NONLINEARITIES IN QUANTUM CORRELATIONS OF TWO TIME-LIKE SEPARATED MEASUREMENTS

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ABSTRACT

NONLINEARITIES IN QUANTUM CORRELATIONS OF TWO TIME-LIKE SEPARATED MEASUREMENTS

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When two time-like separated measurements are done on an observable with 3 or more eigenvalues, such as angular momentum of an spin-1 particle, the result of the measurements correlation is nonlinear in terms of the first measurement outcome while linear in the other. This thesis is the study of this nonlinearity in quantum correlation of such measurements as well as showing the results of simulation on quantum simulator qasm and experiment on the quantum computer ibmq-manila. The results confirm the non-invasive nature of quantum measurements.

Keywords: quantum correlation, Leggett-Garg inequalities, time-like separated measurement, qiskit, ibmq

ÖZ

İKİ ZAMANSAL-AYRIK ÖLÇÜMÜN KUANTUM KORELASYONLARINDAKİ DOĞRUSALSIZLIK

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Bir spin-1 parçacığının açısal momentumu gibi 3 veya daha fazla özdeğere sahip bir gözlemlenebilir üzerinde iki zamansal-ayrık ölçüm yapıldığında, ölçümler korelasyonunun sonucu ilk ölçümün sonucu açısından doğrusal değilken diğerinde doğrusaldır. Bu tez, kuantum simülatörü qasm üzerinde simülasyon sonuçlarını ve kuantum bilgisayar ibmq-manila üzerinde deneyi göstermenin yanı sıra, bu tür ölçümlerin kuantum korelasyonundaki doğrusalsızlığın çalışmasıdır. Sonuçlar, kuantum ölçümlerinin invaziv olmayan doğasını doğrulamaktadır.

Anahtar Kelimeler: kuantum korelasyon, Leggett-Garg eşitsizlikler, zamansal-ayrık ölçüm, qiskit, ibmq

To Papa

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TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

Starting from Bell's inequality [1] there are many inequalities in the study of quantum information theory. And what they do is prove a set of assumptions wrong when they are violated and help us get a better understanding of quantum mechanics. Bell's paper in 1964 [1] showed that if the measurements of two entangled particles depended on hidden variable theories then this would set a constraint on each half of entangled particle, this constraint became known as Bell's inequality. In 1975 Alain Aspect [2] proposed in detail an experimental setup that would test the Bell's inequality and finally between 1980-82 [3] [4] by increasingly complex setups experimentally showed that the Bell's inequality is violated. Violation of Bell's inequality debunks the need for existence of any hidden variable theory and shows that there can be certain non-local actions in quantum theory, that is even when two events are too far away in distant yet close in time so that there can be no communication even at the speed of light, there still can be an interaction between them. To this date there has been several similar inequalities known as Bell-type and they have also been tested experimentally to show that hidden variable theories are inconsistent with the physical world. Next comes the Leggett-Garg's inequalities in 1985 [5], produced by a set of assumptions on macrorealism and non-invasive measurement, and the violation of it in quantum mechanics shows that we cannot understand time evolution classically and distinguishes macrorealistic theories from quantum mechanics. The violation of Leggett-Garg inequalities prove the assumption of non-invasive measurements wrong since it is not a quantum mechanical assumption itself. [6]

Bell's inequality looks at correlation of space-separated particles whereas Leggett-Garg's is based on correlation of time-separated measurements. While looking at similar correlations we notice that if an observable with 3 or more eigenvalues is measured twice with a time-like separation the correlation of the measurements turn out to be nonlinear in the first one. What we do here is looking for a way to quantify this nonlinearity to be able to see it via simulation on a quantum computer. We take angular momentum as such observable and measure it in an arbitrary direction, and immediately after that we measure it in another direction. We repeat this for a set of directions which their unit vectors add up to zero. Even though in such situations the sum of angular momenta is zero, the sum of correlations is not. This can be understood as the fact that there cannot be any non-invasive measurements in quantum mechanics.

In the next chapter, we review the literature related to our discussion. We have a look at definition of measurements in quantum mechanics and explain what is a measurement in the standard basis which would be used in the quantum circuits. Then we introduce the quantum logic gates that will be used in the simulation. We explain Leggett-Garg inequalities, its assumptions of macrorealism and non-invasive measurement and manifestation of its violation.

In Chapter 3, we have a look at correlation of space separated measurements for spin $\frac{1}{2}$ particles or Pauli matrices, then we look at correlation of time separated measurements of the same spin $\frac{1}{2}$ particles and we see a nonlinearity of the correlation in terms of the first measurement. We extend the study to spin-1 particles and measurement of angular momenta and we see the same kind of nonlinearity, we look for ways to quantify this nonlinearity, we consider two different scenarios, one with 3 angles on a surface and one with 4 angles forming a tetrahedron and make theoretical calculations of the expected probabilities for each possible outcome. We will also talk about possible Leggett-Garg type inequalities that can be formed.

In chapter 4 we use the gates we see in chapter 2 and using proper rotations explain how we create two quantum circuits that could be coded via qiskit for the examples we created in our theoretical work. For both cases we give the result of our simulation on ibmq's qasm simulator [7] and also on quantum system ibmq-manila [8]. We also calculate the 5σ for our measurements and show that the results are within an acceptable range of calculations.

And finally in the last chapter we discuss our experimental results, we will consider possible noise sources for the results and argue their agreement with our calculations. We also talk about how the nonlinearity in the correlation we found goes against assumption of non-invasive measurement and conclude this study.

CHAPTER 2

PRELIMINARIES

In this chapter we will talk about and review some concepts that we would be needing in this study. The first section talks about postulate of measurements in quantum mechanics and explains what we mean when we measure a qubit in our circuits. Section 2.2 introduces the quantum logic gates needed for our simulation. And finally the last section 2.3 talks about Leggett-Garg inequalities and the assumptions of the original paper [5].

2.1 Measurement in Quantum Mechanics

One important postulate of quantum mechanics is measurement. When we say measurement, we mean testing or manipulation of system which results in a numerical value. Assuming we repeat the process on an ensemble of equivalent systems then the average of the results of measurements is what quantum mechanics predicts for us i.e., quantum mechanics gives us the probability of each viable result. A closed quantum system is subject to unitary time evolution but when there is an observer interacting with the system this no longer applies. A measurement is a set of operators that when applied to the state of the system gives us a probabilistic result and based on the result leaves the system in a new state. Thus, measurement in quantum mechanics is invasive and collapses the state. A quantum measurement is described by a collection of measurement operators $\{M_m\}$ which act on the state space of the system and the index m refers to possible outcomes of measurement. Let us say the state right before the measurement is $|\Psi\rangle$ then the state after measurement is

$$\frac{M_m\,|\Psi\rangle}{\sqrt{\langle\Psi|M_m^{\dagger}M_m|\Psi\rangle}} \tag{2.1}$$

with the probability of each outcome of measurement m as:

$$p(m) = \langle \Psi | M_m^\dagger M_m | \Psi \rangle \tag{2.2}$$

and

$$\sum_m M_m^\dagger M_m = I \tag{2.3}$$

Which is called the completeness relation and satisfies the fact that sum of probabilities is 1. [9]

Then we have a type of quantum measurement called projective measurement, described by a Hermitian operator on the state space of the system being observed, M, called an observable. With P_m being the projector onto the eigenspace of M with eigenvalue m, the spectral decomposition of our observable is

$$M = \sum_m m P_m \tag{2.4}$$

The eigenvalues m correspond to possible outcome measurements of the observable. As we measure state $|\Psi\rangle$ the probability of measuring m is

$$p(m) = \langle \Psi | P_m | \Psi \rangle \tag{2.5}$$

And for outcome m, the state immediately after measurement collapses to

$$\frac{P_m |\Psi\rangle}{\sqrt{p(m)}} \tag{2.6}$$

An important element that we will use in our quantum circuits is measurement, but in quantum computers and qiskit, measurement is restricted to a projective measurement of a qubit in the standard or computational basis which is represented by a 'meter' symbol on the circuit plot (Figure 2.1). The standard basis states are $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and are eigenvectors of Pauli matrix σ_z. Measurement in the basis states is what is called as measurement in the computational basis. In order to measure in any other basis, one should first rotate the states to the standard basis and then perform a measurement.

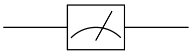


Figure 2.1: Measurement in the computational basis

2.2 Quantum Gates

Similar to how a classic computer uses wires and logic gates in an electrical circuit to transfer and manipulate information. A quantum computer uses a quantum circuit with wires and quantum gates to represent the language of quatum computation and the changes to the system [9]. Here we will have a look at the gates we will be using in our circuit.

