Sigma_i \kappa_i x_i^2)\partial_v h + \partial_v h, \partial_1 h, \dots, \partial_n h)$$

and thus equation (111) becomes

$$\begin{cases} \Sigma_i \kappa_i - 2\Sigma_i \kappa_i x_i \partial_i h - 2\partial_{uu}^2 h + (\Sigma_i \kappa_i x_i^2)\lambda = 0, \\ \partial_{vv}^2 = 0, \\ 2\partial_{uv}^2 h - \lambda = 0, \\ \kappa_j x_j \partial_v h - \partial_{uj}^2 h = 0, \\ \partial_{vj}^2 h = 0, \\ 2\partial_{jj}^2 h - \lambda = 0, \\ \partial_{jk}^2 h = 0. \end{cases} \tag{112}$$

where $j, k = 1, \dots, n, j \neq k$.

Using that $\partial_{vv}^2 h = \partial_{vj}^2 h = \partial_{jk}^2 h = 0, j \neq k$, the function h separates variables $v, x_1, .., x_n$ and the previous system reduces to

$$\begin{cases} \Sigma_i \kappa_i - 2\Sigma_i \kappa_i x_i \partial_i h - 2\partial_{uu}^2 h + (\Sigma_i \kappa_i x_i^2)\lambda = 0, \\ 2\partial_{uv}^h - \lambda = 0, \\ \kappa_j x_j \partial_v h - \partial_{uj}^2 h = 0, \\ 2\partial_{jj}^2 h = 0. \end{cases} \tag{113}$$

where $j = 1, \dots, n$.

Taking the derivative of the third equation gives us $\kappa_j x_j \partial_{uv}^2 h - \partial_{uuj}^3 h = 0$ and by second equation we have

$$2\partial_{uuj}^3 h = \kappa_j x_j \lambda. \tag{114}$$

Next, taking the derivative of the first equation with respect to x_j, we get

$$-2\kappa_j\partial_j h - 2\kappa_j x_j \partial_{jj}^2 h + 2\kappa_j x_j \lambda - 2\partial_{uuj}^3 h = 0.$$

By using the equations (113) and (114) to simplify the previous identity we have

$$\kappa_j \partial_j h = 0.$$

This indicates that h is independent of the variable x_j $(j = 1,..,n)$. Simplify the third equation of (113) we find $\partial_v h = 0$, so that h is the only function of the variable $u,\ h = h(u)$, and by equation (113) it can be reduced to

$$\Sigma\kappa_i - 2h''(u) = 0$$

Taking the integral of the equation above we find

$$h(u) = c_1 + c_2 u + \frac{1}{4}\Sigma_i \kappa_i u^2$$

where c_1 and c_2 are constants as the general solution of the system of the equations (113). It can be seen that $grad\ (h)$ is a null vector field satisfying equation (110) for $\lambda = 0$; indeed $\lambda = 2\frac{div(grad\ (h))}{n+2} = \Delta h = 0.$

2.4. Ricci Solitons on Lightlike Hypersurfaces

Definition 2.5.1. Given a differentiable Riemann manifold M with a linear connection ∇. A tangent vector field v is said concircular with respect to ∇ if

$$\nabla_X v = \mu X,\ X \in \Gamma(TM), \tag{114}$$

where $\mu: M \to \mathbb{R}$ is a smooth function. If $\mu = 0$ or $\mu \neq 0$, the concircular vector field v is said to be non-trivial or trivial 0, respectively (Al-Dayel, 2020).

The concept of concircular vector fields can also be extended to the semi-Riemannian manifolds exactly in the same way (Chen, 2016). In case $\mu = 1$ in the definition above, a concircular vector field v is called a concurrent.

Theorem 2.5.1. Given a lightlike hypersurface (M, g) of semi-Riemannian manifold $(\bar{M}, \bar{g})$. (M, g) is a Ricci soliton iff it satisfies the equation

$$Ric(X,Y) = (\lambda - 2\mu)g(X,Y) - \left(\nabla_{v^T} g\right)(X,Y) - 2g(A_{v^\perp}X,Y) \tag{115}$$

for any vector fields $X, Y \in \Gamma(TM)$.

