INVESTIGATION OF HOMOGENEOUS AND INHOMOGENEOUS PERCOLATION MODELS IN TWO DIMENSIONS

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

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September, 2007

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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September, 2007

In this thesis, we consider some models of percolation in two dimensional spaces. We study some numerical equalities and inequalities for the critical probability, together with a general method for establishing strict inequalities. Then we compare these results for homogeneous and some inhomogeneous percolation models.

Keywords: Percolation, critical probability, FKG inequality, inhomogeneous percolation.

ÖZET

İKİ BOYUTTA HOMOJEN VE İNHOMOJEN SÜZME MODELLERİNİN İNCELENMESİ

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Matematik, Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Azer Kerimov
Eylül, 2007

Bu tezde, iki boyutlu uzayda süzme modellerini inceliyoruz. Bu modellerde kritik olasılık değeri için bazı eşitlik ve eşitsizlikler elde ediyoruz. Bu sonuçları homojen ve inhomojen süzme modellerinde karşılaştırıyoruz.

Anahtar sözcükler: Süzme modelleri, kritik olasılık, FKG eşitsizliği, inhomojen süzme modelleri.

Acknowledgement

First of all, I would like to thank to my thesis supervisor, Assoc. Prof. Dr. Azer Kerimov, for his invaluable guidance throughout this thesis. My work would not have been possible without his motivation and brilliant ideas. I am also appreciative to my thesis jury members, Assoc. Prof. Dr. Ali Aydın Selçuk and Assoc. Prof. Dr. Ali Sinan Sertöz for their helpful comments about my thesis.

I would like to thank to TUBITAK for financial support during the formation of my thesis

I would like to express my deep gratitude to my parents for their precious support during my personal and academic formation.

Finally, I would like to thank to my brothers Melih and Bülent, my friend Fahrettin Cirit, and my band Rocks'N whom we shared good and bad times for many years.

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Chapter 1

Introduction

1.1 Modelling

Suppose we put a large porous stone in water. What is the probability that the center of the stone is wetted? In two dimensions this situation is modelled as the following: Let $\mathbb{Z}^2$ be the plane square lattice and let p be a number satisfying $0 \leq p \leq 1$. We examine each edge of $\mathbb{Z}^2$, and declare these edges to be *open* with probability p and *closed* otherwise, independently of all other edges. The edges of $\mathbb{Z}^2$ represent the passageways in stone, and the parameter p is the proportion of passages which are broad enough to allow water to pass along them.When we put the stone in water, a vertex x inside the stone is wetted if and only if there exists a path in $\mathbb{Z}^2$ from x to some vertex on the boundary of the stone, using open edges only. One of the main concerns of percolation theory is existence of such 'open paths'.

In a such model, the probability that a vertex near the center of the stone is wetted by water permeating into the stone from its surface will behave similarly to the probability that this vertex is the endvertex of an infinite path of open edges in $\mathbb{Z}^2$.

When can such infinite path of open edges exist? If we were able to observe the whole of the infinite lattice $\mathbb{Z}^2$, we would see that all open clusters are finite when p is small, but there exists an infinite open cluster for large values of p. For general case, we begin with some periodic lattice in d dimensions together with a number p satisfying $0 \leq p \leq 1$, and we declare each edge of the lattice to be open with probability p and closed otherwise. This process is called a 'bond' model because the random blockages in the lattice are associated with the *edges.* Another kind of percolation process is the 'site' percolation model, in which the *vertices* rather than the edges are declared to be open or closed randomly.

With few exceptions, we will restrict ourselves to homogeneous and inhomogeneous bond percolation on the d-dimensional cubic lattice $\mathbb{Z}^d$ where $d \geq 2$

Chapter 2

Fundamentel Concepts

2.1 Bond Percolation

In this part, we give basic definitions and notations of bond percolation on $\mathbb{Z}^d$. The letter d stands for the dimension of the process; generally $d \geq 2$, but we assume for the moment that $d \geq 1$. We write $\mathbb{Z} = \{..., -1, 0, 1, ...\}$ for the set of all integers, and $\mathbb{Z}^d$ for the set all vectors $x = (x_1, x_2, ..., x_d)$ with integral coordinates. For $x \in \mathbb{Z}^d$ we generally write x_i for the ith coordinate of x. The distance $\delta(x, y)$ from x to y is defined by

$$\delta(x, y) = \sum_{i=1}^{d} |x_i - y_i| \tag{2.1}$$

and we write $|x|$ for the distance $\delta(0, x)$ from the origin to x.

We may turn $\mathbb{Z}^d$ into a graph, called the *d-dimensional cubic lattice,* by adding edges between all pairs x, y of points of $\mathbb{Z}^d$ with $\delta(x, y) = 1$. We denote this lattice by $\mathbb{L}^d = \left(\mathbb{Z}^d, \mathbb{E}^d\right)$. We can think of $\mathbb{L}^d$ as a graph embedded in $\mathbb{R}^d$, the edges being straight line segments between their end vertices. If $\delta(x,y)=1$, we say that x an y are *adjacent*; and we write $x\sim y$ and we represent the edge from x to y as $\langle x,y\rangle$. The edge e is *incident* to the vertex x if x is an endvertex of e. We denote the origin of $\mathbb{Z}^d$ by 0.

Next we introduce probability. We begin with a family $\mathbf{p}=\left(p(e):e\in\mathbb{E}^d\right)$ with $0\le p(e)\le 1$ for all e. As sample space we take $\Omega=\prod\limits_{e\in\mathbb{E}^d}\{0,1\}$ .We declare edges of $\mathbb{L}^d$ to be *open* with probabilitiy vector $\mathbf{p}$, independetly of all other edges. The appropriate probabilitiy space is $(\Omega,\mathcal{F},P_p)$ where $\mathcal{F}$ to be the σ-field of subsets of Ω generated by the finite-dimensional cylinders and $P_p=\prod\limits_{e\in\mathbb{E}^d}\mu_e$ and;

$$\mu_e(\omega(e)=0)=1-p(e),\qquad \mu_e(\omega(e)=1)=p(e)$$

for each e the value $\omega(e)=0$ corresponds to e being closed, and $\omega(e)=1$ cosrresponds to e being open. .

We write E_p for the corresponding expectation operator. We write $\overline{A}$ (or occosionally A^c) for the complement of an event A, and I_A for the indicator function of A :

$$I_A(\omega)=\begin{cases}1 & \text{if } \omega\in A\\ 0 & \text{if } \omega\notin A\end{cases}.$$

The expression $E_p(X;A)$ denotes the mean of X on the event A; that is to say, $E_p(X;A)=E_p(XI_A)$.

Let f be an edge of $\mathbb{L}^d$. We write P_p^f for Bernoulli product measure on $\prod\limits_{e:e\neq f}\{0,1\}$, the set of configurations af all edges of the lattice other than f. We think of P_p^f as being the measure associated with percolation on $\mathbb{L}^d$ with the edge f deleted.

There is natural partial order on the set Ω of configurations, given by $\omega_1\le\omega_2$ if and only if $\omega_1(e)\le\omega_2(e)$ for all $e\in\mathbb{E}^d$.

There is a one to one correspondence between Ω and the set of subsets of $\mathbb{E}^d$. For $\omega \in \Omega$, we define

$$K(\omega) = \left\{e \in \mathbb{E}^d : \omega(e) = 1\right\} \tag{2.2}$$

Thus $K(\omega)$ is the set open edges of the lattice when the configuration is ω. Clearly, $\omega_1 \leq \omega_2$ iis and only if $K(\omega_1) \leq K(\omega_2)$.

Suppose that $\left(X(e) : e \in \mathbb{E}^d\right)$ is a family of independent random variables indexed by the edge set $\mathbb{E}^d$, where each $X(e)$ is uniformly distributed on $[0,1]$ as the following . Let all $p(e)$ in $\mathbf{p}$ satisfy $0 \leq p(e) \leq 1$ and define $\eta_p(\in \Omega)$ by

$$\eta_p(e) = \begin{cases} 1 & \text{if } X(e) < p(e) \\ 0 & \text{if } X(e) > p(e) \end{cases} \tag{2.3}$$

We say that the edge e is $\mathbf{p}$-*open* if $\eta_p(e) = 1$.We may think of η_p as being the random outcome the bond percolation process on $\mathbb{L}^d$ with edge-probability $\mathbf{p}$. It is clear that $\eta_{p_1} \leq \eta_{p_2}$ whenever $\mathbf{p}_1 \leq \mathbf{p}_2$.

A path of $\mathbb{L}^d$ as an alternating sequence $x_0, e_0, x_1, e_1, ..., e_{n-1}, x_n$ of distinct vertices x_i and edges $e_i = \langle x_i, x_{i+1}\rangle$; such a path has *length* n and is said to connect x_0 to x_n.

A *circuit* of $\mathbb{L}^d$ is an alternating sequence $x_0, e_0, x_1, e_1, ..., e_{n-1}, x_n, e_n, x_0$ of vertices and edges such that $x_0, e_0, ..., e_{n-1}, x_n$ is a path and $e_n = \langle x_n, x_0\rangle$; such a circuit has lenght $n+1$. We call a path or circuit *open* if all of its edges are open, and *closed* if all of its edges are closed. Two subgraphs of $\mathbb{L}^d$ are called *edge-disjoint* if they have no edges in common, and *disjoint* if they have neither edges nor vertices in common.

