# APPLICATIONS OF DUALITY FOR $H_P$ SPACES

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

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Spetember, 2006

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

## APPLICATIONS OF DUALITY FOR $H_P$ SPACES



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We present several classical theorems on Hardy spaces that have in common the idea of duality. Thus, we prove the Rogosinski-Shapiro theorems on realizations of the distance to $H_p$, the Sarason Theorem on the algebra $\mathcal{C}+H_\infty$, V.M. Adamyan, D.Z. Arov, and M.G. Krein Theorem stating that if $F \in L_\infty$ and $\|F-H_\infty\|_\infty < 1$ then $F + H_\infty$ contains an element $\omega$ whose modulus is 1 almost everywhere, D. Marshall Theorem stating that the closed unit ball of $H_\infty$ is the closed convex hull of the set of all Blaschke products, as well as G. Szegö Theorem providing a formula to calculate the distance in $L_p(\mu)$, where $\mu$ is a positive Borel measure on $[-\pi, \pi]$, of the function 1 to the algebra of polynomials vanishing at 0, in terms of the Radon-Nykodim derivative of $\mu$ with respect to the Lebesgue measure.



*Keywords:* Hardy spaces.

# ÖZET

## $H_P$ UZAYLARININ DUALLIKLERİNİN UYGULAMALARI

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Eylül, 2006

Bu tezde, Hardy uzayları üzerinde duallik özelliğyle ortak bir kaç klasik teorem sunulmuştur. Bunun için, $H_p$'deki uzaklık bağlantısıyla Rogosinski-Shapiro teoreminin, $C + H_\infty$ cebiri üzerindeki Sarason teoreminin, V.M. Adamyan, D.Z. Arov, and M.G. Krein tarafından ispatlanan ve eğer, $F \in L_\infty$ ve $\|F - H_\infty\|_\infty < 1$ ise $F + H_\infty$'nin içinde modülüsü hemen hemen her yerde 1 olan bir $\omega$ elementi olduğunu belirten teoremin, $H_\infty$'nin kapal birim küresinin, Blaschke çarpımlarının kapalı konveks örtüsü olduğunu belirten D. Marshall teoreminin ve bize, $\mu$'nün $[-\pi, \pi]$ üzerinde pozitif Borel ölçümü olduğu, $L_p(\mu)$ içinde 1 fonksiyonunun sabit terimi olmayan polinomların cebirine olan uzaklığını $\mu$'nün Lebesgue ölçümüne göre Radon-Nykodim türevi cinsinden veren G. Szegö teoreminin ispatları verilmiştir.

*Anahtar sözcükler*: Hardy uzaylari.

# Acknowledgements

I would like to express my gratefulness to Assoc. Prof. Dr. Aurelian Gheondea who not only served as my supervisor but also guided, encouraged and challenged me through out my thesis.

I also would like to thank Prof. Dr. Mefharet Kocatepe, Assoc. Prof. Dr. Alexander Goncharov and Asst. Prof. Dr. Seçil Gergün for having reviewed my thesis and having made corrections.

I also thank to all my friends and colics who have been with me during my graduate education.

To those of you who I did not specify name I also give my thanks for moving me towards my goal.

# Contents

# Chapter 1

# Introduction

The theory of Hardy spaces $H_p$ on the unit disk (or, the upper half-plane) is difficult to be circumvented to only one domain of analysis, because the techniques used are so diverse, from complex functions, to measure and integration, harmonic analysis, functional analysis and operator theory. The present work has the aim to present a few results from this theory, from the point of view of function theory and functional analysis, following our expositions in the Graduate Seminar on Functional Analysis and Operator Theory at the Department of Mathematics of Bilkent University, under the supervision of Aurelian Gheondea. Thus, we have chosen to present a few classical results that have in common the idea of duality of Hardy spaces. In order to outline the main topics, let us first recall that the Hardy spaces $H_p$ have a duality theory close to that of the Lebesgue spaces $L_p$ but with relevant differences in the sense that the duals of $H_p$ are, in general, factor spaces in $L_q$, where $p$ and $q$ are conjugate numbers in the sense $\frac{1}{p} + \frac{1}{q} = 1$. Other dualities can be expressed in terms of factorizations of the algebra of continuous functions by disk type algebras (continuous functions that admit certain analytic continuations in the disk), cf. Section 3.1.

The first results that we present refer to distances of functions $F \in L_p$ to $H_p$ and the possibility that these distances be actually minima (that is, the infima are attained), a collection of theorems that are attributed to W. Rogosinski and H.S. Shapiro [8], independently obtained by S. Havinson [4].

The second result that we present is a theorem of D. Sarason [9] stating that $\mathcal{C} + H_\infty$ is a (closed) algebra in $L_\infty$. We do this by two different approaches, the latter being an abstract version due to W. Rudin and based on an older idea of L. Zalcman.

Then we present a theorem that belongs to V.M. Adamyan, D.Z. Arov, and M.G. Krein stating that if $F \in L_\infty$ and $\|F - H_\infty\|_\infty < 1$ then $F + H_\infty$ contains an element $\omega$ whose modulus is 1 almost everywhere. The original proof of Adamyan, Arov, and Krein was operator theoretical, but the proof that we present here is functional theoretical and belongs to J. Garnett.

Further, we present the celebrated result of D. Marshall [7] stating that the closed unit ball of $H_\infty$ is the closed convex hull of the set of all Blaschke products, in particular, that the set of Blaschke products is uniformly dense in $H_\infty$. The proof is based on two technical lemmas, one belonging to A. Bernard Lemma and the second of R.G. Douglas and W. Rudin [2].

Finally, given $1 \leq p < \infty$ and a positive measure $\mu$ on $[-\pi, \pi]$, there was a general problem of determining the distance in $L_p(\mu)$ of 1 to the algebra of complex polynomials vanishing at 0. We first prove a theorem due to A.N. Kolmogorov stating that in this problem only the absolutely continuous part of the measure $\mu$ with respect to the Lebesgue measure matters and then, we present the full answer due to G. Szegö that calculates this distance in terms of the Radon-Nykodim derivative of $\mu$ with respect to the Lebesgue measure.

All the above results make the contents of Chapter 3. In order to make the presentation understandable as much as possible and, in the same time, to keep it at a reasonable size, we have included in Chapter 2 a review of the basic definitions and results on Hardy spaces that we use: Poisson representations of harmonic functions in the disk, the Fatou Theorem on the nontangential limit of a harmonic function that is the Poisson transform of a finite signed measure $\mu$ on $[-\pi, \pi]$, the harmonic conjugate, the Theorem of F. and M. Riesz on absolute continuity of Borel measures on $[-\pi, \pi]$ in terms of the Fourier coefficients, Blaschke products, and inner/outer functions and factorizations.

In the preparation of our presentations we have used mainly the monograph of P. Koosis [6] on Hardy spaces, and partially those of P.L. Duren [3] and K. Hoffman [5]. For the prerequisites on functional analysis we refer to the textbook of J.B. Conway [1].

# Chapter 2

# Preliminaries

Before passing to the main chapter we have to give some basic definitions and results. We know any infinitely differentiable function satisfying Laplace equation is called harmonic function and for any function $U(z)$ harmonic in $|z| < R$ we know that there is an analytic function $F(z) = \sum_0^\infty a_n z^n$ such that

$$F(z) = U(z) + iV(z)$$

where $V(z)$ is *harmonic conjugate* of $U(z)$. So we have,

$$U(z) = \Re F(z)$$

and we have a series representation for $U$;

$$U(re^{i\theta}) = \sum_{-\infty}^{\infty} A_n r^{|n|} e^{in\theta} \tag{2.1}$$

which is uniformly convergent on compact subsets of $|z| < R$ and

$$\begin{cases} A_n = \frac{1}{2}a_n, & n > 0, \\ A_0 = \Re a_0 \\ A_n = \frac{1}{2}\overline{a}_{-n}, & n < 0. \end{cases}$$

Moreover we have Poisson's representation: If $U(z)$ is harmonic for $|z| < R$, $R > 1$, and if $0 \leq r < 1$, we have

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-r^2)U(e^{it})}{1 + r^2 - 2r\cos(\theta - t)} dt.$$

Now we can restrict ourselves such that $U(z)$ is harmonic in $|z| < 1$. Then we have some version of Poisson's representation:

**Theorem 2.1.** *Let $p > 1$, let U(z) is harmonic in $|z| < 1$, and suppose that*

$$\int_{-\pi}^{\pi} |U(re^{i\theta})|^p d\theta$$

*called means are uniformly bounded in $\theta$, for $r < 1$. Then there is an $F \in L_p(-\pi, \pi)$ with*

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-r^2)}{1 + r^2 - 2r\cos(\theta - t)} F(t)dt.$$

*for $r < 1$.*

For $p = \infty$ we have almost the same result:

**Theorem 2.2.** *If U(z) is harmonic and bounded in $|z| < 1$, there is an $F \in L_\infty(-\pi, \pi)$ with*

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-r^2)}{1 + r^2 - 2r\cos(\theta - t)} F(t)dt.$$

*for $r < 1$.*

Since $L_1(-\pi, \pi)$ is not a dual of anything things are different for $p = 1$. However using duality between the space of finite signed measures $\mu$ on $[-\pi, \pi]$ and the space of continuous functions on $[-\pi, \pi]$, we have

**Theorem 2.3.** *If U(z) is harmonic in $|z| < 1$, and suppose that the means*

$$\int_{-\pi}^{\pi} |U(re^{i\theta})| d\theta$$

*are uniformly bounded in $\theta$, for $r < 1$. Then there is a finite signed measure $\mu$ on $[-\pi, \pi]$ with*

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) d\mu(t), \quad 0 \leq r < 1.$$

*where*

$$P_r(\theta - t) = \frac{(1-r^2)}{1 + r^2 - 2r\cos\theta}$$

*is called Poisson's Kernel.*

Now our aim is to discuss the pointwise behaviour of $U(z)$ as $z$ tends to points $e^{i\theta}$ on the boundary of the unit circle. We know if $U(z)$ is harmonic in $|z| < 1$ then we have one of the representations given above. However all of them are subsumed in the second one since for if $F \in L_p$ we can take $d\mu\theta = F(\theta)d\theta$, then $\mu$ is a finite signed measure on $[-\pi, \pi]$. In dealing with this measure we introduce the function $\mu(\theta)$ of bounded variation on $[-\pi, \pi]$ given by

$$\mu(\theta) = \int_0^\theta d\mu(\theta).$$

Then we have:

**Theorem 2.4 (Fatou).** *Let $-\pi < \phi_0 < \pi$ and suppose that the derivative $\mu^{'}(\phi_0)$ exists and is finite. Then*

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) d\mu(t)$$

*tends to derivative $\mu^{'}(\phi_0)$ for $re^{i\theta}$ tending to $e^{i\phi_0}$, from within any region of the form $|\theta - \phi_0| \leq c(1 - r)$.*

This means $z = re^{i\theta}$ tends to $e^{i\phi_0}$ from within any sector of opening $< 180°$ having vertex at $e^{i\phi_0}$. In this situation we say that $U(re^{i\theta}) \to \mu^{'}(\phi_0)$ as $re^{i\theta}$ tends to $e^{i\phi_0}$ *non-tangentially*, and write

$$U(re^{i\theta}) \to \mu^{'}(\phi_0) \ \text{ for } \ re^{i\theta} \underset{\angle}{\to} e^{i\phi_0}.$$

Now we can discuss the *harmonic conjugate* we mentioned at the beginning of this chapter. If a harmonic function $U(z)$ is the form (2.1) then harmonic conjugate of $U(z)$ is given by

$$\widetilde{U}(re^{i\theta}) = -\sum_{-\infty}^{\infty} i \mathrm{sgn} n A_n r^{|n|} e^{in\theta},$$

where sgn0 means 0. Similar to the *Poisson's Kernel* we have *conjugate Poisson Kernel* which is given by

$$Q_r(\theta) = \frac{2r\sin\theta}{1 + r^2 - 2r\cos\theta}$$

and we have

**Theorem 2.5.** *If*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} d\mu(t).$$

*with a measure $\mu$ then harmonic conjugate $\widetilde{U}$ is given by*

$$\widetilde{U}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r\sin(\theta-t)}{1+r^2-2r\cos(\theta-t)} d\mu(t).$$

Now let us change our direction and have the famous F. and M.Riesz theorem:

**Theorem 2.6.** *Let $\mu$ be a Borel measure on $[-\pi,\pi]$. If*

$$\int_{-\pi}^{\pi} e^{in\theta} d\mu(\theta) = 0 \; for \; n = 1, 2, 3, \ldots,$$

*then $\mu$ is absolutely continuous with respect to Lebesgue measure.*

Now we are going into $H_p$ functions. However let us start first with $H_1$.

