AN ALGORITHMIC APPROACH TO NASH IMPLEMENTATION WITH TWO INDIVIDUALS

A Master's Thesis

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Ankara
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To my family

AN ALGORITHMIC APPROACH TO
NASH IMPLEMENTATION WITH TWO INDIVIDUALS

The Graduate School of Economics and Social Sciences
of
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by

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

AN ALGORITHMIC APPROACH TO NASH IMPLEMENTATION
WITH TWO INDIVIDUALS

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This thesis investigates the Nash implementation problem in the behavioral domain with two individuals. By strengthening the necessary results in the literature and adopting an algorithmic approach, we provide constructive sufficiency results. Our findings facilitate the design of mechanisms in two individual environments where the individuals' behavior may violate the standard axioms of rationality.

Keywords: Nash Implementation, Behavioral Implementation, Two Individual Consistency

ÖZET

İKİ BİREYLİ NASH UYGULAMASI İÇİN ALGORİTMİK BİR YAKLAŞIM

Uğuz, Güneş
Yüksek Lisans, İktisat Bölümü
Tez Danışmanı: Dr. Öğr. Üyesi Nuh Aygün Dalkıran

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Bu tez, iki bireyin yer aldığı davranışsal Nash uygulama problemini incelemektedir. Literatürdeki gerek koşulları güçlendirerek ve algoritmik bir yaklaşım benimseyerek, mekanizma tasarımı için yeter koşullar sağlanmaktadır. Bulgularımız, bireylerin davranışlarının standart rasyonalite aksiyomlarını ihlal edebileceği iki bireyli ortamlarda mekanizmaların tasarımını kolaylaştırmaktadır.

Anahtar Kelimeler: Nash Uygulaması, Davranışsal Uygulama, İki Bireyli Tutarlılık

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CHAPTER 1

INTRODUCTION

Economics focuses on decision-making and choices under constraints, optimization, and creating incentives for optimal outcomes through policy measures.

Daniel Kahneman and Amos Tversky significantly influenced behavioral economics, a subfield that gained prominence in the late 20th and early 21st centuries. Their work on decision theory and bounded rationality introduced psychological aspects into economic analysis. Kahneman's contributions to understanding decision-making under uncertainty earned him the Nobel Prize in Economic Sciences in 2002. They brought widespread attention to behavioral economics with his book *Thinking, Fast and Slow* (Kahneman, 2011). Richard Thaler, another Nobel laureate, furthered this field by demonstrating practical applications of behavioral economics in public policy and everyday decisions. His collaboration with Cass Sunstein on the book *Nudge* (Thaler and Sunstein, 2008) led to the creation of behavioral insights teams and "nudge units" in governments and organizations worldwide (Halpern, 2015).

Mechanism design, a sub-field of economics and game theory, takes a unique approach. Instead of focusing on existing systems and identifying equilibria, mechanism design starts with desired outcomes and works backward to create mechanisms that reach these outcomes when players act in their self-interest. Mechanism designers explore whether such mechanisms can exist and how to construct them.

This field is crucial in policy-making and economics as it helps develop methods to achieve desired social outcomes. By designing mechanisms, policymakers can incentivize individuals to make choices that align with broader societal goals. Mechanism design concerns creating economic environments and frameworks to allocate resources while achieving social welfare goals.

Research on implementing specific social choice rules that allocate optimal social alternatives has been significant since the late 1980s, thanks to the pioneering work of Hurwicz (1986) and Maskin (1999). In mechanism design, individuals should make choices leading to the desired outcomes without deviations. Hurwicz, Maskin, and Myerson received the Nobel Prize in 2007 for their foundational work in mechanism design. Myerson (1981)'s contributions focused on designing strategy-proof mechanisms that optimize expected revenue, particularly in auction settings, known as optimal mechanisms.

The field has also attracted interest from engineers and AI researchers due to its similarities with algorithmic design. This interest led to studies in Algorithmic Mechanism Design, beginning with Nisan and Ronen (2001) and extending to mechanism design with machine learning Balcan et al. (2005), network design Agarwal and Ergun (2010), and analyzing cooperation and competition among software agents in a truthful manner Wei and Liu (2024).

The study of Nash implementation within the rational domain has yielded valuable insights and findings in economics. Conversely, behavioral economics challenges the reliance on rationality assumptions, asserting that individuals often make decisions based on factors such as external influences and cognitive biases rather than solely assessing the benefits of each choice and focusing purely on self-interest. Acknowledging these irrational decisions helps social planners design policies that consider these factors in individuals' preferences to achieve optimal outcomes. This approach allows policies to be designed to nudge individuals towards desired results, elevating both individual and social welfare. One of the first studies on Nash implementation in the behavioral domain, Korpela (2012), points out the dependence of the results of previous literature on Nash implementation on rationality, particularly the independence of irrelevant alternatives, also known as Sen's α.

de Clippel (2014) has further extended the topic of implementation to the field of behavioral economics and non-rational choices. de Clippel (2014) studies implementation with choice structures, focusing on cases involving three or more individuals. The necessity condition identified by de Clippel (2014) also applies to the case of two individuals. However, de Clippel (2014) does not provide any sufficiency results for the two-individual case.

Computational implementation by Barlo and Dalkıran (2022a) expands theoretical research in this area, computes the domain of preferences for a given mechanism that makes implementation achievable, and checks the requirements sufficient for Nash implementation of any given social choice correspondence. Barlo and Dalkıran (2022a) focus mainly on cases with more than two individuals to generalize Maskin (1999) results. They also provide computational tools to examine the extent of Nash implementation for any SCC by identifying all collections consistent with SCCs to obtain optimal outcomes as Nash equilibria of the mechanism.

In this thesis, we identify the necessary and sufficient conditions for Nash implementation in the behavioral domain with two individuals. The terminology implies settings with two agents; however, it could refer to either settings with two agents or two parties making a bilateral decision. For simplicity, we will refer to settings with two agents and two parties as settings with two individuals, with the understanding that it encompasses both meanings. Utilizing similar methods as in Barlo and Dalkıran (2022a), we develop the necessary and sufficient conditions for Nash implementation of any given social choice correspondence with two individuals. We refined the concept of two-individual consistent pairs of collections of sets, leading us to the concept of two-individual consistent* pairs of collections of sets and two-individual stronger consistent pairs of collections of sets.

A *two-individual consistent collection of sets* is defined as a pair of collections of sets that any social alternative chosen by a social choice correspondence at each state of the world is chosen by both individuals from the corresponding sets the alternative and the state of the world is associated with. However, suppose an alternative is socially optimal in one particular state but not in another state. Then, there is an individual who does not choose this alternative from the set associated with the particular individual, the particular alternative, and the particular state in the other state. Moreover, the sets defined for individuals one and two should have non-empty intersections.

On the other hand, a pair of collection of sets is said to be *two-individual consistent** if they are two-individual consistent, and if the intersection of two sets associated with different agents is a singleton that is not optimal at a state, then one of the agents does not choose this alternative from his associated set at the state the alternative is not socially optimal.

A pair of collection of sets is said to be *two-individual stronger consistent* if they are two-individual consistent and there exists an alternative in the intersection of two sets in the collection associated with individuals one and two, respectively is chosen by both individuals at a state, then this alternative must be chosen by the social choice correspondence at that state as well.

We show that the existence of two-individual consistent* as well as two-individual stronger consistent collections of sets is necessary for Nash-implementability of a social choice correspondence. To turn these necessity results into sufficiency results, we introduce further conditions:

To employ collections of two-individual consistent* sets in a sufficiency result, we require the two-individual consistent* sets to have singleton intersections bilaterally. To this end, we introduce the *singleton intersection property*. Furthermore, to prevent the social alternatives not chosen by f from being attained as a (bad) Nash equilibrium, we introduce the *intersection property**, which ensures that any alternative that appears in a set associated with an agent in the consistency collection also appears in another set that is associated with the other individual. This property implies the equivalence of opportunity sets that can be obtained in the mechanism coincide with the collection of two-individual consistent* sets.

We acknowledge that given a pair of collections that is two-individual consistent* with a social choice correspondence, the singleton intersection property is highly restrictive. This leads us to our other sufficiency result that employs two-individual stronger consistent collections. When singleton intersections are not necessarily the case, we require the existence of a function that indicates which of the alternative in the intersection can be used in constructing a mechanism that Nash-implements the social choice correspondence under consideration. Similar to the case of singleton intersections, this function is essential in designing a mechanism in which the set of all opportunity sets of an agent coincides with the collection of sets that are associated with himself. We refer to the pairs of collections of sets that such a function can be defined on as spanning collections. We show that the existence of a pair of two-individual stronger consistent collections of sets that are also spanning collections is sufficient for Nash implementation.

Recognizing the cases when the use of two-individual stronger consistent sets is not sufficient to design a mechanism Nash-implementing a social choice correspondence alone, we also consider the case when the pair of collection of sets is not a spanning collection. In that case, we require the existence of additional finite collections of sets for both players so that when combined with a pair of collections of two-individual stronger consistent sets becomes an extended spanning collection.

The rest of the thesis is organized as follows: Chapter 2 covers a literature review, preliminaries are presented in Chapter 3, and Chapter 4 contains the main results of this thesis. Finally, Chapter 5 is dedicated to concluding remarks.

CHAPTER 2

LITERATURE REVIEW

In this chapter, we provide a literature review on mechanism design, implementation theory, and aspects of behavioral economics that are relevant to this thesis.

The study of how multiple individuals make collective decisions has become a significant topic in economics, leading to the development of social choice theory. Arrow (1950)'s seminal work outlines the complexities and limitations of voting systems, introducing Arrow's Impossibility Theorem. This theorem states that it is impossible to devise a voting system that satisfies three key properties: non-dictatorship, Pareto efficiency, and independence of irrelevant alternatives. This theorem is crucial as these properties are reasonably desirable for any voting system to ensure it is democratic, considers voters' preferences, and adheres to rationality.

Arrow (1950), and the Impossibility Theorem significantly influenced subsequent research, including the well-known Gibbard-Satterthwaite Theorem, which combines the results of Gibbard (1973) and Satterthwaite (1975). Gibbard (1973) examined non-dictatorial voting systems as social choice rules that can result in three or more social alternatives and found that they are subject to individual manipulation. This means that individuals can vote strategically, rather than truthfully, to alter the social choice in a way that benefits them individually.