Consider the random subgraph of $\mathbb{L}^d$ containing the vertex set $\mathbb{Z}^d$ and the open edges only. The connected components of this graph are called *open clusters.* We write $C(x)$ for the open cluster containing the vertex x, ande we call $C(x)$ the *open cluster* at x. The vertex set of $C(x)$ is the set of all vertices of the lattice which are connected to x by open paths, and the edges of $C(x)$ are the open edges of $\mathbb{L}^d$ which join pairs of such vertices. By the translation invariance of the lattice and of the probability measure P_p, the distribution of $C(x)$ is independent of the choise of x. The open cluster $C(0)$ at the origin is therefore typical of such clusters, and we represent this cluster by the single letter C. Sometimes we shall use the term $C(x)$ to represent the *set of vertices* joined to x by open paths, rather than the graph of this open cluster. We shall be interested the size of $C(x)$, and we denote by $|C(x)|$ the number of vertices in $C(x)$. We note that $C(x) = \{x\}$ whenever x is incident to no open edge.

If A and B are sets of vertices $\mathbb{L}^d$, we shall write $A \Longleftrightarrow B$ if there exists an open path joining some vertex in A to some vertex in B; if $A \cap B \neq \varnothing$ then $A \Longleftrightarrow B$ trivially. We shall write $A \not\Leftrightarrow B$ if there exists no open path from any vertex of A to any vertex of B, and $A \Longleftrightarrow B$ off D if there exists an open path joining some vertex in A to some vertex in B which uses no vertex in the set D.

If A is a set of vertices of the lattice, we write ∂A for the *surface* of A, being the set of vertices in A which are adjecent to some vertex not in A.

A box is a subset of $\mathbb{Z}^d$ of the form;

$B(a,b) = \left\{x \in \mathbb{Z}^d : a_i \leq x_i \leq b_i \text{ for all } i\right\}$, where a and b lie in $\mathbb{Z}^d$; we some-times write

$$B(a,b) = \prod_{i=1}^{n} [a_i, b_i]$$

We denote by $B(n)$ the box with side-length $2n$ and center at the origin;

$$B(n) = [-n,n]^d = \left\{x \in \mathbb{Z}^d : \|x\| \leq n\right\} \tag{2.4}$$

If x is a vertex of the lattice, we write $B(n,x)$ for the box $x + B(n)$ having side-length $2n$ and center at x.

2.2 The Critical Probability

One of the main interests in percolation theory is the *percolation probability* $\theta(p)$, which is the probability that a given vertex belongs to an infinite open cluster. By the translation invariance of the lattice and probability measure, without loss of generality, we can take this vertex to be origin, and thus we define

$$\theta(p) = P_p(|C| = \infty) \tag{2.5}$$

or we may write

$$\theta(p) = 1 - \sum_{n=1}^{\infty} P_p(|C| = n) \tag{2.6}$$

It is clear that $|C| = \infty$ if and only if there exists an infinite sequence $x_0, x_1, ...$ of distinct vertices such that $x_0 = 0$, $x_i \sim x_{i+1}$, and $\langle x_i, x_{i+1} \rangle$ is open for al i. Clearly θ is non-decreasing function of p wilth $\theta(0) = 0$ and $\theta(1) = 1$.

In percolation theory, there exists a critical value $p_c = p_c(d)$ of p such that

$$\theta(p) = \begin{cases} = 0 & \text{if } p < p_c \\ > 0 & \text{if } p > p_c \end{cases}$$

$p_c(d)$ is called the *criticial probability* and is defined by

$$p_c(d) = \sup\{p : \theta(p) = 0\} \tag{2.7}$$

We are not interested in the case of one dimension since, if $p < 1$, there exists infinitely many closed edges of $\mathbb{L}^1$ to the left and right of the origin almost surely, therefore $\theta(p) = 0$ if $p < 1$; thus $p_c(1) = 1$. But the situation is quite different and interesting in two and more dimensions.

The $d-$dimensional lattice $\mathbb{L}^d$ may be embedded in $\mathbb{L}^{d+1}$ in a natural way as the projection of $\mathbb{L}^{d+1}$ onto the subspace generated by the first d coordinates; therefore, the origin of $\mathbb{L}^{d+1}$ belongs to an infinite open cluster for a particular value of p whenever it belongs to an infinite open cluster of the sublattice $\mathbb{L}^d$. Thus $\theta(p) = \theta_d(p)$ is non-decreasing in d, which implies that

$$p_c(d+1) \leq p_c(d) \qquad \text{for } d \geq 1 \tag{2.8}$$

The following theorem states that there exists a non-trivial critical phenomenon in dimension two and more.

Theorem 2.2.1 *([3]) If $d \geq 2$ then $0 < p_c(d) < 1$*

Theorem 2.2.2 *The probability $\psi(p)$ that there exists an infinite open cluster satisfies*

$$\psi(p) = \begin{cases} 0 & \text{if } \theta(p) = 0 \\ 1 & \text{if } \theta(p) > 0 \end{cases}$$

Definition 2.2.1 *$\lambda(d)$ is the connective constant of $\mathbb{L}^d$, given by*

$$\lambda(d) = \lim_{n\to\infty} \left\{\sigma(n)^{1/n}\right\} \tag{2.9}$$

where $\sigma(n)$ is the number of paths (or 'self-avoiding walks) of $\mathbb{L}^d$ having length n and beginning at the origin.

Proof of Theorem 2.2.1 We saw in (2.8) that $p_c(d+1) \leq p_c(d)$, so it is enough to show that $p_c(d) > 0$ for $d \geq 2$, and $p_c(2) < 1$. We prove first that $p_c(d) > 0$ for $d \geq 2$. Consider bond percolation on $\mathbb{L}^d$ when $d \geq 2$. Let $\sigma(n)$ be the number of paths of $\mathbb{L}^d$ which have length n and which begin at the origin, and let $N(n)$ be the number of such paths which are open. Any such path is open with probability p^n, so that

$$E_p(N(n)) = p^n \sigma(n)$$

Now, if the origin belongs to an infinite open cluster then there exist open paths of all lengths beginning at the origin, so that

$$\begin{aligned} \theta(p) &\leq P_p(N(n) \geq 1) \\ &\leq E_p(N(n)) = p^n \sigma(n) \end{aligned} \tag{2.10}$$

for all n.

By the definition of the connective constant $\lambda(d)$ given at (2.9), we have that $\sigma(n) = \{\lambda(d) + \sigma(1)\}^n$ as $n \to \infty$; we subsite this into (2.10) to obtain

$$\begin{aligned} \theta(p) &\leq \{p\lambda(d) + \sigma(1)\}^n \\ &\to 0 \quad \text{if } p\lambda(d) < 1 \end{aligned} \tag{2.11}$$

as $n \to \infty$. Thus we have shown that $p_c(d) \geq \lambda(d)^{-1}$ where $\lambda(d) \leq 2d - 1 < \infty$.

Now, we show that $p_c(2) < 1$. Consider bond percolation on $\mathbb{L}^2$. Let G be a planar graph, drawn in such a way that edges intersect at vertices only. The *planar dual* of G is the graph obtained from G in the following way. We place a vertex in each face of G (including any infinite face which may exist) and join two such vertices by an edge whenever the correspoding face of G share a boundary edge in G. For the sake of definiteness, we take as vertices of this dual lattice the set $\left\{x + \left(\frac{1}{2}, \frac{1}{2}\right) : x \in \mathbb{Z}^2\right\}$ and we join two such neighbouring vertices by a straight line segments of $\mathbb{R}^2$. There is a one to one correspondence between the edges of $\mathbb{L}^2$ and the edges of the dual, since each edge of $\mathbb{L}^2$ is crossed by a unique edge of the dual. We declare en edge of the dual to be open or closed depending respectively on whether it crosses an open or closed edge of $\mathbb{L}^2$. This process creates a bond percolation on the dual lattice with the same edge-probabilty p.

Suppose that the open cluster at the origin of $\mathbb{L}^2$ is finite. We see that the origin is surrounded by closed edges which are blocking of all possible routes from the origin to infinity. We may convince ourselves that the corresponding edges of the dual contain a closed circuit in the dual having the origin of $\mathbb{L}^2$ in its interior. The converse is also true: If the origin lies in the interior of a closed circuit of the dual lattice, then the open cluster at the origin is finite. Therefor $|C| < \infty$ if and only if the origin of $\mathbb{L}^2$ lies in the interior of some closed circuit of the dual.

Let $\rho(n)$ be the number of circuits in the dual which have length n and which contain in their interiors the origin of $\mathbb{L}^2$. We estimate $\rho(n)$ as follows. Each such circuit passes through some vertex of the form $\left(k+\frac{1}{2},\frac{1}{2}\right)$ for same k satisfying $0 \leq k < n$ because; first, it surrounds the origin and therefore passes through $\left(k+\frac{1}{2},\frac{1}{2}\right)$ for some $k \geq 0$ and, secondly, it can not pass through $\left(k+\frac{1}{2},\frac{1}{2}\right)$ where $k \geq n$ since then it would have length at least $2n$. Thus such a circuit contains a self-avoiding walk of length $n-1$ starting from a vertex of the form $\left(k+\frac{1}{2},\frac{1}{2}\right)$ where $0 \leq k < n$. The number of such self-avoiding walk is at most $n\sigma(n-1)$, giving that

$$\rho(n) \leq n\sigma(n-1) \tag{2.12}$$

Let γ be a circuit of the dual containing the origin of $\mathbb{L}^2$ in its interior, and let $M(n)$ be the number of such closed circuit having length n,then

$$\begin{aligned}\sum_{\gamma} P_p(\gamma \text{ is closed }) &\leq \sum_{n=1}^{\infty} q^n n\sigma(n-1) \\ &= \sum_{n=1}^{\infty} q^n \left\{q\lambda(2+\sigma(1))\right\}^{n-1} \\ &< \infty \quad \text{if } q\lambda(2) < 1\end{aligned} \tag{2.13}$$

where $q = 1-p$ and the summation is over all such γ. Furthermore,

$$\sum_{\gamma} P_p(\gamma \text{ is closed }) \to 0 \quad \text{as } q = 1-p \downarrow 0.$$

so that we may find π satisfying $0 < \pi < 1$ such that

$$\sum_{\gamma} P_p\left(\gamma \text{ is closed }\right) \leq \frac{1}{2} \qquad \text{if } \ p > \pi$$

