BISET FUNCTORS AND BRAUER'S INDUCTION THEOREM

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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We introduce two algebras on the endomorphism ring of the direct sum of character rings of groups from some collection. We prove the equality of these algebras to simplify a step in the proof of Brauer's Induction Theorem. We also show that these algebras are isomorphic to the direct sum of character rings of the direct products of the groups from the collection.

Keywords: Brauer's Induction Theorem, Biset functors.

ÖZET

İKİLİ KÜME İZLEÇLERİ VE BRAUER'İN GENİŞLETME TEOREMİ

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Belirli bir topluluğa ait grupların karakter halkalarının direkt toplamının özyapı dönüşüm halkasında iki cebir ortaya koyuyoruz. Bu cebirlerin eşitliğini kanıtlayarak Brauer'in Genişletme Teoremi'nin kanıtındaki bir basamağı basitleştiriyoruz. Ayrıca bu cebirlerin başta bahsettiğimiz topluluğa ait grupların direkt çarpımlarının karakter halkalarının direkt toplamına izomorfik olduğunu gösteriyoruz.

Anahtar sözcükler: Brauer'in Genişletme Teoremi, İkili küme izleçleri .

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Contents

Chapter 1

Introduction

In [1], Bouc brings out biset functors which give rise to concepts of five maps; induction, restriction, isogation, inflation and deflation. On a group G one can find two constructions namely Burnside ring of G and the representation ring of G. These rings shares the same structure obtained from the above maps.

Let $A_{\mathbb{C}}(G)$ denote the character ring of group G and $M = \bigoplus_{H \in \mathcal{K}} A_{\mathbb{C}}(H)$ where $\mathcal{K}$ is a collection of finite groups. We let $E(G,H)$ to be the ring of $\mathbb{C}$ homomorphisms from $A_{\mathbb{C}}(H)$ to $A_{\mathbb{C}}(G)$ and $E = \bigoplus_{G,H \in K} E(G,H)$, ${}^{\oplus}B = \bigoplus_{G,H \in \mathcal{K}} B(G,H)$. Also let ${}^{\oplus}A = \bigoplus_{G,H \in \mathcal{K}} A(G,H)$.

In Chapter 2 we give some preliminaries about dual modules and their characters. We also introduce the standard theory of bisets. We state Brauer's induction theorem and some of the lemma's used in its proof.

In Chapter 3, we introduce embeddings $\nu : M \mapsto E$, $\sigma :^{\oplus} B \mapsto E$ and $\rho :^{\oplus} A \mapsto E$. We show that algebra $\Lambda = \bigoplus_{G,H \in \mathcal{K}} \Lambda(G,H)$, where $\Lambda(G,H)$ is spanned by the elements of the form $\sigma({}_G\mathrm{ind}_H)\nu(a)\sigma({}_H\mathrm{res}_G)$. We also prove that Λ and $\rho({}^{\oplus}A)$ are isomorphic and to indicate why these algebras are of some importance, we use this result to prove a step in the Brauer's Induction Theorem.

In Chapter 4, we investigate the action of ${}^{\oplus}A$ on M. We show that ρ is injective map. Using the result of the chapter 3, we conclude that the three algebras $\Lambda, \rho({}^{\oplus}A)$ and ${}^{\oplus}A$ are isomorphic.

The thesis is motivated by theory of Green functors, as in [2] and [3].

Chapter 2

Some Background on Dual Modules, Theory of Bisets and Brauer's Induction Theorem

2.1 Dual Modules and their characters

For a $\mathbb{C}G$-module M its dual, denoted by M^*, is defined as $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ and its action of G is given by

$$(gp)(m) = p(g^{-1}m) \tag{1.1}$$

for $p \in M^*, m \in M, g \in G$. With this action, M^* becomes a left $\mathbb{C}G$-module. Let $\mathcal{B} = \{m_1, ...m_n\}$ be the $\mathbb{C}$-basis for M. We define the basis $\mathcal{B}^* = \{m_1^*, ...m_n^*\}$ for M^* as follows;

$$m_i^*(m_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Remark. *$\mathcal{B}^*$ forms a basis of M^**