Satterthwaite (1975) extended Arrow's Impossibility Theorem by demonstrating that any voting system with more than three outcomes satisfying the properties Arrow suggested cannot be strategy-proof. The Gibbard-Satterthwaite Theorem thus highlights the trade-off between the fairness of voting systems and the incentivization of voters to reveal their preferences truthfully.

The works of Arrow (1950), Gibbard (1973), and Satterthwaite (1975) play a critical role in establishing the theory of incentives, which is central to the principal-agent model, matching theory (Roth, 1982), and implementation theory, a key element of mechanism design (Maskin et al. (1979); Maskin (1999)), by identifying stable outcomes and mechanisms that ensure individuals reveal their preferences truthfully.

The concept of mechanism design was developed through Hurwicz's research on resource allocation mechanisms (Hurwicz (1960, 1972)). Hurwicz (1986) further focused on the feasibility of achieving efficient outcomes through decentralized mechanisms under complete information and the challenges posed by incomplete information. Hurwicz (1986) was the first to examine irrational societies, where the social choice of a society with rational individuals may not satisfy the basic assumptions of rationality.

One of the most prominent works in mechanism design is Maskin (1999), a revised version of Maskin's Ph.D. dissertation (1977). Maskin (1999) considers a social planner's problem as one where the planner has a welfare goal in a society with multiple individuals with rational preferences, satisfying completeness, and transitivity. In a complete information setting, the true state of the world is commonly known among individuals. However, since the mechanism designer does not know the true state of the world, designing a mechanism that achieves the welfare goal becomes challenging. Danilov (1992) and Maskin (1999) characterized a monotonicity condition as necessary for implementing a social choice correspondence in Nash equilibrium. Maskin (1999) identified the sufficient condition for settings with three or more individuals by introducing the no-veto power condition. For two individuals, Maskin (1999) noted that a social choice correspondence satisfying the Pareto property is dictatorial if and only if it is Nash implementable.

Moore and Repullo (1990) characterized Nash implementation with necessary and sufficient conditions based on specific sets within the set of social alternatives, restricting individuals' choices and optimal outcomes chosen by the social choice correspondence. Dutta and Sen (1991) focused on two-individual settings with rational preferences, determining the necessary and sufficient conditions for Nash implementation.

Sjöström (1991) adopted a constructive approach where Maskin (1999) results were inapplicable, as the outlined conditions were difficult to verify in practice. Sjöström (1991) studied Nash implementation of monotonic social choice correspondences in environments with three or more individuals having single-peaked preferences, even when the no-veto power property was not satisfied. Additionally, Sjöström (1991) examined two-individual settings, highlighting the equivalence of Nash implementation with monotonicity and a boundary condition. In a survey paper, Maskin and Sjöström (2002) gathered together the crucial concepts of implementation theory such as strategy-proofness, virtual implementation (Abreu and Sen, 1991; Matsushima, 1988), renegotiation, Bayesian implementation (Jackson, 1991), and the introduction of the social planner as a player (Baliga et al., 1997). They highlight the fact that two-individual implementation might be more difficult than the case of having three or more individuals due to the lack of a unique deviator from a consensus in the canonical mechanism.[1]

[1]The mechanism defined in the proof of sufficiency theorem in Maskin (1999) is referred to as the canonical mechanism in the implementation literature.

Korpela (2012) extended Hurwicz (1986)'s work by defining Behavioral Nash Equilibrium as a more compatible solution concept with human behavior and analyzing a broader domain of choice sets than Hurwicz (1986). Korpela (2012) highlighted that the full characterization of Nash-implementable SCCs in Moore and Repullo (1990) depends on the existence of a certain class of sets, which are rows of a possible game form, and generalized condition-μ of Moore and Repullo (1990) to condition-λ in a behavioral domain.

Küçükşenel (2012) was one of the first studies to consider mechanism design in a behavioral setting. Küçükşenel (2012) examined Bayesian environments with altruistic agents and characterized interim efficient mechanisms.

de Clippel (2014) extended the analysis of implementation theory without assumptions of rationality after Korpela (2012). de Clippel (2014) introduced the concept of consistency, showing that a consistent collection of sets is necessary for Nash implementation in behavioral domains and that strong consistency and unanimity are sufficient in environments with at least three individuals. He further analyzed specific violations of rationality assumptions, such as status quo bias, endowment, and framing effects, providing prominent examples of these cognitive biases in an implementation setting. Hayashi et al. (2023) introduced a behavioral efficiency concept and further studied behavioral implementation utilizing behavioral strong equilibria.

Barlo and Dalkıran (2009) analyzed implementation via epsilon-Nash equilibrium as a continuation of Dalkıran (2006), where individuals get close to their best responses but do not necessarily achieve them. This paper was one of the first papers in behavioral implementation, allowing for preferences that might not be rational by violating transitivity. Barlo and Dalkıran (2009) introduced the concepts of epsilon-monotonicity and epsilon-limited veto power, providing the necessary and sufficient theorems for implementation in epsilon-Nash equilibrium.

Barlo and Dalkıran (2022a) provided computational tools to compute all consistent collections of sets of a social choice correspondence for Nash implementation. They further developed de Clippel (2014)'s definition of consistency by introducing two-individual consistent collections of sets and proving the necessary conditions for two-individual Nash implementation.

Barlo and Dalkıran (2022b) analyzed the implementation problem when the planner and the individuals have access to incomplete public choice data. In a more recent study, Barlo and Dalkıran (2023a) analyzed behavioral implementation under incomplete information. They characterized the existence of an interim consistent profile of sets of acts as a necessary condition for behavioral implementation. Combining interim consistency with a choice incompatibility condition, they provided a sufficiency theorem for the case of three or more individuals. Barlo and Dalkıran (2023b) extends the analysis of Barlo and Dalkıran (2023a) considering individuals' interim choices over acts and their ex-post choices over deterministic alternatives together, which leads to an analysis of robust behavioral implementation. In the rational domain, the robust mechanism design problem was studied in detail by Bergemann and Morris (2005) in an environment where the common knowledge assumptions are relaxed. Bergemann and Morris (2009) studied the robust full implementation problem using direct mechanisms. Bergemann and Morris (2011) identified a robust monotonicity condition that is necessary and almost sufficient for full robust implementation via general mechanisms. An alternative characterization for robust implementation was recently introduced by Jain et al. (2023).

Gavan and Penta (2023) presented safe implementation, where individuals' deviations can only lead to exogenously given acceptable outcomes. Such an approach accommodates various behavioral considerations and robustness from a different perspective. Their approach parallels Eliaz (2002)'s fault-tolerant implementation where some of the individuals fail to act optimally. A more recent paper that considers another type of robustness is Shoukry (2019) which studied outcome-robust mechanisms, where the equilibrium of a mechanism is sustained even though a minority of players deviate from their equilibrium strategy.

A few other recent works on implementation include Koray and Yildiz (2018), which studied implementation via the right structures, Altun et al. (2023), which studied implementation with a sympathizer, and Barlo et al. (2023), which introduced the concept of anonymous implementation, where individuals end up with the same opportunity sets in equilibrium.

CHAPTER 3

PRELIMINARIES

Preliminaries

This chapter introduces the preliminary definitions, concepts and notations that will be used in the rest of this thesis.

Let $N = \{1, 2\}$ be a pair of individuals who would be selecting an option from a non-empty set of social alternatives X. Let Θ be the set of all possible relevant states of the world. Both of the individuals make non-empty choices from each non-empty subset of X at each state $\theta \in \Theta$. We denote the set of all subsets of X by 2^X. Formally, for every individual $i \in N$, there exists a choice correspondence $C_i^\theta : 2^X \setminus \{\emptyset\} \to 2^X \setminus \{\emptyset\}$ that selects a non-empty subset of each social choice problem $S \subset X$ for any given state θ.

Under rationality, these choices can be obtained through a rational preference relation R_i^θ, that is reflexive, complete, and transitive. When the choices of individual $i \in N$ are rational, we have $C_i^\theta(S) = \{x \in S \mid xR_i^\theta y, \text{ for all } y \in S\}$ for all $i \in N$ and $\theta \in \Theta$.

We would like to emphasize that our focus is on **non-rational choices**, i.e., we assume that individuals' choices fail the weak axiom of revealed preferences (WARP) so that there is no reflexive, complete, and transitive relation that induces these choices.[1]

A **Social Choice Correspondence** (SCC) $f : \Theta \to 2^X \setminus \{\emptyset\}$ describes the 'optimal' alternatives in the eyes of the mechanism designer for each possible state of the world. For each $\theta \in \Theta$, we refer to alternatives that are chosen by SCC f as f-optimal alternatives at θ. We assume that the realized state of the world is commonly known by the agents but unknown to the mechanism designer.

A **mechanism** is denoted by $\mu = (M, g)$ with $M = M_1 \times M_2$ where M_1 and M_2 denote the message spaces of agents 1 and 2 respectively, and $g : M \to X$ is the outcome function that determines the outcome of the mechanism for each possible message profile.

Given a mechanism $\mu = (M, g)$, the **opportunity set** of agent i for the message $m_j \in M_j$ of the other agent is defined by $O_i^\mu(m_j) := \{g(m_i, m_j) \mid m_i \in M_i\}$ for both $i, j = 1, 2$, $i \neq j$. In words, $O_i^\mu(m_j)$ is the set of alternatives that individual i can unilaterally generate when her opponent send the message m_j in the mechanism $\mu = (M, g)$.

A message profile $(m_1^*, m_2^*) \in M$ is said to be a **(behavioral) Nash equilibrium** of mechanism $\mu = (M, g)$ at θ if $g(m_1^*, m_2^*) \in C_1^\theta(O_1^\mu(m_2^*)) \cap C_2^\theta(O_2^\mu(m_1^*))$. The set of all Nash equilibrium outcomes of the mechanism μ at any given state θ is denoted by $NE^\mu(\theta)$.

---

[1] See Sen (1971) for more on the relationship between WARP and individual choices.

