GEODESIC CONNECTEDNESS AND COMPLETENESS OF TWICE WARPED PRODUCTS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

GEODESIC CONNECTEDNESS AND COMPLETENESS OF TWICE WARPED PRODUCTS

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We introduce the semi-Riemannian geometry. We give some results about Riemannian and Lorentzian manifolds. We explained the Lorentzian causality. We focus on the causality of space-times. We define the Lorentzian distance. We compare Lorentzian geometry with Riemannian geometry. We review the Lorentzian warped products and their causality.

We focus on the completeness of Lorentzian manifolds and Lorentzian warped products. Then we focus on geodesic connectedness. We give some information about the geodesic connectedness of multiwarped space-times. Lastly, we analyse the relationship between connectedness and completeness of twice warped products under a given condition.

Keywords: Lorentzian causality, Lorentzian completeness, geodesic connectedness, multiwarped space-times, twice warped products.

ÖZET

İKİLİ BÜKÜLMÜŞ ÇARPIMLARIN JEODEZİK BAĞLANTILILIĞI VE TAMLIĞI

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Yarı-Riemann geometrisini tanttık. Riemann ve Lorentz geometrisiyle ilgili bazı sonuçlar verdik. Lorentz nedenselliğini açıkladık. Uzay-zaman nedenselliği üzerinde yoğunlaştık. Lorentz uzaklığını tanımladık. Lorentz ve Riemann geometrilerini kıyasladık. Lorentz bükülmüş çarpımlar ve nedenselliğini gözden geçirdik.

Lorentz manifoldlarının ve Lorentz bükülmüş çarpımlarının tamlığı üzerinde yoğunlaştık. Sonra jeodezik bağlantılılığa odaklandık. Çoklu bükülmüş uzay-zamanların jeodezik bağlantılılığıyla ilgili bilgi verdik. Son olarak, bağlantılılık ve tamlık teoremlerini ikili bükülmüş çarpımlara bazı şartlarda uyguladık.

Anahtar sözcükler: Lorentz nedenselliği, Lorentz tamlığı, jeodezik bağlantılılık, çoklu bükülmüş uzay-zamanlar, ikili bükülmüş çarpımlar.

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Contents

Chapter 1

Introduction

Riemannian geometry has a wide research area and results. As another field of the semi-Riemannian geometry, Lorentzian geometry is the theory behind general relativity and has lots of similarities with Riemannian geometry. However, because of some differences, Lorentzian geometry follows a different way. Since we define an indefinite metric tensor, we consider three types of curves and subspaces: Spacelike, timelike and lightlike curves and subspaces. This brings about a wider and more complex research on Lorentzian manifolds.

By Hopf-Rinow theorem we know that for any Riemannian manifold, geodesic and metric completeness are equivalent. Also, completeness guarantees the existence of geodesic connectedness. On the other hand, we cannot claim the validity of Hopf-Rinow theorem for Lorentzian manifolds. Metric completeness and geodesic completeness are inequivalent. In addition, unless we consider some constraints, geodesic connectedness and completeness do not have a direct relationship. The aim of this thesis is to study the relationship between geodesic connectedness and completeness of Lorentzian manifolds under a given condition.

In Chapter 2, we recall some basic concepts of semi-Riemannian geometry using [1] and [6]. In Section 2.1, we define connections and the connection coefficients. By these, we give the geodesic equations and some other basic definitions of manifold theory. In Section 2.2, we define metric tensor g and semi-Riemannian manifold M. In Section 2.3, we refer to geodesic connectedness of a Riemannian manifold. The main theorem about completeness of Riemannian manifolds is the Hopf-Rinow theorem and it provides the equivalence of metric and geodesic completeness as well as finite compactness of a Riemannian manifold. In section 2.4, we explain the causality of Lorentzian spaces. A tangent vector v is defined as *spacelike* (respectively, *timelike, lightlike*) if $g(v, v) > 0$ or $v = 0$ (respectively, $g(v, v) < 0$, $g(v, v) = 0$ and $v \neq 0$) and the causal character of the subspaces is determined by the class of the vector. A Lorentzian manifold (M, g) is *time-oriented* if (M, g) admits a timelike vector. In Section 2.5, we introduce the causality of space-times. A space-time is a Lorentzian manifold with a time orientation. In Section 2.6, we give the definition of Lorentzian distance function. In Section 2.7, we compare Riemannian and Lorentzian manifolds by the help of [5].

In Chapter 3, we define and give some properties of warped products (see [1], [6]). In Section 3.1, we introduce Riemannian warped products. Let (M, g) and (H, h) be two Riemannian manifolds and let $f : M \to (0, \infty)$ be a smooth function. A *Riemannian warped product* $(\bar{M}, \bar{g})$ is a product manifold $\bar{M} = M \times H$ with metric tensor $\bar{g} = g \oplus f^2 h$. In Section 3.2, we define Lorentzian warped products. We refer to the causality of Lorentzian warped products. In Section 3.3, we give the definition and causality of multiply warped products (see [12]).

In Chapter 4, we give a detailed explanation and some examples for Lorentzian completeness and geodesic connectedness. For further details, one can read [1], [2], [9], [12]. In Section 4.1, we introduce the geodesic completeness, metric completeness and finite compactness. Geodesic completeness can be separated into three classes called spacelike, timelike and null completeness. Any of these types do not imply the others. In addition, geodesic completeness, metric completeness and finite compactness do not imply each other. Then we give some theorems and lemmas about the completeness of multiply warped products. In Section 4.2, we introduce geodesic connectedness of Lorentzian manifolds. We focus on multiply warped space-times and their geodesic connectedness.

In Chapter 5, we analyze the relationship between geodesic connectedness and completeness of twice warped products. In Section 5.1, we give the connectedness condition for twice warped products by Theorem 3 of [2]. In Section 5.2, we give the completeness conditions using Theorems 4.7 and 4.8 of [12]. In Section 5.3, we compute the relationship between geodesic connectedness and completeness of twice warped products under a given condition.

Chapter 2

Preliminaries

2.1 Connections

In this work we will assume that any manifold M is smooth, paracompact and Hausdorff. TM will denote the tangent bundle, T_pM will denote the tangent space at $p \in M$. For any arbitrary chart (U, x) for M, $(x^1, ..., x^n)$ will denote local coordinates on M and $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$ will denote the natural basis for the tangent space. Details can be found in [1].

Let $p \in M$ and γ be a smooth curve that passes through p. A smooth mapping $X : [a, b] \to TM$ such that $X(t) \in T_{\gamma(t)}M$ for all $t \in [a, b]$ is called a *vector field along* γ. $\mathfrak{X}(M)$ denotes the set of all smooth vector fields defined on M.

Let $\mathfrak{F}(M)$ denote the ring of all smooth real-valued functions on M. For all $f, h \in \mathfrak{F}(M)$ and $X, Y, V, W \in \mathfrak{X}(M)$, a function

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

with properties

1. $\nabla_V(X + Y) = \nabla_V X + \nabla_V Y$

2. $\nabla_{fV+hW}(X) = f\nabla_V X + f\nabla_W X$

3. $\nabla_V(fX) = f\nabla_V X + V(f)X$

is called a *connection*.

The *connection coefficients* Γ_{ij}^k of ∇ with respect to $(x^1, ...x^n)$ are defined by

$$\nabla_{\partial/\partial x^i}\left(\frac{\partial}{\partial x^j}\right) = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k}. \tag{2.1.1}$$

A vector field X along γ is said to move by *parallel translation* along γ if $\nabla_{\gamma'} X(t) = 0$.

Definition 2.1.1. A smooth curve $\gamma : (a, b) \to M$ is said to be a *geodesic* if γ' moves by parallel translation along γ.

Hence γ is called a geodesic if

$$\nabla_{\gamma'}\gamma' = 0. \tag{2.1.2}$$

Equation 2.1.2 is called the *geodesic equation* of γ.

A smooth curve γ that has a reparametrization as a geodesic is called a *pre-geodesic*. A parameter for which γ is a geodesic is called an *affine parameter*. If the domain of some affine parametrization is the whole real line, then a pre-geodesic is called *complete*.

Using the connection coefficients in equation 2.1.1, we can write the *geodesic equations in coordinates*:

$$\frac{d^2 x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx^i}{dt}\frac{dx^j}{dt} = 0.$$

If $X, Y \in \mathfrak{X}(M)$ have local representations

$$X = \sum_{i=1}^{n} X^i(x)\frac{\partial}{\partial x^i} \text{ and } Y = \sum_{i=1}^{n} Y^i(x)\frac{\partial}{\partial x^i},$$

then the local representation of $\nabla_X Y$ is

$$\nabla_X Y = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} X^j\frac{\partial Y^k}{\partial x^j} + \sum_{i,j=1}^{n}\Gamma_{ji}^k X^j Y^i\right)\frac{\partial}{\partial x^k}$$

and $\nabla_X Y$ is called the *covariant derivative* of Y with respect to X.

The *Lie bracket* of ordered pair of X and Y that acts on a smooth function f is defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f))).$$

The local representation of the Lie bracket is

$$[X, Y] = \sum_{i,j=1}^{n}\left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}.$$

If V is a vector field, the derivation L_V such that

1. $L_V(f) = Vf$ for all $f \in \mathfrak{F}(M)$ and
2. $L_V(X) = [V, X]$ for all $X \in \mathfrak{X}(M)$

is called the *Lie derivative* relative to V.

Definition 2.1.2. A vector field X is called a *Killing vector field* if $L_X g = 0$, i.e., the Lie derivative of the metric tensor vanishes.

The function $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

is called as the *torsion tensor* of ∇. In local coordinates, it is expressed as

$$T = \sum_{i,j,k=1}^{n} T_i{}^k{}_j dx^i \otimes \frac{\partial}{\partial x^k} \otimes dx^j,$$

where the *torsion components* are

$$T_{i\ j}^{\ k} = \Gamma^k_{ij} - \Gamma^k_{ji}.$$

The function R which assigns to $X, Y \in \mathfrak{X}(M)$ the f-linear map $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is called *curvature* and defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Let $\omega \in T_p^* M$ be a cotangent vector at p and $x, y, z \in T_p M$ are tangent vectors at p. A (1,3) vector field

$$R(w,x,y,z) = (w, R(X,Y)Z) = w(R(X,Y)Z)$$

is called as the *curvature tensor* and in local coordinates it is expressed as

$$R = \sum_{i,j,k,m=1}^{n} R^i_{jkm} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^m,$$

where the *curvature components* are

$$R^i_{jkm} = \frac{\partial \Gamma^i_{mj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^m} + \sum_{a=1}^{n} (\Gamma^a_{mj} \Gamma^i_{ka} - \Gamma^a_{kj} \Gamma^i_{ma}).$$

2.2 Semi-Riemannian Manifolds

In this section we will use [1] and [6].

Definition 2.2.1. Let b be a symmetric bilinear form on a vector space V. The largest integer that is the dimension of a subspace $W \subset V$ such that $b|W$ is negative definite is called the *index* of b.

Definition 2.2.2. A symmetric nondegenerate $(0,2)$ tensor field g on M of constant index is called a *metric tensor*.

So, for all $p \in M$, $g \in \mathbf{T}^0_2(M)$ smoothly assigns to each p a scalar product g_p and each g_p has the same index. Nondegenerate means that for any $v \in T_p M$, there is some $w \in T_p M$ such that $g_p(v,w) \neq 0$. If g_{ij} are components of g in local coordinates, then nondegeneracy is equivalent to the condition that $det(g_{ij}) \neq 0$.

Definition 2.2.3. A smooth manifold M furnished with a metric tensor g is called a *semi-Riemannian manifold.*

The index ν of g on a semi-Riemannian manifold M, where $0 \leq \nu \leq n = \dim M$, is called the *index of M.* If $\nu = 0$, then M is a Riemannian manifold and for all $p \in M$, the metric induced on the tangent space is Euclidean. If $\nu = 1$, then M is a Lorentzian manifold and for all $p \in M$, the metric induced on the tangent space is Minkowskian.

Example 2.2.4. The dot product on $\mathbb{R}^n$

$$\langle v_p, w_p \rangle = v \cdot w = \sum v^i w^i$$

where $v_p = \sum v^i \partial_i$ is a metric tensor on $\mathbb{R}^n$. Hence, $\mathbb{R}^n$ is a Riemannian manifold, and it is called the *Euclidean n-space.*

Example 2.2.5. For any integer ν with $0 \leq \nu \leq n$, there is a metric tensor

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{\nu} v^i w^i + \sum_{j=\nu+1}^{n} v^j w^j$$

on $\mathbb{R}^n$ of index ν. The resulting semi-Riemannian manifold is called *semi-Euclidean space* and denoted by $\mathbb{R}^n_\nu$. For $\nu = 0$, we observe the metric tensor in Example 2.2.4. For $n \geq 2$ and $\nu = 1$, $\mathbb{R}^n_1$ is called *Minkowski n-space.*

The metric tensor of $\mathbb{R}^n_\nu$ can be written as

$$g = \sum \epsilon_i du^i \otimes du^i,$$

where

$$\epsilon_i = \begin{cases} -1 & \text{for } 1 \leq i \leq \nu, \\ +1 & \text{for } \nu \leq i \leq n. \end{cases}$$

Definition 2.2.6. (i) Two tangent vectors $v, w \in T_pM$ are called *orthogonal* if $g(v, w) = 0$ and denoted by $v \perp w$.

(ii) A vector $v \neq 0$ is called as a *null vector* if $g(v, v) = 0$. The set of all null vectors in T_pM is called *null cone* at $p \in M$.

(iii) If $g(v,v)=\pm 1$, then v is a *unit vector.*

(iv) *Norm* of a vector v is defined as $|v|=\sqrt{|g(v,v)|}$.

Definition 2.2.7. For a semi-Riemannian manifold (M,g), there is a unique connection ∇ called *Levi-Civita connection* such that for all $X,Y,Z\in\mathfrak{X}(M)$,

1. $Z(g(X,Y))=g(\nabla_Z X,Y)+g(X,\nabla_Z Y)$,
2. $[X,Y]=\nabla_X Y-\nabla_Y X$.