Proposition 2.1. *Let G be a group of finite order and M be a $\mathbb{C}G-$module. Then* *we have $\chi_{M^*}(g) = \chi_M(g^{-1})$ where $\chi_{M^*}(g)$ is the character of the representation corresponding to the dual module M^*.*

Proof. Consider matrix representations with respect to a basis $\mathcal{B}$ of M and the dual basis $\mathcal{B}^*$ of M^*. The (i, j)th entry of the matrix of the action of g^{-1} on M (with respect to $\mathcal{B}$) is given by $m_i^*(g^{-1}m_j)$. Hence its trace is

$$\sum_{k=1}^{n} m_k^*(g^{-1}m_k)$$

From the equality 1.1 this sum becomes;

$$\sum_{k=1}^{n} (gm_k^*)(m_k)$$

which is equal to the trace of the matrix for the representation of g on M^* (with respect to $\mathcal{B}^*$) corresponding to $\chi_{M^*}(g)$. □

Detailed information on the above subject can be found in [4].

2.2 Bisets and Double Burnside Group

In this section we will give the standard theory of bisets which can be found in [1].

Definition 2.2. Let G and H be finite groups. Then a (G, H) biset U is a set with left G action and a right H action such that the actions commute, that is for all $h \in H, g \in H$ and for all $u \in U$ we have;

$$(hu)g = h(ug)$$

Definition 2.3. Let U be a (G, H) biset. Then for $u \in U$ the orbit of u in U is the set of elements having the form guh where $g \in G$ and $h \in H$.

From this definition it follows that U is disjoint union of its orbits that is;

$$U = \bigsqcup_{u \in G\backslash U/H} GuH$$

where u runs through the representatives of (G, H) orbits.

Definition 2.4. A (G, H) biset is called transitive if it has single orbit.

One can observe that any (G, H) biset is disjoint union of its transitive (G, H) bisets.

Definition 2.5. Let U be a (G, H) biset then the stabilizer of an element $u \in U$ is the set $U_u = \{g, h \in G \times H \mid guh = u\}$

We denote isomorphism class of a transitive (G, H) biset with point stabilizer L by $\left[\frac{G \times H}{L}\right]$.

Definition 2.6. The Double Burnside group $B(G, H)$ is defined to be the free abelian group on the set of isomorphism classes of (G, H) bisets that is;

$$B(G, H) = \bigoplus_{L \leq_{G,H} G,H} \mathbb{Z}\left[\frac{G \times H}{L}\right]$$

Let $p_1 : G \times H \mapsto G$ and $p_2 : G \times H \mapsto H$ be the canonical projections. Thus, $p_1(g, h) = g$ and $p_2 = h$ for $g \in G$ and $h \in H$. We define the star product $*$ of two subgroups $L \leq G \times H$ and $M \leq H \times K$ as the set of elements having the form (g, k) where $g \in G$ and $k \in K$ such that there exists $h \in H$ with $(g, h) \in L$ and $(h, k) \in M$. Let $\left[\frac{G \times H}{L}\right]$ be isomorphism class of (G, H) biset X with isotropy subgroup L. We have a formula, due to Bouc [3], for the cross product of these transitive bisets given as follows;

Theorem 2.7 (Mackey Product Formula)**.** *Let G, H, K be finite groups and let $L \leq G \times H$ and $M \leq H \times K$. Then*

$$\left[\frac{G \times H}{L}\right]\left[\frac{H \times K}{M}\right] = \sum_{p_2(L)hp_1(M)} \left[\frac{G \times K}{L *^{(x,1)} M}\right]$$

We have the fact that any transitive biset can be written as Mackey product of five particular bisets, which was proven in [1]. Let G be a finite group and K, L be subgroups of G satisfying $K \trianglelefteq L$. Also let H be a group, $\varphi : G \mapsto H$ be an isomorphism and let h, l run over the elements of H, L, respectively. The five bisets are given as follows;