**Definition 2.7.** F(z), analytic for $|z| < 1$ is said to be in $H_1$ if

$$\int_{-\pi}^{\pi} |F(re^{i\theta})| d\theta \text{ is } \textit{bounded} \text{ for } r < 1.$$

For a function in $H_1$ we have the following Poisson representations:

**Theorem 2.8.** *Let $F \in H_1$.*

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} F(e^{it}) dt$$

*where*

$$F(e^{i\theta}) = \lim_{\substack{z \longrightarrow e^{i\theta} \\ \angle}} F(z)$$

.

Also we have important result for functions in $H_1$. This is called uniqueness theorem for $H_1$ functions:

**Theorem 2.9.** *Let $F(z) \in H_1$ and suppose, for a set $E$ of positive measure, that $F(e^{i\theta}) = 0$ for $\theta \in E$. Then $F(z) \equiv 0$.*

Now we can define $H_p$ functions and some properties:

**Definition 2.10.** Let $p > 0$, then $H_p$ is the set of $F(z)$, analytic in $\{|z| < 1\}$ with

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta < \infty.$$

**Definition 2.11.** If $F \in H_p$, we write

$$\|F\|_p = \sup_{r<1} \left( \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta \right)^{1/p}$$

for $p \ge 1$ and

$$\|F\|_p = \sup_{r<1} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta$$

for $0 < p < 1$.

In this case $\| \cdot \|_p$ satisfies triangle inequality in all cases but it is positive homogenous, i.e it is a norm for only $p \ge 1$.

Now, to define norm in $H_p$ we can have:

**Theorem 2.12.** *If $p > 0$ and $F \in H_p$, for almost all $e^{i\theta}$, $\lim F(z)$ for $z \underset{\angle}{\longrightarrow} e^{i\theta}$ exists and is finite,and if we call that limit $F(e^{i\theta})$, we have*

$$\int_{-\pi}^{\pi} |F(e^{i\theta})|^p d\theta = \|F\|_p \ , \quad 0 < p < 1,$$

$$\int_{-\pi}^{\pi} |F(e^{i\theta})|^p d\theta = \|F\|_p^p \ , \quad p \ge 1.$$

We also have a theorem about the convergence in mean to boundary data function:

**Theorem 2.13.** *If $f \in H_p$,*

$$\int_{-\pi}^{\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p d\theta \to 0$$

*as $r \to 1$, where $fe^{i\theta}$ is the $\angle$ boundary value of $f(z)$ which exists a.e. by theorem 2.12.*

Moreover as a last case we have:

**Definition 2.14.** We also have $H_\infty$ which denotes the set of functions *analytic* and *bounded* in $\{|z|<1\}$ with the norm

$$\|F\|_\infty = \sup\{|F(z)|;\ |z|<1\}.$$

We will see factorization of $H_p$ functions but before that we need more definitions:

**Definition 2.15.** A Blaschke product is an analytic function $B(z)$ of the form

$$B(z) = \prod_1^\infty \frac{|z_n|}{z_n}.\frac{z_n - z}{1-\overline{z}_n z}$$

which converges absolutely for $|z|<1$ and $0<|z_n|<1$, for $n = 1, 2, 3, \ldots$

**Definition 2.16.** An inner function is an analytic function g in the unit ball such that $|g(z)| \leq 1$ and $|g(z)| = 1$ a.e. on the unit circle. An outer function is an analytic function $F$ in the unit disc of the form

$$F(z) = \lambda \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} h(\theta)d\theta\right]$$

where $h$ is a real-valued integrable function on the unit circle and $\lambda$ is a complex number of modulus 1.

Now we can have some results.

**Theorem 2.17.** *Let $F(z) \not\equiv 0$ belong to $H_p$. Then there is a Blaschke product B(z) and a $G(z) \in H_p$ with $F(z) = B(z)G(z)$, G(z) without zeros in $\{|z|<1\}$.*

**Theorem 2.18.** *Let $F(z) \not\equiv 0$ belong to $H_p$. Let B(z) be the Blaschke product consisting of the zeros of F(z). Then there is a singular measure $\sigma \geq 0$ on $[-\pi, \pi]$ with*

$$F(z) = B(z)\exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\sigma(t)\right)e^{ic}\cdot\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log|F(e^{it})|dt\right)$$

*for $|z|<1$, where $c$ is a real constant.*

This is called the *canonical representation* of the function $F \not\equiv 0$ in $H_p$. We call

$$I_F(z) = e^{ic}B(z)\exp\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\sigma(t)\right\}$$

the *inner factor* and

$$O_F(z) = \exp\left\{\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log|F(e^{it})|dt\right\}$$

called the *outer factor* of $F(z)$.

**Theorem 2.19.** *Let F and G (both $\not\equiv 0$) belong to $H_p$ spaces. If the ratio of their outer factors $k(z) = O_F(z)/O_G(z)$ has $k(e^{i\theta}) \in L_{p'}$ then $O_F(z)/O_G(z) \in H_{p'}$ where $p'$ is a number greater than p.*

We also have an important result for $H_p$ *spaces* from Beurling:

**Theorem 2.20.** *Let $F = I_F O_F \in H_p$, $p > 0$. Then the closure of F(z). $\{$polynomials in $z\}$ in $H_p$ is precisely $I_F.H_p$.*

We now have a result on approximation of inner functions by Blaschke products:

**Theorem 2.21 (Frostman).** *Let $\omega(z)$ be any inner function. Then given any $\epsilon > 0$, there is a Blashke product B(z) and a real c with*

$$\|\omega - e^{ic}B\|_\infty < \epsilon.$$

Up to know we talked about $H_p$ functions for *unit ball.* We should also talk about $H_p$ functions for upper half plane. In order to study this we should use the conformal mapping

$$z \mapsto w = \frac{i-z}{i+z}$$

onto the unit ball $\{|w| < 1\}$ an we will apply the above results. Let us start with Poisson's formula:

**Theorem 2.22.** *Let V(z) be harmonic and bounded for $\Im z > 0$. Then the limit $V(t) = \lim V(z)$ for $z \underset{\angle}{\rightarrow}$ exists a.e. for $t$ on $\mathbb{R}$, and for $z = x + iy$ and $\Im z > 0$*

$$V(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-t)^2+y^2}V(t)dt.$$

**Theorem 2.23.** *Let V(z) be harmonic and* $\geq 0$ *for* $\Im z > 0$. *Then there is a constant* $\alpha \geq 0$ *and a measure* $\mu \geq 0$ *on* $\mathbb{R}$ *with*

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty,$$

*such that, for* $\Im z > 0$,

$$V(z) = \alpha y + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y d\mu(t)}{(x-t)^2+y^2}.$$

Now let us give the result for harmonic conjugate:

**Theorem 2.24.** *Let,*

$$V(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} V(t)dt.$$

*then we have the following two choices of harmonic conjugate :*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2+y^2} d\mu(t) \quad \textit{and}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{x-t}{(x-t)^2+y^2} + \frac{t}{t^2+1}\right) d\mu(t)$$

For the boundary behaviour we have :

**Theorem 2.25.** *Let,*

$$V(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} V(t)dt.$$

*where* $\mu$ *is a signed measure on* $\mathbb{R}$ *with*

$$\int_{-\infty}^{\infty} \frac{|d\mu(t)|}{1+t^2} < \infty.$$

*Then, at any* $t_0$ *where* $\mu'(t_0)$ *exists and is finite,* $V(z) \to \mu'(t_0)$ *as* $z \underset{\angle}{\to} t_0$.

Now we can define $H_p$ spaces for upper half plane:

**Definition 2.26.** F(z), analytic for $\Im z > 0$, is said to belong to $H_p(\Im z > 0)$, if there is a constant $C < \infty$ such that

$$\int_{-\infty}^{\infty} |F(x+iy)|^p dx \leq C$$

for all $y > 0$. This definition is used for all $p > 0$.

Now we can pass to our main chapter to see dual spaces of $H_p$ spaces and their applications.

# Chapter 3

# Applications of Duality For $H_p$ Spaces

## 3.1 $H_p$ Spaces and Their Duals. Sarason's Theorem

### 3.1.1 Various Spaces and Their Duals

We will first consider the unit circle. If we look at the boundary values $f(e^{i\theta}) \in H_p$, we see that $H_p$ can be considered as a $\|\cdot\|_p$-*closed subspace* of $L_p(-\pi, \pi)$.

In order to describe dual spaces of $H_p$ we first recall some definitions. Firstly, let

$$\mathcal{C} = \{f \text{ continuous on } [-\pi, \pi];\ f(-\pi) = f(\pi)\},$$

and then let $\mathcal{A} = \mathcal{C} \cap H_\infty$ be the set of functions in $\mathcal{C}$ which have analytic extensions to the *open disk* $\{|z < 1\}$ and that give continuous functions on the *closed unit disc.*

Both spaces $\mathcal{A}$ and $\mathcal{C}$ are equipped with the sup-norm $\|\cdot\|_\infty$.

Moreover, let $\mathcal{M}$ be the set of all finite complex-valued Radon measures on the *unit circle* $\{|\zeta| = 1\}$ equipped with the norm

$$\|\mu\| = \int_{-\pi}^{\pi} |d\mu(e^{i\theta})|.$$

Note that by the classical F. Riesz's Theorem, $\mathcal{M}$ is the dual space of $\mathcal{C}$.

Finally we denote

$$H_p(0) = \{f \in H_p; \int_{-\pi}^{\pi} f(e^{i\theta})d\theta = 0\} = zH_p.$$

**Theorem 3.1.** *If $1 < p < \infty$ and $(1/p) + (1/q) = 1$, then:*

**(a)** *The dual of $L_q/H_q$ is $H_p(0)$.*

**(b)** *The dual of $H_p$ is $L_q/H_q(0)$.*

*Proof.* (a) Let $\Lambda$ be a bounded linear functional on $L_q/H_q$. Then, by composing $\Lambda$ with the canonical projection of $L_q$ onto $L_q/H_q$, $\Lambda$ gives a linear functional on $L_q$ with the same norm as on $L_q/H_q$ so, for any $f \in L_q$, we have

$$\Lambda(f) = \Lambda(f + H_q) = \int_{-\pi}^{\pi} f(e^{i\theta})G(e^{i\theta})d\theta, \tag{3.1}$$

where $G$ is some function in $L_p$ with $\|G\|_p$ equal to the norm of $\Lambda$, by the Riesz's Representation Theorem. We have $\Lambda(h) = 0$ for all $h \in H_q$, in particular,

$$\int_{-\pi}^{\pi} e^{in\theta}G(e^{i\theta})d\theta = 0, \quad n = 0, 1, 2, \ldots,$$

so $G(e^{i\theta})$ has a Fourier series of the form

$$\sum_{1}^{\infty} A_n e^{in\theta},$$

and, since $G(e^{i\theta}) \in L_p$, $G(e^{i\theta}) \in e^{i\theta}H_p = H_p(0)$.

Conversely, any function $G(e^{i\theta})$ of this form gives rise to a linear functional $\Lambda$ on $L_q/H_q$ by formula (3.1).