The simplest non-trivial gate is probably the NOT gate or Pauli-X gate, this flips the state from $|0\rangle$ to $|1\rangle$ and vice versa

$$-\boxed{X}- \qquad X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.7}$$

Next gate is the Hadamard, it moves the standard basis states from poles of the Bloch sphere and creates a superposition of $|0\rangle$ and $|1\rangle$. It can be represented by a rotation of π about $(\hat{x} + \hat{z})/\sqrt{2}$ axis.

$$-\boxed{H}- \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.8}$$

also Hadamard takes the states from standard basis to the eigenstates of σ_x

$$H\,|0\rangle = |+\rangle \qquad H\,|1\rangle = |-\rangle \tag{2.9}$$

These were both single qubit gates, meaning that they only act on one single qubit. The most general form of single qubit gates is the U3-gate with the matrix representation

$$-\boxed{U}- \qquad U3\,(\theta, \phi, \lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\phi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix} \tag{2.10}$$

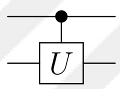
with the θ, ϕ and λ being the Euler angles. This gate is carried out using two X90 or $\pi/2$ pulses in IBM computers [10]

$$U3(\theta, \phi, \lambda) = R_z(\phi)R_x(-\frac{\pi}{2})R_z(\theta)R_x(\frac{\pi}{2})R_z(\lambda) \tag{2.11}$$

From equation 2.10 we can see that

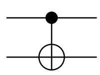
$$X = U3(\pi, 0, 0) \qquad\qquad H = U3(\frac{\pi}{2}, 0, \pi) \tag{2.12}$$

There are also multiple qubit gates that act on 2 or more qubits. We will look at a specific type of multiple qubit gates called controlled gates. In these gates one qubit acts as the control while the others would be the target. In general, for a unitary gate $U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$ we have a Controlled-U gate acting on 2 qubits with the first qubit being control and the second the target as below [11]

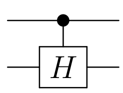


$$CU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{pmatrix} \tag{2.13}$$

This gate performs the U-gate when the control qubit is $|1\rangle$ and leaves the circuit unchanged otherwise. From here we can see that Controlled-NOT and Controlled-Hadamard are



$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.14}$$



$$CH = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{2.15}$$

2.3 Leggett-Garg Inequalities

Legget-Garg inequality named after Anthony James Leggett and Anupam Garg is the result of their 1985 paper "Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks?" [5]. These kind of inequalities are used to tell apart theories that support macrorealism from those supporting quantum mechanics. In their original paper, Leggett and Garg start with making two assumptions:

> 1. Macrorealism(MR): "A macroscopic object, which has available to it two or more macroscopically distinct states, is at any given time in a definite one of those states."
>
> 2. Noninvasive measurability(NIM): "It is possible in principle to determine which of these states the system is in without any effect on the state itself, or on the subsequent system dynamics."

while in a later publication in 2008 Leggett adds a third assumption [12]

> 3. Induction:"Causality propagates only forward in time, that is, an event cannot be offered by any future events, whether or not the latter lie inside its light cone"

They propose a quantum system that; when measured produces a value $Q = \pm 1$, then they measure the system at 2 different times and at least at one other time between those two times. The simplest case would be to measure at t_1,t_2 and t_3 where $t_1 < t_2 < t_3$. Here for an ensemble of systems prepared at an earlier time t_0 we define

> i) Joint probability densities such as $\rho(Q_1, Q_2), \rho(Q_1, Q_2, Q_3)$, which are consistent with one another, that is
>
> $$\sum_{Q_2=\pm 1} \rho(Q_1, Q_2, Q_3) = \rho(Q_1, Q_3) \tag{2.16}$$
>
> ii) Correlation functions as $K_{ij} = \langle Q_i Q_j \rangle$

To deal with the problem of non-invasive measurement, Leggett-Garg gives the idea of an ideal negative measurement. That is we design a device that only interacts strongly with the system when for example $Q = +1$ and does not do anything when $Q = -1$, then we can be sure that the system is in state $Q = -1$

Now let us say that $Q_1 = +1$ then there will be the following possibilities for the measurements

$$\begin{array}{lll} Q_1 = +1 & Q_2 = +1 & Q_3 = +1 \\ Q_1 = +1 & Q_2 = -1 & Q_3 = +1 \\ Q_1 = +1 & Q_2 = -1 & Q_3 = -1 \\ Q_1 = +1 & Q_2 = +1 & Q_3 = -1 \end{array} \tag{2.17}$$

which would give the C_{ij} values as [13]

$$\begin{array}{lll} K_{12} = +1 & K_{23} = +1 & K_{13} = +1 \\ K_{12} = -1 & K_{23} = -1 & K_{13} = +1 \\ K_{12} = -1 & K_{23} = +1 & K_{13} = -1 \\ K_{12} = +1 & K_{23} = -1 & K_{13} = -1 \end{array} \tag{2.18}$$

Writing the same values for $Q_1 = -1$ and even adding a fourth measurement Q_4 we can see that

$$1 + K_{12} + K_{23} + K_{13} \geq 0 \tag{2.19}$$

and

$$|K_{12} + K_{23}| + |K_{14} - K_{24}| \leq 2 \tag{2.20}$$

violation of equations 2.19 and 2.20 by quantum mechanics proves our initial assumptions to be wrong. Since non-invasive measurement is not a quantum mechanical assumption in itself, the violation proves it to be wrong rather than macrorealism.

2.4 Temporal Bell Inequalities

Bell inequalities use space-like separated measurements while Leggett-Garg inequalities use time-like separated measurements. Therefore, Leggett-Garg's are also referred to as temporal Bell inequalities. In an online lecture [6], Sir Anthony Leggett uses this approach and gives a more clear derivation of above inequalities.

As in previous section, consider the two-state system that when measured has a value $Q = \pm 1$ and divide the ensemble of measurements into three:

E_1: Q measured at t_1 and t_2

E_2: Q measured at t_2 and t_3

E_3: Q measured at t_1 and t_3

Then using the experimental average of each of these sub-ensembles we can write

$$K_{exp} = \langle Q_1 Q_2 \rangle_{exp} + \langle Q_2 Q_3 \rangle_{exp} - \langle Q_1 Q_3 \rangle_{exp} \tag{2.21}$$

whereas, as we will also see in section 3.2, the quantum correlation of measurements at t_i and t_j for a simple two-state Hamiltonian $\hat{H} = \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}$ is

$$\langle Q_i Q_j \rangle_{QM} = \cos \Omega (t_j - t_i) \tag{2.22}$$

Building on the fact that macroscopic realism means that all the quantities Q_1, Q_2, Q_3 exist at the same whether they are measured or not and using some algebra Leggett gives

$$Q_1 Q_2 + Q_2 Q_3 - Q_1 Q_3 \leq 1 \tag{2.23}$$

We would have the similar for an ensemble

$$\langle Q_1 Q_2 \rangle_{ens} + \langle Q_2 Q_3 \rangle_{ens} - \langle Q_1 Q_3 \rangle_{ens} \leq 1 \tag{2.24}$$

By the assumption that measurements are non-invasive, $\langle Q_i Q_j \rangle_{exp} = \langle Q_i Q_j \rangle_{ens}$, macroscopic reality gives the inequality

$$K_{MR} \leq 1 \qquad \text{Temporal Bell Inequality} \tag{2.25}$$

Now if we choose $t_2 - t_1 = t_3 - t_2 = \frac{\pi}{3\Omega}$ then using the equation 2.22 we find

$$K_{QM} = \frac{3}{2} \tag{2.26}$$

which violate the temporal Bell inequality.

CHAPTER 3

THEORETICAL CALCULATIONS

This chapter will discuss our theoretical calculations and ideas. First we will consider correlation of two space-like separated measurements of spin 1/2 particles as the simplest case. Then in section 3.2 we calculate the correlation of two time-like separated measurements, again for an observable with two eigenvalues. And in the last section we look at correlation of measurement for a 3 level system ie. an observable with 3 eigenvalues. Then we take angular momentum as an example of such observable and calculate expected results for two different cases.

3.1 Correlation of Space-Like Separated Measurements of Two Qubits

Suppose Alice and Bob have two entangled particles at distant places. Alice does a measurement of n component of pauli spin matrix σ_n^A and Bob measures m component σ_m^B. The matrix for correlation of measurements, C, can be written as

$$C(\hat{n},\hat{m}) = \overline{\sigma_m \sigma_n} = n_i m_j C_{ij} \tag{3.1}$$

while

$$C_{ij} = \left\langle {\sigma_i}^A \otimes {\sigma_j}^B \right\rangle \tag{3.2}$$

For the Bell states we have (first one the singlet, and the triplet)

$$\begin{aligned}
|\Psi_-\rangle &= \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad & C(\Psi_-) &= \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \\
|\Psi_+\rangle &= \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad & C(\Psi_+) &= \begin{bmatrix} +1 & & \\ & +1 & \\ & & -1 \end{bmatrix} \\
|\Phi_+\rangle &= \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}} \qquad & C(\Phi_+) &= \begin{bmatrix} +1 & & \\ & -1 & \\ & & +1 \end{bmatrix} \\
|\Phi_-\rangle &= \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}} \qquad & C(\Phi_-) &= \begin{bmatrix} -1 & & \\ & +1 & \\ & & +1 \end{bmatrix}
\end{aligned} \tag{3.3}$$