Proof. Suppose $\varphi: M \to \bar{M}$ is an isometric immersion. Then, the tangent vector field v on $\bar{M}$ can be written by

$$v = v^T + v^\perp \tag{116}$$

where v^T is the tangential component and $v^\perp$ is the null transversal components of v.

Since v is concircular vector field on $\Gamma(TM)$ then $\nabla_X v = \mu X, X \in \Gamma(TM)$. As a consequence, by Gauss-Weingarten formula, we have

$$\mu X = \bar{\nabla}_X v = \bar{\nabla}_X v^T + \bar{\nabla}_X v^\perp = \nabla_X v^T + B(X, v^T)N - A_{v^\perp}X + \tau(X)N$$

From the equation above we get

$$\nabla_X v^T = A_{v^\perp}X + \mu X \tag{117}$$

$$B(X, v^T)N = \tau(X)N. \tag{118}$$

From equations (53) and (117) we obtain

$$\begin{aligned}\left(\mathcal{L}_{v^T} g\right)(X,Y) &= \left(\nabla_{v^T} g\right)(X,Y) + g(\nabla_X v^T T, Y) + g(X, \nabla_Y v^T) \\ &= \left(\nabla_{v^T} g\right)(X,Y) + 2g(A_{v^\perp}X,Y) + 2\mu g(X,Y)\end{aligned}$$

Therefore, by equation (89), we get equation (115).

From theorem above, we have some corollaries as follow:

Corollary 2.5.1. A lightlike hypersurface (M, g, v^T, λ) is a trivial Ricci soliton iff M is $v^\perp$ umbilical.

Proof. Suppose the lightlike hypersurface (M, g, v^T, λ) is a trivial Ricci soliton. Then, $v^T = 0$, so that $v = v^\perp$ by using (117). As a consequence, by equation (116) we have $A_{v^\perp}X = -\mu X$. Conversely, if M is $v^\perp$ umbilical then it can be expressed by $A_{v^\perp}X = -\mu X$, which implies $v^T = 0$.

Corollary 2.5.2. Every Ricci soliton on a totally umbilical lightlike hypersurface M is trivial.

Corollary 2.5.3. If (M, g, v^T, λ) is a Ricci soliton on an $(m+1)$-totally geodesic lightlike hypersurface M, then its scalar curvature can be given by

$$S = n(\lambda - 2\mu) - 2B(v^T, \xi),\ n + 1 = \dim M. \tag{119}$$

Proof. From equation (115), we have $Ric(X,Y) = (\lambda - 2\mu)g(X,Y) - (\nabla_{v^T} g)(X,Y) - 2g(A_{v^\perp}X, Y)$. This equation is equivalent to $Ric(X,Y) = (\lambda - 2\mu)g(X,Y) - B(v^T, X)\eta(Y) - B(X, v^T)\eta(X) - 2g(h(X,Y), v^\perp)$. Since (M,g) is a totally geodesic lightlike hypersurface then $h(X,Y) = 0$, which implies $g(h(X,Y), v^\perp) = 0$. As a consequence, we have

$$Ric(X,Y) = (\lambda - 2\mu)g(X,Y) - B(v^T, X)\eta(Y) - B(v^T, Y)\eta(X) \tag{120}$$

and taking the trace of the equation (120), yields (119).

Theorem 2.5.2. Let the lightlike hypersurface $(M, g, S(TM))$ be a Ricci soliton with a concircular vector field v on $\Gamma(TM)$. Then, the Ricci tensor of M satisfies

$$Ric(X,Y) = (\lambda - \mu)g(X,Y) - B(v, X)\eta(Y) - B(v, Y)\eta(X), \tag{121}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Using equations (53) and (114) gives us

$$\begin{aligned}(\mathcal{L}_v g)(X,Y) &= B(v,X)\eta(Y) + B(v,Y)\eta(X) + g(\nabla_X v, Y) + g(X, \nabla_Y v)\\ &= B(v,X)\eta(Y) + B(v,Y)\eta(X) + 2\mu g(X,Y).\end{aligned}$$

Subtituting this equation to (89) yields

$$Ric(X,Y) + B(v,X)\eta(Y) + B(v,Y)\eta(X) + 2\mu g(X,Y) = \lambda g(X,Y)$$

which is nothing but (121).