(b) If $G(e^{i\theta}) \in L_q$ and $f \in H_q(0)$ is arbitrary, then the linear form

$$\Lambda h = \int_{-\pi}^{\pi} \left(G(e^{i\theta}) + f(e^{i\theta})\right) h(e^{i\theta}) d\theta,$$

defined for all $h \in H_p$, does not depend on $f \in H_q(0)$. So $G + H_q(0)$ is a bounded linear functional on $H_p$

Conversely, take any linear functional $\Lambda$ on $H_p$. By Hahn-Banach Theorem and the duality of $L_p$ and $L_q$ spaces, we can extend $\Lambda$ to all of $L_p$ and get a function $L \in L_q$ with

$$\Lambda h = \int_{-\pi}^{\pi} L(e^{i\theta}) h(e^{i\theta}) d\theta, \quad h \in L_p.$$

If we restrict this function back to $H_p$, then $\Lambda$ is given by the coset $L + H_q(0)$, by argument as above. □

By similar arguments in the proof of above theorem we have:

**Theorem 3.2.** *The dual of*

$$L_1/H_1 \quad \textit{is} \quad H_\infty(0)$$

*, and the dual of*

$$H_1 \quad \textit{is} \quad L_\infty/H_\infty(0).$$

Now, we have

**Theorem 3.3.** *The dual of*

$$\mathcal{C}/\mathcal{A} \quad \textit{is} \quad H_1(0).$$

*Proof.* Since $\mathcal{M}$ is the dual of $\mathcal{C}$ the dual of $\mathcal{C}/\mathcal{A}$ is

$$\{\mu \in \mathcal{M}; \int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta) = 0. \text{ for } f \in \mathcal{A}\}$$

In particular, for $\mu$ to be in the dual of $\mathcal{C}/\mathcal{A}$, we must have

$$\int_{-\pi}^{\pi} e^{in\theta} d\mu(\theta) = 0, \quad n = 0, 1, 2, \ldots.$$

Therefore, by the theorem of brothers Riesz, $d\mu(\theta) = g(e^{i\theta})d\theta$ for some $g \in L_1(-\pi, \pi)$. For the case when $n = 0$ we have

$$\int_{-\pi}^{\pi} d\mu(\theta) = \int_{-\pi}^{\pi} g(e^{i\theta})d\theta = 0$$

which implies $g \in H_1(0)$.

Conversely, any $g \in H_1(0)$ defines a functional $\Lambda$ on $\mathcal{C}/\mathcal{A}$ by putting, for $\phi \in \mathcal{C}$,

$$\Lambda(\phi + \mathcal{A}) = \int_{-\pi}^{\pi} g(e^{i\theta})\phi(e^{i\theta})d\theta$$

$\square$

**Remark 3.4.** If we define the space $\mathcal{A}(0) = \{e^{i\theta}f(e^{i\theta}); f \in \mathcal{A}\}$, then by the same arguments used in the above proof, we see that $\mathcal{C}/\mathcal{A}(0)$ has dual $H_1$. To sum up, we have

$\mathcal{C}/\mathcal{A}(0)$ *has dual* $H_1$ *which has dual* $L_\infty/H_\infty(0)$

Now we can have our table summarizing these results for the *unit circle*:

**For Unit Circle**

| Space | Dual |
|---|---|
| $H_p, \quad 1 \le p < \infty$ | $L_q/H_q(0), \quad \frac{1}{q} = 1 - \frac{1}{p}$ |
| $H_p(0), \quad 1 \le p < \infty$ | $L_q/H_q, \quad \frac{1}{q} = 1 - \frac{1}{p}$ |
| $L_p/H_p, \quad 1 \le p < \infty$ | $H_q(0), \quad \frac{1}{q} = 1 - \frac{1}{p}$ |
| $L_p/H_p(0), \quad 1 \le p < \infty$ | $H_q, \quad \frac{1}{q} = 1 - \frac{1}{p}$ |
| $\mathcal{C}/\mathcal{A}$ | $H_1(0)$ |
| $\mathcal{C}/\mathcal{A}(0)$ | $H_1$ |

Very similar results holds for the $H_p$ spaces for the upper half plane. However letting

$$\mathcal{C}_0 = \{f \text{ continuous on } \mathbb{R};\ F(x) \to 0,\ \text{as } x \to \mp\infty\}$$

$$\mathcal{A}_0 = \mathcal{C}_0 \cap H_\infty(\Im z > 0)$$

instead of $\mathcal{C}$ and $\mathcal{A}$. Both $\mathcal{C}_0$ and $\mathcal{A}_0$ are equipped with sup norm. With similar proofs we had for the results for the unit circle we have:

**For Upper Half Plane**

| Space | Dual |
|---|---|
| $H_p, \quad 1 \leq p < \infty$ | $L_q/H_q, \quad \frac{1}{q} = 1 - \frac{1}{p}$ |
| $L_p/H_p, \quad 1 \leq p < \infty$ | $H_q, \quad \frac{1}{q} = 1 - \frac{1}{p}$ |
| $\mathcal{C}_0/\mathcal{A}_0$ | $H_1$ |

### 3.1.2 Duality method of Havinson and Rogosinski-Shapiro

In this subsection we will have some theorems about approximation of $H_p$ functions based on duality results given above.

**Theorem 3.5.** *Let $F \in L_p(-\pi, \pi)$, $1 < p < \infty$, and denote*

$$\|F - H_p\|_p = \inf\{\|F - h\|_p;\ \ h \in H_p\}$$

*. Then, with $1/q = 1 - (1/p)$, we have*

**(i)** $\|F - H_p\|_p = \sup\left\{\left|\int_{-\pi}^{\pi} F(e^{i\theta})g(e^{i\theta})d\theta\right|; \quad g \in H_q(0) \text{ and } \|g\|_q = 1\right\}$.

**(ii)** *There is an $h_0 \in H_p$ with $\|F - H_p\|_p = \|F - h_0\|_p$.*

**(iii)** *There is a $g_0 \in H_q(0)$, $\|g_0\|_q = 1$ with*

$$\|F - H_p\|_p = \int_{-\pi}^{\pi} F(e^{i\theta})g_0(e^{i\theta})d\theta.$$

*Proof.* (i) This is a restatement of the duality between $L_p/H_p$ and $H_q(0)$.

(ii) Let $h_n \in H_p$ with $\|F - h_n\|_p \underset{n}{\rightarrow} \|F - H_p\|_p$. Then $\|h_n\|_p$ is bounded since

$$\|h_n\|_p = \|h_n + F - F\|_p \leq \|F - h_n\|_p + \|F\|_p$$

so there is a *subsequence* $\{h_{n_j}\}$ converging weakly to a limit $h_0 \in H_p$. Then $F - h_{n_j} \underset{j}{\rightarrow} F - h_0$ (weakly). It follows that

$$\|F - h_0\|_p \leq \liminf_{j\to\infty} \|F - h_{n_j}\|_p = \|F - H_p\|_p,$$

so $\|F - h_0\|_p = \|F - H_p\|_p$.

(iii) We consider the coset $F + H_p$ in $L_p/H_p$ and by the definition of the quotient norm in $L_p/H_p$ and the Hahn-Banach Theorem, there is a linear functional $\Lambda$ with norm equal to 1 on $L_p/H_p$ such that $\Lambda(F + H_p) = \|F - H_p\|_p$. By (ii) we have

$$\Lambda(f + H_p) = \int_{-\pi}^{\pi} f(e^{i\theta}) g_0(e^{i\theta}) d\theta$$

for all $f \in L_p$ and some $g_0 \in H_q(0)$ of $q$-norm 1. □

**Theorem 3.6.** *Let $F \in L_1(-\pi, \pi)$ Then there is an $h_0 \in H_1$ with*

$$\|F - H_1\|_1 = \|F - h_0\|_1$$

*and there is an $g_0 \in H_\infty(0)$ with $\|g_0\|_\infty = 1$ and*

$$\left(F(e^{i\theta}) - h_0(e^{i\theta})\right) g_0(e^{i\theta}) = |F(e^{i\theta}) - h_0(e^{i\theta})| \quad a.e. \;\; \theta \in [-\pi, \pi]$$

*where*

$$\|F - H_1\|_1 = \inf\{\|F - h\|, \;\; h \in H_1\}$$

*Proof.* For the first part of the statement, it suffices to use the *$w^*$-compactness* of unit ball with respect to duality in remark 3.4 in $H_1$ and the proof of the second part of the previous theorem, to show the existence of such an $h_0$.

Also, as in the proof of the third part in the previous theorem we can get a $g_o \in H_\infty$ with $\|g_0\|_\infty = 1$ and

$$\|F - H_1\|_1 = \|F - h_0\|_1 = \int_{-\pi}^{\pi} F(e^{i\theta}) g_0(e^{i\theta}) d\theta.$$

$$= \int_{-\pi}^{\pi} \left(F(e^{i\theta}) - h_0(e^{i\theta})\right) g_0(e^{i\theta}) d\theta$$

Since $|g_0(e^{i\theta})| \leq 1$ equality of

$$\int_{-\pi}^{\pi} |F(e^{i\theta}) - h_0(e^{i\theta})| d\theta \text{ and } \int_{-\pi}^{\pi} \left(F(e^{i\theta}) - h_0(e^{i\theta})\right) g_0(e^{i\theta}) d\theta$$

shows that $g_0(e^{i\theta})$ has the properties. $\square$

**Theorem 3.7.** *Let $F \in \mathcal{C}$. Then:*

**(i)** $\|F - \mathcal{A}\|_\infty = \|F - H_\infty\|_\infty = \sup\left\{\left|\int_{-\infty}^{\infty} F(e^{i\theta}) g(e^{i\theta}) d\theta\right|; \quad g \in H_1(0) and \|g\|_1 = 1\right\}$

**(ii)** *There is a $g_0 \in H_1(0)$, $\|g_0\|_1 = 1$ with*

$$\|F - \mathcal{A}\|_\infty = \int_{-\pi}^{\pi} F(e^{i\theta}) g_0(e^{i\theta}) d\theta.$$

.

**(iii)** *There is an $h_0 \in H_\infty$ with*

$$|F(e^{i\theta}) - h_0(e^{i\theta})| = \|F - \mathcal{A}\|_\infty \quad a.e.\ \theta \in [-\pi, \pi].$$

*Proof.* Since $H_\infty \supset \mathcal{A}$ so we have $\|F - \mathcal{A}\|_\infty \geq \|F - H_\infty\|_\infty$. Since $H_1(0)$ is the dual of $\mathcal{C}/\mathcal{A}$, we get a $g_0 \in H_1(0)$, $\|g_0\|_1 = 1$ so that (ii) holds. However

$$\int_{-\pi}^{\pi} h(e^{i\theta}) g_0(e^{i\theta}) d\theta = 0$$

for all $h \in H_\infty$, so we have

$$\left|\int_{-\pi}^{\pi} F(e^{i\theta}) g_0(e^{i\theta}) d\theta\right| \leq \|g_0\|_1 \|F - H_\infty\|_\infty.$$

Therefore by the choice of $g_0$, we have that $\|F - \mathcal{A}\|_\infty \leq \|F - H_\infty\|_\infty$ proving (i).

Now by duality results we have $H_\infty$ is the *dual of* $L_1/H_1(0)$ and, using the $w^*$-*compactness* of $H_\infty$, there is an $h_0 \in H_\infty$ with $\|F - H_\infty\|_\infty = \|F - h_0\|_\infty$. Then

$$\|F - h_0\|_\infty = \|F - H_\infty\|_\infty = \|F - \mathcal{A}\|_\infty$$
$$= \int_{-\pi}^{\pi} F(e^{i\theta}) g_0(e^{i\theta}) d\theta = \int_{-\pi}^{\pi} [F(e^{i\theta}) - h_0(e^{i\theta})] g_0(e^{i\theta}) d\theta$$

and, since

$$\int_{-\pi}^{\pi} |g_0(e^{i\theta})| d\theta = 1,$$

we have

$$\int_{-\pi}^{\pi} [F(e^{i\theta}) - h_0(e^{i\theta})] g_0(e^{i\theta}) d\theta = \|F - h_0\|_\infty \int_{-\pi}^{\pi} |g_0(e^{i\theta})| d\theta$$

So $|F(e^{i\theta}) - h_0(e^{i\theta})| = \|F - h_0\|_\infty$ a.e on the support of $g_0$. By theorem 2.9 from preliminaries, $|g_0(e^{i\theta})| > 0$ a.e so we have $|F(e^{i\theta}) - h_0(e^{i\theta})| = \|F - h_0\|_\infty$ a.e. □

**Remark 3.8.** If $F \in \mathcal{C}$, then there is only one $h \in H_\infty$ with $\|F - h\|_\infty = \|F - H_\infty\|_\infty$

Indeed, suppose there were two, say, $h_1$ and $h_2$. Take the $g_0 \in H_1(0)$ as in the theorem 3.7. Then as we get in the proof of theorem 3.7.

$$\int_{-\pi}^{\pi} [F(e^{i\theta}) - h_k(e^{i\theta})] g_0(e^{i\theta}) d\theta = \|F - h_k\|_\infty \int_{-\pi}^{\pi} |g_0(e^{i\theta})| d\theta \quad \text{for } k = 1, 2.$$

and since $|g_0(e^{i\theta})| > 0$ a.e. we must have

$$F(e^{i\theta}) - h_1(e^{i\theta}) = \|F - h_1\|_\infty \frac{|g_0(e^{i\theta})|}{g_0(e^{i\theta})} \quad \text{a.e.}$$

$$F(e^{i\theta}) - h_2(e^{i\theta}) = \|F - h_2\|_\infty \frac{|g_0(e^{i\theta})|}{g_0(e^{i\theta})} \quad \text{a.e.}$$

or, since $\|F - h_1\|_\infty = \|F - h_2\|_\infty = \|F - H_\infty\|_\infty$ we have

$$h_1(e^{i\theta}) = F(e^{i\theta}) - \|F - H_\infty\|_\infty \frac{|g_0(e^{i\theta})|}{g_0(e^{i\theta})} = h_2(e^{i\theta}) \text{ a.e.}$$