We can see that $\det(C) = -1$ for Bell states and more generally for entangled states of 2 qubits of the form

$$|\Psi\rangle = \frac{|\uparrow\uparrow\rangle + \alpha\,|\downarrow\downarrow\rangle}{\sqrt{1+\alpha^2}} \qquad C(\Psi) = \begin{bmatrix} c & & \\ & -c & \\ & & 1 \end{bmatrix} = -c^2 < 0 \tag{3.4}$$

c is the concurrence, a value between 0 and 1, It is equal to 1 for maximally entangled states. It is defined as [14]

$$c(\Psi) = \left|\langle\Psi|\tilde{\Psi}\rangle\right| \tag{3.5}$$

where $|\tilde{\Psi}\rangle$ is the spin flip of $|\Psi\rangle$. For a single qubit pure state the spin flip is $|\tilde{\Psi}\rangle = \sigma_y\,|\Psi^*\rangle$ and for 2qubits or more the same transformation should be applied to each qubit. Therefore the concurrence of states in equation 3.4 is

$$c = \frac{2\alpha}{\sqrt{1+\alpha^2}} \tag{3.6}$$

3.2 Correlation of Time-Like Separated Measurements of A Single Qubit

We want to find the correlation between two consequent measurements on the same state. Let us start with a state $|\Psi(t_1)\rangle$ at time $t = t_1$, first we measure σ_k which would result in a collapse and give values $a_1 = \pm 1$ and leave the collapsed state as $|a_1\rangle$. (subscript 1 to show the first measurement). Then after some time at $t = t_2$ the state has evolved with time and is

$$|\Psi(t_2)\rangle = U(t_2 - t_1)\,|a_1\rangle \tag{3.7}$$

we measure σ_k again at time t_2. In order to find the correlation between the measurements we write is as statistical average of product of 2 measurement results. Note that correlation is not expectation value of Heisenberg picture

$$\overline{a_1 a_2} = \langle \sigma_k(t_2); \sigma_k(t_1)\rangle \neq \langle \sigma_k(t_2)\sigma_k(t_1)\rangle \tag{3.8}$$

$$\begin{aligned}&(\text{prob. of getting } a_1 \text{ for } \sigma_k \text{ at } t_1) = P(a_1) = |\langle a_1|\Psi(t_1)\rangle|^2\\ &(\text{prob. of } a_2 \text{ for } \sigma_k(t_2) \text{ given that } 1^{st} \text{ meas. is } a_1) = P(a_2|a_1) = |\langle a_2|\Psi(t_2)\rangle|^2\end{aligned} \tag{3.9}$$

therefore,

$$\overline{a_1 a_2} = \sum a_2 a_1 P\,(a_2|a_1) P\,(a_1\ ) \tag{3.10}$$

where we have

$$\sum_{a_2} a_2 P(a_2|a_1) = \sum_{a_2} |\langle a_2|\Psi(t_2)\rangle|^2 = \langle \Psi(t_2)|\sigma_k|\Psi(t_2)\rangle \tag{3.11}$$

$$= \langle a_1|U^\dagger(t_2 - t_1)\sigma_k U(t_2 - t_1)|a_1\rangle \tag{3.12}$$

as a result

$$\overline{a_1 a_2} = \sum_{a_1} a_1 \,\langle a_1|U^\dagger(t_2 - t_1)\sigma_k U(t_2 - t_1)|a_1\rangle\, |\langle a_2|\Psi(t_2)\rangle|^2 = \langle A_2; A_1\rangle \tag{3.13}$$

we see that the result is linear in A_2 but not in A_1 (unless $A_1$is dichotomic like σ_k i.e., has eigenvalues ± 1 which is the case for Leggett-Garg inequalities). Taking the Hamiltonian as $\hat{H} = \frac{1}{2}\Omega\ \hat{\sigma_z}$, and for when σ_k is perpendicular to the direction of magnetic field (here $\hat{z}$) we can see that

$$\overline{a_1 a_2} = \langle \sigma_k(t_2); \sigma_k(t_1)\rangle = \cos\Omega(t_2 - t_1) \tag{3.14}$$

3.3 Nonlinearity in Measurement Correlation of Angular Momentum (Spin-1 Particles)

As we saw in equation 3.13, for any observable that has 3 or more eigenvalues the average of the product of results or correlation between two measurements is nonlinear in terms of the first measurement. Our job is to find a way to quantify this nonlinearity. As the simplest case, we consider a 3-level system of spin-1 particles and their angular momentum $\vec{J}$. We can think of a spin-1 system as the addition of 2 spin-$\frac{1}{2}$ particles, therefore the eigenstates of $\vec{J}_z$ correspond to the spin triplet states

$$\begin{aligned}
|1\rangle &= |s=1, m_s=+1\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \\
|0\rangle &= |s=1, m_s=0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle) \\
|\bar{1}\rangle &= |s=1, m_s=-1\rangle = |\downarrow\rangle \otimes |\downarrow\rangle
\end{aligned} \tag{3.15}$$

With the eigenvalues +1, 0 and -1 respectively. And of course there is the spin singlet state for $s = 0$ which we do not need

$$|s=0, m_s=0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle) \tag{3.16}$$

Similar to the previous section we make 2 consecutive measurements on the system but instead of measuring Pauli matrices we would be measuring components of angular moment or in general $\vec{J}_n$, that is for unit vector $\hat{n}$

$$\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k} \qquad n_x^2 + n_y^2 + n_z^2 = 1 \tag{3.17}$$

$\vec{J}_n$ component of angular momentum along $\hat{n}$ is

$$J_n = \vec{J} \cdot \hat{n} = \hat{n} \cdot \vec{J} = J_x n_x + J_y n_y + J_z n_z \tag{3.18}$$

We measure J_n at time t_1 and then measure J_m at time t_2. The system is a spin with magnetic moment, so the Hamiltonian is,

$$H = -\vec{\mu} \cdot \vec{B} = -k\vec{J} \cdot \vec{B} \tag{3.19}$$

and if $\vec{B} = B\hat{b}$ then

$$H = (const)\, J_b \tag{3.20}$$

its effect is spin precession, and after some time state will be still up but along some other direction. As simplest case, we forget about time and therefore direction along magnetic field. We assume that they are measured almost at the same time, immediately after each other.

$$\left\langle J_m(0^+); J_n(0^-)\right\rangle \tag{3.21}$$

Next, we need to find the eigenstates of J_n. We can think of spin-1 particle as composed of 2 spin half particles. For example, if we know the spin up along arbitrary direction for a spin half particle, by adding their spins we can write spin up along any arbitrary direction for spin-1 particle. It is readily seen that spin up and spin up along direction $\hat{n}$ in terms of spin up and down in $\hat{z}$ direction are

$$\begin{aligned}
|\uparrow\rangle_{\hat{n}} &= \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\,|\downarrow\rangle\, e^{i\phi} \\
|\downarrow\rangle_{\hat{n}} &= -\sin\frac{\theta}{2}e^{-i\phi}\,|\uparrow\rangle + \cos\frac{\theta}{2}\,|\downarrow\rangle
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
|1\rangle_{\hat{n}} &= |\uparrow\rangle_{\hat{n}} \otimes |\uparrow\rangle_{\hat{n}} \\
|0\rangle_{\hat{n}} &= \frac{1}{\sqrt{2}}(|\uparrow\rangle_{\hat{n}} \otimes |\downarrow\rangle_{\hat{n}} + |\downarrow\rangle_{\hat{n}} \otimes |\uparrow\rangle_{\hat{n}}) \\
|\bar{1}\rangle_{\hat{n}} &= |\downarrow\rangle_{\hat{n}} \otimes |\downarrow\rangle_{\hat{n}}
\end{aligned} \tag{3.23}$$

after some calculations we find the eigenstates of J_n as

$$\begin{aligned}
|1\rangle_{\hat{n}} &= \cos^2\frac{\theta}{2}e^{-i\phi}\,|1\rangle + \sqrt{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,|0\rangle + \sin^2\frac{\theta}{2}e^{i\phi}\,|\bar{1}\rangle \\
|0\rangle_{\hat{n}} &= \frac{1}{\sqrt{2}}(\sin\theta e^{-i\phi}\,|1\rangle + \sqrt{2}\cos\theta\,|0\rangle + \sin\theta e^{i\phi}\,|\bar{1}\rangle) \\
|\bar{1}\rangle_{\hat{n}} &= \sin^2\frac{\theta}{2}e^{-i\phi}\,|1\rangle - \sqrt{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,|0\rangle + \cos^2\frac{\theta}{2}e^{i\phi}\,|\bar{1}\rangle
\end{aligned} \tag{3.24}$$

Spectral decomposition of J_n is

$$J_n = (+1)\,|1\rangle_{\hat{n}\,\hat{n}}\langle 1| + (0)\,|0\rangle_{\hat{n}\,\hat{n}}\langle 0| + (-1)\,|\bar{1}\rangle_{\hat{n}\,\hat{n}}\langle\bar{1}| \tag{3.25}$$

therefore,

$$\begin{aligned}
J_n &= |1\rangle_{\hat{n}\,\hat{n}}\langle 1| - |\bar{1}\rangle_{\hat{n}\,\hat{n}}\langle\bar{1}| \\
J_n^2 &= |1\rangle_{\hat{n}\,\hat{n}}\langle 1| + |\bar{1}\rangle_{\hat{n}\,\hat{n}}\langle\bar{1}|
\end{aligned} \tag{3.26}$$