Theorem 2.5.2. results the following corollary

Corollary 2.5.4. Let the lightlike hypersurface (M, g, v, λ) be a Ricci soliton. Then, the scalar curvature of M satisfies

$$S = n(\lambda - \mu) - 2nB(v, \xi). \tag{122}$$

Theorem 2.5.3. Let $(M, g, S(TM))$ be a lightlike hypersurface of the semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, (M, g, v, λ) is a Ricci soliton if and only if $(M, g, S(TM))$ is an Einstein hypersurface.

Proof. Let $\{e_1, \dots, e_n, \xi\}$ be a basis on $\Gamma(TM)$ and $\{e_1, \dots, e_n\}$ be an orthonormal basis on $\Gamma(S(TM))$. By using (11) we get

$$Ric(e_i, e_j) = (\lambda - \mu) g(e_i, e_j) - B(v, e_i)\eta(e_j) - B(v, e_j)\eta(e_i)$$

$$= (\lambda - \mu)\delta_{ij} - B(v, e_i) g(N, e_j) - B(v, e_j) g(N, e_i)$$

$$= (\lambda - \mu)\delta_{ij}$$

It is easy to check by using (46a) and (46b) $B(v, e_i) = B(v, e_j) = B(v, \xi) = 0$. So,

$$R(e_i, \xi) = (\lambda - \mu) g(e_i, \xi) - B(v, e_i)\eta(\xi) - B(v, \xi)\eta(e_i)$$

$$= -B(v, e_i) g(N, \xi) - B(v, \xi) g(N, e_i) = 0$$

and

$$R(\xi, \xi) = (\lambda - \mu) g(\xi, \xi) - B(v, \xi)\eta(\xi) - B(v, \xi)\eta(\xi)$$

$$= -B(v, \xi) g(N, \xi) - B(v, \xi) g(N, \xi) = 0$$

Conversely, it is clear that Einstein manifold must be a Ricci soliton.

Corollary 2.5.5. A lightlike hypersurface $(M, g, S(TM))$ is a Ricci soliton with potential vector field v iff μ satisfying (114) is constant. Moreover, $(M, g, S(TM))$ is flat lightlike hypersurface when $\mu = \lambda$.

Theorem 2.5.4. Let the totally geodesic lightlike hypersurface (M, g, v^T, λ) be Ricci soliton. Then, the screen principal curvatures of M are give by

$$k_i = \frac{1}{2}(nc + 2\mu - \lambda) \tag{123}$$

where c denotes the constant curvature of M.

Proof. Suppose M is a lightlike hypersurface of semi-Riemannian manifold $(\bar{M}, \bar{g})$ of dimension (m+2) of constant curvature c. Let $\{\xi, N\}$ be the canonical null pair on M such that 1-form τ vanishes. Because ξ is the eigenvector field of A^*_ξ with respect to the eigenvalue 0 and A_ξ^* is $\Gamma(S(TM))$-valued real symmetric, A_ξ^* has n diagonalizable orthonormal eigenvector field in $S(TM)$. Suppose $\{e_1, \dots, e_n, \xi\}$ is a frame field of eigen vectors of A_ξ^* and $\{e_1, \dots, e_n\}$ is the orthonormal frame field of $S(TM)$. Then, we have,

$$A_\xi^* e_i = k_i e_i, \ 1 \leq i \leq n, \tag{124}$$

where k_i is the screen principal curvature. On the other hand, since M is totally geodesic, then $A_N e_i = 0$.

From equation (56), we have

$$Ric(e_i, e_j) = nc\delta_{ij} \tag{125}$$

and from equation (115),

$$Ric(e_i, e_j) = (\lambda - 2\mu)\delta_{ij} - 2k_i\delta_{ij} \tag{126}$$

is obtained. As a consequence, from (125) and (126) we get

$$k_i = \frac{1}{2}(nc + 2\mu - \lambda)$$

Next, we show that a lightlike hypersurface admitting concircular vector fields is also a gradient Ricci soliton. We also show some properties of the gradient lightlike Ricci soliton.