Induction (G, H)-biset;

$$ {}_G\mathrm{ind}_H := \left[\frac{G \times H}{\{(h,h)\}}\right] $$

Inflation $(L, L/K)$-biset;

$$ {}_L\mathrm{inf}_{L/K} := \left[\frac{L \times L/N}{\{(l, lN)\}}\right] $$

Conjugation (G, H)-biset;

$$ {}_G c_H := \left[\frac{G \times H}{\{(\varphi(h), h)\}}\right] $$

Restriction (H, G)-biset;

$$ {}_H\mathrm{res}_G := \left[\frac{H \times G}{\{(h,h)\}}\right] $$

Deflation $(L/K, L)$-biset;

$$ {}_{L/K}\mathrm{def}_L := \left[\frac{L/K \times L}{\{(lK, l)\}}\right] $$

2.3 Brauer's induction Theorem

Expressing a character of a finite group G as a combination of characters induced from some particular set of subgroups of G has been subject to a number of theorems such as Artin's Induction Theorem, Brauer's Induction Theorem, Serre's Induction Theorem and so on. Here our focus will be on Brauer's which is given below:

Theorem 2.8 (Brauer)**.** *Every complex character of a finite group G is a $\mathbb{Z}-$linear combination of characters induced from linear characters of elementary subgroups of G.*

Definition 2.9. Let G be a finite group and p be a prime dividing the order of G. Then the p-elementary subgroup of G is the direct product of a cyclic group C_n and a p-group where p does not divide n.

Proof of the theorem heavily depends on writing trivial character of the group as a combination of characters induced from elementary subgroups, more precise statement is as follows:

Lemma 2.10. *The 1-character of a group G is an R-linear combination of characters induced from p-elementary subgroups.*

Despite the fact that this lemma covers the hardest part of the proof there is still some work to do. In section 3.3, using bisets we will give a way to move to the next step in the proof which is:

Lemma 2.11. *Every complex character of a finite group G is a $\mathbb{Z}-$linear combination of characters induced from characters of elementary subgroups of G.*

Briefly, In section 3.3 we assume lemma 2.10 and obtain 2.11 . Proof of 2.10 and deriving Brauer's Induction Theorem from 2.11 can be found in [5].

Chapter 3

Proving Brauer's Induction Theorem with Bisets

Let $\mathcal{K}$ be a collection of finite groups. Throughout this chapter $A_{\mathbb{C}}(G)$ will denote the character ring of group G and $M = \bigoplus_{H \in \mathcal{K}} A_{\mathbb{C}}(H)$. We define $E(G, H)$ to be the ring of $\mathbb{C}$ homomorphisms from $A_{\mathbb{C}}(H)$ to $A_{\mathbb{C}}(G)$. We let $E = \bigoplus_{G,H \in K} E(G, H)$. Also let ${}^{\oplus}B = \bigoplus_{G,H \in \mathcal{K}} B(G, H)$ and ${}^{\oplus}A = \bigoplus_{G,H \in \mathcal{K}} A(G, H)$.

3.1 Embedding ${}^{\oplus}B$ and M into E

Consider the following maps;

$$\nu : A_{\mathbb{C}}(G) \to E(G, G)$$
$$\sigma : B(G, H) \to E(G, H)$$

which are defined explicitly as follows;

Let V be a $\mathbb{C}G$-module corresponding to a character v of G. Then ν sends v to the map which takes a $\mathbb{C}G$-module $[M]$, (up to isomorphism) to the module

$[V \otimes_{\mathbb{C}} M]$. More precisely;

$$\nu(v) : [M] \mapsto [V \otimes_{\mathbb{C}} M]$$

Similarly Let X be a (G, H) biset. Then σ sends X to the map which takes a $\mathbb{C}H$-module $[N]$ to the module $[_{\mathbb{C}G}\mathbb{C}X_{\mathbb{C}H} \otimes_{\mathbb{C}H} N]$. So plainly we have;

$$\sigma(X) : [N] \mapsto [_{\mathbb{C}G}\mathbb{C}X_{\mathbb{C}H} \otimes_{\mathbb{C}H} N]$$

