INVENTORY MANAGEMENT PROBLEM FOR TWO SUBSTITUTABLE PRODUCTS: A BAYESIAN APPROACH

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By

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

INVENTORY MANAGEMENT PROBLEM FOR TWO SUBSTITUTABLE PRODUCTS: A BAYESIAN APPROACH

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Accurate estimation of demand and the inventory levels are must for customer satisfaction and the profitability of a business. In this thesis, we consider two main topics: First, the joint estimation of the demand arrival rate, primary and substitute demand rates in a lost sales environment with two products is provided using a Bayesian methodology. It is assumed that demand arrivals for the products follow a Poisson process where the unknown arrival rate is a random variable. An arriving customer requests one of the two products with a certain probability and may substitute the other product if the primary demand is not available. We consider a general mixture gamma density for prior of the Poisson arrival rate and a general prior for the joint distribution of the primary and substitute demand probabilities. It is observed that the resulting marginal posterior and the conditional posterior density of demand arrival rate given the other parameters, after observation of n-period data, are again in the structure of gamma mixtures. Then, we calculated the inventory levels of the products dynamically with the help of first part. Using the updated posterior demand rates on the observed data from the previous period, we revised the inventory levels for the following period to maximize the profit. Finally, some numerical results for both parts are presented to illustrate the performance of the estimations.

Keywords: Inventory Management, Bayesian Estimation, Substitution, Poisson Demand, Lost Sales.

ÖZET

İKAME EDİLEBİLİR İKİ ÜRÜNLÜ BİR ENVANTER PROBLEMİNİN BAYES YAKLAŞIMI İLE YÖNETİMİ

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Talep ve envanter seviyelerinin doğru tahmini müşteri memnuniyeti ve işletmenin karlılığı açısından olmazsa olmazdır. Bu amaçla, iki ana başlık üzerinde çalışılmıştır. İlk olarak, kayıp satışın olduğu iki ürünlü bir ortamda talep geliş oranları, birincil talep oranları ve ikame talep oranları için Bayes metodolojisi kullanılarak bütünleşik bir tahmin yöntemi sunulmuştur. Taleplerin Poisson dağılım ile geldiği ve geliş oranının bilinmeyen bir rassal değişken olduğu varsayılmıştır. Gelen müşteriler belli bir olasılıkla ürünlerden birini tercih etmekte, ilk tercihi yoksa başka bir olasılıkla diğer ürünü almakta ya da hiçbir şey almadan dönmektedir. Poisson geliş oranı için önsel dağılım olarak Genel Gama Bileşimi Dağılımı, birincil talep oranları ve ikame talep oranları için ise özellikle belirtilmemiş dağılımlar seçilmiştir. N-gözlem dönemi boyunca veri toplandıktan sonra, marjinal ve koşullu sonraki dağılımların yine Gama Bileşimi formunda olduğu gözlemlenmiştir. İkinci kısımda, envanter seviyeleri ilk kısımdaki yöntemler kullanılarak dinamik olarak hesaplanmıştır. Güncellenmiş sonraki talep oranları ve gözlemlenen veriler kullanılarak bir sonraki dönem için karı en yüksek seviyeye taşıyan envanter seviyeleri bulunmuştur. Son olarak, her iki ana başlık için de sayısal analizler ve hesaplamalar sunulmuştur. Böylelikle sunulan teorilerin performansı gözlenebilmiştir.

Anahtar sözcükler: Envanter Yönetimi, Bayesyen Tahmin, İkame Etme, Poisson Talep, Kayıp Satış.

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Chapter 1

Introduction

In the past, main stages of a supply chain, that are procurement, production and distribution, are managed independently by keeping huge buffer stocks. However, competitive business environment forces firms to built agile supply chains which quickly respond to variable demand. To stay in the competition, firms need to decrease operation and inventory holding costs while ensuring the customer satisfaction.

Uncertainties emerged by the factors, such as customer demand, product quality or competition, constitutes risk in supply chains. To reduce the risk, retailers intend to order more than they need. An accurate forecast for the future demand is crucial for a firm to manage inventories and actual sales in a successful manner. Both reduction in out-of-stocks and the improvement in on-shelf availability basically depend on this ability. However, decreasing product lifetimes, competitive promotions that effect purchase decisions, proliferation in product types and the substitutions emerged in cases of stock-outs make an accurate estimation of demand more difficult than the past.

The substitution behavior of customers particularly presents the greatest challenge for practitioners. Predicting the customer response in case of not finding the desired product, is really important for the retailers. In this case, there are four possible actions: the customer may postpone buying, decide not to purchase the product, purchase the product from another retailer, or substitute the product with another item [1]. A study by Gruen *et al.* [2] states that 40 percent of customers who are faced with a stock-out situation, buy a substitute product instead.

If a product is not in the portfolio of the retailer, or, although it is in the portfolio, it has stock-out, customers' primary demand cannot be satisfied. Although both may lead to substitution of the product, the first one is a business decision. However, the second case is caused by demand uncertainty and the inventory decisions. In [3], it is stated that 8.3 percent of the stock keeping units that a retailer has are out-of-stock at a particular moment in time. Therefore, understanding the substitution behavior and adopting the inventory policies accordingly are of significant importance to the firms.

Although, using accurate demand and substitution information is crucial for management of inventory systems, estimation of demand and substitution rates in a multi-product environment is not that easy. The methodologies developed for single item settings are not directly applicable; product substitution sometimes confounds observations; and, new synchronized information such as item-specific stock-out times are needed for these settings. For example, consider the joint estimation of demand and substitution rates for a system with two products under a periodic review in which both items are replenished at the beginning of each period. Within a review period, stock-outs may occur until the next replenishment time, especially for fast moving items. When a stock-out occurs, the customer may prefer to buy another product if available. In such a setting, the observed sales of a product do no longer give accurate information for its primary demand since it includes the information for demand of other items when the product assortments include multiple substitutable items. If substitution behavior of the customer is not modeled properly, this may lead to identification problems from a statistical point of view. Furthermore, if the combined demand in the review period, which is the actual demand for a product and the demand transferred to this product due to the stock out of the other, exceeds the available stock available at the beginning of a period, then a well known type of incomplete data, namely *censored* data is observed, which has to be explicitly taken into account for reliable estimates of demand. Thus, aside from resulting in possible lost sales cost and intangibles such as loss of customer trust or erosion of reputation, product substitution results in possible information loss as well.

In this thesis, we provide an inventory management policy using Bayesian approach for the joint estimation of total demand rate, primary and substitute purchase probabilities.We consider a setting of two-products with unit Poisson customer arrivals. Advantage of the Bayesian estimation methodology lies in gathering the prior information available about the parameters with the data revealed over time. For the Poisson arrival rate, we consider a prior distribution expressed as a general mixture of gamma densities and the joint prior distribution of the primary and substitute demands is left general. It is initially assumed that the arrival rate is independent of the primary demand and substitution probabilities. The joint posterior densities are obtained after n-periods of observation. We establish the result that the posterior conditional density of the Poisson arrivals also follow a mixture of gamma densities with parameters modified with respect to the observed data. On the other hand, the marginal joint posterior of the demand and substitution probabilities keep the general form. We also consider a special case of practical interest: the estimation problem for the demand rate when the primary demand and substitution probabilities are known. The case is applicable to the situations where the products are introduced to new markets where the demand rate is unknown due to the particulars of the new market but the substitution behavior may be more apparent from the consumer behavior in the existing markets. Another special case may be where demand rate is known but substitution probabilities are not known. This case may reflect the environments where new products are launched to the market for which marketing research might reveal reliable estimates for the demand size but the effect of substitution may not be that obvious.

Using the results of demand rate estimation, we have developed an algorithm which dynamically updates the initial inventory levels at the begging of each period. Main aim in this section was no different than the goal of every firm: profit maximization. We introduced the costs of substitution, inventory carrying and the lost sales, than try to maximize the profit by updating the inventory levels with revealed data from the prior observations. In our study we incorporate the accumulated data to estimate the unknown distributions of demand rate using a Bayesian perspective. This approach allows the decision makers to use information that become dynamically available in an efficient way. Despite the complexity of functional forms, the Bayesian approach provides a convenient tool for updating the demand estimation by taking into account the dynamic nature of the available data.

The performance of the suggested estimators are evaluated by simulating several realizations and observing the dynamic behavior of the estimators. Our numerical results show that the estimates converge to the target values of the demand parameters after sufficient data has accumulated. Also, parameter analysis are conducted to obverse the effect on the accuracy of our estimation. Finally, another numerical study is provided to evaluate performance of the provided algorithm for the dynamic inventory update policy.

The rest of this thesis is organized as follows. In Chapter 2, we have introduced the literature about inventory management and demand estimation in three main categories, namely; non-Bayesian Perspective, Bayesian Approach and substitution based inventory management. In Chapter 3, preliminaries of the study is provided. We begin with a general description of the Bayesian Estimation.Then, we mentioned about the specifications and essentials of our model setting. In Chapter 4, joint estimation of the primary and substitute demand parameters are formulated. Single period calculations are presented. Then, generalization for multiple period analysis is provided. Finally, base demand rate estimation is studied as a special case. In Chapter 5, we have used the theory provided in Chapter 4 to determine profit maximizing inventory levels. With the help of estimated demand rates, inventory levels for the next period is updated to increase the profit. In Chapter 6, numerical studies are provided for both of the theories mentioned in Chapter 4 and 5. Parameter analysis and numerical computations are conducted and the results are evaluated with different key performance indicators. Finally, in Chapter 7 we give the concluding remarks.

Chapter 2

Literature Review

In this section we provide our findings in the literature about the inventory management under three categories named as non-Bayesian perspective, Bayesian Approach and substitution based inventory management.

2.1 Non-Bayesian Perspective

Regarding demand estimation in the inventory theory literature, there are two basic approaches; namely, the frequentist and the Bayesian.

In the frequentist perspective, Nahmias [4] is the first to analyze the censored demand case with normally distributed demand. Agrawal and Smith [5] then considered the estimation of negative binomial demand, in the presence of censoring induced by lost sales. Anupindi *et al.* [6] tackle this problem for a vending machine environment and propose maximum likelihood estimation for demand rate under stock-outs. They analyze the results based on both perpetual and periodic system data. Although vending machines provide a good example, the model under study is general enough to include any duopoly market with two major product (such as soft-drinks or chocolate brands *etc.*), or problems with single item two locations (such as cash-machines of the same bank located close enough to allow for substitution in heavy population areas).

Demand estimation and assortment optimization under substitution are studied by Kok and Fisher [7]. Their work focuses on an assortment planning model based on product substitution. The authors propose a procedure for estimating the parameters of demand and the substitution rate for various products including the ones that are new on the market. Karabati *et al.* [3] present a method to estimate stock-out-based substitution rates by using point-of-sale data.

In another recent work, Conlon and Mortimer [8] investigate the demand estimation problem under incomplete product availability in the vending industry. They specifically address the bias in demand parameter estimates and propose a method to correct the bias by allowing for changes in product availability under a periodical inventory review. In the proposed model they assess an EM (expectation maximization) algorithm as well as a Bayesian approach to handle missing data. The results demonstrate significant differences in demand estimates reflecting significant impact of stock-outs on profitability.

Musalem *et al.* [9] develop a demand model that takes into account the effect of stock outs on customer choice. The model is based on utility maximization principles and is estimated using sales data from multiple stores.

Vulcano *et al.* [10] consider estimating demand for substitutable products from sales transaction data. They employ the expectation maximization method and discrete choice models using the sales and product availability data, and illustrate the effectiveness of the proposed method using real as well as simulated data sets. Huh, Levi and Orlin [11] use the well-known nonparametric Kaplan-Meier estimator for censored data from statistics, to propose a new class of adaptive data-driven inventory control policies when the demand distribution is not known and only the sales data are available. Finally, Kok *et al.* [12] provides a wide range of other related works in the literature.

2.2 Bayesian Approach

The Bayesian perspective for demand estimation has been adopted by several authors, starting with the early work of Scarf [13, 14] and Iglehart [15]. Later, Azoury [16], Lovejoy [17], Bradford and Sugrue [18], Hill [19, 20] and Eppen and Iyer [21] considered various aspects of inventory management with Bayesian updating of demand distributions.

Lariviere and Porteus [22] considered Bayesian updating with partially observed sales, Ding *et al.* [23] considered the censored demand setting for a perishable product and examined optimal stocking decisions.

Berk, Gürler and Levine [24] revisited the problem when lost demands are not observed and proposed an approximate methodology retaining the conjugacy properties of priors.

Kamath and Pakkala [25] compare Bayesian and non-Bayesian approaches to demand estimation under both stationary and non-stationary demand. It is observed that ignoring prior information could be very costly especially in the case of variable demand rate. The paper strongly recommends considering the Bayesian approach when there is a high uncertainty about the demand.

Chen and Plambeck [26] revisited the periodic review lost sales inventory problem and incorporating the learning about the probability of substitution they show that substitution reduces the Bayesian optimal inventory level in the case that lost sales are observed. They state that reducing the inventory level has two beneficial effects: to observe and learn more about customer substitution behavior and (for a nonperishable product) to reduce the probability of overstocking in subsequent periods.

Yelland [27] compared the forecast performance of three different state-space models, using Bayesian methods.

2.3 Substitution Based Inventory Management

The practical need for demand estimation in the presence of product substitution is best illustrated by the extensive project conducted by Gruen *et al.* [2]. This study has revealed that the average out-of-stocks was 8.3 per cent worldwide and that, in the case of a stock-out, customer response may be buying the same item elsewhere (31 per cent), substituting a different brand (26 per cent), substituting the same brand (19 per cent), delaying the purchase (15 per cent) or not purchasing at all (9 per cent). The same study points out that substitution rates vary significantly from one stock-keeping-unit (SKU) category to another, and that the erroneous store forecasting accounts for 13 per cent of the causes for out of stock, overall worldwide sales losses range from 3.7 to 4.0 per cent, whereas category sales losses vary from 2.1 to 4.5 per cent with more than two-fold difference.

The important role that the presence of substitution among products plays on the stocking decisions and ensuing costs has been well established in the inventory theoretic literature. In one of the earliest works, Ignall and Veinott [28] showed that the optimality of a myopic inventory control policy depends on specific conditions in a periodic review setting. In a single period operating environment with downward substitution among products, Bassok *et al.* [29] illustrated the benefits of taking substitution effects explicitly into consideration for optimal ordering quantities and showed that a greedy allocation policy is optimal. Smith and Agrawal [30] demonstrated the impact of substitution for assortment planning and determination of optimal stocking levels using actual retail sales data described by a negative binomial demand distribution. Ernst and Kamrad [31] supported their analysis of an inventory system with two substitutable products by using sales data explicitly during stock-out occasions. Also for two-product settings, Honhon *et al.* [32] and Gürler and Yilmaz [33] further illustrated the impact of substitution in single period settings. Xu *et al.* [34] established the optimal substitution structure for an infinite horizon periodic review system with two products in the presence of both supplier- and customer-driven substitution.

Within the game theory context, Parlar [35] is among the first who considered a game theoretic analysis of inventory problems with random demands and substitutable products. In a single period setting, he showed that possible product substitution would significantly affect the optimal decisions of inventory managers. In the same vein, Netessine and Rudi [36] studied centralized and competitive inventory models with demand substitution, where unsatisfied demand for a product flow to other products in deterministic proportions. They compared the optimal stocking policies of substitutable products and showed that, when demand is multivariate normal, under centralized inventory management the total profit is decreasing in demand correlation. Recently, Vaagen *et al.* [37] demonstrated in a single-period multi-item substitutable product model that simplifying assumptions on distributions and dependencies can lead to rather poor solutions and profit losses. Similarly, Yücel *et al.* [38] point out that neglecting customer-driven substitution may lead to significantly inefficient assortments.

In Shah and Avittathur [39], single period inventory planning problem with multiple items is studied in a one-way substitution environment. They considered two problems which are the selection of the optimal product portfolio and inventory levels with the objective of profit maximization. Two separate heuristics are proposed to solve two problems. Comparison between the heuristic solutions against the optimal solutions are provided using some numerical examples. Also, a series of sensitivity analysis are conducted to observe the impact of important parameters on retailer profits.

Nagarajan and Rajagopalan [40] considered several inventory management problems under fixed substitution rates. For single-period, two product model, closed-form expressions are provided to obtain the optimal inventory levels. Then, the result is executed on a special N-product model. A dynamic program is developed for the two-product, multiple periods, and the finite-horizon case. They get the result that as substitution rate increases, optimal inventory levels will be lower.

Stavrulaki [41] demonstrated on the joint effect of demand stimulation and product substitution on inventory decisions by considering single-period, two-product setting with two way substitution. They developed two heuristics: No

Substitution Heuristic, ignores the effect of product substation and finds the optimal order quantities only considering the demand stimulation effect and No Stimulation Heuristic, ignores the effect of demand stimulation and finds the optimal order quantities only considering the product substitution effect. They evaluate the impact of varying substitution rates; critical fractile ratios; demand stimulation and variability; and correlated demands on the heuristic performance are evaluated.

Finally, Tan and Karabati [42] studied a multi-product profit maximizing computation method which has a service level constraint. They consider an inventory management problem with Poisson arrival processes, stock-out based dynamic demand substitution, and lost sales. They used a genetic algorithm where the expected inventory levels are estimated with the mean value approximation and expected direct and substitution sales are estimated with the two moment approximation. For two product case formulation is provided.

Chapter 3

Preliminaries

Before formally introducing the Bayesian Approach for estimation of demand in a lost sales environment, we shall briefly explain the classical Bayesian Estimation methodology and our model setting.

3.1 Bayesian Estimation

The main aim of statistical inference is reaching conclusions about a phenomenon using the observed data. Presence of statistical knowledge enlightens some uncertainties involved in a decision problem. These uncertainties can be considered as unknown numerical quantities such as vectors or matrices. Statistics is composed of two main point of views: the frequentist approach and the Bayesian approach. The main difference between the frequentist and Bayesian approaches is the understanding of probability.

The frequentist approach, which is also known as classical approach, interprets probability as a long-run frequency. It employs hypothesis testing and confidence intervals as two of the main ways of inference. According to frequentists , data is a repeatable random sample and the underlying parameters remain constant during this repeatable process. So, there is a frequency.

On the other hand, in Bayesian approach there is an initial belief about the probability before collecting the data. This definition of probability is usually named as subjective probability. It considers data as an observation from a realized sample. Parameters are unknown and described probabilistically. The main aim of a Bayesian analysis is getting the posterior distribution. Posterior distribution is defined as a weighted average of knowledge about the parameters before the data observed (which is represented by the prior distribution) and the information about the parameters contained in the observed data (which is represented by the likelihood function) by Glickman and Dyk [43]. When the posterior distribution is retrieved, the point and interval estimates of parameters can be computed and the prediction for future data can be made.

A typical Bayesian analysis is composed of four stages: data modeling, prior distribution selection, posterior distribution calculation and posterior distribution analysis.

The initial step for a Bayesian analysis is deciding on a probability model for the data. If the parameters are known, this step will be choosing a probability distribution for the data. It will be a conditional probability function of observed data, given the parameter set. If the observed data values are independent, say $y_1, ..., y_n$; and the unknown parameter vector is θ , then the probability function would be $p(y_i|\theta)$.

After choosing the probability model, a prior distribution for the unknown model parameters need to be declared. Prior distribution represents the current state of knowledge about the model parameters before the data is observed. While selecting the prior distribution, previous studies, opinions of the experts, or research intuitions about the data may guide one. The main point of choosing the prior distribution is representing the prior belief about the data model. When prior beliefs are available, they may be converted to moments such as mean and variance, to fit a common distribution to these moments. Different choices of prior distribution can result in different inference from the same data.

Once the data is observed, the likelihood function is built up. Likelihood is used to describe a function of a parameter vector for a given outcome. When assuming that θ is fixed, integrating the probability function over all possible values of y gives one in total. If the same function is considered when data is fixed and the parameter is the variable, it is called the likelihood. Values of θ where the likelihood is high are those that have a high probability of producing the observed data. Assume that the observed data values $y = (y_1, ..., y_n)$ are obtained, the likelihood function will be

$$L(\theta|y) = p(y_1, ..., y_n|\theta) = \prod p(y_i|\theta)$$

In Bayesian analysis, likelihood function has all of the information which is obtained from the data about the parameter set, θ . After computing the likelihood function, Bayes' theorem is applied to obtain the posterior distribution $p(\theta|y)$:

$$p(\theta|y) = \frac{p(\theta)p(\theta|y)}{\int p(\theta)p(y|\theta)d\theta} = \frac{p(\theta)L(\theta|y)}{p(y)} \propto p(\theta)L(\theta|y)$$

where $\propto$ denotes proportionality.

When treating the above product as a function of θ with y fixed, it should integrate to one for the left-hand-side of the equation to be a proper probability density for θ. Therefore, in theory, to obtain the posterior distribution, the prior distribution is multiplied by the likelihood and a constant that empowers the expression to integrate to one is determined. An easy way to compute the posterior distribution is simplifying the multiplicative constants in the likelihood and the prior distribution, then determining the normalizing constant [43].

Once the posterior distribution is obtained, inferential analysis can be done. The mean or the mode of the posterior distribution is commonly used as the point estimates of parameters. Interval estimates can be computed by calculating the end points of an interval which corresponds to specified percentiles of the posterior distribution. Besides, some predictive inference for a future observation of data can be made using the posterior distribution. Main principle of this predictions is stated that future data is independent of past data conditional on the parameters by Glickman and Dyk [43].

3.2 Model Description

In this section, specifics of our model is introduced. Consider a retailer with two products A and B. The retailer carries inventories of these products, which are replenished periodically at intervals of equal length, τ. The initial stock levels of products A and B at the beginning of period $j(\geq 1)$ are denoted by S_{Aj} and S_{Bj}, respectively. An order-up-to inventory control policy is employed by the retailer. Customers arrive to the system with a Poisson process with a rate of Λ. An arriving customer demands one unit of a particular product with a certain probability. Let P_A denote the primary demand probability for product A, and P_B for B. For generality, we assume that $P_A + P_B = 1$. Under the Poisson assumption for demand arrivals, for a given total demand rate, the demand processes for individual products are independent Poisson processes with rates $\Lambda_A = \Lambda P_A$ and $\Lambda_B = \Lambda P_B$ respectively, if both products are on hand. If the product of the first choice is not available but the other is on hand, the customer buys the other product with some substitution probability. Thus, the actual sales rate of a product is increased in general. γ_A is the probability that an arriving customer substitutes product A in place of B if B is not available; γ_B is similarly defined for product B. Then, an arriving customer purchases product A with probability $P_{A\bar{B}} = P_A + \gamma_A P_B$ due to a possible substitution of the unavailable product B; the demand rate for A becomes $\Lambda_{A\bar{B}} = \Lambda P_{A\bar{B}}$. Similarly, when only product B is available, an arriving customer purchases product B with probability $P_{B\bar{A}}$, where $P_{B\bar{A}} = P_B + \gamma_B P_A$, and the demand rate for B becomes $\Lambda_{B\bar{A}} = \Lambda P_{B\bar{A}}$.

Our aim is to estimate the parameter set that describes the system dynamics, $\Psi = (\Lambda, P_A, P_B, \gamma_A, \gamma_B)$, dynamically over time as sales data accumulate in the course of the operation of the inventory system. To this end, we propose a dynamic Bayesian approach for estimation.

Our analysis rests on the following observation: Within any period, the demand process with stock-out based substitution discussed above behaves as a non-homogeneous Poisson process with random switching times. This general problem has been studied by Anupindi et al. [6] in the context of a vending machine where two products are sold and maximum likelihood estimation methods are employed. In our work we consider a similar model but we consider a different approach for estimation purposes. In particular, taking into account the dynamic nature of the data accumulation process, we propose a dynamic Bayesian approach for estimation, which allows for updating the demand and substitution rates according to the most recently available data.