It follows from the previous remarks that

$$\begin{aligned} P_p\left(|C| = \infty\right) &= P_p\left(M\left(n\right) = 0 \ \text{ for all } n\right) \\ &= 1 - P_p\left(M\left(n\right) \geq 1 \ \text{ for some } n\right) \\ &\geq 1 - \sum_{\gamma} P_p\left(\gamma \text{ is closed }\right) \\ &\geq \frac{1}{2} \qquad\qquad \text{if } \ p > \pi \end{aligned}$$

giving that $p_c\left(2\right) \leq \pi$

Now we are going to deduce that $p_c\left(2\right) \leq 1 - \lambda\left(2\right)^{-1}$. Let m be a positive integer. Let F_m be the event that there exists a closed dual circuit containing the box $B\left(m\right)$ in its interior, and let G_m be the event that all edges of $B\left(m\right)$ are open. These two events are independent, since they are defined in terms of disjoint sets of edges. Now, similarly to (2.13)

$$P_p\left(F_m\right) \leq P_p\left(\sum_{n=4m}^{\infty} M\left(n\right) \geq 1\right) \leq \sum_{n=4m}^{\infty} q^n n\sigma\left(n-1\right)$$

Much as before, if $q < \lambda\left(2\right)^{-1}$, we may find m such that $P_p\left(F_m\right) < \frac{1}{2}$, and we choose m accordingly. Assume now that G_m occurs but F_m does not, the non-occurrence of F_m implies that some vertex of $B\left(m\right)$ lies in an infinite open path. Together with the occurence of G_m, this implies that $|C| = \infty$. Therefore, using the independence of F_m and G_m,

$$\theta\left(p\right) \geq P_p\left(\overline{F_m} \cap G_m\right) = P_p\left(\overline{F_m}\right) P_p\left(G_m\right) \geq \frac{1}{2} P_p\left(G_m\right) > 0$$

if $q < \lambda\left(2\right)^{-1}$.

Proof of Theorem 2.2.2 Observe that the event;

$\left\{\mathbb{L}^d \text{ contains an infinite open cluster}\right\}$ does not depend upon the states of any finite collection of edges. By the usual zero-one law ([12]), ψ takes the values 0 and 1 only. If $\theta(p) = 0$ then

$$\psi(p) \leq \sum_{x \in \mathbb{Z}^d} P_p(|C(x)| = \infty) = 0$$

On the other hand, if $\theta(p) > 0$ then

$$\psi(p) \geq P_p(|C| = \infty) > 0$$

so that $\psi(p) = 1$ by the zero-one law.

2.3 Site Percolation

In this case, we set *vertex set* of the lattice $\mathbb{L}^d$ *open* with parobabilty $\mathbf{p}$,different vertices receive independent designations. We shall write θ^{site} or θ^{bond} and similarly p_c^{site} or p_c^{bond} to understand the difference for bond and site percolation. The *covering graph* of a graph G is the graph G_c defined as follows. For each edge of G there corresponds a distinct vertex of G_c, and two such vertices are adjacent if and only if the correspoding edges of G share an endvertex. Suppose we are provied with a bond percolation process on G. We call a vertex of G_c *open* if and only if the corresponding edge of G is open. This induces a site percolation process on G_c. Furthermore, it is clear that every path of open edges in G corresponds to a path of open vertices in G_c

Let us now consider an arbitrary infinite connected graph $G = (V, E)$. Let 0 denote a specified vertex of G which we call the origin. We define $\theta^{bond}(p)$ $\left(\text{respectively } \theta^{site}(p)\right)$ to be the probability that 0 lies in an infinite open cluster of G in a bond percolation (respectively site percolation) process on G having parameter p. Clearly $\theta^{bond}(p)$ and $\theta^{site}(p)$ are non-decreasing functions of p, and the bond and site critical probabilities are given by

$$\begin{aligned} p_c^{bond} &= p_c^{bond}(G) = \sup\left\{p : \theta^{bond}(p) = 0\right\}, \\ p_c^{site} &= p_c^{site}(G) = \sup\left\{p : \theta^{site}(p) = 0\right\} \end{aligned}$$

We have from the above considerations that

$$p_c^{bond}(G) = p_c^{site}(G_c) \tag{2.14}$$

Now it is natural to ask wheter there exists a relationship between the two critical points of given graph G.

Theorem 2.3.1 *([3]) Let $G = (V, E)$ be an infinite connected graph with countably many edges, origin 0, and maximum vertex degree $\Delta\,(< \infty)$. The critical probabilities of G satisfy*

$$\frac{1}{\Delta - 1} \leq p_c^{bond} \leq p_c^{site} \leq 1 - \left(1 - p_c^{bond}\right)^{\Delta} \tag{2.15}$$

Proof. The first inequality of (2.15) follows by counting paths, as in (2.10)-(2.11). Therefore we turn immediately to the remaining two inequalities. In order to obtain these, we shall prove a certain stochastic inequality. Given two random subsets X, Y of V with associated expection operator E, we write $X \leq_{st} Y$ and say that X is *stochastically dominated* by Y, if

$$E\left(f\left(X\right)\right) \leq E\left(f\left(Y\right)\right)$$

for all bounded, measurable functions f satisfiying $f(A) \leq f(B)$ if $A \subseteq B \subseteq V$.

Let $C^{bond}(p)$ be a random subset of V having the law of the cluster of bond percolation at the origin; let $C^{site}(p)$ be a random subset having the law of the cluster of the percolation at the origin *conditional on* 0 being an open vertex. We claim that

$$C^{site}(p) \leq_{site} C^{bond}(p) \tag{2.16}$$

and that

$$C^{bond}(p) \leq_{site} C^{site}(p') \qquad \text{where } p' = 1-(1-p)^{\Delta} \tag{2.17}$$

Since

$$\begin{aligned} \theta^{bond}(p) &= P_p\left(\left|C^{bond}(p)\right| = \infty\right), \\ p^{-1}\theta^{site}(p) &= P_p\left(\left|C^{site}(p)\right| = \infty\right) \end{aligned}$$

the remaining claims of (2.15) will follow from (2.16)-(2.17). Indeed, (2.16)-(2.17) imply that

$$\frac{\theta^{site}(p)}{p} \leq \theta^{bond}(p) \leq \frac{\theta^{site}(p')}{p'} \qquad \text{where } p' = 1-(1-p)^{\Delta} \tag{2.18}$$

Which is slightly stronger that the remainig parts of (2.15).

We construct appropriate couplings of the bond and site models in order to prove (2.16)-(2.17). Let $\omega \in \{0,1\}^E$ be a realization of a bond percolation process on $G = (V, E)$ having density p. We may build the cluster at the origin in the following standart manner. Let $e_1, e_2, ...$ be a fixed ordering of E. At each stage k of the inductive construction, we shall have a pair (A_k, B_k) where $A_k \subseteq V$, $B_k \subseteq E$. Initially we set $A_0 = \{0\}$, $B_0 = \varnothing$. Having found (A_k, B_k) for some k, we define (A_{k+1}, B_{k+1}) as follows. We find earliest edge e in the ordering of E having the following properties: $e \notin B_k$, and e is incident with exactly one vertex of A_k, say the vertex x. We now set

$$A_{k+1} = \begin{cases} A_k & \text{if } e \text{ is closed,} \\ A_k \cup \{y\} & \text{if } e \text{ is open,} \end{cases} \tag{2.19}$$

$$B_{k+1} = \begin{cases} B_k \cup \{e\} & \text{if } e \text{ is closed,} \\ B_k & \text{if } e \text{ is open,} \end{cases} \tag{2.20}$$

where $e = \langle x, y \rangle$. If no such edge e exists, we declare $(A_{k+1}, B_{k+1}) = (A_k, B_k)$. The sets A_k, B_k are non-decreasing, and the open cluster at the origin is given by $A_\infty = \lim_{k\to\infty} A_k$.

We now augment the above construction in the following way. We colour the vertex 0 *red.* Furthermore, to obtain the edge e given above, we colur the vertex y red if e is open, and *black* otherwise. We specify that each vertex is coloured at most once in the construction, in the sense that any vertex y which is obtained at two or more stages is coloured at two more stages is coloured in perpetuity according to the first colour it receives.

Let $A_\infty(\text{red})$ be the set of points connected to the origin by red paths of G

(that is, by paths all of whose vertices are red). We make two claims concerning $A_\infty(\text{red})$:

(i) it is the case that $A_\infty(\text{red}) \subseteq A_\infty$, and all neighbours of vertices in $A_\infty(\text{red})$ which do not lie $A_\infty(\text{red})$ are black;

(ii) $A_\infty(\text{red})$ has the same distribution as $C^{site}(p)$;

Claim (i) is straightforward. In order to be coloured red, a vertex is necessarily connected to the origin by a path of open edges. Furthermore, since all edges with exactly one endvertex in A_∞ are closed, all neighbours of $A_\infty(\text{red})$ which are not themselves coloured red are necessarily black.

We sketch an explanation of claim (ii). Whenever a vertex is coloured either red or black, it is coloured red with probabilty p independently of all earlier colourings.