**Remark 3.9.** Suppose $F \in L_\infty$ but $F \notin \mathcal{C}$. Using *$w^*$-compactness* of $H_\infty$ and above results we can get an $h_0 \in H_\infty$ with

$$\|F - h_0\|_\infty = \|F - H_\infty\|_\infty = \sup\left\{ \left| \int_{-\pi}^{\pi} F(e^{i\theta}) g(e^{i\theta}) d\theta \right|; \quad g \in H_1(0) \text{ and } \|g\|_1 = 1 \right\}.$$

### 3.1.3 Sarason's Theorem

In the first subsection we have proved that

*$\mathcal{C}/\mathcal{A}(0)$ has dual $H_1$ which has dual $L_\infty/H_\infty(0)$.*

Thus, if $B$ is the Banach Space $\mathcal{C}/\mathcal{A}(0)$, then $B^{**}$ is $L_\infty/H_\infty(0)$. $B$ has a canonical isometric image in $B^{**}$ obtained by identifying linear functionals over $B^*$. In this case, the element of $L_\infty/H_\infty(0)$ corresponding to $F \in \mathcal{C}$, is the coset $\Phi + H_\infty(0)$, where $\Phi \in L_\infty$ is determined by

$$\int_{-\pi}^{\pi} F(e^{i\theta})g(e^{i\theta})d\theta = \int_{-\pi}^{\pi} g(e^{i\theta})\Phi(e^{i\theta})d\theta$$

for all $g \in H_1$. This holds iff $\Phi \in F + H_\infty(0)$, i.e in the canonical embedding of $\mathcal{C}/\mathcal{A}(0)$ in $L_\infty/H_\infty(0)$, $F + \mathcal{A}(0)$, $F \in \mathcal{C}$ corresponds to $F + H_\infty(0)$. Therefore, the image of $\mathcal{C}/\mathcal{A}(0)$ in $L_\infty/H_\infty(0)$ under this embedding is

$$\Xi = \{F + H_\infty(0);\ F \in \mathcal{C}\}$$

In particular $\Xi$ is $\|\ \cdot\|_\infty -$ *closed* in the quotient space $L_\infty/H_\infty(0)$.
Since the canonical homomorphism $\Theta : L_\infty \mapsto L_\infty/H_\infty(0)$ is continuous we have $\Theta^{-1}(\Xi)$ is $\|\ \cdot\|_\infty$-closed in $L_\infty$. But $\Theta^{-1}(\xi) = \mathcal{C} + H_\infty(0) = \mathcal{C} + H_\infty$, since $1 \in \mathcal{C}$. Therefore we have

**Theorem 3.10 (Sarason).** *$\mathcal{C} + H_\infty$ is $\|.\|_\infty -$ closed.*

Now we can prove:

**Theorem 3.11 (Sarason).** *If $F$ and $G \in \mathcal{C} + H_\infty$ then $FG \in \mathcal{C} + H_\infty$ i.e, $\mathcal{C} + H_\infty$ is an algebra.*

*Proof.* It is enough to show that if $F \in \mathcal{C}$ and $G \in H_\infty$, then $FG \in \mathcal{C} + H_\infty$. Since $F \in \mathcal{C}$ by the Weierstrass theorem we can find $F_N(e^{i\theta})$ of the form

$$\sum_{-N}^{N} A_k(N)e^{ik\theta}$$

such that $F_N \to F$ *uniformly*. Then $F_NG \to FG$ *uniformly* so if we can show that each $F_NG \in \mathcal{C} + H_\infty$, then $FG \in \mathcal{C} + H_\infty$ by theorem 3.10 .
Let

$$G(z) = \sum_{0}^{\infty} a_n z^n \in H_\infty$$

and define

$$G_N(z) = \sum_N^{\infty} a_n z^n \in H_\infty.$$

Clearly $G_N(z) \in H_\infty$ and

$$F_N(e^{i\theta})G_N(e^{i\theta}) = \sum_{-N}^{N} A_n(N)e^{in\theta}G_N(e^{i\theta}).$$

Since $e^{in\theta}G_N(e^{i\theta}) \in H_\infty$ for $n = -N,\ -N+1, \dots$ we get $F_N(e^{i\theta})G_N(e^{i\theta}) \in H_\infty$. Also we have $F_N(G - G_N) \in \mathcal{C}$ since

$$F_N(G - G_N) = \left(\sum_{-N}^{N} A_n(N)e^{in\theta}\right)\left(\sum_{0}^{N} a_n e^{in\theta}\right)$$

is clearly a trigonometric polynomial. So we have $F_N G = F_N(G - G_N) + F_N G \in \mathcal{C} + H_\infty$. □

In the following we describe an abstract and operator theoretical approach to theorem 3.10

**Theorem 3.12.** *Let $(B, \|\cdot\|)$ be a Banach space with $E$ and $F$ norm-closed subspaces of $B$. Suppose there is a family $\mathcal{F}$ of linear operators on $B$ with the following properties:*

**(i)** $\|T\| \leq M < \infty$ *for all* $T \in \mathcal{F}$.

**(ii)** $TB \subseteq E$ *for each* $T \in \mathcal{F}$.

**(iii)** $TF \subseteq F$ *for each* $T \in \mathcal{F}$.

**(iv)** *If* $u \in E$*and* $\epsilon > 0$, *there is a* $T \in \mathcal{F}$ *with* $\|Tu - u\| < \epsilon$.

*Then E+F is norm-closed in B.*

*Proof.* Let $x$ be in the norm closure of $E + F$. Then we can find $u_n \in E, v_n \in F$, with $\|u_n + v_n\| \leq 2^{-n}$ for $n \geq 2$ and

$$x = \sum_{n=1}^{\infty}(u_n + v_n).$$

Write $x_n = u_n + v_n$. Then $x = \sum_{n=1}^{\infty} x_n$, and we have

$$x_n = (u_n - T_n u_n + T_n x_n) + (v_n - T_n v_n)$$

where for each $n$, $T_n \in \mathcal{F}$ is chosen so that $\|u_n - T_n u_n\| < 2^{-n}$. Now, define $\widetilde{u}_n = u_n - T_n u_n + T_n x_n$. Since $T_n u_n, T_n x_n \in E$ by condition (ii) we get $\widetilde{u}_n \in E$ and we have $\|\widetilde{u}_n\| \leq 2^{-n} + \|T_n x_n\| \leq 2^{-n}(1+M)$ for $n \geq 2$ by (i) and $\|x_n\| \leq 2^{-n}$ for $n \geq 2$. Similarly since $T_n v_n \in F$ by (iii) so $\widetilde{v}_n \in F$ and $\|\widetilde{v}_n\| \leq \|\widetilde{u}_n\| + \|x_n\| \leq 2^{-n}(2+M)$ for $n \geq 2$. Therefore $\sum_{n=1}^{\infty} \widetilde{u}_n$ converges, to, say $u \in E$ because $E$ is closed, and $\sum_{n=1}^{\infty} \widetilde{v}_n$ converges to $v \in F$ since $F$ is closed. So we have,

$$x = \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (\widetilde{u}_n + \widetilde{v}_n) = u + v \in E + F. \qquad \square$$

Now we can state the theorem 3.10 of this subsection as a corollary of this theorem:

**Corollary 3.13.** *$\mathcal{C} + H_\infty$ is $\|\cdot\|_\infty -$ closed.*

*Proof.* Take $B = L_\infty$, $E = \mathcal{C}$, $F = H_\infty$, and let $\mathcal{F} = \{T_N;\ N = 1, 2, ...\}$ where, if $F \in L_\infty$ and $f(e^{i\theta}) \sim \sum_{-\infty}^{\infty} A_n e^{in\theta}$, we have

$$(T_N F)(e^{i\theta}) = \sum_{-N}^{N} \left(1 - \frac{|n|}{N}\right) A_n e^{in\theta}.$$

Then $\|T_N\| \leq 1$ and for each $F \in \mathcal{C}$, $\|T_N F - F\|_\infty \underset{n}{\rightarrow} 0$. Also we have $T_N L_\infty \subseteq \mathcal{C}$ and $T_N H_\infty \subseteq H_\infty$. So by the previous theorem $E + F = \mathcal{C} + H_\infty$ is *norm-closed* in $B = L_\infty$. $\square$

## 3.2 Marshall's Theorem

### 3.2.1 A result of Adamyan, Arov and Krein

In the previous section we have seen that given $F \in L_\infty$, the coset $F + H_\infty$ contains an element of constant modulus equal to $\|F - H_\infty\|_\infty$ under certain circumstances.

Now we are interested which elements of constant modulus occur in $F + H_\infty$.

**Theorem 3.14.** *If $F \in L_\infty$ and $\|F - H_\infty\|_\infty < 1$, then $F + H_\infty$ contains an element $\omega$ with $|\omega(e^{i\theta})| \equiv 1$ a.e.*

*Proof.* We will look for the $\omega \in F + H_\infty$ with $\|\omega\|_\infty \leq 1$ which maximizes

$$\left| \int_{-\pi}^{\pi} \omega(e^{i\theta}) d\theta \right|$$

and we will show such $\omega$ does the job.

Now let,

$$a = \sup \left\{ \left| \int_{-\pi}^{\pi} \omega(e^{i\theta}) d\theta \right| ; \omega \in F + H_\infty, \|\omega\|_\infty \leq 1 \right\}$$

There is indeed an $\omega \in F + H_\infty$, $\|\omega\|_\infty \leq 1$ with

$$\left| \int_{-\pi}^{\pi} \omega(e^{i\theta}) d\theta \right| = a.$$

For, if we take $\omega_n \in F + H_\infty$, $\|\omega_n\|_\infty \leq 1$ with

$$\left| \int_{-\pi}^{\pi} \omega_n(e^{i\theta}) d\theta \right| \underset{n}{\to} a,$$

we can let $\omega$ be a $w^*$-limit (in $L_\infty$) of some $w^*$-*convergent subsequence* of the $\omega_n$, and then $\|\omega\|_\infty \leq 1$, while

$$\left| \int_{-\pi}^{\pi} \omega(e^{i\theta}) d\theta \right| = a$$

since $1 \in L_\infty$.

Therefore, we can suppose that

$$\int_{-\pi}^{\pi} \omega(e^{i\theta}) d\theta = a$$

since we can attain this by working with $e^{i\gamma}F$ instead of $F$ where $\gamma$ is a real constant.

Now, firstly let us show $\|\omega\|_\infty = 1$. Assume that $\|\omega\|_\infty = 1 - \epsilon$ with $\epsilon > 0$. Then we have $\omega + \epsilon \in \omega + H_\infty = F + H_\infty$, $\|\omega + \epsilon\|_\infty \leq 1$ and

$$\int_{-\pi}^{\pi} (\omega(e^{i\theta}) + \epsilon) d\theta = a + 2\pi\epsilon > a,$$

which is a contradiction to the choice of $\omega$.

Secondly, let us show that $\|\omega - H_\infty(0)\|_\infty = 1$. For, otherwise, $\|\omega - H_\infty(0)\|_\infty < 1$, and then there is an $h \in H_\infty(0)$ with $\|\omega - h\|_\infty = 1 - \epsilon$, $\epsilon > 0$. Then $-h + \epsilon \in H_\infty$ so

$$\omega - h + \epsilon \in F + H_\infty, \quad \|\omega - h + \epsilon\| \leq 1,$$

and since $h \in H_\infty(0)$,

$$\int_{-\pi}^{\pi} (\omega(e^{i\theta}) - h(e^{i\theta}) + \epsilon) d\theta = \int_{-\pi}^{\pi} \omega(e^{i\theta}) d\theta + 2\pi\epsilon = a + 2\pi\epsilon > a$$

again contradiction with the choice of $\omega \in F + H_\infty$.