We are now ready to define Nash implementability in behavioral domains:

Definition 1 (Nash Implementability) *An SCC $f : \Theta \to 2^X \setminus \{\emptyset\}$ is said to be* ***Nash implementable*** *if there exists a mechanism $\mu = (M, g)$ such that for all $\theta \in \Theta$, we have*

(i) $\forall x \in f(\theta), \exists m^\star$ *such that* $g(m^\star) = x$*,* $m^\star$ *is a Nash Equilibrium of* μ *at* θ*,*

(ii) $\forall m^\star \in M$ *such that* $m^\star$ *is a Nash Equilibrium of* μ *at* θ*,* $g(m^\star) \in f(\theta)$*.*

When the above conditions hold for mechanism $\mu = (M, g)$, we say that μ ***implements SCC f in Nash equilibrium****.*

de Clippel (2014) identified the following condition called **consistency** as a necessary condition for Nash implementability of an SCC.

Definition 2 (Consistency) *For a given SCC $f : \Theta \to 2^X \setminus \{\emptyset\}$, we say that a collection of sets $\mathbb{S} := \{S_i(x, \theta) \mid i \in N, \theta \in \Theta, x \in f(\theta)\}$ is* ***consistent with SCC*** *f if for any $\theta, \theta' \in \Theta$, we have*

(i) *if* $x \in f(\theta)$*, then* $x \in \bigcap_{i \in N} C_i^{\theta}(S_i(x, \theta))$*,*

(ii) *if* $x \in f(\theta) \setminus f(\theta')$*, then* $x \notin \bigcap_{i \in N} C_i^{\theta'}(S_i(x, \theta))$*.*

In words, the existence of a consistent collection for an SCC f ensures that the social alternatives chosen by the SCC are also chosen by each agent at each state from the set associated with this agent, the f-optimal alternative, and this state. Moreover, if an alternative is f-optimal at some state θ but not at another state θ', there is an agent who does not choose this alternative at θ' from the set associated with this individual, this alternative and state θ.

Barlo and Dalkıran (2022a) highlight that the existence of a consistent collection as a necessary condition for Nash implementability can be slightly strengthened by the existence of a two-individual consistent collection when there are only two individuals in the society, as in the setup of this thesis:

Definition 3 (Two-Individual Consistency) *For a given SCC $f : \Theta \to 2^X \setminus \{\emptyset\}$, a pair of profile of sets $(\mathbb{S}_1, \mathbb{S}_2)$ with $\mathbb{S}_1 := \{S_1(x,\theta) \mid \theta \in \Theta, x \in f(\theta)\}$ and $\mathbb{S}_2 := \{S_2(x,\theta) \mid \theta \in \Theta, x \in f(\theta)\}$ is said to be **two-individual consistent with SCC** f if for any $\theta, \theta' \in \Theta$, and any $x, y \in X$, we have*

(i) *If $x \in f(\theta)$, then $x \in C_1^{\theta}(S_1(x,\theta)) \cap C_2^{\theta}(S_2(x,\theta))$,*

(ii) *If $x \in f(\theta) \setminus f(\theta')$, then there is $j \in \{1,2\}$ such that $x \notin C_j^{\theta'}(S_j(x,\theta))$,*

(iii) *If $x \in f(\theta)$ and $y \in f(\theta')$, then $S_1(x,\theta) \cap S_2(y,\theta') \neq \emptyset$.*

Two-individual consistency improves upon consistency by adding the extra requirement that for any two f-optimal alternatives that are chosen by the SCC at different states, the sets associated with these alternatives but different individuals must have a non-empty intersection.

CHAPTER 4

MAIN RESULTS

In this chapter, we present the main results of this thesis. In particular, we strengthen the necessity results identified by de Clippel (2014) and Barlo and Dalkıran (2022a) and provide new sufficiency results for Nash implementation with two individuals whose choices need not be rational.

We begin by slightly strengthening the two-individual consistency as a necessary condition: We provide a stronger two-individual consistency condition, which we call **Two-Individual Consistency***, existence of which is necessary for Nash implementability of an SCC with two individuals. The difference from two-individual consistency concerns the case when the intersection of two sets associated with different agents is a singleton. If a social alternative is the unique element in the intersection of two sets associated with different individuals, then for any state at which this unique element is not f-optimal, there must be an individual who does not select this element from his corresponding set at the particular state:

Definition 4 (Two-Individual Consistency*) *For a given SCC $f : \Theta \to 2^X \setminus \{\emptyset\}$, a pair of collections of sets $(\mathbb{S}_1, \mathbb{S}_2)$ with $\mathbb{S}_1 := \{S_1(x,\theta) \mid \theta \in \Theta, x \in f(\theta)\}$ and $\mathbb{S}_2 := \{S_2(x,\theta) \mid \theta \in \Theta, x \in f(\theta)\}$ is* ***two-individual consistent* with SCC*** *f if for any $\theta, \theta', \hat{\theta} \in \Theta$, $x, y, z \in X$, we have*

(i) If $x \in f(\theta)$, then $x \in C_1^{\theta}(S_1(x,\theta)) \cap C_2^{\theta}(S_2(x,\theta))$,

(ii) If $x \in f(\theta) \setminus f(\theta')$, then there is $j \in \{1,2\}$ such that $x \notin C_j^{\theta'}(S_j(x,\theta))$,

(iii) If $x \in f(\theta)$ and $y \in f(\theta')$, then $S_1(x,\theta) \cap S_2(y,\theta') \neq \emptyset$,

(iv) If $S_1(x,\theta) \cap S_2(y,\theta') = \{z\}$ and $z \notin f(\hat{\theta})$, then either $z \notin C_1^{\hat{\theta}}(S_1(x,\theta))$ or $z \notin C_2^{\hat{\theta}}(S_2(y,\theta'))$.

We now present our necessity result involving two individual consistency*:

Theorem 1 (Necessity of two-individual consistency*) *If $f : \Theta \to 2^X \setminus \{\emptyset\}$ is Nash implementable, then there exists a pair of collections $\mathbb{S}_1$ and $\mathbb{S}_2$ that is two-individual consistent* with SCC f.*

Proof. Let $f : \Theta \to 2^X \setminus \{\emptyset\}$ be an SCC, and let $\mu = (M, g)$ implement f in Nash equilibrium where $M = (M_1, M_2)$. For any $\theta \in \Theta$ and for all $x \in f(\theta)$, there exists (m_1^x, m_2^x) which is a Nash Equilibrium of μ at θ such that $g(m_1^x, m_2^x) = x$. Hence, $x \in C_1^{\theta}(O_1^{\mu}(m_2^x))$ and $x \in C_2^{\theta}(O_2^{\mu}(m_1^x))$. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be defined such that for all $\theta \in \Theta$ and $x \in f(\theta)$, $S_1(x,\theta) := O_1^{\mu}(m_2^x)$ and $S_2(x,\theta) := O_2^{\mu}(m_1^x)$. Then (i) of two-individual consistency* trivially follows.

For some $\theta, \theta' \in \Theta$, let $x \in f(\theta) \setminus f(\theta')$. Suppose that $x \in C_1^{\theta'}(O_1^{\mu}(m_2^x))$ and $x \in C_2^{\theta'}(O_2^{\mu}(m_1^x))$. Then, $g(m_1^x, m_2^x) \in C_1^{\theta'}(O_1^{\mu}(m_2^x)) \cap C_1^{\theta'}(O_2^{\mu}(m_1^x))$. So, (m_1^x, m_2^x) is a Nash Equilibrium of μ at θ'. But then, by (ii) of Nash Implementability, we must have $x \in f(\theta')$, which contradicts with $x \in f(\theta) \setminus f(\theta')$; so the supposition must be wrong. Hence, there is $j \in \{1,2\}$ such that $g(m_1^x, m_2^x) = x \notin C_j^{\theta'}(O_j^{\mu}(m_i^x))$ where $i \in \{1,2\}$ and $i \neq j$, which implies (ii) of two-individual consistency*.

To obtain (iii) of two-individual consistency*, let $x \in f(\theta)$ and $y \in f(\theta')$ for some $\theta, \theta' \in \Theta$. Then, there exists $m^x \in M$ and $m^y \in M$ such that $g(m^x) = x \in C_1^{\theta}(S_1(x,\theta)) \cap C_2^{\theta}(S_2(x,\theta))$ and $g(m^y) = y \in C_1^{\theta'}(S_1(y,\theta')) \cap C_2^{\theta'}(S_2(y,\theta'))$. Note that $S_1(x,\theta) = O_1^{\mu}(m_2^x) = \{g(m_1, m_2^x) \mid m_1 \in M_1\}$ and $S_2(y,\theta') = O_2^{\mu}(m_1^y) = \{g(m_1^y, m_2) \mid m_2 \in M_2\}$. Consider $g(m_1^y, m_2^x) = z$. Since $z \in S_1(x,\theta)$ and $z \in S_2(y,\theta')$, we have $z \in S_1(x,\theta) \cap S_2(y,\theta')$, i.e., $S_1(x,\theta) \cap S_2(y,\theta') \neq \emptyset$.

Finally, let $S_1(x,\theta) \cap S_2(y,\theta') = \{z\}$ and $z \notin f(\hat{\theta})$ for some $\hat{\theta} \in \Theta$. Since $S_1(x,\theta) = O_1^{\mu}(m_2^x) = \{g(m_1, m_2^x) \mid m_1 \in M_1\}$, and $S_2(y,\theta') = O_2^{\mu}(m_1^y) = \{g(m_1^y, m_2) \mid m_2 \in M_2\}$, it must be $g(m_1^y, m_2^x) = z \in (S_1(x,\theta) \cap (S_2(y,\theta'))$. Suppose for a contradiction, $z \in C_1^{\hat{\theta}}(S_1(x,\theta))$ and $z \in C_2^{\hat{\theta}}(S_2(y,\theta'))$, then $g(m_1^y, m_2^x) = z \in C_1^{\hat{\theta}}(O_1^{\mu}(m_2^x)) \cap C_2^{\hat{\theta}}(O_2^{\mu}(m_1^y))$ and hence $z \in NE^{\mu}(\hat{\theta})$. This implies, by (ii) of Nash implementability, we must have $z \in f(\hat{\theta})$, a contradiction. Thus, if for some $\hat{\theta} \in \Theta$, $z \notin f(\hat{\theta})$ then either $z \notin C_1^{\hat{\theta}}(S_1(x,\theta))$ or $z \notin C_2^{\hat{\theta}}(S_2(y,\theta'))$ implying (iv) of the two-individual consistency*.