In order to find the correlation between two measurements of angular momentum we write

$$\langle J_m; J_n\rangle = \sum a \langle a_n|J_m|a_n\rangle \left|\langle a_n|\Psi\,(0)\rangle\right|^2 = \sum a\hat{m}\cdot\left\langle \vec{J}\right\rangle |\langle a_n|\Psi(0)\rangle|^2 \tag{3.27}$$

where $\left\langle \vec{J}\right\rangle$ and $a^2 = 0$ or 1 so it simplifies to

$$\langle J_m; J_n\rangle = \hat{m}\cdot\hat{n}\left(\left|{}_{\hat{n}}\langle 1|\Psi(0)\rangle\right|^2 + \left|{}_{\hat{n}}\langle \bar{1}|\Psi(0)\rangle\right|^2\right) \tag{3.28}$$

now referring to equation 3.26 it is easily seen that

$$\langle J_m; J_n\rangle = \hat{m}\cdot\hat{n}\left\langle {J_n}^2\right\rangle \tag{3.29}$$

Therefore we can write

$$\begin{aligned}\sum_i \langle J_m; J_{n_i}\rangle &= \sum_i \left\langle (\hat{m}\cdot\hat{n}_i)(\hat{n}_i\cdot\vec{J})(\hat{n}_i\cdot\vec{J})\right\rangle \\ &= \left\langle (\hat{m}\otimes\vec{J}\otimes\vec{J})\cdot(\sum_i \hat{n}_i\otimes\hat{n}_i\otimes\hat{n}_i)\right\rangle\end{aligned} \tag{3.30}$$

even if $\sum_i c_i\hat{n}_i = 0$, we have $\sum_i c_i\hat{n}_i\otimes\hat{n}_i\otimes\hat{n}_i \neq 0$ so we can conclude that for $\sum_i \vec{J_{n_i}} = 0$

$$\sum_i \langle J_m; J_{n_i}\rangle \neq 0 \tag{3.31}$$

Even though we can see this nonlinearity in any set of $\hat{n}_i$ directions, in order to see the nonlinearity better, we will pick two cases with the most symmetry. The first case is N=3 unit vectors that lie on a surface and have a $120°$ with each other. The second case is for N=4 unit vectors in space that form a tetrahedron. In both cases, the $\hat{m}$ direction of second measurement and also the initial state $|\Psi\rangle$ are chosen in a way that would maximize the outcome.

We also note that looking at equation 3.29 it can be seen that for a completely unpolarized state the nonlinearity is lost. An unpolarized state is one for which we an equal distribution of possible states so its density matrix would be

$$\rho = \frac{1}{3}\sum_a |a_{\hat{n}}\rangle\langle a_{\hat{n}}| = \frac{1}{3}\mathbb{I} \tag{3.32}$$

The expectation of J_n^2 is independent of $\hat{n}$ and always equal to $\frac{2}{3}$ therefore, in such case the correlation is linear in terms of n as well.

3.3.1 J_n Measurement in 3 Directions on x-y Plane

The first case to consider is that of N=3 unit vectors that lie on a surface. But before we set any values for the angles, we need to find the maximum value of correlation. First we look at the matrices for J_n and J_n^2 are

$$J_n = \begin{bmatrix} \cos\theta & \frac{1}{\sqrt{2}}\sin\theta e^{-i\phi} & 0 \\ \frac{1}{\sqrt{2}}\sin\theta e^{i\phi} & 0 & \frac{1}{\sqrt{2}}\sin\theta e^{-i\phi} \\ 0 & \frac{1}{\sqrt{2}}\sin\theta e^{i\phi} & -\cos\theta \end{bmatrix} \tag{3.33}$$

$$J_n^2 = \begin{bmatrix} \frac{1+\cos^2\theta}{2} & \frac{1}{2\sqrt{2}}\sin 2\theta e^{-i\phi} & \frac{\sin^2\theta}{2}e^{-2i\phi} \\ \frac{1}{2\sqrt{2}}\sin 2\theta e^{i\phi} & \sin^2\theta & \frac{1}{2\sqrt{2}}\sin 2\theta e^{-i\phi} \\ \frac{\sin^2\theta}{2}e^{2i\phi} & \frac{1}{2\sqrt{2}}\sin 2\theta e^{i\phi} & \frac{1+\cos^2\theta}{2} \end{bmatrix} \tag{3.34}$$

Since we are on the $x - y$ axis, this means $\theta = \frac{\pi}{2}$ and simplifies the matrices

$$J_n = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & e^{-i\phi} & 0 \\ e^{i\phi} & 0 & e^{-i\phi} \\ 0 & e^{i\phi} & 0 \end{bmatrix} \tag{3.35}$$

$$J_n^2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & e^{-2i\phi} \\ 0 & 2 & 0 \\ e^{2i\phi} & 0 & 1 \end{bmatrix} \tag{3.36}$$

If we name the angle of $\hat{m}$ as ϕ_m , then $\hat{m} \cdot \hat{n}_i = \cos(\phi_m - \phi_i)$ and taking the state as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{i\phi_m}{2}} \\ 0 \\ e^{-\frac{i\phi_m}{2}} \end{pmatrix} \tag{3.37}$$

we can find the expected result of sum of correlations as an expectation value of $\frac{3}{4}$

$$\sum \langle J_m ; J_{n_i} \rangle = \sum \left\langle \cos\left(\phi_m - \phi_i\right) {J_{n_i}}^2 \right\rangle$$
$$= \frac{3}{4} \begin{bmatrix} 1 & 0 & e^{i\phi_m} \\ 0 & 2 & 0 \\ e^{-i\phi_m} & 0 & 1 \end{bmatrix} \quad \text{expectation value is } \frac{3}{4} \tag{3.38}$$

which means the maximum value of sum of correlations for directions adding up to zero on $x - y$ plane is $\frac{3}{4}$. The direction $\hat{m}$ can be any direction on $x - y$ plane, which we will choose it to be $\hat{x}$, and the initial state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{3.39}$$

Let us also find the value of each correlation for now called as $\Gamma_i \equiv \langle J_m ; J_{n_i} \rangle$.

$$\langle \Psi | \Gamma_i | \Psi \rangle = \frac{\cos\left(\phi_m - \phi_i\right)\left(1 + \cos\left(\phi_m + 2\phi_i\right)\right)}{2} \tag{3.40}$$

By plugging in our angles $\phi_m = 0$, $\phi_1 = 0$, $\phi_2 = \frac{2\pi}{3}$ and $\phi_3 = \frac{4\pi}{3}$ we have,

$$\begin{aligned} \langle J_m ; J_{n_1} \rangle &= 1 \\ \langle J_m ; J_{n_2} \rangle &= -\frac{1}{8} \\ \langle J_m ; J_{n_3} \rangle &= -\frac{1}{8} \end{aligned} \tag{3.41}$$

Next, we want to see the probabilities of each measurement outcome. First measurement is J_{n_i} in the state Ψ and it has three possible outcomes. The probability of each possible outcome is as follows

$$\begin{aligned} \mathbb{P}(J_n = +1) &= \left|{}_{\hat{n}}\langle 1 | \Psi \rangle\right|^2 = \frac{\cos^2\phi_n}{2} \\ \mathbb{P}(J_n = 0) &= \left|{}_{\hat{n}}\langle 0 | \Psi \rangle\right|^2 = \sin^2\phi_n \\ \mathbb{P}(J_n = +1) &= \left|{}_{\hat{n}}\langle \bar{1} | \Psi \rangle\right|^2 = \frac{\cos^2\phi_n}{2} \end{aligned} \tag{3.42}$$

For $\phi_n = \pm\frac{2\pi}{3}$ they give probabilities $\frac{1}{8}$,$\frac{6}{8}$,$\frac{1}{8}$ respectively. Next, we measure J_m or in our case J_x, so the conditional probabilities for $\phi_n = \pm\frac{2\pi}{3}$ are

$$\begin{aligned}
\mathbb{P}(00|00) &= \left|{}_{\hat{m}}\langle 1|1\rangle_{\hat{n}}\right|^2 = 1/16 \\
\mathbb{P}(01|01) &= \left|{}_{\hat{m}}\langle 0|0\rangle_{\hat{n}}\right|^2 = 4/16 \\
\mathbb{P}(01|00) &= \left|{}_{\hat{m}}\langle 0|1\rangle_{\hat{n}}\right|^2 = 3/16 \\
\mathbb{P}(00|11) &= \left|{}_{\hat{m}}\langle 1|\bar{1}\rangle_{\hat{n}}\right|^2 = 9/16 \\
\mathbb{P}(01|11) &= \left|{}_{\hat{m}}\langle 0|\bar{1}\rangle_{\hat{n}}\right|^2 = 6/16
\end{aligned} \tag{3.43}$$

Having in mind that for joint probabilities we have need to multiply conditional probabilities with the probability of each outcome of J_n. It can also easily be seen that for $\phi_n = 0$ there is a 50-50 probability for outcome of J_n to be ± 1. And the conditional probabilities are

$$\mathbb{P}(00|00) = \left|{}_{\hat{m}}\langle 1|1\rangle_{\hat{n}}\right|^2 = 1 \qquad \mathbb{P}(11|11) = \left|{}_{\hat{m}}\langle \bar{1}|\bar{1}\rangle_{\hat{n}}\right|^2 = 1 \tag{3.44}$$