Lemma 3.1. *ν is injective*

Proof. Let V and W be $\mathbb{C}G$-modules. Then ν will send them to the map which takes a $\mathbb{C}G$-module M to the module $[V \otimes_{\mathbb{C}} M]$ and $[W \otimes_{\mathbb{C}} M]$. The characters corresponding to these modules are $\chi_{[V\otimes_{\mathbb{C}}M]} = \chi_V \chi_M$ and $\chi_{[W\otimes_{\mathbb{C}}M]} = \chi_W \chi_M$. For ν to be injective we must have $\chi_V \chi_M \neq \chi_W \chi_M$ that is $\chi_V \neq \chi_W$ which is the case when V and W are different modules. $\square$

3.2 The algebra Λ

Having defined σ and ν we can introduce the following algebras; Let a and b be characters of H and $\beta(G, H)$ be the algebra generated by finite products of $\sigma(_{\mathbb{C}G}X_{\mathbb{C}H})$ and $\nu(a)$ in any order. Moreover let $\Lambda(G, H)$ be the algebra consisting of only the product of the form $\sigma(_G\mathrm{ind}_H)\nu(a)\sigma(_H\mathrm{res}_G)$. We let $\beta = \bigoplus_{G,H\in\mathcal{K}} \beta(G, H)$ and $\Lambda = \bigoplus_{G,H\in\mathcal{K}} \Lambda(G, H)$. For following lemmas let α be a homomorphism from G to H . Also let $[N]$ and $[V]$ be the $\mathbb{C}H$-modules corresponding to b and a.

Lemma 3.2. $\sigma(_G\mathrm{res}^{\alpha}_H)\nu(a) = \nu(_G\mathrm{res}^{\alpha}_H(a))\sigma(_G\mathrm{res}^{\alpha}_H)$

Proof. We will show that for all $b \in A_{\mathbb{C}}(H)$, we have;

$$\sigma(_G\mathrm{res}^{\alpha}_H)\nu(a)b = \nu(_G\mathrm{res}^{\alpha}_H(a))\sigma(_G\mathrm{res}^{\alpha}_H)b$$

from there the result will follow. We have

$$\sigma(_G\mathrm{res}^{\alpha}_H)\nu([V]\,[N]) = \sigma(_G\mathrm{res}^{\alpha}_H)([V \otimes N]) = [_G\mathrm{res}_H(V) \otimes_{\mathbb{C}} \mathrm{res}_H(N)]$$
$$= [_G\mathrm{res}_H(V)]\,[_G\mathrm{res}_H(N)] = \nu(_G\mathrm{res}^{\alpha}_H(V))\,[_G\mathrm{res}^{\alpha}_H(N)] = \nu(_G\mathrm{res}^{\alpha}_H(a))\sigma(_G\mathrm{res}^{\alpha}_H)b$$

$\square$

Lemma 3.3. $\nu(a)\sigma({}_G\mathrm{ind}_H^\alpha) = \sigma({}_G\mathrm{ind}_H^\alpha)\nu({}_H\mathrm{res}_G^\alpha(a))$

Proof. We will follow the above logic again;

$$\begin{aligned}\nu(a)\sigma({}_G\mathrm{ind}_H^\alpha)b = [V] \otimes_{\mathbb{C}} [{}_G\mathrm{ind}_H^\alpha N] = [{}_G\mathrm{ind}_H^\alpha({}_H\mathrm{res}_G^\alpha V)] \otimes_{\mathbb{C}} [\mathrm{ind}_H^\alpha N] \\ = \sigma({}_G\mathrm{ind}_H^\alpha)\,[{}_H\mathrm{res}_G^\alpha V] \otimes [N] = \sigma({}_G\mathrm{ind}_H^\alpha)\nu({}_H\mathrm{res}_G^\alpha(a))b\end{aligned}$$