Therefore, ($\mathbb{S}_1$, $\mathbb{S}_2$) is two-individual consistent* with SCC f. ■

To provide a relatively straightforward sufficiency result employing two-individual consistent* pair of collections, we introduce the following properties:

Definition 5 (Singleton Intersection Property) *A given pair of collections of sets $\mathbb{S}_1$ and $\mathbb{S}_2$ satisfy **singleton intersection property** if for all $x, y \in X$, $\theta, \theta' \in \Theta$ such that either $x \neq y$ or $\theta \neq \theta'$, we have $S_1(x,\theta) \cap S_2(y,\theta') = \{z\}$ for some $z \in X$.*

In words, the singleton intersection property holds when the intersection of a set in $\mathbb{S}_1$ with another set in $\mathbb{S}_2$ is a singleton whenever these two sets are not associated with the same alternative and the same state of the world.

Definition 6 *Intersection Property* A pair of collections of sets $\mathbb{S}_1$ and $\mathbb{S}_2$ two-individual consistent* with SCC f satisfy* ***intersection property**** *if for any $i = 1, 2$ and $j \neq i$, for all $z \in S_i(x, \theta)$ with $z \neq x$, there is $S_j(y, \theta')$ with either $x \neq y$ or $\theta \neq \theta'$ such that $z \in S_j(y, \theta')$.*

Intersection property* ensures that any element in a set associated with an agent in a two-individual consistent* collection also appears in another set in this two-individual consistent* that is associated with the other individual.

We note that the singleton intersection property and the intersection property* are independent from each other as can be seen in the following examples:

Example 1. Singleton intersection property does not imply intersection property*: Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be a pair of collections of sets two-individual consistent* with SCC f, also let $a, b, c \in X$ and $\theta_1, \theta_2 \in \Theta$. Suppose that $\mathbb{S}_1 = \{S_1(a, \theta_1), S_1(b, \theta_2)\}$, $\mathbb{S}_2 = \{S_2(a, \theta_1), S_2(b, \theta_2)\}$ where $a \in f(\theta_1)$ and $b \in f(\theta_2)$ such that $S_1(a, \theta_1) = \{a, c\}$, $S_1(b, \theta_2) = \{b\}$ and $S_2(a, \theta_1) = \{a, b, c\}$, $S_2(b, \theta_2) = \{b, c\}$.

Observe that $S_1(a, \theta_1) \cap S_2(b, \theta_2) = \{a, c\} \cap \{b, c\} = \{c\}$ and $S_1(b, \theta_2) \cap S_2(a, \theta_1) = \{b\} \cap \{a, b, c\} = \{b\}$, which means that for any pair of $S_1(x, \theta), S_2(y, \theta')$ where either $x \neq y$ or $\theta \neq \theta'$, $S_1(x, \theta) \cap S_2(y, \theta')$ is a singleton. Thus, $\mathbb{S}_1$ and $\mathbb{S}_2$ satisfy singleton intersection property.

Nevertheless, for $c \in S_2(a, \theta_1)$, there exists no $S_1(x, \theta)$ with either $a \neq x$ or $\theta_1 \neq \theta$ such that $c \in S_1(x, \theta)$. Hence, $\mathbb{S}_1$ and $\mathbb{S}_2$ do not satisfy the intersection property* in this example.

Example 2. Intersection property* does not imply singleton intersection property: Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be a pair of collections of sets two-individual consistent* with an SCC f, and let $a, b, c, d \in X$ and $\theta_1, \theta_2, \theta_3 \in \Theta$, $a \in f(\theta_1), b \in f(\theta_2), c \in f(\theta_3)$.

Suppose that $\mathbb{S}_1 = \{S_1(a, \theta_1), S_1(b, \theta_2), S_1(c, \theta_3)\}$ and $\mathbb{S}_2 = \{S_2(a, \theta_1), S_2(b, \theta_2), S_2(c, \theta_3)\}$ are such that $S_1(a, \theta_1) = \{a, b, c\}$, $S_1(b, \theta_2) = \{b, c, d\}$, $S_1(c, \theta_3) = \{b, c\}$ and $S_2(a, \theta_1) = \{a, b, c\}$, $S_2(b, \theta_2) = \{b, c\}$, $S_2(c, \theta_3) = \{c, d\}$.

Observe that for $b \in S_1(a, \theta_1)$, $b \in S_2(b, \theta_2)$; for $c \in S_1(a, \theta_1)$, $c \in S_2(c, \theta_3)$; for $c \in S_1(b, \theta_2)$, $c \in S_2(a, \theta_1)$; for $d \in S_1(b, \theta_2)$, $d \in S_2(c, \theta_3)$; for $b \in S_1(c, \theta_3)$, $b \in S_2(a, \theta_1)$; for $b \in S_2(a, \theta_1)$, $b \in S_1(b, \theta_2)$; for $c \in S_2(a, \theta_1)$, $c \in S_1(b, \theta_2)$; for $c \in S_2(b, \theta_2)$, $c \in S_1(a, \theta_1)$; and for $d \in S_2(c, \theta_3)$, $d \in S_1(b, \theta_2)$. Therefore, the intersection property* holds for $\mathbb{S}_1$ and $\mathbb{S}_2$.

However, since $S_1(a, \theta_1) \cap S_2(b, \theta_2) = \{a, b, c\} \cap \{b, c\} = \{b, c\}$, which is not a singleton, $\mathbb{S}_1$ and $\mathbb{S}_2$ fails to satisfy singleton intersection property.

Next, we exemplify how to construct a mechanism by employing a pair of two-individual consistent* collections that satisfy the singleton intersection property as well as the intersection property*.

Example 3. Let $X = \{a, b, c, d\}$ and $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Consider SCC $f : \Theta \to 2^X \setminus \{\emptyset\}$ such that $f(\theta_1) = a$, $f(\theta_2) = b$, $f(\theta_3) = c$. Suppose $\mathbb{S}_1 = \{S_1(a, \theta_1), S_1(b, \theta_2), S_1(c, \theta_3)\}$ and $\mathbb{S}_2 = \{S_2(a, \theta_1), S_2(b, \theta_2), S_2(c, \theta_3)\}$ are a pair of collections that are two-individual consistent* with SCC f where

$S_1(a, \theta_1) = \{a, b, c\}$, $S_1(b, \theta_2) = \{b, d\}$, $S_1(c, \theta_3) = \{a, c, d\}$, and

$S_2(a, \theta_1) = \{a, b\}$, $S_2(b, \theta_2) = \{b, d\}$, $S_2(c, \theta_3) = \{c, d\}$.

Notice that the singleton intersection property and the intersection property* hold for $\mathbb{S}_1$ and $\mathbb{S}_2$ in this example. Because $\mathbb{S}_1$ and $\mathbb{S}_2$ are two-individual consistent* with SCC f, we must have $a \in C_1^{\theta_1}(S_1(a, \theta_1)),\ b \in C_1^{\theta_2}(S_1(b, \theta_2)),\ c \in C_1^{\theta_3}(S_1(c, \theta_3))$ and $a \in C_2^{\theta_1}(S_2(a, \theta_1)),\ b \in C_2^{\theta_2}(S_2(b, \theta_2)),\ c \in C_2^{\theta_3}(S_2(c, \theta_3))$.

Let us construct a mechanism $\mu = (M, g)$ by introducing a row for each set in $\mathbb{S}_2$ and a column for each set in $\mathbb{S}_1$ following the same order of f-optimal alternatives and states of the world for both agents. The goal is to obtain the sets in rows and columns as opportunity sets for the agents.[1] We can now place the socially optimal alternatives at the diagonal of the resulting game form and fill the rest by employing the singleton intersection property. The intersection property* ensures that no alternative is left behind, i.e., the set associated with each row becomes an opportunity set for agent two, and the set associated with each column becomes an opportunity set for agent one, as desired.

	$S_1(a,\theta_1) = \{a,b,c\}$	$S_1(b,\theta_2) = \{b,d\}$	$S_1(c,\theta_3) = \{a,c,d\}$
$S_2(a,\theta_1) = \{a,b\}$	a	b	a
$S_2(b,\theta_2) = \{b,d\}$	b	b	d
$S_2(c,\theta_3) = \{c,d\}$	c	d	c

Figure 1: Mechanism designed to Nash implement the SCC f with two-individual consistency*

We note that this is the unique game form that can be constructed in this example due to the singleton intersection property.

In order to check for Nash implementation, consider the following state-by-state best response analysis: With a slight abuse of notation, we refer to each message of an agent as the opportunity set that it generates for the other agent. Furthermore, we write $S_i \in BR_i^{\theta}(S_j)$ whenever $g(S_i, S_j) \in C_i^{\theta}(S_i)$ for both $i, j = 1, 2$ with $i \neq j$.

[1]Any two-individual mechanism can be regarded as each agent choosing an opportunity set for the other agent.

Let us first show that each f-optimal alternative is attained under Nash equilibrium in each state of the world:

At the state of the world θ_1, $BR_1^{\theta_1}(\{a,b,c\}) = \{a,b\}$ since $g(\{a,b\},\{a,b,c\}) = a \in C_1^{\theta_1}(\{a,b,c\})$; $BR_2^{\theta_1}(\{a,b\}) = \{\{a,b,c\},\{a,c,d\}\}$ since $g(\{a,b\},\{a,b,c\}) = g(\{a,b\},\{a,c,d\}) = a \in C_2^{\theta_1}(\{a,b\})$. Therefore, $(\{a,b\},\{a,b,c\})$ is a Nash Equilibrium at θ_1 that attains the f-optimal outcome a at θ_1.

At the state of the world θ_2, $BR_1^{\theta_2}(\{b,d\}) = \{\{a,b\},\{b,d\}\}$ since $g(\{a,b\},\{b,d\}) = g(\{b,d\},\{b,d\}) = b \in C_1^{\theta_2}(\{b,d\})$; $BR_2^{\theta_2}(\{b,d\}) = \{\{a,b,c\},\{b,d\}\}$ since $g(\{b,d\},\{a,b,c\}) = g(\{b,d\},\{b,d\}) = b \in C_2^{\theta_2}(\{b,d\})$. Hence, $(\{b,d\},\{b,d\})$ is a Nash Equilibrium at θ_2 attaining f-optimal outcome b at θ_2.