Chapter 4

Joint Estimation of Primary and Substitute Demand Parameters

At the end of any review period, there may be five different types of realizations that can be observed, with respect to the stock-out status of the products and their time ordering: (i) A period where none of the products stock out is of Type-N; (ii) a period where only product A stocks out is of Type-A; (iii) a period where only product B stocks out is of Type-B; (iv) a period where both products stock-out and the stockout time of A precedes that of B is of Type-AB; (v) and the reverse is of Type-BA. By definition, in a Type-N realization, none of the items experiences a stock-out and the stock levels at the end of the period are strictly positive for both products. In all other period types, at least one of the products stock out. Let τ_A and τ_B denote the stock out times of the products A and B respectively. Then, for the Type-N, Type-A, Type-B, Type-AB, and Type-BA periods, the stock-out times of the products are ordered as $\tau < \tau_A, \tau_B$, $\tau_B < \tau < \tau_A$, $\tau_A < \tau < \tau_B$, $\tau_B < \tau_A < \tau$, $\tau_A < \tau_B < \tau$, respectively.

We assume that sales data is observed at the end of the periods. In general, for period $j(\geq 1)$, observed data consists of two sets: sales quantities, and (possible) stock-out times. (i) In the first set, let $S_j = (X_j, Y_j, Z_j, W_j)$ denote the sales data, where X_j denotes the sales in the period arising from primary demand for product A, Y_j denotes the sales arising from primary demand for product B, Z_j denotes the sales for product A arising from the combined demand for A and B after B has stocked out, and, finally, W_j denotes the sales for product B arising from the combined demand for A and B after A has stocked out. Note here that in the combined sales, information is not available about how many of the sales are for the primary demand and how many are substitutes. With this notation, if $S = (S_{A_j}, y, 0, w)$, this indicates that a Type-A period has been observed, in which product A stocks out. The primary demand for product B until A stocks out is y, no items are purchased of product A as a substitute for B and after A stocks out, w items are sold of product B where the sales are generated from the combined primary demand for B and the flow from the stocked-out product A. (ii) In the second set, let $T_j = (T_{AB_j}, T_{A_j}, T_{B_j})$ denote the durations of the time intervals over which products are available in j^{th} period. T_{AB_j} is the time interval where both products are available. For T_{A_j} units of time only product A is available and for T_{B_j} units of time only product B is available. T_j is also a random vector, the structure of which depends on the observed type of the period. In particular, for the Type-N, Type-A and Type-AB periods, the structures of this vector are given as $T_j = (\tau, 0, 0)$, $T_j = (\tau_A, 0, \tau - \tau_A)$ and $T_j = (\tau_A, 0, \tau_B - \tau_A)$, respectively.

Hence, we can express the observed data for period j as $D_j = (X_j, Y_j, Z_j, W_j, T_{ABj}, T_{Aj}, T_{Bj}) \equiv (S_j, T_j)$ and denote a realization of this random vector as $D = (x_j, y_j, z_j, w_j, t_{ABj}, t_{Aj}, t_{Bj})$. To avoid cumbersome notation to describe each type of available data, we use $\mathbf{D}$ in shorthand to denote all the information accumulated by the time when the estimation is done.

As mentioned before, in Bayesian methodology, it is assumed that, prior to any analysis based on the observations, there is an initial belief (or probabilistic statement) regarding the values that an unknown parameter may take on, known as the *prior* of that entity. It is common to use *conjugate* priors; that is, distributions which retain their family through updating based on the observations. For the rate of the Poisson distribution the conjugate family is gamma. Berger[2] studied a comprehensive treatment of Bayesian methodology. For our initial joint prior for Ψ, the customer arrival rate, Λ and the probabilities, $R = (P_A, P_B, \gamma_A, \gamma_B)$ are assumed independent. For Λ, we consider a more general distribution than the gamma family and assume a gamma mixture prior. A gamma mixture density is defined as follows. Let $\eta = \{\eta_i, i \geq 1\}$ $\theta = \{\theta_i, i \geq 1\}$, and $\mathbf{c} = \{c_i, i \geq 1\}$ be non-negative sequence of real numbers such that $\sum_i c_i = 1$. Then, a gamma mixture density g is written as

$$g(\lambda|\eta, \theta, c) = \sum_i c_i q(\eta_i, \theta_i)\lambda^{\eta_i - 1} e^{-\theta_i \lambda} \equiv \Gamma(\eta, \theta, c) \tag{4.1}$$

where $q(\eta_i, \theta_i) = \frac{\theta_i^{\eta_i}}{\Gamma(\eta_i)}$ is the normalizing constant for the ith gamma kernel $\lambda^{\eta_i - 1} e^{-\theta_i \lambda}$, and c is the sequence of mixing coefficients. We refer to the above distribution as a gamma mixture with parameters η, θ, c.

Specifically, for the arrival rate Λ, we assume that the *prior*, $\pi_{0,\Lambda}$ is a gamma mixture with parameters η_0, θ_0, c_0 and denote this density function as $\pi_{0,\Lambda} \equiv \Gamma(\eta_0, \theta_0, c_0)$ where $\eta_0 = \{\eta_{0,i}, i \geq 1\}$ and the others are similarly defined. It will be illustrated in the next sections that under the observation scheme of the present model with two substitutable products, the conjugacy property of the gamma mixture distribution is retained. We make no assumptions about the distribution of the prior for the probabilities, $\pi_{0,R}$.

4.1 First Period Analysis

We start the Bayesian estimation with the first period. Suppressing the period index, $D = (x, y, z, w, t_{AB}, t_A, t_B)$ denotes the data that became available after the first period. The initial prior distribution of Ψ denoted by $\pi_{0,\Psi}$ is updated to find the *posterior* density of Ψ, given D. This posterior density, denoted by $\pi_{1,\Psi}(\cdot|D)$ will be the prior density for the second period. We have

$$\pi_{0,\Psi}(\lambda, a, b, \alpha, \beta) = \pi_{0,\Lambda}(\lambda) \times \pi_{0,R}(a, b, \alpha, \beta).$$

Note that since we assume Poisson arrivals, for the assessment of the likelihood functions the time interval over which arrivals occur and the rates of arrivals are crucial. As the arrival rates are described as fractions of a general rate λ, the total arrival rate during a period can be described as

$$\lambda[t_{AB}(a+b)+t_A(a+\alpha b)+t_B(b+\beta a)] \equiv \lambda L \tag{4.2}$$

where L considered as the total time over which demand demand arrivals are observed. In other words, the total time in a period where at least one of the products is available in the stocks.

Letting $L=[t_{AB}(a+b)+t_A(a+\alpha b)+t_B(b+\beta a)]$, the likelihood is written as

$$l(x,y,z,w|\Psi) \propto \frac{1}{x!y!z!w!}\lambda^{x+y+z+w}e^{-\lambda L}a^x b^y (b+\beta a)^z (a+\alpha b)^w \tag{4.3}$$

For brevity, let $\mathbf{v}=(\lambda,a,b,\alpha,\beta)$ and $\mathbf{r}=(a,b,\alpha,\beta)$. Henceforth, we use these vectors either explicitly or sometimes in the vector form with a slight abuse of notation when multiple integrals are involved. Using the Bayes framework, we multiply the above likelihood by the joint prior and obtain the posterior distribution of Ψ as

$$\begin{aligned}\pi_{1,\Psi}(\lambda,a,b,\alpha,\beta|D) &= \frac{l(x,y,z,w|\Psi)\pi_{0,\Psi}(\lambda,a,b,\alpha,\beta|D)}{\int_0^\infty l(x,y,z,w)|\Psi=\mathbf{v})\pi_{0,\Psi}(\mathbf{v}|\mathbf{D})d\mathbf{v}} \\ &\equiv \frac{K_1(\lambda,a,b,\alpha,\beta)}{\int K_1(\mathbf{v})d\mathbf{v}} \end{aligned}\tag{4.4}$$

where,

$$K_1(\lambda,a,b,\alpha,\beta)=\sum_i c_{0,i}\lambda^{x+y+z+w+\eta_{0,i}-1}e^{-\lambda(L+\theta_{0,i})}a^x b^y (b+\beta a)^z (a+\alpha b)^w \pi_{0,R}(a,b,\alpha,\beta) \tag{4.5}$$

We can further restructure the posterior density to write it in the form of the marginal posterior of R and the marginal posterior of Λ given R. To this end we observe that K_1 function above involves the kernel of a gamma densities in the summation, however missing the normalizing constants. We therefore multiply and divide these densities to get gamma densities and then adjust the mixing parameters to obtain a mixture gamma density. In particular, we write the updated parameters of the gamma mixing densities and mixing coefficients as

$$\eta_{1,i} = x + y + z + w + \eta_{0,i}, \quad \theta_{1,i} = L + \theta_{0,i}, \quad \text{and} \qquad c_{1,i} = \frac{c_{0,i}/q(\eta_{1,i}, \theta_{1,i})}{\sum_j c_{0,j}/q(\eta_{1,j}, \theta_{1,j})}$$

Now, letting $\eta_1 = \{\eta_{i,1}\}$ and similarly defining θ_1 we have

$$\begin{aligned} \pi_{1,\Psi}(\lambda, a, b, \alpha, \beta | D) &= \pi_{1,\Lambda}(\lambda | a, b, \alpha, \beta) \times \pi_{1,R}(a, b, \alpha, \beta) \\ &= \Gamma(\eta_1, \theta_1, c_1) \times \frac{a^x b^y (b + \beta a)^z (a + \alpha b)^w \pi_{0,R}(a, b, \alpha, \beta)}{\int a^x b^y (b + \beta a)^z (a + \alpha b)^w \pi_{0,R}(\mathbf{r}) d\mathbf{r}} \end{aligned} \tag{4.6}$$

Some fundamental remarks are in order:

(i) The above structure implies that the marginal posterior density, $\pi_{1,R}$, of R is given in a closed explicit form.

(ii) Without specifying the structure of $\pi_{0,R}(a, b, \alpha, \beta)$, it is not possible to provide a more explicit expression for the marginal density of Λ.

(iii) However, most importantly, the conditional distribution of Λ given R is a gamma mixture density where both the mixing coefficients and the parameters of the mixing densities are updated with the observed data.

Hence, we observe that the conjugacy property of gamma mixtures are retained for the conditional distribution of Λ given R.

In order to elucidate some of the important characteristics of the joint posterior density, we visit the special cases arising from the type of the observed data. These cases also illustrate in detail how the general likelihood expression above has been obtained.

Type-N Period: If the first period is of Type-N, the observed demands are such that no stock-out occurs. That is, $S = (x, y, 0, 0)$, $T = (\tau, 0, 0)$ and

$L = \tau(a+b)$. The likelihood of this event is written as

$$l(x,y)|\Psi) \propto \frac{1}{x!y!}\lambda^{x+y}e^{-\lambda\tau(a+b)}a^x b^y \tag{4.7}$$

Then, the updated parameters of the gamma mixing densities and the updated mixing coefficients are given as

$$\eta_{1,i} = \eta_{0,i} + x + y, \ \ \theta_{1,i} = \theta_{0,i} + L, \ \ , c_{1,i} = \frac{c_{0,i}/q(\eta_{1,i},\theta_{1,i})}{\sum_j c_{0,j}/q(\eta_{1,j},\theta_{1,j})}.$$

Letting again $\eta_1 = \{\eta_{1,i}\}$, and similarly defining θ_1 and c_1, we can write

$$\begin{aligned} \pi_{1,\Psi}(\lambda,a,b,\alpha,\beta|D) &= \pi_{1,\Lambda}(\lambda) \times \pi_{1,R}(a,b,\alpha,\beta) \\ &= \Gamma(\eta_1,\theta_1,c_1) \times \frac{a^x b^y \pi_{0,R}(a,b,\alpha,\beta)}{\int a^x b^y \pi_{0,R}(\mathbf{r})d\mathbf{r}}. \end{aligned} \tag{4.8}$$

When the initial joint prior density for Ψ is chosen as a product of a gamma mixture density (for Λ) and an unpsecified general density (for R), if there is no stockout resulting in any substitution, the joint posterior reduces to the product of the marginal posteriors, as well. This implies the independence of Λ and R. Furthermore, the posterior marginal distribution of Λ remains as a mixture of gamma densities where the mixing coefficients do not change, but the parameters of the mixing gamma densities are updated. That is, the joint posterior obtained retains the structure of the initial joint prior in this case.

Type-A Period : If the first period is of Type-A, stock-out occurs only for product A at time T_A. Up to this instance, the primary demand rates for the products are Λ_A and Λ_B. However, after T_A, the demand rate for product B shifts to $\Lambda_{B\bar{A}}$. In this case, the observed data become $S = (S_A, y, 0, w)$ $T = (\tau_A, 0, \tau - \tau_A)$ and $L = \tau_A(a+b) + (b+\beta a)(\tau - \tau_A)$. The likelihood can be written as

$$l(x,y,z,w)|\Psi) \propto \frac{1}{S_A!y!w!}\lambda^{S_A+y+w}e^{-\lambda[\tau_A(a+b)+(b+\beta a)(\tau-\tau_A)]}a^x b^y (b+\beta a)^z (a+\alpha b)^w \tag{4.9}$$

and the updated parameters are given by

$$\eta_{1,i} = \eta_{0,i} + S_A + y + w, \;\; \theta_{1,i} = \theta_{0,i} + L, \;\; c_{1,i} = \frac{c_{0,i}/q(\eta_{1,i}, \theta_{1,i})}{\sum_i c_{0,i}/q(\eta_{1,i}, \theta_{1,i})}$$

Then,

$$\begin{aligned} \pi_{1,\Psi}(\lambda, a, b, \alpha, \beta|\mathbf{D}) &= \pi_{1,\Lambda|R}(\lambda|a, b, \alpha, \beta) \times \pi_{1,R}(a, b, \alpha, \beta) \\ &= \Gamma(\eta_1, \theta_1, c_1) \times \frac{a^{S_A} b^y (b+\beta a)^w (\pi_{0,R}(a, b, \alpha, \beta)}{\int a^x b^y \pi_{0,R}(\mathbf{r}) d\mathbf{r}}. \end{aligned} \tag{4.10}$$

Type-B Period: This period is similar to Type-A period, where the roles of products A and B are switched, hence it is not elaborated further.

Type-AB Period: In this type, first the product A stocks out at time τ_A and then the product B stocks out at time $\tau_B < \tau$. Let y be the amount of product B sold in $[0, \tau_A]$. Then, $S_B - y$ items of product B will be sold in the interval $[\tau_A, \tau_B)$. In this case $S = (S_A, y, 0, S_B - y)$, $T = (\tau_A, 0, \tau_B - \tau_A)$ and $L = \tau_A + (b + \beta a)(\tau - \tau_A)$ so that the likelihood is

$$l(x, y, z, w)|\Psi) = \frac{1}{S_A! y! (S_B - y)!} \lambda^{S_A + S_B} e^{-\lambda(T_A + (b+\beta a)(\tau - \tau_A))} a^x b^y (b+\beta a)^z (a+\alpha b)^w \tag{4.11}$$

We update the parameters of the posterior densities as

$$\eta_{1,i} = \eta_{0,i} + S_A + S_B; \;\; \theta_{1,i} = \theta_{0,i} + L; \;\; c_{1,i} = \frac{c_{0,i}/q(\eta_{1,i}, \theta_{1,i})}{\sum_j c_{0,j}/q(\eta_{1,j}, \theta_{1,j})}$$

Then, we obtain the posterior density as

$$\begin{aligned} \pi_{1,\Psi}(\lambda, a, b, \alpha, \beta|\mathbf{D}) &= \pi_{1,R}(a, b, \alpha, \beta) \pi_{1,\Lambda|R}(\lambda|a, b, \alpha, \beta) \\ &= \Gamma(\eta_1, \theta_1, c_1) \times \frac{a^{S_A} b^y (b+\beta a)^{S_B - y} \pi_{0,R}(a, b, \alpha, \beta)}{\int a_A^S b^y (b+\beta a)^{S_B - y} \pi_{0,R}(\mathbf{r}) d\mathbf{r}} \end{aligned} \tag{4.12}$$

Type-BA Period: This period is similar to the Type-AB period with the roles of products A and B switched, hence also omitted for further details.

Remark: For the last four cases, we observe that the joint posterior density of Λ and R can no longer be written as the product of marginals, implying that Λ and R are no longer independent. However, this joint density can still be written as the product of a gamma mixture density (conditional density of Λ given R, π) and the posterior marginal density of R.

4.2 Multiple Period Analysis

From the second period onwards, starting with the prior structure specified above, the posterior density obtained for any period j, will be of the same form. Recalling that $\mathbf{D}= \{S_j = (x_j, y_j, z_j, w_j), T_j = (t_{ABj}, t_{Aj}, t_{Bj}), j = 1, ..., n\}$ denotes all the accumulated information over the $n-$period observation span, this result is formally stated in the following.

Theorem 1: The joint posterior distribution of Ψ after n periods of observation is given by

$$\begin{aligned} \pi_{n,\Psi}(\lambda, a, b, \alpha, \beta|\mathbf{D}) &= \frac{K_n(\lambda, a, b, \alpha, \beta)}{\int_{\mathbf{v}} K_n(\mathbf{v})d\mathbf{v}} \\ &= \frac{K_{n,\Lambda|R}(\lambda, a, b, \alpha, \beta)K_{n,R}(\lambda, a, b, \alpha, \beta)}{\int_{\lambda,\mathbf{r}} K_{n,\Lambda,R}(\lambda, \mathbf{r})K_{n,R}(\mathbf{r})dud\mathbf{r}} \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} K_n(\lambda, a, b, \alpha, \beta) &= \sum_i c_{0,i}\lambda^{\alpha_{n,i}-1}e^{-\lambda\beta_{n,i}}a^{X^n}b^{Y^n}(b+\beta a)^{Z^n}(a+\alpha b)^{W^n}\pi_{0,R}(a, b, \alpha, \beta) \\ K_{n,\Lambda,R}(\lambda, a, b, \alpha, \beta) &= \sum_i c_{0,i}\lambda^{\alpha_{n,i}-1}e^{-\lambda\beta_{n,i}} \\ K_{n,R}(a, b, \alpha, \beta) &= a^{X^n}b^{Y^n}(b+\beta a)^{Z^n}(a+\alpha b)^{W^n}\pi_{0,R}(a, b, \alpha, \beta) \end{aligned} \tag{4.14}$$

with

$$\begin{aligned} \eta_{n,i} &= \eta_{0,i} + \sum_{j=1}^{n}(x_j + y_j + z_j + w_j) \\ \theta_{n,i} &= \theta_{0,i} + \sum_{j=1}^{n} t_{ABj}(a+b) + t_{Aj}(a+\alpha b) + t_{Bj}(b+\beta a)) \\ X^n &= \sum_{j=1}^{n} x_j,\ \ Y^n = \sum_{j=1}^{n} y_j,\ \ Z^n = \sum_{j=1}^{n} z_j,\ \ W^n = \sum_{j=1}^{n} w_j \end{aligned} \tag{4.15}$$

Proof: First we note that in the one period analysis the updated parameters of the gamma mixing densities and mixing coefficients can be written as

$$\eta_{1,i} = x + y + z + w + \eta_{0,i}, \quad \theta_{1,i} = L + \theta_{0,i}, \quad \text{and} \qquad c_{1,i} = \frac{c_{0,i}/q(\eta_{1,i}, \theta_{1,i})}{\sum_j c_{0,j}/q(\eta_{1,j}, \theta_{1,j})}$$

The first period analysis reveals the fundamental updating characteristics:

(i) When the initial joint prior for Λ and R is chosen as the product of a gamma mixture density (for Λ) and another general density (for R), the mixing parameters for Λ and R are updated in an additive fashion (through the sufficient statistics) depending on the type of the period observed.

(ii) For all period types, the joint posterior density obtained for Λ and R also retains the same structure - the product of a gamma mixture and a general density.

By induction, from the second period onwards, starting with the prior structure specified above, the posterior density obtained for any period j, will be of the same form.

The structure of the above joint density allows us a further decomposition. In particular, we observe from (4.13) that the joint density is written as the product of two terms, one solely a function of the R ($K_{n,R}$)and the other a function of both R and Λ ($K_{n,\Lambda,R}$), the latter being in the form of a mixture gamma kernel. We exploit this structure to get mixture of gamma densities after multiplying and dividing by proper constants, and we obtain the following result that explicitly gives the mixture gamma density of the conditional marginal posterior for the Poisson arrival rate.

Corollary 1: The joint posterior of Ψ at the end of the nth period is the product of the marginal posterior of R and the conditional posterior of Λ given R, as written below

$$\begin{aligned} \pi_{n,\Psi}(\lambda, a, b, \alpha, \beta|\mathbf{D}) &= \pi_{n,\Lambda|R}(\lambda) \times \pi_{n,R}(a, b, \alpha, \beta) \\ &= \Gamma(\eta_n, \theta_n, c_n) \times \frac{K_{n,R}(\mathbf{r})}{\int K_{n,R}(\mathbf{r}) d\mathbf{r}} \end{aligned} \tag{4.16}$$

where

$$c_{n,i} = \frac{c_{0,i}/q(\eta_{n,i}, \theta_{n,i})}{\sum_j c_{0,j}/q(\eta_{n,j}, \theta_{n,j})}$$

The above corollary provides us with a form useful in practice for computing the characteristics of the marginal densities such as the mean, variance and higher moments of the quantities of interest.

Thus, we have provided the Bayes estimation methodology to find the joint posterior distribution of the vector $\Psi = (\Lambda, P_A, P_B, \gamma_A, \gamma_B) \equiv (\Lambda, R)$ where the base demand rate Λ is assumed to have a mixture of gamma densities as its prior distribution but the prior distributions for the substitution probabilities are left as general functions. In their generality, Theorem 1 and, subsequently, Corollary 1 comprise our main result for the joint estimation of primary and substitute demand rates.

4.3 Special Case 1: Base Demand Rate Estimation

Above we have provided the Bayes estimation methodology to find the joint posterior distribution of the vector $\Psi = (\Lambda, P_A, P_B, \gamma_A, \gamma_B) \equiv (\Lambda, R)$ where the base demand rate Λ is assumed to have a mixture of gamma densities as its prior distribution but the prior distributions for the substitution probabilities are left as general functions.

In this section, we elaborate two special cases that are commonly encountered in practice and provide results that are directly implementable by practitioners. We consider a case where the customer choices in terms of primary and substitute demands are known but the total demand rate Λ is not known. This case corresponds to the scenario where existing products are introduced to new markets. For the existing products, previous market experience would provide information about customer choices and their substitution behavior. Hence, a highly reliable estimate for R would be readily available. The uncertainty would be in the possible size of the market; that is, the customer arrival rate, Λ. For this special case, some numerical studies are provided in Chapter 6 to illustrate the performance of the suggested estimators for the demand rate using simulated samples. We consider the case when primary demand and substitution probabilities $(P_A, P_B, \gamma_A, \gamma_B)$ are known but the customer arrival rate Λ. In the numerical examples in Chapter 6, we illustrate the convergence of the mean of the posterior distributions to the assumed true mean and the speed of convergence.

This special case corresponds to obtaining the conditional posterior of Λ given R, $\pi_{n,\Lambda|R}$ and directly follows from Corollary 1. Since $\pi_{n,\Lambda|R}$ as provided in (4.16) is a mixture of gamma densities given by $\Gamma(\eta_n, \theta_n, c_n)$, letting $(a, b, \alpha, \beta) \equiv R = (P_A, P_B, \gamma_A, \gamma_B)$ the parameters of the mixture of gamma density is given by

$$\begin{aligned}
\eta_{n,i} &= \eta_{0,i} + \sum_{i=1}^{n}(x_i + y_i + z_i + w_i) \\
\theta_{n,i} &= \theta_{0,i} + \sum_{i=1}^{n} t_{ABi}(P_A + P_B) + t_{Ai}(P_A + \gamma_A P_B) + t_{Bi}(P_B + \gamma_B P_A) \\
X^n &= \sum_{i=1}^{n} x_i,\ Y^n = \sum_{i=1}^{n} y_i,\ Z^n = \sum_{i=1}^{n} z_i,\ W^n = \sum_{i=1}^{n} w_i
\end{aligned} \tag{4.17}$$

As mentioned earlier this form allows us to calculate some basic characteristics easily. In particular, the mean μ_n of the nth posterior density for Λ is computed as

$$\mu_n = \sum_{i=1}^{\infty} c_{n,i}\eta_{n,i}/\theta_{n,i} \tag{4.18}$$

4.4 Special Case 2: Substitute Demand Rate Estimation

Next, we address the estimation of substitution probabilities when arrival rate $\Lambda = \lambda$ and primary demands $P_A = a, P_B = b$ for products A and B are known and only the substitution probabilities are not known. Hence, the Bayesian estimation is applied to $R = (\gamma_A, \gamma_B)$. Recall that when product A stocks out, the demand for product B becomes $P_{B\bar{A}} = P_B + \gamma_B P_A$, and similarly the inflated demand for product A is $P_{A\bar{B}} = P_A + \gamma_A P_B$ when B is not available.