$\square$

Lemma 3.4. $\nu({}_G\mathrm{ind}_H^\alpha(b)) = \sigma({}_G\mathrm{ind}_H^\alpha)\nu(b)\sigma({}_H\mathrm{res}_G^\alpha)$

Proof. Let $c = [M]$ be a $\mathbb{C}H-$module. Then

$$\begin{aligned}\nu({}_G\mathrm{ind}_H^\alpha(b))c = [{}_G\mathrm{ind}_H^\alpha N] \otimes [M] = [{}_G\mathrm{ind}_H^\alpha N] \otimes [{}_G\mathrm{ind}_H^\alpha({}_H\mathrm{res}_G^\alpha V)M] \\ = \sigma({}_G\mathrm{ind}_H^\alpha)\nu(b)\sigma({}_H\mathrm{res}_G^\alpha)c\end{aligned}$$

$\square$

Theorem 3.5. *We have $\beta = \Lambda$.*

Proof. From the above lemmas it is evident that any element of β can be transformed into a linear combination of terms having the form $\sigma({}_G\mathrm{ind}_H)\nu(a)\sigma({}_H\mathrm{res}_G)$ which is in Λ. $\square$

3.3 Proving Brauer's Induction Theorem with the help of Λ

Let N be an $\mathbb{C}H$-module also let P be a (G, H)-module. Then there is a character x of $A_{\mathbb{C}}(G, H)$ corresponding to P. Therefore we have the following map;

$$\rho :^{\oplus} A \mapsto E$$

where

$$\rho(P)\,[N] = [{}_{\mathbb{C}G}P \otimes_{\mathbb{C}H} N] \tag{3.1}$$

This defines an action of ${}^{\oplus}A$ on the direct sum $\bigoplus\limits_{G} A_{\mathbb{C}}(G)$. To check whether it is faithful, we need to calculate the character corresponding to the module $[{}_{\mathbb{C}G}P \otimes_{\mathbb{C}H} N]$ which will be done in the next chapter.

Theorem 3.6. $\Lambda = \rho({}^{\oplus}A)$

Proof. Let $\{I_i\}_i$ be the set of elementary subgroups mentioned in the Lemma 2.10 and l_i be the corresponding characters. Then we have;

$$\nu(1_{A_{\mathbb{C}}(G)}) = \nu(\sum_i {}_G\text{ind}_{I_i}(l_i)) = \sum_i \nu({}_G\text{ind}_{I_i}(l_i)) = \sum_i \sigma({}_G\text{ind}^{\varphi}_{I_i})\nu(l_i)\sigma({}_{I_i}\text{res}^{\varphi}_{G})$$

Take $a \in A_{\mathbb{C}}(G)$. Then;

$$\nu(a)\nu(1_{A_{\mathbb{C}}(G)}) = \sum_i \nu(a)\sigma({}_G\text{ind}^{\varphi}_{I_i})\nu(l_i)\sigma({}_{I_i}\text{res}^{\varphi}_{G})$$

Using the Lemma 3.3 this can be transformed into following form:

$$\begin{aligned}\nu(a)\nu(1_{A_{\mathbb{C}}(G)}) &= \sum_i \sigma({}_G\text{ind}^{\varphi}_{I_i})\nu({}_H\text{res}^{\varphi}_{G}(a))\nu(l_i)\sigma({}_{I_i}\text{res}^{\varphi}_{G}) \\ &= \sum_i \sigma({}_G\text{ind}^{\varphi}_{I_i})\nu({}_H\text{res}^{\varphi}_{G}(a)l_i)\sigma({}_{I_i}\text{res}^{\varphi}_{G})\end{aligned}$$

which is in $\Lambda(G, H)$. This shows that image of every character of G under ν is in the algebra $\Lambda(G, H)$. By making the above calculations for all pairs of groups (G, H) we get $\Lambda = \rho({}^{\oplus}A)$ □

Theorem 3.7. *Theorem 3.6 can be used to prove Brauer's Induction Theorem*

Proof. We will obtain Lemma 2.11 from Theorem 3.6

Let a be a character of finite group G. Then we have

$$\nu(a) = \sum_i \sigma({}_G\text{ind}^{\varphi}_{I_i})\nu({}_H\text{res}^{\varphi}_{G}(a)l_i)\sigma({}_{I_i}\text{res}^{\varphi}_{G})$$