At the state of the world θ_3, $BR_1^{\theta_3}(\{a,c,d\}) = \{c,d\}$ since $g(\{c,d\},\{a,c,d\}) = c \in C_1^{\theta_3}(\{a,c,d\})$; $BR_2^{\theta_3}(\{c,d\}) = \{\{a,b,c\},\{a,c,d\}\}$ since $g(\{c,d\},\{a,b,c\}) = g(\{c,d\},\{a,c,d\}) = c \in C_2^{\theta_3}(\{c,d\})$. Thus, $(\{c,d\},\{a,c,d\})$ is a Nash Equilibrium at θ_3 that obtains f-optimal outcome c at θ_3.

Next, let us show that there is no bad Nash equilibrium in any state of the world, i.e., the outcomes that are not f-optimal in a state cannot arise as a Nash equilibrium:

At the state of the world θ_1, observe that $(\{b,d\},\{a,b,c\})$ is not a Nash Equilibrium at θ_1 as $BR_1^{\theta_1}(\{a,b,c\}) = \{a,b\}$. Similarly, $(\{c,d\},\{a,b,c\})$ is not a Nash Equilibrium at θ_1. $(\{a,b\},\{b,d\})$ is not a Nash Equilibrium either as $BR_2^{\theta_1}(\{a,b\}) = \{\{a,b,c\},\{a,c,d\}\}$. Since $g(\{b,d\},\{b,d\}) = b \notin f(\theta_1)$, either $b \notin C_1^{\theta_1}(S_1(b,\theta_2))$ or $b \notin C_2^{\theta_1}(S_2(a,\theta_1))$ due to (iv) of two-individual consistency*. Thus either $\{b,d\} \notin BR_1^{\theta_1}(\{b,d\})$ or $\{b,d\} \notin BR_2^{\theta_1}(\{b,d\})$. So, $(\{b,d\},\{b,d\})$ cannot be a Nash Equilibrium either. $g(\{c,d\},\{b,d\}) = d \notin f(\theta_1)$ requires either $\{c,d\} \notin BR_1^{\theta_1}(\{b,d\})$ or $\{b,d\} \notin BR_2^{\theta_1}(\{c,d\})$ due to (iv) of two-individual consistency*. So, $(\{c,d\},\{b,d\})$ cannot be a Nash Equilibrium.

Similarly, $g(\{b,d\},\{a,c,d\}) = d \notin f(\theta_1)$ implies either $\{b,d\} \notin BR_1^{\theta_1}(\{a,c,d\})$ or $\{b,d\} \notin BR_2^{\theta_1}(\{a,c,d\})$ implying that $(\{b,d\},\{a,c,d\})$ is not a Nash Equilibrium. Lastly, $g(\{c,d\},\{a,c,d\}) = c \notin f(\theta_1)$ requires either $\{c,d\} \notin BR_1^{\theta_1}(\{a,c,d\})$ or $\{a,c,d\} \notin BR_2^{\theta_1}(\{c,d\})$ due to (iv) of two-individual consistency*. Then, $(\{c,d\},\{a,c,d\})$ cannot be a Nash Equilibrium as well.

<u>At the state of the world θ_2</u>, $(\{c,d\},\{b,d\})$ is not a Nash Equilibrium at θ_2 as $BR_1^{\theta_2}(\{b,d\}) = \{\{a,b\},\{b,d\}\}$. Likewise, $(\{b,d\},\{a,c,d\})$ is not a Nash Equilibrium at θ_2 either as $BR_2^{\theta_2}(\{b,d\}) = \{\{a,b,c\},\{b,d\}\}$. Since $g(\{a,b\},\{a,b,c\}) = a \notin f(\theta_2)$, either $a \notin C_1^{\theta_2}(S_1(a,\theta_1))$ or $a \notin C_2^{\theta_2}(S_2(a,\theta_1))$ due to (iv) of two-individual consistency*. Thus, either $\{a,b\} \notin BR_1^{\theta_2}(\{a,b,c\})$ or $\{a,b,c\} \notin BR_2^{\theta_2}(\{a,b\})$. So, $(\{a,b\},\{a,b,c\})$ cannot be a Nash Equilibrium. Because $g(\{c,d\},\{a,b,c\}) = c \notin f(\theta_2)$, either $c \notin C_1^{\theta_2}(S_1(a,\theta_1))$ or $c \notin C_2^{\theta_2}(S_2(c,\theta_3))$ due to (iv) of two-individual consistency*. Thus either $\{c,d\} \notin BR_1^{\theta_2}(\{a,b,c\})$ or $\{a,b,c\} \notin BR_2^{\theta_2}(\{c,d\})$. Therefore, $(\{c,d\},\{a,b,c\})$ cannot be a Nash Equilibrium. Since $g(\{a,b\},\{a,c,d\}) = a \notin f(\theta_2)$, either $a \notin C_1^{\theta_2}(S_1(c,\theta_3))$ or $a \notin C_2^{\theta_2}(S_2(a,\theta_1))$ due to (iv) of two-individual consistency*. Thus, either $\{a,b\} \notin BR_1^{\theta_2}(\{a,c,d\})$ or $\{a,c,d\} \notin BR_2^{\theta_2}(\{a,b\})$. So, $(\{a,b\},\{a,b,c\})$ is not a Nash Equilibrium. Since $g(\{c,d\},\{a,c,d\}) = c \notin f(\theta_2)$, either $c \notin C_1^{\theta_2}(S_1(c,\theta_3))$ or $c \notin C_2^{\theta_2}(S_2(c,\theta_3))$ due to (iv) of two-individual consistency*. So, either $\{c,d\} \notin BR_1^{\theta_2}(\{a,c,d\})$ or $\{a,c,d\} \notin BR_2^{\theta_2}(\{c,d\})$. Hence, $(\{c,d\},\{a,c,d\})$ cannot be a Nash Equilibrium as well.

<u>At the state of the world θ_3</u>, $(\{a,b\},\{a,c,d\})$ and $(\{b,d\},\{a,c,d\})$ are not Nash Equilibria at θ_3 as $BR_1^{\theta_3}(\{a,c,d\}) = \{c,d\}$. $(\{c,d\},\{b,d\})$ is not a Nash Equilibrium at θ_3 either because $BR_2^{\theta_3}(\{c,d\}) = \{\{a,b,c\},\{a,c,d\}\}$. Since $g(\{a,b\},\{a,b,c\}) = a \notin f(\theta_3)$, either $a \notin C_1^{\theta_3}(S_1(a,\theta_1))$ or $a \notin C_2^{\theta_3}(S_2(a,\theta_1))$ due to (iv) of two-individual consistency*. Hence, either $\{a,b\} \notin BR_1^{\theta_3}(\{a,b,c\})$ or $\{a,b,c\} \notin BR_2^{\theta_3}(\{a,b\})$. So, $(\{a,b\},\{a,b,c\})$ is not a Nash Equilibrium. Since $g(\{b,d\},\{a,b,c\}) = b \notin f(\theta_3)$, either $b \notin C_1^{\theta_3}(S_1(a,\theta_1))$ or $b \notin C_2^{\theta_3}(S_2(b,\theta_2))$ due

to (iv) of two-individual consistency*. Thus, either $\{b, d\} \notin BR_1^{\theta_3}(\{a, b, c\})$ or $\{a, b, c\} \notin BR_2^{\theta_3}(\{b, d\})$. Hence, $(\{b, d\}, \{a, b, c\})$ cannot be a Nash Equilibrium. Since $g(\{a, b\}, \{b, d\}) = b \notin f(\theta_3)$, either $b \notin C_1^{\theta_3}(S_1(b, \theta_2))$ or $b \notin C_2^{\theta_3}(S_2(a, \theta_1))$ due to (iv) of two-individual consistency*. So, either $\{a, b\} \notin BR_1^{\theta_3}(\{b, d\})$ or $\{b, d\} \notin BR_2^{\theta_3}(\{a, b\})$. Thus, $(\{a, b\}, \{b, d\})$ cannot be a Nash Equilibrium. Finally, since $g(\{b, d\}, \{b, d\}) = b \notin f(\theta_3)$, either $b \notin C_1^{\theta_3}(S_1(b, \theta_2))$ or $b \notin C_2^{\theta_3}(S_2(b, \theta_2))$ due to (iv) of two-individual consistency*. Thus either $\{b, d\} \notin BR_1^{\theta_3}(\{b, d\})$ or $\{b, d\} \notin BR_2^{\theta_3}(\{b, d\})$. Therefore, $(\{b, d\}, \{b, d\})$ is also not a Nash Equilibrium.

The above analysis shows that (i) and (ii) of Nash Implementability are satisfied in this example; thus, μ Nash implements SCC f.

We are now ready to present our sufficiency theorem employing two-individual consistent* collections together with singleton intersection property and intersection property*:

Theorem 2 (Sufficiency with two-individual consistency*) *If a pair of collections $\mathbb{S}_1$ and $\mathbb{S}_2$ two-individual consistent* with SCC f satisfies the singleton intersection property and the intersection property*, then f is Nash Implementable.*

Proof. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be a two-individual consistent* pair of collection of sets with an SCC f which satisfy the singleton intersection property and intersection property*. The proof is constructive.

Consider the mechanism $\mu = (M, g)$ where $M = \mathbb{S}_2 \times \mathbb{S}_1$, i.e., $M_1 = \mathbb{S}_2$ and $M_2 = \mathbb{S}_1$, and the outcome function is as follows

$$g(S_2(y, \theta'), S_1(x, \theta)) = \begin{cases} x & \text{if } x = y \text{ and } \theta = \theta' \\ z & \text{otherwise} \end{cases}, \text{ where } S_2(y, \theta') \cap S_1(x, \theta) = \{z\} \in X.$$

Recall that, due to singleton intersection property, for any $\theta, \theta' \in \Theta$ and for all $x \in f(\theta)$ and $y \in f(\theta')$, $S_1(x,\theta) \cap S_2(y,\theta') = \{z\}$ for some $z \in X$ when either $x \neq y$ or $\theta \neq \theta'$. That is, the mechanism is well-defined.