Theorem 2.5.5. The lightlike hypersurface (M, g, v, λ) satisfies the condition of a gradient Ricci soliton, where

$$f = \frac{1}{2\mu}\|v\|^2 \tag{127}$$

is the potential function and v is a concircular vector field.

Proof. For any $X \in \Gamma(TM)$, we have

$$Xf = \nabla_X\left(\frac{1}{2\mu}\|v\|^2\right) = \frac{1}{\mu}, \bar{g}(\nabla_X v, v) = \bar{g}(X, v).$$

This implies $v = grad(f)$. Therefore, the lightlike hypersurface endowed wıth a concircular vector field is gradient Ricci soliton with potential function $f = \frac{1}{2}\bar{g}(v, v)$, which is nothing but (127).

Corollary 2.5.6. A lightlike hypersurface$(M, g, S(TM))$ with concircular potential vector field v satisfies the condition of a gradient Ricci soliton iff it is an Einstein lightlike hypersurface.

Proof. Since $v = grad(f)$ then equation (89) can be written by

$$Ric(X,Y) + Hess_f(X,Y) = \lambda g(X,Y) \tag{128}$$

Therefore, we find

$$Ric(X,Y) + g(\nabla_X \nabla f, Y) = \lambda g(X,Y)$$

$$Ric(X,Y) + g(\nabla_X v, Y) = \lambda g(X,Y)$$

$$Ric(X,Y) + \mu g(X,Y) = \lambda g(X,Y)$$

$$Ric(X,Y) = (\lambda - \mu) g(X,Y)$$

From corollary (255), μ is constant. As a consequence $\lambda - \mu$ is constant. Conversely, let $(M, g, S(TM))$ be an Einstein lightlike hypersurface. Then, there exist a constant k suct that $Ric(X,Y) = kg(X,Y)$. By expressing k as subtraction of two constants, it is easy to find (128), so that (M, g, f, λ) is a gradient Ricci soliton.

Corollary 2.5.7. If the lightlike hypersurface (M, g, f, λ) of the semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a gradient Ricci soliton, then

$$Ric(X,Y) = (\lambda - \mu) g(X,Y). \tag{129}$$

Theorem 2.5.6. A gradient Ricci soliton on a lightlike hypersurface (M, g, f, λ) is trivial if and only if it satisfies:

(i) $\|v\| = \text{constant}$

(ii) Concircular vector field v is normal to M.

Proof. Suppose (M, g, f, λ) is a gradient Ricci soliton. By Theorem 2.5.3, we find that M is an Einstein hypersurface and by corollary 2.5.5, μ is constant. As consequence, we have the statement (i) by applying (30). Next, taking the derivative of $\bar{g}(v, v)$ in the direction of X gives $0 = X\bar{g}(v, v) = 2\mu\bar{g}(X, v)$. This gives statement (ii). Conversely, it is clear that if the statement (i) and (ii) hold then (M, g, f, λ) is a trivial gradient Ricci soliton.

Theorem 2.5.7. If a ligtlike hypersurface (M, g) with potential vector field v is a gradient Ricci soliton, then the potential function of (M, g, f, λ) can be expressed as

$$f = \frac{1}{4\mu - 2\lambda}\big(C - n(\lambda - \mu)\big), \tag{130}$$

where C is a real constant.

Proof. From theorem 2.5.5 we have $\|grad(f)\|^2 = 2\mu f$. Therefore, by using equation (92), we find

$$n(\lambda - \mu) + 4\mu f - 2\lambda f = C$$

$$n(\lambda - \mu) + (4\mu - 2\lambda)f = C$$

$$f = \frac{1}{4\mu - 2\lambda}\big(C - n(\lambda - \mu)\big).$$

Next, we assume that $\big(M, g, S(TM)\big)$ is totally umbilical screen homothetic lightlike hypersurface. This condition provides easiness since the screen distribution is non-degenerate.