Table 3.1: probability of each outcome of measurement

outcome	J_n	J_m	probability for $\phi_n = \pm\frac{2\pi}{3}$	probability for $\phi_n = 0$
0000	+1	+1	$\frac{1}{128} \cong 0.008$	$\frac{1}{2}$
0001	+1	0	$\frac{6}{128} \cong 0.047$	0
0011	+1	-1	$\frac{9}{128} \cong 0.070$	0
0100	0	+1	$\frac{36}{128} \cong 0.281$	0
0101	0	0	$\frac{24}{128} \cong 0.188$	0
0111	0	-1	$\frac{36}{128} \cong 0.281$	0
1100	-1	+1	$\frac{9}{128} \cong 0.070$	0
1101	-1	0	$\frac{6}{128} \cong 0.047$	0
1111	-1	-1	$\frac{1}{128} \cong 0.008$	$\frac{1}{2}$

Just as in equation 3.10 we can write the value of 3.41 in terms of outcome probabilities.

$$\begin{aligned}
\langle J_m; J_{n_i}\rangle &= \sum mnP(m,n) \\
&= P(0000) - P(0011) - P(1100) + P(1111)
\end{aligned} \tag{3.45}$$

We can see that the probabilities of table 3.1 agree with 3.41 equations.

3.3.2 J_n Measurements in 4 Directions That Form a Tetrahedron

Let us do a similar measurement for the 4 unit vectors that form a tetrahedron as in figure 3.1. The unit vectors are

$$\begin{aligned}
&\hat{n}_3 = \hat{z} \\
&\hat{n}_\alpha = \frac{2\sqrt{2}}{3}(\frac{\cos 2\pi\alpha}{3}\hat{x} + \frac{\sin 2\pi\alpha}{3}\hat{y}) - \frac{1}{3}\hat{z} \qquad \text{for } \alpha = 0, 1, 2
\end{aligned} \tag{3.46}$$

The unit vectors and their corresponding components of angular momentum add up to zero

$$\sum_{\alpha=0}^{3} J_{n_\alpha} = 0 \tag{3.47}$$

But

$$\sum_{\alpha=0}^{3} \langle J_m; J_{n_\alpha}\rangle = \hat{m} \cdot \langle\Psi|\vec{G}|\Psi\rangle \tag{3.48}$$

where $\vec{G}$ is as below

$$\vec{G} = \sum_{\alpha=0}^{3} \hat{n}_\alpha {J_{n_\alpha}}^2 = \frac{4}{9}\begin{bmatrix} \hat{z} & -\hat{e}_- & 2\hat{e}_+ \\ -\hat{e}_+ & -2\hat{z} & \hat{e}_- \\ 2\hat{e}_- & \hat{e}_+ & \hat{z} \end{bmatrix} \qquad (\hat{e}_\pm = \frac{\hat{x}\pm i\hat{y}}{\sqrt{2}}) \tag{3.49}$$

First we choose an $\hat{m}$ direction and calculate the eigenvectors of $\vec{G}$. We choose the eigenvector corresponding to the smallest or largest eigenvalue as $|\Psi\rangle$ then choosing any of the unit vectors or $\hat{m} = \hat{n}_i$ will give us the extremum value. We choose $\hat{m} = \hat{z}$ We find the minimum and the corresponding initial state as

$$\sum_{\alpha=0}^{3} \langle J_z; J_{n_\alpha}\rangle = -\frac{8}{9} \qquad |\Psi\rangle = |0\rangle_m \tag{3.50}$$

Also for the expected value of each set of measurements using eq.3.34 and taking

$$|\Psi\rangle = \begin{pmatrix} 0 \\ e^{\frac{i\phi_m}{2}} \\ 0 \end{pmatrix} \tag{3.51}$$

we have,

$$\langle J_z; J_{n_\alpha}\rangle = \cos\theta \left\langle {J_{n_\alpha}}^2\right\rangle = \cos\theta \sin^2\theta \tag{3.52}$$

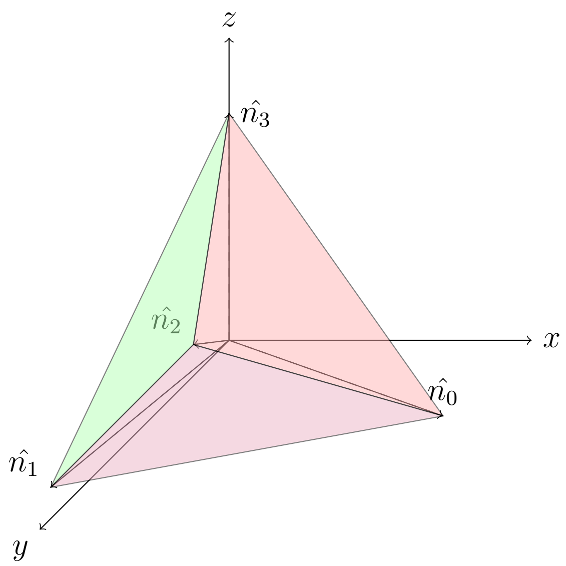


Figure 3.1: 4 unit vectors towards the corners of a tetrahedron

Therefore

$$
\begin{aligned}
\langle J_z ; J_{n_0} \rangle &= \frac{-8}{27} \\
\langle J_z ; J_{n_1} \rangle &= \frac{-8}{27} \\
\langle J_z ; J_{n_2} \rangle &= \frac{-8}{27} \\
\langle J_z ; J_{n_3} \rangle &= 0
\end{aligned}
\tag{3.53}
$$

As in the previous section, in order to see the probabilities of each outcome, we look at probable outcomes of J_n first

$$
\begin{aligned}
\mathbb{P}(J_n = +1) &= |{}_{\hat{n}}\langle 1|\Psi\rangle|^2 = \frac{1}{2}(\sin\phi - \cos\theta\cos\phi)^2 \\
\mathbb{P}(J_n = 0) &= |{}_{\hat{n}}\langle 0|\Psi\rangle|^2 = \sin\theta\cos\phi^2 \\
\mathbb{P}(J_n = +1) &= |{}_{\hat{n}}\langle \bar{1}|\Psi\rangle|^2 = \frac{1}{2}(\sin\phi + \cos\theta\cos\phi)^2
\end{aligned}
\tag{3.54}
$$

It can be seen that for the $\hat{n}_4$ case the probability of J_n is 1 and for other n_i the probabilities are $\frac{4}{9}, \frac{1}{9}, \frac{4}{9}$ respectively. Then we calculate the conditional probabilities.

For $\hat{n}_4$ $\mathbb{P}(01|01) = |_{\hat{m}}\langle 0|0\rangle_{\hat{n}}|^2 = 1$, and for the rest of directions we have,

$$\begin{aligned}
\mathbb{P}(00|00) &= |_{\hat{m}}\langle 1|1\rangle_{\hat{n}}|^2 = 1/9 \\
\mathbb{P}(01|01) &= |_{\hat{m}}\langle 0|0\rangle_{\hat{n}}|^2 = 1/9 \\
\mathbb{P}(01|00) &= |_{\hat{m}}\langle 0|1\rangle_{\hat{n}}|^2 = 4/9 \\
\mathbb{P}(00|11) &= |_{\hat{m}}\langle 1|\bar{1}\rangle_{\hat{n}}|^2 = 4/9 \\
\mathbb{P}(01|11) &= |_{\hat{m}}\langle 0|\bar{1}\rangle_{\hat{n}}|^2 = 4/9
\end{aligned} \tag{3.55}$$

The joint probabilities are shown in table 3.2.

Table 3.2: probability of each outcome of measurement

outcome	J_n	J_m	probability for $\hat{n_0}, \hat{n_1}, \hat{n_2}$
0000	+1	+1	$\frac{4}{81} \cong 0.049$
0001	+1	0	$\frac{16}{81} \cong 0.198$
0011	+1	-1	$\frac{16}{81} \cong 0.198$
0100	0	+1	$\frac{4}{81} \cong 0.049$
0101	0	0	$\frac{1}{81} \cong 0.012$
0111	0	-1	$\frac{4}{81} \cong 0.049$
1100	-1	+1	$\frac{16}{81} \cong 0.198$
1101	-1	0	$\frac{16}{81} \cong 0.198$
1111	-1	-1	$\frac{4}{81} \cong 0.049$

3.4 Nonlinearity in Correlation of Two Time-like Measurements and Leggett-Garg Inequalities

As we saw in section 2.3 and 2.4, Leggett-Garg inequality is a way of showing that the assumption of non-invasive measurement does not hold in quantum mechanics by violation of the inequality. So one might wonder if this nonlinearity could give a value higher than $\frac{3}{2}$ of equation 2.26. Using 2.25 for 3 consecutive measurements in time we have the simplest case of Leggett-Garg inequality as below

$$\langle J_n(t_3); J_m(t_2)\rangle + \langle J_m(t_2); J_l(t_1)\rangle - \langle J_n(t_3); J_l(t_1)\rangle \leq 1 \tag{3.56}$$

By plugging in values, it can be seen that the inequality is violated but the left side is again the maximum $\frac{3}{2}$. Therefore, the nonlinearity in quantum correlation of two time-like measurements for an observable with more than 2 eigenvalues can be considered another way to show the same thing as Leggett-Garg inequality does, that is the assumption of non-invasive measurements is wrong.