The intuition is the same as in the example we provide above; each agent chooses an opportunity set for the other agent, these opportunity sets have singleton intersections, and each opportunity set of an agent is spanned by the collection of opportunity sets of the other agent.

We now show that this mechanism Nash implements SCC f via two claims:

Claim 1 *For all $\theta \in \Theta$, $f(\theta) \subseteq NE^{\mu}(\theta)$.*

Proof: For any $\theta \in \Theta$, take any $x \in f(\theta)$. Observe that, if $x \in f(\theta)$, then $x \in C_1^{\theta}(S_1(x,\theta)) \cap C_2^{\theta}(S_2(x,\theta))$ by (i) of two-individual consistency*. Consider the message profile $m^* = (m_1^*, m_2^*)$ such that $g(m_1^*, m_2^*) = x$ and observe that for $i, j = 1, 2$ with $i \neq j$, $m_j^* = S_i(x,\theta)$, then $O_i^{\mu}(m_j^*) = O_i^{\mu}(S_i(x,\theta)) = S_i(x,\theta)$, which follows from the following observation: Due to the intersection property*, for any $z \neq x$ with $z \in S_i(x,\theta) = m_j^*$, there is $S_j(y,\theta')$ with either $y \neq x$ or $\theta' \neq \theta$ such that $z \in S_j(y,\theta')$, and hence, if $m_i = S_j(y,\theta')$, then $g(m_i, m_j^*) = z$ thanks to the singleton intersection property, i.e., $z \in O_i^{\mu}(m_j^*)$. Therefore, $x \in C_1^{\theta}(S_1(x,\theta)) \cap C_2^{\theta}(S_2(x,\theta)) = C_1^{\theta}(O_1^{\mu}(S_2(x,\theta)) \cap C_2^{\theta}(O_2^{\mu}(S_1(x,\theta))$. Since $x = g(m_1^*, m_2^*) = g(S_2(x,\theta), S_1(x,\theta)) \in C_1^{\theta}(O_1^{\mu}(S_1(x,\theta))$ and $g(S_2(x,\theta), S_1(x,\theta)) \in C_2^{\theta}(O_2^{\mu}(S_2(x,\theta)), (m_1^*, m_2^*)$ is a Nash equilibrium of μ at θ, i.e., $g(S_2(x,\theta), S_1(x,\theta)) = x \in NE^{\mu}(\theta)$. ■

Claim 2 *For all $\theta \in \Theta$, $NE^{\mu}(\theta) \subseteq f(\theta)$.*

Proof: For any $\theta \in \Theta$, take any $z \in NE^{\mu}(\theta)$. Then there exists $S_1(x,\theta') \in \mathbb{S}_1$ and $S_2(y,\theta'') \in \mathbb{S}_2$ such that $g(S_2(y,\theta''), S_1(x,\theta')) = z$. If $z = x$ and $\theta' = \theta$ or $z = y$ and $\theta'' = \theta$, then we are trivially done as $S_i(x,\theta) \in \mathbb{S}_i$ implies $z \in f(\theta)$. Further, we cannot have $z \neq x$ and $g(S_2(y,\theta''), S_1(x,\theta')) = z$ whenever $x = y$ and $\theta' = \theta''$

due to the construction of mechanism μ. Then, $z \neq x$ and $g(S_2(y, \theta''), S_1(x, \theta')) = z$ implies either $x \neq y$ or $\theta' \neq \theta''$. But then, the single intersection property implies that $S_1(x, \theta') \cap S_2(y, \theta'') = \{z\}$. Since $z \in NE^\mu(\theta)$, $z \in C_1^\theta(O_1^\mu(S_1(x, \theta')))$ and $z \in C_2^\theta(O_2^\mu(S_2(y, \theta'')))$, which implies $z \in C_1^\theta(S_2(y, \theta''))$ and $z \in C_2^\theta(S_1(x, \theta'))$. It follows from (iv) of two-individual consistency* that it must be $z \in f(\theta)$. ■

Combining the claims above, for all $\theta \in \Theta$, we have $f(\theta) = NE^\mu(\theta)$. Therefore, f is Nash Implementable by μ if $\mathbb{S}_1$ and $\mathbb{S}_2$ are two-individual consistent* sets satisfying the single intersection property and the intersection property*. ■

Having shown the necessity and sufficiency results with two-individual consistency* when the singleton intersection property and the intersection property* hold, we now turn to the cases where these properties are not necessarily satisfied.

To this end, we first strengthen our necessity result by obtaining a stronger two-individual consistency:

Definition 7 (Two-Individual Stronger Consistency) *A pair of collections of sets $\mathbb{S}_1$ and $\mathbb{S}_2$ is **two-individual stronger consistent with SCC** f if for any $\theta, \theta' \in \Theta$, and any $x, y \in X$,*

(i) If $x \in f(\theta)$, then $x \in C_1^\theta(S_1(x, \theta)) \cap C_2^\theta(S_2(x, \theta))$,

(ii) If $x \in f(\theta) \setminus f(\theta')$, then there is $j \in \{1, 2\}$ such that $x \notin C_j^{\theta'}(S_j(x, \theta))$,

(iii) There exists $z \in S_1(x, \theta) \cap S_2(y, \theta')$ such that if for any θ'', $z \in C_1^{\theta''}(S_1(x, \theta)) \cap C_2^{\theta''}(S_2(y, \theta'))$, then $z \in f(\theta'')$.

We note that two-individual stronger consistency implies two-individual consistency* as (iii) of two-individual stronger consistency implies both (iii) and (iv) of two-individual consistency*.

Next, we present our necessity result with two-individual stronger consistency:

Theorem 3 (Necessity of two-individual stronger consistency) *If* $f : \Theta \rightarrow 2^X \setminus \{\emptyset\}$ *is Nash implementable, then there exists a pair of collections* $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *that is two-individual stronger consistent with SCC* f*.*

Proof. Let $f : \Theta \rightarrow 2^X \setminus \{\emptyset\}$ be an SCC, and let $\mu = (M, g)$ implement f in Nash equilibrium where $M = (M_1, M_2)$. For any $\theta \in \Theta$ and for all $x \in f(\theta)$, there exists (m_1^x, m_2^x) which is a Nash Equilibrium of μ at θ such that $g(m_1^x, m_2^x) = x$. Hence, $x \in C_1^\theta(O_1^\mu(m_2^x))$ and $x \in C_2^\theta(O_2^\mu(m_1^x))$. Following the same steps as in the proof for Theorem 1, we obtain (i) and (ii), which coincide for two-individual consistency* and two-individual stronger consistency.

To obtain (ii) of two-individual stronger consistency, let (m_1^x, m_2^x) be the Nash equilibrium at θ such that $g(m_1^x, m_2^x) = x$, (m_1^y, m_2^y) be the Nash equilibrium at θ' such that $g(m_1^y, m_2^y) = y$, and let $g(m_1^y, m_2^x) = z$. Then, $z \in S_1(x, \theta) \cap S_2(y, \theta')$. If $z \in C_1^{\theta''}(O_1^\mu(m_2^x)) \cap C_2^{\theta''}(O_2^\mu(m_1^y))$, then (m_1^y, m_2^x) must be a Nash equilibrium of μ at θ'' with $z \in C_1^{\theta''}(S_1(x, \theta)) \cap C_2^{\theta''}(S_2(y, \theta'))$. Thus, by (ii) of Nash implementability, we must have $z \in f(\theta'')$, which implies (iii) of two-individual stronger consistency holds as well.

Therefore, ($\mathbb{S}_1$, $\mathbb{S}_2$) must be two-individual stronger consistent with SCC f. ■

To turn our necessity result with two-individual stronger consistency into a sufficiency result, we need the following:

Definition 8 (Spanning Collection) *We say that* $(\mathbb{S}_1, \mathbb{S}_2)$ *is a* ***spanning pair of collection of sets*** *if for all* $\theta, \theta' \in \Theta$*,* $x \in f(\theta)$*, and* $y \in f(\theta')$*, we have*

$$S_1(x,\theta) = \bigcup_{S_2(y,\theta') \in \mathbb{S}_2} \{z(S_1(x,\theta), S_2(y,\theta'))\} \text{ and}$$

$$S_2(y,\theta') = \bigcup_{S_1(x,\theta) \in \mathbb{S}_1} \{z(S_1(x,\theta), S_2(y,\theta'))\}$$

for some function $z : \mathbb{S}_1 \times \mathbb{S}_2 \to X$ *that satisfies*

(i) $z(S_1(x,\theta), S_2(y,\theta')) \in S_1(x,\theta) \cap S_2(y,\theta')$ *and*

(ii) *if for any* $\theta'' \in \Theta$*,* $z(S_1(x,\theta), S_2(y,\theta')) \in C_1^{\theta''}(S_1(x,\theta)) \cap C_2^{\theta''}(S_2(y,\theta'))$*, then* $z(S_1(x,\theta), S_2(y,\theta')) \in f(\theta'')$*.*

We are ready to present our second sufficiency result, which shows that a pair of two-individual stronger consistent collection that is also a spanning collection suffices for Nash implementation.

Theorem 4 (Sufficiency with two-individual stronger consistency) *If* $(\mathbb{S}_1, \mathbb{S}_2)$ *is a pair of collections of sets that is two-individual stronger consistent with SCC* f *and a spanning collection, then* f *is Nash Implementable.*

Proof. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two-individual stronger consistent collections of sets with SCC f and $(\mathbb{S}_1, \mathbb{S}_2)$ be a spanning collection. The proof is constructive and relatively straightforward.

Let us construct mechanism $\mu = (M, g)$ slightly modifying the one in the proof of our first sufficiency result, Theorem 2: Let $M_1 = \mathbb{S}_2$ and $M_2 = \mathbb{S}_1$ and the outcome function be given by

$$g(S_2(y,\theta'), S_1(x,\theta)) = \begin{cases} x & \text{if } x = y \text{ and } \theta = \theta', \\ z(S_1(x,\theta), S_2(y,\theta')) & \text{otherwise.} \end{cases}$$

The mechanism is well-defined thanks to the existence of the function described in the definition spanning collection.