Let g^{ij} represents the (2,0) tensor given by

$$\sum_{a=1}^{n} g^{ia}g_{aj}=\delta^i_j$$

for $1\leq i,j\leq n$. The *connection coefficients* of a semi-Riemannian manifold is given by

$$\Gamma^k_{ij}=\frac{1}{2}\sum_{a=1}^{n} g^{ak}\left(\frac{\partial g_{ia}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^a}+\frac{\partial g_{aj}}{\partial x^i}\right).$$

Definition 2.2.8. The *Riemann-Christoffel tensor* $\tilde{R}$ is a (0,4) tensor such that

$$\tilde{R}(W,Z,X,Y)=g(W,R(X,Y)Z).$$

The components of Riemann-Christoffel tensor, or *covariant curvature components* are

$$R_{ijkm}=\sum_{a=1}^{n} g_{ai}R^a_{jkm}.$$

Definition 2.2.9. For each $p\in M$, the *Ricci curvature* is a symmetric bilinear map $\mathrm{Ric}_p : T_pM\times T_pM\to\mathbb{R}$ defined as

$$\begin{aligned}\mathrm{Ric}(v,w)&=\sum_{i=1}^{n} g(e_i,e_i)\tilde{R}(e_i,v,e_i,w)\\&=\sum_{i=1}^{n} g(e_i,e_i)g(R(e_i,w)v,e_i),\end{aligned}\qquad(2.2.1)$$

where $\{e_1,...,e_n\}$ is an orthonormal basis for T_pM.

If v and w are expressed in local coordinates such that

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} \text{ and } w = \sum_{i=1}^{n} w^i \frac{\partial}{\partial x^i},$$

then the Ricci curvature can be expressed as

$$\mathrm{Ric}(v, w) = \sum_{i,j=1}^{n} R_{ij} v^i w^j,$$

where

$$R_{ij} = \sum_{a=1}^{n} R^a_{iaj}$$

are the *Ricci curvature components.*

Definition 2.2.10. If $f \in \mathfrak{F}(M)$, df is a (0,1) tensor field on M. For any vector field Y, the *gradient* of f, denoted by $\mathrm{grad} f$, is the (1,0) tensor field defined by

$$Y(f) = df(Y) = g(\mathrm{grad} f, Y).$$

In local coordinates,

$$\mathrm{grad} f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

Definition 2.2.11. The Hessian is a symmetric (0,2) tensor field defined as

$$H^f = \nabla(\nabla f),$$

i.e., the second covariant differential of f.

The formula of H^f related to $\mathrm{grad} f$ for arbitrary vector fields X and Y is

$$H^f(X, Y) = g(\nabla_X(\mathrm{grad} f), Y).$$

Definition 2.2.12. The *Laplacian* is defined as

$$\Delta f = div(gradf),$$

where $div(\mathrm{grad} f)$ denotes the divergence of the gradient of f.

Definition 2.2.13. The trace of the Ricci curvature is called *scalar curvature* and denoted by τ. That is,

$$\tau = \sum_{a=1}^{n} R_a^a.$$

Scalar curvature can be defined by

$$\tau = \sum_{i=1}^{n} g(e_i, e_i)\,\mathrm{Ric}(e_i, e_i),$$

where $e_1, ..., e_n$ is an orthonormal basis.

Corollary 2.2.14. *[6]* $d\tau = 2div\mathrm{Ric}$.

For indefinite metric tensors, we can define three types of vector classes.

Definition 2.2.15. A tangent vector v to M is

1. *spacelike* if $g(v, v) > 0$ or $v = 0$,
2. *null* or *lightlike* if $g(v, v) = 0$ and $v \neq 0$,
3. *timelike* if $g(v, v) < 0$.

Null and timelike vectors are also called *nonspacelike* or *causal*.

A curve γ on M is *spacelike* (respectively, *timelike, null*) if all of its tangent vectors γ' are spacelike (respectively, timelike, null).

Lemma 2.2.16. *(Lemma 2.1, p.26) [1] Let V be a real vector space of dimension $n \geq 2$. Let g and h be two nondefinite nondegenerate arbitrary inner products on V. Suppose for any $v \in V$ that*

$$g(v, v) = 0 \iff h(v, v) = 0.$$

Then

1. g has the same index as either h or $-h$,

2. *there exists a number* $\lambda \neq 0$ *such that for any* $v, w \in V$

$$h(v,w) = \lambda g(v,w).$$

Furthermore, if the index of g *is* ν *and* $\nu \neq n - \nu$*, then*

1. $\lambda > 0$ *if* g *and* h *have the same index,*

2. $\lambda < 0$ *if* g *and* $-h$ *have the same index.*

Corollary 2.2.17. *(Corollary 2.2, p.28) [1] Let* V *be a real vector space of dimension* $n \geq 3$*. Let* g *and* h *be two inner products on* V *of index 1. Suppose* g *and* h *have the same null vectors. Then there exists a constant* $\lambda > 0$ *such that* $h = \lambda g$*.*

Theorem 2.2.18. *(Theorem 2.3, page 28) [1] Let* M *be a smooth manifold of dimension* $n \geq 3$*. Let* g *and* h *be two Lorentzian metrics on* M*. Suppose for all* $v \in TM$ *that* g *and* h *satisfy the condition*

$$g(v,v) = 0 \iff h(v,v) = 0.$$

Then, there exists a smooth function $\Omega : M \to (0,\infty)$ *such that for any* $v, w \in TM$

$$h(v,w) = \Omega g(v,w).$$

Definition 2.2.19. Let $F : (M,g) \to (N,h)$ be a global diffeomorphism between two semi-Riemannian manifolds. If there exists a smooth function $\Omega : M \to (0,\infty)$ such that for all $v, w \in T_pM$ and $p \in M$

$$h(F_*v, F_*w) = \Omega(p)g(v,w),$$

then F is called a *global conformal transformation.*

Corollary 2.2.20. *(Corollary 2.4, p.29) [1] Let* (M,g) *and* (N,h) *be two Lorentzian manifolds of dimension* $n \geq 3$*. If* $F : M \to N$ *satisfies*

$$g(v,v) = 0 \iff h(F_*v, F_*v) = 0,$$

then F *is a global conformal transformation of* (M,g) *onto* (N,h)*.*

2.2.1 Geodesics on a Semi-Riemannian Manifold

Definition 2.2.21. Let $\gamma : M \to (0, \infty)$ be a curve. Then the *speed* of γ is defined as γ'. If $|\gamma'| = 1$, then γ is said to have *unit speed parametrization* (or *arc length parametrization*).

Definition 2.2.22. Let M be a semi-Riemannian manifold and $\gamma : [a, b] \to M$ be a piecewise smooth curve segment in M. The arc length of γ is

$$L(\gamma) = \int_a^b |\gamma'(s)| ds.$$

Lemma 2.2.23. *(Lemma 12, p.131) [6]*

1. *A monotone reparametrization does not change the length of a piecewise smooth curve segment.*
2. *If γ is a curve segment and $|\gamma| > 0$, then there is a strictly increasing reparametrization function ϕ such that $\tilde{\gamma} = \gamma(\phi)$ is a unit speed curve.*

Proposition 2.2.24. *(Proposition 24 p.68) [6] Let $v \in T_pM$. Then there is a unique geodesic γ_v in M such that*

1. *The initial velocity of γ_v is v, i.e. $\gamma_v'(0) = v$.*
2. *The domain I_v of γ_v is the largest possible. If $\alpha : J \to M$ is a geodesic with initial velocity v, then $J \subset I$ and $\alpha = \gamma_v|J$.*

Because of 2., the geodesic γ_v is said to be *maximal* or *geodesically inextendible.*

Definition 2.2.25. Let $p \in M$ and $\mathcal{D}_p$ be the set of vectors $v \in T_pM$ such that the inextendible geodesic γ_v is defined at least on $[0, 1]$. The *exponential map* of M at p is the function

$$exp_p : \mathcal{D}_p \to M$$

such that $exp_p(v) = \gamma_v(1)$ for all $v \in \mathcal{D}_p$.

exp_p carries lines through the origin of T_pM to geodesics of M through p.

Proposition 2.2.26. *(Proposition 30, p.71) [6] Let $p \in M$. Then there exists a neighborhood $\tilde{\mathcal{U}}$ of 0 in T_pM on which the exponential map exp_p is a diffeomorphism onto a neighborhood $\mathcal{U}$ of p in M.*

Let $S \subset V$ where V is a vector space. If $v \in S$ implies for all $0 \leq t \leq 1$, then S is called *starshaped* about 0. If $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are defined as in Proposition 2.2.18 and $\tilde{\mathcal{U}}$ is starshaped about 0, then $\mathcal{U}$ is a *normal neighborhood* of p.

Let $q \in \mathcal{U}$, and $v = exp_p^{-1}(q)$. Clearly, $v \in \tilde{\mathcal{U}}$ and the map $\rho(t) = tv$ lies in $\tilde{\mathcal{U}}$. Then, the geodesic segment $\sigma = exp_p \circ \rho$ which lies in $\mathcal{U}$ and runs from p to q. This geodesic segment is called as *radial geodesic*.

Proposition 2.2.27. *(Proposition 31, p.72) [6] Let $\mathcal{U}$ be a normal neighborhood of $p \in M$. Then for any $q \in \mathcal{U}$, there exists a unique radial geodesic $\sigma : [0,1] \to \mathcal{U}$ from p to q in $\mathcal{U}$. Furthermore, $\sigma'(0) = exp_p^{-1}(p) \in \tilde{\mathcal{U}}$.*

Definition 2.2.28. Let $\mathcal{U}$ be a normal neighborhood of a point $p \in M$. For $q \in \mathcal{U}$, the function

$$r(q) = |exp_p^{-1}(q)|$$

is called as *radius function* of M at p.

Lemma 2.2.29. *(Lemma 13, p.132) [6] Let r be the radius function, $\mathcal{U}$ be a normal neighborhood of p and let $q \in \mathcal{U}$. If σ is the radial geodesic from p to q, then $L(\sigma) = r(q)$.*

2.2.2 Sectional Curvature

Let (M, g) be a semi-Riemannian manifold. A *plane section* is a two-dimensional linear subspace E of T_pM. For Lorentzian manifolds, degenerate planes are called *null* (or *lightlike*).

Let E be a plane section of a Lorentzian manifold and $\{v, w\}$ be a basis for E. Then we have the following classification:

1. timelike plane if $g(v,v)g(w,w)-[g(v,w)]^2<0$,
2. degenerate plane if $g(v,v)g(w,w)-[g(v,w)]^2=0$,
3. spacelike plane if $g(v,v)g(w,w)-[g(v,w)]^2>0$.

Let E be a nondegenerate plane section with basis $\{v,w\}$, where

$$v=\sum v^i\frac{\partial}{\partial x^i} \text{ and } w=\sum w^i\frac{\partial}{\partial x^i}.$$

The *sectional curvature* of E is given by

$$\begin{aligned} K(p,E) &= \frac{g(R(w,v)v,w)}{g(v,v)g(w,w)-[g(v,w)]^2} \\ &= \frac{\tilde{R}(w,v,w,v)}{g(v,v)g(w,w)-[g(v,w)]^2} \\ &= \frac{\sum R_{ijkm}w^iv^jw^kv^m}{\sum g_{ij}v^iv^jg_{km}w^kw^m-[\sum g_{ij}v^iw^j]^2}. \end{aligned} \tag{2.2.2}$$

Let w be a unit vector at $p\in M$. For $\{e_1,...,e_{n-1},e_n=w\}$ an orthonormal basis, let E_i =span$\{e_i,w\}$, $1\leq i\leq n-1$. If (M,g) is a Riemannian manifold, then by equations 2.2.1 and 2.2.2,

$$\text{Ric}(w,w)=\sum_{i=1}^{n-1}g(R(e_i,w)w,e_i)=\sum_{i01}^{n-1}K(p,E_i).$$

If (M,g) is a Lorentzian manifold and w is a unit timelike vector, then by equations 2.2.1 and 2.2.2,

$$\text{Ric}(w,w)=\sum_{i=1}^{n-1}g(R(e_i,w)w,e_i)=-\sum_{i01}^{n-1}K(p,E_i).$$

Let $W=\sum_{a=1}^{n}W^a\frac{\partial}{\partial x^a}$ be a tangent vector. Then W^a are called *contravariant components*. The values $W_b=\sum_{a=1}^{n}g_{ab}W^a$ are *covariant components*. So, $\sum_{b=1}^{n}W_bdx^b$ is cotangent to W. If

$$\sum_{c,d=1}^{n}W^cW^dW_{[a}R_{b]cd[e}W_{f]}\neq 0,$$

then the *generic condition* is said to be satisfied for a vector W at $p \in M$. If this condition does not hold, then W is *nongeneric*.

Lemma 2.2.30. *(Lemma 2.5, p.33) [1] If R_{abcd} are the components of the curvature tensor with respect to an orthonormal basis $\{v_1, ..., v_n\}$ of T_pM and, then $W = v_n$ satisfies the generic condition if and only if there exist b and e with $1 \leq b, e \leq n-1$ and $R_{bnne} = 0$.*

Proposition 2.2.31. *(Proposition 2.6, p.34) [1] Let $W \in T_pM$ be a nonnull vector. Then, for all nondegenerate plane section E containing W, the sectional curvature $K(p, E)$ vanishes if and only if W is nongeneric.*

Proposition 2.2.32. *(Proposition 2.8, p.36) [1] Let $W \in T_pM$ be a nonnull vector with $\mathrm{Ric}(W, W) \neq 0$. Then, W is generic.*

2.3 Riemannian Manifolds

Lemma 2.3.1. *(Lemma 14, p.133) [6] Let M be a Riemannian manifold, $\mathcal{U}$ be a normal neighborhood of $p \in M$ and let $q \in \mathcal{U}$. Then the radial geodesic segment $\sigma : [0, 1] \to \mathcal{U}$ from p to q is the unique shortest curve in $\mathcal{U}$.*

Definition 2.3.2. Let (M, g) be a Riemannian manifold. Let $\Omega_{p,q}$ be the set of piecewise smooth curves in M from p to q. The *Riemannian distance* $d : M \times M \to [0, \infty)$ is defined by

$$d(p, q) = \inf\{L(\gamma) : \gamma \in \Omega_{p,q}\} \geq 0.$$

The function $d : M \times M \to [0, \infty)$ has the following properties:

1. $d(p, q) = 0 \iff p = q$,
2. $d(p, q) = d(q, p)$ for all $p, q \in M$,
3. $d(p, q) \leq d(p, r) + d(r, q)$ for all $p, q, r \in M$.