CHAPTER 4

EXPERIMENTAL RESULTS

In order to see the results of measurement experimentally, we will use $qiskit$ a module based on python designed to run quantum simulations and we use $ibmq$ as a provider for quantum computer simulators, first we use quantum $qasm$ simulator, which is a classical computer that simulates a quantum computer with minimal noise. And then we run the same code on $ibmq - manila$ the quantum system available with the lowest CNOT noise.

4.1 Measuring J_n in 3 Directions on x-y Plane

The first part of our circuit is to prepare the initial state that is equation 3.37

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{4.1}$$

All qubits in qiskit are by default start at $|0\rangle$ unless initialized or set to any other value, therefore we start by $|00\rangle$ and apply a Hadamard to the first qubit

$$|00\rangle \xrightarrow{H} |{+0}\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}} \tag{4.2}$$

then we apply a CNOT with the first qubit being the target

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{CNOT} \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{4.3}$$

Figure 4.1 show the part of circuit that prepares the initial state.

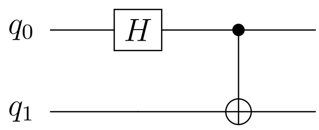


Figure 4.1: Initial State for 3 Directions on x-y Plane

Then we to take the standard basis states to the eigenstates of angular momentum J_z, that is

$$\begin{array}{ccc} |m_1 m_2\rangle & & |jm\rangle \\ |\uparrow\uparrow\rangle & & |\uparrow\uparrow\rangle \equiv |j=1, m=1\rangle \\ |\uparrow\downarrow\rangle & \xrightarrow{\Omega} & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \equiv |j=1, m=0\rangle \\ |\downarrow\uparrow\rangle & & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \equiv |j=0, m=0\rangle \\ |\downarrow\downarrow\rangle & & |\downarrow\downarrow\rangle \equiv |j=1, m=-1\rangle \end{array}$$

To find the gate or gate combination Ω, first we apply a CNOT (2.14) on the initial qubits.

$$\begin{array}{ccc} |00\rangle & & |00\rangle \\ |01\rangle & \xrightarrow{CNOT} & |01\rangle \\ |10\rangle & & |11\rangle \\ |11\rangle & & |10\rangle \end{array}$$

Then we apply an upside-down Controlled-Hadamard (2.15) that is the second qubit is the control.

$$\begin{array}{ccc} |00\rangle & & |00\rangle \\ |01\rangle & \xrightarrow{CH} & |+1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \\ |11\rangle & & |-1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \\ |10\rangle & & |10\rangle \end{array}$$

And a final CNOT will give us what we need

$$\begin{array}{ccc} |00\rangle & & |00\rangle \\ |+1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) & \xrightarrow{CNOT} & \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |-1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) & & \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ |10\rangle & & |11\rangle \end{array}$$

Figure 4.2 shows this circuit. Note that because of the symmetry, the same circuit takes the eigenstates of J_z back to the standard basis.

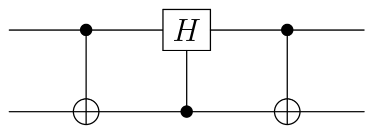


Figure 4.2: The circuit taking standard basis to spin-1 basis.The input is 00, 01 or 11 and the output is $|m\rangle_z$

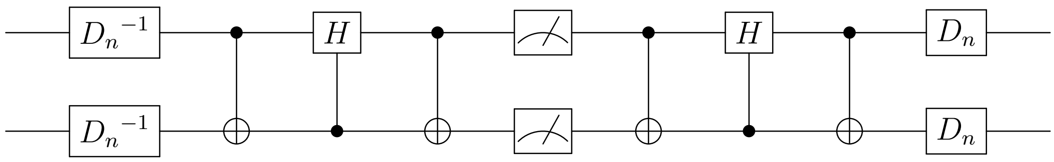


Figure 4.3: Circuit for J_n measurement

In the next step we want to transform the qubits from $\hat{z}$ direction to $\hat{n}$, we do this by applying a rotation on each of the qubits.

$$(D_n \otimes D_n)\,|m\rangle_z = |m\rangle_n \tag{4.4}$$

What we see in fig.4.3 is that the circuit rotates the qubits from $\hat{n}$ to $\hat{z}$ direction and then decodes from J_z basis to the computational basis to take a measurement since it is the basis that our quantum computers measure in them. Note that at this point a collapse of quantum state has occurred. Immediately after our first measurement we need to take the second measurement, so we do the inverse of the operations to take back the states to J_n direction. Then we use another transformation that would take the states to $\hat{m}$ direction or for our case $\hat{x}$ direction and decode to have the qubits in the computational basis. What we described can be seen in fig. 4.4.

In order to find the transformations we need, we refer to the most general unitary gate, U3-gate, which we saw its matrix representation in equation 2.10 as below

$$U3\,(\theta, \phi, \lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\phi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix}$$

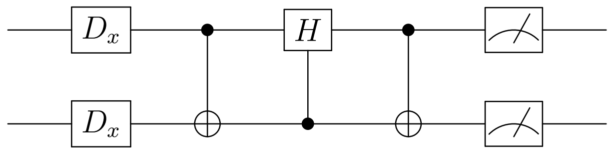


Figure 4.4: Circuit for J_x measurement

it has the inverse

$$U3^{-1}(\theta,\phi,\lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ -e^{-i\lambda}\sin\frac{\theta}{2} & e^{-i(\phi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix} \tag{4.5}$$

we can easily see that

$$U3^{-1}(\theta,\phi,\lambda) = U3(-\theta,-\lambda,-\phi) \tag{4.6}$$

Therefore transformation that rotates J_z to the direction of J_n is the unitary matrix

$$D_n = U3\left(\frac{\pi}{2},\phi_n,0\right) \tag{4.7}$$

with the inverse rotating J_n to J_z and being the first unitary gate in our circuit as

$$D_n{}^{-1} = U3\left(-\frac{\pi}{2},0,-\phi_n\right) \tag{4.8}$$

also the third unitary rotates the states from J_n to J_x direction as below

$$D_x{}^{-1} = U3(-\frac{\pi}{2},0,0) \tag{4.9}$$

Figure 4.5 is an example of how the complete circuit looks like in qiskit. The circuit also has 4 classical bits, it takes the result of first measurement and writes it in the first and second bits whereas the result of second measurement is written on third and fourth bits.

Running these circuits on qasm for a total of 10000 shots and 10 runs we find that the results are within the acceptable 5σ range of our calculations. Figure 4.6 shows these results. Note that the outcomes should be read from top to bottom in qiskit.

The average of 10 runs each consisting of 10000 shots on qasm are as follows

$$\begin{aligned} \overline{\langle J_x; J_{n_1}\rangle} &= 1.0000 \\ \overline{\langle J_x; J_{n_2}\rangle} &= -0.12604 \\ \overline{\langle J_x; J_{n_3}\rangle} &= -0.12515 \end{aligned} \tag{4.10}$$

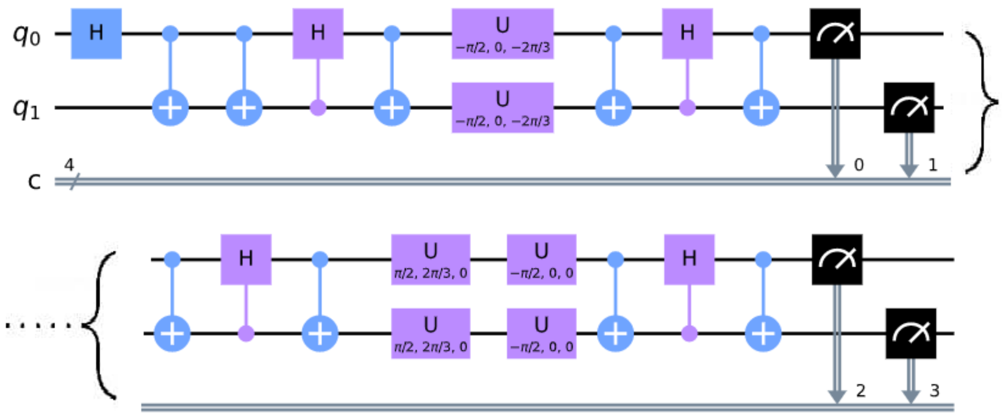


Figure 4.5: qiskit circuit for the measurement in $\hat{n}_2$ direction in $3\hat{n}_i$ case

Let's name the summation of correlators as Γ

$$\Gamma = \langle J_x; J_{n_1}\rangle + \langle J_x; J_{n_2}\rangle + \langle J_x; J_{n_3}\rangle \tag{4.11}$$

Then the standard deviation of Γ is

$$\sigma^2 = V = \frac{1}{N-1}\sum_{N=1}^{10}\left(\Gamma_i - \overline{\Gamma}\right)^2 \tag{4.12}$$

where

$$\begin{aligned} \overline{\Gamma} &= 0.74881 \\ \sigma &= 0.00545 \\ 5\sigma &= 0.02726 \end{aligned} \tag{4.13}$$

The results of qasm simulator agree with our theoretical values nicely, but when we run the same circuit on an actual quantum computer, the actual noise of circuit changes the results. Figure 4.7 shows the result for the same circuit ran on ibmq-manila.