The substitution probabilities can be updated only when a product is sold-out and in the remaining part of the period customers substitute the other product for their primary demand. Then, for example, the distribution of γ_B can be updated only when relevant information is provided by a Type-A or a Type-AB period is observed where product A is depleted. Similarly, for γ_A only Type-B and Type-BA periods provide information. Let us start again with the first period and suppose the observed data is given as $D = (x, y, z, w, t_{AB}, t_{AB}, t_B)$.

As before , we have $L = [t_{AB}(a+b)+t_A(a+\alpha b)+t_B(b+\beta a)]$, and the likelihood satisfies

$$\begin{aligned} l(z, w|\Psi) &\propto e^{-\lambda[t_A(a+\alpha b)+t_B(b+\beta a)]}(a+\alpha b)^z(b+\beta a)^w \\ &= e^{-\lambda t_A(a+\alpha b)}(a+\alpha b)^z \times e^{-\lambda t_B(b+\beta a)}(b+\beta a)^w \qquad (4.19) \end{aligned}$$

We apply the standard calculations for Bayesian methodology to obtain the posterior density. In particular, the above conditional density is multiplied by the joint prior $\pi_{0,R^*}(\alpha, \beta)$ for (γ_A, γ_B) to obtain the joint density of the substitution probabilities and the observed data. Then, the ratio of this joint density to the marginal likelihood of the observed data (obtained by integrating the joint density w.r.t. α, β) is constructed to obtain the first posterior as for $\alpha, \beta \in (0, 1)$

$$\pi_{1,R}(\alpha, \beta|\mathbf{D}) = \frac{e^{-\lambda t_A(a+\alpha b)}(a+\alpha b)^z e^{-\lambda t_B(b+\beta a)}(b+\beta a)^w \pi_{0,R}(\alpha, \beta)}{\int\int e^{-\lambda t_A(a+\alpha b)}(a+\alpha b)^z e^{-\lambda t_B(b+\beta a)}(b+\beta a)^w \pi_{0,R}(\alpha, \beta) d\alpha d\beta}$$

(4.20)

From the structure of the joint posterior given above, we observe the following. If γ_A and γ_B are initially independent, that is, if we can write $\pi_{0,R}(\alpha, \beta) = \pi_{0,\gamma_A}(\alpha)\pi_{0,\gamma_B}(\beta)$, then, the joint posterior can be written as the product of the marginal posteriors, which indicates the independence of the substitution probabilities throughout the observation process. Otherwise, the dependent structure will be retained throughout the following periods; however, still a manageable structure is obtained for the posterior after n-periods since the likelihood is written in a separable product form, as given in Corollary 1.

Chapter 5

Dynamic Estimation of Stock Levels

From inventory management perspective, the objective is to determine the stock levels of the products in an optimal way with respect to a suitable criteria. In typical cases, this criteria is the maximization of the expected profit of period under a given set of cost parameters, such as the holding, lost sales, and possibly substitution costs. Therefore, accurate estimation of demand rates becomes a crucial part of the decision making problem.

Once an updated demand distribution is provided at the beginning of each period, we may assume that the optimum stocking levels for products A and B are determined by the inventory manager. Hence, in order to allow for an estimation method in a more general setting, we assume below that the starting inventory levels of the products may change from one period to the other.

Using the Bayesian Methodology, explained in Chapter 4, we can decide the stock levels that maximizes the total profit of a period. Then, if we arrange our stock levels with respect to optimal values of the previous period, lost sales and the substitution costs will be less than no information environment. For this approach, we need to define the profit per period, first.

5.1 Computation of Expected Profit

We assume that, profit of a product is composed of three components that are: revenue (PR_i), holding cost (HC_i) and the substitution cost (SC_{ij}). Since, z and w are defined as the sales of product A and B, arising from the combined demand of both products after B or A has stock out, respectively, we need to separate the sales arose of primary demand of the product and the ones emerged by substitution. We have stated that the total demand rate of product A, after B has stock out, is $P_{A\bar{B}} = P_A + \gamma_A P_B$. So, it is assumed that the expected sales comes from the primary demand of A is $z\frac{P_A}{P_A+\gamma_A P_B}$. Expected sales due to the substitution of B is $z\frac{\gamma_A P_B}{P_A+\gamma_A P_B}$. Same is true for w with $w\frac{P_B}{P_B+\gamma_B P_A}$ and $w\frac{\gamma_B P_A}{P_B+\gamma_B P_A}$ respectively.

To compute the expected profit of a period, first, profit with respect to period type is multiplied by the likelihood of that type. Then, all five type of period profits are summed up. We can formulate it as follows:

$\sum_i \left[\sum_x \sum_y \sum_w \sum_z (E[\textit{Profit of Period Type-i}] \times \textit{likelihood of Period Type-i}) \right]$
where $i = Type-N, Type-A, Type-B, Type-AB, Type-BA$

5.1.1 Single Period Analysis

Profit and likelihood functions for five period types are computed as follows:

Type-N Period: If the period is of Type-N, the observed demands are such that no stock-out occurs. That is, $S = (x, y, 0, 0)$, $T = (\tau, 0, 0)$ and $L = \tau(a + b)$. The likelihood of this event is written as

$$l((x,y)|\Psi) = \frac{1}{x!y!}\lambda^{x+y}e^{-\lambda\tau(a+b)}a^x b^y \tau^{x+y} \tag{5.1}$$

Expected profit of a type-N Period is

$$E[Profit_N|\Psi] = \sum_{y=0}^{S_B-1}\sum_{x=0}^{S_A-1}[(PR_A x + PR_B y - HC_A(S_A - x)$$

$$-HC_B(S_B - y)) \times l((x,y)|\Psi)] \tag{5.2}$$

Type-A Period : If the period is of Type-A, stock-out occurs only for product A at time τ_A. Up to this instance, the primary demand rates of the products are Λ_A and Λ_B. However, after τ_A, the demand rate for product B shifts to $\Lambda_{B\bar{A}}$. In this case, the observed data become $S = (S_A, y, 0, w)$ $T = (\tau_A, 0, \tau - \tau_A)$ and $L = \tau_A(a+b) + (b+\beta a)(\tau - \tau_A)$. The likelihood can be written as

$$l((S_A, y, w)|\Psi) = \int_0^\tau \frac{1}{(S_A-1)!y!w!}\lambda^{S_A+y+w}e^{-\lambda[\tau_A(a+b)+(\tau-\tau_A)(b+\beta a)]}a^{(S_A)}b^y(b+\beta a)^w \times \tau_A^{S_A+y-1}(\tau-\tau_A)^w d\tau_A \tag{5.3}$$

Then, expected profit is

$$E[Profit_A|\Psi] = \sum_{w=0}^{S_B-y-1}\sum_{y=0}^{S_B-1}[(PR_A(S_A) + PR_B(y+w) - HC_B(S_B-y-w) - SC_{AB}(w\frac{\beta a}{b+\beta a})) \times l((S_A, y, w)|\Psi)] \tag{5.4}$$

Type-B Period: This period is similar to Type-A period, where the roles of products A and B are switched. The likelihood can be written as

$$l((x, S_B, z)|\Psi) = \int_0^\tau \frac{1}{(S_B-1)!x!z!}\lambda^{S_B+x+z}e^{-\lambda[\tau_B(a+b)+(\tau-\tau_B)(a+\alpha b)]} \times a^x b^{(S_B)}(a+\alpha b)^z \tau_B^{S_B+z-1}(\tau-\tau_B)^z d\tau_B \tag{5.5}$$

Expected profit of the type-B period is

$$E[Profit_B|\Psi] = \sum_{z=0}^{S_A-x-1}\sum_{x=0}^{S_A-1}[(PR_A(x+z) + PR_B(S_B) - HC_A(S_A-x-z) - SC_{BA}(z\frac{\alpha b}{b+\alpha b})) \times l((S_B, x, z)|\Psi)] \tag{5.6}$$

Type-AB Period: In this type, first the product A stocks out at time τ_A and then the product B stocks out at time $\tau_B < \tau$. Let y be the amount of product B sold in $[0, \tau_A]$. Then, $S_B - y$ items of product B will be sold in the interval $[\tau_A, \tau_B)$. In this case $S = (S_A, y, 0, S_B - y)$, $T = (\tau_A, 0, \tau_B - \tau_A)$ and $L = \tau_A + (b + \beta a)(\tau - \tau_A)$ so that the likelihood is

$$l((S_A, y, w)|\Psi) = \int_0^{\tau} \int_{\tau_A}^{\tau} \frac{1}{(S_A - 1)!y!(S_B - y - 1)!} \lambda^{S_A + S_B} e^{-\lambda(\tau_A + (\tau_B - \tau_A)(b + \beta a))} \times a^{S_A} b^y (b + \beta a)^{S_B - y} \tau_A^{S_A + y - 1} (\tau_B - \tau_A)^{S_B - y - 1} d\tau_B d\tau_A \tag{5.7}$$

Expected profit is

$$E[Profit_{AB}|\Psi] = \sum_{w=0}^{S_B - y} \sum_{y=0}^{S_B - 1} [(PR_A S_A + PR_B S_B - SC_{AB}(S_B - y)\frac{\beta a}{b + \beta a}) \times l((S_A, y, z)|\Psi)] \tag{5.8}$$

Type-BA Period: This period is similar to the Type-AB period with the roles of products A and B switched. The likelihood is

$$l((x, S_B, z)|\Psi) = \int_0^{\tau} \int_{\tau_B}^{\tau} \frac{1}{(S_B - 1)!x!(S_A - x - 1)!} \lambda^{S_A + S_B} e^{-\lambda(\tau_B + (\tau_A - \tau_B)(a + \alpha b))} \times a^x b^{S_B} (a + \alpha b)^{S_A - x} \tau_B^{S_B + x - 1} (\tau_A - \tau_B)^{S_A - x - 1} d\tau_A d\tau_B \tag{5.9}$$

Expected profit is

$$E[Profit_{AB}|\Psi] = \sum_{z=0}^{S_A - x} \sum_{x=0}^{S_A - 1} [(PR_A S_A + PR_B S_B - SC_{BA}(S_A - x)\frac{\alpha b}{a + \alpha b}) \times l((x, S_B, w)|\Psi)] \tag{5.10}$$

So, total expected profit will be the summation over all period types:

$$E[Profit|\Psi] = \sum E[Profit] I[PeriodType] \tag{5.11}$$

where I is the indicator function for the period type.

5.2 Dynamic Update of Inventory Levels

After defining the expected profit function, we can set an algorithm to calculate the initial stock levels dynamically. Combining the known parameters, obtained data over periods, and the Bayesian Estimation, we aim to maximize the profit per period and in parallel, the total profit over all periods. Updating is done on the mean of the posterior distributions.

We construct the profit maximizing algorithm as follows:

`Step 1`: Given the initial parameter setting $\Psi_0 = (\lambda, a, b, \alpha, \beta)$ and τ; compute the optimal initial inventory levels $S_0^* = (S_{A0}^*, S_{B0}^*)$ that maximizes the $E[Profit|\Psi]$.

`Step 2`: Using obtained S_0^* from `Step 1` and the simulation data, get Ψ_1 by applying the Bayesian parameter estimation approach stated in Chapter(4.1) for one period.

`Step 3`: Use Ψ_1 to compute the expected profit. Decide the profit maximizing S_1^*.

`Step 4`: Repeat `Step 2` and `Step 3` to get Ψ_n and S_n^* for n period of observations.

Following the steps 1 through 4, we first compute an initial inventory level that provides the maximum profit for the given parameter set. Then, Bayesian approach helps us to evaluate and update the inventory levels for the following periods using the provided past data. Every period, we use updated posterior demand rate to calculate the expected profit. Then, again search for the best values of the inventory level for each product.

The above algorithm assumes that the products are perishable. It means, products are usable within the period. The reasoning behind holding a one-period estimation at `Step 2` is this assumption. At the end of each period, we need to renew all of the products that are on hand. Cost of renewal is expressed with the holding cost term in expected profit expression.

Chapter 6

Numerical Study

In this chapter, we provide the results of the numerical experiments conducted to analyze the impacts of parameters on the estimations and the stock level update policy we discussed in previous chapters. The numerical study findings and discussions are provided under two sections.

In Section 6.1, the effects of the problem parameters, such as prior distribution parameters, initial inventory levels or substitution parameters, on the estimated base demand rate is investigated.

In Section 6.2, we conduct a numerical experiment in order to observe the changes in the optimal stock levels and cost when the theory in Chapter 5 is implemented. Profits computed by using fixed stock levels and optimal stock levels are compared. Then, effect of different substitution probabilities and the prior distributions on expected profit is observed.

In our numerical experiments, all of the expressions are evaluated in MATLAB.

6.1 Parameter Sensitivity in Base Demand Rate Estimation

In this section, we illustrate the convergence of the mean of the posterior distribution to the assumed true mean and the speed of convergence. In all of our numerical examples, the customer arrivals are generated from a Poisson process with arrival rate of one per hour ($\lambda = 1$) and a period is taken to consist of 24 hours ($\tau = 24$). It is assumed that stocks are replenished the beginning of each period to their prescribed maximum levels.

To illustrate the estimation of the arrival rate Λ, with known product preference and substitution rates, we considered the following setting with simulated samples. Then, the impacts of four different factors on the computed mean value of Λ is evaluated separately.

6.1.1 Priority Sensitivity

In this section, we examined the impact of prior distribution selection on the mean of the posterior density. For this approach, different mixing densities and weights are chosen for the prior mixture of gamma distributions. The impact on the posterior mean is evaluated in terms of cumulative mean absolute deviation ($CMAD$). $CMAD$ is sum of the absolute difference between estimated mean and actual mean over all periods. ($CMAD = \sum_i (\lambda_i - \lambda)/n$)

In this setting, we took the following set of parameters to generate the simulation data: $r = (a, b, \alpha, \beta) = (0.5; 0.5; 0.3; 0.3)$. Initial stock levels of both products A and B at the beginning of each period are taken as 12 ($S_A = S_B = 12$). We have chosen five different η_0, θ_0 combinations for the prior mixture of gamma densities. Then, for each η_0, θ_0 combination, 9 different values of mixture weight $c_0 = (c_{01}; c_{02})$ has been evaluated. Five different realizations are generated for each η_0, θ_0, c_0 combination. As it can be seen on Table 6.1, different expected values of Λ is tried to be covered by prior selection.

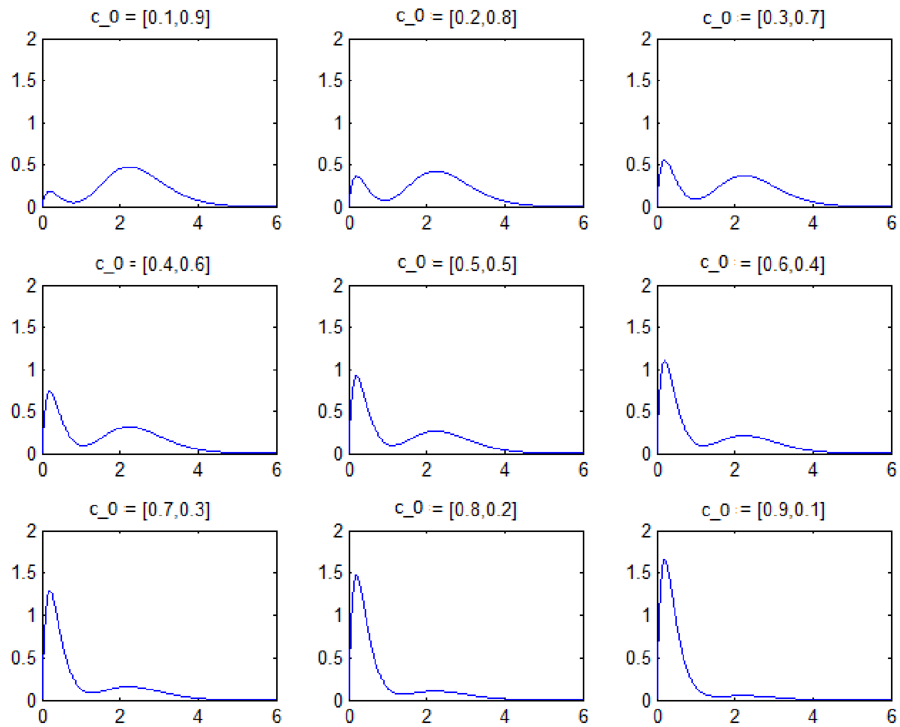


Figure 6.1: Prior Distribution Graph of $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$ for different c_0 values

Figure 6.1 and Figure 6.9 shows how mixture of gamma priors with $\eta_0 = [2; 10], \theta_0 = [5; 4]$ and $\eta_0 = [20; 10], \theta_0 = [6; 7]$changes with respect to different mixing weights, c_0, respectively. The mixing parameters not only has an effect on mean and variance of the distribution, but also they change the shape of the distribution. With the same η_0 and θ_0 values, uni-modal, bimodal or skewed curves can be obtained.

Our aim in this section is keeping track of the estimated posteriors' quality for different mixing weights . As we apply the Bayesian Analysis to five different simulation realizations which generated according to selected prior parameter, change in the mean of posterior demand distributions over consecutive periods can be seen on Figure 6.3. It is clearly seen that posterior means converge to the true mean of the distribution as number of observations increase.

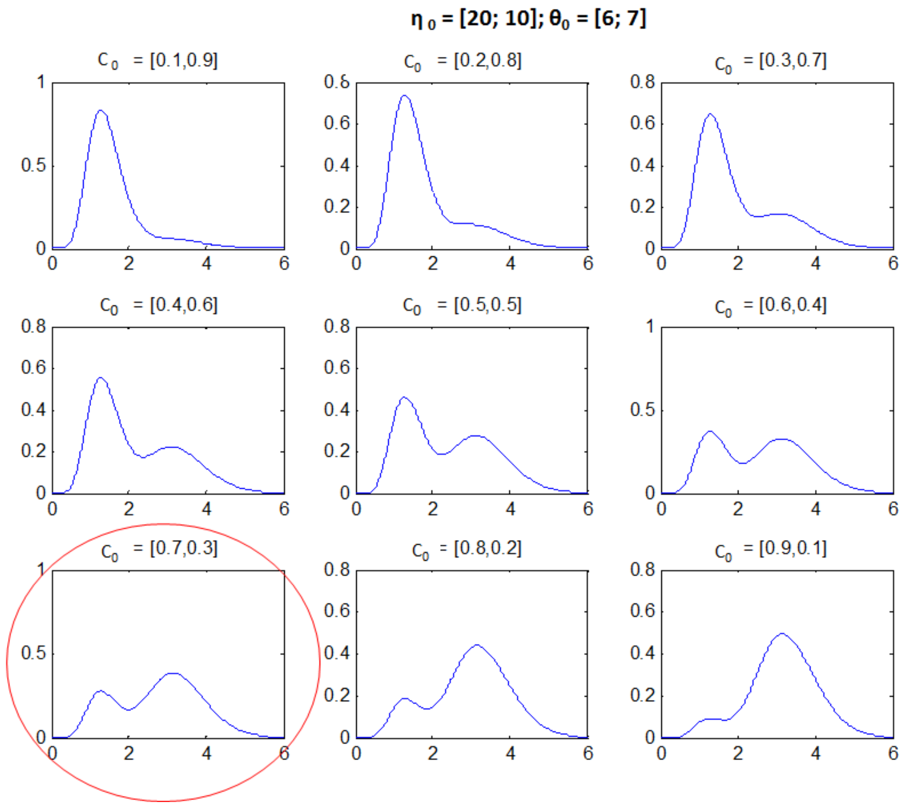


Figure 6.2: Prior Distribution Graph of $\eta_0 = [20; 10], \theta_0 = [6; 7]$ for different c_0 values

As stated before, our key performance indicator here is CMAD values for different c_0 values. We have calculated two different CMAD values for each c_0 in five realizations. First value, $CMAD_0$, corresponds to the overall value throughout the periods. Second value, $CMAD'$ is calculated regarding the periods after 15^{th}. Since, the variance until period 15 is really high for all of the simulation realizations, we want to see the difference between these values.

Table 6.2 and Table 6.3 show the calculated $CMAD_0$ and $CMAD'$ values. Cells with minimum values are filled on the Table. At first sight, it seems that c_0's that minimize $CMAD_0$ and $CMAD'$ are different than each other for the same realization.Except that Realization 2, $CMAD'$ values are always less than $CMAD_0$.