Definition 2.5.2. A lightlike hypersurface $\big(M, g, S(TM)\big)$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be locally screen conformal on any coordinate neighborhood $\mathcal{U}$ if

$$A_N = \varphi A_\xi^* \tag{131}$$

where A_N and A_ξ^* are the shape operators of M and $S(TM)$, respectively and φ is a non-zero function on $\mathcal{U}$ which is smooth everywhere.

It can be seen that (131) is equivalent to

$$C(X, PY) = \varphi B(X, Y), \tag{132}$$

for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$. In case φ is a non-constant on M, then M is said to be screen homothetic. Furthermore, in case of $\varphi = 0$ or $C = A_N = 0$, M is said to be totally geodesic.

If semi-Riemannian manifold $(\bar{M}, \bar{g})$ has constant curvature c, then we have

$$g(R(X, Y)Z, W) = cg(Y, Z)g(X, W) - g(X, Z)g(Y, W).$$

Let $\dim(\bar{M}) = n + 2$. Locally, $\bar{R}ic$ is given by

$$\bar{R}ic(X,Y) = \sum_{i=1}^{n+2} \epsilon_i g(R(e_i, X)Y, e_i),$$

where $\{e_1, \dots, e_{n+2}\}$ is an orthonormal frame field of $\bar{M}$.

As a consequence, we get

$$Ric(X,Y) = (n+1)cg(X,Y). \tag{133}$$

Therefore, we have the following lemma

Lemma 2.5.1. Given a semi-Riemannian manifold $(\bar{M}, \bar{g})$ of dimension $(n+2)$ with constant curvature. Then, $(\bar{M}, \bar{g}\,)$ is an Einstein manifold.

Theorem 2.5.8. Given a totally umbilical screen homothetic lightlike hypersurface $\big(M, g, S(TM)\big)$ of semi-Riemannian manifold M of constant curvature of dimension (n+2). Then, M is a Ricci soliton iff we can find a constant λ satisfying

$$\lambda = nc - \alpha F - n\varphi F^2 + 2\mu, \tag{134}$$

where c is the constant curvature of $\bar{M}$, φ is the smooth function on M, F is a smooth function and α denotes the mean curvature of M.

Proof. Let M be a totally umbilical screen homothetic lightlike hypersurface such that the equations (131) and (132) hold. Consider $\{e_1, \dots, e_n, \xi\}$ as a basis of TM and $\{e_1, \dots, e_n\}$ as a basis of $S(TM)$. Using equation (56), we have

$$Ric\big(e_i, e_j\big) = ncg\big(e_i, e_j\big) - F\alpha g\big(e_i, e_j\big) + \textstyle\sum_{i=1}^{n} \epsilon_i F g\big(e_i, e_j\big)\varphi F g\big(e_i, e_j\big)$$

$$= nc\delta_{ij} - \alpha F\delta_{ij} + n\varphi F^2\delta_{ij},$$

$$Ric(e_i, \xi) = ncg(e_i, \xi) - F\alpha g(e_i, \xi) + \textstyle\sum_{i=1}^{n} \epsilon_i F g(e_i, \xi)\varphi F g(e_i, \xi) = 0,$$

$$Ric(\xi, \xi) = ncg(\xi, \xi) - F\alpha g(\xi, \xi) + \textstyle\sum_{i=1}^{n} \epsilon_i F g(\xi, \xi)\varphi F g(\xi, \xi) = 0.$$

Then, using (52) we have the Lie derivative of the metric tensor as follow

$$(\mathcal{L}_v g)\big(e_i, e_j\big) = B\big(v, e_j\big)g(N, e_i) + B(e_i, N)g\big(N, e_j\big) + 2\mu g\big(e_i, e_j\big) = 2\mu\delta_{ij},$$

$$(\mathcal{L}_v g)(e_i, \xi) = B(v, e_i)g(N, \xi) + B(e_i, N)g(N, \xi) + 2\mu g(e_i, \xi) = 0,$$

$$(\mathcal{L}_v g)(\xi, \xi) = B(v, \xi)g(N, \xi) + B(\xi, N)g(N, \xi) + 2\mu g(\xi, \xi) = 0.$$

Therefore,

$$Ric_{ij} + \mathcal{L}_v g_{ij} = nc\delta_{ij} - \alpha F\delta_{ij} + n\varphi F^2\delta_{ij} + 2\mu\delta_{ij}. \tag{135}$$

is obtained. If the given condition holds, i.e., we have λ satisfying the equation (134), then equation (135) gives $Ric_{ij} + \mathcal{L}_v g_{ij} = \lambda g_{ij}$ and hence, totally umbilical screen homothetic lightlike hypersurface M is a Ricci soliton with potential v.