Thus, (M, d) is a metric space.

The set $\mathcal{N}_\epsilon(p) = \{q \in M : d(p,q) < \epsilon\}$ is called the *ϵ-neighborhood of p*.

Proposition 2.3.3. *(Proposition 16) [6] Let M be a Riemannian manifold and $p \in M$. Then,*

1. *$\mathcal{N}_\epsilon(p)$ is normal for $\epsilon > 0$ sufficiently small.*
2. *Let $\mathcal{N}_\epsilon(p)$ be normal and $q \in \mathcal{N}_\epsilon(p)$. Then the radial geodesic σ from p to q is the unique shortest curve in M from p to q and*

$$L(\sigma) = r(q) = d(p,q).$$

A curve segment γ from p to q is called a *minimizing curve segment from p to q* if $L(\gamma) = d(p,q)$.

2.3.1 Geodesic Completeness of Riemannian Manifolds

Definition 2.3.4. A Riemannian manifold (M, g) is called *geodesically complete* if for all $p \in M$, the exponential map exp_p is defined in T_pM.

For the proofs of the following, [6] should be checked.

Theorem 2.3.5 (Hopf-Rinow)**.** *(Theorem 21, p.138) [6] For a connected Riemannian manifold M, the following conditions are equivalent:*

1. *M is complete as a metric space under Riemannian distance d.*
2. *There exists a point $p \in M$ from which M is geodesically complete.*
3. *M is geodesically complete.*
4. *Every closed bounded subset of M is compact.*

Proposition 2.3.6. *(Proposition 22, p.138) [6] If M is a complete connected Riemannian manifold, then for any $p, q \in M$, there exists a minimizing geodesic segment which joins p and q.*

A Riemannian manifold M that has a minimizing geodesic segment defined as in Proposition 2.3.6, is called *convex*. If we throw the minimizing condition, we will call it *geodesically connected*. So, by Proposition 2.3.6, every complete Riemannian manifold is convex.

Corollary 2.3.7. *(Corollary 23, p.138) [6] A compact Riemannian manifold is complete.*

2.4 Lorentzian Causality

This section contains a summary of the references [1], [6] and [7].

Let V be a Lorentzian subspace and let W is a subspace of V. If g is a scalar product on V, then one and only one of the following types is valid for W:

1. $g|W$ is positive definite. Then W is *spacelike*.
2. $g||W$ is nondegenerate of index 1. Then W is *timelike*.
3. $g|W$ is degenerate. Then W is *null* or *lightlike*.

The type of W is called as the *causal character of W*. If $v \in V$, then the causal character of the subspace $span\{v\}$ is the same as the causal character of the vector v.

Proposition 2.4.1. *(Proposition 1.1.3, p.20) [7] Let $W \in V$ be a subspace.*

1. *W is timelike (spacelike) if and only if $W^{\perp}$ is spacelike (timelike).*
2. *W is lightlike if and only if $W \cap W^{\perp} \neq \{0\}$ if and only if $W^{\perp}$ is lightlike.*

Lemma 2.4.2. *(Lemma 26, p.141) [6] If v is a timelike (respectively, spacelike, lightlike) vector, then $span\{v\}^\perp$ is spacelike (respectively, timelike, lightlike) subspace and $V = span\{v\} \oplus span\{v\}^\perp$.*

Corollary 2.4.3. *(Corollary 1.1.5, p.20) [7] Two lightlike vectors are orthogonal if and only if they are collinear.*

Lemma 2.4.4. *(Lemma 27, p.141) [6] If W is a subspace of dimension ≥ 2 in a Lorentzian vector space, then the following are equivalent:*

1. *W is timelike, so as a vector space, W is Lorentzian.*
2. *W contains a timelike vector.*
3. *W contains two linearly independent lightlike vectors.*

Lemma 2.4.5. *(Lemma 28, p.142) [6] If W is a subspace of a Lorentzian vector space, then the following are equivalent.*

1. *W is lightlike, i.e, degenerate.*
2. *W contains a lightlike vector but does not contain a timelike vector.*
3. *$W \cap \Lambda = L - \{0\}$, where Λ is the nullcone of V and L is a one-dimensional subspace.*

Let $\mathcal{T}$ denote the set of all timelike vectors in a Lorentzian vector space V. Then for any $v \in \mathcal{T}$,

$$C(v) = \{w \in \mathcal{T} : g(v, w) < 0\}.$$

is the *timecone* of V containing v. The *opposite* timecone is

$$C(-v) = -C(v) = \{w \in \mathcal{T} : g(v, w) > 0\}.$$

Lemma 2.4.6. *(Lemma 29, p.143) [6] Two timelike vectors v and w in a Lorentz vector space V are in the same timecone iff $g(v, w) < 0$. So, $w \in C(v)$ if and only if $v \in C(w)$ if and only if $C(v) = C(w)$*

Proposition 2.4.7. *(Lemma 30, p.143) [6] Let V be a Lorentzian vector space and let v and w be two timelike vectors in V. Then*

1. $|g(v,w)| \geq |v||w|$, *with equality if and only if v and w are collinear.*
2. *Let v and w be two timelike vectors in the same timecone. Then $|v| + |w| \leq |v+w|$, with equality if and only if v and w are collinear.*
3. *If v and w are in the same timecone of V, then there is a unique number $\phi \geq 0$ such that*
$$g(v,w) = -|v||w| \cosh \phi$$

The number ϕ in (3) above is called the *hyperbolic angle* between v and w.

A vector field X on a manifold M is called *timelike* if $g(X,X) < 0$ at all $p \in M$. A Lorentzian manifold (M,g) is called *time oriented* if (M,g) admits a timelike vector field $X \in \mathfrak{X}(M)$.

The vector field X divides all causal vectors into two distinct classes:

Definition 2.4.8. Let M be a time-orientable Lorentzian manifold and X be a smooth vector field that fixes a time orientation on M. For any $p \in M$, a nonzero causal vector $v \in T_pM$ is called *future directed* (respectively, *past directed*) if $g(X,v) < 0$ (respectively, $g(X,v) > 0$).

2.5 Causality Theory of Space-Times

For a detailed research, one can use [1].

Definition 2.5.1. A *space-time* is a Lorentzian manifold with a time orientation.

Let (M,g) be a space-time and $p, q \in M$. We will write

1. $p \ll q$ if there is a smooth future directed timelike curve from p to q,

2. $p \leqslant q$ if either there is a smooth future directed nonspacelike curve from p to q or $p = q$.

Let $p, q, r \in M$. If $p \ll q$ and $q \ll r$ (respectively, $p \leqslant q$ and $q \leqslant r$), then $p \ll r$ (respectively, $p \leqslant r$). If $p \ll q$ and $q \leqslant r$ (or if $p \leqslant q$ and $q \ll r$), then $p \ll r$.

Definition 2.5.2. Let (M, g) be a space-time and $p, q \in M$.

1. The *chronological past* of p is defined as $I^-(p) = \{q \in M : q \ll p\}$,
2. The *chronological future* of p is defined as $I^+(p) = \{q \in M : p \ll q\}$,
3. The *causal past* of p is defined as $J^-(p) = \{q \in M : q \leqslant p\}$,
4. The *causal future* of p is defined as $J^+(p) = \{q \in M : p \leqslant q\}$.

Lemma 2.5.3. *(Lemma 3.5, p.56) [1] For any $p \in M$, $I^-(p)$ and $I^+(p)$ are open sets of M.*

In general, $J^-(p)$ and $J^+(p)$ are neither open nor closed.

Definition 2.5.4. If $p \in I^+(p)$, then there is a closed timelike curve through p. Such a space-time is said to have a *causality violation*.

Definition 2.5.5. Let (M, g) be a space-time.

1. (M, g) is called *chronological* if it does not contain any closed timelike curves.
2. (M, g) is called *causal* if it does not contain any closed nonspacelike curves.

Proposition 2.5.6. *(Proposition 3.10, p.58) [1] If a compact space-time (M, g) contains a closed timelike curve, it cannot be chronological.*

Definition 2.5.7. Let $p, q \in M$. If either $I^-(p) = I^-(q)$ or $I^+(p) = I^+(q)$ implies $p = q$, then the space-time (M, g) is called *distinguishing*.

Definition 2.5.8. Let $U \subset M$ be an open set. If no nonspacelike curve intersects U in a disconnected set, then U is called *causally convex.*

Definition 2.5.9. Let $p \in M$. If p has arbitrarily small causally convex neighborhoods, then the space-time (M, g) is called *strongly causal at p.* If (M, g) is strongly causal at every point of M, then the space-time is called *strongly causal.*

Definition 2.5.10. Let $\gamma : [a, b) \to M$ be a curve in M and let $p \in M$. If

$$\lim_{t \to b^-} \gamma(t) = p,$$

then p is called the *endpoint* of γ corresponding to $t = b$.

Definition 2.5.11. A nonspacelike curve which has no future (respectively, past) endpoint is called *future* (respectively, *past*) *inextendible.* A nonspacelike curve γ which is both future and past inextendible is said to be *inextendible.*

Definition 2.5.12. An inextendible curve γ is called *imprisoned* if it has a compact closure.

Let $\gamma : [a, b) \to M$ be a future directed nonspacelike curve and K be a compact set. If there is some $t_0 < b$ such that $\gamma \in K$ for all $t_0 < t < b$, then γ is called *future imprisoned* in K. If there is an infinite sequence $t_n \uparrow b$ with $\gamma(t_n) \in K$ for all n, then γ is called *partially future imprisoned* in K.

Proposition 2.5.13. *(Proposition 3.13, p.63) [1] If (M, g) is a strongly causal space-time, then there is no inextendible nonspacelike curve which is future or past imprisoned in any compact set.*

Let $\mathrm{Lor}(M)$ denote the space of all Lorentzian metrics on M. Let $\mathcal{B} = \{B_i\}$ be a fixed countable covering of M by coordinate neighborhoods such that each compact subset of M intersects only finitely many of B_i's and with the property that the closure of B_i lies in a coordinate of M. The *fine C^r topologies* on $\mathrm{Lor}(M)$ can be defined by using the finite covering $\mathcal{B}$.

Definition 2.5.14. Let $U(g)$ be a fine C^0 neighborhood of g in $\mathrm{Lor}(M)$ such that each g_1 is causal. Then then the space-time (M, g) is called *stably causal.*

Definition 2.5.15. If for each pair $p, q \in M$ the set $J^+(p) \cap J^-(q)$ is compact, then a strongly causal space-time (M, g) is called *globally hyperbolic.*

A space-time is (M, g) is *causally simple* if it is distinguishing and $J^+(p)$ and $J^-(p)$ are closed in M.

Proposition 2.5.16. *(Proposition 3.16) [1] If a space-time is globally hyperbolic, then it is causally simple.*

Remark 2.5.17. For a space-time, globally hyperbolic $\Longrightarrow$ causally simple $\Longrightarrow$ stably causal $\Longrightarrow$ strongly causal $\Longrightarrow$ distinguishing $\Longrightarrow$ causal $\Longrightarrow$ chronological.

Theorem 2.5.18. *(Theorem 3.18, p.66) [1] Let (M, g) be globally hyperbolic and $p, q \in M$ with $p \leqslant q$. Then there is a nonspacelike geodesic from p to q with length greater than or equal to the length of any other future directed nonspacelike curve from p to q.*

Definition 2.5.19. Let $p \in M$. The space-time (M, g) is called *vicious* at p if $I^+(p) \cup I^-(p) = M$. (M, g) is called *totally vicious* if it is vicious at every point of M.

2.6 Lorentzian Distance

In this section we will review some facts from [1].

Let (M, g) be a Lorentzian manifold of dimension $n \geq 2$ and $p, q \in M$ with $p \leqslant q$. $\Omega_{p,q}$ will denote the set of all future directed piecewise smooth nonspacelike curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Let $\gamma \in \Omega_{p,q}$ and let $0 = t_0 < ... < t_{n-1} < t_n = 1$ be a partition such that $\gamma|(t_i, t_{i-1})$ is smooth for $i = 0, 1, ..., n - 1$.

The *Lorentzian arc length* $L = L_g : \Omega_{p,q} \to \mathbb{R}$ is defined as

$$L(\gamma) = L_g(\gamma) = \sum_{i=0}^{n-1} \int_{t_{i+1}}^{t_i} \sqrt{-g(\gamma'(t), \gamma'(t))}dt.$$

Definition 2.6.1. For any $p \in M$ the *Lorentzian distance* $d = d(g) : M \times M \to \mathbb{R} \cup \{\infty\}$ of (M, g) is

$$d(p,q) = \begin{cases} 0 & \text{if } q \notin J^+(p) \\ \sup\{L_g(\gamma) : \gamma \in \Omega_{p,q}\} & \text{if } q \in J^+(p). \end{cases}$$

Lemma 2.6.2. *(Lemma 4.2, p.137) [1] For any space-time (M, g),*

1. *If $p \in I^+(p)$, then $d(p,p) = \infty$. So, either $d(p,p) = 0$ or $d(p,p) = \infty$ for all $p \in M$.*
2. *$d(p,q) = \infty$ for all $p, q \in M$ iff (M, g) is totally vicious.*
3. *If (M, g) is vicious at p, then (M, g) is totally vicious.*

Remarks 2.6.3.
1. If $p \neq q$ and $d(p,q)$ and $d(q,p)$ are finite, then either $d(p,q) = 0$ or $d(q,p) = 0$.
2. If $d(p,q) > 0$ and $d(q,p) > 0$, then $d(p,q) = d(q,p) = \infty$.