From Lemma 3.4 we get

$$\nu(a) = \sum_i \nu({}_G\mathrm{ind}^{\varphi}_{I_i}({}_{I_i}\mathrm{res}^{\varphi}_{G}(a) l_i))$$

By injectivity of ν we have

$$a = \sum_i {}_G\mathrm{ind}^{\varphi}_{I_i}({}_{I_i}\mathrm{res}^{\varphi}_{G}(a) l_i) \tag{3.2}$$

This completes the proof since ${}_{I_i}\mathrm{res}^{\varphi}_{G}(a) l_i$ is a character of the elementary subgroup I_i. $\square$

Chapter 4

The action of $\bigoplus A$ on M

4.1 The character $\chi_{\mathbb{C}G P \otimes_{\mathbb{C}H} N}$ in terms of χ_P and χ_N

Let G be a finite group and $\mathbb{F}$ be an arbitrary field. For $\mathbb{F}G$, define $\mathrm{Aug}(\mathbb{F}G)$ to be $\mathbb{F}G$-module generated by elements of the form $\sum_{g \in G} \lambda_g g$ where $\lambda_g \in \mathbb{F}$ and $\sum_{g \in G} \lambda_g = 0$.

For an $\mathbb{F}G$-module N define $\mathrm{Aug}(N)$ to be the product $\mathrm{Aug}(\mathbb{F}G)N$. Equivalently $\mathrm{Aug}(N)$ is the $\mathbb{F}G$-module generated by elements of the form $(1-g)n$ where $g \in G, n \in N$.

Let $N_G = N/\mathrm{Aug}(N)$. Then by Maschke's theorem we have

$$N = N^G \bigoplus \mathrm{Aug}(N)$$

hence we get $N^G = N_G$.

Definition 4.1. Let G be a finite group and M a $\mathbb{C}G$-module. Then the opposite module M^{op} of M is defined to be the right $\mathbb{C}G$-module such that M^{op} is equal to M as a $\mathbb{C}$ vector space and $mg = g^{-1}m$ for $g \in G$ and $m \in M$.

Now we will make use of above observations to prove following lemma;

Lemma 4.2. *Let P and Q be $\mathbb{C}G$-modules. Then $P^* \otimes_{\mathbb{C}} Q \cong Hom_{\mathbb{C}}(P, Q)$*

Proof. Define a linear map

$$\alpha : P^* \otimes_{\mathbb{C}G} Q \mapsto \mathrm{Hom}_{\mathbb{C}}(P, Q)$$

which is specified as

$$\pi \otimes q \mapsto \varphi_{\pi,q} \in \mathrm{Hom}_{\mathbb{C}}(P, Q)$$

where

$$\varphi_{\pi,q}(p) = \langle \pi \mid p \rangle \, q$$

We must show that α is a map of $\mathbb{C}G$-modules, that is, for $g \in G$, $\alpha(g(\pi \otimes q)) = g \circ \varphi_{\pi,q}$. We have

$$g(\pi \otimes q) = g\pi \otimes gq$$

This has $\varphi_{g\pi,gq}$ as image under α. More explicitly we have

$$\varphi_{g\pi,gq}(p) = \langle g\pi \mid p \rangle \, gq = \left\langle \pi \mid g^{-1}p \right\rangle gq = g \circ \varphi_{\pi,q}(p)$$

This ends the proof since we have shown that α is a map of $\mathbb{C}G$-modules and is injective by definition. □

Lemma 4.3. *Let P and Q be $\mathbb{C}G$-modules. Then $P^{*op} \otimes_{\mathbb{C}G} Q \cong Hom_{\mathbb{C}G}(P, Q)$*

Proof. We have

$$P^{*op} \otimes_{\mathbb{C}G} Q = P^{*op} \otimes_{\mathbb{C}} Q / \left\langle \pi \otimes q - \pi g^{-1} \otimes gq \right\rangle$$