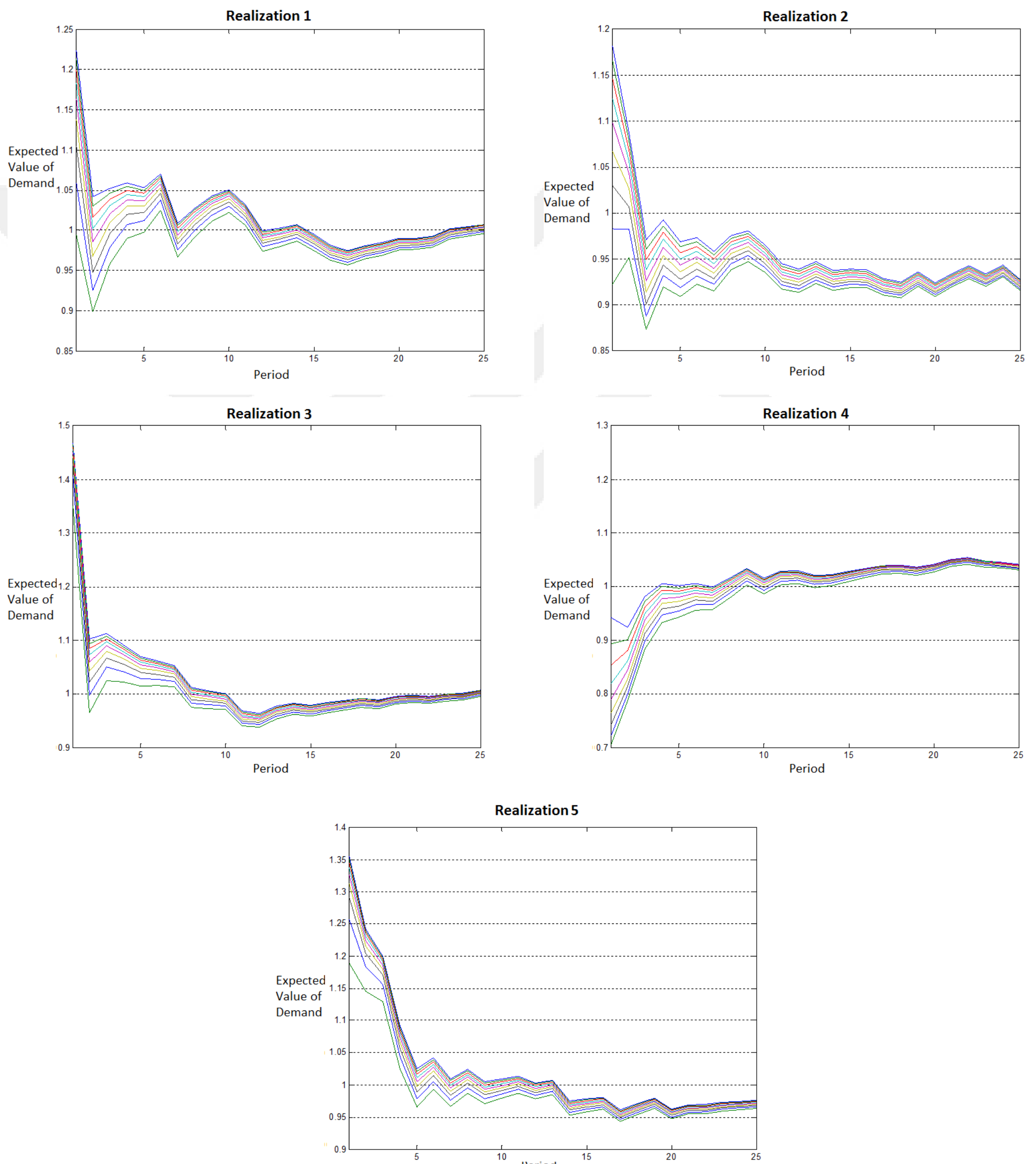


Figure 6.3: Mean of posterior demand distributions over time for 5 different realizations where $\eta_0 = [2; 10], \theta_0 = [5; 4]$

$c_0 =$	(0,1;0,9)	(0,2;0,8)	(0,3;0,7)	(0,4;0,6)	(0,5;0,5)	(0,6;0,4)	(0,7;0,3)	(0,8;0,2)	(0,9;0,1)
η_0=[1,2];θ_0=[2,4]	0,50	0,50	0,50	0,50	0,50	0,50	0,50	0,50	0,50
η_0=[3,5];θ_0=[3,5]	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00
η_0=[20,10];θ_0=[6,7]	1,62	1,81	2,00	2,19	2,38	2,57	2,76	2,95	3,14
η_0=[2,10];θ_0=[5,4]	2,29	2,08	1,87	1,66	1,45	1,24	1,03	0,82	0,61
η_0=[3,6];θ_0=[6,3]	1,85	1,70	1,55	1,40	1,25	1,10	1,95	0,80	0,65

Table 6.1: Prior Mean Values for Different Parameter Sets of Prior Distribution

$c_0 =$	(0,1;0,9)	(0,2;0,8)	(0,3;0,7)	(0,4;0,6)	(0,5;0,5)	(0,6;0,4)	(0,7;0,3)	(0,8;0,2)	(0,9;0,1)
Realization 1	0,031804	0,030093	0,02819	0,026222	0,025249	0,024322	0,023377	0,022843	0,023644
Realization 2	0,057667	0,059345	0,061053	0,062767	0,064459	0,066088	0,067589	0,07165	0,07977
Realization 3	0,046973	0,046455	0,045898	0,045346	0,04473	0,044092	0,043002	0,041448	0,04029
Realization 4	0,031635	0,033903	0,036457	0,038807	0,040818	0,042558	0,044368	0,046093	0,04744
Realization 5	0,054306	0,05353	0,052668	0,051952	0,051354	0,050978	0,050778	0,050175	0,04816

Table 6.2: $CMAD_0$ Values for Different Realizations and c_0 values, where $\eta_0 = [2;10], \theta_0 = [5;4]$

$c_0 =$	(0,1;0,9)	(0,2;0,8)	(0,3;0,7)	(0,4;0,6)	(0,5;0,5)	(0,6;0,4)	(0,7;0,3)	(0,8;0,2)	(0,9;0,1)
Realization 1	0,0096222	0,010049	0,010661	0,011639	0,012759	0,014467	0,016671	0,019535	0,023172
Realization 2	0,064511	0,066264	0,068115	0,070073	0,07215	0,074351	0,076695	0,079194	0,081867
Realization 3	0,01201	0,013034	0,014285	0,015725	0,017379	0,019277	0,021491	0,02441	0,027944
Realization 4	0,037904	0,037054	0,036055	0,034861	0,033411	0,031612	0,029319	0,026295	0,022315
Realization 5	0,024448	0,025317	0,02639	0,027765	0,029346	0,031489	0,033964	0,036857	0,040287

Table 6.3: CMAD' Values for Different Realizations and c_0 values, where $\eta_0 = [2; 10], \theta_0 = [5; 4]$

6.1.2 Initial Inventory Levels

From the different values of initial mixture weight we choose $c_0 = (0,2; 0,8)$ with mean $2,08$. In order to see the effect of different inventory levels on posterior mean, we take the values of $\lambda\tau$, $\lambda\tau \pm \sigma$ and $\lambda\tau \pm 2\sigma$. We have generated two different simulation samples. Figures 6.4, 6.5, 6.6, 6.7 and 6.8 has been generated consequently.

The direct inference can be done using the posterior mean graphs is that estimation quality decreases when the information is not complete. In the cases of $E[\Lambda] - \sigma$ and $E[\Lambda] - 2\sigma$, we face with censored data. We know that demand is more than the initial inventory levels, but we have no chance to observe it. Therefore, we cannot reach the true mean using those initial inventory levels. Estimation quality is less than 50 in $E[\Lambda] - 2\sigma$ case.

On the other hand, it is not easy to make a judgment about how $E[\Lambda] + \sigma$ and $E[\Lambda] + 2\sigma$ cases effect the estimation accuracy. We cannot state that the inventory levels above the expected mean have an influence on the estimation accuracy. Somewhat over estimation is observed when initial stocks are high.

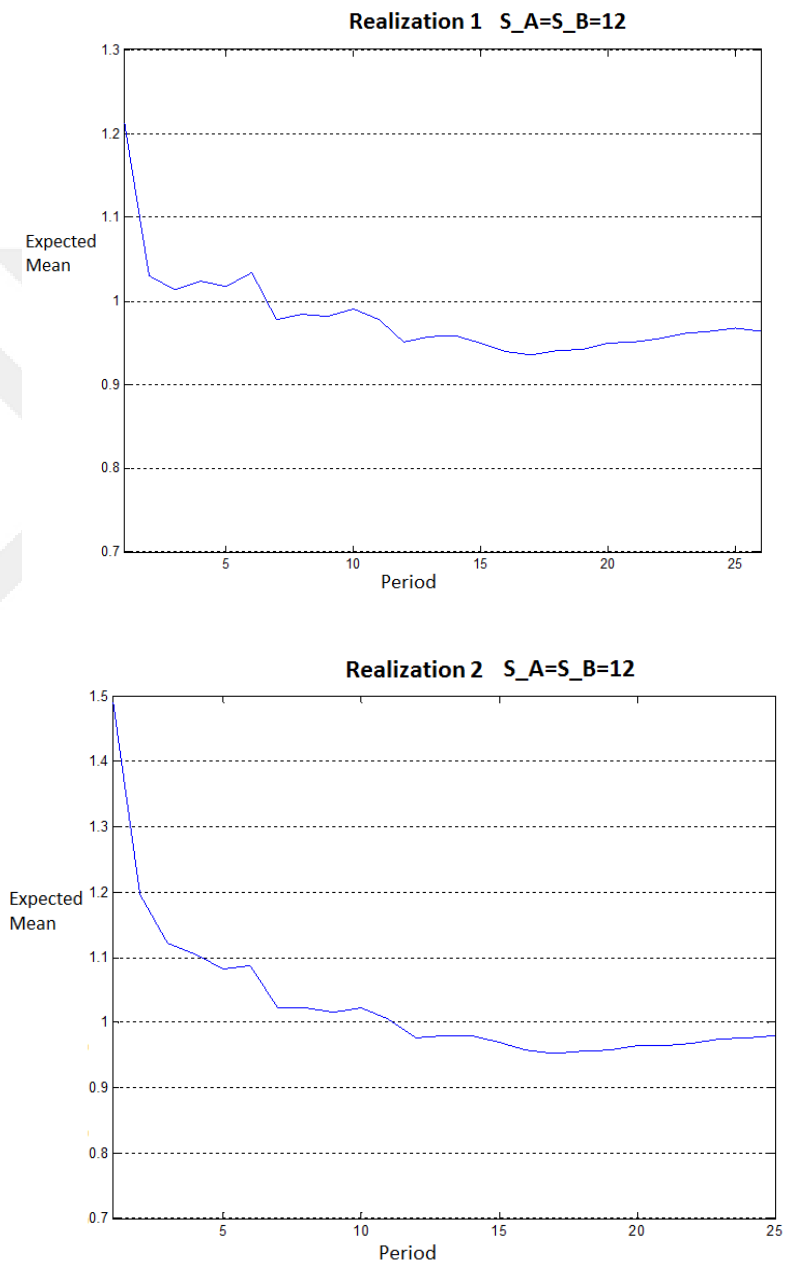


Figure 6.4: Mean of posterior demand distributions over time for the initial inventory levels $S_A = S_B = Mean$

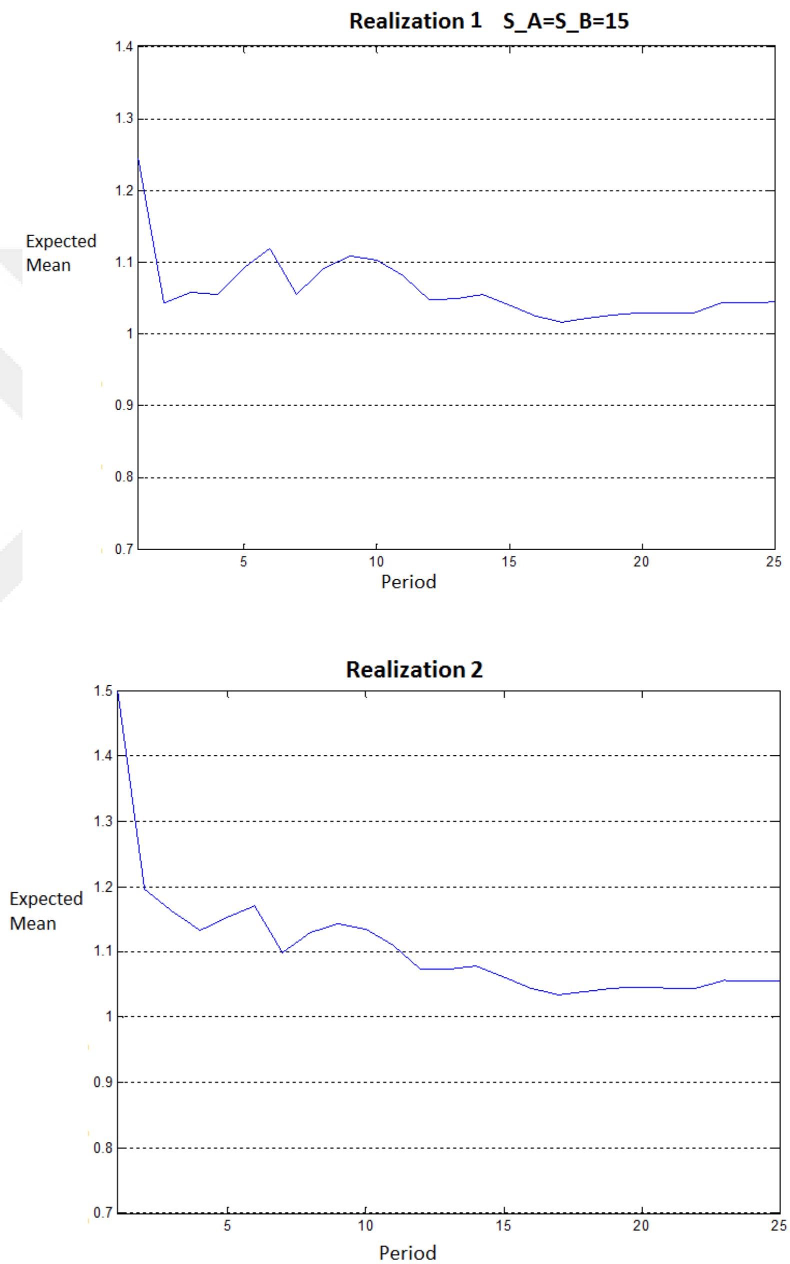


Figure 6.5: Mean of posterior demand distributions over time for the initial inventory levels $S_A = S_B = Mean + \sigma$

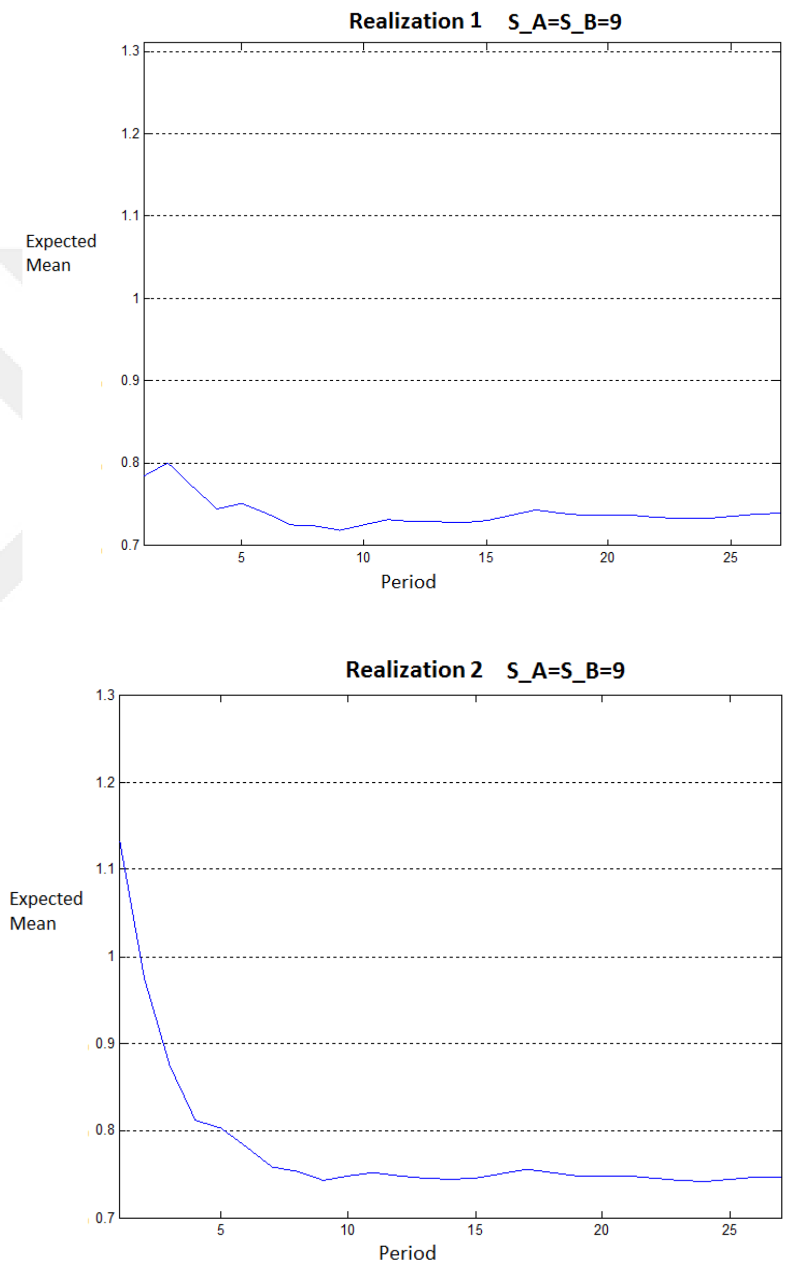


Figure 6.6: Mean of posterior demand distributions over time for the initial inventory levels $S_A = S_B = Mean - \sigma$

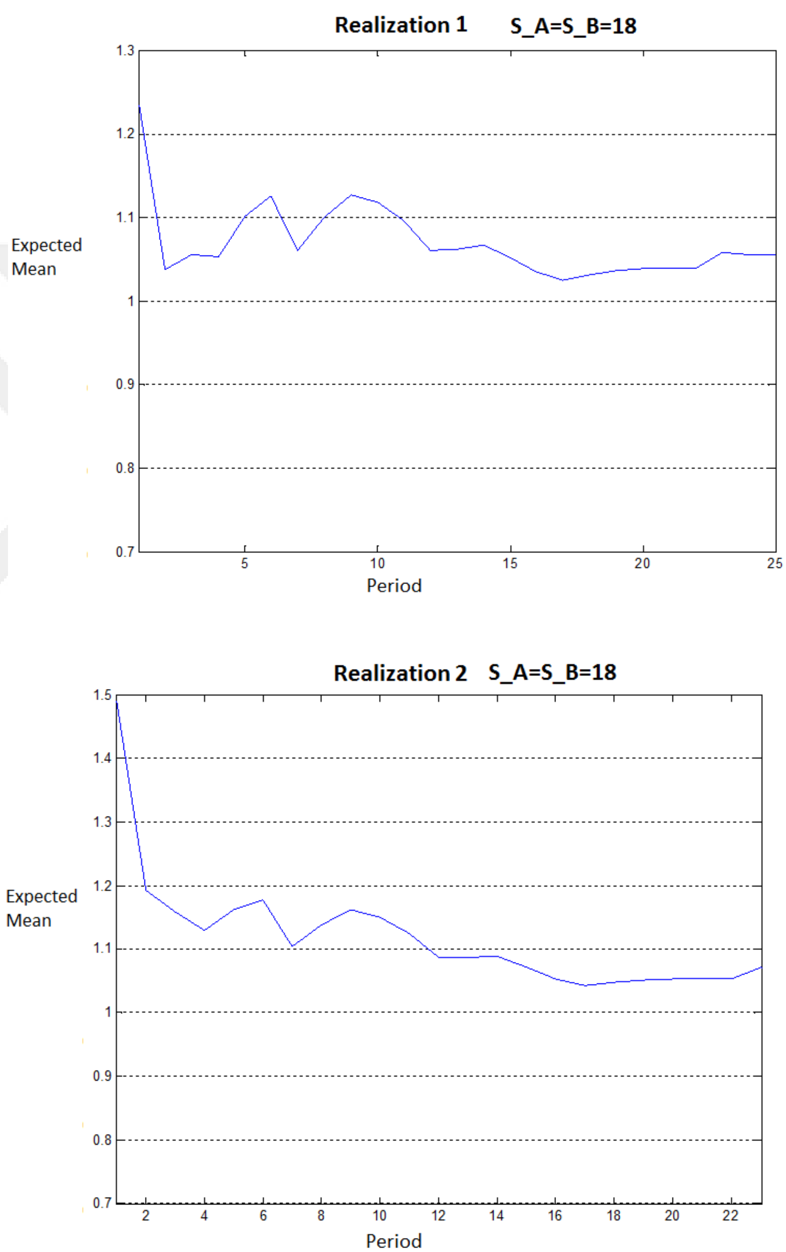


Figure 6.7: Mean of posterior demand distributions over time for the initial inventory levels $S_A = S_B = Mean + 2\sigma$

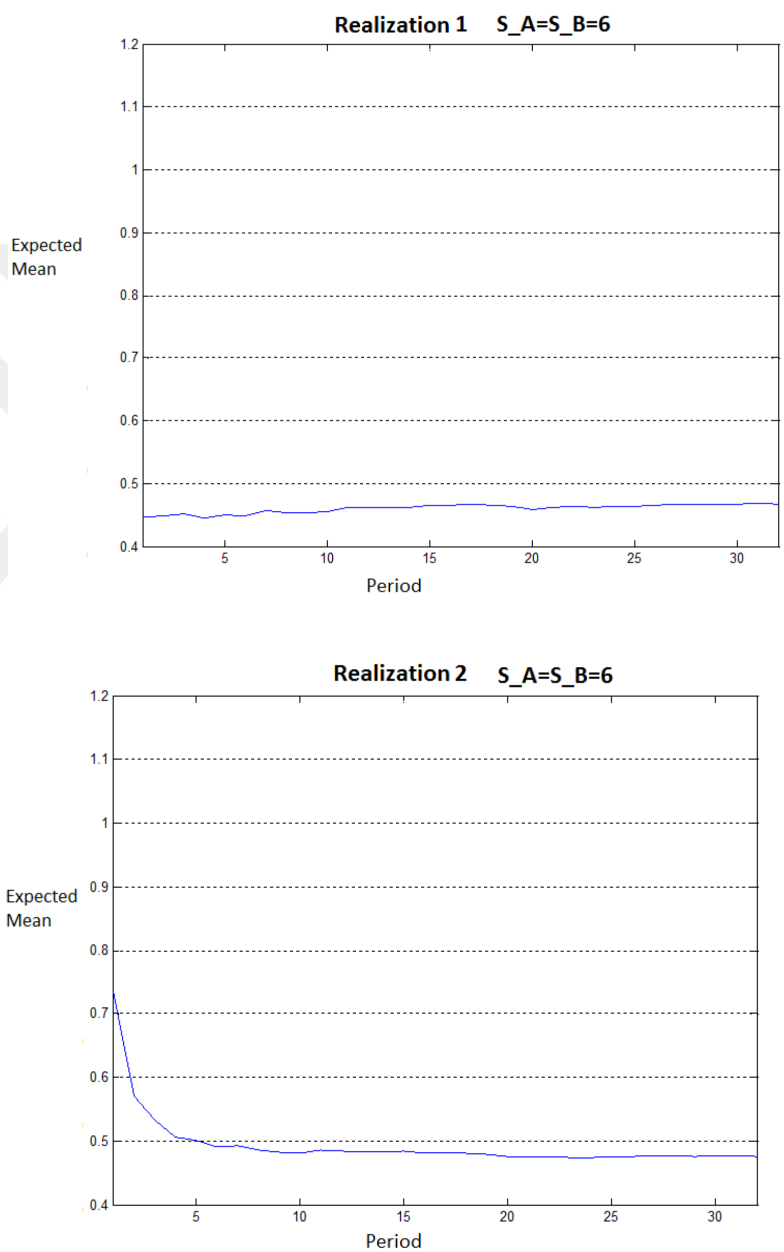


Figure 6.8: Mean of posterior demand distributions over time for the initial inventory levels $S_A = S_B = Mean - 2\sigma$

6.1.3 Impact of Substitution on Estimation of Demand Rate

In this scenario, we have evaluated the impact of different substitution probabilities on posterior mean accuracy. We have selected $\eta_0 = [20; 10], \theta_0 = [6; 7]$ with $c_0 = [0.7; 0.3]$. Selected prior distribution which is marked, can be seen on Figure 6.9. The initial inventory levels S_A and S_B are set to expected value 12. We look for different α and β values for these initial parameters.

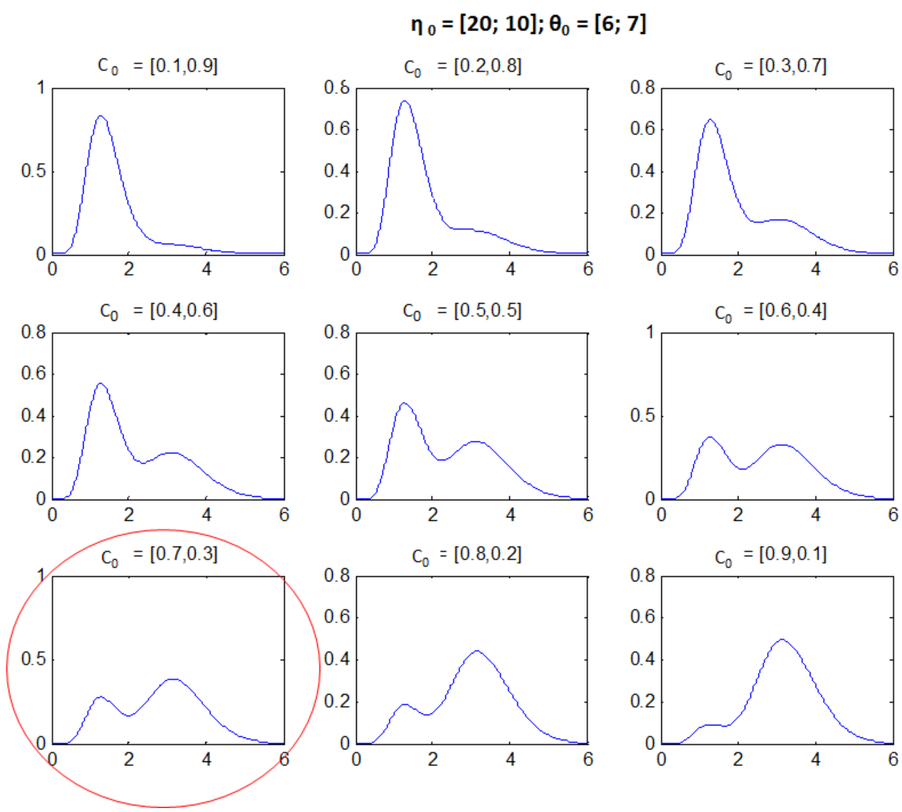


Figure 6.9: Prior Distribution Graph of $\eta_0 = [20; 10], \theta_0 = [6; 7]$ for different c_0 values

Figures 6.10, 6.11 and 6.12 show the change in posterior mean over time with various substitution rate probabilities. As we can see over them, as the substitution probabilities decrease, posterior mean value converges to the true mean. This result is not surprising. Since, the concept of substitution bring extra variability to the model, estimation accuracy is influenced by it. As the substitution rate increase the overall mean seems to be underestimated.