Definition 2.6.4. If for any $p, q \in M$ with $d(p,q) < \infty$, then the space-time (M, g) is said to satisfy the *finite distance condition.*

Corollary 2.6.5. *(Corollary 4.7, p.142) [1] Let (M, g) be a globally hyperbolic space-time. Then $d : M \times M \to \mathbb{R}$ is continuous and (M, g) satisfies the finite distance condition.*

Definition 2.6.6. Let $p, q \in M$ with $p \leqslant q$ and $p \neq q$. If $\gamma \in \Omega_{p,q}$ and $L(\gamma) = d(p,q)$, then γ is called *maximal.*

Corollary 2.6.7. *(Corollary 4.17, p.147) [1] If $p \leqslant q$ but $p \not\ll q$, then there exists a maximal null geodesic which joins p to q.*

Lemma 2.6.8. *(Lemma 4.23, p.158) [1] Let (M, g) be a space-time and $p, q \in M$.*

1. *$q \in I^+(p)$ iff $d(p,q) > 0$.*
2. *$d(p,q) = \infty$ iff (M, g) is totally vicious.*

3. *d is 0 on the diagonal $\Delta(M) = \{(p,p) : p \in M\}$ of $M \times M$ iff (M,g) is chronological.*

4. *There is some $x \in M$ such that exactly one of $d(p,x)$ and $d(q,x)$ (respectively, $d(x,p)$ and $d(x,q)$) is zero iff (M,g) is future (respectively, past) distinguishing.*

5. *There exists a neighborhood U of g in the fine C^0 topology on $\mathrm{Lor}(M)$ such that for all $g' \in U$ and $p \in M$ $d(g')(p,p) = 0$iff (M,g) is stably causal.*

Definition 2.6.9. Let $\gamma : [a,b] \to (M,g)$ be a future directed nonspacelike curve. If $L(\gamma) = d(\gamma(a), \gamma(b))$ and $L(\gamma|[s,t]) = d(\gamma(s), \gamma(t))$ for all $s, t \in [a,b]$, then γ is called *maximal segment.*

Lemma 2.6.10. *(Lemma 4.36, p.168) [1] If $\gamma : [0,1] \to (M,g)$ is a maximal null geodesic segment and $I(\gamma)$ denotes the domain of γ extended to be a future inextendible null geodesic starting from 0. Then*

1. *For $r \in J^+(\gamma(s)) \cap J^-(\gamma(t))$ and s, t with $0 \leq s < t \leq 1$,*

$$d(\gamma(s), r) = d(r, \gamma(t)) = 0$$

   and r lies on $\gamma(I(\gamma))$.

2. *If $p, q \in J^+(\gamma(s)) \cap J^-(\gamma(t))$, then $d(p,q) = 0$,*

3. *Chronology holds at all points of $J^+(\gamma(0)) \cap J^-(\gamma(1))$.*

Lemma 2.6.11. *(Lemma 4.37, p.169) [1] If $\gamma : [0,1] \to (M,g)$ is a causal maximal segment, then*

1. *For $p, q \in J^+(\gamma(0)) \cap J^-(\gamma(1))$ with $p \leqslant q$, the distance $d(p,q)$ is finite.*

2. *Chronology holds at all points of $J^+(\gamma(0)) \cap J^-(\gamma(1))$.*

Lemma 2.6.12. *(Lemma 4.38, p.170) [1] If $\gamma : [0,1] \to (M,g)$ is a maximal timelike segment, then causality holds at all points of $\gamma([0,1])$.*

Lemma 2.6.13. *(Lemma 4.39, p.170) [1] Let $p \in (M, g)$. The strong causality of (M, g) fails at p iff there exists a point $q \in J^{-}(p)$ with $q \neq p$ such that for $x, y \in M$ $x \leqslant p$ and $q \leqslant y$ implies $x \leqslant y$..*

Proposition 2.6.14. *(Proposition 4.40, p.170) [1] If $\gamma : [0, a] \to (M, g)$ is a maximal timelike geodesic segment, then for any $t_0 \in [0, a]$, the causality holds at $p = \gamma(t_0)$.*

2.7 Riemannian and Lorentzian Geometry

In this section we will give a brief comparison of Riemannian and Lorentzian geometry.

1. The Riemannian metrics are always positive definite while the Lorentzian metrics are indefinite.
2. Lorentzian manifolds have the concept of spacelike, timelike and lightlike vectors and curves.
3. A Lorentzian manifold can have time orientation.
4. For two vectors, Riemannian geometry gives triangle inequality and the angle betveen them. In Lorentzian geometry, reverse triangle inequality holds and the hyperbolic angle between two vectors can be defined.
5. Lorentzian distance $d(p, q)$ from the point p to the point q is defined as 0 unless there is a future directed piecewise smooth nonspacelike curve from p to q.
6. The statements of Hopf-Rinow theorem do not hold in Lorentzian geometry. For example (see [5]), let $\mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus with the projection of the metric $g = -2dudv + (\cos^4(v) - 1)du^2$. Let $u = x + t$, $v = x - t$. Then the geodesic $\gamma(t) = (1/t - t, \arctan(t))$ is incomplete. So, compactness does not imply completeness.

7. Lorentzian completeness can be studied in weaker classes (timelike, null and spacelike geodesic conpleteness) as well as stronger classes (b.a completeness, b-completeness will be given in Section 4.1.).

Chapter 3

Warped Products

3.1 Riemannian Warped Products

A detailed explanation about this subject can be found in [1] and [6].

Let (M, g) and (H, h) be Riemannian manifolds. Then there is a natural product metric g_0 on the product manifold $M \times H$ such that $(M \times H, g_0)$ is also a Riemannian manifold.

If $f : M \to (0, \infty)$ is a smooth function, then the product manifold

$$(M \times H, g \oplus f^2 h)$$

is called a *warped product* and shown as $M \times_f H$. The function f is called the *warping function*.

M is called the *base* of the warped product and H is called the *fiber* of the warped product. The *fibers* and *leaves* of the warped product are defined as $p_1 \times H = \pi^{-1}(p_1)$ and $M \times p_2 = \eta^{-1}(p_2)$, respectively, where $p_1 \in M$ and $p_2 \in H$. The vectors tangent to fibers are *vertical* and the vectors tangent to leaves are *horizontal*.

Definition 3.1.1. Let $M \times H$ be any product manifold.

1. If $X \in \mathfrak{X}(M)$, then the *lift of* X to $M \times H$ is $\tilde{X} \in \mathfrak{X}(M \times H)$ such that $d\pi(\tilde{X}) = X$ and $d\eta(\tilde{X}) = 0$.

2. If $Y \in \mathfrak{X}(H)$, then the *lift of* Y $M \times H$ to $M \times H$ is $\tilde{Y} \in \mathfrak{X}(M \times H)$ such that $d\pi(\tilde{Y}) = 0$ and $d\eta(\tilde{Y}) = Y$.

The set of all horizontal lifts is denoted as $\mathfrak{L}(M)$ and the set of all vertical lifts is denoted as $\mathfrak{L}(H)$.

It is proven by Bishop and O'Neill that if (M, g) and (H, h) are complete Riemannian manifolds $M \times_f H$ is a complete Riemannian manifold.

Proposition 3.1.2. *(Proposition 35, p.206) [6] Let* $X, Y \in \mathfrak{L}(M)$ *and* $V, W \in \mathfrak{L}(H)$*. Then*

1. $D_X Y \in \mathfrak{L}(M)$ *is the lift of* $D_X Y$ *on* M*.*

2. $D_X V = D_V X = (Xf/f)V$

3. $\mathrm{nor} D_V W = II(V, W) = -(\langle V, W \rangle / f)\mathrm{grad} f$*.*

4. $\tan D_V W \in \mathfrak{L}(H)$ *is the lift of* $\nabla_V W$ *on* H*.*

Corollary 3.1.3. *(Corollary 36, p.207) [6] The leaves* $M \times q$ *are totally geodesic and the fibers* $p \times H$ *are totally umbilic.*

Proposition 3.1.4. *(Proposition 38, p.208) [6] Let* $\gamma = (\gamma_1, \gamma_2)$ *be a curve in* $M \times_f H$*.* γ *is a geodesic iff*

1. $\gamma_1'' = (\gamma_2', \gamma_2') f \circ \gamma_1 \mathrm{grad} f$ *in* M*,*

2. $\gamma_2'' = \dfrac{-2}{f \circ \gamma_1} \dfrac{d(f \circ \gamma_1)}{ds} \gamma_2'$ *in* H*.*

Let A be a covariant tensor on M. The *lift* $\tilde{A}$ of A to $M \times_f H$ is $\pi_*(A)$. When $A : \mathfrak{X}(M) \times ... \times \mathfrak{X}(M)$ is a (1,s) tensor, for $v_1, ..., v_s \in T_{(p,q)}(\bar{M})$ we can define $\tilde{A}(v_1, ..., v_s)$ as the horizontal vector at (p, q) that projects to $A(d\pi v_1, ..., d\pi v_s)$ in

T_pM. Then $\tilde{A}$ is zero on the vectors when any one of them is vertical. These definitions are also valid for lifts from H. The lifts to $M \times_f H$ of the Riemannian curvature tensors of M and H are expressed as ${}^M R$ and ${}^H R$, respectively.

Proposition 3.1.5. *(Proposition 42, p.210) [6] Let R be the Riemannian curvature tensor of $M \times_f H$. If $X, Y, Z \in \mathfrak{L}(M)$ and $U, V, W \in \mathfrak{L}(H)$, then*

1. $R_{XY}Z \in \mathfrak{L}(M)$ *is the lift of* ${}^M R_{XY}Z$ *on* M.
2. $R_{VX}Y = (H^f(X, Y)/f)V$.
3. $R_{XY}V = R_{VW}X = 0$.
4. $R_{XV}W = (\langle V, W\rangle/f)D_x(\text{grad} f)$.
5. $R_{VW}U = {}^H R_{VW}U - (\langle \text{grad} f, \text{grad} f\rangle/f^2)(\langle V, U\rangle W - \langle W, U\rangle V)$.

Equations 1. to 5. are called *tensor equations.*

In the following corollary, we will consider the Ricci curvature of a warped product. Let ${}^M Ric$ be the lift of the Ricci curvature of M and ${}^H Ric$ be the lift of the Ricci curvature of H.

Corollary 3.1.6. *(Corollary 43, p.211) [6] If $d = \dim H > 1$, X, Y are horizontal and V, W are vertical on $M \times_f H$, then*

1. $Ric(X, Y) = {}^M Ric(X, Y) - (d/f)H^f(X, Y)$.
2. $Ric(X, V) = 0$.
3. $Ric(V, W) = {}^H Ric(V, W) - \langle V, W\rangle f^{\#}$ *where*
$$f^{\#} = \frac{\Delta f}{f} + (d-1)\frac{\langle \text{grad} f, \text{grad} f\rangle}{f^2}.$$
and Δf is the Laplacian on M.

Corollary 3.1.7. *(Exersice 13, p.214) [6] Let M be a Riemannian manifold and H be a semi-Riemannian manifold with dimension d. Then, the scalar curvature of $M \times_f H$ is*

$$\tau =^M S +^H S/f^2 - 2d\Delta f/f - d(d-1)\langle \text{grad} f, \text{grad} f\rangle/f^2.$$

3.2 Lorentzian Warped Products

In this section we will restrict our attention on Lorentzian warped products analyzed in [1].

Let $\pi : M \times H \to M$ and $\eta : M \times H \to H$ be the projection maps. These maps are given $\pi(m,h) = m$ and $\eta(m,h) = h$ where $(m,h) \in M \times H$.

Definition 3.2.1. Let (M,g) be a Lorentzian manifold, (H,h) be a Riemannian manifold and $f : M \to (0,\infty)$ be a smooth function. The manifold

$$\bar{M} = M \times H$$

with the Lorentzian metric

$$\bar{g} = g \oplus f^2 h,$$

where $\bar{p} \in M$ and $v, w \in T_{\bar{p}}M$ is called the *Lorentzian warped product* and denoted by $M \times_f H$.

3.2.1 Causality

Lemma 3.2.2. *(Lemma 3.54, p.96) [1] The warped product $M \times_f H$ is time oriented iff either (M,g) is a one-dimensional manifold with a negative definite metric or (M,g) is time oriented where $dim M \geq 2$.*

Lemma 3.2.3. *(Lemma 3.55, p.96) [1]*
Let $M = (a,b)$ with $-\infty \leq a < b \leq \infty$ be given the metric $-dt^2$ and (H,h) be any Riemannian manifold. Then the warped product $(\bar{M},\bar{g})$ is stably causal for any smooth function $f : M \to (0,\infty)$.

Corollary 3.2.4. *(Corollary 3.56, p.97) [1]*
Let $M = (a,b)$ with $-\infty \leq a < b \leq \infty$ be given the metric $-dt^2$ and (H,h) be any Riemannian manifold. Then, the warped product $(M \times_f H, \bar{g})$ is chronological, causal, distinguishing and strongly causal for any smooth function
$f : M \to (0,\infty)$.

Lemma 3.2.5. *(Lemma 3.59, p.98) [1] If* $p = (p_1, p_2)$ *and* $q = (q_1, q_2)$ *are in* $(\bar{M}, \bar{g})$ *and* $p \ll q$ *(respectively,* $p \leqslant q$*) in* $(\bar{M}, \bar{g})$*, then* $p_1 \ll q_1$ *(respectively,* $p_1 \leqslant q_1$*) in* (M, g)*.*

Lemma 3.2.6. *(Lemma 3.60, p.99) [1] Let* $p = (p_1, b)$ *and* $q = (q_1, b)$ *are in the same leaf* $\eta^{-1}(b)$ *of* $(\bar{M}, \bar{g})$*. Then,* $p_1 \ll q_1$ *(respectively,* $p_1 \leqslant q_1$*) in* (M, g) *iff* $p \ll q$*(respectively,* $p \leqslant q$*) in* $(\bar{M}, \bar{g})$*.*

Proposition 3.2.7. *(Lemma 3.61-62-64, p.100-101) [1] Let* (M, g) *be a space-time and* (H, h) *be a Riemannian manifold.* (M, g) *is chronological (respectively, causal, strongly causal, stably causal) iff* $(\bar{M}, \bar{g})$ *is chronological (respectively, causal, strongly causal, stably causal).*

Theorem 3.2.8. *(Theorem 3.66, p.101) [1]*
Let $M = (a, b)$ *with* $-\infty \leq a < b \leq \infty$ *be given the metric* $-dt^2$ *and* (H, h) *be a Riemannian manifold.* (H, h) *is complete iff* $(\bar{M}, \bar{g})$ *is globally hyperbolic.*

Theorem 3.2.9. *(Theorem 3.67, p.101) [1] If* (H, h) *is a Riemannian manifold and* $\mathbb{R} \times H$ *is given the metric* $-dt^2 \oplus h$*. Then, the following are equivalent:*

1. (H, h) *is geodesically complete.*
2. $(\mathbb{R} \times H, -dt^2 \oplus h)$ *is geodesically complete.*
3. $(\mathbb{R} \times H, -dt^2 \oplus h)$ *is globally hyperbolic.*

Theorem 3.2.10. *(Theorem 3.68, p.101) [1] Let* (M, g) *be a space-time and* (H, h) *be a Riemannian manifold.* $(\bar{M}, \bar{g})$ *is globally hyperbolic iff*

1. (M, g) *is globally hyperbolic.*
2. (H, h) *is a complete Riemannian manifold.*

3.3 Multiply Warped Products

We will review [12].