Since P^{op} and P correspond to the same module, we get

$$P^{*op} \otimes_{\mathbb{C}G} Q = (P^* \otimes_{\mathbb{C}} Q)_G = P^* \otimes_{\mathbb{C}} Q / \langle \pi \otimes q - g\pi \otimes gq \rangle$$

Using Mashcke's Theorem obtain

$$(P^* \otimes_{\mathbb{C}} Q)_G = ((P^* \otimes_{\mathbb{C}} Q)^G) = \mathrm{Hom}_{\mathbb{C}G}(P, Q)$$

which completes the proof. □

Lemma 4.4. *Let S and T be simple $\mathbb{C}G$-modules. Then*

$$S^{op}\otimes_{\mathbb{C}G}T \cong \begin{cases} \mathbb{C} & if S^* \cong T \\ 0 & otherwise \end{cases}$$

Proof. Suppose $S^* \ncong T$. Let e_s be the primitive idempotent of $Z(\mathbb{C}G)$(the centre of the group algebra $\mathbb{C}G$) such that for $s \in S$ we have $e_s s = s$. Write

$$e_s = \frac{\dim S}{|G|}\sum_{g\in G}\chi_s(g^{-1})g$$

Then for $s \in S$ and $t \in T$ we have the following;

$$s \otimes_{\mathbb{C}G} t = e_s s \otimes_{\mathbb{C}G} t = (\frac{\dim S}{|G|}\sum_{g\in G}\chi_s(g^{-1})g)s \otimes_{\mathbb{C}G} t$$

Since s is in the opposite module S^{op} this becomes

$$s \otimes_{\mathbb{C}G} t = s(\frac{\dim S}{|G|}\sum_{g\in G}\chi_s(g^{-1})g^{-1}) \otimes_{\mathbb{C}G} t$$

$$= s(\frac{\dim S}{|G|}\sum_{g\in G}\chi_s^*(g)g^{-1}) \otimes_{\mathbb{C}G} t = se_s^* \otimes_{\mathbb{C}G} t = s \otimes_{\mathbb{C}G} e_{s^*}t$$

From the assumption $S^* \ncong T$ we get $s \otimes_{\mathbb{C}G} t = s \otimes_{\mathbb{C}G} 0 = 0$. Now Suppose $S^* \cong T$. Then by the previous lemma we have $S^{op} \otimes_{\mathbb{C}G} T \cong \mathrm{Hom}_{\mathbb{C}G}(S^*, T)$ so from Schur's lemma we get $S^{op} \otimes_{\mathbb{C}G} T \cong \mathbb{C}$. ☐

Theorem 4.5. *Let M be a $\mathbb{C}(G, H)$-module and W a $\mathbb{C}H$-module. Then for the character corresponding to the module $M \otimes_{\mathbb{C}H} W$ we have*

$$\chi_{M\otimes_{\mathbb{C}H}W}(g) = \frac{1}{|H|}\sum_{h\in H}\chi_M(g,h)\chi_W(h)$$

Proof. For some $\mathbb{C}G$-module U and some $\mathbb{C}H$-module V we have $M = U \otimes V$. From lemma 4.4 we get

$$U \otimes V \otimes_{\mathbb{C}H} W = \begin{cases} U \otimes \mathbb{C} & \text{when } V^{*^{op}} \cong W \\ 0 & \text{otherwise} \end{cases}$$