Conversely, assume that (M, g, v, λ) is a Ricci soliton, then by direct computation we get equation (134) easily.

Corollary 2.5.8. Let the totally umbilical screen homothetic lightlike hypersurface M of semi-Riemannian manifold $\bar{M}$ of dimension $(n+2)$ be totally geodesic. Then, $(\bar{M}, g, v, \lambda)$ is a Ricci soliton if and only if we can find a constant λ such that

$$\lambda = nc - \alpha F + 2\mu \tag{136}$$

where c is the constant curvature of $\bar{M}$, φ is the smooth function on $M$and α stand for the mean curvature of M.

Theorem 2.5.9. Let $\big(M, g, S(TM)\big)$ be a totally umbilical screen homothetic lightlike hypersurface of semi-Riemannian manifold $\bar{M}$ of ddimension $(n+2)$. Then, M is a gradient Ricci soliton if and only if there exists a smooth function f on M such that

$$\lambda = \frac{n^2c + n\alpha F + n^2\varphi F^2 - C}{2f} + \mu, \tag{137}$$

where c is the constant curvature of $\bar{M}$, φ is the smooth function on $M$and α denotes the mean curvature of M.

Proof. Given a totally umbilical screen homothetic lightlike hypersurface $\big(M, g, S(TM)\big)$ of semi-Riemannian manifold of constant curvature. Suppose (M, g, f, λ) is a graddient Ricci soliton.

By theorem 2.5.5,

$$\|grad(f)\|^2 = \|v\| = 2\mu f$$

is obtained. From theorem 2.5.8, the scalar curvature of M is

$$S = \sum_{i=1}^{n} Ric(e_i, e_i) + Ric(\xi, \xi) = n(nc + \alpha F + n\varphi F^2).$$

As a consequence, substituting these two equations into (91) yields

$$n(nc + \alpha F + n\varphi F^2) + 2\mu f - 2\lambda f = 0$$

So that, it is easy to find (137).

Conversely, since $S + \|grad(f)\|^2 = n(nc + \alpha F + n\varphi F^2) + 2\mu f$, if there exist λ such that (137) holds, then we have $S + \|grad(f)\|^2 - 2\lambda f = C$. Hence, totally umbilical screen homothetic lightlike hypersurface (M, g, f, λ) is a gradient Ricci soliton.

Corollary 2.5.9. Let the totally umbilical screen homothetic lightlike hypersurface $\big(M, g, S(TM)\big)$ of $(n+2)$ semi-Riemannian manifold $\bar{M}$ be totally geodesic. Then, (M, g, f, λ) is a gradient Ricci soliton if and only if there exists a constant λ satisfying

$$\lambda = \frac{n^2c + n\alpha F - C}{2f} + \mu, \tag{138}$$

where c is the constant curvature of$\bar{M}$, φ is the smooth function on M and α denotes the mean curvature of M.

)

3. CONCLUSSION

From the result and discussion in the previous section, we have the following conclusions:

1. Egorov space $(\mathbb{R}^{n+2}, g_f)$ of Lorentzian manifolds is Ricci solitons and gradient Ricci solitons.
2. Every Cahen-Wallach symmetric space is Ricci solitons.
3. Lightlike hypersurface (M, g, v^T, λ) is a trivial Ricci soliton if and only if M is $v^\perp$ umbilical.
4. Every Ricci solitons on totally umbilical lightlike hyperspace are trivial.
5. A lightlike hypersurface (M, g, v, λ) is Ricci solitons if and only if $(M, g, S(TM))$ is an Einstein hypersurface.
6. Lightlike hypersurface (M, g, f, λ) is gradient Ricci solitons if and only if $(M, g, S(TM))$ is an Einstein hypersurface.