The following claims imply together that mechanism μ Nash implements SCC f:

Claim 3 *For all $\theta \in \Theta$, $f(\theta) \subseteq NE^{\mu}(\theta)$.*

Proof: The proof involves similar steps to the proof of Claim 1: For any $\theta \in \Theta$, take any $x \in f(\theta)$. Observe that $x \in C_1^{\theta}(S_1(x,\theta)) \cap C_2^{\theta}(S_2(x,\theta))$ by (i) of two-individual stronger consistency. Consider the message profile $m^* = (m_1^*, m_2^*)$ with $m_1^* = S_2(x,\theta)$ and $m_2^* = S_1(x,\theta)$. Then, $g(m_1^*, m_2^*) = x$. It follows from the fact that $(\mathbb{S}_1, \mathbb{S}_2)$ is a spanning collection and $g(S_2(y,\theta'), S_1(x,\theta)) = z(S_1(x,\theta), S_2(y,\theta'))$ whenever either $x \neq y$ or $\theta \neq \theta'$ that for each $i,j = 1,2$ with $i \neq j$, $m_j^* = S_i(x,\theta)$, we have $O_i^{\mu}(m_j^*) = O_i^{\mu}(S_i(x,\theta)) = S_i(x,\theta)$. Therefore, $x \in C_1^{\theta}(S_1(x,\theta)) \cap C_2^{\theta}(S_2(x,\theta)) = C_1^{\theta}(O_1^{\mu}(S_2(x,\theta)) \cap C_2^{\theta}(O_2^{\mu}(S_1(x,\theta))$, i.e., $x \in NE^{\mu}(\theta)$. ■

Claim 4 *For all $\theta \in \Theta$, $NE^{\mu}(\theta) \subseteq f(\theta)$.*

Proof: For any $\theta \in \Theta$, take any $z \in NE^{\mu}(\theta)$. Then, there exists $S_1(x,\theta') \in \mathbb{S}_1$ and $S_2(y,\theta'') \in \mathbb{S}_2$ such that $g(S_2(y,\theta''), S_1(x,\theta')) = z$. If either $z = x$ and $\theta' = \theta$ or $z = y$ and $\theta'' = \theta$, then the proof follows trivially because $S_i(z,\theta) \in \mathbb{S}_i$ implies $z \in f(\theta)$. Therefore, by the construction of the mechanism, we must have $g(S_2(y,\theta''), S_1(x,\theta')) = z$ implying either $x \neq y$ or $\theta' \neq \theta''$ and hence $z = z(S_1(x,\theta'), S_2(y,\theta''))$ where $z : \mathbb{S}_1 \times \mathbb{S}_2 \to X$ is the function described in the definition of spanning collection. Furthermore, since $z \in NE^{\mu}(\theta)$, we have $z \in C_1^{\theta}(O_1^{\mu}(S_1(x,\theta')))$ and $z \in C_2^{\theta}(O_2^{\mu}(S_2(y,\theta'')))$, which implies $z \in C_1^{\theta}(S_2(y,\theta''))$ and $z \in C_2^{\theta}(S_1(x,\theta'))$. It follows from property (ii) of the function $z : \mathbb{S}_1 \times \mathbb{S}_2 \to X$ given in the definition of spanning collection that it must be $z \in f(\theta)$. ■

The two claims above imply that mechanism μ Nash implements SCC f. ■

Below, we provide an example where one can employ our sufficiency result with two-individual stronger consistency given above.

Example 4. Let $X = \{a, b, c, d\}$ and $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Consider an SCC $f : \Theta \to 2^X \setminus \{\emptyset\}$ with $f(\theta_1) = a$, $f(\theta_2) = b$, $f(\theta_3) = c$. Suppose $\mathbb{S}_1 = \{S_1(a, \theta_1),\ S_1(b, \theta_2),\ S_1(c, \theta_3)\}$ and $\mathbb{S}_2 = \{S_2(a, \theta_1),\ S_2(b, \theta_2),\ S_2(c, \theta_3)\}$ are a pair of collections that is two-individual stronger consistent with SCC f and also spanning a collection such that $S_1(a, \theta_1) = \{a, b, d\}$, $S_1(b, \theta_2) = \{a, b\}$, $S_1(c, \theta_3) = \{c, d\}$ and $S_2(a, \theta_1) = \{a, d\}$, $S_2(b, \theta_2) = \{b, c\}$, $S_2(c, \theta_3) = \{a, c, d\}$. We note that since $S_1(a, \theta_1,) \cap S_2(c, \theta_3) = \{a, b, d\} \cap \{a, c, d\} = \{a, d\}$, singleton intersection property does not hold in this example. Let $z : \mathbb{S}_1 \times \mathbb{S}_2 \to X$ be such that $z(S_1(a, \theta_1,), S_2(c, \theta_3)) = z(\{a, b, d\}, \{a, c, d\}) = d$.

Similar to Example 3, we can construct the mechanism that Nash implements f by placing the sets in $\mathbb{S}_2$ as rows and sets in $\mathbb{S}_1$ so that the f-optimal alternatives at each state, a, b, c, are in the diagonal entries of the corresponding game form, thanks to two-individual stronger consistency:

	$S_1(a, \theta_1)$	$S_1(b, \theta_2)$	$S_1(c, \theta_3)$
$S_1(a, \theta_1)$	a	$z(S_1(b, \theta_2), S_2(a, \theta_1))$	$z(S_1(c, \theta_3), S_2(a, \theta_1))$
$S_2(b, \theta_2)$	$z(S_1(a, \theta_1), S_2(b, \theta_2))$	b	$z(S_1(c, \theta_3), S_2(b, \theta_2))$
$S_2(c, \theta_3)$	$z(S_1(a, \theta_1), S_2(c, \theta_3))$	$z(S_1(b, \theta_2), S_2(c, \theta_3))$	c

Figure 2: Mechanism designed to Nash implement the SCC f with two-individual stronger consistency

Let us first replace the messages with the actual sets that they correspond to:

	$\{a, b, d\}$	$\{a, b\}$	$\{c, d\}$
$\{a, d\}$	a	$z(S_1(b, \theta_2), S_2(a, \theta_1))$	$z(S_1(c, \theta_3), S_2(a, \theta_1))$
$\{b, c\}$	$z(S_1(a, \theta_1), S_2(b, \theta_2))$	b	$z(S_1(c, \theta_3), S_2(b, \theta_2))$
$\{a, c, d\}$	$z(S_1(a, \theta_1), S_2(c, \theta_3))$	$z(S_1(b, \theta_2), S_2(c, \theta_3))$	c

Figure 3: Mechanism designed to Nash implement the SCC f with two-individual stronger consistency

It follows from the defining properties of $z : \mathbb{S}_1 \times \mathbb{S}_2 \to X$ that we must have $z(\{a,b\},\{a,d\}) = a$, $z(\{c,d\},\{a,d\}) = d$, $z(\{a,b,d\},\{b,c\}) = b$, $z(\{c,d\},\{b,c\}) = c$, and $z(\{a,b\},\{a,c,d\}) = a$ as $z(S_1(x,\theta), S_2(y,\theta')) \in S_1(x,\theta) \cap S_2(y,\theta')$ for all $x, y \in X = \{a,b,c,d\}$ and all $\theta, \theta' \in \Theta = \{\theta_1, \theta_2, \theta_3\}$. Furthermore, we were given that $z(S_1(a,\theta_1,), S_2(c,\theta_3)) = z(\{a,b,d\},\{a,c,d\}) = d$. This leads us to the following mechanism:

	$\{a,b,d\}$	$\{a,b\}$	$\{c,d\}$
$\{a,d\}$	a	a	d
$\{b,c\}$	b	b	c
$\{a,c,d\}$	d	a	c

Figure 4: The mechanism that Nash implements SCC f

To see why this mechanism Nash implements SCC f, one can do a similar best-reply analysis as in Example 3. On the other hand, it is straightforward to see that (i) of two-individual stronger consistency implies that the diagonal entries of the mechanism above induce the Nash equilibria corresponding to f-optimal alternatives in each state. It follows from (ii) of two-individual stronger consistency that the diagonal entries are Nash equilibria only for the states in which they are f-optimal. As for the off-diagonal alternatives, the key condition that eliminates bad equilibria is the property (ii) described in the definition of spanning collection that the function $z : \mathbb{S}_1 \times \mathbb{S}_2 \to X$ satisfies.

Both in Theorems 2 and Theorem 4, the proof constructs a mechanism in which there cannot be an opportunity set that is not an element in the two-individual consistent* or two-individual stronger consistent pair of collections. However, it is possible that in a mechanism, there might be messages that are not used as part of a Nash equilibria in any state of the world. Such messages might lead to opportunity sets that do not appear in consistent collections. Below, we extend the above sufficiency result to an if-and-only-if result in such cases.

First, we need to slightly modify our definition of a spanning collection:

Definition 9 (Extended Spanning Consistent Collection) *Let* $(\mathbb{S}_1, \mathbb{S}_2)$ *be a two-individual stronger consistent collection of sets with SCC* f*. If there exists (possibly empty) collections of sets* $\mathcal{S}_1 = \{S_{1,k}\}_{k\in\{1,\dots,K\}}$ *and* $\mathcal{S}_2 = \{S_{2,l}\}_{l\in\{1,\dots,L\}}$ *such that* $(\mathbb{S}_1 \cup \mathcal{S}_1, \mathbb{S}_2 \cup \mathcal{S}_2)$ *is a spanning collection of sets* $(\mathbb{S}_1 \cup \mathcal{S}_1, \mathbb{S}_2 \cup \mathcal{S}_2)$ *is said to be **an extended pair of spanning consistent collections of sets.***

We now present the characterization of Nash implementation involving an extended pair of spanning consistent collections in a finite environment:

Theorem 5 (Characterization with extended spanning collections) *An SCC* f *is Nash Implementable by a finite mechanism if and only if there exist two-individual stronger consistent pair of sets* $(\mathbb{S}_1, \mathbb{S}_2)$ *and a pair of finite family of sets* $(\mathcal{S}_1, \mathcal{S}_2)$ *such that* $(\mathbb{S}_1 \cup \mathcal{S}_1, \mathbb{S}_2 \cup \mathcal{S}_2)$ *is an extended pair of spanning consistent collections of sets.*

Proof. First, we consider the necessity for Nash implementation. Suppose an SCC $f : \Theta \to 2^X \setminus \{\emptyset\}$ is implemented by $\mu = (M, g)$ in Nash equilibrium where $M = (M_1, M_2)$ is a finite set. For any $\theta \in \Theta$ and for all $x \in f(\theta)$, there exists (m_1^x, m_2^x) which is a Nash Equilibrium of μ at θ such that $g(m_1^x, m_2^x) = x$. Hence, $x \in C_1^\theta(O_1^\mu(m_2^x))$ and $x \in C_2^\theta(O_2^\mu(m_1^x))$. Following the same steps in the proof of Theorem 4, we obtain the existence of two-individual stronger consistent sets $(\mathbb{S}_1, \mathbb{S}_2)$ that satisfy (i), (ii) and (iii) in the definition of two-individual stronger consistency. If $(\mathbb{S}_1, \mathbb{S}_2)$ is an (extended) pair of spanning consistent collections, the proof follows trivially.