Now, since $\|\omega - H_\infty(0)\|_\infty = 1$, and using duality results we have proven, there is a sequence of $f_n \in H_1$, $\|f_n\| = 1$, with

$$\int_{-\pi}^{\pi} \omega(e^{i\theta}) f_n(e^{i\theta}) d\theta \underset{n}{\rightarrow} 1$$

We must have $|f_n(0)| \geq c$ for some $c > 0$. Indeed, otherwise without loss of generality, let us assume $f_n(0) = c_n$ with $c_n \underset{n}{\rightarrow} 0$. Then $f_n - c_n \in H_1(0)$, $\|f_n - c_n\|_1 \underset{n}{\rightarrow} 1$, whilst

$$\int_{-\pi}^{\pi} \omega(e^{i\theta})(f_n(e^{i\theta}) - c_n) d\theta \underset{n}{\rightarrow} 1 \ .$$

Therefore $\|\omega - H_\infty\|_\infty \geq 1$ because of the duality. So we have $\|F - H_\infty\|_\infty \geq 1$ since $\omega \in F + H_\infty$, which contradicts with the hypothesis $\|F - H_\infty\|_\infty < 1$, i.e. we have $|f_n(0)| \geq c > 0$

Now using this inequality we can show that $|\omega(e^{i\theta})| \equiv 1$ a.e. Assume the contrary. Then, for some $\lambda < 1$ there is a measurable set E, $|E| > 0$, with $|\omega(e^{i\theta})| \leq \lambda$ on E. Without loss of generality assume $|E| < 2\pi$ so that $|E^c| > 0$. Since, in any case, $|\omega(e^{i\theta})| \leq 1$ and $\|f_n\|_1 = 1$ we have

$$\left| \int_{-\pi}^{\pi} \omega(e^{i\theta}) f_n(e^{i\theta}) d\theta \right| \leq \lambda \int_E |f_n(e^{i\theta})| d\theta + \left(1 - \int_E |f_n(e^{i\theta})| d\theta\right),$$

and since the left side tends to 1 as $n$ approaches infinity, we must have

$$\int_E |f_n(e^{i\theta})| d\theta \underset{n}{\rightarrow} 0.$$

Therefore

$$\frac{1}{|E|}\int_E \log|f_n(e^{i\theta})|d\theta \le \log\left(\int_E \log|f_n(e^{i\theta})|d\theta\right) \underset{n}{\to} -\infty.$$

At the same time we have

$$\frac{1}{|E^c|}\int_{E^c} \log|f_n(e^{i\theta})|d\theta \le \log\left(\frac{\|f_n\|_1}{|E^c|}\right) = \log\left(\frac{1}{|E^c|}\right) < \infty,$$

so finally

$$\int_{-\pi}^{\pi} log|f_n(e^{i\theta})|d\theta \underset{n}{\to} -\infty$$

Therefore the sequence of the *outer factors* of $f_n$ already tend to 0 at the origin, so surely $|f_n(0)| \underset{n}{\to} 0$. However, this is a contradiction with the fact that$|f_n(0)| \ge c > 0$. So we have $|\omega(e^{i\theta})| \equiv 1$ a.e. □

**Corollary 3.15.** *Let $f \in H_\infty$, let $\|f\|_\infty < 1$, and let $\Omega$ be any inner function. Then there is another inner function $\omega \in f + \Omega H_\infty$.*

*Proof.* Apply theorem 3.14 with $F(e^{i\theta}) = f(e^{i\theta})/\Omega(e^{i\theta})$. Then $\|F\|_\infty < 1$, so $\|F - H_\infty\|_\infty < 1$, and there is an $h \in H_\infty$ with $|F+h| \equiv 1$ a.e. Then $f+\Omega h \in H_\infty$ and $|f+\Omega h| \equiv 1$ a.e. So $f+\Omega h$ is also an inner function. □

### 3.2.2 Marshall's Theorem

In this subsection we will investigate the apporximation of $H_\infty$ functions by *Blaschke products*. There is a theorem related to this given by D. Marshall. Before Marshall's theorem let us present an analogous proposition about $L_\infty$

**Theorem 3.16.** *Let $f \in L_\infty(-\pi,\pi)$ and $\|f\|_\infty \le 1$. Then given any $\epsilon > 0$ we can find $u_1, \dots u_n \in L_\infty$ with $|u_k(\theta)| \equiv 1$ a.e and numbers $\lambda_1, \dots \lambda_n \ge 0$, $\sum_k \lambda_k = 1$, with*

$$\left\| f - \sum_k \lambda_k u_k \right\|_\infty < \epsilon.$$

This means that the *norm-closed convex hull* of the set of unimodular functions in $L_\infty$ is the unit ball of $L_\infty$.

*Proof.* Without loss of generality, we can assume $\|f\|_\infty \leq 1-(\epsilon/2)$, otherwise we work with $(1-(\epsilon/2))f$ instead of f and observe that $\|f-(1-(\epsilon/2)f)\|_\infty \leq (\epsilon/2)$. Then we may assume $|f(\theta)| \leq 1-(\epsilon/2)$ everywhere, so by Cauchy theorem,

$$f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+f(\theta)}{1+\overline{f(\theta)}e^{it}}dt.$$

Now each of the functions

$$u_t = \frac{e^{it}+f(\theta)}{1+\overline{f(\theta)}e^{it}}$$

is in $L_\infty$ and clearly $|u_t(\theta)| \equiv 1$. Also, since $\|f\|_\infty \leq 1-(\epsilon/2)$, we have

$$\|u_t - u_{t'}\|_\infty < \frac{\epsilon}{2}$$

if $|t-t^{'}| < \delta$, $\delta$ depending on $\epsilon$. So for large N,

$$\left\| \frac{1}{2\pi}\int_{-\pi}^{\pi} u_t dt - \frac{1}{N}\sum_{k=1}^{N} u_{2\pi k/N} \right\|_\infty < \frac{\epsilon}{2}.$$

Therefore

$$\left\| f - \frac{1}{N}\sum_{k=1}^{N} u_{2\pi k/N} \right\|_\infty < \frac{\epsilon}{2}.$$

□

This is the analogous result of Marshall's conjecture that the *norm-closed convex hull* of the set of inner functions is the unit ball of $H_\infty$. Before getting into the theorem we have a lemma.

**Lemma 3.17.** *Let $u \in L_\infty$, $|u(\zeta)| \equiv 1$ a.e for $|\zeta| = 1$. Then there are inner functions $\omega$ and $\Omega$ in $H_\infty$ with*

$$\left| u(\zeta) - \frac{\omega(\zeta)}{\Omega(\zeta)} \right| < \epsilon \quad a.e,\ |\zeta| = 1.$$

*Proof.* Let $E_k$ denote the set

$$\frac{2\pi}{N}(k-1) \leq \arg\, u(\zeta) < \frac{2\pi}{N}k, \quad k = 1, \ldots, N$$

Then $E_k^{\prime}s$ are clearly subsets of $|\zeta| = 1$, they are disjoint and add up to the unit circumference. Put

$$u_k(\zeta) = \begin{cases} e^{2ki\pi/N}, & \zeta \in E_k, \\ 1, & \zeta \notin E_k. \end{cases}$$

Then clearly each $u_k$ is unimodular, takes only two values and

$$\|u - u_1 \cdot u_2 \cdots u_N\|_\infty \leq \frac{2\pi}{N}.$$

Since product of inner functions is also an inner function it is enough to establish the result for each $u_k$.

Now let $u$ be the unimodular function taking only two values say 1 and $\gamma$, $|\gamma| = 1$,$\gamma \neq 1$, with

$$u(\zeta) = 1, \quad \zeta \in E,$$
$$u(\zeta) = \gamma, \quad \zeta \in E^c.$$

Using the Poisson's formula construct $V(z)$ bounded and harmonic in $\{|z| < 1\}$ having nontangential boundary values

$$V(\zeta) = 0 \text{ a.e on } E,$$
$$V(\zeta) = -K, \text{ a.e on } E^c.$$

Then define $h = \exp(V + i\widetilde{V})$. Clearly $h$ is analytic and bounded in $|z| < 1$, and takes values in the open ring $e^{-K} < |h| < 1$. On E, $|h| = 1$ a.e and on $E^c$ $|h| = e^{-K}$ a.e.

Now let $\Phi_K$ be a conformal mapping of the ring $e^{-K} < |w| < 1$ onto the infinite domain obtained from $\mathbb{C} \cup \{\infty\}$ by removing therefrom two segments $[-\epsilon, 0]$ and $[l, l']$ with

$$l = i\frac{1-\gamma}{1+\gamma}$$

and $l' > l$. We furthermore want $\Phi_K$ so as to map $|w| = 1$ onto $[-\epsilon, 0]$ and map $|w| = e^{-K}$ onto $[l, l']$.

In this mapping we can choose $\epsilon$ as small as we want but $l' > l$ depends on the radius $e^{-K}$. However, it is clear that $l$ tends to $l'$ as K approaches infinity, i.e. $\Phi_K$ tends to conformal mapping of $\{|w| < 1\}$ onto $(\mathbb{C} \cup \{\infty\}) \sim [-\epsilon, 0]$ which takes 0 to $l$. So, given $\epsilon > 0$ let us fix K so large that $l' < l + \epsilon$.

Put

$$\Psi(w) = \frac{i - \Phi_k(w)}{i + \Phi_k(w)}$$

for $e^{-K} \leq |w| \leq 1$. $\Psi$ takes the ring $e^{-K} < |w| < 1$ conformally on to the complement, in $\mathbb{C} \cup \{\infty\}$, of the two arcs

$$\frac{i+\epsilon}{i-\epsilon}, 1 \text{ and } \gamma, \gamma'$$

lying on the unit circle. Here we have $\gamma' = (i-l')/(i+l')$ and $\gamma = (i-l)/i+l$. Also clearly these two arcs have diameters less than $2\epsilon$.

Now let us go back to function $h$ constructed above. We have $\Psi(h(\zeta))$ lies on the arc

$$\frac{i+\epsilon}{i-\epsilon}, 1$$

for almost all $\zeta \in E$, and on the arc

$$\gamma, \gamma'$$

for almost all $\zeta$ in $E^c$. Therefore, $|\Psi(h(\zeta)) - u(\zeta)| < 2\epsilon$ a.e on $|\zeta| = 1$, and $|\Psi(h(\zeta))| = 1$ a.e., so it is unimodular.

Moreover, $\Psi(h(\zeta))$ is meromorphic for $|z| < 1$. Indeed, $\Psi$ is conformal and there is only one point c, $e^{-K} < |c| < 1$ with $\Psi(c) = \infty$, and it has a simple pole at c. $\Psi$ is regular elsewhere. So, it is regular at the points $z$, $|z| < 1$, where $h(z) \neq c$ and has a pole at $z$, $h(z) = c$. The order of the pole is equal to the order of zero that $h(z) - c$ has there.

However $h(z) - c$ is in $H_\infty$. By theorem 2.18 we can write

$$h(z) - c = \Omega(z)O(z)$$

with $\Omega(z)$ is inner and O(z) is an outer factor and we have:

$$|O(\zeta)| = |h(\zeta) - c| \geq 1 - |c| > 0 \quad \text{a.e for } \zeta \text{ in E}$$

$$|O(\zeta)| = |h(\zeta) - c| \geq |c| - e^{-K} > 0 \quad \text{a.e for } \zeta \text{ in } E^c.$$

Thus $|O(\zeta)|$ is bounded below a.e. on $|\zeta| = 1$. So by theorem 2.19 $1/O(\zeta) \in H_\infty$ meaning $|\Omega(z)/(h(z) - c)|$ is bounded above in $|z| < 1$.

Now let $\omega(z) = \Omega(z)\Psi(h(z))$. This function is analytic in $|z| < 1$ because any poles of $\Psi(h(z))$ is cancelled by corresponding zeros of $\Omega(z)$, i.e inner factor of $h(z) - c$. It is also bounded in $|z| < 1$. Indeed, take any small $\delta > 0$, if $e^{-K} < |w| < 1$ and $|w - c| \geq \delta$, $\Psi(w)$ is bounded, hence $\Psi(h(z))$ and therefore $\Omega(z)\Psi(h(z))$ is bounded on the set $z$, $|h(z) - c| \geq \delta$. If $|w - c| < \delta$ we have $|\Psi(w)| \leq A/|w - c|$, so if $|h(z) - c| < \delta$

$$|\Omega(z)\Psi(h(z))| \leq A|\Omega_z/(h(z) - c)|.$$

However, we have just seen that the expression on the right side is bounded in $|z| < 1$. Therefore $\omega(z)$ is bounded above, hence it is in $H_\infty$.