4. RECOMENDATIONS

Ricci solitons on semi-Riemannian geometry is a new theory in both mathematics, which can be extended to other fields, especially physics. Therefore, this thesis can be a reference for further study to find new properties of applications of Ricci solitons on semi-Riemannian manifold, especially in lightlike case.

5. REFERENCES

Al-Dayel I., Deshmukh S. and Belova O., 2020. A Notable Property of Concircular Vector Fields on a Riemannian Manifold. Mathematics, 8, 4, 469-478.

Andrews, B. and Hopper, C., 2011. Ricci Flow in Riemannian Geometry: A Complete Proof of the Differentiable 1/4-Pinching Sphere Theorem. Springer, Germany.

Atindogbe, C. and Duggal, K.L., 2004. Confomal Screen Lightlike Hypersurfaces, International Journal of Pure and Applied Mathematics, 11, 421-442.

Atindogbe, C., Harouna, M. M. and Tossa, J., 2014. Lightlike Hypersurfaces in Lorentzian Manifolds with Constant Screen Principal Curvatures, African Diaspora Journal of Mathematics, 16, 2, 31-45.

Batat, W., Brozos V., Miguel, Garcia-Rio, E. and Gavino-Fernandez, S., 2011. Ricci Solitons on Lorentzian Manifolds with Large Isometry Groups. Bulletin of the London Mathematical Society, 43, 1219-1227.

Brendle, S., 2010. Ricci Flow and Sphere Theorem. American Mathematical Society, Rhode Island.

Brickell, F. and Yano, K., 1974. Concurrent vector field and Minkowski structures. Kodai math. sem. rep., 26, 22-28.

Calın, O. and Chang, D.C, 2005. Geometric Mechanics on Riemann Manifolds; Application to Partial Differential Equations. Birkhäuser. Boston, USA

Calvaruso, G. and Eduardo, G. R., 2010. Algebraic Properties of Curvature Operatörs in Lorentzian Manifolds with Large Isometry Groups, Symmetry, Integrability and Geometry: Methods and Applications, 6, 1-8.

Chandrasekhar, S., 1983. The mathematical theory of the black holes, Oxford university Press, New York.

Chen, B.Y., 1973. Geometry of Submanifold, Marcel Dekker Inc., New York.

Chen, B.Y. and Deshmukh, S., 2015. Ricci Solitons and Concurrents Vector Fields. Balkan Journal of Gemoetry and Its Application, 20, 1, 14-25.

Chen, B.Y. and Deshmukh, S., 2015. Ricci Solitons and Concurrent Vector Field, Balkan Journal of Geometry and Its Applications, 20, 6, 14-25.

Chen, B.Y., 2015. Some Results on Concircular Vector Fields and Their Applications to Ricci Solitons, Bull. Korean Math. Soc., 52, 5, 1535-1547.

Chen, B.Y., 2016. Concircular Vector Fields and Pseudo-Kaehler Manifold. Kragujevac Journal of Mathematics, 40, 1, 7-14.

Chen, B.Y., Ricci Solitons on Riemannian Submanifolds. Proceesings of The Conference RIGA 2014, Riemannian Geometry And Applications to Engineering and Economics. Bucharest, Romania.

Chern, S. S., Cen, W. H. and Lam, K. S., 2000. Lectures on Differential Geometry, World Scientific, Singapore.

Chow, B., Chu, S., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F. and Ni, L., 2007. The Ricci Flow Techniques and Applications, Part I: Geometric Aspects, American Mathematical Society. USA.

Chow, B., Knopf, D., 2004. The Ricci Flow: An Introduction, American Mathematical Society, Rhode Island.

Chrusciel, P.T., Recent results in mathematical relativity, Classical quantum gravity, 2005, 1-17.

Duggal, K.L. and Bejancu, A., Lightlike submanifolds of semi-Riemannian manifold and application, Kluwer Academic Publisher, The Netherland, 1996.

Duggal, K.L. and Jin, D.H., 2007. Null curves and hypersurfaces of semi-Riemannian manifolds, World Scientific, Singapore.

Duggal, K.L. and Sahin, B., 2010. Differential geometry of Lightlike submanifolds, Birkhauser, Berlin.