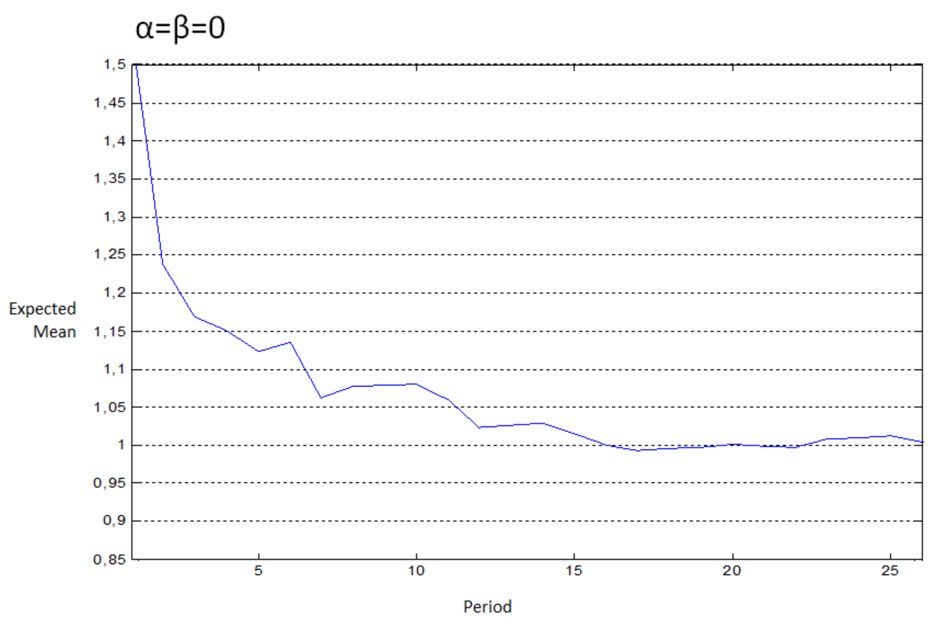


Figure 6.10: Change in the Posterior Mean over Period when there is no substitution($\alpha = \beta = 0$)

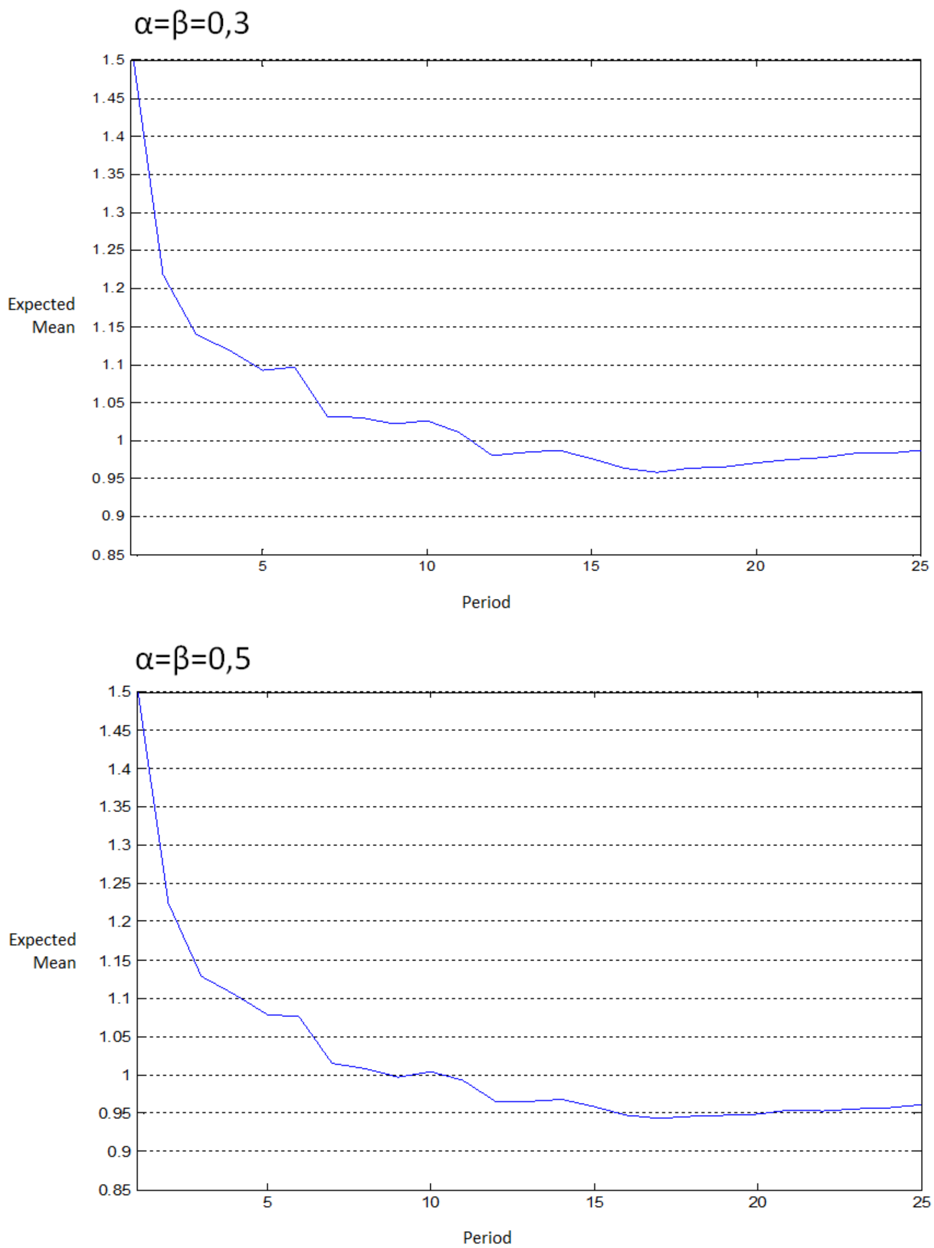


Figure 6.11: Change in the Posterior Mean over Period when $\alpha = \beta = 0.3$ and $\alpha = \beta = 0.5$

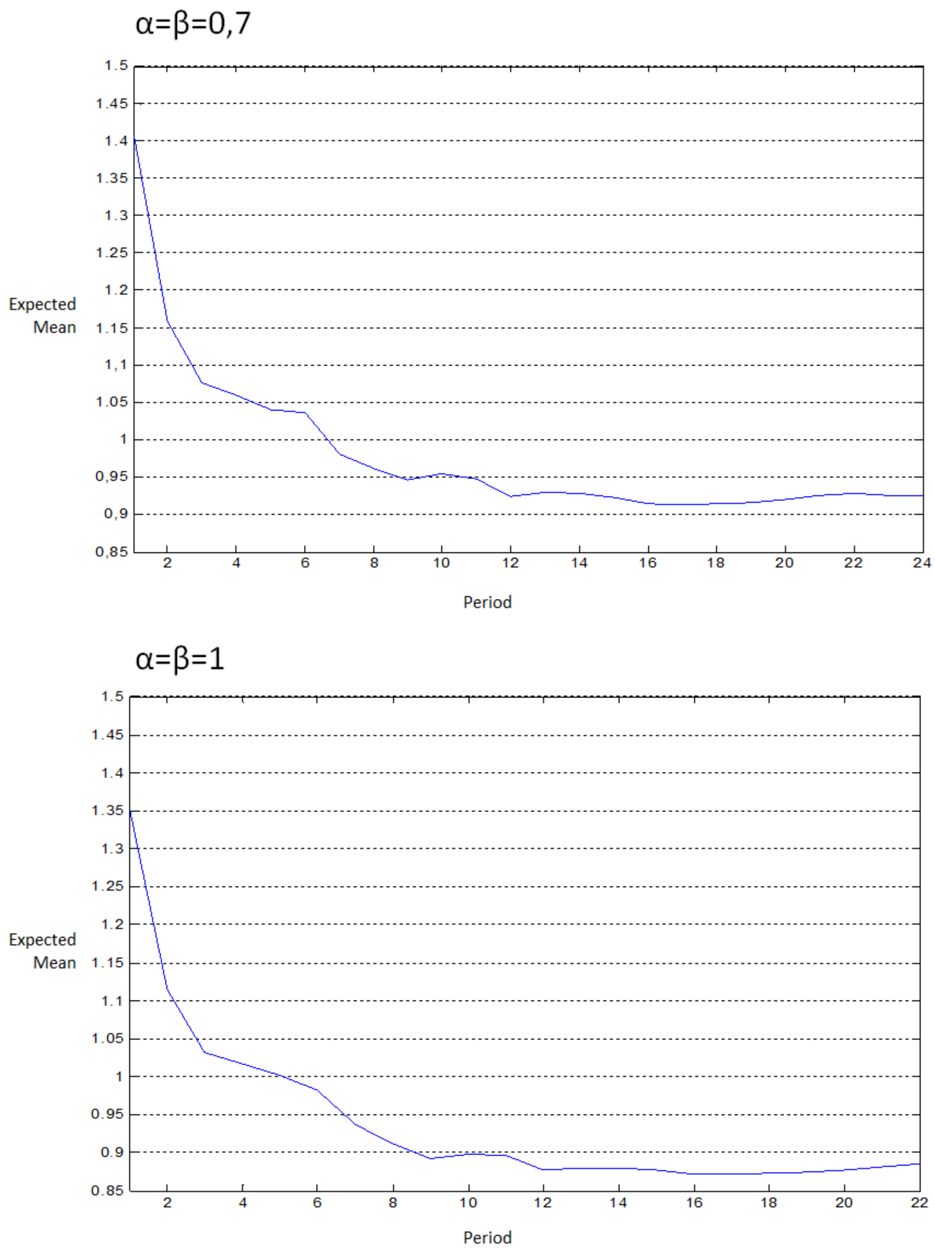


Figure 6.12: Change in the Posterior Mean over Period when $\alpha = \beta = 0.7$ and $\alpha = \beta = 1$

6.2 Dynamic Estimation of Stock Levels

In this section, we apply the Algorithm given in Chapter 5, to a simulation data. Then, change in the profit maximizing inventory levels for each period are evaluated. The impact of different initial inventory levels, at the beginning of each review period, on expected profit is studied. Then, parameter analysis are conducted based on different substitution rates and different prior selections.

6.2.1 Computation of Profit Maximizing Inventory Levels

In order to generate the simulation data, we took the following set of parameters: $\lambda = 1$, $r = (a, b, \alpha, \beta) = (0.6; 0.4; 0.4; 0.4)$, $\tau = 24$, $PR_A = 50, PR_B = 60, HC_A = 10, HC_B = 10, SC_{AB} = 20, SC_{BA} = 20$. Our prior distribution parameters are selected as $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$ and $c_0 = [0.4; 0.6]$ which has a mean of 1.6599 and the variance of 1.4649.

As it is stated in Chapter 5, first step of the numerical study is calculating $S_0^* = (S_{A0}^*, S_{B0}^*)$ value with respect to given parameter set. For this approach, we search for the maximum value of the expected profit for different S_A and S_B values. Execution of the first step gives us the value that $S_0^* = (11, 8)$.

Then, we conduct a single-period Bayesian Estimation to compute the updated demand rate. As the loop stated in the algorithm is completed for three different simulation data sets, obtained demand rate values, expected maximum profits and the optimal inventory levels for each period are listed on Table 6.4, Table 6.5 and Table 6.6 on page 53.

If we analyze the results of our numerical study, we can focus on two main points: (i)Change in the Profit Maximizing Inventory Levels , (ii) Expected Profit Values for Different Initial Inventory Level Policies.

In Figure 6.14 on page 56 , change in the inventory that which maximize the expected profit for updated demand rate is shown for three different simulation realizations. If we watch the change over time or compare the profit maximizing inventory levels with S_0^* values, we cannot observe a specific pattern. They are really dependent on the simulation data. As it is seen on Figure 6.14 , they are positively correlated. They generally decrease or increase at the same time. However, we cannot observe a tendency to converge some value in 20 periods.

Since, it takes so much time to make so many iterations with all of the simulation data, we have selected one of the data sets, namely sata set 3, and conduct a 40 periods ($n = 40$) of analysis for it. However, results obtained are not so different than the $n = 20$ case. On Figure 6.13, it is seen that again there is no convergence in the inventory levels.

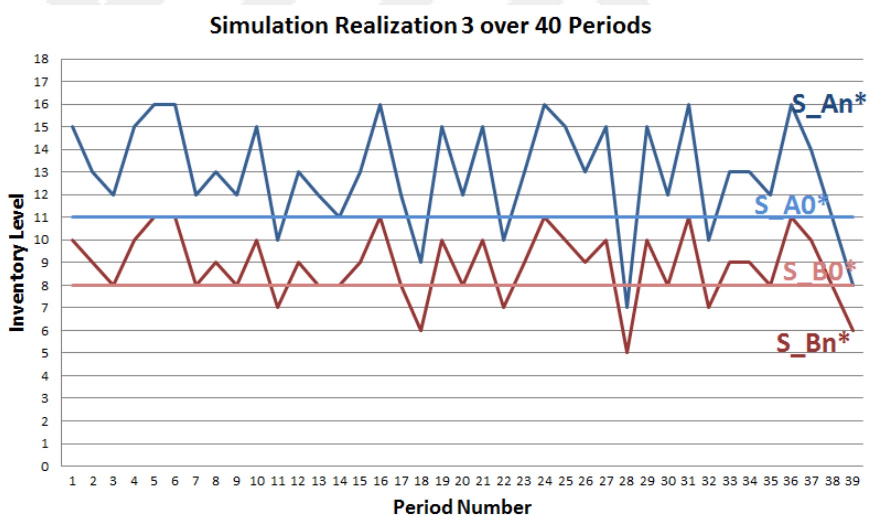


Figure 6.13: Change in the Profit Maximizing Inventory Levels over 40 Review Periods for Simulation Realization 3

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
λ_n	1,34	1,26	1,31	0,88	0,82	1,32	0,95	1,27	0,99	1,2	1,24	1,03	1,24	0,59	1,23	1,18	0,7	1,35	1,36	0,7
$E[Profit_1^*]$	184,64	216,24	208,12	213,53	174,24	167,78	214,7	180,8	209,21	183,62	204,64	206,78	187,44	207,02	140,86	206,2	202,82	154,76	216,82	217,92
S_{An}^*	11	15	14	15	10	9	15	10	14	10	13	13	12	13	7	13	13	7	15	15
S_{Bn}^*	8	10	10	10	7	6	10	7	10	7	9	9	8	9	5	9	9	5	10	10

Table 6.4: Updated Demand Values, Inventory Levels and Maximum Expected Profit for Simulation Realization 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
λ_n	1,25	1,13	1,01	0,85	1,38	1,3	1,31	1,34	0,88	1,39	1,06	1,19	0,82	0,76	0,76	0,82	1,17	1,2	0,85	1,12
$E[Profit_1^*]$	184,64	207,27	196,72	185,28	170,93	218,68	212,76	213,05	216,18	174,31	219,66	191,66	203,69	168,07	160,29	160,29	168,07	201,97	204,56	170,97
S_{An}^*	11	13	12	11	9	15	15	15	15	10	16	12	13	9	8	8	9	13	13	9
S_{Bn}^*	8	9	8	8	6	10	10	10	10	7	11	8	9	6	6	6	6	9	9	6

Table 6.5: Updated Demand Values, Inventory Levels and Maximum Expected Profit for Simulation Realization 2

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
λ_n	1,33	1,24	1,09	1,35	1,41	1,38	1,09	1,21	1,08	1,32	0,88	1,22	1,05	1,02	1,24	1,38	1,09	0,85	1,32	1,04
$E[Profit_1^*]$	184,64	215,04	207,16	194,21	216,78	221,68	219,19	194,49	204,73	193,72	214,03	174,31	205,51	190,66	186,86	206,92	219,05	193,89	170,73	213,91
S_{An}^*	11	15	13	12	15	16	16	12	13	12	15	10	13	12	11	13	16	12	9	15
S_{Bn}^*	8	10	9	8	10	11	11	8	9	8	10	7	9	8	8	9	11	8	6	10

Table 6.6: Updated Demand Values, Inventory Levels and Maximum Expected Profit for Simulation Realization 3

To show the effect of updated inventory levels, we computed the expected profit with a constant value, S_0^* that maximizes the profit for initial parameters of period zero. Figure 6.15 compares the expected profit values over period for updated inventory level case vs. constant inventory levels. As it is seen, for every simulation realization, updated inventory levels gives a better result for expected profit computation for each period. Therefore, we can state that our algorithm improves the expected profit levels in a clear manner.

In order to support the above statement, we have computed the expected profit levels for four different inventory level policies again:
(i) Inventory Levels are updated with respected to the algorithm
(ii) Inventory levels are set to S_0^*
(iii) Inventory levels are set to constant expected levels which are $S_A = \lambda a \tau$ and $S_B = \lambda b \tau$, with respect to the initial parameter set
(iv) Inventory levels are assumed equal and set to $S_A = S_B = \lambda \tau / 2$

In (i), we conduct a one-period Bayesian Estimation which means both the initial belief about the distribution and a one-period data observation is used for all periods. In (ii), we decide the stock levels only using the information that comes from period zero. In (iii), expected mean values computed from the primary demand rates are used as stock levels. Finally in (iv), stock levels are assumed to be equal. Neither the observed data, nor the expectations about the primary demands are used. (iv) can be thought as a case with no information, a blind choice.

Results of the four policies are shown on Figure 6.16. Figure directly shows that updating the inventory levels with respect to previous periods state of knowledge gives the maximum profit levels compered to other three cases.

However, making direct inferences for the other three cases by looking at the Figure 6.16 is not that easy. Different policies are better for different periods according to the figure. So, we have compared the absolute differences from the maximum profit for each period. Those values are shown on Figure 6.17.

Then, to make an overall statement for all periods, we have compared the sum of profits on Table 6.7 and the sum of absolute differences from the maximum profit over all periods on Table 6.8 on page 60.

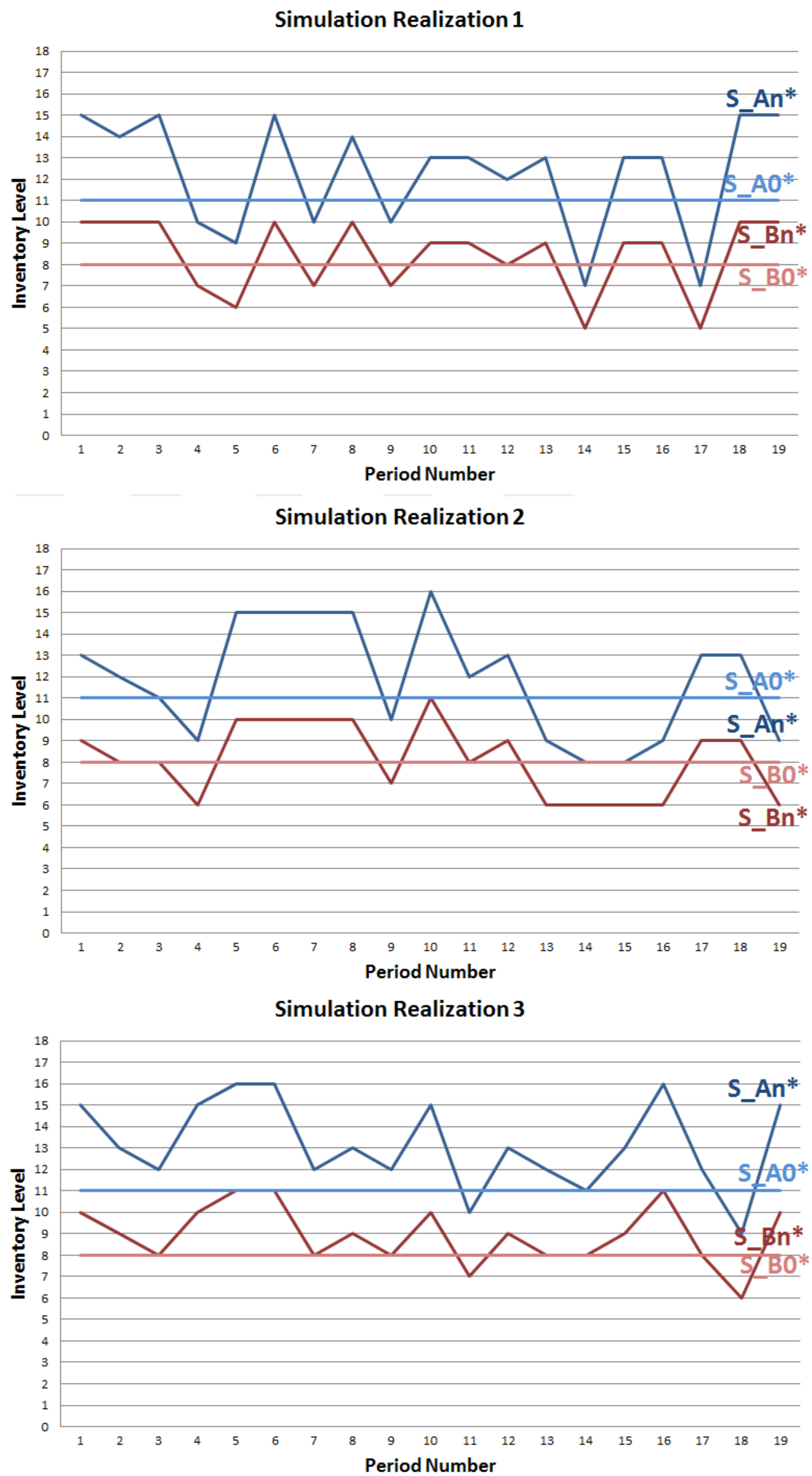


Figure 6.14: Change in the Profit Maximizing Inventory Levels over Review Periods for Three Different Simulation Realizations