The derivation of (2.17) is smilar. We start with a directed version of G, namely the directed graph $\overline{G} = (V, E)$ obtained from G by replacing each edge $e = \langle x, y \rangle$ by two directed edges, one in each direction, and denoted respectively by $\{x, y\}$ and $[y, x\rangle$ .We now let $\overline{\omega} \in \{0,1\}^{\overline{E}}$ be a realization of an (oriented) bond percolation process on $\overline{G}$ having density p.

We colour the origin *green.* We colour a vertex $x\,(\neq 0)$ *green* if at least one edge f of the form $[y, x\rangle$ satisfies $\overline{\omega}\,(f) = 1$; otherwise we colour x *black.* Then

$$P_p\,(x \text{ is green}) = 1 - (1-p)^{\rho(x)} \leq 1 - (1-p)^{\Delta} \tag{2.21}$$

where $\rho\,(x)$ is the degree of x, and $\Delta = \max_x \rho\,(x)$.

We now build a copy A_∞ of $C^{bond}\,(p)$ more or less as described in (2.19)-(2.20). The only difference is that, on considering the edge $e = \langle x, y \rangle$ where $x \in A_k$, $y \notin A_k$, we declare e to be open for the purpose of (2.19)-(2.20) if and only if $\overline{\omega}\,([x, y\rangle) = 1$. Finally, we let A_∞ (green) be the set of points connected to the origin by green paths. It may be seen that $A_\infty\,(\text{green}) \supseteq A_\infty$. Furthermore, by (2.21), $A_\infty\,(\text{green})$ is stochastically dominated $C^{site}\,(p')$ where $p' = 1 - (1-p)^{\Delta}$ .Inequality (2.17) follows. ■

Chapter 3

Correlation Inequalities

3.1 Increasing Events

The event A in $\mathcal{F}$ is called *increasing if* $I_A(\omega) \leq I_A(\omega')$ whenever $\omega \leq \omega'$, where I_A is the indicator function of A. We call A *decreasing* if complement $\overline{A}$ is increasing.

More generally, a random variable N on the measurable pair $(\Omega, \mathcal{F})$ is called *increasing* if $N(\omega) \leq N(\omega')$ whenever $\omega \leq \omega'$; N is called *decreasing* if-N is increasing.

As simple (and canonical) examples of increasing events and random variables, consider the event $A(x, y)$ that there exists an open path joining x to y, and the number $N(x, y)$ of diferent open paths from x to y.

Theorem 3.1.1 *If N is an increasing random variable on $(\Omega, \mathcal{F})$, then*

$$E_{p_1}(N) \leq E_{p_2}(N) \quad \textit{whenever } p_1 \leq p_2 \tag{3.1}$$

so long as these mean values exist. If A is an increasing event in $\mathcal{F}$, then

$$P_{p_1}(A) \leq P_{p_2}(A) \quad \textit{whenever } p_1 \leq p_2 \tag{3.2}$$

3.2 The FKG Inequality

Theorem 3.2.1 ***FKG inequality***

(a) *If X and Y are increasing random variables such that $E_p(X^2) < \infty$ and $E_p(Y^2) < \infty$, then*

$$E_p(XY) \geq E_p(X)\, E_p(Y) \tag{3.3}$$

(b) *If A and B are increasing events then*

$$P_p(A \cap B) \geq P_p(A)\, P_p(B) \tag{3.4}$$

Similar inequalities are valid for decreasing random variables and events. For example, if X and Y are both decreasing then–X and –Y are increasing, giving that

$$E_p(XY) \leq E_p(X)\, E_p(Y)$$

Example 1 *Let $G = (V, E)$ be an infinite connected graph with countably many edges, and consider a bond percolation process on G. For any vertex x, we write $\theta(p, x)$ for the probabilty that x lies in an infinite open cluster, and*

$$p_c(x) = \sup\{p : \theta(p, x) = 0\}$$

for the associated critical probabilty. We have by the FKG inequality that

$$\theta(p, x) \geq P_p(\{x \Longleftrightarrow y\} \cap \{y \Longleftrightarrow \infty\}) \geq P_p(x \Longleftrightarrow y)\, \theta(p, y)$$

whence $p_c(x) \leq p_c(y)$. The latter inequality holds also with x and y interchanged. A similar argument is valid for site percolation, and we arrive at the following theorem.

Theorem 3.2.2 *Let g be a connected graph with countably many edges. The values of the bond critical probability $p_c^{bond}(x)$ and the site critical probabilty $p_c^{site}(x)$ are independent of the choice of initial vertex x.*

Proof of FKG inequality Enough to prove part (a), since part (b) follows if we apply the first part to the indicator functions of A and B. Firs we prove part (a) for random variables X and Y which are defined in terms of the states of only finitely many edges; later we shall remove this restriction.

Suppose that X and Y are increasing random variables which depend only on the states of the edges $e_1, e_2, ..., e_n,$. We proceed by induction on n. First, suppose $n = 1$. Then X and Y are functions of the state $\omega(e_1)$ of e_1, which takes the values 0 and 1 with probabilities $1 - p$ and p, respectively. Now,

$$\{X(\omega_1) - X(\omega_2)\}\{Y(\omega_1) - Y(\omega_2)\} \geq 0$$

for all pairs ω_1, ω_2 each taking the value 0 or 1; this is trivial if $\omega_1 = \omega_2$, and follows from the monotonicity of X and Y otherwise. Thus

$$\begin{aligned} 0 &\leq \sum_{\omega_1,\omega_2} \{X(\omega_1) - X(\omega_2)\}\{Y(\omega_1) - Y(\omega_2)\} \\ &\quad \times P_p(\omega(e_1) = \omega_1) P_p(\omega(e_1) = \omega_2) \\ &= 2\{E_p(XY) - E_p(X) E_p(Y)\} \end{aligned}$$

as required. Suppose now that the result is valid for values of n satisfying $n < k$, and that X and Y are increasing functions of the states

$\omega(e_1), \omega(e_2), ..., \omega(e_k)$ of the edges $e_1, e_2, ..., e_k$. Then

$$\begin{aligned} E_p(XY) &= E_p\left(E_p\left(XY \mid \omega(e_1), \omega(e_2), ..., \omega(e_{k-1})\right)\right) \\ &\geq E_p\begin{pmatrix} \left(E_p\left(X \mid \omega(e_1), \omega(e_2), ..., \omega(e_{k-1})\right)\right) \times \\ \left(E_p\left(Y \mid \omega(e_1), \omega(e_2), ..., \omega(e_{k-1})\right)\right) \end{pmatrix} \end{aligned}$$

since, for given $\omega(e_1), \omega(e_2), ..., \omega(e_{k-1})$, its the case that X and Y are increasing in the single variable $\omega(e_k)$. Now,

$E_p(X \mid \omega(e_1), \omega(e_2), ..., \omega(e_{k-1}))$ is an increasing function of the states of $k-1$ edges, as is the corresponding function of Y.

It follows from the induction hypothesis that the last mean value above is no smaller than the product of the means, whence

$$\begin{aligned} E_p(XY) &\geq E_p\left(E_p\left(X \mid \omega(e_1), \omega(e_2), ..., \omega(e_{k-1})\right)\right) \times \\ &\quad E_p\left(E_p\left(Y \mid \omega(e_1), \omega(e_2), ..., \omega(e_{k-1})\right)\right) \\ &= E_p(X) E_p(Y) \end{aligned}$$

We now lift the condition that X and Y depend on the states of only finitely many edges. Suppose that X and Y are increasing random variables with finite second moments. Let $e_1, e_2, ...$ be a (fixed) ordering of the edges of $\mathbb{L}^d$, and define

$$\begin{aligned} X_n &= E_p\left(X \mid \omega(e_1), \omega(e_2), ..., \omega(e_n)\right), \\ Y_n &= E_p\left(Y \mid \omega(e_1), \omega(e_2), ..., \omega(e_n)\right) \end{aligned}$$

Now X_n and Y_n are increasing functions of the states of $e_1, e_2, ..., e_n$, and therefore,

$$E_p(X_n Y_n) \geq E_p(X_n) E_p(Y_n) \tag{3.5}$$

by the discussion above. As a consequence of the martingale convergence theorem we have that, as $n \to \infty$,

$$X_n \to X \text{ and } Y_n \to Y \quad P_p\text{-as and in } L^2(P_p)$$

whence

$$E_p(X_n) \to E_p(X) \quad \text{and} \quad E_p(Y_n) \to E_p(Y) \quad \text{as } n \to \infty \tag{3.6}$$

Also, by the triangle and Cauchy-Schwarz inequalities,

$$\begin{aligned} E_p|X_nY_n - XY| &\leq E_p(|(X_n - X)Y_n| + |X(Y_n - Y)|) \\ &\leq \sqrt{E_p\left((X_n - X)^2\right)E_p(Y_n^2)} + \\ &\quad \sqrt{E_p(X_n^2)E_p\left((Y_n - Y)^2\right)} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

so that $E_p(X_nY_n) \to E_p(XY)$. We take the limit as $n \to \infty$ in (3.5) to obtain the result.

3.3 The BK Inequality

Let $e_1, e_2, ..., e_n$ be n distinct edges of $\mathbb{L}^2$, and A and B be increasing events which depend on the vector $\omega = (\omega(e_1), \omega(e_2), ..., \omega(e_n))$ of the states of these edges only. Each such ω is specified uniquely by the set $K(\omega) = \{e_i : \omega(e_i) = 1\}$ of edges with state 1. We define the event $A \circ B$ to be the set of all ω for which there exists a subset H of $K(\omega)$ such that ω', determined by $K(\omega') = H$, belongs to A, and ω'', determined by $K(\omega'') = K(\omega) \backslash H$, belongs to B. We say that $A \circ B$ is the event that A and B 'occur disjointly'. Thus $A \circ B$ is the set of configurations ω for which there exists disjoints sets of open edges with the property that the first such that guarentees the occurrence of A and the second guarentees the occurrence of B.