The average of 10 runs each consisting of 8000 shots on ibmq-manila are as follows

$$\begin{aligned} \overline{\langle J_x; J_{n_1}\rangle} &= 0.82057 \\ \overline{\langle J_x; J_{n_2}\rangle} &= -0.10504 \\ \overline{\langle J_x; J_{n_3}\rangle} &= -0.11375 \end{aligned} \tag{4.14}$$

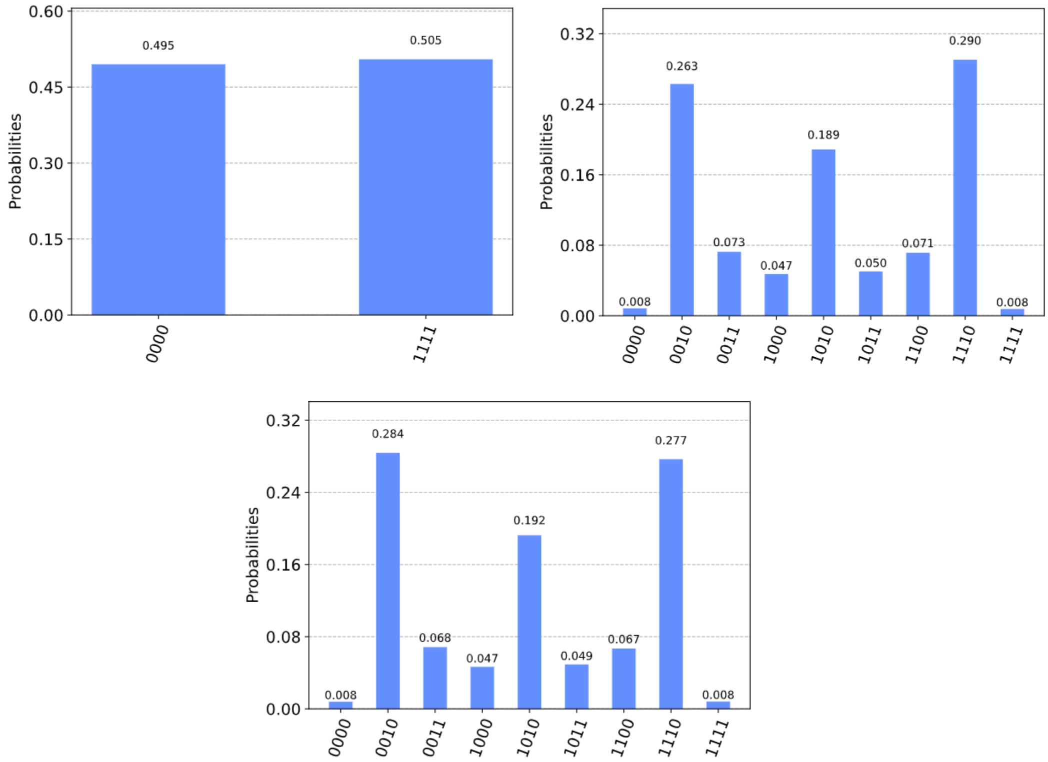


Figure 4.6: Probabilities of each outcome for $\phi_1 = 0$, $\phi_2 = 2\pi/3$ $\phi_3 = 4\pi/3$ respectively by qasm

And the mean and standard deviation of Γ are

$$\begin{aligned} \overline{\Gamma} &= 0.60179 \\ \sigma &= 0.03283 \\ 5\sigma &= 0.16416 \end{aligned} \tag{4.15}$$

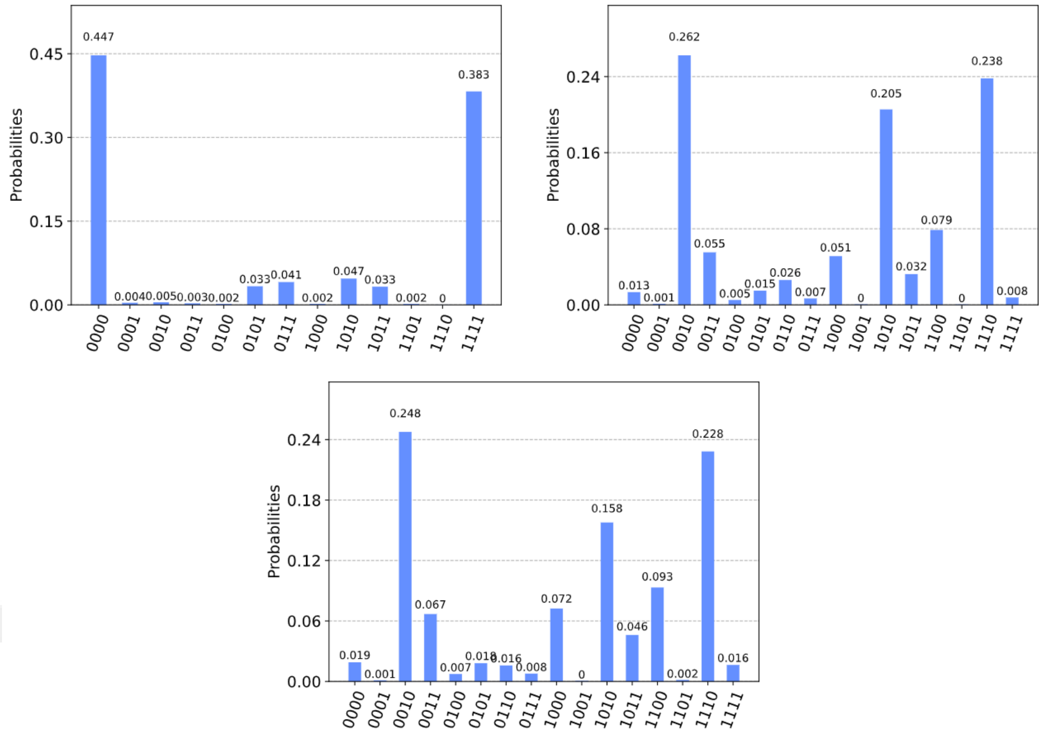


Figure 4.7: Probabilities of each outcome for $\phi_1 = 0$, $\phi_2 = 2\pi/3$ $\phi_3 = 4\pi/3$ respectively by ibmq-manila

4.2 Measuring J_n in 4 Directions That Form a Tetrahedron

The biggest part of the circuit for the tetrahedron case is the same as 3n directions one. The thing that changes is the initial state which is $|\Psi\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and of course the angles of the unitary gates. To prepare the initial state we only need an x-gate as in figure 4.8.

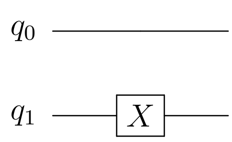


Figure 4.8: Initial state for tetrahedron case

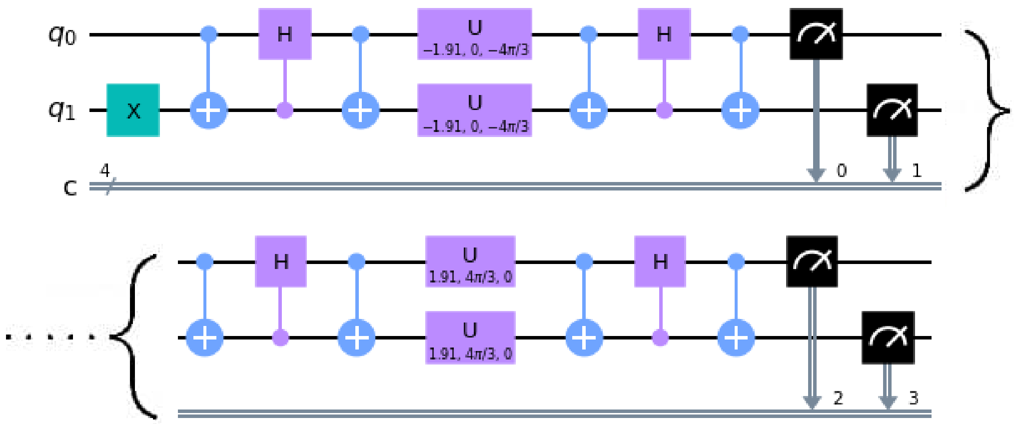


Figure 4.9: qiskit circuit for the measurement in $\hat{n}_3$ direction in tetrahedron case

As for the transformations, the unitary taking J_z to J_n

$$D_n = U3(\arccos\frac{-1}{3}, \phi_n, 0) \tag{4.16}$$

with the inverse taking J_n to J_z

$$D_n{}^{-1} = U3(-\arccos\frac{-1}{3}, 0, -\phi_n) \tag{4.17}$$

Note that there is no need for a third unitary since our second measurement is in J_z direction. Figure 4.9 shows an example of the whole circuit in qiskit.