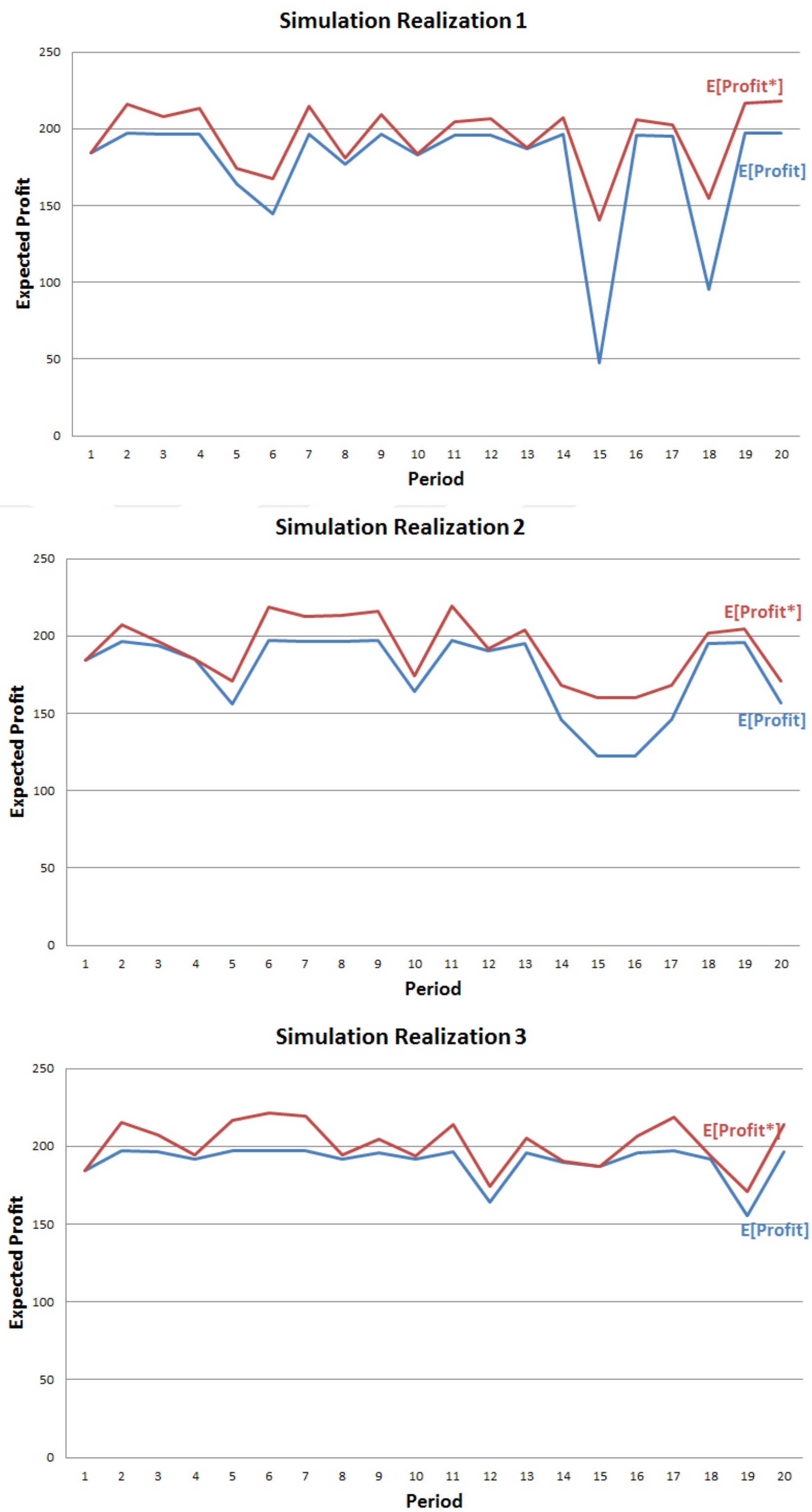


Figure 6.15: Expected Profit Values for Dynamically Calculated vs. Constant (S_0^*) Inventory Levels for Three Different Simulation Realizations

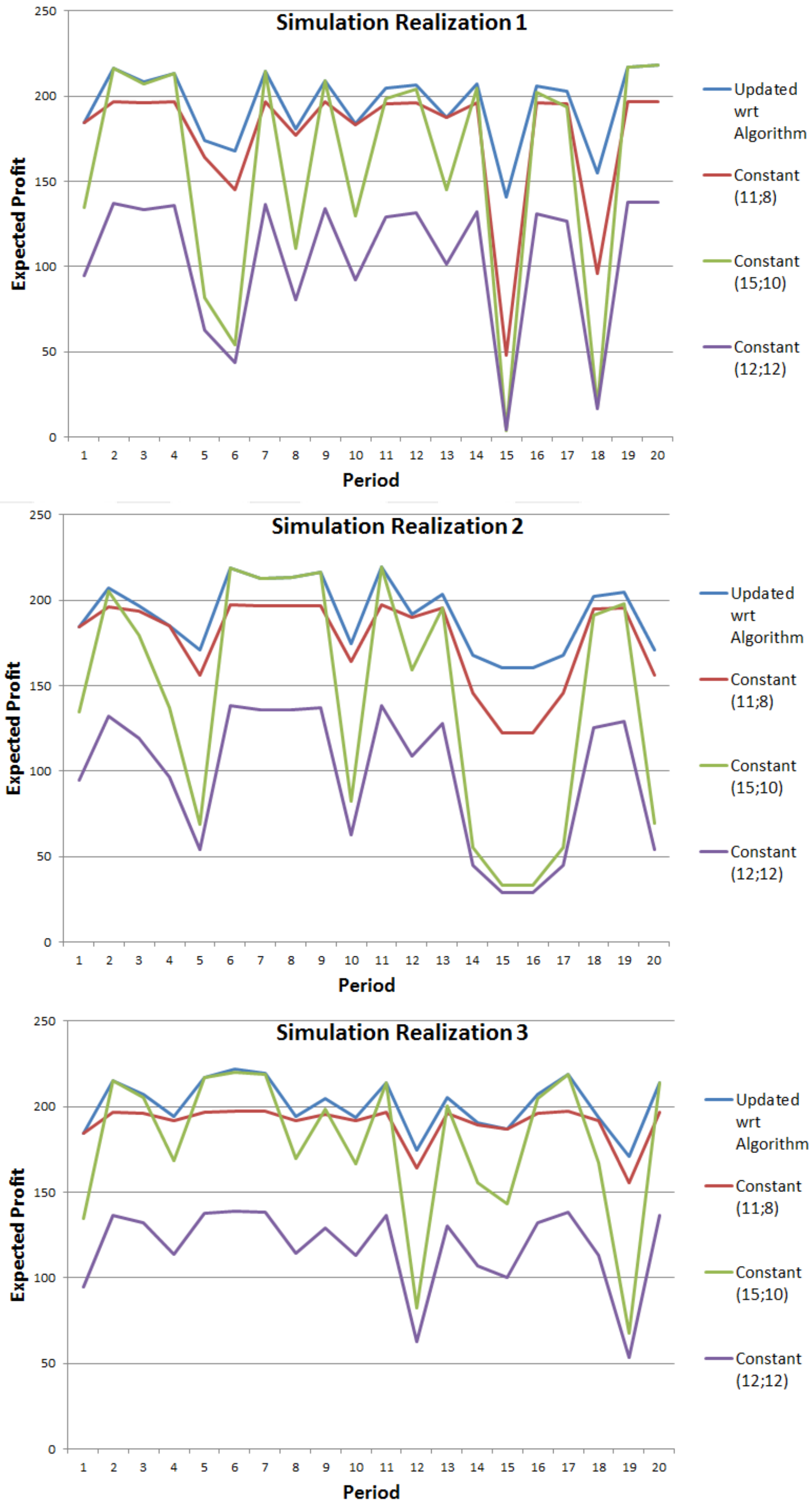


Figure 6.16: Expected Profit Values for Different Initial Inventory Level Policies for Three Different Simulation Realizations

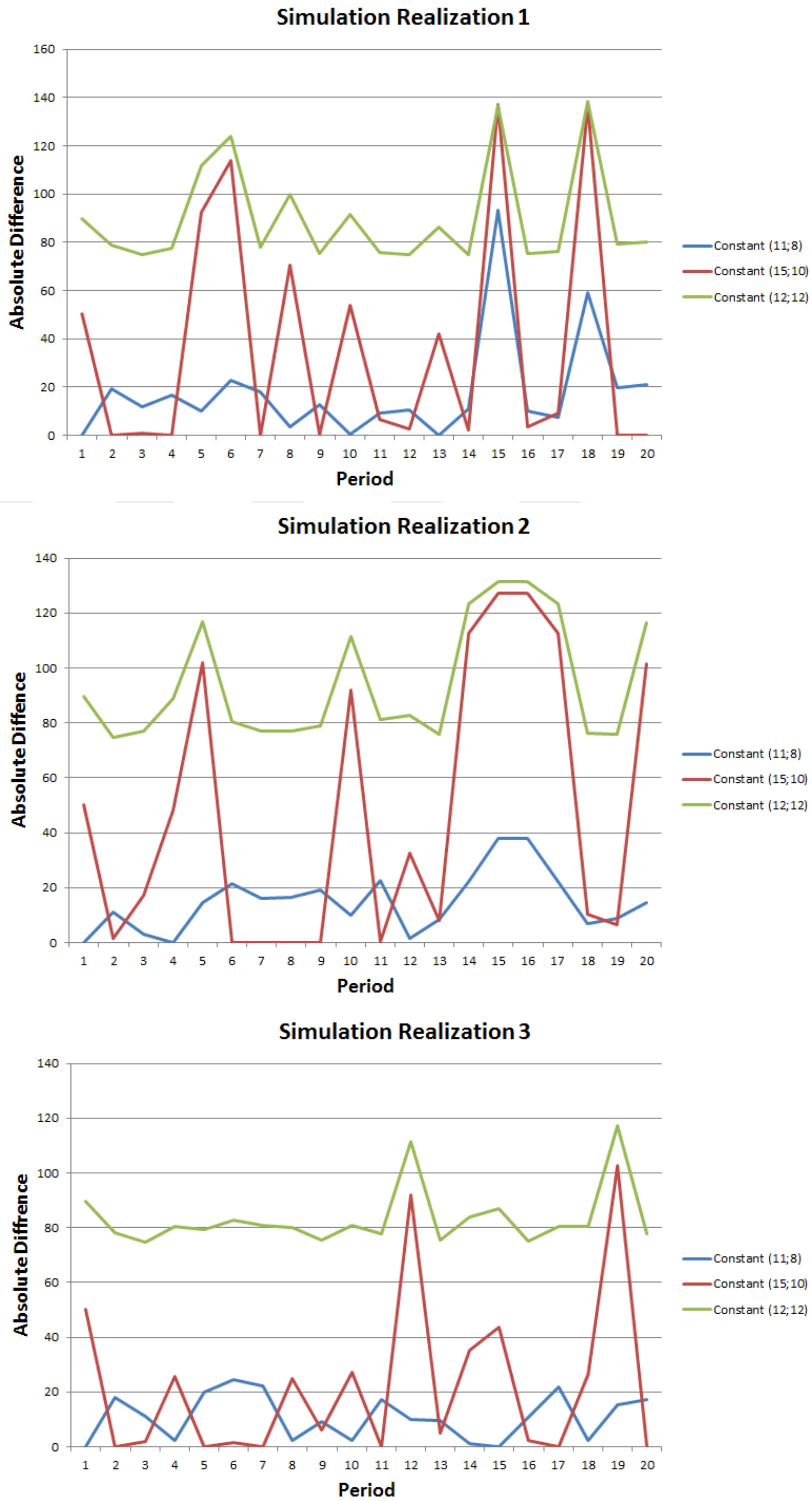


Figure 6.17: Absolute Difference from the Maximum Profit Level for Constant Inventory Level Policies for Three Different Simulation Realizations

	Realization 1	Realization 2	Realization 3
Constant S_0^*	356,72	295,7	216,97
Constant $S_A = \lambda a\tau, S_B = \lambda b\tau$	721,9	951,05	445,00
Constant $S_A = S_B = \lambda\tau/2$	1798,52	1890,65	1668,68

Table 6.7: Absolute Sum of the Differences from the Maximum Profit Levels for Constant Inventory Levels

	Realization 1	Realization 2	Realization 3
Constant S_0^*	3541,43	3533,35	3810,52
Constant $S_A = \lambda a\tau, S_B = \lambda b\tau$	3176,25	2878	3582,50
Constant $S_A = S_B = \lambda\tau/2$	2099,63	1938,39	2358,81

Table 6.8: Total Profit of all periods for Constant Inventory Levels

As it is seen on Table 6.7 and Table 6.8, the best policy of constant inventory levels is setting them to S_0^* value for all of the simulation realizations. Results show us that the more information we have about the data, the more we will earn.

6.2.2 The Effect of Different Substitution Probabilities

In this section, we want to see whether substitution probabilities has an impact on maximum profit and the profit maximizing inventory levels. In previous section, substitution probabilities α and β are assumed to be equal to 0.4. We want to check the extreme cases with no substitution and certain substitution. So, we have generated three more D_j. $D = (x_j, y_j, z_j, w_j, t_{AB_j}, t_{A_j}, t_{B_j})$ data sets for both $\alpha = \beta = 0$ and $\alpha = \beta = 1$ cases using the same data seed. This implies that primary demand or the stock-out times of the products do not change.

In the first place, we executed the algorithm on data and check the profit maximizing inventory levels. As Figure 6.18 shows, inventory levels of both products with no substitution is the highest. As substitution probability increases, inventory levels tend to decrease generally.

Then, maximum profit levels are taken into consideration. In Figure 6.19, calculated profit values of updated inventory level policy is shown. Although there is not a big difference between the values, with no substitution case we earn more. Table 6.9 shows the values of maximum profit and the total profit over all period for each substitution rate. With no substitution, we earn 5 percent more compared to the certain substitution case. However, these values highly dependent on the per unit cost of substitution, holding and per unit profit. So, we cannot state that substituting means gaining less. With a different set of cost parameters, we may obtain different results.

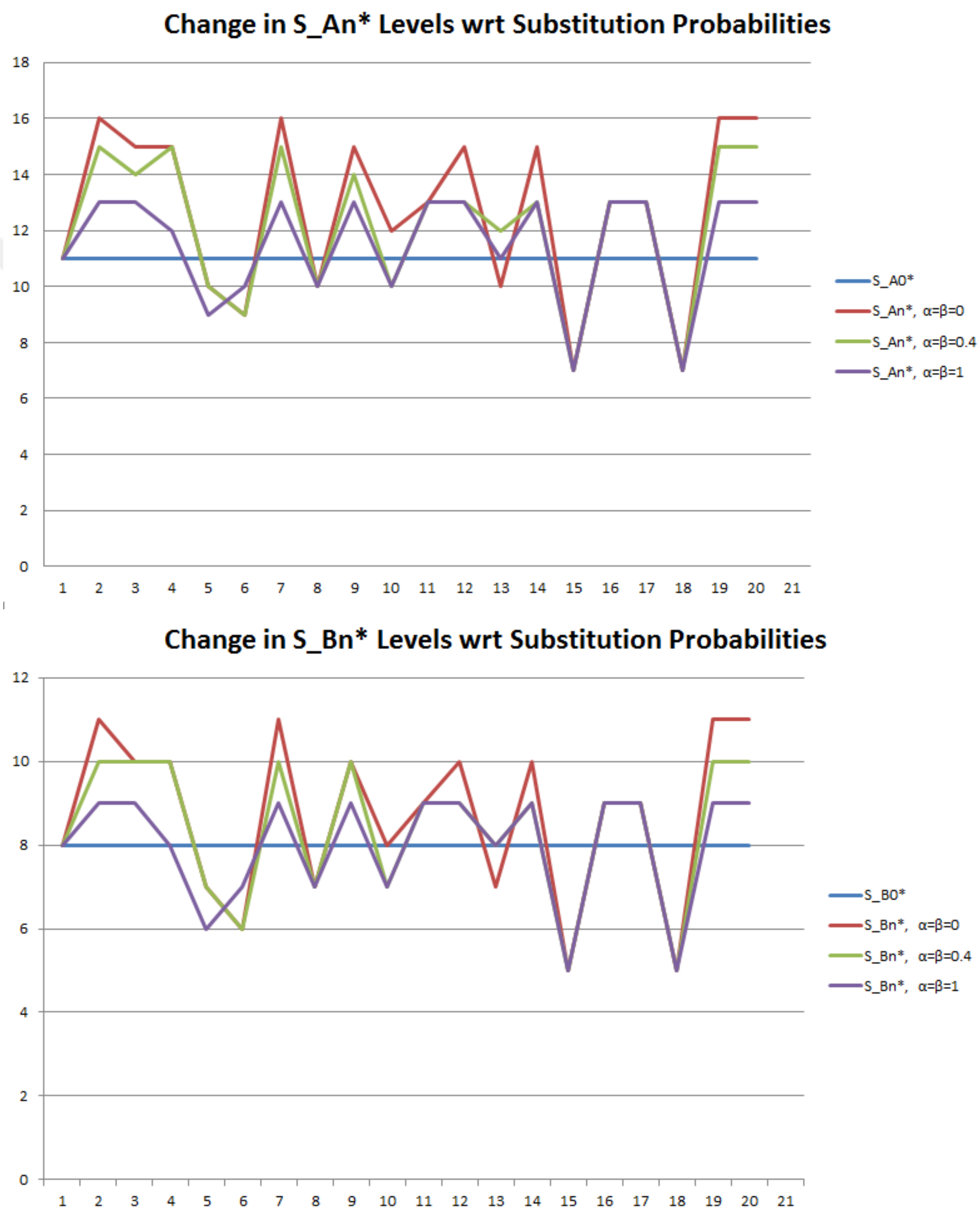


Figure 6.18: Profit Maximizing Inventory Levels for $\alpha = \beta = 0, 0.4$, and 1 over time, Simulation Realization 1

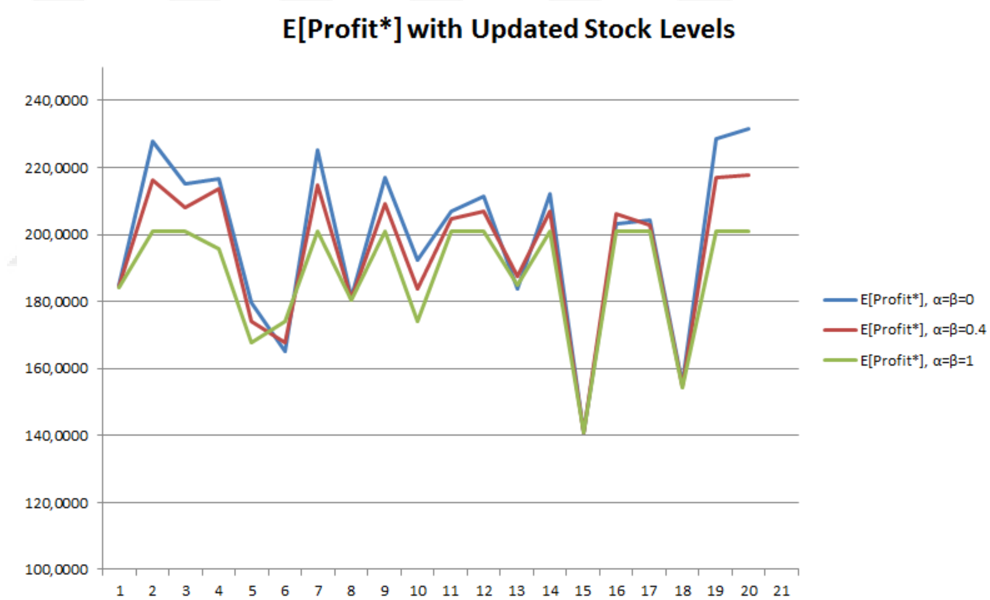


Figure 6.19: Change in the Expected Maximum Profit with Updated Stock Levels, for $\alpha = \beta = 0, 0.4$, and 1 over time, Simulation Realization 1

Period	$E[Profit^*], \alpha = \beta = 0$	$E[Profit^*], \alpha = \beta = 0.4$	$E[Profit^*], \alpha = \beta = 1$
1	185,05	184,64	184,24
2	227,67	216,24	200,88
3	215,22	208,12	200,88
4	216,71	213,53	195,58
5	179,83	174,24	167,63
6	165,2	167,78	173,96
7	225,23	214,7	200,88
8	181,34	180,8	180,3
9	216,92	209,21	200,88
10	192,42	183,62	173,96
11	207,03	204,64	200,88
12	211,43	206,78	200,88
13	183,82	187,44	184,87
14	212,31	207,02	200,88
15	140,8	140,86	140,77
16	203,26	206,2	200,88
17	204,29	202,82	200,88
18	155,23	154,76	154,24
19	228,53	216,82	200,88
20	231,73	217,92	200,88
SUM	3983,99	3898,15	3765,27

Table 6.9: Expected Maximum Profit Values for Different Substitution Rates

Figure 6.20 shows the expected profits for different initial inventory level policies as discussed in the previous section. Results are in parallel with $\alpha = \beta = 0,4$ case for both no substitution and certain substation cases. We expect to gain more as we update the initial inventory levels. We expecet to gain least where the initial choice of inventory levels does not reflect the reality of observed data as in Constant Inv Levels $S_A = S_B = 12$ case. Table 6.10 shows the expected profit values for diferent inventory policies.

	$\alpha = \beta = 0$	$\alpha = \beta = 0.4$	$\alpha = \beta = 1$
Updated S_0^* values	3983,9924	3898,1513	3765,2662
Constant S_0^*	3561,2184	3541,4297	3491,5748
Constant $S_A = \lambda a\tau, S_B = \lambda b\tau$	3260,5474	3176,2472	2886,7886
Constant $S_A = S_B = \lambda\tau/2$	2140,8472	2099,6267	1953,5474

Table 6.10: Total Profit of all periods for Different Inventory Policies

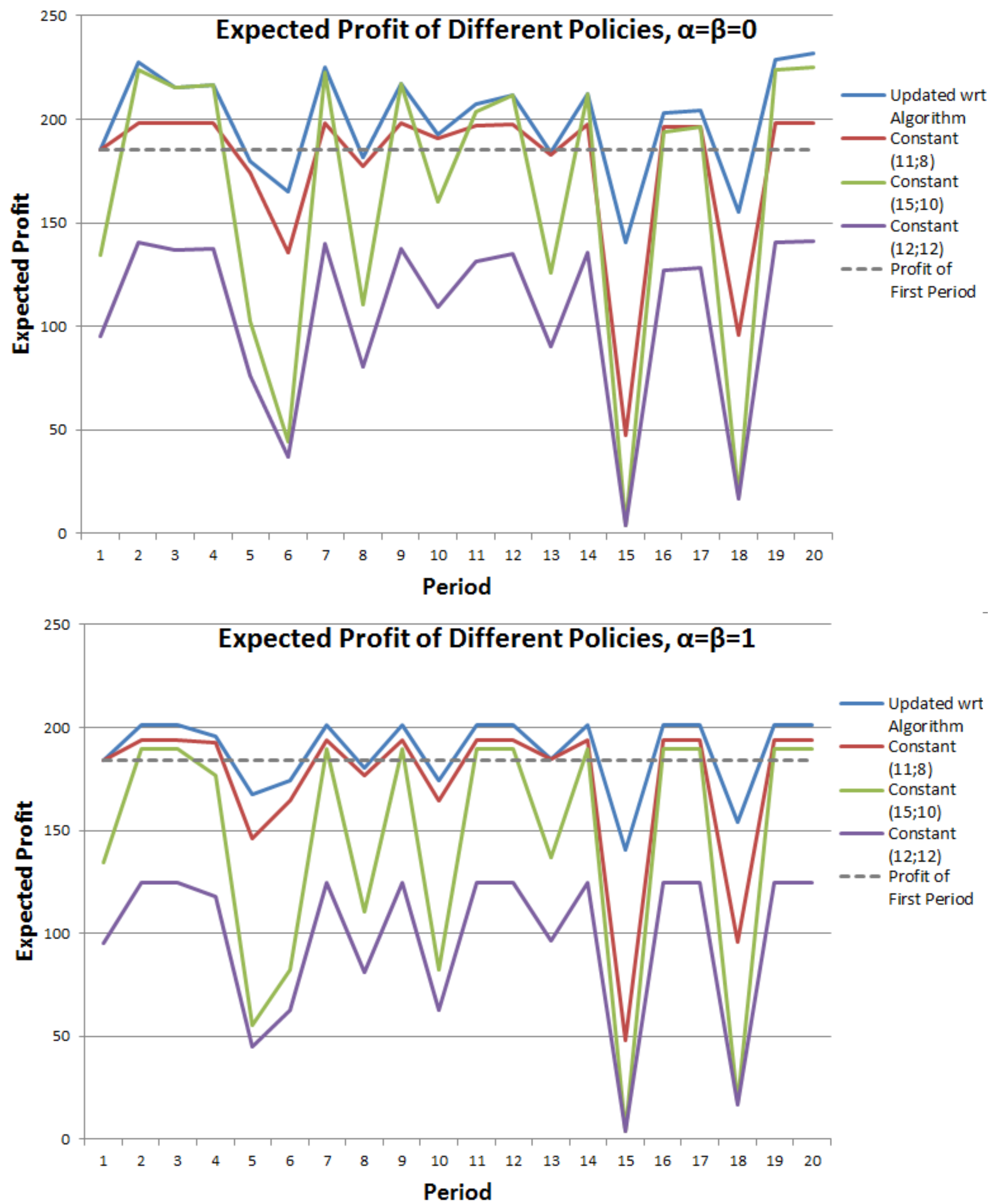


Figure 6.20: Expected Profit Levels for Different Inventory Level Policies, where $\alpha = \beta = 0$ and 1

6.2.3 The Effect of Different Prior Selections

In this section, we want to see the impact of prior distribution parameters on expected cost and cost maximizing inventory levels. In Section 6.2.1, our prior distribution parameters were selected as $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$ and $c_0 = [0.4; 0.6]$. The distribution mean was 1.6599 and the variance was 1.4649.