Theorem 3.3.1 ***BK İnequality.*** *([6]) If A and B are increasing events $\mathcal{G}$, then*

$$P(A \circ B) \leq P(A)P(B) \tag{3.7}$$

Theorem 3.3.2 *Consider bond percolation on $\mathbb{L}^d$ and A and B be increasing events defined in terms of the states of only finitely many edges. Then*

$$P_p (A \circ B) \leq P_p (A) P_p (B) \tag{3.8}$$

3.4 Rate of Change of $\mathbf{P_p(A)}$

Now we will try to estimate the rate of change of $P_p(A)$ as a function of p. First, in comparing $P_p (A)$ and $P_{p+\delta} (A)$, it is useful to construct the two processes having densites p and $p + \delta$ one the same probability space in the usual way. Let $\left\{X (e) : e \in \mathbb{E}^d\right\}$ be independent random variables having the uniform distribution on $[0, 1]$, and define $\eta_p (e) = 1$ if $X (e) < p$ and $\eta_p (e) = 0$ otherwise. For increasing events A,

$$P_{p+\delta} (A) - P_p (A) = P \left(\eta_p \notin A,\ \eta_{p+\delta} \in A\right) \tag{3.9}$$

If $\eta_p \notin A$ but $\eta_{p+\delta} \in A$, there exists edges e satisfying $\eta_p (e) = 0$, $\eta_{p+\delta} (e) = 1$, which is to say that $p \leq X (e) < p + \delta$. Let $E_{p,\delta}$ be the set of such edges, and assume that A depends on the states of only finitely many edges. It is clear that $P (|E_{p,\delta}| \geq 2) = o (\delta) \downarrow 0$,and so we shall neglect the posibility that there exists two or more edges in $E_{p,\delta}$. If e is the unique edge satisfying $p \leq X (e) < p + \delta$, then e must be 'essential' for A in the sense that $\eta_p \notin A$ but $\eta'_p \in A$ where η'_p is obtained from η_p by changing of e from 0 to 1. The 'essentalness' of e does not depend on the state of e, so that each edge e contributes roughly an amount.

$$P (p \leq X (e) < p + \delta) P_p (e \text{ is 'essential' for } A) = \delta P_p (e \text{ is 'essential' for } A) \tag{3.10}$$

towards the quantity in (3.9). We divide by δ and take the limit as $\delta \downarrow 0$ to obtain without rigour the formula

$$\frac{d}{dp} P_p (A) = \sum_e P_p (e \text{ is 'essential' for } A)$$

We now derive a formula, valid for all increasing events A which depend on only finitely many edges.

Definition 3.4.1 *Let A be an event, not necessarily increasing, and let ω be a configuration. We call the edge e pivotal for the pair (A, ω) if $I_A(\omega) \neq I_A(\omega')$, where ω' is the configuration which agress with ω on all edges except e ,and $\omega'(e) = 1 - \omega(e)$. Thus e is pivotal for (A, ω) if the occurrence or non-occurrence of A depends crucilly on the state of e. The event that 'e is pivotal for A' is the set of configurations ω for which e pivotal for (A, ω). We note that this event depends only on the states of edges other than e; it is independent of the state of e itself. We shall be interested particularly in increasing events A; for such an event A; an edge e is pivotal if and only if A does not occur when e is closed but A does occur when e is open.*

Theorem 3.4.1 ***Russo's formula*** *([10]) Let A be an increasing event defined in terms of the states of only finitely many edges of $\mathbb{L}^d$. Then*

$$\frac{d}{dp} P_p(A) = E_p(N(A)) \tag{3.11}$$

where $N(A)$ is the number of edges which are pivotal for A.

Formula (3.11) may be writen as

$$\frac{d}{dp} P_p(A) = \sum_{e \in \mathbb{E}^d} P_p(e \text{ is pivotal for } A) \tag{3.12}$$

Such formula are not generally valid for event which depend on more than finitely many edges; indeed $P_p(A)$ need not be differentiable for all values of p. For general increasing events A, the best that the method allows is a lower bound on the right-hand derivative of $P_p(A)$:

$$\liminf_{\delta \downarrow 0} \left(P_{p+\delta}(A) - P_p(A)\right) \geq E_p(N(A)) \tag{3.13}$$

Proof of Russo's formula and Equation 3.13 Suppose A to be an increasing event and let $\mathbf{p} = \left(p(e) : e \in \mathbb{E}^d\right)$ be a collection of numbers satisfying $0 \leq p(e) \leq 1$ for all e. Let $\left(X(e) : e \in \mathbb{E}^d\right)$ be independent random variables each having the uniform distribution on $[0,1]$. We construct the configuration η_p on $\mathbb{E}^d$ by defining $\eta_p(e) = 1$ if $X(e) < p(e)$ and $\eta_{\mathbf{p}}(e) = 0$ otherwise. Writing $P_{\mathbf{p}}$ for the probability measure on Ω in which the state $\omega(e)$ of the edge e equals 1 with probability $p(e)$, we have that

$$P_{\mathbf{p}}(A) = P\left(\eta_{\mathbf{p}} \in A\right)$$

as usual. Now, choose an edge f and define $\mathbf{p}' = \left(p'(e) : e \in \mathbb{E}^d\right)$ by

$$p'(e) = \begin{cases} p(e) & \text{if } e \neq f \\ p'(e) & \text{if } e = f \end{cases}$$

$\mathbf{p}$ and $\mathbf{p}'$ may differ only at the edge f. Now, ig $p(f) \leq p'(f)$ then

$$\begin{aligned} P_{\mathbf{p}'}(A) - P_{\mathbf{p}}(A) &= P\left(\eta_{\mathbf{p}} \notin A,\ \eta_{\mathbf{p}'}(A)\right) \\ &= \{p'(f) - p(f)\} P_{\mathbf{p}}(f \text{ is pivotal for } A) \end{aligned}$$

by the discussion leading to (3.10). We divide by $p'(f) - p(f)$ and take the limit as $p'(f) - p(f) \to 0$ to obtain

$$\frac{\partial}{\partial p(f)} P_p(A) = P_{\mathbf{p}}(f \text{ is pivotal for } A)$$

So far we have assumed nothing about A save that it be increasing. If A depends on finitely many edges only, then $P_{\mathbf{p}}(A)$ is a function of a finite collection $(p(f_i) : 1 \leq i \leq m)$ of edge-probabilities. and the chain rule gives that

$$\begin{aligned} \frac{d}{dp} P_p(A) &= \sum_{i=1}^{m} \frac{\partial}{\partial p(f_i)} P_p(A) \mid_{p=(p,p,\ldots,p)} \\ &= \sum_{i=1}^{m} P_p(f_i \text{ is pivotal for } A) \\ &= E_p(N(A)) \end{aligned}$$

as required. If, on other hand, A depends on infinitely many edges, we let E be a finite collection of edges and define

$$\mathbf{p}_E(e) = \begin{cases} p & \text{if } e \notin E \\ p+\delta & \text{if } e \in E \end{cases}$$

where $0 \leq p \leq p+\delta \leq 1$. Now, A is increasing, so that

$$P_{p+\delta}(A) \geq P_{\mathbf{p}_E}(A)$$

and hence

$$\begin{aligned} \frac{1}{\delta}\left(P_{p+\delta}(A) - P_p(A)\right) &\geq \frac{1}{\delta}\left(P_{\mathbf{p}_E}(A) - P_p(A)\right) \\ &\to \sum_{e\in\mathbb{E}} P_p(e \text{ is pivotal for } A) \end{aligned}$$

as $\delta \downarrow 0$. we let $E \uparrow \mathbb{E}^d$ to obtain (3.13).

Remark 1 *([3],[10]) The Russo's theorem may be extended in various ways. For $\omega \in \Omega$ and $A, B \subseteq \mathbb{E}^d$ with $A \cap B = \varnothing$, let ω_B^A be the configuration given by*

$$\omega_B^A(e) = \begin{cases} \omega(e) & \textit{if } e \notin A \cup B \\ 1 & \textit{if } e \in A \\ 0 & \textit{if } e \in B \end{cases}$$

For simplicity, we abbreviate singleton sets $A = \{f\}$ by f, and pairs $A = \{e, f\}$ by ef. For a random variable X, we define the increment ot X at e by

$$\delta_e X(\omega) = X(\omega^e) - X(\omega_e)$$

Theorem 3.4.2 *let X be a random varable which is defined in terms of the states only finitely many edges of $\mathbb{L}^d$. Then*

$$\frac{d}{dp}E_p[A] = \sum_{e\in\mathbb{E}} E_p(\delta_e X)$$

Turning to second derivative, it follows from the theorem that

$$\frac{d^2}{dp^2}E_p[A] = \sum_{e,f\in\mathbb{E}} E_p(\delta_e\delta_f X)$$

3.5 Other Inequalities

Let G be a finite graph, and declare each edge of G to be open with probabilty p, independently of all other edges. Let x and y specified vertices of G and let $h(p)$ be the probability that there exists an open path of G from x to y. In the study of h as a function of p, we present some result here, in form suitable for application to percolation.

Let E be a finite set of edges of $\mathbb{L}^d$. We shall confine ourselves to events which depend only on the edges in E. We write $cov_p(X,Y)$ for the covariance of two random variables X and Y under the measure P_p.

Theorem 3.5.1 *([3]) Let A be an event which depends only on the edges in E, and let N be the (random) number of edges of E which are open. Then*

$$\frac{d}{dp}P_p(A) = \frac{1}{p(1-p)}cov_p(N, I_A) \quad \textit{for } 0 < p < 1 \tag{3.14}$$

The following is an immediate corollary.