Also, for almost all $\zeta$ we have

$$|\omega(\zeta)| = |\Omega(\zeta)||\Psi(h(\zeta))| = 1.$$

Therefore $\omega$ is an inner function and since $\Psi(h(z)) = \omega(z)/\Omega(z)$ we have

$$\left| u(\zeta) - \frac{|\omega(\zeta)|}{|\Omega(\zeta)|} \right| = |u(\zeta) - \Psi(h(\zeta))| < 2\epsilon \quad \text{a.e,} \ |\zeta| = 1.$$

Therefore, we established the result for each factor $u_k$ with $2\epsilon$ instead of $\epsilon$, but this means the lemma is proved. □

Remembering Theorem 3.16 and using this lemma we get

**Corollary 3.18.** *Let $f \in L_\infty$, $\|f\|_\infty \leq 1$ and let $\epsilon > 0$. Then we can find inner funtions $\omega_1, \dots, \omega_n$, $\Omega_1, \dots, \Omega_n$ and numbers $\lambda_k > 0$, $\sum_1^n \lambda_k = 1$ with*

$$\left| f(\zeta) - \sum_1^n \lambda_k \omega_k(\zeta)/\Omega_k(\zeta) \right| < \epsilon \quad a.e, \ |\zeta| = 1.$$

**Remark 3.19.** Since product of inner functions is also a inner function we can take all $\Omega_k$ equal, in order to get a common denominator.

Now we have another lemma

**Lemma 3.20.** *Let $f \in H_\infty$. Then we can find inner functions $\Omega, \omega, \omega_1, \dots, \omega_n$ and real constants $a, a_1, \dots, a_n$ such that*

**(i)** *$g = (a\omega + a_1\omega_1 + \ldots + \omega_n)/\Omega$ is in $H_\infty$;*

**(ii)** $\|f - g\|_\infty < 2\epsilon$.

*Proof.* Without loss of generality, $\|f\|_\infty \leq 1$. Then by Corollary 3.18 and Remark 3.19 we get inner functions $\omega_1, \ldots, \omega_n, \Omega$ and numbers $\lambda_k > 0$ with

$$\left| f(\zeta) - \sum_1^n \lambda_k \omega_k(\zeta)/\Omega_k(\zeta) \right| < \frac{\epsilon}{2} \quad \text{a.e for } |\zeta| = 1 \tag{3.2}$$

Call

$$F(\zeta) = \sum_1^n \lambda_k \omega_k(\zeta)/\Omega_k(\zeta)$$

Since $f \in H_\infty$, (3.2) shows that $\|F - H_\infty\|_\infty < \epsilon$. Therefore, by Theorem 3.14 there is a $g \in H_\infty$ with $|g(\zeta) - F(\zeta)| \equiv \epsilon$ a.e. So, we have

$$\|f - g\|_\infty \leq \|f - F\|_\infty + \|F - g\|_\infty < 2\epsilon.$$

Also,

$$\Omega(\zeta)g(\zeta) - \Omega(\zeta)F(\zeta) \in H_\infty,$$

because

$$\Omega F = \sum_k \lambda_k \omega_k \in H_\infty,$$

and since $|\Omega(\zeta)| \equiv 1$ a.e for $|\zeta| = 1$, $\Omega(\zeta)g(\zeta) - \Omega(\zeta)F(\zeta)$ must be equal to $\epsilon\omega(\zeta)$, with an inner function $\omega$, being in $H_\infty$ and of constant modulus $\epsilon$ a.e. on $|\zeta| = 1$. Therefore

$$g(\zeta) = \epsilon \frac{\omega(\zeta)}{\Omega(\zeta)} + \sum_1^n \lambda_k \frac{\omega_k(\zeta)}{\Omega_k(\zeta)} \qquad \square$$

**Theorem 3.21 (Marshall).** *Let $f \in H_\infty$ and $\|f\|_\infty \leq 1$. Given $\epsilon > 0$, we can find inner functions, $u_1, \ldots, u_n$ and positive numbers $\lambda_1, \ldots, \lambda_n$, $\sum_1^n \lambda_k = 1$ with*

$$\left\| f - \sum_k \lambda_k u_k \right\|_\infty < 4\epsilon.$$

*Proof.* Without loss of generality, we may suppose $\|f\|_\infty \leq 1 - 2\epsilon$, otherwise we may work with $(1 - 2\epsilon)f$ instead of $f$.By Lemma 3.20 we can find a $g \in H_\infty$ of the form

$$\sum_k a_k \omega_k/\Omega,$$

with real constants $a_k$ and inner functions $\omega_k, \Omega$, such that $\|f - g\|_\infty < \epsilon$. In particular, $\|g\|_\infty < 1 - \epsilon$. Now, since for $|\zeta| = 1$ we have $|\omega_k(\zeta)| \equiv 1$ and $|\Omega(\zeta)| \equiv 1$ a.e we have

$$\overline{g(\zeta)} = \sum_k a_k \frac{\Omega(\zeta)}{\omega_k(\zeta)} \quad \text{a.e.}$$

Let $\Omega_1$ be the product of $\omega_k$'s so it is a inner function. So we have $\Omega_1(\zeta)\overline{g(\zeta)}$ in $H_\infty$.

We have $g \in H_\infty$ and since $|g(\zeta)| \leq 1 - \epsilon$ a.e we can apply Cauchy's theorem to have

$$g(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\gamma e^{it} + g(\zeta)}{1 + \overline{g(\zeta)}\gamma e^{it}} dt,$$

$\gamma$ being any number of modulus 1. Take $\gamma = \Omega_1(\zeta)$ to get

$$g(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Omega_1(\zeta)e^{it} + g(\zeta)}{1 + \overline{g(\zeta)}\Omega_1(\zeta)e^{it}} dt.$$

Denote by

$$u_t(\zeta) = \frac{\Omega_1(\zeta)e^{it} + g(\zeta)}{1 + \overline{g(\zeta)}\Omega_1(\zeta)e^{it}}.$$

Each function $u_t \in H_\infty$, because all of $\Omega_1,\ g$ and $\Omega_1\overline{g}$ are in $H_\infty$, and because $\|\Omega_1\overline{g}\|_\infty \leq 1 - \epsilon < 1$. Also, since $|\Omega_1| = 1$ a.e for $|\zeta| = 1$, we have $|u_t(\zeta)| = 1$ a.e for $|\zeta| = 1$, meaning that $u_t$'s are inner.

By the same arguments we have used to prove Theorem 3.16, we get

$$\left\| g - \frac{1}{N} \sum_{k=1}^{N} u_{2\pi k/N} \right\|_\infty < \epsilon\ ,$$

if N is large.

Therefore,

$$\left\| f - \frac{1}{N} \sum_{k=1}^{N} u_{2\pi k/N} \right\|_\infty < 2\epsilon$$

for the function $f$ with $\|f\|_\infty < 1 - 2\epsilon$, but if $\|f\|_\infty \leq 1$ we get the result with $4\epsilon$. $\square$

Recall from preliminaries that any inner function can be approximated by Blaschke products. Combining this result with above theorem we have Marshall's Theorem:

**Theorem 3.22.** *Let $f \in H_\infty$, $\|f\|_\infty \leq 1$. Then there are Blaschke products $B_1, \ldots, B_n$ and positive numbers $\lambda_k$, $\sum_1^n \lambda_k = 1$, such that for any $\epsilon > 0$*

$$\left\| f - \sum_k \lambda_k B_k \right\|_\infty < \epsilon$$

Thus, the *unit ball* in $H_\infty$ is the *norm-closed convex hull* of the set of Blaschke products.

## 3.3 Szegö's Theorem

In this section we will investigate how small we can make

$$\int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p d\mu(\theta) \quad \text{for } P \in \mathbb{P}(0),$$

for $1 \leq p < \infty$, where $\mu$ is a finite positive measure on $[-\pi, \pi]$ and $\mathbb{P}(0)$ denotes the class of polynomials without constant term, i.e. $P(z)$ with $P(0) = 0$.

**Theorem 3.23.** *If $\sigma$ is a positive finite singular measure, then*

$$\inf \left\{ \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p d\sigma(\theta) \;\middle|\; P \in \mathbb{P}(0) \right\} = 0$$

*Proof.* Assume the infimum is strictly positive. Then, for

$$\frac{1}{p} + \frac{1}{q} = 1,$$

there is a $G \in L_q(d\sigma)$ with

$$\int_{-\pi}^{\pi} G(e^{i\theta}) e^{in\theta} d\sigma(\theta) = 0, \quad n = 1, 2, \ldots, \tag{3.3}$$

but

$$\int_{-\pi}^{\pi} G(e^{i\theta}) 1 d\sigma(\theta) > 0.$$

By Hölder Inequality, $G \in L_1(d\sigma)$, so $ds(\theta) = G(e^{i\theta})d\sigma(\theta)$ is a finite Radon measure on $[-\pi, \pi]$ whose support is contained in a set of Lebesgue measure zero, i.e $ds(\theta)$ is singular. Then (3.3) becomes

$$\int_{-\pi}^{\pi} e^{in\theta} ds(\theta) = 0, \quad n = 1, 2, \dots,$$

Therefore, by brothers Rieszs' theorem we have $ds(\theta)$ is absolutely continuous. This implies $ds(\theta) = 0$. However, on the other hand

$$\int_{-\pi}^{\pi} 1.ds(\theta) > 0,$$

and we have a contradiction. □

**Theorem 3.24 (Kolmogorov).** *Let $d\mu(\theta) = w(\theta)d\theta + d\sigma(\theta)$, with $w \in L_1$, $w \geq 0$, $d\sigma \geq 0$ and $\sigma$ is singular. Let $1 \leq p < \infty$. Then*

$$\inf_{P \in \mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w(\theta) d(\theta) = \inf_{p \in \mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p d\mu(\theta).$$

This means that only the absolutely continuous part of $\mu$ matters.

*Proof.* It is clear that

$$\inf_{P \in \mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w(\theta) d(\theta) \leq \inf_{P \in \mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p d\mu(\theta).$$

So we need to prove only the inverse inequality.

Let

$$K = \inf_{p \in \mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w(\theta) d(\theta).$$

Then, there is a $P \in \mathbb{P}(0)$ with

$$\int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w(\theta) d(\theta) < K + \epsilon.$$

It suffices to find a $Q \in \mathbb{P}(0)$ with

$$\int_{-\pi}^{\pi} |1 - Q(e^{i\theta})|^p (w(\theta) d(\theta) + d\sigma(\theta)) < K + 4\epsilon.$$

First of all, by Theorem 3.23 we can find $P_1 \in \mathbb{P}(0)$ with

$$\int_{-\pi}^{\pi} |1 - P_1(e^{i\theta})|^p d\sigma(\theta) < \epsilon.$$

Let $P_2(e^{i\theta}) = P_1(e^{i\theta}) - P(e^{i\theta})$; then $P_2(0) = 0$, i.e it is in $\mathbb{P}(0)$ and

$$\int_{-\pi}^{\pi} |1 - P(e^{i\theta}) - P_2(e^{i\theta})|^p d\sigma(\theta) < \epsilon \ .$$

Now we will take a closed set $E \subseteq \text{supp}(\sigma)$, with

$$\int_{E^c} \left(1 + |P(e^{i\theta})| + |P_2(e^{i\theta})|\right)^p d\sigma(\theta) < \epsilon \ .$$

This is possible because $d\sigma$ is singular, $|E| = 0$. Therefore, we can find $h \in \mathcal{A}$ with $h(e^{i\theta}) \equiv 1$ for $e^{i\theta} \in E$ and $|h(e^{i\theta})| < 1$ if $e^{i\theta} \notin E$. So, for $n = 1, 2, \ldots,$

$$\int_{E} |1 - P(e^{i\theta}) - [h(e^{i\theta})]^n P_2(e^{i\theta})|^p d\sigma(\theta) = \int_{-\pi}^{\pi} |1 - P(e^{i\theta}) - P_2(e^{i\theta})|^p d\sigma < \epsilon \ .$$

In any case we have $|h(e^{i\theta})| \leq 1$ so by the choice of E, we have

$$\int_{-\pi}^{\pi} |1 - P(e^{i\theta}) - [h(e^{i\theta})]^n P_2(e^{i\theta})|^p d\sigma(\theta) \leq \epsilon + \int_{E^c} \left(1 + |P(e^{i\theta})| + |P_2(e^{i\theta})|\right)^p d\sigma < 2\epsilon.$$

Since $|h(e^{i\theta})| < 1$ outside E, hence a.e.