Hence $M\otimes_{\mathbb{C}H}W$ is a $\mathbb{C}G$-module and for $g \in G$

$$\chi_{M\otimes_{\mathbb{C}H}W}(g) = \begin{cases} \chi_U(g) & \text{if } V^{*^{op}} \cong W \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\chi_{M\otimes_{\mathbb{C}H}W}(g) = \chi_U(g)\langle V^{*^{op}}, W\rangle = \chi_U(g)\frac{1}{|H|}\sum_{h\in H}\chi_{V^{*^{op}}}(h)\overline{\chi_W(h)}$$

$$= \chi_U(g)\frac{1}{|H|}\sum_{h\in H}\chi_{V^{op}}(h^{-1})\overline{\chi_W(h)} = \chi_U(g)\frac{1}{|H|}\sum_{h\in H}\chi_V(h)\chi_W(h)$$

$$= \frac{1}{|H|}\sum_{h\in H}\chi_U\chi_V(h)\chi_W(h) = \frac{1}{|H|}\sum_{h\in H}\chi_{U\times V}(g,h)\chi_W(h) = \frac{1}{|H|}\sum_{h\in H}\chi_M(g,h)\chi_W(h)$$

which finishes the proof. □

4.2 Injectivity of ρ

To check if its possible to replace $\rho({}^{\oplus}A)$ with ${}^{\oplus}A$, we must show that both algebras have the same dimension. The logic we follow would be to investigate whether the map ρ is injective.

Theorem 4.6. *The map ρ is injective.*

Proof. Let $P, Q \in {}^{\oplus}A$ be arbitrary $\mathbb{C}(G,H)$-modules. Then for an H-module N we have the following action;

$$\rho({}_{\mathbb{C}G}P_{\mathbb{C}H}).[N]$$

by 3.1 it becomes

$$={}_{\mathbb{C}G}P_{\mathbb{C}H}\otimes_{\mathbb{C}H}N$$

Showing ρ is injective is equivalent with proving that there is some $\mathbb{C}H$-module N such that

$${}_{\mathbb{C}G}P_{\mathbb{C}H}\otimes_{\mathbb{C}H}N \ncong {}_{\mathbb{C}G}Q_{\mathbb{C}H}\otimes_{\mathbb{C}H}N$$

Equivalently for some $g \in G$

$$\chi_{P \otimes_{\mathbb{C}H} N}(g) \neq \chi_{Q \otimes_{\mathbb{C}H} N}(g)$$

From theorem 4.5 the character $\chi_{P \otimes_{\mathbb{C}H} N}(g)$ is equal to

$$\frac{1}{|H|} \sum_{h \in H} \chi_P(g,h)\chi_N(h)$$

Suppose for every $g \in G$ and every $\mathbb{C}H$-module N we have

$$\frac{1}{|H|} \sum_{h \in H} \chi_P(g,h)\chi_N(h) = \frac{1}{|H|} \sum_{h \in H} \chi_Q(g,h)\chi_N(h)$$

or

$$\frac{1}{|H|} \sum_{h \in H} (\chi_P(g,h) - \chi_Q(g,h))\chi_N(h) = 0$$

Let the number of conjugacy classes of H be l. Then treating $(\chi_P(g,h) - \chi_Q(g,h))$ in above equality as unknowns we get a system of l linear equations with l unknowns, which must have unique solution. Hence we must have

$$(\chi_P(g,h) - \chi_Q(g,h)) = 0$$

So the assumption holds only if P is isomorphic to Q. Therefore we can derive the result that ρ is injective. $\square$

Therefore;

Theorem 4.7. $\Lambda \cong {}^{\oplus}A$

Which means that we manage to give three different descriptions for the algebra ${}^{\oplus}A$. Firstly we have ${}^{\oplus}A = \bigoplus_{G,H \in \mathcal{K}} A(G,H)$. Secondly, from above theorem we get ${}^{\oplus}A = \bigoplus_{G,H \in \mathcal{K}} \Lambda(G,H)$, where $\Lambda(G,H)$ is generated by linear combinations of the terms of the form $\sigma({}_G\mathrm{ind}_H)\nu(a)\sigma({}_H\mathrm{res}_G)$. Lastly, in 3.6 we proved that $\Lambda = \rho({}^{\oplus}A)$, therefore ${}^{\oplus}A$ can also be seen as an algebra on E generated by the elements in the image of ρ where $\rho(P)\,[N] = [{}_{\mathbb{C}G}P \otimes_{\mathbb{C}H} N]$ for a $\mathbb{C}H$-module N is and $P \in {}^{\oplus}A$.

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