Theorem 3.5.2 *([13]) Let A be an event which depends only on the edges in E, and suppose that $0 < p < 1$. Then*

(a)

$$\frac{d}{dp}P_p(A) \leq \sqrt{\frac{mP_p(A)(1-P_p(A))}{p(1-p)}} \quad \textit{where } m = |E|,$$

(b) *if A is increasing, we have that*

$$\frac{d}{dp}P_p(A) \geq \frac{P_p(A)(1-P_p(A))}{p(1-p)} \tag{3.15}$$

Theorem 3.5.3 *([13]) Let A be an increasing event which depends on only finitely many edges of $\mathbb{L}^d$, and suppose that $0<p<1$. Then $\log P_p(A)/\log p$ is a non-increasing function of p.*

This theorem amounts to saying that $h(p) = P_p(A)$ satisfies.

$$h(p^\gamma) \leq h(p)^\gamma \quad \text{if } 0<p<1 \text{ and } \gamma \geq 1 \tag{3.16}$$

whenever A is increasing. Rewriting the conclusion of the theorem in terms of the derivative of $\log P_p/\log p$, we obtain

$$\frac{d}{dp}P_p(A) \geq \frac{P_p(A)\log P_p(A)}{p\log p} \tag{3.17}$$

which is an improvement over inequality (3.15) whenever $P_p(A) < p$.

Let r be a positive integer. For any configuration ω of edge-states, we define the *sphere* with radius r and centre at ω by

$$S_r(\omega) = \left\{\omega' \in \Omega : \sum_{e\in\mathbb{E}^d} |\omega'(e)-\omega(e)| \leq r\right\};$$

$S_r(\omega)$ is the collection of configurations which differ from ω on at most r edges. For any event A, we define the *interior* of A with depth r by

$$I_r(A) = \{\omega \in \Omega : S_r(\omega) \subseteq A\};$$

thus $I_r(A)$ is the set of configurations in A which are still A even if we perturb the states of up to r edges.

Theorem 3.5.4 *([3]) Let A be an increasing event and let r be a positive integer. Then*

$$1 - P_{p_2}\left(I_r\left(A\right)\right) \leq \left(\frac{p_2}{p_2 - p_1}\right)^r \left\{1 - P_{p_1}\left(A\right)\right\} \tag{3.18}$$

whenever $0 \leq p_1 < p_2 \leq 1$.

3.6 Fınıte Inhomogeneous Case

Now, a question may arise in the situation that what happens in a finite inhomogeneous case, i.e. if we take a finite subset $\mathcal{F}$ in $\mathbb{Z}^2$ and change "the probability of being open" of the bonds in this set ?

Observe that a percolation from a fixed vertex occurs with positive probability, if and only if for any finite $\mathcal{F}$ of vertices, the probability of an open connection from $\mathcal{F}$ to infinity, which stays outside $\mathcal{F}$ (except for its initial point) is positive ([6]). This in turn follows from the deterministic fact that if a fixed point is connected to infinity by an open path, then any finite set F is connected to infinity outside F (with the exception of the initial point).

Hence the critical probability and the results depending on thid probability remain the same. In chapter 5, we will consider inhomogeneous bond percolation with some special cases in more detail.

Chapter 4

Critical Probabilities

4.1 Equalities and Inequalities in Percolation

For a given graph G , it is natural to look for an exact calculation of $p_c(G)$, but there seems no reason to expect a closed form for $p_c(G)$ unless G has special structure. Indeed, the value of $p_c(G)$ have no special numerical features in general. Some of the exceptional cases are:

$$\begin{array}{rccl} \text{square lattice} & p_c & = & \dfrac{1}{2} \\ \text{triangular lattice} & p_c & = & 2\sin(\pi/18) \\ \text{hexagonal lattice} & p_c & = & 1-2\sin(\pi/18) \end{array}$$

Given a planar lattice $\mathcal{L}$ and its lattice $\mathcal{L}_d$,

$$p_c(\mathcal{L}) + p_c(\mathcal{L}_d) = 1 \tag{4.1}$$

subject to certain conditions of symmetry on $\mathcal{L}$.

Equation (4.1) is equivalent to the following statement:

$$p < p_c(\mathcal{L}) \quad \text{if and only if} \quad 1-p < p_c(\mathcal{L}_d)$$

which tells the following: If $p > p_c(\mathcal{L})$, there exists (almost surely) an infinite open cluster of $\mathcal{L}$, and infinite cluster occupy a strictly positive density of space. If there is an unique such infinite cluster then this cluster extends throughout space, and precludes the existence of an infinite closed cluster of $\mathcal{L}_d$; therefore $1-p < p_c(\mathcal{L})$. Conversely, if $p < p_c(\mathcal{L})$, all open cluster of $\mathcal{L}$ are (almost surely) finite, and the intervening space should contain an infinite closed cluster of $\mathcal{L}_d$; therefore, $1-p > p_c(\mathcal{L})$.

We can construct a matching pair $\mathcal{G}_1$, $\mathcal{G}_2$ of graphs in two dimensions as in the following. We begin with an infinite planar graph G with $'$origin$'$ 0, and we select some arbitrary family $\mathcal{F}$ of face of G. We obtain $\mathcal{G}_1$ (respectively $\mathcal{G}_2$) from G by adding all diagonals to all faces in $\mathcal{F}$ (respectively all faces not in $\mathcal{F}$). The graphs G, $\mathcal{G}_1$,$\mathcal{G}_2$ have the same vertex sets, so, a site percolation process on G inducess site percolation procesess on $\mathcal{G}_1$ and $\mathcal{G}_2$. If the origin 0 belongs to a finite open cluster of $\mathcal{G}_1$, then the external (vertex) boundary of this cluster forms a closed circuit of $\mathcal{G}_2$.We say that $\mathcal{G}_1$ is *self-matching* if $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic graphs. Note that, if G is a triangulation (every face of G is a triangle), then $G = \mathcal{G}_1$=$\mathcal{G}_2$, and in this case G is self-matching. The triangular lattice $\mathbb{T}$ is an example of a self-matching lattice.

Let $\mathcal{G}_1$,$\mathcal{G}_2$ be a matching pair of lattices in two dimensions. Subject to assumptions on the pair $\mathcal{G}_1$,$\mathcal{G}_2$, one may on occasion be able to justify the relation

$$p_c^{site}(\mathcal{G}_1) + p_c^{site}(\mathcal{G}_2) = 1;$$

One may deduce that the triangular lattice $\mathbb{T}$, being self-matching, has site critical probability $p_c^{site}(\mathbb{T}) = \frac{1}{2}$, Indeed, it is belived that $p_c^{site} = \frac{1}{2}$ for a broad family of $'$reasonable$'$ triangulations of the plane.

In the absence of a general method for computing critical percolation probabilities, we may have cause to seek inequalities.

4.2 Strict Inequalities

It is clear that If $\mathcal{L}$ is a sublattice of the lattice $\mathcal{L}'$ (written $\mathcal{L} \subseteq \mathcal{L}'$) then their critical probabilities satisfy $p_c(\mathcal{L}) \geq p_c(\mathcal{L}')$, since any infinite open cluster of $\mathcal{L}$ is contained in some infinite open cluster of $\mathcal{L}'$. But when does the *strict* inequality $p_c(\mathcal{L}) > p_c(\mathcal{L}')$ hold?

Example 2 *The triangular lattice $\mathbb{T}$ may be obtained by adding diagonals across the squares of the square lattice $\mathbb{L}^2$, Since any infinite open cluster of $\mathbb{L}^2$ is contained in an infinite open cluster of $\mathbb{T}$, it follows that $p_c(\mathbb{T}) \leq p_c(\mathbb{L}^2)$, but does strict inequality hold?*

First we embed the problem in a two-parameter system. Let $p, s \in [0,1]^2$. We declare each edge of $\mathbb{L}^2$ to be open with-probability p, and each further edge of $\mathbb{T}$ to be open with probability s. Writing $P_{p,s}$ for the associated measure, define

$$\theta(p,s) = P_{p,s}(0 \Longleftrightarrow \infty) \tag{4.2}$$

We propose to establish certain differential inequalities which will imply that $\partial\theta/\partial p$ are comparable, uniformly on any closed subset of the interior $(0,1)^2$ of the parameter space. This cannot itself be literally achieved, since we have insufficient information about the differetiability of θ. Therefore, we shall approximate θ by a finite-volume quantity θ_n, and shall work with the partial derivatives of θ_n.

Let $B(n) = [-n,n]^d$, and define

$$\theta_n(p,s) = P_{p,s}(0 \Longleftrightarrow \partial B(n)) \tag{4.3}$$

Note that θ_n is a polynomial in p and s, and that

$$\theta_n(p,s) \downarrow \theta(p,s) \quad \textit{as } n \to \infty$$

Lemma 1 *There exists a positive integer L and a continuous function α mapping $(0,1)^2$ to $(0,\infty)$ such that*

$$\alpha(p,s)^{-1} \frac{\partial}{\partial p}\theta_n(p,s) \geq \frac{\partial}{\partial s}\theta_n(p,s) \geq \alpha(p,s) \frac{\partial}{\partial p}\theta_n(p,s) \tag{4.4}$$

for $0 < p,\ s < 1$ and $n \geq L$.