$$1 - P(e^{i\theta}) - (h(e^{i\theta}))^n P_2(e^{i\theta}) \underset{n}{\rightarrow} 1 - P(e^{i\theta}) \quad \text{a.e.} \ ,$$

whence, since $w(\theta) \in L_1$, by Lebesgue Dominated Convergence Theorem we have

$$\int_{-\pi}^{\pi} |1 - P(e^{i\theta}) - (h(e^{i\theta}))^n P_2(e^{i\theta})|^p w(\theta) d(\theta) \underset{n}{\rightarrow} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w(\theta) d(\theta) < K + \epsilon \ .$$

Therefore, for sufficiently large $n$ we have

$$\int_{-\pi}^{\pi} |1 - P(e^{i\theta}) - (h(e^{i\theta}))^n P_2(e^{i\theta})|^p w(\theta) d(\theta) < K + 2\epsilon \ .$$

Thence

$$\int_{-\pi}^{\pi} |1 - P(e^{i\theta}) - (h(e^{i\theta}))^n P_2(e^{i\theta})|^p d\mu(\theta) < K + 2\epsilon + 2\epsilon = K + 4\epsilon \ .$$

Now, since $h \in \mathcal{A}$ we can approximate it uniformly by polynomials

$$\sum_0^N a_k e^{ik\theta}$$

as closely as we want. We have $\left(\sum_0^N a_k e^{ik\theta}\right)^n P_2(e^{i\theta}) \in \mathbb{P}(0)$ since $P_2 \in \mathbb{P}(0)$, and if $\sum_0^N a_k e^{ik\theta}$ is close enough to $h$ we have

$$\int_{-\pi}^{\pi} |1 - Q(e^{i\theta})|^p d\mu(\theta) < K + 4\epsilon \ ,$$

with $Q \in \mathbb{P}(0)$, since

$$Q(e^{i\theta}) = P(e^{i\theta}) + \left(\sum_0^N a_k e^{ik\theta}\right)^n P_2(e^{i\theta})$$

and $P, P_2 \in \mathbb{P}(0)$. □

Now our study reduces to determine

$$\inf_{P \in \mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w(\theta) d(\theta)$$

with $w \in L_1(-\pi, \pi),\ w \geq 0$. This is solved by a theorem by Szegö:

**Theorem 3.25.** *If* $1 \leq p < \infty$

$$\inf_{P \in \mathbb{P}(0)} \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w(\theta) d(\theta) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(\theta) d\theta\right) \ .$$

*Proof.* Firstly suppose that

$$\int_{-\pi}^{\pi} \log^- w(\theta) d\theta < \infty,$$

so that $\log w(\theta) \in L_1(-\pi, \pi)$, because, since $w \in L_1$ it is clear that $\int_{-\pi}^{\pi} \log^+ w(\theta) d\theta < \infty$. We will work with

$$w_1(\theta) = w(\theta) \cdot \exp\left(-\frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(t) dt\right)$$

and let

$$K = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(t) dt\right)$$

to have $w(\theta) = Kw_1(\theta)$ with

$$\int_{-\pi}^{\pi} \log w_1(\theta) d\theta = 0 \ .$$

Now we have to show that

$$\inf_{p \in \mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w_1(\theta) d(\theta) = 2\pi \ .$$

Since $\log w_1(\theta) \in L_1(-\pi, \pi)$, we can form an analytic function

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log w_1(t) dt, \quad |z| < 1;$$

so we have

$$f(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log w_1(t) dt = 0 \ .$$

Actually we have $e^{f(z)} \in H_1$ indeed, we have

$$\begin{aligned} |e^{f(z)}| = e^{\Re f(z)} &= \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re\left\{ \frac{e^{it} + z}{e^{it} - z} \right\} \log w_1(t) dt \right\} \\ &= \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) \log w_1(t) dt \right\} . \end{aligned}$$

we know Poisson's Kernel $P_r(\theta - t)$ is non-negative and $\frac{1}{2\pi} P_r(\theta - t) dt$ is a positive measure of mass 1. Therefore we have:

$$|e^{f(z)}| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) w_1(t) dt$$

i.e.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{f(z)}| d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} w_1(t) dt$$

resulting $e^{f(z)} \in H_1$ since $w_1 \in L_1$. Hence we have $\exp(f(z)/p) \in H_p$ and it is outer by definition. Also since $e^{f(z)} \in H_1$ and outer we have

$$|e^{f(e^{i\theta})}| = w_1(\theta) \quad \text{a.e}$$

equivalently

$$|\exp(f(e^{i\theta})/p)| = (w_1(\theta))^{1/p} \quad \text{a.e.}$$

If $P \in \mathbb{P}(0)$, $G(z) = (1 - P(z)) \exp(f(z)/p)$ is also in $H_p$, and $G(0) = 1$ since $f(0) = 0$. So, for $r < 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(re^{i\theta})|^p d\theta \geq 1.$$

Since $G(re^{i\theta}) \to G(e^{i\theta})$ in the $L_p$ norm as $r \to 1$, we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{i\theta})|^p d\theta \geq 1,$$

implying,

$$\int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w_1(\theta) d(\theta) \geq 2\pi$$

for any $P \in \mathbb{P}$. So the infimum is greater than $2\pi$.

To show the reverse inequality, since $\exp(f(z)/p)$ is outer, by Beurling's Theorem, there is a sequence of polynomials $Q_n(z)$ with

$$Q_n(z) \exp(f(z)/p) \underset{n}{\to} 1$$

in the $H_p$ norm and, since $\exp(f(0)/p) = 1$, we must have $Q_n(0) \underset{n}{\to} 1$, so we have

$$\frac{Q_n(z)}{Q_n(0)} \exp \frac{f(z)}{p} \underset{n}{\to} 1$$

in the $H_p$ norm. We can write $Q_n(z)/Q_n(0) = 1 - P_n(z)$ with $P_n \in \mathbb{P}(0)$, i.e.

$$\int_{-\pi}^{\pi} |1 - P_n(e^{i\theta})|^p w_1(\theta) d(\theta) = \int_{-\pi}^{\pi} |1 - P_n(e^{i\theta})|^p |e^{f(e^{i\theta})}| d(\theta) \underset{n}{\to} \int_{-\pi}^{\pi} 1^p d(\theta) = 2\pi,$$

implying the infimum is less than $2\pi$.

So, the theorem is proved for

$$\int_{-\pi}^{\pi} \log^- w(\theta) d\theta < \infty \,.$$

If now

$$\int_{-\pi}^{\pi} \log w(\theta) d\theta = -\infty,$$

let us take

$$w_n(\theta) = \max(w(\theta), 1/n) \,.$$

Then for each $n$ $\log w_n \in L_1$. We have

$$\int_{-\pi}^{\pi} \log w_n(\theta) d\theta \underset{n}{\to} -\infty \ .$$

For each n, $w_n(\theta) \geq w(\theta)$, so that

$$\inf_{P\in\mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w(\theta) d(\theta) \leq \inf_{P\in\mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w_n(\theta) d(\theta)$$

$$= 2\pi \exp\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log w_n(\theta) d\theta \right) \ .$$

This is true for all n so letting $n \to \infty$ we get

$$\inf_{p\in\mathbb{P}(0)} \int_{-\pi}^{\pi} |1 - P(e^{i\theta})|^p w(\theta) d(\theta) = 0 \ .$$

But, in this case, this equals to

$$2\pi \exp\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(\theta) d\theta \right) \ . \qquad \square$$

## 3.4 The Helson-Szegö Theorem

In this section we will see the characterization of a finite positive measure $\mu$ on $[-\pi, \pi]$ having the property that

$$\int_{-\pi}^{\pi} |\widetilde{T}(\theta)|^2 d\mu(\theta) \leq \text{const.} \int_{-\pi}^{\pi} |T(\theta)|^2 d\mu(\theta)$$

where $T(\theta)$ is a trigonometric polynomial and $\widetilde{T}(\theta)$ is its harmonic conjugate.

**Definition 3.26.** A positive measure $\mu$ is called Helson-Szegö measure if

$$\int_{-\pi}^{\pi} |\widetilde{T}(\theta)|^2 d\mu(\theta) \leq \text{const.} \int_{-\pi}^{\pi} |T(\theta)|^2 d\mu(\theta)$$

for all trigonometric polynomials T.

**Theorem 3.27.** *A Helson-Szegö measure is absolutely continuous.*

*Proof.* Suppose E is closed, $|E| = 0$, but $\mu(E) > 0$. As we used in the proof Theorem 3.24 there exist $h \in \mathcal{A}$ with $h(e^{i\theta}) \equiv 1$ on E and $h(e^{i\theta}) < 1$ elsewhere. We have $h(0) = a$ with $|a| < 1$. Put $F_n(z) = (h(z))^n - a^n$. Then $F_n \in \mathcal{A}$ and $F_n(0) = 0$ so $\Re F_n = -\widetilde{\Im F_n}$. If $\mu$ is Helson-Szegö, we have

$$\int_{-\pi}^{\pi} |(\Re F_n)(e^{i\theta})|^2 d\mu(\theta) \leq \text{C.} \int_{-\pi}^{\pi} |(\Im F_n)(e^{i\theta})|^2 d\mu(\theta) \tag{3.4}$$

because $F_n \in \mathcal{A}$, so for each $n$ there is a sequence of trigonometric polynomials $T_m$ with $T_m \underset{m}{\longrightarrow} (\Im F_n)(e^{i\theta})$ uniformly and $\widetilde{T}_m \underset{m}{\longrightarrow} -(\Re F_n)(e^{i\theta})$ uniformly.

Now on the set E we have,

$$(\Re F_n)(e^{i\theta}) = 1 - \Re a^n \underset{n}{\to} 1$$

so,

$$\liminf_{n \to \infty} \int_{-\pi}^{\pi} |(\Re F_n)(e^{i\theta})|^2 d\mu(\theta) \geq \mu(E) > 0.$$

However,

$$|(\Im F_n)(e^{i\theta})| \leq 1 + |a|^n < 2,$$

whilst, for $e^{i\theta} \in E$,

$$(\Im F_n)(e^{i\theta}) = -\Im a^n \underset{n}{\to} 0,$$

and for $e^{i\theta} \notin E$,

$$|(\Im F_n)(e^{i\theta})| \leq |h(e^{i\theta})|^n + |\Im a^n| \underset{n}{\to} 0\ .$$

Therefore $(\Im F_n)(e^{i\theta}) \underset{n}{\to} 0$ boundedly and everywhere, hence

$$\int_{-\pi}^{\pi} |(\Im F_n)(e^{i\theta})|^2 d\mu(\theta) \underset{n}{\to} 0\ .$$

This contradicts with (3.4). □

**Theorem 3.28.** *A nonzero Helson-Szegö measure is necessarily of the form $d\mu(\theta) = w(\theta)d\theta$ with $w \geq 0$ in $L_1(-\pi, \pi)$ and $\int_{-\pi}^{\pi} \log w(\theta)d\theta > -\infty$.*

*Proof.* By theorem we have $d\mu(\theta) = w(\theta)d\theta$ with $w \geq 0$ and $w \in L_1$. Suppose

$$\int_{-\pi}^{\pi} \log w(\theta)d\theta = -\infty;$$

then we have sequence of trigonometric polynomials $P_n \in \mathbb{P}(0)$ such that,

$$\int_{-\pi}^{\pi} |1 - P_n(\theta)|^2 w(\theta) d\theta \underset{n}{\rightarrow} 0.$$

Put $T_n(\theta) = 1 - P_n(\theta)$. Then $P_n = \widetilde{\widetilde{T}}_n$, so if $w(\theta)d\theta$ is a Helson-Szegö measure,

$$\int_{-\pi}^{\pi} |P_n(\theta)|^2 w(\theta) d\theta = \int_{-\pi}^{\pi} |\widetilde{\widetilde{T}}_n(\theta)|^2 w(\theta) d\theta \leq C. \int_{-\pi}^{\pi} |\widetilde{T}_n(\theta)|^2 w(\theta) d\theta$$

$$\leq C^2. \int_{-\pi}^{\pi} |T_n(\theta)|^2 w(\theta) d\theta \underset{n}{\rightarrow} 0.$$

So,

$$\int_{-\pi}^{\pi} |P_n(\theta)|^2 w(\theta) d\theta \underset{n}{\rightarrow} 0,$$

and finally,

$$\int_{-\pi}^{\pi} 1^2 w(\theta) d\theta = 0.$$

or $w(\theta) = 0$ a.e. So we get a contradiction using Szegö's theorem. □

Therefore, our study reduces to investigate measures of the form $w(\theta)d\theta$ with $w \geq 0$ and in $L_1$ having property $\int_{-\pi}^{\pi} \log w(\theta) d\theta > -\infty$. Now we have a definition:

**Definition 3.29.** If $T(\theta) = \sum_n a_n e^{in\theta}$ is a trigonometric polynomial, call

$$(\Pi T)(\theta) = \sum_{n>0} a_n e^{in\theta}.$$

Also we have following notations:

$$\langle f, g \rangle_w = \int_{-\pi}^{\pi} f(\theta) \overline{g(\theta)} w(\theta) d\theta,$$

$$\|f\|_w = \sqrt{\int_{-\pi}^{\pi} |f(\theta)|^2 w(\theta) d\theta}.$$

Now we have two lemmas;

**Lemma 3.30.**

$$\|\widetilde{T}\|_w \leq K \|T\|_w$$

*for all trigonometric polynomials $T$ and some $K$ iff*

$$\|\Pi T\|_w \leq C \|T\|_w$$

*for all such $T$ and some $C$.*

*Proof.* First of all observe that for all trigonometric polynomial $T(\theta) = \sum_n a_n e^{in\theta}$ we have,

$$\overline{T(\theta)} = \sum_n \overline{a_{-n}} e^{in\theta},$$

so,

$$\overline{(\Pi\overline{T})(\theta)} = \sum_{n<0} a_n e^{in\theta},$$

and finally,

$$\widetilde{T} = -i\Pi T + i\overline{(\Pi\overline{T})}.$$

and clearly we have $\|\overline{T}\|_w = \|T\|_w$, so if $\|\Pi T\|_w \leq C\|T\|_w$,

$$\|\widetilde{T}\|_w \leq \|\Pi T\|_w + \|\Pi\overline{T}\|_w \leq C\|T\|_w + C\|T\|_w = 2C\|T\|_w.$$

Now observe that, for all T we have $2\Pi T = -\widetilde{\widetilde{T}} + i\widetilde{T}$, so if $\|\widetilde{T}\|_w \leq K\|T\|_w$,

$$\|\Pi T\|_w \leq \frac{K^2 + K}{2}\|T\|_w. \qquad \square$$

**Lemma 3.31.** *$w(\theta)d\theta$ is a Helson-Szegö measure iff there is a $\rho < 1$ such that, whenever $P, Q \in \mathbb{P}(0)$,*

$$\left|\Re\int_{-\pi}^{\pi} P(e^{i\theta}) \cdot e^{-i\theta} Q(e^{i\theta}) w(\theta) d\theta\right| \leq \rho\|P\|_w\|Q\|_w. \tag{3.5}$$

*Proof.* By Lemma 3.30, $w(\theta)d\theta$ is a Helson-Szegö iff $\|\Pi T\|_w \leq C\|T\|_w$ for each trigonometric polynomial T. Any such T can be written as $P(\theta) + e^{i\theta}\overline{Q(e^{i\theta})}$ where $P, Q \in \mathbb{P}(0)$ and $P = \Pi T$. For such T we have

$$\|T\|_w^2 = \|P\|_w^2 + \|Q\|_w^2 + 2\Re\int_{-\pi}^{\pi} P(e^{i\theta}) e^{-i\theta} Q(e^{i\theta}) w(\theta) d\theta. \tag{3.6}$$

If the inequality (3.5) holds then (3.6) is,

$$\geq \|P\|_w^2 + \|Q\|_w^2 - 2\rho\|P\|_w^2\|Q\|_w^2 = (1-\rho^2)\|P\|_w^2 + (\rho\|P\|_w^2 - \|Q\|_w^2)^2$$

$$\geq (1-\rho^2)\|P\|_w^2 = (1-\rho^2)\|\Pi T\|_w^2,$$

proving

$$\|\Pi T\|_w^2 \leq \frac{1}{1-\rho^2}\|T\|_w^2.$$

Conversely, if $w(\theta)d\theta$ is a Helson-Szegö measure i.e $\|P\|_w^2 \leq C^2\|T\|_w^2$ where we can assume C is greater than 1, when $\|P\|_w^2 = \|Q\|_w^2 = 1$, we have $\|T\|_w^2 \geq 1/C^2$. So using (3.6) we get,

$$\|T\|_w^2 = 1 + 1 + 2\Re \int_{-\pi}^{\pi} P(e^{i\theta})e^{-i\theta}Q(e^{i\theta})w(\theta)d\theta \geq \frac{1}{C^2},$$

$$\Re \int_{-\pi}^{\pi} P(e^{i\theta})e^{-i\theta}Q(e^{i\theta})w(\theta)d\theta \geq -\left(1 - \frac{1}{2C^2}\right).$$

Repeating same argument with $-P$ instead of $P$ we get,

$$\Re \int_{-\pi}^{\pi} P(e^{i\theta})e^{-i\theta}Q(e^{i\theta})w(\theta)d\theta \leq 1 - \frac{1}{2C^2},$$

this proves (3.5) with $\rho = 1 - 1/2C^2$. □

Now we can have Helson-Szegö's result:

**Theorem 3.32 (Helson-Szegö).** *A measure $d\mu$ such that*

$$\int_{-\pi}^{\pi} |\widetilde{T}(\theta)|^2 d\mu(\theta) \leq C \int_{-\pi}^{\pi} |T(\theta)|^2 d\mu(\theta)$$

*for all trigonometric polynomials $T$ is necessarily of the form*

$$\boxed{\begin{array}{c} d\mu(\theta) = e^{u(\theta)+\widetilde{v}(\theta)}d\theta \\ \textit{where } u \textit{ and } v \textit{ are real valued, } |u(\theta)| \textit{ is bounded, and} \\ |v(\theta)| \leq \frac{\pi}{2} - \epsilon, \quad \epsilon > 0 \end{array}}$$

*Conversely, if $d\mu(\theta) = e^{u(\theta)+\widetilde{v}(\theta)}d\theta$ as given above, then the above inequality holds for some C.*

*Proof.* By theorems 3.4 and 3.28 we can restrict to examine $d\mu(\theta) = w(\theta)d\theta$ with $w \in L_1$, and

$$\int_{-\pi}^{\pi} \log w(\theta)d\theta > -\infty.$$

Take any such $w$ and form

$$\phi(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log\sqrt{w(t)}dt\right);$$

clearly $\phi$ is an outer function and it is in $H_2$ because $w \in L_1$, and by boundary behaviour of harmonic functions

$$|\phi(e^{i\theta})| = w(\theta) \quad \text{a.e.}$$

Call

$$\frac{w(\theta)}{(\phi(e^{i\theta}))^2} = e^{iv(\theta)}.$$

By Lemma 3.31, $w(\theta)d\theta$ is Helson-Szegö iff, for some $\rho < 1$, $P, q \in \mathbb{P}(0)$ and $\|P\|_w \leq 1$, $\|Q\|_w \leq 1$ imply

$$\left|\Re \int_{-\pi}^{\pi} \phi(e^{i\theta})P(e^{i\theta}).\phi(e^{i\theta})e^{-i\theta}Q(e^{i\theta})e^{iv(\theta)}d\theta\right| \leq \ \rho.$$

By replacing $P$ with $e^{i\gamma}$ and varying $\gamma$ through real values, we get

$$\left|\int_{-\pi}^{\pi} \phi(e^{i\theta})P(e^{i\theta}).\phi(e^{i\theta})e^{-i\theta}Q(e^{i\theta})e^{iv(\theta)}d\theta\right| \leq \ \rho\|\phi P\|_2\|\phi Q\|_2$$

for all $P, Q \in \mathbb{P}(0)$. Now by Beurling's Theorem $\phi\mathbb{P}(0)$ *is dense in* $H_2(0)$ and $e^{-i\theta}\phi(e^{i\theta}).\mathbb{P}(0)$ is dense in $H_2$. So above condition is equivalent to

$$\left|\int_{-\pi}^{\pi} e^{iv(\theta)}F(e^{i\theta})G(e^{i\theta})d\theta\right| \leq \ \rho\|F\|_2\|G\|_2$$

for all $F \in H_2(0)$ and $G \in H_2$.

We can have any $f \in H_1(0)$ can be written as $FG$ with $F \in H_2(0)$, $G \in H_2$ and $\|F\|_2 = \|G\|_2 = \sqrt{\|f\|_1}$.

So, finally $w(\theta)d\theta$ is a Helson-Szegö measure iff, for some $\rho < 1$

$$\left|\int_{-\pi}^{\pi} e^{iv(\theta)}f(\theta)d\theta\right| \leq \ \rho\|f\|_1, \quad f \in H_1(0).$$

By subsection (3.1.2) this equivalent to

$$\|e^{iv(\theta)} - H_\infty\|_\infty \leq \rho < 1. \tag{3.7}$$

Suppose now that $w_0(\theta) = e^{\tilde{v}(\theta)}$ with

$$|v(\theta)| \leq \frac{\pi}{2} - \epsilon, \quad \epsilon > 0.$$

Then, as we constructed in the beginning of the proof the corresponding outer function $\phi_0(e^{i\theta})$ satisfies $(\phi_0(e^{i\theta}))^2 = e^{\widetilde{v}(\theta)-iv(\theta)}$, so $w_0(\theta)/\phi_0(e^{i\theta}))^2 = e^{iv(\theta)}$.

Since $|v(\theta)| \leq (\pi/2) - \epsilon$, we have

$$|e^{iv(\theta)} - \sin\epsilon| \leq \cos\epsilon < 1,$$

and the constant $\sin\epsilon$ is in $H_\infty$, so (3.7) is fulfilled with $\rho = \cos\epsilon < 1$. Then, $\|\widetilde{T}\|_{w_0} \leq K_0\|T\|_{w_0}$. If now $w(\theta) = e^{u(\theta)}w_0(\theta)$, and $-c \leq u(\theta) \leq c$, we still have

$$\|\widetilde{T}\|_w \leq e^{2c}\|T\|_w.$$

So our condition is sufficient.

For if the (3.7) holds we get an $h \in H_\infty$ with

$$\left|\frac{w(\theta)}{(\phi(e^{i\theta}))^2} - h(e^{i\theta})\right| = |e^{iv(\theta)} - h(e^{i\theta})| \leq \rho < 1.$$

Multiplying by $|\phi(e^{i\theta})|^2 = w(e^{i\theta})$, we almost everywhere get,

$$|(\phi(e^{i\theta}))^2 h(e^{i\theta}) - w(\theta)| \leq \rho w(\theta).$$

where $w(\theta) > 0$ a.e. because $\int_{-\pi}^{\pi} \log w(\theta)d\theta > -\infty$. So the boundary values $(\phi(e^{i\theta}))^2 h(e^{i\theta})$ of the $H_1$ function $\phi^2 h$ belongs almost everywhere to the sector $|\text{arg}W| \leq \arcsin\rho < (\pi/2)$, and by Poisson formula for $H_1$ functions given in Theorem 2.8, the values $(\phi(z))^2 h(z)$ also belong to that sector for $|z| < 1$. In particular, $\Re\phi^2 h \geq 0$ meaning $\phi^2 h \geq 0$ is outer. So it is of the form $e^{i\gamma}e^{u+i\widetilde{u}}$ where $\gamma$ is real. We now have $|\widetilde{u}(e^{i\theta} + \gamma)| \leq \arcsin\rho$, so by we can take harmonic conjugate

$$\widetilde{\widetilde{u} + \gamma} = -u + u(0).$$

Call $-\widetilde{u}(e^{i\theta} - \gamma) = v_1(\theta)$. Then $|v_1(\theta)| \leq \arcsin\rho$ and $u(e^{i\theta})\widetilde{v_1}(\theta) + c$, $c$ real constant, so

$$|h(e^{i\theta})||\phi(e^{i\theta})|^2 = e^{\widetilde{v_1}(\theta)+c}.$$

Finally,

$$|e^{v(\theta)} - h(e^{i\theta})| \leq \rho < 1$$

makes

$$\frac{1}{1+\rho} \leq \frac{1}{|h(e^{i\theta})|} \leq \frac{1}{1-\rho},$$

so we can call

$$\frac{e^c}{|h(e^{i\theta})|} = e^{u_1(\theta)}$$

with a bounded real $u_1$. Then

$$w(\theta) = |\phi(e^{i\theta})|^2 = e^{u_1(\theta)+\tilde{v}_1(\theta)}$$

with $|v_1(\theta)| \leq \arcsin\rho < \pi/2$. $\square$

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