Definition 3.3.1. Let (M, g) and (H_i, h_i) be semi-Riemannian manifolds and let $f_i : M \to (0, \infty)$ be smooth functions where $i \in \{1, 2, ..., m\}$. The product manifold $\bar{M} = M \times H_1 \times ... \times H_m$ with the metric $\bar{g} = g \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m$ is called a *multiply warped product* and written as $M \times_{f_1} H_1 \times ... \times_{f_m} H_m$.

Remark 3.3.2. Let (M, g) and (H_i, h_i) be Riemannian manifolds. Then $(M \times H_1 \times ... \times H_m, g \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m)$ is also a Riemannian manifold.

Definition 3.3.3. Let (H_i, h_i) be all Riemannian manifolds and (M, g) be either a Lorentzian manifold or a one-dimensional manifold with a negative metric $-dt^2$. Then $(M \times H_1 \times ... \times H_m), g \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m)$ is called a *Lorentzian doubly product.*

Remark 3.3.4. Let $(M \times H_1 \times ... \times H_m, g \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m)$ be a semi-Riemannian warped product. Then any geodesic $\gamma : I \subset \mathbb{R} \to \bar{M}$ where $\gamma = (\alpha, \beta_1, ..., \beta_m)$ satisfies the following:

1. $\beta_i : I \to H_i$ where $i \in \{1, ..., m\}$ is a pre-geodesic in H_i.
2. $(f_i \circ \alpha)^4 h_i(\beta_i', \beta_i') \equiv c_i$ for all $i \in \{1, ..., m\}$.
3. There is a point $t_0 \in I$ such that $\alpha(t_0) = 0$ and $c_i = 0$ for all $i \in \{1, ..., m\}$ or $\alpha(t_0) = 0$ and $grad_M(f_i)(\alpha(t_0)) = 0$ for all $i \in \{1, ..., m\}$ if and only if α is a constant.
4. $c_i = 0$ if and only if β_i is constant for some $i \in \{1, ..., m\}$.

3.3.1 Causality

Theorem 3.3.5. *(Theorem 3.1, p.293) [12] If $\bar{M} = (a, b) \times H_1 \times ... \times H_m$ is a Lorentzian multiply warped product with the metric $\bar{g} = -dt^2 \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m$ and $-\infty \leq a < b \leq \infty$, then $(\bar{M}, \bar{g})$ is stably causal.*

Theorem 3.3.6. *(Theorem 3.2, p.293) [12] If $\bar{M} = M \times H_1 \times ... \times H_m$ is a Lorentzian multiply warped product with the metric $\bar{g} = g \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m$, then we have the following:*

1. *The space-time (M, g) is chronological (respectively, causal) if and only if $(\bar{M}, \bar{g})$ is chronological (respectively, causal).*
2. *The space-time (M, g) is strongly causal (respectively, stably causal) if and only if $(\bar{M}, \bar{g})$ is strongly causal (respectively, stably causal).*

Theorem 3.3.7. *(Theorem 3.3, p.293) [12] Let $\bar{M} = M \times H_1 \times ... \times H_m$ be a Lorentzian multiply warped product with the metric $\bar{g} = g \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m$ and let (H_i, h_i) be complete for any i where $1 \leq i \leq m$. Then $(\bar{M}, \bar{g})$ is globally hyperbolic if and only if (M, g) is globally hyperbolic.*

Corollary 3.3.8. *(Corollary 3.4, p.293) [12] Let $\bar{M} = (a, b) \times H_1 \times ... \times H_m$ be a Lorentzian multiply warped product with the metric $\bar{g} = -dt^2 \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m$. Then (H_i, h_i) is complete for any i where $1 \leq i \leq m$ if and only if $(\bar{M}, \bar{g})$ is globally hyperbolic.*

Chapter 4

Lorentzian Completeness and Connectedness

4.1 Completeness

For any Riemannian manifold, we know by Hopf-Rinow theorem that metric completeness and geodesic completeness are equivalent. Also, the existence of these conditions implies the existence of minimal geodesics.

In this section we will see the analogous conditions for Lorentzian manifolds. One can check [1] and [12] for details.

4.1.1 Existence of Maximal Geodesic Segments

In the Lorentzian case, geodesic completeness does not imply the existence of maximal geodesic segments. The following is an example which shows this fact.

Example 4.1.1. [1] The universal covering (M, g) of two-dimensional anti-de Sitter space is a geodesically complete space-time where

$$M = \{(x, t) \in \mathbb{R}^2 : -\pi/2 < x < \pi/2\}$$

with the metric

$$ds^2 = \sec^2 x(-dt^2 + dx^2).$$

There are two points $p, q \in M$ such that $p \ll q$ but all future timelike geodesics from p are focused at the future timelike conjugate point $r \in M$. So, there is no timelike geodesic from p to q.

Now we will consider the class of globally hyperbolic space-times.

Theorem 4.1.2. *(Theorem 6.1, p.200) [1] Let (M, g) be a globally hyperbolic space-time. Then for $p, q \in M$ with $q \in J^+(p)$, there exists a maximal geodesic segment $\gamma \in \Omega_{p,q}$.*

4.1.2 Geodesic Completeness

Since null geodesics have zero length, they cannot be parametrized by arc length. In this subsection we will use affine parameters.

Definition 4.1.3. If a geodesic γ in (M, g) with affine parameter t can be extended to be defined for $-\infty < t < \infty$, then this geodesic is called *complete*.

Let s and t be affine parameters for a curve $\gamma : I \to M$. Then by geodesic equations, there exists $a, b \in \mathbb{R}$ such that $s(t) = at + b$ for all $t \in I$. So the completeness or incompleteness of a geodesic is independent of the choice of affine parameter.

Definition 4.1.4. A space-time (M, g) all of whose timelike (respectively, null, nonspacelike, spacelike) inextendible geodesics are complete is called *timelike* (respectively, *null, nonspacelike, spacelike*) *geodesically complete.*

Lemma 4.1.5. *(p.154) [6] A spacelike or timelike pregeodesic $\gamma : [0, b) \to M$ has infinite length if and only if it is complete.*

Example 4.1.6. [6] Consider the semicircular pregeodesic $\gamma(t) = (\sin t, \cos t)$, $0 \le t \le \pi/2$ of the Poincaré half-plane with metric $g(\gamma', \gamma') = \sec^2 t$. Then

$$L(\gamma) = \int_0^{\pi/2} \sec^2 t dt = \infty.$$

So the Poincaré half-plane is geodesically complete.

If all inextendible geodesics are complete, then the space-time (M, g) is called *geodesically complete.*

The following theorem shows the relationship between any to types of geodesical completeness.

Theorem 4.1.7. *(Theorem 6.4, p.203) [1] Any two of timelike, null and spacelike geodesic completeness are not logically equivalent.*

Example 4.1.8. [1] Consider the Minkowski two space $(\mathbb{R}^2, g)$ with the metric $g = dx^2 - dt^2$. Let $\phi : \mathbb{R}^2 \to (0, \infty)$ be a smooth function which conformally changes the metric g to $\tilde{g} = \phi g$ for $\mathbb{R}^2$ and which has the following properties:

1. $\phi(x, t) = 1$ if $x \leq -1$ or $x \geq 1$,
2. $\phi(x, t) = \phi(-x, t)$ for all $(x, t) \in \mathbb{R}^2$,
3. $\phi(0, t)$ goes to zero like t^{-4} as $t \to \infty$

So, $(\mathbb{R}^2, \tilde{g})$ is null and spacelike complete but timelike incomplete.

Theorem 4.1.9. *(Theorem 6.5, p.206) [1] Let (M, g) be distinguishing, strongly causal, stably causal or globally hyperbolic. Then there is a smooth conformal factor $\Omega : M \to (0, \infty)$ such that $(M, \Omega g)$ is null and timelike complete.*

Definition 4.1.10. Let $\gamma : J \to M$ be a C^2 curve with $g(\gamma'(t), \gamma'(t)) = -1$ for all $t \in J$. γ is said to have *bounded acceleration* if there exists a number $B > 0$ such that $|g(\nabla_{\gamma'}\gamma'(t), \nabla_{\gamma'}\gamma'(t)) \leq B$ for all $t \in J$.

Definition 4.1.11. A space-time (M, g) all of whose future (respectively, past) directed, future (respectively, past) inextendible, unit speed, C^2 timelike curves with bounded acceleration have infinite length is said to be *b.a. complete.*

Remarks 4.1.12.

1. Geodesic completeness does not imply b.a. completeness, even for globally hyperbolic space-times.

2. B.a. completeness implies timelike geodesic completeness.

3. B.a. completeness does not imply spacelike geodesic completeness.

Definition 4.1.13. Let $V : I \to TM$ be a smooth vector field along γ. The *generalized affine parameter* $\mu = \mu(\gamma, E_1, ..., E_n)$ is

$$\mu(t) = \int_{t_0}^{t} \sqrt{\sum_{i=1}^{n} [V^i(s)]^2 ds}, \quad t \in I,$$

where $V^i : I \to \mathbb{R}$ for $1 \leq i \leq n$.

Definition 4.1.14. Let (M, g) be a space-time such that every C^1 curve of finite arc length as measured by a generalized affine parameter has an endpoint in M. Then (M, g) is called a *b-complete space-time.*

Remark 4.1.15. B-completeness implies b.a. completeness but b.a. completeness does not imply b-completeness.

4.1.3 Metric Completeness

The Hopf-Rinow theorem does not hold for Lorentzian manifolds. We need two definitions to talk about the metric completeness of Lorentzian manifolds.

Definition 4.1.16. Let $\{x_n\}$ be a sequence with $x_n \ll x_{n+m}$ for $n, m = 1, 2, ...$ and $d(x_n, x_{n+m}) \leq B_n$ (or $x_{n+m} \ll x_n$ and $d(x_{n+m}, x_n) \leq B_n$ for all $m \geq 0$, where $B_n \to 0$ as $n \to \infty$. If this sequence is convergent, then The causal space-time (M, g) is called as *timelike Cauchy complete.*

Definition 4.1.17. Let (M, g) be a space-time. If for any $B > 0$ and for each sequence of points $\{x_n\}$ with $p \ll q \leqslant x_n$ and $d(p, x_n) \leq B$ for all n (or $x_n \leqslant q \ll p$ and $d(x_n, p) \leq B$) $\{x_n\}$ has an accumulation point in M, then (M, g) is called as *finitely compact.*

Although finite compactness requires all closed metric balls of a Riemannian manifold be compact, in Lorentzian case, the subsets $\{q \in J+(p) : d(p, q) \leq \epsilon\}$ of a space-time are noncompact in general.

Lemma 4.1.18. *(Lemma 6.11, p.211) [1] For a globally hyperbolic space-time (M, g), the following are equivalent:*

1. *(M, g) is finitely compact.*
2. *For any $B > 0$, $\{x \in M : p \ll q \leqslant x, d(p, x) \leq B\}$ is compact for all $p, q \in M$ with $q \in I^+(p)$ and $\{x \in M : x \leqslant q \ll p, d(x, p) \leq B\}$ is compact for all $p, q \in M$ with $p \in I^+(q)$.*

Theorem 4.1.19. *(Theorem 6.12, p.212) [1] Let (M, g) be a globally hyperbolic space-time.*

1. *(M, g) is timelike Cauchy complete iff (M, g) is finitely compact.*
2. *If (M, g) is nonspacelike geodesically complete, then (M, g) is timelike Cauchy complete and hence finitely compact.*

Remarks 4.1.20.

1. Finite compactness does not imply timelike geodesic completeness.
2. Timelike Cauchy completeness does not imply timelike geodesic completeness.

The above remarks is also valid for globally hyperbolic space-times.

Example 4.1.21. [1] The globally hyperbolic space-time given in Example 4.1.6 is finitely compact but timelike geodesically incomplete.

Let (M, g) be a Riemannian manifold. Then we have the following:

Theorem 4.1.22. *[1] Let (N, h) be a complete Riemannian manifold and $F : M \to N$ be an embedding of M where $F(M)$ is a closed subset of N. Then M is a complete Riemannian manifold using the induced metric.*

In the Lorentzian case, closed embedded submanifolds need not be complete.

Example 4.1.23. [1] The (x, y) plane with the usual Minkowski metric $\eta = dx^2 - dy^2$ is a complete Lorentzian manifold.On the other hand, let

$$F(x) = \left(x, \int_0^{|x|} (1 - e^{-t})^1/2dt \right)$$

be an embedding of $\mathbb{R}^1$. It is an incomplete closed spacelike submanifold of the Minkowski space.