6.2.3.1 The Effect of Prior Mean

Firstly, the same shape and scale parameters with a different mixture weight is selected. We change the c_0 value as $[0.9; 0.1]$. This situation changes the prior mean as 0.61 and the variance as 0.5313. Then, we conduct the same analysis on this prior.

Figure 6.21 shows the change in profit maximizing inventory levels for the selected prior. As the initial mean value(0.61) is really small compared to the true mean ($\Lambda = 1$), profit maximizing inventory levels of the first period are very low. As it can be seen on Figure 6.21, S^*_{A0} and S^*_{B0} values are lower bounds for the following periods.

Figure 6.22 shows that profit maximizing inventory levels diminish as the prior mean of the distribution decreased. In our initial prior choice ($\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$, $c_0 = [0.4; 0.6]$), mean of the distribution was 1.6599. In the new prior ($\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$, $c_0 = [0.9; 0.1]$) mean is 0.61. As we can see from the figure, S^*_{An} and S^*_{Bn} values of $c_0 = [0.4; 0.6]$ are always higher than $c_0 = [0.9; 0.1]$ case.

Expected maximum profit values for both cases are shown on Figure 6.23. Figure indicates that, expected maximum profits are always higher for the initial prior choice.

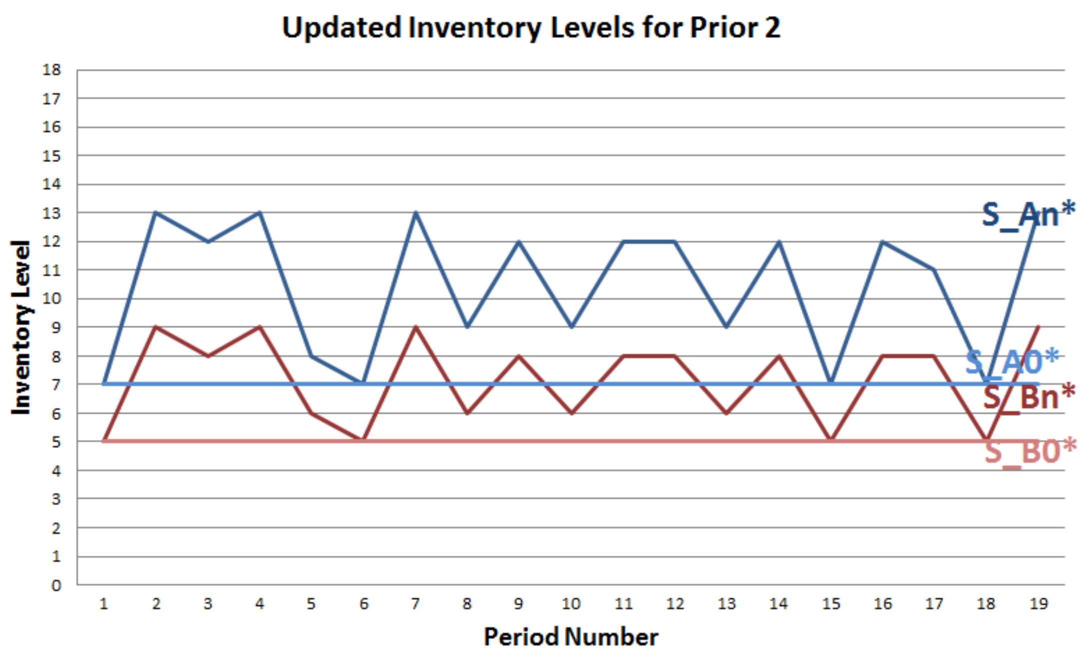


Figure 6.21: Change in the Profit Maximizing Inventory Levels for $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$ and $c_0 = [0.9; 0.1]$

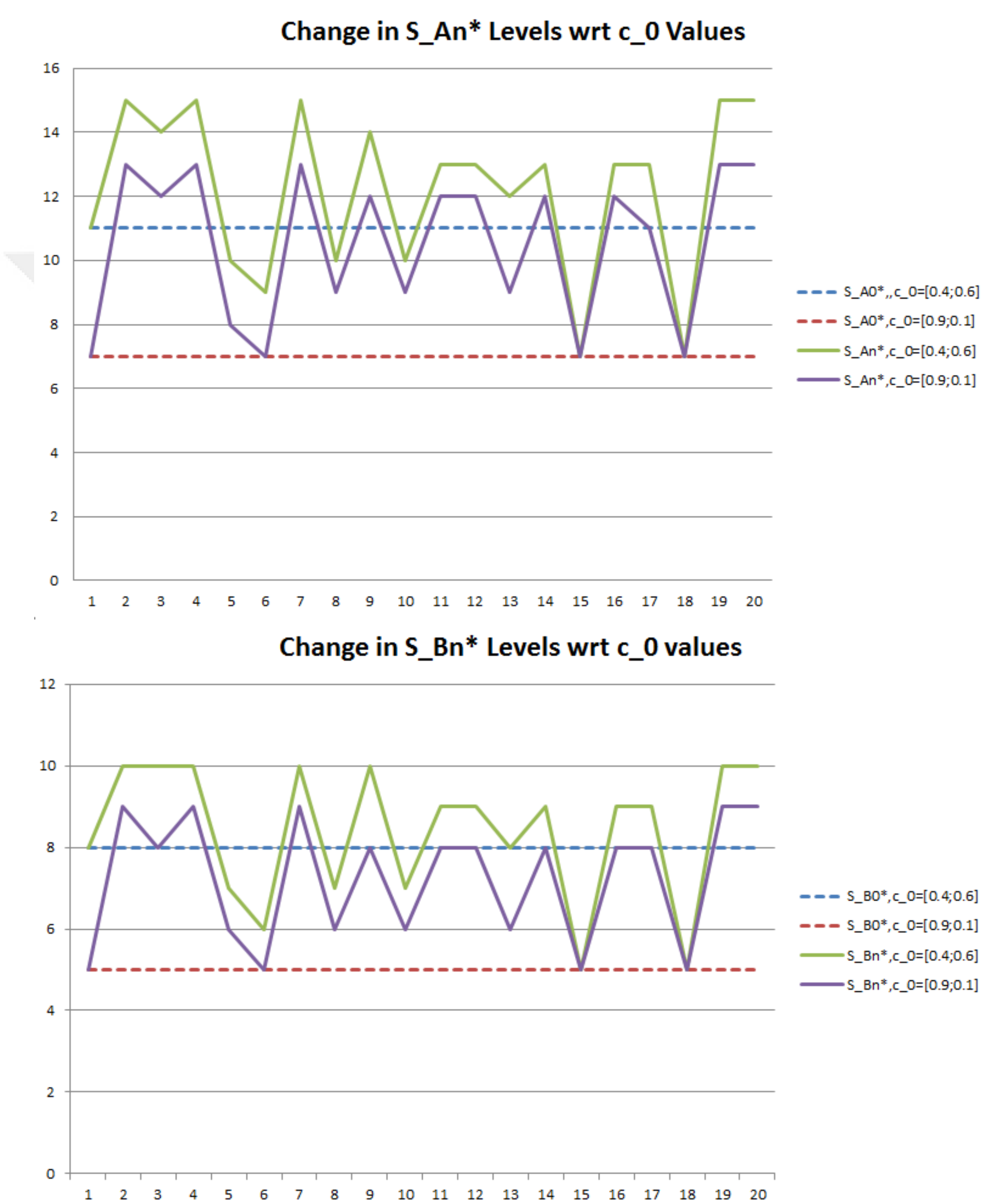


Figure 6.22: Change in the Profit Maximizing Inventory Levels for $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$ with $c_0 = [0.4; 0.6]$ and $c_0 = [0.9; 0.1]$

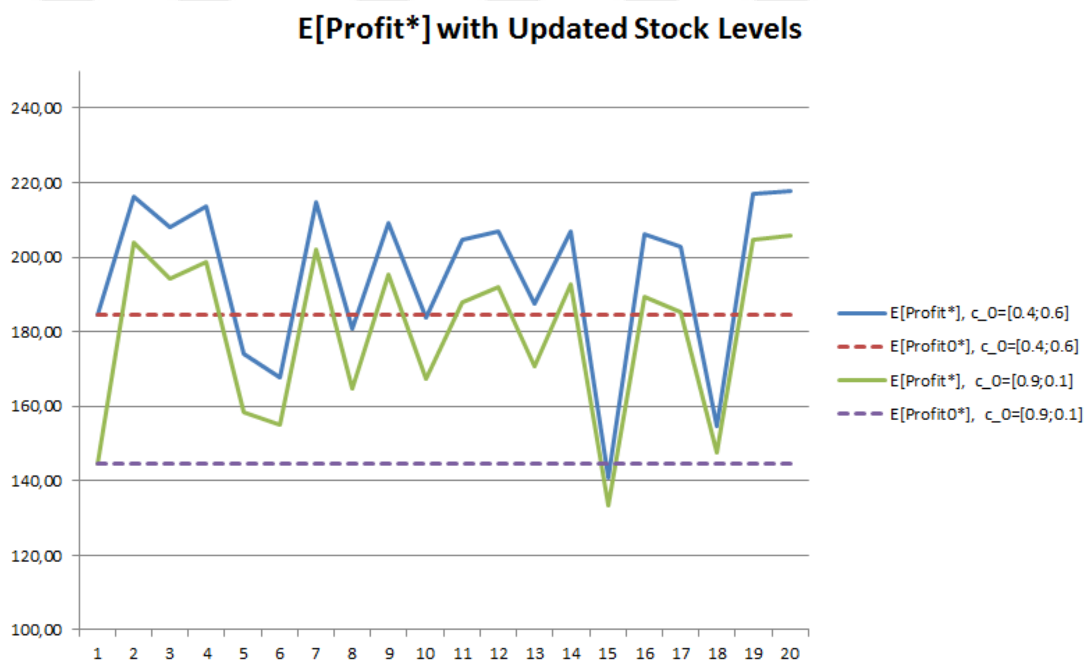


Figure 6.23: Expected Maximum Profit Values for $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$ with $c_0 = [0.4; 0.6]$ and $c_0 = [0.9; 0.1]$

Suppose, we have choose the prior parameter values as : $\eta_0 = [20; 10]$, $\theta_0 = [6; 7]$ and $c_0 = [0.9; 0.1]$. Shape of the distribution can be seen on Figure 6.9 on page 47. In the prior analysis above, we have changed only the mixture weights of the prior distribution. If also change the shape and scale parameters, we will obtain the following results.

Figure 6.24 shows the change in profit maximizing inventory levels for the selected prior. As the initial mean value(3.1425) is very big compared to the true mean ($\Lambda = 1$), profit maximizing inventory levels of the first period are very high. As it can be seen on Figure 6.24, S^*_{A0} and S^*_{B0} values are lower bounds for the following periods.

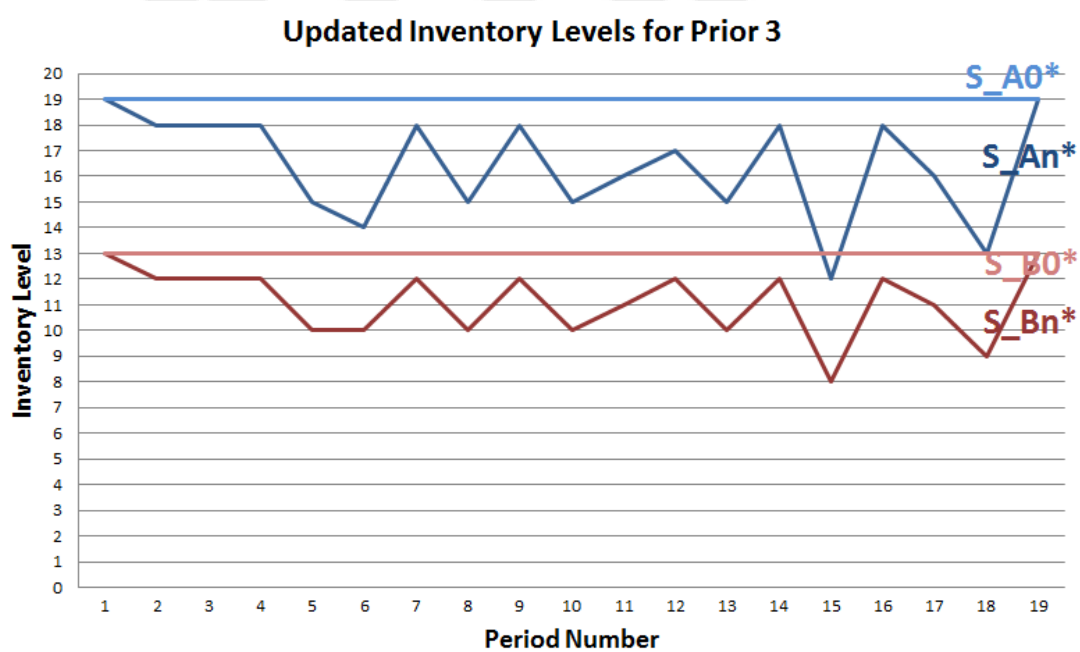


Figure 6.24: Change in the Profit Maximizing Inventory Levels for $\eta_0 = [20; 10]$, $\theta_0 = [6; 7]$ and $c_0 = [0.9; 0.1]$

If we want to compare the profit maximizing inventory levels with our initial prior choice ($\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$ and $c_0 = [0.4; 0.6]$), inventory levels raised as the prior mean of the distribution increased. Figure 6.25, shows the change in profit maximizing inventories for both cases.

Expected maximum profit values for both cases are shown on Figure 6.26.

Figure indicates that, expected maximum profits are always less for the initial prior choice.

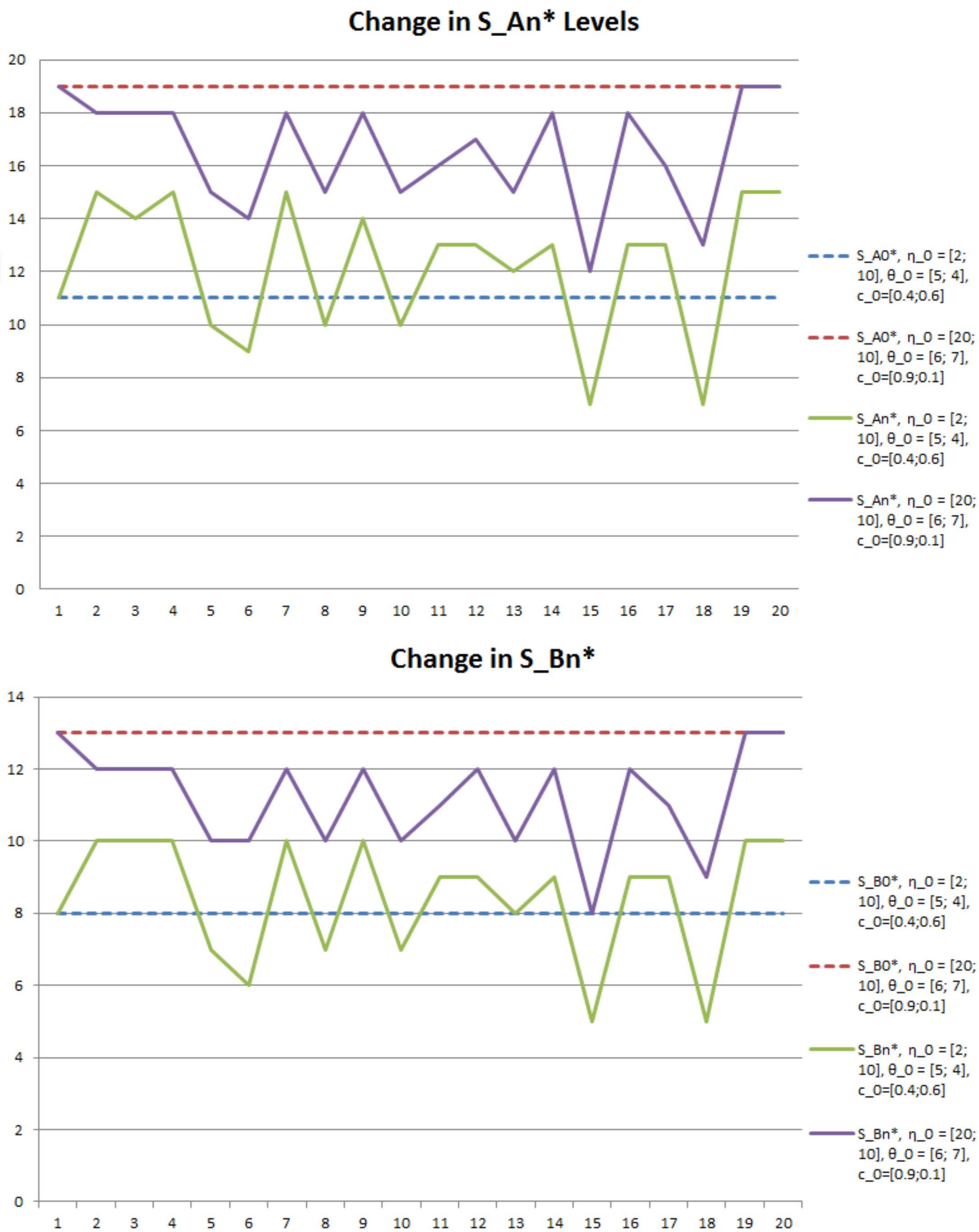


Figure 6.25: Change in the Profit Maximizing Inventory Levels for $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$, $c_0 = [0.4; 0.6]$ and $\eta_0 = [20; 10]$, $\theta_0 = [6; 7]$, $c_0 = [0.9; 0.1]$

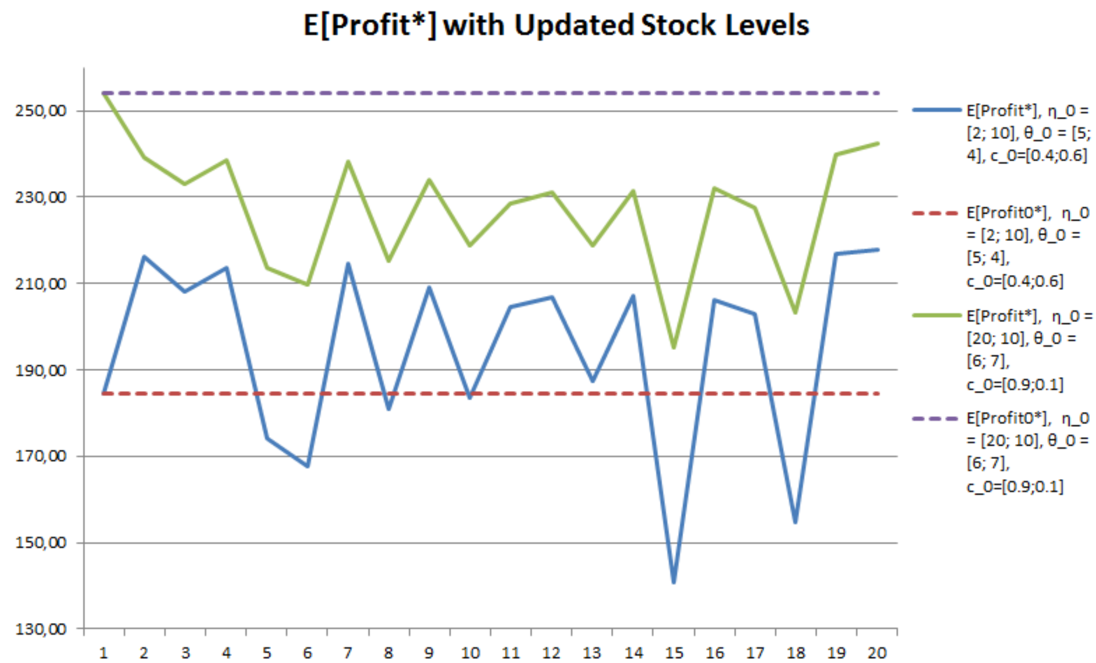


Figure 6.26: Expected Maximum Profit Values for $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$, $c_0 = [0.4; 0.6]$ and $\eta_0 = [20; 10]$, $\theta_0 = [6; 7]$, $c_0 = [0.9; 0.1]$

6.2.3.2 The Effect of Prior Distribution Characteristics

In previous analysis we see the impact of prior mean on profit levels and profit maximizing inventory values. We concluded that if mean value of the prior distribution increases, inventory levels increase too. Now, we want to see the effect of prior distribution characteristics such as uni-modality, bi-modality or skewness.

For this approach, we have choose the prior parameter values as : $\eta_0 = [5;10]$, $\theta_0 = [3;6]$ and $c_0 = [0.4;0.6]$. Mean value of the prior distribution is equal to $1,6$ as in the initial case we have selected. Shape of the distribution can be seen on Figure 6.27 on page 76. As it can be seen on the Figure, it is a uni-modal distribution where the initial case was a bi-modal one.

If we compare the profit maximizing inventory levels with our initial prior choice ($\eta_0 = [2;10]$, $\theta_0 = [5;4]$ and $c_0 = [0.4;0.6]$), inventory levels are generally lower than the initial case. In some periods, they coincide with the initial case. There exist one-unit difference at most over all periods. Figure 6.28, shows the change in profit maximizing inventories for both cases. Variance of the prior selection $\eta_0 = [5;10]$, $\theta_0 = [3;6]$ and $c_0 = [0.4;0.6]$ is equal to 0.3888. In our initial selection, the variance was 1.4649. So, the lower inventory levels may be the result of lower variance.

Expected maximum profit values for both cases are shown on Figure 6.29. Figure indicates that, expected maximum profits are generally lower than the initial choice. Also, the total profit for the initial choice was $3898,15$ while the new total profit is equal to $3802,16$. However, we can see that variance in the maximum profit values are less than the initial case.