If $(\mathbb{S}_1, \mathbb{S}_2)$ is not an extended pair of spanning consistent collection itself, then we can construct $\mathcal{S}_1 = \{S_{1,k}\}_{k\in\{1,\dots,K\}}$ and $\mathcal{S}_2 = \{S_{2,l}\}_{l\in\{1,\dots,L\}}$ for some finite $K \in \mathbb{N}$ and $L \in \mathbb{N}$ as follows so that $(\mathbb{S}_1 \cup \mathcal{S}_1, \mathbb{S}_2 \cup \mathcal{S}_2)$ is a spanning collection of sets: For every $x \in f(\theta)$, fix a message profile (m_1^x, m_2^x) such that (m_1^x, m_2^x) is a Nash equilibrium of μ at θ with $g(m_1^x, m_2^x) = x$ so that $O_i^\mu(m_j^x) = S_i(x, \theta)$. Such a message profile exists due to (i) of two-individual stronger consistency.

Let $\tilde{M}_1 = \{m_1 \in M_1 \mid O_2^\mu(m_1) \notin \mathbb{S}_2$ or $O_2^\mu(m_1) = S_2(x,\theta)$ for some $x \in f(\theta)$ but $\exists m_2' \in M_2$ such that $g(m_1, m_2') \neq g(m_1^x, m_2')\}$ and $\tilde{M}_2 = \{m_2 \in M_2 \mid O_1^\mu(m_2) \notin \mathbb{S}_1$ or $O_1^\mu(m_2) = S_1(x,\theta)$ for some $x \in f(\theta)$ but $\exists m_1' \in M_1$ such that $g(m_1', m_2) \neq g(m_1', m_2^x)\}$. Let $\tilde{M}_1 = \{\tilde{m}_1^1, \tilde{m}_1^2, ..., \tilde{m}_1^L\}$ and $\tilde{M}_2 = \{\tilde{m}_2^1, \tilde{m}_2^2, ..., \tilde{m}_2^K\}$. Then define $\tilde{S}_{1,k} = O_1^\mu(\tilde{m}_2^k)$ and $\tilde{S}_{2,l} = O_2^\mu(\tilde{m}_1^l)$ for all $k \in \{1, ..., K\}, l \in \{1, ..., L\}$.

Next, we turn to the sufficiency for Nash Implementation. Let $(\mathbb{S}_1 \cup \mathcal{S}_1, \mathbb{S}_2 \cup \mathcal{S}_2)$ be an extended pair of spanning consistent collection of sets with an SCC f.

Thanks to $(\mathbb{S}_1 \cup \mathcal{S}_1, \mathbb{S}_2 \cup \mathcal{S}_2)$ being an extended pair of spanning consistent collection of sets. There is a function $z : \mathbb{S}_1 \cup \mathcal{S}_1 \times \mathbb{S}_2 \cup \mathcal{S}_2 \to X$ such that (i) $z(S_1, S_2) \in S_1 \cap S_2$ and (ii) if for any $\theta'' \in \Theta$, $z(S_1, S_2) \in C_1^{\theta''}(S_1) \cap C_2^{\theta''}(S_2)$, then $z(S_1, S_2) \in f(\theta'')$ for each pair of sets S_1, S_2 with $S_1 \in \mathbb{S}_1 \cup \mathcal{S}_1$ and $S_2 \in \mathbb{S}_2 \cup \mathcal{S}_2$.

Let us define the mechanism $\mu = (M, g)$ as follows: Let $M_1 = \mathbb{S}_2 \cup \mathcal{S}_2$ and $M_2 = \mathbb{S}_1 \cup \mathcal{S}_1$, and the outcome function $g : M \to X$ be given by

$$g(S_2, S_1) = \begin{cases} x & \text{if } S_1 = S_1(x,\theta) \in \mathbb{S}_1,\ S_2 = S_2(y,\theta') \in \mathbb{S}_2;\ x = y,\ \theta = \theta', \\ z(S_1, S_2) & \text{otherwise.} \end{cases}$$

We would like to emphasize that we construct the above mechanism so that $O_1^\mu(S_1(x,\theta)) = S_1(x,\theta)$ and $O_2^\mu(S_2(x,\theta)) = S_2(x,\theta)$. Furthermore, as before, we have $O_i^\mu(S_i) = S_i$ for both $i = 1, 2$, thanks to $(\mathbb{S}_1 \cup \mathcal{S}_1, \mathbb{S}_2 \cup \mathcal{S}_2)$ being an extended pair of spanning consistent collection of sets.

We complete the proof of the sufficiency part with the following claims.

Claim 5 *For all $\theta \in \Theta$, $f(\theta) \subseteq NE^\mu(\theta)$.*

Proof: For any $\theta \in \Theta$, let $x \in f(\theta)$. Consider the message profile $(m_1^*, m_2^*) = (S_2(x,\theta), S_1(x,\theta))$. Then, by construction, $g(S_2(x,\theta), S_1(x,\theta)) = x$. It follows from (i) of two-individual stronger consistency that $x \in C_1^\theta(S_1(x,\theta)) \cap C_2^\theta(S_2(x,\theta))$. Because $O_1^\mu(S_1(x,\theta) = S_1(x,\theta)$ and $O_2^\mu(S_2(x,\theta) = S_2(x,\theta)$, we have $x \in C_1^\theta$ $(O_1^\mu(S_1(x,\theta)) \cap C_2^\theta(O_2^\mu(S_2(x,\theta))$, which implies (m_1^*, m_2^*) is a Nash Equilibrium of μ at θ with $g(m_1^*, m_2^*) = x$, i.e., $x \in NE^\mu(\theta)$. ■

Claim 6 *For all $\theta \in \Theta$, $NE^\mu(\theta) \subseteq f(\theta)$.*

Proof: For any $\theta \in \Theta$, let $x \in NE^\mu(\theta)$. Then, there is $m^* = (m_1^*, m_2^*)$, a Nash equilibrium of μ at θ such that $g(m^*) = x$. Let $m_1^* = S_2^*$ and $m_2^* = S_1^*$. If $S_2^* = S_2(x,\theta)$ and $S_1^* = S_1(x,\theta)$, then, by construction, $O_1^\mu(m_2^*) = S_1(x,\theta)$ and $O_2^\mu(m_1^*) = S_2(x,\theta)$ and $x \in f(\theta)$ follows from two-individual stronger consistency. Otherwise, $g(m^*) = g(S_2^*, S_1^*) = z(S_1^*, S_2^*)$. Since m^* is a Nash Equilibrium of the μ at θ, $g(m^*) = x \in C_1^\theta(O_1^\mu(m_2^*)) \cap C_2^\theta(O_2^\mu(m_1^*)) = C_1^\theta(S_1^*) \cap C_2^\theta(S_2^*)$. Since $S_1^* \in \mathbb{S}_1 \cup \mathcal{S}_1$ and $S_2^* \in \mathbb{S}_2 \cup \mathcal{S}_2$ and $(\mathbb{S}_1 \cup \mathcal{S}_1, \mathbb{S}_2 \cup \mathcal{S}_2)$ is an extended pair of spanning consistent collection of sets, it follows from the defining properties of the function $z : \mathbb{S}_1 \cup \mathcal{S}_1 \times \mathbb{S}_2 \cup \mathcal{S}_2 \to X$ that $x \in f(\theta)$ as $x = z(S_1^*, S_2^*) \in C_1^\theta(S_1^*) \cap C_2^\theta(S_2^*)$. ■ ■

CHAPTER 5

CONCLUSION

In this thesis, we focus on behavioral settings with two individuals and prove necessity and sufficiency theorems for Nash implementation. This thesis extends the current literature on behavioral implementation by providing results that can be used to incorporate behavioral considerations in environments with two individuals.

We begin by strengthening some conditions in the existing literature and integrating them into the behavioral domain. Our contributions include the revision of two-individual consistency and the introduction of two-individual consistency* and two-individual stronger consistency. We show that the existence of these sets is necessary for (behavioral) Nash implementation of a given social choice correspondence.

To establish the sufficiency part of Nash implementation, we introduce the concepts of singleton intersection property, intersection property*, and spanning collection. Our first sufficiency result indicates that two-individual consistency* combined with the singleton intersection property and intersection property* is sufficient for Nash implementation. On the other hand, our second sufficiency result employs a two-individual stronger consistent collection that is also a spanning collection.

We also provide a necessary and sufficient condition for Nash's implementation of a social choice correspondence by a finite mechanism. This is achieved when a two-individual stronger consistent collection of sets is extended with a pair of finite families of sets to form an extended pair of spanning consistent collections.

In this thesis, we take an algorithmic approach to study Nash implementation with two individuals. However, our results lack a concrete algorithm that constructs a mechanism that implements a given social choice correspondence in Nash equilibrium in finite time. We believe future research can explore our conditions to formulate such an algorithm. Additionally, our results facilitate computational methods that can help the planner design practical mechanisms to (behaviorally) Nash implement her goals in two individual settings. It is our hope that our results will pave the path toward behavioral algorithmic mechanism design to study two individual design problems where these individuals may not act according to classical rationality assumptions.

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