4.1.4 Completeness of Lorentzian Warped Products

In the Riemannian case, we have the following theorem:

Theorem 4.1.24. *(Theorem 4.14-15, p.300) [12] Let $\bar{M} = M \times H_1 \times ... \times H_m$ be a Riemannian multiply warped product with the metric $g \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m$. Then $(\bar{M}, \bar{g})$ is a complete Riemannian manifold if and only if (M, g) and (H_i, h_i) are complete Riemannian manifolds for all $i \in \{1, ..., m\}$.*

In this section we will focus on the Lorentzian multiply warped products of the form

$$\bar{M} = (a, b) \times H_1 \times ... \times H_m$$

where $-\infty \leq a < b \leq \infty$, with the metric tensor

$$\bar{g} = -dt^2 \oplus f_1^2 h_1 \oplus ... \oplus f_m^2 h_m.$$

Theorem 4.1.25. *(Theorem 4.2, p.295) [12] Let $(\bar{M}, \bar{g})$ be a Lorentzian multiply warped product. Then we have the following:*

1. *$(\bar{M}, \bar{g})$ is future directed null geodesic past incomplete if $\lim\limits_{t \to a^+} \int_t^c f_i(s) ds < \infty$ for some $i \in \{1, ..., m\}$ and for some $c \in (a, b)$,*
2. *$(\bar{M}, \bar{g})$ is future directed null geodesic future incomplete if $\lim\limits_{t \to b^-} \int_c^t f_i(s) ds < \infty$ for some $i \in \{1, ..., m\}$ and for some $c \in (a, b)$.*

Theorem 4.1.26. *(Theorem 4.3, p.296) [12] Let $(\bar{M}, \bar{g})$ be a Lorentzian multiply warped product. Then we have the following:*

1. $(\bar{M}, \bar{g})$ *is future directed timelike geodesic past incomplete if* $\lim_{t \to a^+} \int_t^c \sqrt{1 + f_i^2(s)}ds < \infty$ *for some* $i \in \{1, ..., m\}$ *and for some* $c \in (a, b)$,

2. $(\bar{M}, \bar{g})$ *is future directed timelike geodesic future incomplete if* $\lim_{t \to b^-} \int_c^t \sqrt{1 + f_i^2(s)}ds < \infty$ *for some* $i \in \{1, ..., m\}$ *and for some* $c \in (a, b)$.

Theorem 4.1.27. *(Theorem 4.4, p.296) [12] Let* $(\bar{M}, \bar{g})$ *be a Lorentzian multiply warped product. Then we have the following:*

1. $(\bar{M}, \bar{g})$ *is future directed spacelike geodesic past incomplete if* $\lim_{t \to a^+} \int_t^c f_i(s)ds < \infty$ *and* f_i *is a bounded function on* (a, c) *for some* $i \in \{1, ..., m\}$ *and for some* $c \in (a, b)$,

2. $(\bar{M}, \bar{g})$ *is future directed spacelike geodesic future incomplete if* $\lim_{t \to b^-} \int_c^t f_i(s)ds < \infty$ *and* f_i *is a bounded function on* (c, b) *for some* $i \in \{1, ..., m\}$ *and for some* $c \in (a, b)$.

Lemma 4.1.28. *(Lemma 4.5, p.296) [12] If* $(\bar{M}, \bar{g})$ *is a Lorentzian multiply warped product,* $\gamma : [0, \delta) \to M$ *where* $\gamma = (\alpha, \beta_1, ..., \beta_m)$ *is a geodesic for some* $\delta > 0$ *and* (H_i, h_i) *are complete for all* $i \in \{1, ..., m\}$, *then the following are equivalent:*

1. γ *is extendible as a geodesic past* δ.

2. α *is continously extendible to* δ.

3. $\alpha'[0, \delta)$ *is a compact subset of* TM.

4. $\alpha[0, \delta)$ *is a compact subset of* M.

Lemma 4.1.29. *(Lemma 4.5, p.296) [12] Let* $(\bar{M}, \bar{g})$ *be a Lorentzian multiply warped product,* $\gamma : [0, \delta) \to M$ *where* $\gamma = (\alpha, \beta_1, ..., \beta_m)$ *is a future directed geodesic,* $D = \bar{g}(\gamma', \gamma')$ *and* $(f_i \circ \alpha)^4 h_i(\beta_i', \beta_i') = c_i$. *Then we have the following:*

1. *If* $\lim_{t \to b-} \int_c^t (-D + \sum_{i=1}^m c_i / f_i^2(s))^{-1/2} ds = \infty$ *for some* $c \in (a, b)$, *then* γ *is a future complete geodesic.*

2. *If* $\lim\limits_{t\to a+}\int_t^c(-D+\sum_{i=1}^m c_i/f_i^2(s))^{-1/2}ds=\infty$ *for some* $c\in(a,b)$*, then* γ *is a past complete geodesic.*

Let $\mathcal{B}=\{f_1,...,f_m\}$ and let $\{\bar{f}_1,...,\bar{f}_k\}$ be some subset of $\mathcal{B}$ for some $k\in\{1,...,m\}$. Then we define

1. $f[\bar{f}_1,...,\bar{f}_k]=\prod\limits_{i=1}^{k}\bar{f}_i$ and
2. $h[\bar{f}_1,...,\bar{f}_k]=\sum_{i=1}^k\bar{f}_1^2...\bar{f}_{i-1}^2\bar{f}_{i+1}^2...\bar{f}_k^2$.

We will assume that $h[\bar{f}_1]=1$ for any $\bar{f}_1$.

For the following results let $D=1$ for spacelike geodesics, $D=0$ for null geodesics and $D=-1$ for timelike geodesics.

Theorem 4.1.30. *(Theorem 4.7, p.297) [12] Let* $(\bar{M},\bar{g})$ *be a Lorentzian multiply warped product. Suppose that* (H_i,h_i) *is complete for all* $i\in\{1,...,m\}$ *and any* $\mathcal{B}=\{f_1,...,f_m\}$*. Then*

1. *All future directed null geodesics are future complete if and only if* $\lim\limits_{t\to b-}\int_c^t f[\bar{f}_1,...,\bar{f}_k](s)/\sqrt{h[\bar{f}_1,...,\bar{f}_k](s)}ds=\infty$ *for some* $c\in(a,b)$*, for all* $k\in\{1,...,m\}$ *and for all subsets* $\{\bar{f}_1,...,\bar{f}_k\}$ *of* $\mathcal{B}$*.*
2. *All future directed null geodesics are past complete if and only if* $\lim\limits_{t\to a+}\int_t^c f[\bar{f}_1,...,\bar{f}_k](s)/\sqrt{h[\bar{f}_1,...,\bar{f}_k](s)}ds=\infty$ *for some* $c\in(a,b)$*, for all* $k\in\{1,...,m\}$ *and for all subsets* $\{\bar{f}_1,...,\bar{f}_k\}$ *of* $\mathcal{B}$*.*

Theorem 4.1.31. *(Theorem 4.8, p.298) [12] Let* $(\bar{M},\bar{g})$ *be a Lorentzian multiply warped product. Suppose that* (H_i,h_i) *is complete for all* $i\in\{1,...,m\}$ *and any* $\mathcal{B}=\{f_1,...,f_m\}$*. Then*

1. *All future directed timelike geodesics are future complete if and only if* $\lim\limits_{t\to b-}\int_c^t f[\bar{f}_1,...,\bar{f}_k](s)/\sqrt{f[\bar{f}_1,...,\bar{f}_k]^2(s)+h[\bar{f}_1,...,\bar{f}_k](s)}ds=\infty$ *for some* $c\in(a,b)$*, for all* $k\in\{1,...,m\}$ *and for all subsets* $\{\bar{f}_1,...,\bar{f}_k\}$ *of* $\mathcal{B}$*.*

2. *All future directed timelike geodesics are past complete if and only if* $\lim\limits_{t \to a+} \int_t^c f[\bar{f}_1, ..., \bar{f}_k](s)/\sqrt{f[\bar{f}_1, ..., \bar{f}_k](s) + h[\bar{f}_1, ..., \bar{f}_k](s)}ds = \infty$ *for some* $c \in (a, b)$*, for all* $k \in \{1, ..., m\}$ *and for all subsets* $\{\bar{f}_1, ..., \bar{f}_k\}$ *of* $\mathcal{B}$*.*

Theorem 4.1.32. *(Theorem 4.9, p.298) [12] Let* $(\bar{M}, \bar{g})$ *be a Lorentzian multiply warped product. Suppose that* (H_i, h_i) *is complete for all* $i \in \{1, ..., m\}$ *and any* $\mathcal{B} = \{f_1, ..., f_m\}$*. Then*

1. *All future directed spacelike geodesics are future complete if and only if either* $\lim\limits_{t \to b-} \int_c^t f[\bar{f}_1, ..., \bar{f}_k](s)/\sqrt{h[\bar{f}_1, ..., \bar{f}_k](s)}ds = \infty$ *or* $f[\bar{f}_1, ..., \bar{f}_k](s)/\sqrt{h[\bar{f}_1, ..., \bar{f}_k](s)}$ *is an unbounded function on* (c, b) *for some* $c \in (a, b)$*, for all* $k \in \{1, ..., m\}$ *and for all subsets* $\{\bar{f}_1, ..., \bar{f}_k\}$ *of* $\mathcal{B}$*.*

2. *All future directed spacelike geodesics are past complete if and only if either* $\lim\limits_{t \to a+} \int_t^c f[\bar{f}_1, ..., \bar{f}_k](s)/\sqrt{h[\bar{f}_1, ..., \bar{f}_k](s)}ds = \infty$ *or* $f[\bar{f}_1, ..., \bar{f}_k](s)/\sqrt{h[\bar{f}_1, ..., \bar{f}_k](s)}$ *is an unbounded function on* (a, c) *for some* $c \in (a, b)$*, for all* $k \in \{1, ..., m\}$ *and for all subsets* $\{\bar{f}_1, ..., \bar{f}_k\}$ *of* $\mathcal{B}$*.*

Corollary 4.1.33. *(Corollary 4.10, p.298) [12] Let* $(\bar{M}, \bar{g})$ *be a Lorentzian multiply warped product. Suppose that* (H_i, h_i) *is complete for all* $i \in \{1, ..., m\}$*.*

1. *If* $(\bar{M}, \bar{g})$ *is timelike complete, then* $(\bar{M}, \bar{g})$ *is null complete.*

2. $0 < \inf(f_i) < \sup(f_i) < 0$ *for any* $i \in \{1, ..., m\}$*, then* $(\bar{M}, \bar{g})$ *is timelike complete iff* $(\bar{M}, \bar{g})$*.*

Theorem 4.1.34. *(Theorem 4.12, p.299) [12] Let* $(\bar{M}, \bar{g})$ *be a Lorentzian multiply warped product. Then* (H_i, h_i) *is a complete Riemannian manifold for all* $i \in \{1, ..., m\}$ *if* $(\bar{M}, \bar{g})$ *is spacelike, null or timelike complete.*

Proposition 4.1.35. *(Proposition 4.13, p.300) [12] Let* $(\bar{M}, \bar{g})$ *be a Lorentzian multiply warped product. If* $(\bar{M}, \bar{g})$ *is spacelike (respectively null, timelike) complete, then* (M, g) *is spacelike (respectively null, timelike) complete.*

4.2 Geodesic Connectedness

By Hopf-Rinow Theorem it is known that a complete Riemannian manifold is convex and hence geodesically connected. However, neither completeness nor compactness implies geodesic connectedness of a Lorentzian manifold. In this section we will give some conditions of Lorentzian geodesic connectedness by [2] and [9].

Definition 4.2.1. Let (M, g) be a Lorentzian manifold. If any $p, q \in M$ can be joined by a causal geodesic, then (M, g) is called *geodesically connected*.

Definition 4.2.2. Let $\mathcal{S}$ be a plane with a Lorentzian metric. If there exists a diffeomorphism of $\mathcal{S}$ onto $\mathbb{R}^2$ which takes null geodesics into an axis-parallel line, then $\mathcal{S}$ is called *normal*.

Theorem 4.2.3. *(Theorem 5, p.3093) [9]*

1. *A normal Lorentzian plane is geodesically connected.*
2. *The universal covering of a complete torus with a Killing vector field $K \not\equiv 0$ is normal and hence, it is geodesically connected.*

The pseudosphere $\mathbb{S}^n_\nu$ is the set of spacelike vectors of norm 1 in $\mathbb{R}^n_\nu$. The Lorentzian pseudosphere $\mathbb{S}^n_1$ is called as *de Sitter* space-time. The Lorentzian pseudosphere is globally hyperbolic.

Remark 4.2.4. If $\nu > 0$, then $\mathbb{S}^n_\nu$ is not geodesically connected.

Theorem 4.2.5. *(Theorem 7, p.3094) [9] Let $n \geq 2$ and $p, q \in \mathbb{S}^n_\nu$ with $\langle p, q\rangle_1 > -1$ where $\langle ., .\rangle_1$ is the usual Lorentzian inner product of $\mathbb{R}^{n+1}_1$. Then, p and q are connectable by a geodesic.*

A Lorentzian manifold (M, g) which has a globally defined timelike Killing vector field K is called *stationary*.

Let $M = \mathbb{R} \times M_0$ with where M_0 is arbitrary manifold and let

$$g = -bdt^2 + 2\omega \otimes dt + g_0$$

where

1. b is a positive function on M_0,
2. ω is a 1-form on M_0 and
3. g_0 is a Riemannian metric.

Then M is called a *standard stationary space-time.*

Theorem 4.2.6. *(Theorem 8, p.3096) [9] A standard stationary space-time is geodesically connected if the following conditions hold:*

1. *g_0 is complete.*
2. *$0 < \inf(b) \leq \sup(b) < \infty$.*
3. *The g_0-norm of $\omega(x)$ has a sublinear growth in M, i.e, an upper bound of the norm of $\omega(x)$ is $Ad_0(x, p_0)^a + B$, where A,B are some real numbers, $a \in (0, 1)$, $p_0 \in M_0$ and d_0 is the g_0-distance on M_0.*

4.2.1 Multiwarped Space-times

A Lorentzian multiply warped product of the form

$$\bar{M} = (a, b) \times H_1 \times ... \times H_m \text{ with } \bar{g} = -dt^2 + f_1^2 h_1 + ... + f_m^2 h_m$$

is called a *multiwarped space-time.*

If $m = 1$, then $\bar{M}$ is called as the *Generalized Robertson-Walker* space-time.

Let $x \in H_1 \times ... \times H_m$ and $I = (a, b)$. The set $L[x] = \{(t, x) : t \in I\}$ is the *line* at x.

Let $m = 1$. If any point $z = (t, x) \in I \times F_1$ and any line $L[x']$ can be joined by both a future and a past directed causal curve, then the space-time is geodesically connected. Equivalently, the spacetime is geodesically connected if

$$\int_a^c f_1^{-1} = \int_c^b f_1^{-1} = \infty \tag{4.2.1}$$

for some $c \in I$. [2]

Example 4.2.7. [2] De Sitter space-time is a generalized Robert-Walker space-time with $f_1 = cosh$. It does not satisfy the condition 4.2.1 and it is not geodesically connected.