Fialkow, A., 1939. Conformal Geodesics. Trans. Amer. Math. Soc., 45, 3, 443-473.

Fischer, A., 2004. An introduction to conformal Ricci flow, Classical and Quantum Gravity, 21. S171-S218. arXiv:math/0312519 Doi. 10.1088/0264-9381/21/3/011

Güneş, R., Şahin, B. and Kılıç, E., 2003. On Lightlike Hypersurfaces of A Semi-Riemannian Space Form. Turkish Journal of Mathematics, 27, 283-297.

Hacısalihoğlu, H. H., 2000. Diferensiyel Geometri, Ankara Üniversitesi Fen Fakultesi, Ankara.

Hawking, S. W., 1973. The event horizon. In black holes, C. DeWitt and B. DeWitt eds., Gordon and Breach, New York.

Jin, D. H, 2012. Lightlike Hypersurfaces of A Semi-Riemannian Manifold of Quasi-Constant Curvature, Commun. Korean Math. Soc., 27, 4, 763-770.

Kılıç, E., Gülbahar, M. and Kavuk, E., 2021. Concurrent Vector Fields On Lightlike Hypersurfaces, Mathematics, 9, 59, 1-16.

Kuhnel, W., 2013. Differential Geometry Curves-Surface-Manifold. American Mathematical Society, Rhode Island.

Lee, J. M., Introduction to Smooth Manifolds, Springer, New York , 2013

Lee. J. M., 2009. Manifold and Differential Geometry. American Mathematical Society, Rhode Island.

Mackenzie, D. and Cipra, B., What's Happening in the Mathematical Sciences, Vol. 6, American Mathematical Society, Pennsylvania, 2006.

Muller, R., 2006. Differential Harnack Inequalities and the Ricci Flow, European Mathematical Society, Freiburg.

O'Neil, B., 1983. Semi-Riemann Geometry with Applications to Relativity, Academic Press, New York.

Perelman, G., 2002. The Entrophy Formula for the Ricci Flow and its Geometric Applications. Zbl 1130.53001; arXiv math/0211159.

Renteln, P., 2014. Manifolds, Tensors and Forms: An Introduction to Mathematicians and Physicist, Cambridge University Press.

Sodais, H.A., Deshmukh, S. and Alodan, H., 2015. Hypersurfaces of Euclidean Space as Gradient Ricci Solitons, Annals of the Alexandru Ioan Cuza University – Mathematics, 61, 2, 437-444.

Şahin, B., 2007. Lightlike Hypersurfaces of Semi-Riemannian Spaces Satisfying Curvature Conditions of Semisymmmetry Type, Turkish Journal of Mathematics, 31, 139-162.

Şahin, B., 2012. Manifoldların Diferensiyel Geometrisi. Nobel Akademik Yayıncılık Eğitim Danışmanlık Tic. Ltd. Şti. Ankara.

Takeno, H., 1967. Concircular Scalar Field in Spherically Symmetric Space-Times I. Tensor (N.S.), 20, 2, 167-176.

Vazquez, M. B., Calvaruso, G., Rio, E. G. and Fernandez, S. G., 2012. Three-Dimensional Lorentzian Homogeneous Ricci Solitons, Israel journal of mathematics, 188, 385-403.

Yano, K., 1940. Concircular Geometry I. Concircular Transformations, Proc. Imp. Acad. Tokyo, 16, 195-200.

Yıldız, C., 2005. Genel Topoloji. Gazi Kitabevi Tiç. Ltd. Şti. Ankara.

CURRICULUM VITAE

Arfah Arfah was born in Labbakkang, a small village in South Sulawesi Province, Indonesia. He started his elementary education in SDN Labbakkang and then continued to SMP N 1 Pallangga and SMAN 1 Bajeng. In 2013, He began his studies in the Department of Mathematics of State University of Makassar, Indonesia and completed his undergraduate program in 2017. In 2018, he granted a scholarship from Turkish government to study Turkish in Foreign Language Department of Karadeniz Technical University and did his master in the Department of Mathematics, Institute of Science of Karadeniz Technical University. In 2021, he completed his master of science in mathematics.