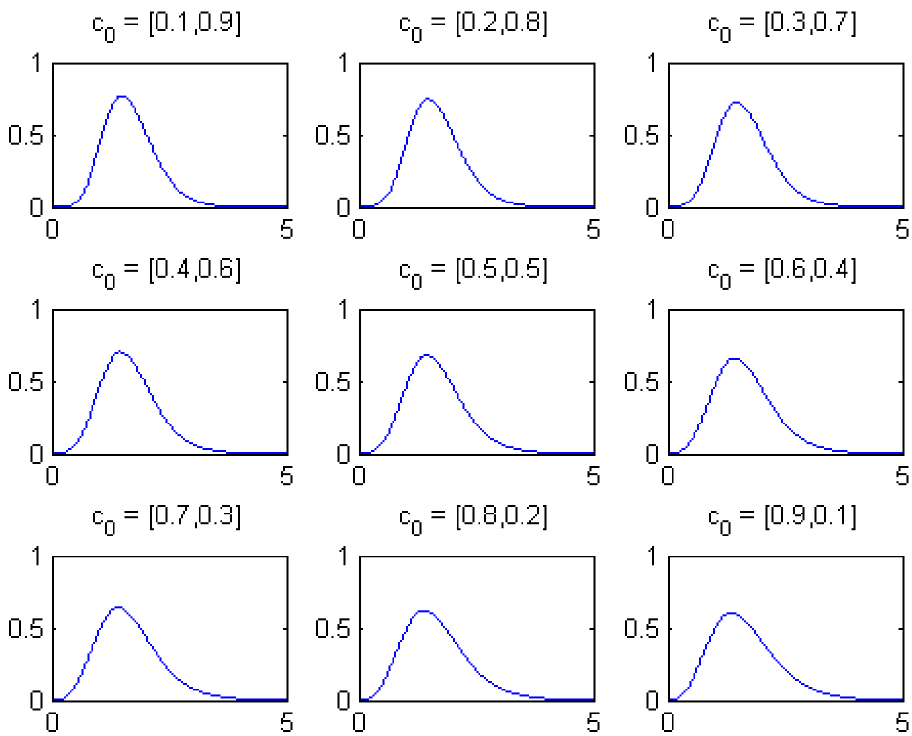
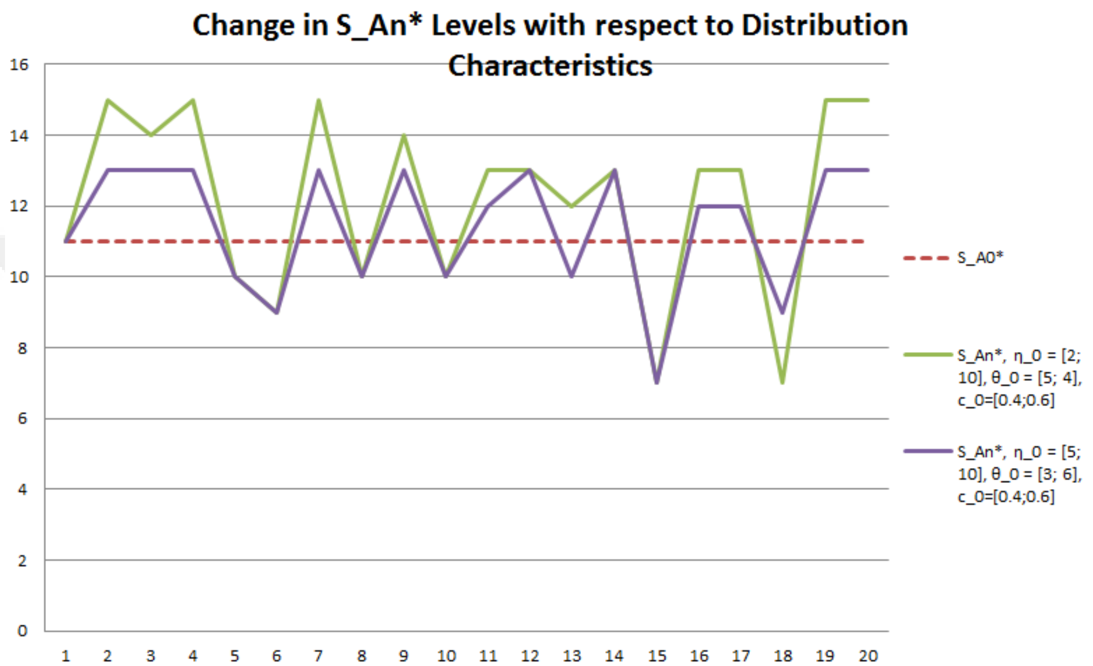


Figure 6.27: Prior Distribution Graph of $\eta_0 = [5; 10]$, $\theta_0 = [3; 6]$ for different c_0 values



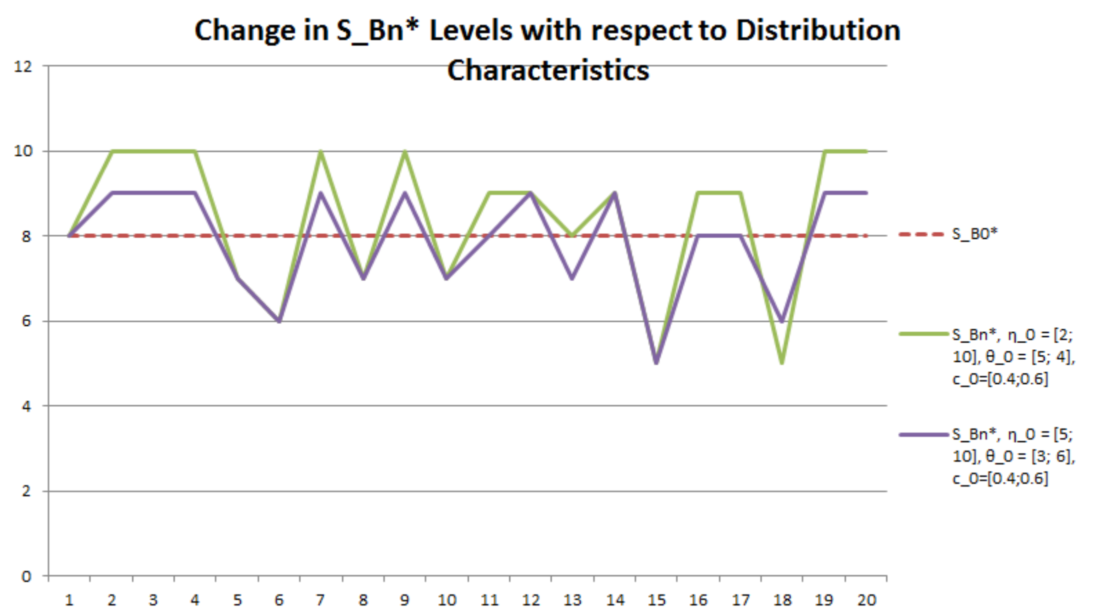


Figure 6.28: Change in the Profit Maximizing Inventory Levels for $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$, $c_0 = [0.4; 0.6]$ and $\eta_0 = [5; 10]$, $\theta_0 = [3; 6]$ and $c_0 = [0.4; 0.6]$

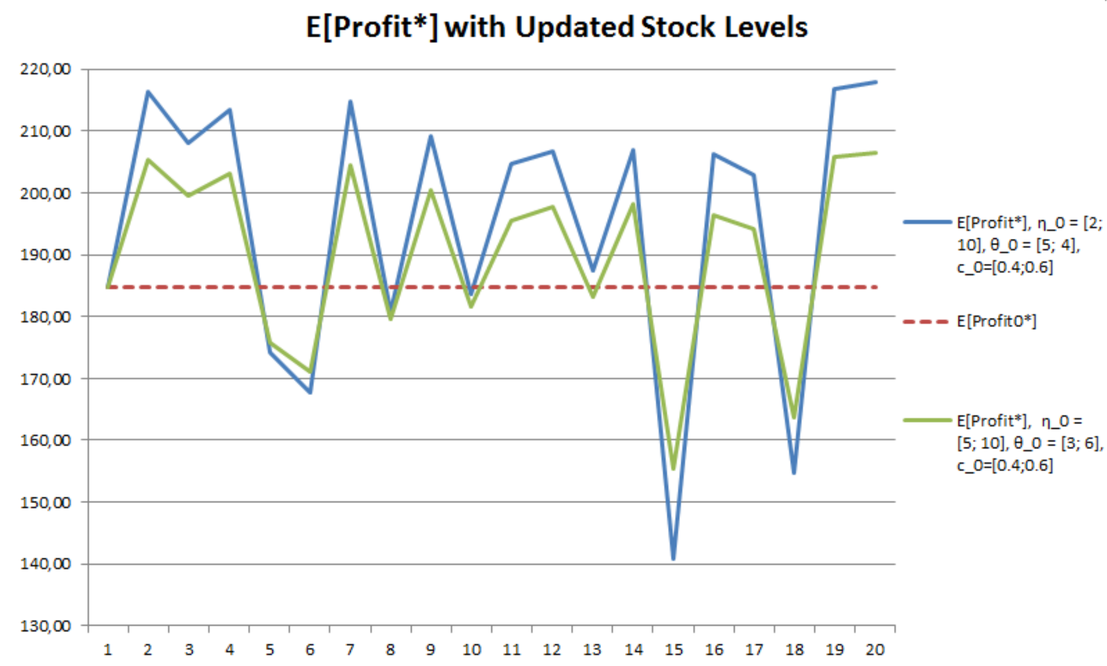


Figure 6.29: Expected Maximum Profit Values for $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$, $c_0 = [0.4; 0.6]$ and $\eta_0 = [5; 10]$, $\theta_0 = [3; 6]$ and $c_0 = [0.4; 0.6]$

In order to strengthen our theory that prior variance affects the inventory levels, we choose a prior distribution with a higher variance than the initial case. Prior parameter values are selected as : $\eta_0 = [2; 2.5]$, $\theta_0 = [5; 1]$ and $c_0 = [0.4; 0.6]$. Mean value of the prior distribution is equal to $1,6$ as in the initial case we have selected. Variance of the new prior is equal to 2.5886. In our initial selection, the variance was 1.4649.Shape of the distribution can be seen on Figure 6.30 on page 80. As it can be seen on the Figure, it is a skewed distribution compared to the initial case.

If we compare the profit maximizing inventory levels with our initial prior choice ($\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$ and $c_0 = [0.4; 0.6]$), inventory levels are again generally lower than the initial case. In some periods, they coincide with the initial case. There exist one-unit difference at most over all periods. Figure 6.31, shows the change in profit maximizing inventories for both cases. Therefore, we cannot state that prior distribution's shape or variance has a direct impact on profit maximizing inventory levels.

Expected maximum profit values for both cases are shown on Figure 6.32. Figure indicates that, expected maximum profits are generally lower than the initial choice. Also, the total profit for the initial choice was $3898, 15$ while the new total profit is equal to $3784, 70$.

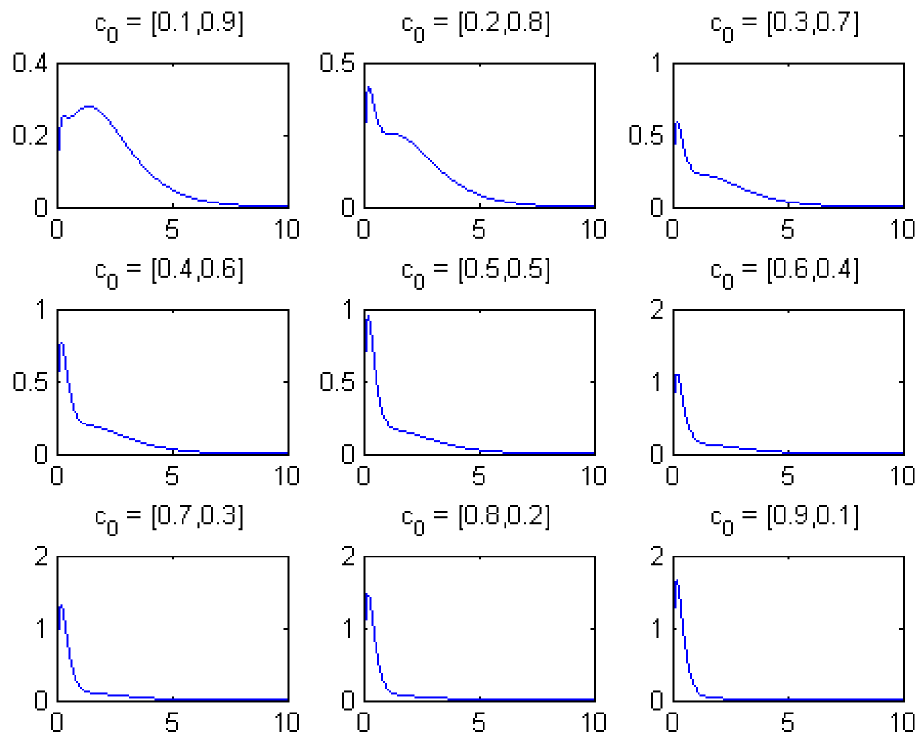


Figure 6.30: Prior Distribution Graph of $\eta_0 = [2; 2.5]$, $\theta_0 = [5; 1]$ for different c_0 values

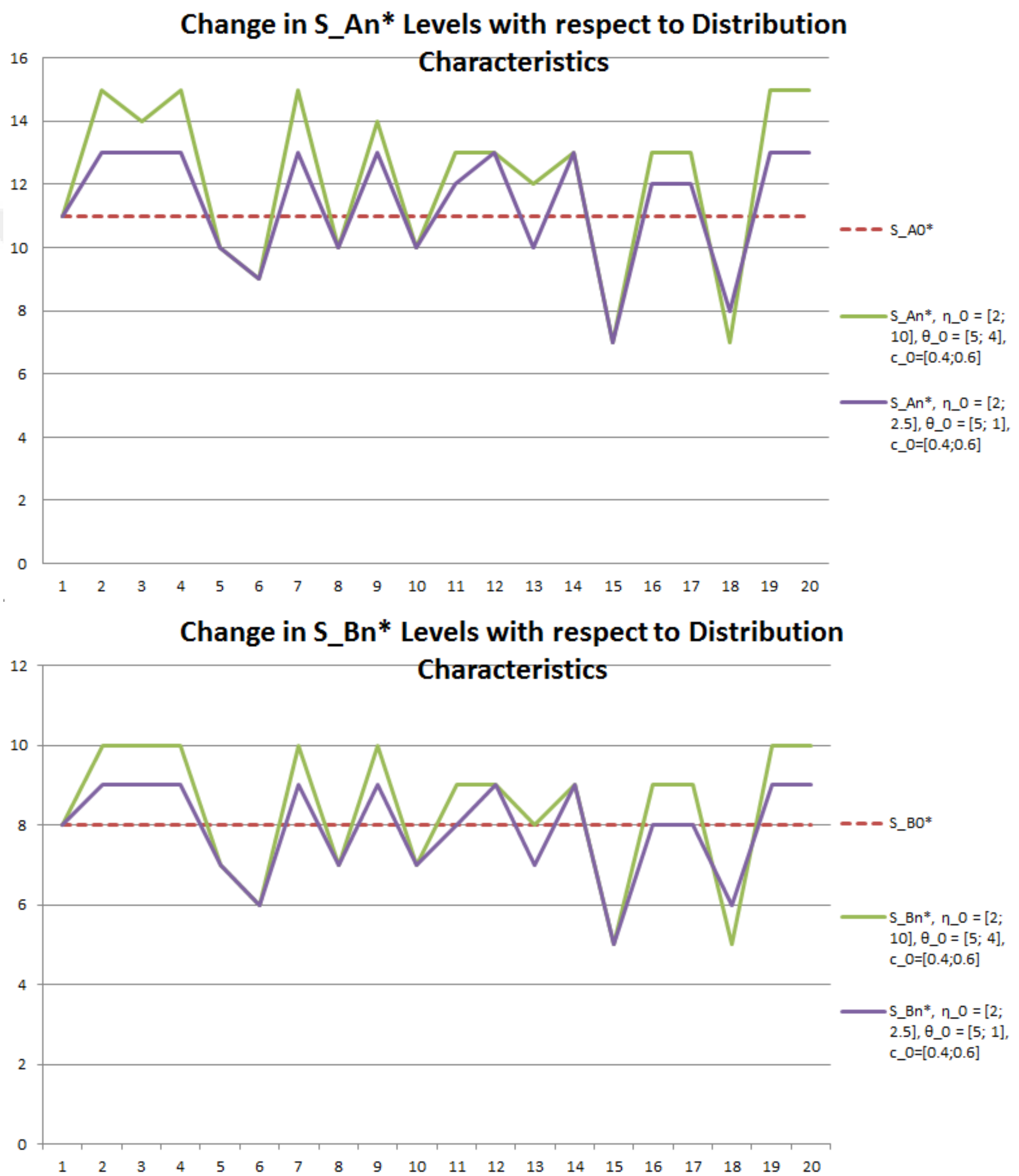


Figure 6.31: Change in the Profit Maximizing Inventory Levels for $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$, $c_0 = [0.4; 0.6]$ and $\eta_0 = [2; 2.5]$, $\theta_0 = [5; 1]$ and $c_0 = [0.4; 0.6]$

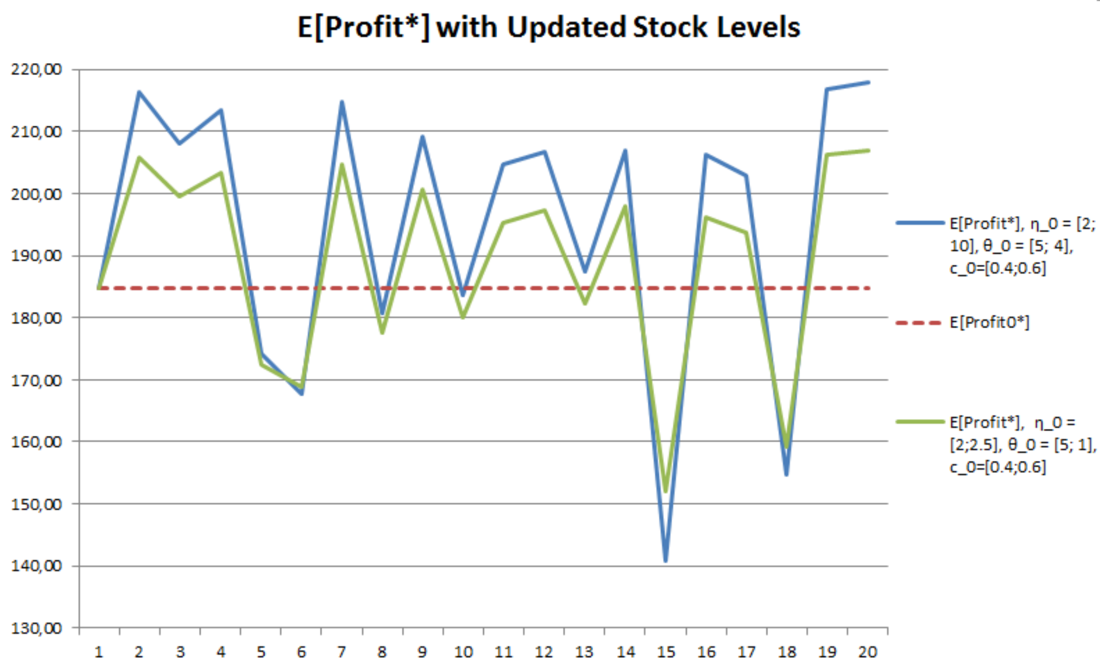


Figure 6.32: Expected Maximum Profit Values for $\eta_0 = [2; 10]$, $\theta_0 = [5; 4]$, $c_0 = [0.4; 0.6]$ and $\eta_0 = [2; 2.5]$, $\theta_0 = [5; 1]$ and $c_0 = [0.4; 0.6]$

Chapter 7

Conclusion

In this thesis, we considered the joint estimation of demand arrival rate, primary and substituted demand probabilities in an environment with two products using a Bayesian methodology. The demand arrivals are assumed to be generated by a Poisson distribution where an arriving customer want to purchase one of the products with a certain probability. If the first preferred product is not available in the stock the demand may flow to the other product if it is available. The unmet demands are lost and not observable. It is assumed that the inventory of the products are monitored periodically and at the end of a period, the sales data and the stock-out times become available. Based on this information, we propose a dynamic estimation method using a Bayes perspective. The prior distribution for the rate of the Poisson arrivals is taken as a general gamma mixture and the joint prior distribution of the primary and substitute demands is taken as a general distribution. To the best of our knowledge, this is the first attempt to estimate all distributions of the demand parameters simultaneously with a Bayesian perspective. The Bayesian approach benefits from its dynamic nature which allows for the adjustment of the probability distributions of the quantities of interest as more and more data become available.

We provide the explicit expressions for the joint posterior density of the demand arrival rate, primary and substitute demand probabilities under a general gamma mixture prior for demand arrival rate, where the distributions of the remaining quantities are left general. It is illustrated that the conditional density of the demand arrival rate given the primary and substitute demand probabilities have the form of a mixture of gamma densities where the mixture coefficients and the parameters of the mixing densities are updated according to the observed data. In the particular cases when either the total demand rate or the primary and the substitute demand probabilities are known, the posterior densities also take well defined forms mixture of densities. The proposed methods and their short run and long run behavior are also illustrated by numerical examples using simulated data.

Using the Bayesian methodology, we have derived the total expected profit formulations per period. Then, we search for the profit maximizing inventory levels for each period. We generated an algorithm for this approach. Using the simulation data and the dynamic inventory level updating algorithm, we conducted some numerical studies. As a result, we show that updated inventory policy performs better than any constant inventory policy in all cases.

Moreover, the effect of substitution probabilities and the prior selection on expected profit and the profit maximizing inventory levels are also investigated. Because of our cost parameter preferences, we get the result that no substitution is more profitable than the substitution cases. Finally, we investigated the effect of prior selection on profit maximizing inventory levels and the profit levels. According to our analysis, both the inventory levels and profit have positive correlation with the prior mean value.If we choose a prior with lower mean, inventory levels and profit decrease and vice versa.

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Appendix A

Table of Parameters

S_{A_j} / S_{B_j} **:** Stock level for product A / B at the beginning of period j
Λ : Rate of customer arrivals which follows a Poisson process
P_A / P_B : Primary demand probability for product A / B
γ_A / γ_B : probability that an arriving customer substitutes product A in place of B/A if B is not available. (Similarly defined for γ_B)
Λ_A / Λ_B : Independent primary demand arrival rates for each product
Ψ : Parameter set that describes the system dynamics. $\Psi = (\Lambda, P_A, P_B, \gamma_A, \gamma_B)$
τ : Replenishment period
τ_A / τ_B : stock out times of products A and B respectively
X_j : Sales for product A arising from primary demand at period j.
Y_j : Sales for product B arising from primary demand at period j.
Z_j : Sales for product A, arising from the combined demand for A and B, after B has stock out at period j.
W_j : Sales for product B, arising from the combined demand for A and B, after A has stock out at period j.
S_j : Sales data set at period j. $S_j = (\ X_j, Y_j, Z_j, W_j)$
T_{AB_j} : Length of the time interval over which both products are available at period j.
T_{A_j} : Length of the time interval over which only product A is available at period j.

T_{B_j} : Length of the time interval over which only product B is available at period j.

T_j : Time data set at period j. T_j=$(T_{AB_j}, T_{A_j}, T_{B_j})$

D_j : Observed data at period j. $D_j = (\ X_j, Y_j, Z_j, W_j, T_{AB_j}, T_{A_j}, T_{B_j})$

D : A realization of D_j. $D = (\ x_j, y_j, z_j, w_j, t_{AB_j}, t_{A_j}, t_{B_j})$

λ : Variable that corresponds to demand

$\Gamma(\eta, \theta, c)$: Gamma mixture density with parameters η, θ, c.

η_0, θ_0, c_0 : Parameters of the prior gamma mixture.

g : Gamma mixture density function.

q : Normalizing function for the gamma kernell. $q(\eta_i, \theta_i) = \frac{{\theta_i}^{\eta_i}}{\Gamma(\eta_i)}$

$\pi_{0,\Psi}$: Prior density function of the parameter set Ψ that describes the system dynamics.

$\pi_{0,\Lambda} = \Gamma(\eta_0, \theta_0, c_0)$

a, b, α, β : Realizations of $P_A, P_B, \gamma_A, \gamma_B$

$v = (\lambda, a, b, \alpha, \beta)$

R : Vector of prior and substitution probabilities. $R = (\ P_A, P_B, \gamma_A, \gamma_B)$

r : A realization of R $r = (a, b, \alpha, \beta)$

L : Realized demand rate over a period $L = \tau * \Lambda$

l : the likelihood function. $l(x, y, z, w|\Psi)$

K : A function of v vector

X^n : $\sum x_j$

Y^n : $\sum y_j$

Z^n : $\sum z_j$

W^n : $\sum w_j$

μ_n : Mean of the n^{th} posterior density for Λ

p_i : Profit per unit for product i. $i = A, B$

hc_i : Holding cost per unit for product i. $i = A, B$

sc_i : Substitution cost per unit for product i. $i = A, B$

I_i : Inventory on hand at the end of the period for product i. $i = A, B$