Theorem 4.2.1 *([13]) $p_c(\mathbb{T}) < p_c(\mathbb{L}^2)$*

Proof We can show that there exists a $'$critical curve$'$ in $(p,s)-$space, separating the regime where $\theta(p,s) = 0$ from that when $\theta(p,s) > 0$. Suppose that this critical curve may be written in the form $h(p,s) = 0$ for some increasing and continouosly differentiable function h satisfying $h(p,s) = \theta(p,s)$ whenever $\theta(p,s) > 0$. It segment, and we shall prove this by working with the gradient vector

$$\overline{v}h = \left(\frac{\partial h}{\partial p}, \frac{\partial h}{\partial s}\right),$$

We take some liberties with (4.4) in the limit as $n \to \infty$, and deduce that

$$\overline{v}h.(0,1) = \frac{\partial h}{\partial s} \geq \alpha(p,s)\frac{\partial h}{\partial p}$$

whence

$$\frac{1}{|\overline{v}h|}\frac{\partial h}{\partial s} = \left\{\left(\frac{\partial h}{\partial p} : \frac{\partial h}{\partial s}\right)^2 + 1\right\}^{-\frac{1}{2}} \geq \frac{\alpha}{\sqrt{\alpha^2+1}}$$

which is bounded away from 0 on any closed subset of $(0,1)^2$. This indicates required that the critical curve has no vertical segment.

Let η be positive and small, and find $\gamma\,(>0)$ such that $\alpha\,(p,s) \geq \gamma$ on $[\eta, 1-\eta^2]$. Let $\psi \in [0, \pi/2]$ satisfy $\tan\psi = \gamma^{-1}$

At the point $(p,s) \in [\eta, 1-\eta^2]$, the rate of change of $\theta_n\,(p,s)$ in the direction

$(\cos\psi, -\sin\psi)$ satisfies

$$\begin{aligned} \overline{v}\theta_n\,(\cos\psi, -\sin\psi) &= \frac{\partial\theta_n}{\partial p}\cos\psi - \frac{\partial\theta_n}{\partial s}\sin\psi \\ &\leq \frac{\partial\theta_n}{\partial p}\,(\cos\psi, -\sin\psi) = 0 \end{aligned} \tag{4.5}$$

by (4.4), since $\tan\psi = \gamma^{-1}$.

Suppose now that $(a,b) \in [2\eta, 1-2\eta^2]$. Let

$$(a', b') = (a,b) + \eta\,(\cos\psi, -\sin\psi)$$

noting that $(a', b') \in [\eta, 1-\eta^2]$. By integrating (4.5) along the line segment joining (a,b) to (a',b'), we obtain that

$$\theta\,(a',b') = \lim_{n\to\infty}\theta_n\,(a',b') \leq \lim_{n\to\infty}\theta\,(a',b') = \theta\,(a,b) \tag{4.6}$$

Let η be small and positive. Take $(a,b) = (p_c\,(\mathbb{T}) - \zeta, p_c\,(\mathbb{T}) - \zeta)$ and define (a',b') as above. We choose ζ sufficiently small that $(a,b)\,(a',b') \in [2\eta, 1-2\eta^2]$, and that $a' > p_c\,(\mathbb{T})$. The above calculation implies that

$$\theta\,(a',0) \leq \theta\,(a',b') \leq \theta\,(a,b) = 0 \tag{4.7}$$

whence $p_c\,(\mathbb{L}^2) \geq a' > p_c\,(\mathbb{T})$

4.3 Enhancements and Related Results

In a simplest way we can define an $'$enhancement$'$ as a systematic addition of connections according to local rules. We wonder if an enhancement can create an infinite cluster when previously was none?

The answer can be negative. For example: join any two neighbours of $\mathbb{L}^d$ with probability $\frac{1}{2}p_c$ whenever they have no incident open edges. Such an enhancement creates extra connections but creates (almost surely) no extra infinite cluster.

Here is a proper definition of the concept of enhancement for bond percolation on $\mathbb{L}^d$ with parameter p. Let R be a positive integer, and let $\mathcal{G}$ be the set of all simple graphs on the vertex set $B = B(R)$. Note that the set of open edges of any configuration $\omega\,(\in \Omega)$ generates a number of $\mathcal{G}$ denoted ω_B; $\mathcal{G}$ contains in addition many graphs not obtainable in this way. Let F be a function which associates with every ω_B a graph in $\mathcal{G}$. We call R the $'$enhancement range$'$ and F the $'$enhancement function$'$. In the remainder of this part, we denote by $e + x$ the translate of an edge e by the vector x; similarly, $G + x$ denotes the translate by x of the graph G on the vertex set $\mathbb{Z}^d$.

We may consider making an enhancement at each vertex x of $\mathbb{L}^d$, and we will do this in a stochastic way. To this end, we provide ourselves with a vector $\eta = \left(\eta(x) : x \in \mathbb{Z}^d\right)$ Iying in the space $\Xi = \{0,1\}^{\mathbb{Z}^d.}$ We shall interpret the value $\eta(x) = 1$ as meaning that the enhancement at the vertex x is $'$activated$'$.

For each $x \in \mathbb{Z}^d$, we observe the configuration ω on the box $x + B$, and we write $F(x,\omega)$ for the associated evaluation of F. So, we set $F(x,\omega) = F\left((\tau_x\omega)_B\right)$ where τ_x is the shift operator to be the graph

$$G^{enh}(\omega,\eta) = G(\omega) \cup \left\{ \bigcup_{x:\eta(x)=1} \{x + F(x,\omega)\} \right\} \tag{4.8}$$

where $G(\omega)$ is the graph of open edges under ω. In writing the union of graphs, we mean the graph with vertex set $\mathbb{Z}^d$ having the union of teh appropriate edge sets; whereever this union contains two or mode edges between the same pair of vertices, these edges are allowed to coalesce into a single edge.

Thus we associate with pair $(\omega, \eta) \in \Omega \times \Xi$ an enhanced graph $G^{enh}(\omega, \eta)$ We endow the sample space $\Omega \times \Xi$ with the product probability measure $P_{p,s}$, and we refer to the parameter s as the *density* of the enhancement.

We call the enhancement function F *essantial* if there exists a configuration $\omega\,(\in \Omega)$ such that $G(\omega) \cup F(\omega)$ contains a doubly-infinite path but $G(\omega)$ contains no such path. Here are two examples of this definition

(i) Suppose that F has the effect of adding an edge joining and any given unit vector whenever these two vertices are isolated in $G(\omega)$. In this case, F is not essential.

(ii) If, on the order hand, F adds such an edge whether or not the edvertices are isolated, then F is indeed essential.

We call the enhancement function F *monotonic* if for all η and all $\omega \leq \omega'$, the graph $G^{enh}(\omega, \eta)$ is a subgraph of $G^{enh}(\omega', \eta)$. For F to be monotonic it suffices that $\omega_B \cup F(\omega_B)$ be a subgraph of $\omega'_B \cup F(\omega'_B)$ whenever $\omega \leq \omega'$.

The *enhanced percolation probability* is defined as

$$\theta^{enh}(p, s) = P_{p,s}\left(0 \text{ belongs to an infinite cluster of } G^{enh}\right) \tag{4.9}$$

and *enhancement critical point* is given by

$$p_c^{enh}(F, s) = \inf\left\{p : \theta^{enh}(p, s) > 0\right\} \tag{4.10}$$

We note from (4.8) that θ^{enh} is non-decreasing in s. If F is monotonic then, by Theorem (3.1.1), θ^{enh} is non-decreasing in p also, whence

$$\theta^{enh}(p, s) = \begin{cases} = 0 & \text{if } \; p < p_c^{enh}(F, s), \\ > 0 & \text{if } \; p > p_c^{enh}(F, s) \end{cases}$$

If F is not monotonic, there will generally by ambiguity the correct definition of the critical point.

Theorem 4.3.1 *Let $s > 0$. If the enhancement function F is essential, then $p_c^{enh}(F, s) < p_c$.*

Here are some examples of Theorem (4.3.1) and related arguments.

A. *Entanglements.* Consider bond percolation on the three-dimensional cubic lattice $\mathbb{L}^3$. Whenever we see two interlinking 2×2 open squares, we join them by an edge. It is easy to see that this enhancement is essential and therefore it shifts the critical point downwards. Any reasonale definition of entanglement would require that two such interlocking squares be entagled, and it would follow that $p_c^{ent} < p_c$.

B. *Site percolation.* The condition of $'$essentialness$'$ was formulated above for bond percolation, and is replaced as follows for site percolation. We say that the realization $\xi \{0, 1\}^{\mathbb{Z}^d}$ of site percolation contains a doubly-infinite *self-repelling* path if there exists a doubly-infinite open path none of whose vertices is adjacent to any other vertex of the path except for its two neighbours in the path. An enhancement of site percolation is called *essential* if there exists a configuration ξ containing no doubly-infinite self-repelling path, but such that the enhanced configuration obtained by activating the enhancement at the origin does indeed contain such a path.

4.4 Bond and Site Critical Probabilities

As we saw earlier, for any connected graph G,we have $p_c^{bond}(G) \leq p_c^{site}(G)$,but when does strict inequality hold here?. We observe a special and important case of this situation.

Theorem 4.4.1 *([13],[11]) Consider $\mathbb{L}^d$ with $d \geq 2$. We have that $p_c^{bond} < p_c^{site}$*

Lemma 2 *We have that $\theta^{enh}(p, p^2) \leq \theta^{bond}(p)$.*

Theorem (4.4.1) follows easily from this lemma, as follows. Let s satisfy $\sqrt{s} = \frac{1}{2}p_c^{site}$, It follows from the appropriate form of Theorem (4.3.1) that there exists $\pi(s)$ satisfying $\pi(s) < p_c^{site}$ such that $\theta^{enh}(p, s) > 0$ for all $p > \pi(s)$. Let p *satisfy*

$$\max\left\{\pi(s), \sqrt{s}\right\} < p < p_c^{site}$$

Since $p^2 > s$, we have that $\theta^{enh}(p, p^2) \geq \theta^{enh}(p, s) > 0$.Therefore, by Lemma 2, $\theta^{bond}(p) > 0$, Whence $p_c^{site} > p \geq p_c^{bond}$ as required.