Definition 4.2.8. If any two points of a Riemannian manifold can be joined by a distance minimizing geodesic (not necessarily unique), then that manifold is called *weakly convex*. If this geodesic is unique, then that manifold is called *strongly convex*.

Theorem 4.2.9. *(Theorem 1, p.3) [2] If any point of multiwarped space-time $(\bar{M}, \bar{g})$ where (H_i, h_i) is weakly convex for all $i \in \{1, ..., m\}$ can be joined by both a future and a past directed causal curve, then $(\bar{M}, \bar{g})$ is geodesically connected.*

Proposition 4.2.10. *(Proposition 1, p.9) [2] Let $(\bar{M}, \bar{g})$ be geodesically connected and $H_{r+1}, ..., H_m$ be strongly convex for some $r \in \{1, ..., m-1\}$. Then $(a, b) \times H_1 \times ... \times H_r$ is geodesically connected with the metric $-dt^2 + f_1^2 g_1 + ... + f_r^2 g_r$.*

Remark 4.2.11. If the strong convexity condition is replaced by weak convexity, then Proposition 4.2.10 does not hold. See Example 4.2.24.

Theorem 4.2.12. *(Theorem 2, p.13) [2] Let $(\bar{M}, \bar{g})$ be a multiwarped space-time and let (H_i, h_i) be weakly convex for all $i \in \{1, ..., m\}$. Let $p = (t_0, x_1, ..., x_m)$ and $q = (t'_0, x'_1, ..., x'_m)$ be two distinct points in $\bar{M}$ and $t_0 \leq t'_0$. Then the following conditions are equivalent:*

1. *There exists a timelike geodesic which joins p and q.*

2. *There exists $c_1, ..., c_m \geq 0$ with $c_1 + ... + c_m = 1$ such that*

$$\sqrt{c_i} \int_{t_0}^{t'_0} f_i^{-2} \left(\frac{c_1}{f_1^2} + ... + \frac{c_m}{f_m^2} \right) \geq l_i \tag{4.2.2}$$

for all i, where $l_i = d_i(x_i, x_i')$ and equality holds in jth equation if and only if $c_j = 0$.

3. $q \in I^+(p)$.

Theorem 4.2.13. *(Theorem 2, p.13) [2] Let $(\bar{M}, \bar{g})$ be a multiwarped space-time and let (H_i, h_i) be weakly convex for all $i \in \{1, ..., m\}$. Let $p = (t_0, x_1, ..., x_m)$ and $q = (t_0', x_1', ..., x_m')$ be two distinct points in $\bar{M}$ and $t_0 \leq t_0'$. Then the following conditions are equivalent:*

1. *There exists a timelike or null geodesic which joins p and q.*
2. *There exists $c_1, ..., c_m \geq 0$ with $c_1 + ... + c_m = 1$ such that the equation 4.2.2 hold for all $i \in \{1, ..., m\}$.*
3. $q \in J^+(p)$.

Theorem 4.2.14. *(Theorem 9, p.3098) [9]*

1. *If (H_i, h_i) is convex for all $i \in \{1, ..., m\}$, then any two causally related points of the multiwarped space-time $(\bar{M}, \bar{g})$ can be joined by a causal geodesic.*
2. *The multiwarped space-time $(\bar{M}, \bar{g})$ is geodesically connected if (H_i, h_i) is geodesically connected for all $i \in \{1, ..., m\}$ and*

$$\int_c^b f_i^{-2}(f_1^{-2} + ... + f_m^{-2})^{-1/2} = \infty$$
$$\int_a^c f_i^{-2}(f_1^{-2} + ... + f_m^{-2})^{-1/2} = \infty \tag{4.2.3}$$

for all $i \in \{1, ..., m\}$ and for some $c \in (a, b)$.

Theorem 4.2.15. *(Theorem 3, p.14) [2] Let (H_i, h_i) for all $i \in \{1, ..., n\}$ be weakly convex fibers of the multiwarped spacetime $(\bar{M}, \bar{g})$. Then $(\bar{M}, \bar{g})$ is geodesically convex if it satisfies*

$$\int_c^b f_i^{-2}(f_1^{-2} + ... + f_m^{-2} + 1)^{-1/2} = \infty$$
$$\int_a^c f_i^{-2}(f_1^{-2} + ... + f_m^{-2} + 1)^{-1/2} = \infty \tag{4.2.4}$$

for all $i \in \{1, ..., m\}$ and for some $c \in (a, b)$.

Lemma 4.2.16. *(Lemma 4, p.14) [2] For all $c_1, ..., c_m \geq 0$, $c_1 + ... + c_m = 1$, the equations 4.2.4 imply*

$$\int_c^b f_i^{-2}\left(\frac{c_1}{f_1^2} + ... + \frac{c_m}{f_m^2} - D\right)$$
$$\int_a^{c'} f_i^{-2}\left(\frac{c_1}{f_1^2} + ... + \frac{c_m}{f_m^2} - D\right) \tag{4.2.5}$$

for all $i \in \{1, ..., m\}$ and $c, c' \in I$, $D \in \mathbb{R}$ such that the denominators of the integrals do not vanish.

At the extremes of the interval I, the behavior of the functions

$$f^{\hat{c}}(t) = \sum_{i=1}^{m} \frac{c_i}{f_i^2(t)}$$

is important to show geodesic connectedness.

Definition 4.2.17. Let $f : I \to \mathbb{R}$ be a smooth function, $m_b = \liminf t \to bf(t)$ and $m_a = \liminf t \to af(t)$.

1. When $b < \infty$ (respectively, $a > -\infty$), there exists $\epsilon > 0$ such that if $0 < \epsilon' < \epsilon$, then $f(b - \epsilon') > m_b$ (respectively, $f(a + \epsilon') > m_a$).

2. When $b = \infty$ (respectively, $a = -\infty$), there exists $M > 0$ such that if $M' > M$, then $f(M') > m_b$ (respectively, $f(-M') > m_a$).

Then the extreme b (respectively, a) is called a *(strict) relative minimum* of f.

Remark 4.2.18. Let $c_1, ..., c_m \geq 0$, $c_1 + ... + c_m = 1$ such that $f^{\hat{c}}$ reaches a relative minimum in b (respectively, in a). Then $\int_c^b f_i^{-2}(\frac{c_1}{f_1^2} + ... + \frac{c_m}{f_m^2} - m_b)^{-1/2} = \infty$ (respectively, $\int_a^c f_i^{-2}(\frac{c_1}{f_1^2} + ... + \frac{c_m}{f_m^2} - m_a)^{-1/2} = \infty$) holds for all i anc some $c \in I$ close to b (respectively, a). Here *close to* b (respectively, a) means $c \in (b - \epsilon, b)$ (respectively, $c \in (a, a + \epsilon)$) if the extreme is finite or $c > M$ (respectively, $c < -M$) if the extreme is infinite, where ϵ and M are as in Definition 4.2.17.

Theorem 4.2.19. *(Theorem 5, p.26) [2] Let (H_i, h_i) be weakly convex for all i. Then any multiwarped space-time $(I \times H_1 \times H_m, \bar{g})$ verifying the condition in Remark 4.2.18 is geodesically connected.*

Example 4.2.20. [2] Consider $\mathbb{R} \times (r_-, r_+) \times \mathbb{S}^2$ with the metric

$$-\left(1-\frac{2k}{r}+\frac{e^2}{r^2}\right)ds^2 + -\left(1-\frac{2k}{r}+\frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2),$$

where

- e, k are constants, k is positive, $e \neq 0$, $e^2 < k^2$,
- $r_- = k - (k^2 - e^2)^{1/2}$,$r_+ = k + (k^2 - e^2)^{1/2}$, $r \in (r_-, r_+)$,
- $s \in \mathbb{R}$ and
- θ, ϕ are usual spherical coordinates.

This space-time is called *Schwarzschild space-time* and its metric is the intermediate zone of usual Reissner-Nordström space-time. Since $\frac{1}{f_1^2(t)}$ diverges on the extremes of the interval and has relative minima which satisfy the condition in Remark 4.2.18, this space-time is geodesically connected.

Lemma 4.2.21. *(Lemma 13, p.27) [2] Let $(I \times H_1 \times H_2, \bar{g})$ be a multiwarped space-time and let (H_1, h_1), (H_2, h_2) be weakly convex. Assume that condition in Remark 4.2.18 holds for all $c \in [0, 1]$ except for c=1 and i=1. Then the multiwarped space-time is geodesically connected if for all $x_2, x_2' \in H_2$, there exists a sequence $\{\gamma_2^- n\}_{n \in \mathbb{N}}$ of geodesics of H_2 joining them with diverging lengths L_m.*

Example 4.2.22. [2] When $e = 0$ in Example 4.2.20, the condition in Remark 4.2.18 does not hold for f_1 However, by Lemma 4.2.21, the multiwarped space-time in Example 4.2.20 with $e = 0$ is geodesically connected.

Example 4.2.23. [2] Schwarzschild space-time in Example 4.2.20 is geodesically connected since H_2 is a sphere and any $x_2, x_2' \in H_2$ can be joined by a closed geodesic.

Example 4.2.24. [2] Schwarzchild black hole $(I \times H_1 \times H_2, -dt^2 + f_1^2 h_1 + f_2^2 h_2)$ is geodesically connected but its part $(I \times H_1, -dt^2 + f_1^2 h_1)$ is not geodesically connected by considering the condition in Lemma 4.2.21 for the fiber (H_1, h_1).

Chapter 5

Twice Warped Products

In this chapter, we will analyse geodesic connectedness and completeness of the multiwarped space-times when $m = 2$. We call such products as *twice warped space-times.*

5.1 Connectedness

By Theorem 4.2.15, we get the following result:

Corollary 5.1.1. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ and let $c \in I = (a, b)$. Then, $(\bar{M}, \bar{g})$ is geodesically connected if (H_1, h_1) and (H_2, h_2) are weakly convex and the following holds:*

1. $\lim_{t \to b^-} \int_c^t \frac{f_2}{f_1} \frac{1}{\sqrt{f_1^2 + f_2^2 + f_1^2 f_2^2}} ds = \infty,$
2. $\lim_{t \to b^-} \int_c^t \frac{f_1}{f_2} \frac{1}{\sqrt{f_1^2 + f_2^2 + f_1^2 f_2^2}} ds = \infty,$
3. $\lim_{t \to a^+} \int_t^c \frac{f_2}{f_1} \frac{1}{\sqrt{f_1^2 + f_2^2 + f_1^2 f_2^2}} ds = \infty,$
4. $\lim_{t \to a^+} \int_t^c \frac{f_1}{f_2} \frac{1}{\sqrt{f_1^2 + f_2^2 + f_1^2 f_2^2}} ds = \infty.$

5.2 Completeness

By Theorem 4.1.30 we get the following results:

Corollary 5.2.1. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ and let $c \in I = (a, b)$. Then all future directed null geodesics are future complete if and only if*

1. $\lim\limits_{t \to b-} \int_c^t f_1 ds = \infty$,
2. $\lim\limits_{t \to b-} \int_c^t f_2 ds = \infty$,
3. $\lim\limits_{t \to b-} \int_c^t \frac{f_1 f_2}{\sqrt{f_1^2 + f_2^2}} ds = \infty$.

Corollary 5.2.2. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ and let $c \in I = (a, b)$. Then all future directed null geodesics are past complete if and only if*

1. $\lim\limits_{t \to a+} \int_t^c f_1 ds = \infty$,
2. $\lim\limits_{t \to a+} \int_t^c f_2 ds = \infty$,
3. $\lim\limits_{t \to a+} \int_t^c \frac{f_1 f_2}{\sqrt{f_1^2 + f_2^2}} ds = \infty$.

By Theorem 4.1.31, we get the following results:

Corollary 5.2.3. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ and let $c \in I = (a, b)$. Then all future directed timelike geodesics are future complete if and only if*

1. $\lim\limits_{t \to b-} \int_c^t \frac{f_1}{\sqrt{f_1^2 + 1}} ds = \infty$,
2. $\lim\limits_{t \to b-} \int_c^t \frac{f_2}{\sqrt{f_2^2 + 1}} ds = \infty$,
3. $\lim\limits_{t \to b-} \int_c^t \frac{f_1 f_2}{\sqrt{f_1^2 + f_2^2 + f_1^2 f_2^2}} ds = \infty$.

Corollary 5.2.4. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ and let $c \in I = (a, b)$. Then all future directed timelike geodesics are past complete if and only if*

1. $\lim\limits_{t \to a+} \int_t^c \frac{f_1}{\sqrt{f_1^2+1}} ds = \infty$,

2. $\lim\limits_{t \to a+} \int_t^c \frac{f_2}{\sqrt{f_2^2+1}} ds = \infty$,

3. $\lim\limits_{t \to a+} \int_t^c \frac{f_1 f_2}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} ds = \infty$.

5.3 Relationship between Completeness and Connectedness

Case 1. Suppose $\lim\limits_{s \to a^+} \frac{f_1}{f_2} = K \neq 0$. (*)

Corollary 5.3.1. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $a > -\infty$, $b < \infty$ be geodesically connected. Then $(\bar{M}, \bar{g})$ can be neither null past complete nor timelike past complete.*

Proof. By Corollary 5.1.1, we can choose some $c \in (a, b)$ where $\int_c^b \frac{f_2}{f_1} \frac{1}{\sqrt{f_1^2+f_2^2}} ds = \infty$. Since $a > -\infty$, $b < \infty$, the integrand must be unbounded. We can choose any $c_1 \in (a, b)$ such that $c < c_1 < b$. Also, since (*) and f_1, f_2 are continuous, $\frac{f_1}{f_2}$ is continuous and bounded in (a, c). So, $\frac{1}{f_1^2+f_2^2+f_1^2 f_2^2}$ must be unbounded. So, $f_1, f_2 \to 0$ as $s \to a^+$.