Just as in the previous case we run each set of measurements 10 times for 10000 shots each on qasm simulator and see that the results agree with our theoretical values nicely. The figure 4.10 shows an output for the qasm simulation.

The average of 10 runs each consisting of 10000 shots on qasm are as follows

$$\begin{aligned} \overline{\langle J_z; J_{n_0}\rangle} &= -0.29388 \\ \overline{\langle J_z; J_{n_1}\rangle} &= -0.29995 \\ \overline{\langle J_z; J_{n_2}\rangle} &= -0.29665 \\ \overline{\langle J_z; J_{n_3}\rangle} &= 0 \end{aligned} \tag{4.18}$$

Naming the summation of correlators as Γ we have

$$\begin{aligned} \overline{\Gamma} &= -0.89048 \\ \sigma &= 0.00994 \\ 5\sigma &= 0.04972 \end{aligned} \tag{4.19}$$

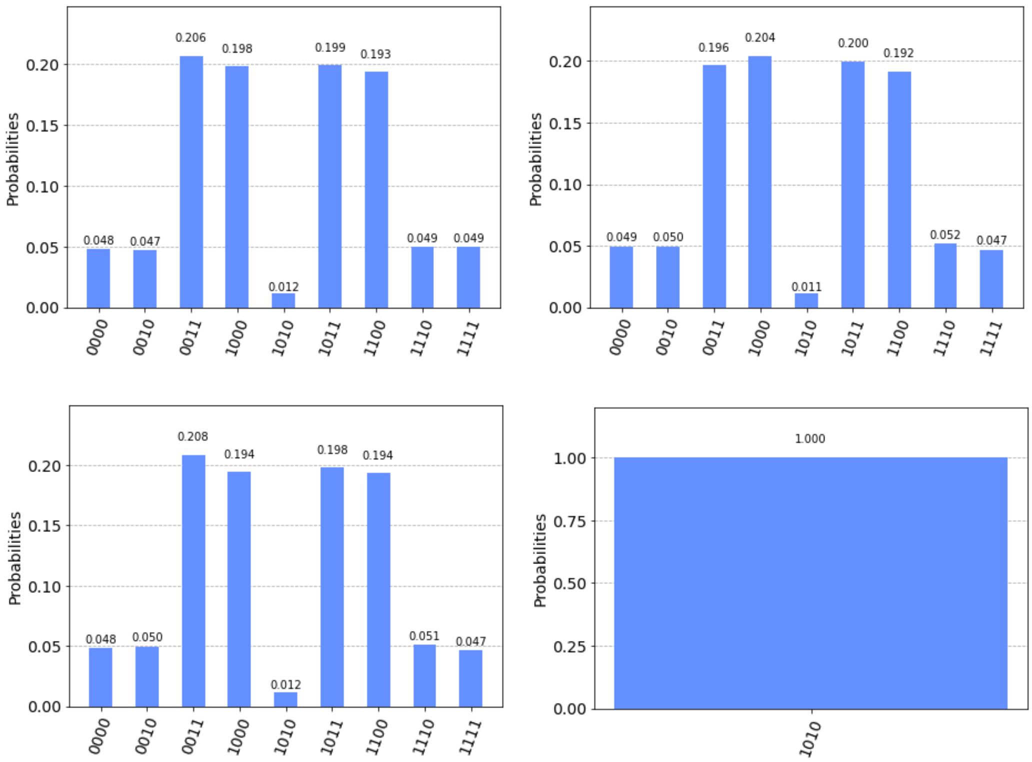


Figure 4.10: Probabilities of each outcome for $\hat{n}_i$ for i=0 to 3 respectively by qasm

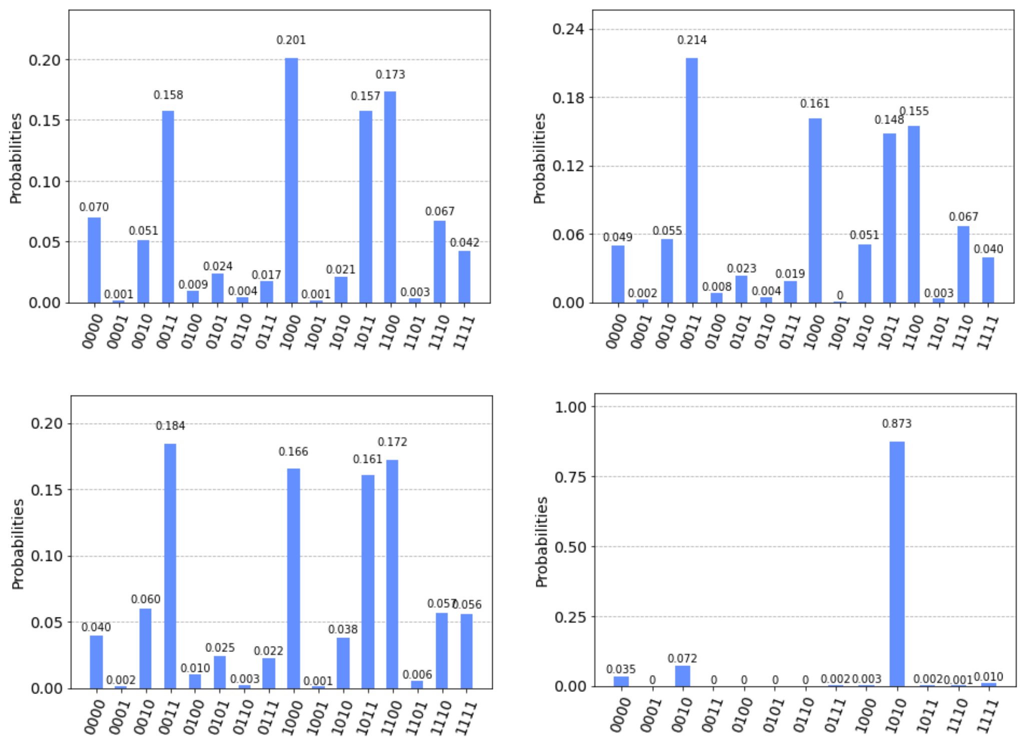


Figure 4.11: Probabilities of each outcome for $\hat{n}_i$ for i=0 to 3 respectively by ibmq-manila

And the average of 10 runs each for 8000 shots on ibmq-manila is

$$\begin{aligned}
\overline{\langle J_z; J_{n_0} \rangle} &= -0.20941 \\
\overline{\langle J_z; J_{n_1} \rangle} &= -0.20035 \\
\overline{\langle J_z; J_{n_2} \rangle} &= -0.22831 \\
\overline{\langle J_z; J_{n_3} \rangle} &= 0.04716
\end{aligned} \tag{4.20}$$

we also have

$$\begin{aligned}
\overline{\Gamma} &= -0.64534 \\
\sigma &= 0.051597 \\
5\sigma &= 0.25799
\end{aligned} \tag{4.21}$$

Figure 4.11 shows an example of ibmq-manila output for our set of measurements

CHAPTER 5

CONCLUSIONS

In this study we started with correlation of measurement of two entangled spin 1/2 particles or space-like separated measurements, we extended the study to time-like separated measurements and observed that for cases that are not dichotomic i.e. observables with 3 or more eigenvalues, the average of product of outcomes or the correlation function is nonlinear in terms of the first measurement in time. Then by taking the measurements to be very close in time we assumed that spin precession does not affect the system.

Then we looked for a way to quantify this nonlinearity and concluded that if we have a set of measurements in the direction of $\hat{n}_i$ for which the summation of unit vectors are zero, then summation of angular momenta in those directions are zero as well but the correlation of them with measurement in a second arbitrary direction is not zero. To be able to show our results quantitatively we chose two symmetric cases of unit vectors for measurements. Firstly, 3 vectors on a surface with the second measurement on the same surface and secondly, 4 vectors in space that would form a tetrahedron. For both cases we calculated the expectation of correlation function and all the outcome possibilities. In chapter 4 we designed a circuit on qiskit and simulated our measurements both on qasm, a classical computer simulating quantum computers and on ibmq-manila, a quantum computer.

The results that we had were within the 5σ range of theoretical values in all cases. But we saw a rather great difference between the values of quantum computer $ibmq-manila$ and that of theoretical values or even $qasm$. This noise is mostly caused by the number of CNOT gates that we have in our circuit. That is the reason, as we mentioned before, that we chose $ibmq-manila$ the system available with the lowest

CNOT noise value. A possible solution could be a circuit that does the same job but with a lesser number of gates and CNOTs. There was also deviation from mean in the results which was mostly observed when measurements were done on the same quantum system but at different times and/or days, this is probably caused by the condition of the system itself.

We talked about how the assumption of non-invasive measurability is proved wrong by the violation of Leggett-Garg inequalities. We can say that by proving the nonlinearity in the correlation of two time-like measurements we have also confirmed that measurements are not non-invasive and that this is another way of showing the same thing as Leggett-Garg inequalities. And finally, we looked at whether or not the maximum value of temporal bell inequality 2.25 that the quantum mechanics gives can be exceeded by this nonlinearity, and we found the maximum to be the same as the one quantum mechanics offers which is $\frac{3}{2}$.

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