We end this section with an exact calculation of bond critical probability in $\mathbb{Z}^2$

Theorem 4.4.2 *([7],[14]) The critical probalility of bond percolation on $\mathbb{Z}^2$equals $\frac{1}{2}$.Furthermore, $\theta\left(\frac{1}{2}\right) = 0$*

We begin with *setting* $p = \frac{1}{2}$. *Let* $T(n)$ be the box $T(n) = [0, n]^2$, find N;

$$P_{\frac{1}{2}}(\partial T(n) \leftrightarrow \infty) \;>\; 1 \;-\; \frac{1}{8^4} \qquad for \;\; n \geq \; N.$$

We set $n = N + 1$. Writing A^l, A^r, A^t, A^b for the (respective) events that the left, right, top, bottom, sides of $T(n)$ are joined to ∞ off $T(n)$, we have by the FKG inequality that

$$\begin{aligned} P_{\frac{1}{2}}\left(T\left(n\right) \not\leftrightarrow \infty\right) &= P_{\frac{1}{2}}(\overline{A^l} \cap \overline{A^r} \cap \overline{A^t} \cap \overline{A^b}) \\ &\geq P_{\frac{1}{2}}(\overline{A^l})P(\overline{A^r})P(\overline{A^t})P(\overline{A^b}) \\ &= P_{\frac{1}{2}}\overline{(A^g)}^4 \end{aligned}$$

by symmetry, for g = l,r,t,b. therefore

$$P_{\frac{1}{2}}(A^g) \geq 1 - (1 - P_{\frac{1}{2}}(T(n) \leftrightarrow \infty))^{1/4} > \frac{7}{8}.$$

Now we move to the dual box, with vertex set

T(n)$_d = \left\{x + (\frac{1}{2,}, \frac{1}{2}) : 0 \leq x_1, x_2 < n\right\}$.

Let $A_d^l, A_d^r, A_d^t, A_d^b$ denote the (respective) events that the left, right, top, bottom sides of $T(n)_d$ are joined to ∞ by a closed dual path off $T(n)_d$. Since each edge of the dual is closed with probability $\frac{1}{2}$, we have that

$$P_{\frac{1}{2}}(A_d^g) > \frac{7}{8} \qquad \text{for g = 1,r,t,b.}$$

Consider the event $A = A^l \cap A^r \cap A_d^t \cap A_d^b$. Clearly $P_{\frac{1}{2}}(\overline{A}) \leq \frac{1}{2}$, so that $P\frac{1}{2}(A) \geq \frac{1}{2}$.However, on A, either L^2 has two infinite open clusters, or its dual has two infinite closed clusters. Each event has probability 0 by uniqueness of infinite cluster, a contradiction. We deduce that $\theta(\frac{1}{2}) = 0$, implying that $p_c \geq \frac{1}{2}$.

Next we prove that $p_c \leq \frac{1}{2}$.Suppose instead that $p_c > \frac{1}{2}$, so that

$$P_{\frac{1}{2}}(0 \leftrightarrow \partial B(n)) \leq e^{-\gamma n} \qquad \text{for all } n,$$

for some $\gamma > 0$.

Let $S(n)$ be the graph with vertex set in the following;

$\{x \in Z^2 : 0 \leq x_1 \leq n+1, 0 \leq x_2 \leq n\}$ and edge set containing all edges inherited from L^2except those in either the left side or the right side of $S(n)$. Denote by A the event that there is an open path joining the left side and right side of $S(n)$. Using duality, if A does not occur, then the top side of the dual of $S(n)$ is joined to the bottom side by a closed dual path. Since the dual of $S(n)$ is isomorphic to $S(n)$, and since P=$\frac{1}{2}$, it follows that $P_{\frac{1}{2}}(A) = \frac{1}{2}$. However, the above inequality applied for the event A gives a contradiction for large n. We deduce that $P_c \leq \frac{1}{2}$.

Chapter 5

Inhomogeneous Percolation Models

Up to this chapter we mostly considered homogeneous percolation in $\mathbb{Z}^d$,and at the end of chapter 3, we discussed the "finite inhomogeneous" percolation model in $\mathbb{Z}^2$and saw that the critical probability and related results still remains the same in this model. Now we move onto the case where inhomogeneity is infinite. We will consider a special model described as following and compare the results with the homogeneous model. We consider the following inhomogeneous nearest neighbour independent bond percolation model on $\mathbb{Z}^2$: each bond in $\{0\}\ X\ \mathbb{Z}$ is open (respectively, closed) with probability p_1 (respectively, $1-p_1$) and each remaning bond in $\mathbb{Z}^2$ is open (respectively closed) with probability p_2 (respectively $1-p_2$): all bonds of $\mathbb{Z}^2$ are independent of each other ([9]). We call this a (p_1, p_2) model and shall assume that $0 < p_1, p_2 < 1$ unless stated otherwise. The resulting probability measure will be denoted by P_{p_1,p_2} and expectation with respect to P_{p_1,p_2} denoted by E_{p_1,p_2} . Two sites, x and y, of $\mathbb{Z}^2$ are said to be connected if there is a path of open bonds in $\mathbb{Z}^2$ from x to y, and we denote this event by $x \leftrightarrow y$. The open cluster $C(x)$ of x is defined to be the random set of sites connected to x. The number of sites in $C(x)$ is denoted by $|C(x)|$. The percolation probability θ is then defined by

$$\theta(p_1, p_2) = P_{p_1,p_2}(|C(o)| = \infty), \tag{5.1}$$

where o is the origin of $\mathbb{Z}^2$. Although the probability $P_{p_1,p_2}(|C(o)| = \infty)$ depends on the site x because of the model, it is easy to show [By FKG inequality] that either $P_{p_1,p_2}(|C(o)| = \infty) > 0$ for every x in $\mathbb{Z}^2$ or it is identically 0. Percolation occurs if $\theta(p_1, p_2) > 0$. When $p_1 = p_2 = p$, the model becomes the standart homogeneous one. As mentioned above, it is a fundamental result that there exists a critical value $p_c = p_c(\mathbb{Z}^2)$ in (0,1) such that ([3],[6])

$$\begin{aligned} \theta(p,p) &= 0 \text{ if } p < p_c \\ \theta(p,p) &> 0 \text{ if } p > p_c \end{aligned} \tag{5.2}$$

It has been long conjectured that

$$\theta(p_c, p_c) = 0. \tag{5.3}$$

However, (5.3) is only proved for $d = 2$ and for sufficiently large d ([5], [7])

In this section, we consider the model in which p_2 is fix to be $p_c = p_c(\mathbb{Z}^2)$ while $p_1$varies. we will show that for d sufficiently large,

$$\theta(p_1, p_c) = 0 \text{ for any } p_1 \in [0, 1). \tag{5.4}$$

Consider the following two theorems and their corollary after introducing some more notation. Define

$$\tau_{p_1,p_2}(x, y) = P_{p_1,p_2}(x \leftrightarrow y)$$

to be the connectivity function between two sites x and y of $\mathbb{Z}^d$ and write $\tau_{p_c}(x,y)$ for $\tau_{p_c,p_c}(x,y)$, the critical connectivity function in the homogeneous model. Note that by uniqueness of the infinite cluster and the FKG inequalities,

$$\begin{aligned}
\tau_{p_c}(x,y) &= P_{p_c,p_c}(C(x)=C(y)) \\
&\geq P_{p_c,p_c}(C(x)=C(y), |C(x)|=\infty=|C(y)|) \\
&= P_{p_c,p_c}(|C(x)|=\infty, |C(y)|=\infty) \\
&\geq P_{p_c,p_c}(|C(x)|=\infty)P_{p_c,p_c}(|C(y)|=\infty) = [\theta(p_c,p_c)]^2;
\end{aligned}$$

thus the hypotheses of our claim automatically imply that $\theta(p_c,p_c)=0$.

We claim the following; let $d \geq 2$.

$If\ \sum \tau\ _{p_c}((0,0,...,1),(0,...,0,m)) < \infty$, then $\theta(p_1,p_c) = 0\quad for\ any\ p_1 \in [0,1)$, .as a corollary we see that, $If\ d$ is sufficiently large, $\theta(p_1,p_c)=0\ for\ any\ p_1 \in [0,1)$;

Now, let's consider the case that for any fixed p_1 in $(p_c,1)$ (with p_2 as the single parameter). In [8], it has been shown that this model has the same critical value as that of the homogeneous model on $\mathbb{Z}^d$. That is, for any $p_1 \in (p_c,1)$

$$\theta(p_1,p_2)=0 \text{ if } p_2<p_c, \tag{5.5a}$$

$$\theta(p_1,p_2)>0\ if\ \ p_2>p_c. \tag{5.5b}$$

[Of course, (5.5b) is obvious.] .

What happens for the case $0 \leq p_1 \leq p_c$?

Observe that (5.5) is also valid for $0 \leq p_1 \leq p_c$. The $p_1 < p_c$ case of (5.5a) is obvious while the $p_1 < p_c$ case of (5.5b) follows from the fact that the critical probability for the half space equals the critical probability for the full space ([4] , [5]); this is so because $\theta(0, p_2)$ is positive if there is percolation in a half space at $p = p_2$. Finally, the $p_1 = p_c$ case follows by simple comparisons to the $p_1 > p_c$ and $p_1 < p_c$ cases.

As we see that, similar to the finite inhomogeneous case, the critical probability and related results remain the same in this special infinite inhomogeneous percolation model.

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