Null past completeness fails since $\int_a^c f_1 ds < \infty$. Timelike past completeness fails since $\frac{f_1}{\sqrt{f_1^2+1}} = \sqrt{1 - \frac{1}{f_1^2+1}} = \to 0$ as $s \to a^+$, i.e, $\int_a^c \frac{f_1}{\sqrt{f_1^2+1}} ds < \infty$. □

Corollary 5.3.2. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $a > -\infty$, $b < \infty$ be null past complete. Then $(\bar{M}, \bar{g})$ cannot be geodesically connected.*

Proof. By Corollary 5.2.2, $f_1, f_2 \to \infty$ and $\frac{f_1 f_2}{\sqrt{f_1^2+f_2^2}} \to \infty$ as $s \to a^+$. Then, $\lim\limits_{s \to a^+} \frac{f_1}{f_2} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = \sqrt{\frac{1}{f_2^2(1+f_2/f_1+f_2^2)}} = 0$. So $\int_a^c \frac{f_1}{f_2} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} < \infty$. So, geodesic connectedness fails. □

Corollary 5.3.3. *Any $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $a > -\infty$, $b < \infty$ and the condition (*) cannot be timelike past complete.*

Proof. Since $\frac{f_1}{\sqrt{f_1^2+1}} < \infty$, $\int_a^c \frac{f_1}{\sqrt{f_1^2+1}} < \infty$, i.e, $(\bar{M}, \bar{g})$ cannot be timelike past complete. □

Corollary 5.3.4. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (-\infty, b), b \leq \infty$ and $\lim\limits_{s \to -\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} \to \infty$. Then geodesic connectedness of $(\bar{M}, \bar{g})$ does not imply either null past completeness or timelike past completeness of $(\bar{M}, \bar{g})$.*

Example 5.3.5. Let $f_1, f_2 : (-\infty, 0)$ be defined as $f_1(s) = \frac{1}{x^2}, f_2(s) = \frac{1}{2x^2}$. Then $\lim\limits_{s \to -\infty} \frac{f_1}{f_2} = 2$. Then, $\int_{-\infty}^c \frac{f_1}{f_2} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} ds = \infty$ but $\int_{-\infty}^c f_1 ds < \infty$ for all c. So, $\bar{M}$ is not null past complete. Similarly $\int_{-\infty}^c \frac{f_1}{\sqrt{f_1^2+1}} ds < \infty$ for all c. So, $(\bar{M}, \bar{g})$ is not timelike past complete.

Corollary 5.3.6. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (-\infty, b), b \leq \infty$ and $\lim\limits_{s \to -\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = L < \infty$. Then geodesic connectedness of $(\bar{M}, \bar{g})$ implies null past completeness of $(\bar{M}, \bar{g})$.*

Proof. Let $\lim\limits_{s \to -\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = L > 0$, then $f_1, f_2 < \infty$. By Condition (*), none of f_1 and f_2 goes to 0 as $s \to -\infty$. Then, $\frac{f_1 f_2}{\sqrt{f_1^2+f_2^2}} = \frac{f_2}{1+f_1^2/f_2^2} > 0$. So, the integrals in Corollary 5.2.2 diverges, i.e. $(\bar{M}, \bar{g})$ is null past complete. □

Corollary 5.3.7. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (-\infty, b), b \leq \infty$ and $\lim\limits_{s \to -\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = L < \infty$. Then geodesic connectedness of $(\bar{M}, \bar{g})$ implies timelike past completeness of $(\bar{M}, \bar{g})$.*

Proof. Let $\lim\limits_{s \to -\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = L > 0$, then $f_1, f_2 < \infty$. By Condition (*), none of f_1 and f_2 goes to 0 as $s \to -\infty$. Then, $\frac{f_1}{\sqrt{f_1^2+1}} > 0$, $\frac{f_2}{\sqrt{f_2^2+1}} > 0$ and $\frac{f_1 f_2}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = \frac{1}{\sqrt{1/f_1^2+1/f_2^2+1}} > 0$. So, the integrals in Corollary 5.2.4 diverges, i.e. $(\bar{M}, \bar{g})$ is timelike past complete. □

Corollary 5.3.8. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (-\infty, b), b \leq \infty$ and $\lim\limits_{s \to -\infty} f_1 = \infty$, $\lim\limits_{s \to -\infty} f_2 = \infty$. Then null past completeness of $(\bar{M}, \bar{g})$ does not imply geodesic connectedness of $(\bar{M}, \bar{g})$.*

Example 5.3.9. Let $f_1, f_2 : (-\infty, 0) \to (0, \infty)$ be defined as $f_1(s) = s^2$, $f_2(s) = 2s^2$. Then, $\int_{-\infty}^c \frac{f_1}{f_2} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} < \infty$ for all $c \in (-\infty, 0)$. So $(\bar{M}, \bar{g})$ is not geodesically connected.

Corollary 5.3.10. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (-\infty, b), b \leq \infty$ and $\lim_{s\to-\infty} f_1 = l_1 < \infty$, $\lim_{s\to-\infty} f_2 = l_2 < \infty$. Then null past completeness of $(\bar{M}, \bar{g})$ implies the conditions 3. and 4. in Corollary 5.1.1.*

Proof. Since $\lim_{s\to-\infty} f_1 = l_1 < \infty$, $\lim_{s\to-\infty} f_2 = l_2 < \infty$, $\frac{1}{\sqrt{f_1^2+f_2^2+f_1^2f_2^2}} \to l > 0$ as $s \to -\infty$. So, the integrals in 3. and 4. of Corollary 5.1.1 diverges. $\square$

Corollary 5.3.11. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (-\infty, b), b \leq \infty$ and $\lim_{s\to-\infty} f_1 = \infty$, $\lim_{s\to-\infty} f_2 = \infty$. Then timelike past completeness of $(\bar{M}, \bar{g})$ does not imply geodesic connectedness of $(\bar{M}, \bar{g})$.*

Example 5.3.12. Let $f_1, f_2 : (-\infty, 0) \to (0, \infty)$ be defined as $f_1(s) = -s$, $f_2(s) = s^4$.Then, $\int_{-\infty}^{c} \frac{f_1}{f_2} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2f_2^2}} < \infty$ for all $c \in (-\infty, 0)$. So $(\bar{M}, \bar{g})$ is not geodesically connected.

Corollary 5.3.13. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (-\infty, b), b \leq \infty$ and $\lim_{s\to-\infty} f_1 = l_1 < \infty$, $\lim_{s\to-\infty} f_2 = l_2 < \infty$. Then timelike past completeness of $(\bar{M}, \bar{g})$ implies the conditions 3. and 4. in Corollary 5.1.1.*

Proof. Similar to the proof of Corollary 5.3.10. $\square$

Case 2. Suppose $\lim_{s\to b^-} \frac{f_1}{f_2} = \bar{K} \neq 0$. (**) Then, $\lim_{s\to b^-} \frac{f_2}{f_1} = \frac{1}{\bar{K}} \neq 0$.

Corollary 5.3.14. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $a > -\infty$, $b < \infty$ be geodesically connected. Then $(\bar{M}, \bar{g})$ can be neither null future complete nor timelike future complete.*

Proof. By Corollary 5.1.1, we can choose some $c \in (a, b)$ where $\int_a^c \frac{f_2}{f_1} \frac{1}{\sqrt{f_1^2+f_2^2}} ds = \infty$. Since $a > -\infty$, $b < \infty$, the integrand must be unbounded. We can choose any $c_1 \in (a, b)$ such that $a < c_1 < c$. Also, since (*) and f_1, f_2 are continuous, $\frac{f_1}{f_2}$ is continuous and bounded in (c, b). So, $\frac{1}{f_1^2+f_2^2+f_1^2f_2^2}$ must be unbounded. So, $f_1, f_2 \to 0$ as $s \to b^-$. Null past completeness fails since $\int_a^c f_1 ds < \infty$. Timelike past completeness fails since $\frac{f_1}{\sqrt{f_1^2+1}} = \sqrt{1 - \frac{1}{f_1^2+1}} = \to 0$ as $s \to b^-$, i.e, $\int_c^b \frac{f_1}{\sqrt{f_1^2+1}} ds < \infty$. $\square$

Corollary 5.3.15. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $a > -\infty$, $b < \infty$ be null future complete. Then $(\bar{M}, \bar{g})$ cannot be geodesically connected.*

Proof. By Corollary 5.2.2, $f_1, f_2 \to \infty$ and $\frac{f_1 f_2}{\sqrt{f_1^2+f_2^2}} \to \infty$ as $s \to a^+$. Then $\lim_{s\to a^+} \frac{f_1}{f_2}\frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = \sqrt{\frac{1}{f_2^2(1+f_2/f_1+f_2^2)}} = 0$. So $\int_a^c \frac{f_1}{f_2}\frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} < \infty$. So, geodesic connectedness fails. $\square$

Corollary 5.3.16. *Any $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $a > -\infty$, $b < \infty$ and the condition (**) cannot be timelike future complete.*

Proof. Since $\frac{f_1}{\sqrt{f_1^2+1}} < \infty$, $\int_c^b \frac{f_1}{\sqrt{f_1^2+1}} < \infty$, i.e, $(\bar{M}, \bar{g})$ cannot be timelike future complete. $\square$

Corollary 5.3.17. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (a, \infty), a \geq -\infty$ and $\lim_{s\to\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} \to \infty$. Then geodesic connectedness of $(\bar{M}, \bar{g})$ does not imply either null future completeness or timelike future completeness of $(\bar{M}, \bar{g})$.*

Example 5.3.18. Let $f_1, f_2 : (0, \infty) \to (0, \infty)$ be defined as $f_1(s) = \frac{1}{x^2}$, $f_2(s) = \frac{1}{2x^2}$. Then $\lim_{s\to\infty} \frac{f_1}{f_2} = 2$. Then, $\int_c^\infty \frac{f_1}{f_2}\frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} ds = \infty$ but $\int_{-\infty}^c f_1 ds < \infty$ for all $c \in (0, \infty)$. So, $\bar{M}$ is not null future complete. Similarly $\int_c^\infty \frac{f_1}{\sqrt{f_1^2+1}} ds < \infty$ for all $c \in (0, \infty)$. So, $(\bar{M}, \bar{g})$ is not timelike future complete.

Corollary 5.3.19. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (a, \infty), a \geq -\infty$ and $\lim_{s\to\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = L < \infty$. Then geodesic connectedness of $(\bar{M}, \bar{g})$ implies null future completeness of $(\bar{M}, \bar{g})$.*

Proof. Let $\lim_{s\to\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = L > 0$, then $f_1, f_2 < \infty$. By Condition (**), none of f_1 and f_2 goes to 0 as $s \to -\infty$. Then, $\frac{f_1 f_2}{\sqrt{f_1^2+f_2^2}} = \frac{f_2}{1+f_1^2/f_2^2} > 0$. So, the integrals in Corollary 5.2.1 diverges, i.e. $(\bar{M}, \bar{g})$ is null future complete. $\square$

Corollary 5.3.20. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (a, \infty), a \geq -\infty$ and $\lim_{s\to\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = L < \infty$. Then geodesic connectedness of $(\bar{M}, \bar{g})$ implies time-like future completeness of $(\bar{M}, \bar{g})$.*

Proof. Let $\lim_{s\to\infty} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = L > 0$, then $f_1, f_2 < \infty$. By Condition (**), none of f_1 and f_2 goes to 0 as $s \to \infty$. Then, $\frac{f_1}{\sqrt{f_1^2+1}} > 0$, $\frac{f_2}{\sqrt{f_2^2+1}} > 0$ and $\frac{f_1 f_2}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} = \frac{1}{\sqrt{1/f_1^2+1/f_2^2+1}} > 0$. So, the integrals in Corollary 5.2.3 diverges, i.e. $(\bar{M}, \bar{g})$ is timelike future complete. □

Corollary 5.3.21. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (a, \infty), a \geq -\infty$ and $\lim_{s\to-\infty} f_1 = \infty$, $\lim_{s\to-\infty} f_2 = \infty$. Then null future completeness of $(\bar{M}, \bar{g})$ does not imply geodesic connectedness of $(\bar{M}, \bar{g})$.*

Example 5.3.22. Let $f_1, f_2 : (0, \infty) \to (0, \infty)$ be defined as $f_1(s) = s^2$, $f_2(s) = 2s^2$. Then, $\int_{-\infty}^{c} \frac{f_1}{f_2} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} < \infty$ for all $c \in (-\infty, 0)$. So $(\bar{M}, \bar{g})$ is not geodesically connected.

Corollary 5.3.23. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (a, \infty), a \geq -\infty$ and $\lim_{s\to-\infty} f_1 = l_1 < \infty$, $\lim_{s\to-\infty} f_2 = l_2 < \infty$. Then null future completeness of $(\bar{M}, \bar{g})$ implies the conditions 1. and 2. in Corollary 5.1.1.*

Proof. Since $\lim_{s\to\infty} f_1 = l_1 < \infty$, $\lim_{s\to\infty} f_2 = l_2 < \infty$, $\frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} \to l > 0$ as $s \to \infty$. So, the integrals in 1. and 2. of Corollary 5.1.1 diverges. □

Corollary 5.3.24. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (a, \infty), a \geq -\infty$ and $\lim_{s\to\infty} f_1 = \infty$, $\lim_{s\to\infty} f_2 = \infty$. Then timelike future completeness of $\bar{M}$ does not imply geodesic connectedness of $(\bar{M}, \bar{g})$.*

Example 5.3.25. Let $f_1, f_2 : (0, \infty) \to (0, \infty)$ be defined as $f_1(s) = s$, $f_2(s) = s^4$.Then, $\int_{-\infty}^{c} \frac{f_1}{f_2} \frac{1}{\sqrt{f_1^2+f_2^2+f_1^2 f_2^2}} < \infty$ for all $c \in (-\infty, 0)$. So $(\bar{M}, \bar{g})$ is not geodesically connected.

Corollary 5.3.26. *Let $\bar{M} = I \times_{f_1} H_1 \times_{f_2} H_2$ with $I = (a, \infty), a \geq -\infty$ and $\lim_{s\to\infty} f_1 = l_1 < \infty$, $\lim_{s\to\infty} f_2 = l_2 < \infty$. Then timelike future completeness of $(\bar{M}, \bar{g})$ implies the conditions 1. and 2. in Corollary 5.1.1.*

Proof. Similar to the proof of Corollary 5.3.23. □

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